ESTIMATING THE ARCH PARAMETERS BY SOLVING LINEAR EQUATIONS

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Abstract. This paper discusses the asymptotics of two-stage least squares estimator of the parameters of ARCH models. The estimator is easy to obtain since it involves solving two sets of linear equations. At the same time, the estimator has the same asymptotic efficiency as that of the widely used quasi-maximum likelihood estimator. Simulation results show that, even for small sample size, the performance of our estimator compared to the quasi-maximum likelihood estimator is better.

Keywords. Quasi maximum likelihood estimation; ARCH models; stationary and ergodic process; Martingale central limit theorem.

1. INTRODUCTION

Recent years have seen an exponential growth in the applications of autoregressive conditional heteroscedastic (ARCH) models to economics and finance. Introduced in his seminal paper by Engle (1982), ARCH models have been applied to numerous economic and financial data to model the volatility; the strong dependence of the instantaneous variability of a time series on its own past. A survey of the already vast literature can be had from the survey papers of Bollerslev *et al.* (1992), Bera and Higgins (1993), Bollerslev *et al.* (1994), Shephard (1996) and the book by Gouriéroux (1997), among others.

Several different models in the ARCH literature are all known as ARCH models. In this article, by an ARCH model, we mean the linear ARCH model of order p ($p \geqslant 1$) introduced by Engle (1982). Here one observes $\{X_i; 1 - p \leqslant i \leqslant n\}$ satisfying

$$X_i = \sigma_{i-1}(\beta)\epsilon_i \qquad 1 \leqslant i \leqslant n \tag{1}$$

where $\beta' = [\beta_0, \beta_1, \cdots \beta_p]$ is the unknown parameter with $\beta_0 > 0$, $\beta_j \geqslant 0, 1 \leqslant j \leqslant p$; $\sigma_{i-1}(\beta) = \{\beta_0 + \beta_1 X_{i-1}^2 + \cdots + \beta_p X_{i-p}^2\}^{1/2}$; $\{\epsilon_i; 1 \leqslant i \leqslant n\}$ are independent and identically distributed (i.i.d) with zero mean, unit variance and finite fourth moment; and $\{\epsilon_i; 1 \leqslant i \leqslant n\}$ are independent of $\{X_i; 1 - p \leqslant i \leqslant 0\}$.

In this paper, we are concerned with the problem of estimating β. Here, β can be thought as a parameter associated with variance. For another type of autoregression model with ARCH errors, see Pantula (1988) where estimation of the autoregressive parameters is considered.

Standard assumptions which facilitate the derivation of asymptotic properties of estimators are the stationarity and ergodicity of the process. We also assume that the process $\{X_i; 1 - p \leqslant i\}$ is *stationary* and *ergodic*. When $p = 1$, it follows from Nelson (1990) that a sufficient condition for the stationarity is $E\{\log(\beta_1 \epsilon_1^2)\} < 0$. For general p, a sufficient condition for stationarity and ergodicity of the process based on Lyapunov exponent is given by Bougerol and Picard (1992, Thm 1.3).

One of the most commonly used estimation procedures for ARCH models is the Gaussian likelihood approach. In this approach, the estimator is obtained as a maximizer of the logarithm of a Gaussian likelihood function. The resulting estimator is called the quasi-maximum likelihood estimator (QMLE). This yields a consistent estimator even when the conditional error density is nonnormal. The consistency and asymptotic normality of the QMLE was established by Weiss (1986). It is also known to be asymptotically fully efficient when the error distribution is Gaussian.

However, the QMLE does not admit a closed-form expression. It is also not computationally easy to obtain. The likelihood tends to be flat unless n is very large. A discussion of this problem can be found in Shephard (1996). As is the case with any estimator defined through an optimization problem, one needs to specify a definite rule to choose the estimator in case of multiple solutions.

The purpose of this note is to propose an estimator of the ARCH parameter which has a closed-form expression, which is computationally easy and which, at the same time, compares favourably with the QMLE. The computations involve solving two sets of linear equations and there is no nonlinear optimization to be performed. This estimator turns out to have the same limiting dispersion as the QMLE and hence is also fully efficient when the error distribution is Gaussian. This is not so surprising since the standard theory would suggest that one step of a quasi-Newton method from a consistent estimator would deliver the same result. Interestingly, simulation results show that our estimator performs better than the QMLE even for small sample size such as $n = 30$ when the errors are standard normal. This is even true when the error distribution is normalized (to have zero mean unit variance) double exponential or t-distribution although the simulation results are exhibited only for standard normal error.

2. THE ESTIMATOR AND ITS ASYMPTOTIC DISTRIBUTION

To motivate the estimator, let $Y_i = X_i^2, 1 - p \leqslant i \leqslant n$,

$$\mathbf{Z}_{i-1} = [1, Y_{i-1}, \ldots, Y_{i-p}]' = [1, X_{i-1}^2, \ldots, X_{i-p}^2]'$$

and $\eta_i = \epsilon_i^2 - 1$, $1 \leqslant i \leqslant n$. Then

$$\sigma_{i-1}^2(\beta) = \mathbf{Z}_{i-1}'\beta \tag{2}$$

and squaring (1) and using the form of $\sigma_{i-1}^2(\beta)$ in (2), gives

$$Y_i = \mathbf{Z}_{i-1}'\beta + \sigma_{i-1}^2(\beta)\eta_i \qquad 1 \leqslant i \leqslant n \tag{3}$$

where

$$\mathrm{E}\{\sigma_{i-1}^2(\beta)\eta_i\} = \mathrm{E}\{\sigma_{i-1}^2(\beta)\}\mathrm{E}(\eta_i) = 0 \qquad 1 \leqslant i \leqslant n.$$

Equation (3) is similar to a standard linear autoregressive model with centred errors, except that there is a multiplicative random scaling involved in the errors. Ignoring the randomness of $\sigma_{i-1}^2(\beta)$ and also the presence of β in it, one can obtain a (preliminary) least squares estimator of β as

$$\hat{\boldsymbol{\beta}}_{\mathrm{pr}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} \tag{4}$$

where $\mathbf{Z}$ is the matrix of order $n \times (1 + p)$ with ith row equal to $\mathbf{Z}_{i-1}'$ and $\mathbf{Y}$ is the vector with ith entry Y_i, $1 \leqslant i \leqslant n$. Note that this estimator does not take into account the heterogeneity in the errors and, hence, serves only as a preliminary estimate. The asymptotic distribution of the estimator can be easily derived by using an appropriate central limit theorem and, as expected, its efficiency is smaller than that of the QMLE.

Next we use $\hat{\boldsymbol{\beta}}_{\mathrm{pr}}$ defined in (4) to construct an improved estimator $\hat{\boldsymbol{\beta}}$ of β as follows. Note that dividing (3) by $\sigma_{i-1}^2(\beta)$, we obtain

$$\frac{Y_i}{\sigma_{i-1}^2(\beta)} = \left\{\frac{\mathbf{Z}_{i-1}}{\sigma_{i-1}^2(\beta)}\right\}'\beta + \eta_i.$$

In this expression, if we replace $\sigma_{i-1}^2(\beta)$ by $\sigma_{i-1}^2(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})$, we obtain

$$\frac{Y_i}{\sigma_{i-1}^2(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})} \approx \left\{\frac{\mathbf{Z}_{i-1}}{\sigma_{i-1}^2(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})}\right\}'\beta + \eta_i.$$

If we again ignore the randomness present in $\sigma_{i-1}^2(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})$, then this can now be visualized as a standard linear autoregression model with homoscedastic errors. Thus, we estimate β by the least squares method, yielding the estimator

$$\hat{\boldsymbol{\beta}} = \left[\sum_{i=1}^{n}\left\{\frac{\mathbf{Z}_{i-1}\mathbf{Z}_{i-1}'}{\sigma_{i-1}^4(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})}\right\}\right]^{-1}\left[\sum_{i=1}^{n}\left\{\frac{\mathbf{Z}_{i-1}Y_i}{\sigma_{i-1}^4(\hat{\boldsymbol{\beta}}_{\mathrm{pr}})}\right\}\right].$$

Recall that we have assumed that the process is stationary and ergodic. First, we shall state a Lemma which deals with $\hat{\boldsymbol{\beta}}_{\rm pr}$ given in (4). Here we assume the moment conditions: for all $1 \leqslant j, k, l, m \leqslant p$,

$$\mathrm{E}\{Y_j Y_k Y_l Y_m\} < \infty. \tag{5}$$

Note that (5) ensures that $\mathrm{E}(\mathbf{Z}_0 \mathbf{Z}_0')$ and $\mathrm{E}\{(\beta' \mathbf{Z}_0)^2 \mathbf{Z}_0 \mathbf{Z}_0'\}$ are all finite. When the errors are standard normal, sufficient condition for the existence of higher moments of Ys in terms of the parameter β is given by Engle (1982, Thms 1 and 2).

Lemma. *In addition to the assumptions of model* (1), *assume that* (5) *holds. Then*

$$n^{1/2}(\hat{\boldsymbol{\beta}}_{\rm pr} - \beta) \to \mathbf{N}\Big[0, \mathrm{Var}(\epsilon_1^2)\{\mathrm{E}(\mathbf{Z}_0\mathbf{Z}_0')\}^{-1}\mathrm{E}\{(\beta'(\mathbf{Z}_0)^2\mathbf{Z}_0\mathbf{Z}_0'\}\{\mathrm{E}(\mathbf{Z}_0\mathbf{Z}_0')\}^{-1}\Big] \tag{6}$$

in distribution.

The following theorem gives the asymptotic distribution of $\hat{\boldsymbol{\beta}}$ which is the main result of the paper. Here, we assume only this property of $\hat{\boldsymbol{\beta}}_{\rm pr}$:

$$n^{1/2}(\hat{\boldsymbol{\beta}}_{\rm pr} - \beta) = O_p(1) \tag{7}$$

and this moment property of $\{X_i; 1-p \leqslant i \leqslant 0\}$: for all $1 \leqslant j, k, l \leqslant p$,

$$\mathrm{E}\left\{\frac{Y_{-j}Y_{-k}Y_{-l}}{(\beta'\mathbf{Z}_0)^3}\right\} < \infty. \tag{8}$$

Note that the conditions of the above Lemma implies (7). Also note that in the asymptotics of the QMLE, Weiss (1986) assumes that for all $1 \leqslant j, k \leqslant p$, $\mathrm{E}\{Y_{-j}Y_{-k}/(\beta'\mathbf{Z}_0)^2\} < \infty$. When $\beta_j > 0 \ \forall\ 1 \leqslant j \leqslant p$, (8) is automatically satisfied since the functions $y \to y/(\beta_0 + \beta_j y)$ are bounded. The proofs for the Lemma and the Theorem are given in Section 4.

Theorem. *In addition to the assumptions of model (1), suppose that (7) and (8) hold. Then, in distribution,*

$$n^{1/2}(\hat{\boldsymbol{\beta}} - \beta) \to \mathbf{N}\big[0, \mathrm{Var}(\epsilon_1^2)\big\{\mathrm{E}\{\mathbf{Z}_0\mathbf{Z}_0'(\beta'\mathbf{Z}_0)^{-2}\}\big\}^{-1}\big]. \tag{9}$$

3. PERFORMANCE COMPARISON ON SIMULATED DATA

Table I presents some simulation results for the ARCH model in (1) of order $p = 1$ with standard normal error. Here, for each of the tth trial $(1 \leqslant t \leqslant 10)$, a particular pair of (β_0, β_1) is randomly selected from χ^2 distribution and fixed throughout that trial. Then, at the kth iteration where $1 \leqslant k \leqslant 1000$,

TABLE I

VALUES OF S_0, S_1 S_{0Q} AND S_{1Q} WITH STANDARD NORMAL ERRORS

Trial	β_0	β_1	S_0	S_{0Q}	S_1	S_{1Q}
1	0.35463	0.11346	0.01159	0.01649	0.04819	0.13894
2	0.69867	0.38751	0.07856	0.08463	0.04429	0.07058
3	1.82138	0.04873	0.28108	0.36670	0.02815	0.07021
4	0.72256	0.25672	0.06668	0.06757	0.03894	0.07285
5	1.10568	0.27014	0.14951	0.17849	0.03318	0.08187
6	0.89437	0.41837	0.10735	0.11150	0.05529	0.11589
7	1.18623	0.40427	0.30077	0.31742	0.05844	0.10209
8	0.61056	0.25444	0.04714	0.06219	0.03584	0.04743
9	0.84292	0.02405	0.06183	0.07347	0.03721	0.12475
10	1.83434	0.29898	0.43420	0.60428	0.04377	0.09289

$\{X_i; 0 \leqslant i \leqslant n = 30\}$ are observed and, based on them, estimators of (β_0, β_1) are calculated using both linear equation method and QML method. Let

$$S_0 = \sum_{i=1}^{1000} \frac{\left(\beta_0 - \hat{\boldsymbol{\beta}}_{0k}\right)^2}{1000}$$

and

$$S_1 = \sum_{k=1}^{1000} \frac{\left(\beta_1 - \hat{\boldsymbol{\beta}}_{1k}\right)^2}{1000}$$

where the estimators $\hat{\boldsymbol{\beta}}_{0k}$ and $\hat{\boldsymbol{\beta}}_{1k}$ are obtained through the linear equation method at the kth iteration. Similarly, S_{0Q} and S_{1Q} are defined when the estimators are obtained through the QML method. As seen from Table I, even with a sample of size as small as 30 and standard normal errors, for each trial with randomly chosen parameter, $S_0 < S_{0Q}$ and $S_1 < S_{1Q}$. Thus, the linear equation method performs better even for small sample size. The entire program is written in Splus.

4. THE PROOFS

In this section we give complete proofs of the Lemma and the Theorem.

PROOF OF LEMMA. From (3) and (4),

$$n^{1/2}(\hat{\boldsymbol{\beta}}_{\text{pr}} - \beta) = \left[n^{-1} \sum_{i=1}^{n} \mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}\right]^{-1} \left[n^{-1/2} \sum_{i=1}^{n} \mathbf{Z}_{i-1}\sigma^2_{i-1}(\beta)\eta_i\right]. \qquad (10)$$

By the stationarity and ergodicity, $n^{-1}\sum_{i=1}^{n}\mathbf{Z}_{i-1}\mathbf{Z}_{i-1}'$ converges almost surely to $\mathrm{E}(\mathbf{Z}_0\mathbf{Z}_0')$ which is finite by (5).

To conclude that the sum $\sum_{i=1}^{n} n^{-1/2}\mathbf{Z}_{i-1}\sigma_{i-1}^2(\beta)\eta_i$ converges to a normal vector with mean zero and dispersion $\mathrm{Var}(\eta_1)\mathrm{E}\{\sigma_0^4(\beta)\mathbf{Z}_0\mathbf{Z}_0'\}$, it is enough to show that $\forall l \in \mathbb{R}^p - \{0\}$,

$$n^{-1/2}\sum_{i=1}^{n} l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\eta_i \to \mathrm{N}\Big[0, \mathrm{Var}(\eta_1)\mathrm{E}\{(l'\mathbf{Z}_0)^2(\beta'\mathbf{Z}_0)^2\}\Big]. \tag{11}$$

To show this, we verify that the conditions of the Martingale CLT (Hall and Heyde, 1980, Cor. 3.1) are satisfied.

Consider the increasing sequence of sigma-fields $\{F_i = \sigma < \mathbf{Z}_0, \ldots, \mathbf{Z}_i >; i \geqslant 0\}$. Then $\{\mathbf{Z}_{i-1}\sigma_{i-1}^2(\beta)\eta_i, F_i; i \geqslant 1\}$ is a martingale difference sequence.

Now we verify the conditional Lindeberg condition only since the other conditions can be verified easily. Let E_{i-1} denote the conditional expectation with respect to

$$F_{i-1} = \sigma\langle \mathbf{Z}_0, \ldots, \mathbf{Z}_{i-1}\rangle \qquad i \geqslant 1.$$

We show that $\forall \epsilon > 0$,

$$\begin{aligned} L_n(\epsilon) &= n^{-1}\sum_{i=1}^{n}\{l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\}^2 E_{i-1}\eta_i^2 I(|\eta_i l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}| > n^{1/2}\epsilon^2) \\ &= o_p(1). \end{aligned} \tag{12}$$

Note that

$$\begin{aligned} &L_n(\epsilon) \\ &\leqslant n^{-1}\sum_{i=1}^{n}\{l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\}^2 E_{i-1}[\eta_i^2\{I(|\eta_i| > n^{1/4}\epsilon) + I(|l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}| > n^{1/4}\epsilon)\}] \\ &= n^{-1}\sum_{i=1}^{n}\{l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\}^2 E\{\eta_1^2 I(|\eta_1| > n^{1/4}\epsilon)\} \\ &\quad + E(\eta_1^2)n^{-1}\sum_{i=1}^{n}\{l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\}^2 I(|l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}| > n^{1/4}\epsilon) \\ &= S_1 + E(\eta_1^2)S_2, \text{ say.} \end{aligned}$$

Since

$$n^{-1}\sum_{i=1}^{n}\{l'\mathbf{Z}_{i-1}\beta'\mathbf{Z}_{i-1}\}^2 = \mathrm{E}\{l'\mathbf{Z}_0\beta'\mathbf{Z}_0\}^2 + o_p(1)$$

and $\mathrm{E}\eta_1^2 < \infty$, we get $S_1 = o_p(1)$. Observe that $S_2 \geqslant 0$ with

$$\mathrm{E}S_2 = \mathrm{E}\{l'\mathbf{Z}_0\beta'\mathbf{Z}_0\}^2 I(|l'\mathbf{Z}_0\beta'\mathbf{Z}_0| > n^{1/4}\epsilon)\} = o(1)$$

by (5). Hence (12) is satisfied and (11) is established.

PROOF OF THEOREM. As in (10),

$$n^{1/2}(\hat{\beta}-\beta) = \left[n^{-1}\sum_{i=1}^{n}\frac{\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}}{\sigma^4_{i-1}(\hat{\beta}_{\mathrm{pr}})}\right]^{-1}\left[n^{-1/2}\sum_{i=1}^{n}\frac{\mathbf{Z}_{i-1}\sigma^2_{i-1}(\beta)\eta_i}{\sigma^4_{i-1}(\hat{\beta}_{\mathrm{pr}})}\right].$$

Next we show that

$$n^{-1}\sum_{i=1}^{n}\left\{\frac{1}{\sigma^4_{i-1}(\hat{\beta}_{\mathrm{pr}})}-\frac{1}{\sigma^4_{i-1}(\beta)}\right\}\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1} = o_p(1) \tag{13}$$

and

$$n^{-1/2}\sum_{i=1}^{n}\sigma^2_{i-1}(\beta)\left\{\frac{1}{\sigma^4_{i-1}(\hat{\beta}_{\mathrm{pr}})}-\frac{1}{\sigma^4_{i-1}(\beta)}\right\}\mathbf{Z}_{i-1}\eta_i = o_p(1). \tag{14}$$

Then it will follow that

$$n^{1/2}(\hat{\beta}-\beta) = \left[n^{-1}\sum_{i=1}^{n}\frac{\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}}{\sigma^4_{i-1}(\beta)}\right]^{-1}\left[n^{-1/2}\sum_{i=1}^{n}\frac{\mathbf{Z}_{i-1}\sigma^2_{i-1}(\beta)\eta_i}{\sigma^4_{i-1}(\beta)}\right] + o_p(1)$$

and hence, again by the Martingale CLT, (9) will follow similar to (6).

To prove (13) and (14), we use

(i) the mean value theorem where for $u, v > 0$,

$$\frac{1}{u^2}-\frac{1}{v^2}=\frac{-2(u-v)}{\chi^3} \tag{15}$$

with χ satisfying $0 < 1/\chi \leqslant (1/v)\{1+(v/u)\}$;

(ii) the two-step Taylor's formula where for $u, v > 0$,

$$\frac{1}{u^2}-\frac{1}{v^2}=\frac{-2(u-v)}{v^3}+\frac{3(u-v)^2}{\xi^4} \tag{16}$$

with ξ satisfying $0 < 1/\xi \leqslant (1/v)\{1+(v/u)\}$; and

(iii) if $\mathbf{U}=[u_1,\ldots,u_k]'$, $\mathbf{V}=[v_1,\ldots,v_k]'$ and $\mathbf{W}$ are vectors with all entries nonnegative and $k \geqslant 1$ is an integer then (by induction on k)

$$\frac{\mathbf{W}'\mathbf{V}}{\mathbf{W}'\mathbf{U}} \leqslant 1+\frac{v_1}{u_1}+\cdots+\frac{v_k}{u_k} \tag{17}$$

where we define $v_j/u_j = 0$ if $u_j = 0 = v_j$.

In particular, when (15) and (16) are used with $u=\hat{\beta}'_{\mathrm{pr}}\mathbf{Z}_{i-1}$ and $v=\beta'\mathbf{Z}_{i-1}$, then, by (17), the intermediate points χ_{in} or ξ_{in} satisfy

$$
\begin{aligned}
\frac{1}{\chi_{\text{in}}} &\leqslant \frac{1}{\mathbf{Z}'_{i-1}\beta}\left[1 + \frac{\mathbf{Z}'_{i-1}\beta}{\mathbf{Z}'_{i-1}\hat{\beta}_{\text{pr}}}\right] \\
&\leqslant \frac{1}{\mathbf{Z}'_{i-1}\beta}\left[1 + \left\{1 + \frac{\beta_0}{\hat{\beta}_{0\text{pr}}} + \cdots + \frac{\beta_p}{\hat{\beta}_{p\text{pr}}}\right\}\right]
\end{aligned} \tag{18}
$$

where $\hat{\beta}_{j\text{pr}}$ is the jth entry of $\hat{\beta}_{\text{pr}}, 0 \leqslant j \leqslant p$ and by assumption (7),

$$
\left[1 + \left\{1 + \frac{\beta_0}{\hat{\beta}_{0\text{pr}}} + \cdots + \frac{\beta_p}{\hat{\beta}_{p\text{pr}}}\right\}\right] = O_p(1).
$$

Write $\mathbf{B}_n = [b_{n0}, \ldots, b_{np}]' = n^{1/2}(\hat{\beta}_{\text{pr}} - \beta) = O_p(1)$. For proving (13), note that by (15)

$$
\begin{aligned}
& n^{-1}\sum_{i=1}^{n}\left\{\frac{1}{(\mathbf{Z}'_{i-1}\hat{\beta}_{\text{pr}})^2} - \frac{1}{(\mathbf{Z}'_{i-1}\beta)^2}\right\}\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1} \\
&= -2n^{-3/2}\sum_{i=1}^{n} n^{1/2}(\hat{\beta}_{\text{pr}} - \beta)'\frac{\mathbf{Z}_{i-1}\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}}{\chi^3_{i,n}} \\
&= -2b_{n0}n^{-3/2}\sum_{i=1}^{n}\frac{\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}}{\chi^3_{i,n}} - 2\sum_{j=1}^{p} b_{nj}n^{-3/2}\sum_{i=1}^{n}\frac{Y_{i-j}\mathbf{Z}_{i-1}\mathbf{Z}'_{i-1}}{\chi^3_{i,n}} \\
&= -2\mathbf{T}_1 - 2\mathbf{T}_2, \text{say.}
\end{aligned}
$$

Since from (18),

$$
\frac{1}{\chi^3_{i,n}} \leqslant \beta_0^{-3}\left[1 + \left\{1 + \frac{\beta_0}{\hat{\beta}_{0\text{pr}}} + \cdots + \frac{\beta_p}{\hat{\beta}_{p\text{pr}}}\right\}\right]^3
$$

which is free from 'i' and is $O_p(1)$ and the $\{\mathbf{Z}_{i-1}\}$ are stationary, $\mathbf{T}_1 = o_p(1)$.

For $\mathbf{T}_2$, note that by (18), a typical (l, k)th entry inside the summation with respect to 'i' is

$$
\begin{aligned}
\sum_{i=1}^{n}\frac{Y_{i-j}Y_{i-k}Y_{i-l}}{\chi^3_{\text{in}}} &\leqslant \sum_{i=1}^{n}\left\{\frac{Y_{i-j}Y_{i-k}Y_{i-l}}{(\mathbf{Z}'_{i-1}\beta)^3}\right\}\left[1 + \left\{\frac{\mathbf{Z}'_{i-1}\beta}{\mathbf{Z}'_{i-1}\hat{\beta}_{\text{pr}}}\right\}\right]^3 \\
&\leqslant \left[1 + \left\{1 + \frac{\beta_0}{\hat{\beta}_{0\text{pr}}} + \cdots + \frac{\beta_p}{\hat{\beta}_{p\text{pr}}}\right\}\right]^3\sum_{i=1}^{n}\frac{Y_{i-j}Y_{i-k}Y_{i-l}}{(\mathbf{Z}'_{i-1}\beta)^3}.
\end{aligned}
$$

Since

$$
n^{-3/2}\mathrm{E}\left\{\sum_{i=1}^{n}\frac{Y_{i-j}Y_{i-k}Y_{i-l}}{(\mathbf{Z}'_{i-1}\beta)^3}\right\} = o(1)
$$

we get $\mathbf{T}_2 = o_p(1)$.

For proving (14) note that by (16),

$$\begin{aligned} n^{-1/2}\sum_{i=1}^{n}\sigma_{i-1}^{2}(\beta)&\left\{\frac{1}{(\mathbf{Z}_{i-1}'\hat{\beta}_{\mathrm{pr}})^{2}}-\frac{1}{(\mathbf{Z}_{i-1}'\beta)^{2}}\right\}\mathbf{Z}_{i-1}\eta_{i}\\ &=-2n^{-1}\sum_{i=1}^{n}\sigma_{i-1}^{2}(\beta)n^{1/2}(\hat{\beta}_{\mathrm{pr}}-\beta)'\frac{\mathbf{Z}_{i-1}\mathbf{Z}_{i-1}\eta_{i}}{(\mathbf{Z}_{i-1}'\beta)^{3}}\\ &\quad+3n^{-3/2}\sum_{i=1}^{n}\sigma_{i-1}^{2}(\beta)\{n^{1/2}(\hat{\beta}_{\mathrm{pr}}-\beta)'\mathbf{Z}_{i-1}\}^{2}\frac{\mathbf{Z}_{i-1}\eta_{i}}{\xi_{i,n}^{4}}\\ &=-2\mathbf{T}_{3}+3\mathbf{T}_{4},\text{say}. \end{aligned}$$

Since $\mathrm{E}(\eta_1)=0$, using techniques similar to the proof of $\mathbf{T}_1=o_p(1)$ and $\mathbf{T}_2=o_p(1)$, we get $\mathbf{T}_3=o_p(1)$.

Write $\mathbf{T}_4=\mathbf{T}_{41}+\mathbf{T}_{42}$ where $\mathbf{T}_{41}$ is the sum over all those j and k such that at least one of j and k equals zero and

$$\mathbf{T}_{42}=\sum_{j=1}^{p}\sum_{k=1}^{p}b_{nj}b_{nk}n^{-3/2}\sum_{i=1}^{n}\frac{Y_{i-j}Y_{i-k}\mathbf{Z}_{i-1}\eta_{i}\mathbf{Z}_{i-1}'\beta}{\xi_{\mathrm{in}}^{4}}.$$

Similar to the proof of $\mathbf{T}_1=o_p(1)$, we get $\mathbf{T}_{41}=o_p(1)$.

To handle $\mathbf{T}_{42}$, note that by (18), the absolute value of a typical lth entry involving the sum with respect to 'i' of the vector $\mathbf{T}_{42}$ is bounded above by

$$O_p(1)\times n^{-3/2}\sum_{i=1}^{n}\frac{Y_{i-j}Y_{i-k}Y_{i-l}|\eta_i|}{(\mathbf{Z}_{i-1}'\beta)^{3}}$$

and

$$\begin{aligned} n^{-3/2}\sum_{i=1}^{n}\mathrm{E}&\left\{\frac{Y_{i-j}Y_{i-k}Y_{i-l}|\eta_i|}{(\mathbf{Z}_{i-1}'\beta)^{3}}\right\}\\ &=\mathrm{E}(|\eta_1|)n^{-1/2}\mathrm{E}\left\{\frac{Y_{-j}Y_{-k}Y_{-l}}{(\mathbf{Z}_0'\beta)^{3}}\right\}\\ &=o(1) \end{aligned}$$

by (8). Hence $\mathbf{T}_{42}=o_p(1)$.

This completes the proof of the Theorem.

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