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ESTIMATION BY TWO-MOMENTS METHOD FOR GENERALIZED POWER SERIES DISTRIBUTION AND CERTAIN APPLICATIONS

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SUMMARY. In this paper, the 'Two-Moments Method' of estimation of the parameter of a generalized power series distribution is presented. The important properties of this method of estimation are discussed. Some special applications are studied.

1. INTRODUCTION

Let T be an arbitrary non-null subset of non-negative integers and define the generating function

$$f(\theta) = \sum_{x \epsilon T} a_x \theta^x \qquad \ldots \quad (1.1)$$

with $a_x \geqslant 0$, $\theta \geqslant 0$, so that $f(\theta)$ is positive, finite and differentiable.

Then we can define a random variable X taking non-negative integral values in T with probabilities

$$P_x = \text{prob } \{X = x\} = \frac{a_x \theta^x}{f(\theta)} \quad x \epsilon T \qquad \ldots \quad (1.2)$$

and call this distribution a generalized power series distribution (gpsd). We add here that we call the set of admissible values of the parameter θ of gpsd as the parameter space Θ of the gpsd. Also, we refer to set T of values of random variable X defined by the gpsd, as the range T of the gpsd.

The author (1959, 1961) has shown that the standard discrete distributions like the Binomial, Poisson, Negative Binomial and Logarithmic Series distributions can be obtained as special cases of the gpsd by proper choice of the range T and the generating function $f(\theta)$. The method of maximum likelihood for estimation on the basis of samples from gpsd's which are either complete, truncated or censored has also been discussed. In this paper, we suggest what we call "Two-Moments Method" for estimation of the parameter of the gpsd and investigate its important properties and study certain applications.

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2. Estimation by two-moments method for a gpsd

Range T consisting of consecutive integers with positive probabilities. Consider the gpsd (1.2) with finite or infinite range $T = (c, c+1, \ldots, d)$ with positive probabilities; that is, consider the gpsd

$$P_x = \text{prob}\{X=x\} = \frac{a_x\theta^x}{f(\theta)} \quad \ldots \quad (2.1)$$

where $x \epsilon T = (c, c+1, \ldots, d)$, d finite or infinite

$$f(\theta) = \sum_{x=c}^{d} a_x\theta^x \quad a_x > 0. \quad \ldots \quad (2.2)$$

For gpsd (2.1), it is easy to see that

$$\mu = \theta G_{01}+cP_c \quad \ldots \quad (2.3)$$

and

$$m_2 = \mu+\theta G_{11}+c(c-1)P_c \quad \ldots \quad (2.4)$$

where

$$G_{ij} = \sum_{x=c}^{d-1} x^i \left[\frac{(x+1)a_{x+1}}{a_x} \right]^j P_x. \quad \ldots \quad (2.5)$$

Further, from (2.3) and (2.4) we have

$$\frac{m_2-\mu-\theta G_{11}}{\mu-\theta G_{01}} = c-1 \quad \text{when } c \neq 0 \quad \ldots \quad (2.6)$$

which when solved for θ gives the identity

$$\theta = \frac{m_2-c\mu}{G_{11}-(c-1)G_{01}} \quad \text{when } c \neq 0. \quad \ldots \quad (2.7)$$

From (2.3), we have the identity

$$\theta = \frac{\mu}{G_{01}} \quad \text{when} \quad c = 0. \quad \ldots \quad (2.8)$$

The identities (2.7) and (2.8) can be made use of in estimating θ. One has only to compute

$$S_i = \sum_{x=c}^{d} x^i n_x \qquad i = 1, 2 \quad \ldots \quad (2.9)$$

and

$$g_{ij} = \sum_{x=c}^{d-1} x^i \left[\frac{(x+1)a_{x+1}}{a_x} \right]^j n_x \cdot i = 0, 1;\ j = 1 \quad \ldots \quad (2.10)$$

from the sample, and then

$$t = \frac{S_2-cS_1}{g_{11}-(c-1)g_{01}} \quad \text{when } c \neq 0 \quad \ldots \quad (2.11)$$

or

$$t = \frac{S_1}{g_{01}} \qquad \text{when } c = 0, \quad \ldots \quad (2.12)$$

can be taken as an estimate for θ. Because we use the first two moments for the estimation of the single parameter, we call the estimate t as "two-moments estimate" and the method as "two-moments method."

Proceeding along the same lines as Patil (1961) one gets to terms of order $\frac{1}{N}$,

$$b(t) = E(t)-\theta = \frac{1}{NG^2}\,(\theta\sigma_{22}-\sigma_{12}) \qquad \ldots \quad (2.13)$$

and

$$\text{var}\,(t) = \frac{1}{NG^2}\,[\sigma_{11}-2\theta\,\sigma_{12}+\theta^2\,\sigma_{22}] \qquad \ldots \quad (2.14)$$

where

(i) for $c \neq 0$,

$$G = G_{11}-(c-1)G_{01}$$

$$\sigma_{11} = (m_4-m_2^2)+c^2(m_2-\mu^2)-2c(m_3-\mu m_2)$$

$$\sigma_{12} = (G_{31}-m_2G_{11})-c(G_{21}-\mu G_{11})-(c-1)(G_{21}-m_2G_{01})+c(c-1)(G_{11}-\mu G_{01})$$

$$\sigma_{22} = (G_{22}-G_{11}^2)+(c-1)^2(G_{02}-G_{01}^2)-2(c-1)(G_{12}-G_{01}G_{11})$$

and

(ii) for $c = 0$,

$$G = G_{01}$$

$$\sigma_{11} = m_2-\mu^2$$

$$\sigma_{12} = G_{11}-\mu G_{01}$$

$$\sigma_{22} = G_{02}-G_{01}^2.$$

3. Estimation by the two-moments method for a truncated gpsd*

Consider the gpsd (2.1) truncated to

$$T^* = (c^\cdot,\ c^\cdot+1,\ \ldots,\ \dot{d}),\ \dot{d} \neq d \text{ when } d \text{ finite.}$$

The truncated gspd can be written as

$$P_x^* = \text{prob}\,\{X^* = x\} = \frac{a_x\theta^x}{f^*(\theta)}\ x\epsilon T^* \qquad \ldots \quad (3.1)$$

where

$$f^*(\theta) = \sum_{x=c^\cdot}^{\dot d} a_x\theta^x. \qquad \ldots \quad (3.2)$$

For this distribution it is easy to see that

$$\mu^*-\theta H_{01} = c^\cdot P^*_{c^\cdot}-(\dot{d}+1)P^*_{\dot{d}+1} \qquad \ldots \quad (3.3)$$

and

$$m_2^*-\mu^*-\theta H_{11} = c^\cdot(c^\cdot-1)P^*_{c^\cdot}-\dot{d}(\dot{d}+1)P^*_{\dot{d}+1} \qquad \ldots \quad (3.4)$$

where

$$H_{ij} = \sum_{x=c^\cdot}^{\dot d} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j P_x^*, \qquad \ldots \quad (3.5)$$

* The work of this section was carried out while the author was at the University of Michigan.

For estimation purposes, we consider the following four mutually exclusive and exhaustive cases:

Case (1): $c^{\cdot} = 0$ and $\dot{d}$ finite

Case (2): $c^{\cdot} = 0$ and $\dot{d}$ infinite

Case (3): $c^{\cdot} \neq 0$ and $\dot{d}$ infinite

Case (4): $c^{\cdot} \neq 0$ and $\dot{d}$ finite.

Case (1) : $c^{\cdot} = 0$ and $\dot{d}$ finite. From (3.3) and (3.4), we have the identity

$$\theta = \frac{m_2^{*}-(\dot{d}+1)\mu^{*}}{H_{11}-\dot{d}H_{01}} \qquad \dots \quad (3.6)$$

which we utilize to estimate θ. We have only to compute

$$S_i = \Sigma\, x^i n_x \qquad i = 1, 2 \qquad \dots \quad (3.7)$$

and

$$h_{ij} = \sum_{x=\dot{c}}^{\dot{d}} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j n_x \quad i = 0, 1 \,;\, j = 1 \qquad \dots \quad (3.8)$$

from the sample and then

$$t^{*} = \frac{S_2-(\dot{d}+1)S_1}{h_{11}-\dot{d}h_{01}} \qquad \dots \quad (3.9)$$

can be taken as an estimate for θ. The estimate t^{*} makes use of the (additional) information that the sample is taken from some known gpsd and truncated to the one under consideration. The estimate t of Section 2 above does not require, and hence, does not make use of this information. The formula for the bias and variance of t^{*} can be written down to order $1/N$ as :

$$b(t^{*}) = \frac{1}{NH^2}(\theta\sigma_{22}^{*}-\sigma_{12}^{*}) \qquad \dots \quad (3.10)$$

and

$$\text{var}\,(t^{*}) = \frac{1}{NH^2}(\sigma_{11}^{*}-2\theta\sigma_{12}^{*}+\theta^2\sigma_{22}^{*}) \qquad \dots \quad (3.11)$$

where

$$H = H_{11}-\dot{d}H_{01}$$

$$\sigma_{11}^{*} = (m_4^{*}-m_2^{*2})+(\dot{d}+1)^2(m_2^{*}-\mu^{*2})-2(\dot{d}+1)(m_3^{*}-\mu^{*}m_2^{*})$$

$$\sigma_{12}^{*} = (H_{31}-m_2^{*}H_{11})-(\dot{d}+1)(H_{21}-\mu^{*}H_{11})-\dot{d}(H_{21}-m_2^{*}H_{01})+\dot{d}(\dot{d}+1)(H_{11}-\mu^{*}H_{01})$$

$$\sigma_{22}^{*} = (H_{22}-H_{11}^2)+\dot{d}^2(H_{02}-H_{01}^2)-2\dot{d}(H_{12}-H_{01}H_{11}).$$

Case (2) and *Case* (3) : $c' = 0$, $\dot{d}$ infinite and $c' \neq 0$, $\dot{d}$ infinite. It can be easily verified in these cases that

$$H_{ij} = G_{ij}$$

and, hence,

$$t^* = t.$$

Thus, we have the same treatment as in Section 2.

Here also, we observe that if we allow $\dot{d} = d$—even when d is finite—and use $a_{d+1} = 0$ formally, we again get $H_{ij} = G_{ij}$ and $t^* = t$. This observation is specially important in the case of the binomial distribution.

Case (4) : $c' \neq 0$ and $\dot{d}$ finite. It may be noted that the t^* estimate is not available in this case. However, the t estimate still works, and the estimate of θ can thus be obtained by employing two-moments method.

4. An upper bound for bias per unit standard error for two-moments estimates

We first establish a general result true for the bias of an estimate of a certain type. Let the probability distribution, from which a sample $x_1, x_2, \ldots, x_n$ is drawn, be a general distribution of a random variable X with a single parameter θ.

Let t_1 and t_2 be two statistics based on the sample such that

$$\frac{E(t_1)}{E(t_2)} = \frac{E[t_1(x_1, x_2, \ldots, x_n)]}{E[t_2(x_1, x_2, \ldots, x_n)]} = \theta$$

for all θ in the parameter space of the given distribution.

Consider the estimate $s = \dfrac{t_1}{t_2}$ to estimate θ. To find the bias in s per unit standard error of s, we have

$$\text{cov}(s, t_2) = E(st_2) - E(s)E(t_2) = E(t_1) - E(s)E(t_2) = [\theta - E(s)]\, E(t_2). \quad \ldots \quad (4.1)$$

Now

$$|\text{cov}(s, t_2)| \leqslant \sigma(s)\,\sigma(t_2) \quad \ldots \quad (4.2)$$

where σ denotes the standard error. Therefore, from (4.1) we have

$$\left|\frac{E(s)-\theta}{\sigma(s)}\right| \leqslant \left|\frac{\sigma(t_2)}{E(t_2)}\right| = |\text{c.v.}(t_2)| \quad \ldots \quad (4.3)$$

where c.v.(t_2) is the coefficient of variation of t_2. Thus, for the bias in s we have

$$\left|\frac{b(s)}{\sigma(s)}\right| \leqslant |\text{c.v.}(t_2)| \quad \ldots \quad (4.4)$$

In particular, when t_2 is a constant, we have an unbiased estimate for θ.

It may be noted that two-moments estimates for the parameter θ of gpsd's, which we discussed earlier, are estimates of the actual type of the estimate s that we have discussed in this section. Hence, the result in (4.4) also applies to them.

5. Estimation by the two-moment method for singly truncated binomial distribution

Fisher (1936) and Haldane (1932, 1938) discussed uses of the truncated binomial distribution. For instance, in problems of human genetics, in estimating the proportion of albino children produced by couples capable of producing albinos, sampling has necessarily to be restricted to families having at least one albino child. Finney (1949) has cited some more applications. Fisher and Haldane derived the maximum likelihood procedure to estimate the parameter π. Patil (1959) gave a direct method to obtain the maximum likelihood estimate. Moore (1954) suggested a simple "ratio-estimate" based on an identity between binomial probabilities. For a slightly different problem, where, in a sample from a complete binomial distribution, the frequencies in some lowest classes are missing, Rider (1955) suggested a method of estimation, which uses first two moments of the complete binomial and leads to a linear equation.

The probability law of the binomial distribution truncated at c on the left can be written as

$$b^*(x, \pi, n) = (B^*(c, \pi, n))^{-1}\binom{n}{x}\pi^x(1-\pi)^{n-x} \quad x = c, c+1, \ldots, n \qquad \ldots \quad (5.1)$$

where

$$B^*(c, \pi, n) = \sum_{x=c}^{n} \binom{n}{x}\pi^x(1-\pi)^{n-x}. \qquad \ldots \quad (5.2)$$

The first two moments about the origin of (5.1), then, are

$$\mu^* = \mu^*(c, \pi, n) = n\pi\, B^*(c-1, \pi, n-1)/B^*(c, \pi, n) \qquad \ldots \quad (5.3)$$

and

$$m_2^* = m_2^*(c, \pi, n) = \mu^*(c, \pi, n)\{1+\mu^*(c-1, \pi, n-1)\}. \qquad \ldots \quad (5.4)$$

The case of truncation to the right can be dealt with in a similar way by replacing π by $1-\pi$ and the truncation point c by $n-c$.

Proceeding on lines in Section 3, one gets in this case

$$\theta = \frac{\pi}{1-\pi} = \frac{m_2^*-c\mu^*}{H_{11}-(c-1)H_{01}} \qquad \ldots \quad (5.5)$$

where μ^* and m_2^* are defined by (5.3) and (5.4) respectively, and H_{11} and H_{01} reduce to

$$H_{11} = n\mu^*-m_2^*$$

$$H_{01} = n-\mu^*$$

then (5.5) gives

$$\pi = \frac{m_2^*-c\mu^*}{(n-1)\mu^*-n(c-1)} \qquad \ldots \quad (5.6)$$

so that, on the basis of a random sample of size N with n_x as the frequency of x drawn from (5.1), the estimate for π can be written as

$$t = \frac{S_2 - cS_1}{(n-1)S_1 - n(c-1)N} \qquad \dots \quad (5.7)$$

where $S_1 = \Sigma \, x n_x$

and $S_2 = \Sigma \, x^2 n_x$.

It is obvious that (5.7) is quite simple. On the other hand, the estimate obtained from (5.7) is likely to be inefficient. It is important, therefore, to investigate the loss in efficiency due to the use of (5.7) instead of the maximum likelihood estimate.

To find the asymptotic variance of the two-moments estimate t of π, one gets on some simplification

$$\text{var } (t) = \frac{1}{NH^2} (\sigma^*_{11} + \pi^2 \sigma^*_{22} - 2\pi\sigma^*_{12}) \qquad \dots \quad (5.8)$$

where

$$H = (n-1)\mu^* - n(c-1)$$

$$\sigma^*_{11} = (m^*_4 - m^{*2}_2) + c^2(m^*_2 - \mu^{*2}) - 2c(m^*_3 - \mu^* m^*_2)$$

$$\sigma^*_{22} = (n-1)^2(m^*_2 - \mu^{*2})$$

and

$$\sigma^*_{12} = (n-1)(m^*_3 - \mu^* m^*_2) - c(m^*_2 - \mu^{*2})$$

where m^*_r is the r-th theoretical moment of (5.1) about the origin. Thus,

$$\text{var } (t) = \frac{1}{N\{(n-1)\mu^* - n(c-1)\}^2} [(m^*_4 - m^{*2}_2) + \{(n-1)\pi + c\}^2 .$$

$$(m^*_2 - \mu^{*2}) - 2\{(n-1)\pi + c\}(m^*_3 - \mu^* m^*_2)]. \qquad \dots \quad (5.9)$$

The asymptotic efficiency of t is then given by

$$\text{eff } (t) = \text{var } (\hat{\pi}) / \text{var } (t) \qquad \dots \quad (5.10)$$

where $\hat{\pi}$ is the maximum likelihood estimate of π with variance given by

$$\text{var } (\hat{\pi}) = \frac{[\pi(1-\pi)]^2}{N\mu^*_2} \qquad \dots \quad (5.11)$$

where μ^*_2 is the variance of (5.1).

The special cases of some importance in genetics are $c = 1$ and $\pi = 1/4$, $1/2$, or $3/4$. The efficiency of the Two-Moments Estimate (TM) relative to the Maximum Likelihood (ML) in these cases is tabulated below.

TABLE 1. ASYMPTOTIC EFFICIENCY OF TM FOR $c = 1$

n	efficiency $\pi = 1/4$	1/2	3/4
3	.925	.875	.875
4	.871	.818	.859
5	.817	.795	.870
6	.800	.780	.886
7	.781	.794	.901
8	.766	.803	.913
9	.755	.814	.923
10	.749	.823	.931

General examination of the table shows that TM is fairly efficient in the cases cited. A closer look shows that the efficiency of TM in case of $\pi = 1/2$ and $\pi = 3/4$ decreases in the beginning with n, reaches a minimum and then increases with increasing values of n. For $\pi = 1/4$, however, the efficiency decreases throughout. Let us compute, therefore, the efficiency of TM for higher values of n. The following gives the results obtained for $n = 11(1)15$.

ASYMPTOTIC EFFICIENCY OF TM
FOR $c = 1$ AND $\pi = 1/4$

n	efficiency
11	.746
12	.744
13	.745
14	.747
15	.750

Thus, in case of $\pi = 1/4$ also, the efficiency reaches a minimum and then increases with increasing n. It is interesting to note that in all these cases the efficiency of TM has reached the minimum at $n = 3/\pi$.

Following Section 3, one gets to order $1/N$, the amount of bias of t(TM) as follows for $c = 1$:

$$b(t) = \frac{1}{N}\,\frac{\mu_2^{*2}+\mu^*m_2^*-m_3^*}{(n-1)\mu^{*2}} = \frac{B(t)}{N}\,. \qquad \ldots \quad (5.12)$$

The table given below provides the value of $B(t)$ for $\pi = 1/4$, $1/2$ and $3/4$.

TABLE 2 : N (AMOUNT OF BIAS TO ORDER $1/N$) OF TM

n	$\pi = 1/4$	1/2	3/4
3	−.2412	−.2717	−.1763
4	−.2152	−.1940	−.1290
5	−.1806	−.1748	−.1004
6	−.1715	−.1398	−.0818
7	−.1535	−.1230	−.0689
8	−.1379	−.1063	−.0595
9	−.1187	−.0934	−.0524
10	−.1097	−.0748	−.0468

Table 2 shows that the amount of bias of two-moments estimate is rather small and one need not worry much about it, especially when one knows that the maximum likelihood estimate is also a biased one.

Illustrative example. The detailed computation procedure of evaluating the two-moments estimate discussed above will be illustrated with reference to K. Pearson's data on albinism in man. The table below gives the number of families (n_x) each of five children having exactly x albino children in the family, ($x = 1, 2, 3, 4, 5$).

number of albinos in family (x)	1	2	3	4	5
number of families (n_x)	25	23	10	1	1

If π is the probability for a child to be an albino, we may accept the truncated binomial model :

$$\frac{\binom{n}{x}\pi^x(1-\pi)^{n-x}}{1-(1-\pi)^n} \quad x = 1, 2, \ldots, n,$$

for the probability of x albinos in a family of n. Here $n = 5$, and the problem is to estimate π on the basis of the data given in the table above.

To compute the two-moments estimate for π, we require $S_1 = \Sigma x n_x = 110$ and $S_2 = \Sigma x^2 n_x = 248$. Then the estimate is

$$t = \frac{1}{n-1}\left[\frac{S_2}{S_1} - 1\right]$$

$$= \frac{1}{4}\left[\frac{248}{100} - 1\right] = 0.3136.$$

To compute the variance of t we require

$$\mu^* = \frac{n\pi}{1-(1-\pi)^n} = 1.95280$$

$$m_2^* = \mu^*[(n-1)\pi+1] = 4.40239$$

$$m_3^* = \mu^*[(n-1)\pi+1+(n-1)\pi\{(n-2)\pi+2\}] = 11.60615$$

$$m_4^* = \mu^*[(n-1)\pi+1+3(n-1)\pi\{(n-2)\pi+2\}+(n-1)\pi.\ (n-2)\pi\{(n-3)\pi+3\}]$$

$$= 32{\cdot}06830$$

all evaluated by taking 0.3136 as the estimate for π. The variance of t is estimated from the formula

$$\text{var}\,(t) = \frac{1}{N\{(n-1)\mu^*\}^2}\,(m_4^* - m_2^{*2}) + \{(n-1)\pi+1\}^2(m_2^* - \mu^{*2}) - 2\{(n-1)\pi+1\}(m_3^* - \mu^* m_2^*)\}$$

$$= 0{\cdot}0012066$$

so that the standard error is S.E.$(t) = 0.03474$. Incidentally, the maximum likelihood estimate $\hat{\pi}$ comes out to be in this case, $\hat{\pi} = 0.3088$ with S.E. $(\hat{\pi}) = 0{\cdot}03210$.

6. Estimation by the two-moments method for singly truncated Poisson distribution

Problems of estimation in a truncated Poisson distribution with known truncation points have been discussed by various authors. The case of truncation on the left has been considered by David and Johnson (1948) who gave the maximum likelihood estimate; Plackett (1953) gave a simple and highly efficient ratio-estimate, and Rider (1953) used the first two moments. Truncation on the right has been discussed by Tippett (1932), Bliss (1948), and Moore (1952). Tippett derived the maximum likelihood solution, Bliss developed an approximation to it, and Moore suggested a simple ratio estimate. For both types of truncations, the author (1959) has provided neat and compact equations for estimation by the method of maximum likelihood. He has also presented numerical tables and some suitable charts to facilitate the solution of these equations in certain special cases. In this section, we study the Two-Moments Method of estimation as applied to singly truncated Poisson distributions.

The probability law of the singly truncated Poisson distribution with truncation point on the right at d can be written as :

$$p^*(x, \mu) = [p(d, \mu)]^{-1} e^{-\mu} \frac{\mu^x}{x!} \qquad x = 0, 1, 2, \ldots, d \qquad \ldots \quad (6.1)$$

where
$$p(d, \mu) = \sum_{x=0}^{d} e^{-\mu} \frac{\mu^x}{x!} . \qquad \ldots \quad (6.2)$$

The first two moments about the origin of (6.1) can be written down as

$$\mu^* = \mu^*(d, \mu) = \mu \, p(d-1, \mu)/p(d, \mu) \qquad \ldots \quad (6.3)$$

and
$$m_2^* = m_2^*(d, \mu) = \mu^*(d, \mu)[1+\mu^*(d-1, \mu)]. \qquad \ldots \quad (6.4)$$

Proceeding as in Section 3, one gets in this case

$$\theta = \mu = \frac{m_2^*-(d+1)\mu^*}{H_{11}-dH_{01}} \qquad \ldots \quad (6.5)$$

where μ^* and m_2^* are defined by (6.3) and (6.4), respectively, and H_{11} and H_{01} reduce to

$$H_{11} = \mu^*$$

$$H_{01} = 1.$$

Then (6.5) gives
$$\mu = \frac{m_2^*-(d+1)\mu^*}{\mu^*-d} \qquad \ldots \quad (6.6)$$

so that, on the basis of a random sample of size N with n_x as the frequency of x drawn from (6.1), the two-moments estimate for μ becomes

$$t = \frac{S_2-(d+1)S_1}{S_1-dN} , \qquad \ldots \quad (6.7)$$

where
$$S_1 = \Sigma\, xn_x \quad \text{and} \quad S_2 = \Sigma\, x^2n_x.$$

To find the asymptotic variance of the two-moments estimate (TM) given by (6.7) one gets on simplification,

$$\text{var}\ (t) = \frac{1}{NH^2}(\sigma_{11}^*+\mu^2\sigma_{22}^*-2\mu\sigma_{12}^*) \qquad \ldots \quad (6.8)$$

where

$$H = \mu^*-d$$
$$\sigma_{11}^* = (m_4^*-m_2^{*2})+(d+1)^2(m_2-\mu^{*2})-2(d+1)(m_3^*-\mu^*m_2^*) \quad \dots \quad (6.9)$$
$$\sigma_{22}^* = m_2^*-\mu^{*2}$$

and
$$\sigma_{12}^* = (m_3^*-\mu^*m_2^*)-(d+1)(m_2^*-\mu^{*2})$$

where m_r^* is the r-th theoretical moment of (6.1) about origin.

Thus

$$\text{var}(t) = \frac{1}{N(\mu^*-d)^2}[(m_4^*-m_2^{*2})+(\mu+d+1)^2(m_2^*-\mu^{*2})-2(\mu+d+1)(m_3^*-\mu^*m^{*2})] \quad \dots \quad (6.10)$$

The asymptotic efficiency of t is then given by

$$\text{eff}(t) = \text{var}(\hat{\mu})/\text{var}(t) \quad \dots \quad (6.11)$$

where $\hat{\mu}$ is the maximum likelihood estimate of μ with variance given by

$$\text{var}(\hat{\mu}) = \frac{\mu^2}{N\mu_2^*} \quad \dots \quad (6.12)$$

where μ_2^* is the variance of (6.1).

The following table gives the asymptotic efficiency of t relative to $\hat{\mu}$ for values of $d = 5$ with $\mu = .25, .5(.5)\,2.5$, and $d = 10$ with $\mu = .5(.5)5$.

TABLE 3. EFFICIENCY OF TM

μ	.25	.50	1.00	1.50	2.00	2.50
			Case (i) $d = 5$			
eff	.978	.954	.904	.867	.850	.838

μ	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
					Case (ii) $d = 10$					
eff	.980	.988	.960	.942	.920	.897	.874	.855	.835	.815

Thus, the asymptotic efficiency of TM is not less than 80 per cent in the above cases, and one may therefore use TM to estimate μ in such problems.

The amount of bias, to order $1/N$, of the TM comes out to be

$$b(t) = (\mu\,\sigma_{22}^*-\sigma_{12}^*)/NH^2 = \frac{B(t)}{N} \quad \dots \quad (6.13)$$

where H, σ_{22}^* and σ_{12}^* are defined by (6.9). The following table gives $B(t)$ for values of $d = 5$ with $\mu = .25, .5(.5)2.5$ and $d = 10$ with $\mu = .5(.5)5$.

TABLE 4. N (AMOUNT OF BIAS TO ORDER $1/N$ OF TM)

μ	.25	.50	1.00	1.50	2.00	2.50
			Case (i) $d = 5$			
$B(t)$	.0526	.1111	.2498	.4260	.6507	.9181

μ	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
					Case (ii) $d = 10$					
$B(t)$	.0526	.1111	.1765	.2500	.3333	.4284	.5547	.6640	.8093	.9786

Examination of Table 4 shows that the amount of bias involved in TM is rather very small.

The probability law of the singly truncated Poisson distribution with truncation point on the left at c can be written as

$$p^*(x, \mu) = [P^*(c, \mu)]^{-1} e^{-\mu} \frac{\mu^x}{x!} \quad x = c, c+1, \ldots \infty \qquad \ldots \quad (6.14)$$

where

$$P^*(c, \mu) = \sum_{x=c}^{\infty} e^{-\mu} \frac{\mu^x}{x!} . \qquad \ldots \quad (6.15)$$

The first two moments about origin of (6.14) can be written down as

$$\mu^* = \mu^*(c, \mu) = \mu P^*(c-1, \mu)/P^*(c, \mu) \qquad \ldots \quad (6.16)$$

and

$$m_2^* = m_2^*(c, \mu) = \mu^*(c, \mu)[1+\mu^*(c-1, \mu)]. \qquad \ldots \quad (6.17)$$

For a slightly different problem, where in a sample from a complete Poisson distribution, the frequencies for some lowest "counts" are missing, Rider (1953) suggested a method of estimation which uses first two moments of the complete Poisson and leads to a linear equation.

The Two-Moments Method discussed in Section 3 gives the estimate for μ in this case as

$$t = \frac{S_2 - cS_1}{S_1-(c-1)N} \qquad \ldots \quad (6.18)$$

where

$$S_1 = \sum_{i=1}^{N} x_i \text{ and } S_2 = \sum_{i=1}^{N} x_i^2$$

are based on the random sample $x_i (i = 1, 2, \ldots, N)$ of size N drawn from (6.14).

To find the asymptotic variance of t, one gets on simplification,

$$\text{var}(t) = \frac{1}{N[\mu^*-(c-1)]^3} [(m_4^*-m_2^{*2})+(\mu+c)^2(m_2^*-\mu^{*2})-2(\mu+c)(m_3^*-\mu^* m_2^*)] \quad \ldots \quad (6.19)$$

where m_r^* is the r-th moment of (6.14) about origin.

The asymptotic efficiency of t is then given by

$$\text{eff}(t) = \text{var}(\hat{\mu})/\text{var}(t)$$

where $\hat{\mu}$ is the maximum likelihood estimate of μ with

$$\text{var}(\hat{\mu}) = \frac{\mu^2}{N\mu_2^*} \qquad \ldots \quad (6.20)$$

where μ_2^* is the variance of (6.14).

The case of single truncation on the left at $c = 1$ is of practical importance. David and Johnson (1952) studied the efficiency for this particular case. The following is the table of eff (t) computed by them.

TABLE 5. EFFICIENCY OF TM FOR $c = 1$

μ	.5	1.0	1.5	2.0	2.5	3.0	4.0
eff.	.87	.80	.75	.73	.71	.71	.72

Source : David and Johnson (1952).

Thus, the efficiency of TM is not less than 70 per cent for $c = 1$ with $\mu = .5(.5)4.0$.

One gets to order $1/N$ the amount of bias of t (TM) as follows :

$$b(t) = (\mu\sigma_{22}^* - \sigma_{12}^*)/NH^2 = \frac{B(t)}{N} \qquad \ldots \quad (6.21)$$

where

$$H = \mu^* - (c-1)$$
$$\sigma_{22}^* = m_2^* - \mu^{*2} \qquad \ldots \quad (6.22)$$
$$\sigma_{12}^* = (m_3^* - \mu^* m_2^*) - c(m_2^* - \mu^{*2}).$$

The following table gives $B(t)$ for $\mu = .5(.5)\,4.0$.

TABLE 6. N (AMOUNT OF BIAS TO ORDER $1/N$ OF TM) FOR $c = 1$

μ	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$B(t)$	−.3935	−.6321	−.6373	−.8647	−.9147	−.9502	−.9698	−.9817

Illustrative example. The detailed computation procedure of evaluating the TM is illustrated with reference to data collected by Varley (1949) to study population balance in the Knapweed Gall-fly. The table below gives the number of flower-heads (n_x) each having exactly x gall-cells ($x = 1, 2, \ldots$).

number of gall-cells in a flower-head (x)	1	2	3	4	5	6	7	8	9	10
number of flower heads (n_x)	287	272	196	79	29	20	2	0	1	0

Assuming the truncated Poisson model

$$\frac{\mu^x}{x!(e^\mu - 1)} \qquad x = 1, 2, \ldots$$

for the probability of x gall-cells in a flower-head, the problem is to estimate μ on the basis of the given data.

We have $S_1 = 2023$, $S_2 = 6027$, $N = 886$. Therefore the TM is

$$t = \frac{S_2}{S_1} - 1 = \frac{6027}{2023} - 1 = 1.9792.$$

To compute the variance of t, taking 1.9792 as the estimate for μ, we have

$$\mu^* = \frac{\mu e^{\mu}}{e^{\mu}-1} = 2.2965$$

$$m_2^* = \mu^*(1+\mu) = 6.8417$$

$$\mu_2^* = m_2^* - \mu^{*2} = 1.5678$$

$$m_3^* = \mu(\mu^* + m_2^*) + (1+\mu)\mu_2^* = 22.7571$$

and

$$m_4^* = \mu[\mu^* + m_2^* + 2\mu_2^* + m_3^* - 2\mu(1+\mu)] + (m_3^* - \mu m_2^*)(1+2\mu) = 91.6896.$$

The variance of t is then estimated from the formula,

$$\text{var}(t) = \frac{1}{N\mu^{*2}}\,[(m_4^* - m_2^{*2}) + (1+\mu)^2.\,\mu_2^* - 2(1+\mu)(m_3^* - \mu^* m_2^*)] = 0.003600$$

so that the standard error is S.E. $(t) = 0.0600$.

Incidentally, the maximum likelihood estimate comes out to be

$$\hat{\mu} = 1.9623 \text{ with S.E. } (\hat{\mu}) = 0.0529.$$

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