EFFECTIVE FIELD THEORY FOR NONCOMMUTATIVE SPACETIME: A TOY MODEL

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Abstract:
A novel geometric model of a noncommutative plane has been constructed. We demonstrate that it can be construed as a toy model for describing and explaining the basic features of physics in a noncommutative spacetime from a field theory perspective. The noncommutativity is induced internally through constraints and does not require external interactions. We show that the noncommutative space-time is to be interpreted as having an *internal* angular momentum throughout. Subsequently, the elementary excitations - *i.e.* point particles - living on this plane are endowed with a *spin*. This is explicitly demonstrated for the zero-momentum Fourier mode. The study of these excitations reveals in a natural way various *stringy* signatures of a noncommutative quantum theory, such as dipolar nature of the basic excitations [7] and momentum dependent shifts in the interaction point [8]. The observation [9] that noncommutative and ordinary field theories are alternative descriptions of the same underlying theory, is corroborated here by showing that they are gauge equivalent.

Also, treating the present model as an explicit example, we show that, even classically, in the presence of additional constraints, (besides the usual ones due to reparameterization invariances), the equivalence between Nambu-Goto and Polyakov formulations is subtle.

Key Words: Noncommutative spacetime, Constraints, Reparameterization invariant theory, Spinning particle.

Non-Commutativity in Quantum Field Theory (QFT) has a long history. It was originally postulated by Snyder [1] as a means of a (Lorentz invariant) regularization to cure the short distance singularities. In a different scenario, Connes [2] introduced Non-Commutative (NC) geometry by extending the standard differentiable manifold to a mixed one with an additional discrete NC manifold, where the Higgs field, (and subsequently the Higgs potential), appear as part of the gauge field structure. For various reasons the above formulation did not gain much popularity.

With the advent of D-brane solutions [3] in open string theory, the role of noncommutativity in space-time [4] has gained importance since it reflects the low energy stringy behavior in a (constant) background field. The effective NC gauge theory is more tractable than the original string theory as the former is (computationally) very close to field theory in ordinary space-time. The NC is induced in the D-branes when the open string endpoints are on the branes, in the presence of a two-form background field, via a qualitative change in the boundary conditions [5, 6].

A very interesting feature of NC theory is that in the charged sector, the basic excitations act as *dipoles* [7] in $U(1)$ gauge interactions. Another peculiarity is that a consistent quantization requires a momentum dependent *shift* in the interaction point [8], leading to a specific kind of non-locality. A further crucial observation of [9] is that ordinary and NC gauge theories are *alternative* descriptions, (dictated by the choice of regularization), of the same quantum theory.

The goal of this Letter is to remove some of the mysteries surrounding NC theory and to establish the noncommutativity from fundamental principles. In the present work we demonstrate that the above mentioned alien features of an NC QFT can be accommodated in a natural way, staying within the realm of conventional QFT on a differentiable manifold. Thus their origin can be studied in a unified manner at a deeper level. [1] This requires the construction of a noncommutative space-time hypersurface on which a field theory can be studied directly. Essentially the present work is a toy model of such a surface.

Here we have initiated the study of a relativistic, reparameterization invariant theory of such a noncommutative *surface*, both in the Nambu-Goto and Polyakov formalism. We stress that, on this plane, the string induced effects of noncommutativity mentioned above, are generated and can be interpreted in a natural way. The NC is induced internally through constraints and does not require an external interaction. Obviously, it can be reduced to conventional NC theory for the sake of comparison. On this NC surface, we show that "point" particles will behave like dipoles [7] in electromagnetic interaction. Occurrence of the non-locality [8], via the momentum dependent phase shift in the interaction vertex, is explained naturally. Also, we demonstrate that in the Polyakov formulation, the NC is induced, depending on our choice of gauge. This is reminiscent of the equivalence between the NC and ordinary gauge theories [9]. In the present setup, the celebrated Seiberg-Witten map [9] might possibly be interpreted as a gauge transformation.

The above has been achieved by the extension of the target space $X^\mu(\sigma,\tau)$ to$(X^\mu(\sigma,\tau), N^\nu(\sigma,\tau))$ where $N^\mu N_\mu = 1$. Hence the extended manifold is a bundle of unit spheres over the flat $X^\mu(\sigma,\tau)$ manifold. In physical terms, an internal angular momentum has been generated by N^ν throughout the space-time in a reparameterization invariant way. This leads to a *spin* in

[1] Generally, one considers the effective theory as an NC field theory on the brane and simply postulates the fundamental NC brane coordinate relation $[X^\mu, X^\nu] = i\theta^{\mu\nu}$ and proceeds. Also the properties [7, 8] are stated without explanation.

the fundamental point particle excitations, resulting in a magnetic moment and subsequently the above observations follow smoothly. Indeed, this is very appealing since spin is a very fundamental and well studied geometrical property. This enlargement is, in spirit, somewhat akin to the Superspace formulation. Indeed, it would be interesting if a field theory can be developed in this extended manifold and matched to the conventional NC theory. Construction of the model and its applications constitute the first part of the paper.

In the second part, we comment on the established notion, (which is used throughout in string theory context [10]), that *classically* the Nambu-Goto (square root action) and Polyakov (induced metric action) forms are completely equivalent. We explicitly demonstrate that in the presence of constraints, showing the equivalence between the theories resulting from the above two pictures raises subtle points and in fact the Polyakov form happens to be more general than the Nambu-Goto form, in the sense that the NC structure is fixed in the Nambu-Goto form, whereas it is gauge choice dependent in the Polyakov form. Lastly, we make a brief observation on the quantum theory.

As mentioned before, our model consists of the coordinate $X^\mu(\sigma,\tau)$ coupled to another field $N^\mu(\sigma,\tau)$ in a reparaterization invariant way. The N^μ manifold is compact with $N^\mu N_\mu = 1$. Similar to the extension of the relativistic point particle action to the string action, our model can be thought of as an extension of the relativistic *spinning* point particle [11, 12] with the spatial parameter σ being infinite in extent as well. N^μ imparts the spin.

Let us consider the Lagrangian,

$$\mathcal{L} = 2[(\dot{X}\dot{N})(X'N') - (\dot{X}N')(X'\dot{N})]^{\frac{1}{2}} \equiv 2\mathcal{A}, \tag{1}$$

where $(\dot{X}N') \equiv \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\sigma}$ etc. and the conjugate momenta are

$$P^\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}_\mu} = \frac{1}{\mathcal{A}}[(X'N')\dot{N}^\mu - (X'\dot{N})N'^\mu]\ ;\ P_N^\mu = \frac{\partial\mathcal{L}}{\partial\dot{N}_\mu} = \frac{1}{\mathcal{A}}[(X'N')\dot{X}^\mu - (\dot{X}N')X'^\mu], \tag{2}$$

The primary constraints in the theory, indicating reparamerization invariance, are

$$P^\mu X'_\mu \equiv 0\ ;\ P_N^\mu N'_\mu \equiv 0\ . \tag{3}$$

The resulting First Class Constraint (FCC) algebra [2] (or Schwinger algebra) is given by,

$$\chi_{1,2} = (PX') \pm (P_N N')$$

$$\{\chi_1(\sigma),\chi_1(\sigma')\} = (\chi_1(\sigma)+\chi_1(\sigma'))\delta'(\sigma-\sigma')\ ;\ \{\chi_2(\sigma),\chi_2(\sigma')\} = (\chi_1(\sigma)+\chi_1(\sigma'))\delta'(\sigma-\sigma'),$$
$$\{\chi_1(\sigma),\chi_2(\sigma')\} = (\chi_2(\sigma)+\chi_2(\sigma'))\delta'(\sigma-\sigma'), \tag{4}$$

where the Poisson brackets are,

$$\{X^\mu(\sigma),P^\nu(\sigma')\} = g^{\mu\nu}\delta(\sigma-\sigma')\ ,\ \{N^\mu(\sigma),P_N^\nu(\sigma')\} = g^{\mu\nu}\delta(\sigma-\sigma'). \tag{5}$$

[2] In the Dirac Hamiltonian constraint analysis [13] the commuting (in the Poisson Bracket sense) constraints are termed as First Class Constraints (FCC) and the non-commuting ones as Second Class Constraints (SCC). The former signals gauge invariance and the latter modify the symplectic structure from Poisson Brackets to Dirac Brackets.

In a geometric interpretation, just as the string action constitutes the elementary area in the $\sigma-\tau$ plane, the action from (1) can be worked out to represent an area in the $\sigma-\tau$ plane (see appendix). So far, this is a free theory with (decoupled) wave equations satisfied by X^μ and N^μ. This will be shown later.

The N^μ manifold is now compactified by invoking the constraint $\chi_3 \equiv N^2-1 \approx 0$ via the multiplier λ,

$$\mathcal{L} = 2\mathcal{A} + \lambda(N^2-1). \tag{6}$$

As stated before, the model is motivated by an earlier spinning particle model [12], which was later amended [11] to a first order Hamiltonian form. χ_3 leads to more primary constraints, the full set being,

$$\chi_1 \ ; \ \chi_2 \ ; \ \chi_3 \equiv N^2-1 \ ; \ \psi_1 \equiv (PN) \ ; \ \psi_2 \equiv (PP_N) - (X'N') \ , \tag{7}$$

out of which χs and ψ are FCCs and SCCs [13] respectively. The canonical Hamiltonian vanishes on the constraint surface, as it should in a reparameterization invariant theory. The following definition of Dirac Bracket [13],

$$\{A,B\}_{DB} = \{A,B\} - \{A,\psi_i\}\{\psi_i,\psi_j\}^{-1}\{\psi_j,B\}; \quad i,j=1,2 \ , \tag{8}$$

yields the new symplectic structure [3] ,

$$\{X_\mu(\sigma),X_\nu(\sigma')\} = \frac{P_{N\mu}N_\nu - P_{N\nu}N_\mu}{P^2+N'^2}\delta \ ; \ \{X_\mu(\sigma),P_\nu(\sigma')\} = g_{\mu\nu}\delta - \frac{N_\mu N'_\nu}{P^2+N'^2}(\sigma)\delta'$$

$$\{N_\mu(\sigma),P_{N\nu}(\sigma')\} = (g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2+N'^2})\delta \ ; \ \{P_{N\mu}(\sigma),P_{N\nu}(\sigma')\} = (\frac{P_\mu X'_\nu}{P^2+N'^2}(\sigma) + \frac{P_\nu X'_\mu}{P^2+N'^2}(\sigma'))\delta' \tag{9}$$

$$\{X_\mu(\sigma),N_\nu(\sigma')\} = -\frac{N_\mu P_\nu}{P^2+N'^2}\delta \ ; \ \{X_\mu(\sigma),P_{N\nu}(\sigma')\} = -\frac{P_{N\mu}P_\nu}{P^2+N'^2}\delta - \frac{N_\mu X'_\nu}{P^2+N'^2}(\sigma)\delta' \ ,$$

$$\{P_{N\nu}(\sigma),P_\mu(\sigma')\} = \frac{P_\nu N'_\mu}{P^2+N'^2}(\sigma)\delta'. \tag{10}$$

The notations in the above are $\delta' \equiv \partial_\sigma\delta(\sigma-\sigma')$, $\frac{A(\sigma)}{B(\sigma)} = \frac{A}{B}(\sigma)$. Using the above brackets it is straightforward to check that

$$J_{\mu\nu} = \int d\sigma(P_\mu X_\nu - P_\nu X_\mu + P_{N\mu}N_\nu - P_{N\nu}N_\mu) \tag{11}$$

generates the angular momentum algebra and transforms the vectors properly,

$$\{J_{\mu\nu},V_\alpha(\sigma)\} = g_{\nu\alpha}V_\mu(\sigma) - g_{\mu\alpha}V_\nu(\sigma), \tag{12}$$

where $V_\alpha \equiv X_\alpha, P_\alpha, N_\alpha, P_{N_\alpha}$. Clearly the spin contribution is coming from the N^μ sector. Thus we have constructed the NC space-time X^μ as is evident from the non-vanishing $\{X,X\}$ bracket. The NC factor is not a constant, which would have violated Lorentz invariance. Introduction

[3] Unless otherwise stated, henceforth all $\{,\}$ are Dirac brackets.

of spin has turned the ordinary space-time into an NC one on the $\sigma - \tau$ plane. Thus, we have succeeded in constructing a model for the noncommutative plan and (10) is the cherished form of the noncommutative structure. This is the major result of the present work.

A quite involved computation reveals that the Schwinger algebra (4) is intact if one exploits the new symplectic structure given in (9,10). This establishes the fact that the diffeomorphism symmetry is *not* destroyed by the introduction of a constraint (χ_3 in the present case) from outside. We will use this idea in a crucial way later in the Polyakov formulation.

Now that the field theoretic model for the NC plane is at hand, we can check whether it leads to some of the intriguing features [7, 8] of elementary (NC) excitations, in a fundamental way. Note that although these observations [7, 8] have been made in connection with string theory, what really matters is the underlying NC spacetime structure. In our more general field theoretic formulation of the NC spacetime, these results are reproduced naturally. To discuss the manifestations of noncommutativity in our toy model, we concentrate on the (so called) point particles living on the NC plane constructed here to analyze how this NC has affected them. However, deriving the particle properties from the involved symplectic structure given in (9,10), this turns out to be non-trivial.

For a conventional field theory, with the canonical equal-time Poisson bracket

$$\{Q(\sigma), P(\sigma')\} = \delta(\sigma - \sigma'), \tag{13}$$

the particle-like properties are revealed upon Fourier mode expansion,

$$Q(\sigma, \tau) = \frac{1}{\sqrt{L}} \sum_k q_k(\tau) e^{ik\sigma}, \tag{14}$$

where for convenience we have confined the system inside L. Using the identity for discrete k

$$\frac{1}{L} \int e^{i(k-k')\sigma} d\sigma = \delta_{k,k'}, \tag{15}$$

the discrete mode q_k and its conjugate momentum p_k are expressed as

$$q_k(\tau) = \frac{1}{\sqrt{L}} \int e^{-ik\sigma} Q(\sigma, \tau) d\sigma, \quad p_k(\tau) = \frac{1}{\sqrt{L}} \int e^{ik\sigma} P(\sigma, \tau) d\sigma, \tag{16}$$

it is easy to verify that

$$\{q_k, p_{k'}\} = \delta_{k,k'}. \tag{17}$$

In the above, all the modes are decoupled and behave in an identical canonical fashion. Notice that for the $k = 0$ oscillator, this result is obtained trivially. By utilizing the following relations,

$$q_0(\tau) = \frac{1}{\sqrt{L}} \int Q(\sigma, \tau) d\sigma, \quad p_0(\tau) = \frac{1}{\sqrt{L}} \int P(\sigma, \tau) d\sigma, \tag{18}$$

we can derive,

$$\{q_0, p_0\} = \frac{1}{L} \int d\sigma d\sigma' \{Q(\sigma), P(\sigma')\} = \frac{1}{L} \int d\sigma = 1. \tag{19}$$

In the present case, performing a similar analysis for the $k = 0$ modes of all the field variables, we find that (9,10) reduces to a much simpler algebra,

$$\{x_\mu, x_\nu\} = \frac{p_{n_\mu} n_\nu - p_{n_\nu} n_\mu}{p^2} \; ; \; \{x_\mu, p_\nu\} = g_{\mu\nu} \; ; \; \{n_\mu, p_{n_\nu}\} = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}),$$

$$\{x_\mu, n_\nu\} = -\frac{n_\mu p_\nu}{p^2} \; ; \; \{x_\mu, p_{n_\nu}\} = -\frac{p_{n_\mu} p_\nu}{p^2}, \tag{20}$$

where $x^\mu(\tau) \equiv x_0^\mu(\tau) = \frac{1}{L}\int d\sigma X_\mu(\sigma, \tau)$ etc.. Naively this is obtained from (9,10) by integrating out σ, which amounts to dropping the σ-derivative terms (since $k = 0$) and δ'. Clearly the brackets for the non-zero k-modes are more complicated.

Notice that the above set (20) is identical to the Dirac Brackets given in the corrected version of the spinning particle model [11]. This algebra can be thought of to be originated from the first order Lagrangian posited in [11] with p^2 denoting the mass of the particle. In this sense, our model in 2+1-dimensions can be interpreted as a *field theoretic model for the anyon*, excitations having arbitrary spin and statistics [11, 12]. Similar type of spin induced NC in a variant [14] of the present model has been discussed in [15].

The $\{x, x\}$-noncommutativity that appears in (20) is actually a generalization of the more restricted form that is commonly used, where $\{x_\mu, x_\nu\} = \theta_{\mu\nu}$, $\theta_{\mu\nu}$ being a constant c-number tensor. However, non-constant and operatorial noncommutativity, of a different form, have also appeared in [16]. On the other hand, as we discuss below, reducing the noncommutativity in our model to the above constant form is quite subtle.

Since we have already related our brackets (9,10) to that of the spinning particle model symplectic structure [11], let us rely more heavily on [11] where a first order Hamiltonian formalism has been discussed. Leaving out the details, (which are provided in [11]), the equations of motion are obtained as,

$$\dot{p}_\mu = 0 \; , \; \dot{n}_\mu = 0 \; \rightarrow p_\mu(\tau) \equiv p_\mu^0, \; n_\mu(\tau) \equiv n_\mu^0. \tag{21}$$

One further gets,

$$\dot{x}_\mu = 2\Lambda p_\mu^0 \; , \; \dot{p_{n\mu}} = -\lambda n_\mu^0, \tag{22}$$

with Λ and λ being arbitrary. The equations (22) are integrated to

$$x_\mu(\tau) = \alpha(\tau) p_\mu^0 + x_\mu^0 \; , \; p_{n\mu}(\tau) = \beta(\tau) n_\mu^0 + p_{n\mu}^0, \tag{23}$$

where $\dot{\alpha} = 2\Lambda$, $\dot{\beta} = -\lambda$. Exploiting the relation (23) in the noncommutativity bracket, we get

$$\{x_\mu, x_\nu\} = \frac{p_{n\mu} n_\nu - p_{n\nu} n_\mu}{p^2} = \frac{1}{p^2}[(\beta n_\mu^0 + p_{n\mu}{}^0) n_\nu^0 - (\beta n_\nu^0 + p_{n\nu}{}^0) n_\mu^0]$$

$$= \frac{p_{n\mu}{}^0 n_\nu^0 - p_{n\nu}{}^0 n_\mu^0}{p^2}. \tag{24}$$

Clearly the right hand side is time independent. Finally, using (23), we derive the complete symplectic structure that is stable under time (τ) translation,

$$\{x_\mu^0, x_\nu^0\} = \frac{p_{n\mu}{}^0 n_\nu^0 - p_{n\nu}{}^0 n_\mu^0}{p^2} \; ; \; \{x_\mu^0, p_\nu^0\} = g_{\mu\nu} \; ; \; \{n_\mu^0, p_{n\nu}{}^0\} = (g_{\mu\nu} - \frac{p_\mu^0 p_\nu^0}{p^2}),$$

$$\{x_\mu^0, n_\nu^0\} = -\frac{n_\mu^0 p_\nu^0}{p^2} \; ; \; \{x_\mu^0, p_{n\nu}{}^0\} = -\frac{p_{n\mu}{}^0 p_\nu^0}{p^2}. \tag{25}$$

The constant c-number NC parameter $\{x_\mu, x_\nu\} = \theta_{\mu\nu}$ is finally generated by considering the spontaneous symmetry breaking reasoning, originally developed in the context of Standard

Model extension [17], and later used in demonstrating the violation of Lorentz invariance in NC field theory [18]. Here, the vector fields attain a non-vanishing expectation value in the vacuum at low energy.[4]

Let us now discuss the effects of the noncommutativity exhibited in (25) in the quantum theory. However, (25) is not convenient for the conventional quantization programme of elevating the classical brackets to quantum commutators via the correspondence principle. To facilitate this, we move on to a canonical (q^μ, Q^ν) setup with

$$[q,q] = [Q,Q] = 0 \ ; \ [q^\mu, Q^\nu] = ig^{\mu\nu}$$

and solve the non-trivial spacetime algebra in (25) by rewriting [19],

$$x^\mu = q^\mu - \frac{1}{2}\theta^{\mu\nu}Q_\nu \ , \ p^\mu = Q^\mu. \tag{26}$$

Thus the original theory should be reexpressed in the q, Q variables in the quantized version. This momentum dependent shift, a hallmark of NC quantum theory [8], appears here as a prerequisite for quantization.

Returning to more down to earth physics, the 3+1-dimensional Coulomb potential due to a point charge e in the noncommutative spacetime now turns out to be

$$\frac{e}{\mid x_i \mid} = \frac{e}{[(q_i^2 - \theta^{ij}q_iQ_j + O(\theta^2)]^{\frac{1}{2}}} \approx \frac{e}{\mid q_i \mid}(1 - \frac{\theta^{ij}q_iQ_j}{2q_i^2} + O(\theta^2)). \tag{27}$$

For simplicity, we have considered $\theta^{0i} = 0$ in the above relation. Clearly the second term in (27) reflects the dipole nature of the excitation, with a dipole moment $d_i = -\frac{1}{2}e\theta_{ij}Q^j$ [7].

The dipole feature is also apparent if we place the point charge e in an external electrostatic potential ϕ. The energy W of the system in the NC plane is

$$W = e\phi(x_i) = e\phi(q_i - \frac{1}{2}\theta_{ij}Q^j) = e\phi(q) - \frac{1}{2}e\theta_{ij}Q^j\partial^i\phi(q) + O(\theta^2) = e\phi(q) - d_iE^i + O(\theta^2), \tag{28}$$

where $E^i = -\partial^i\phi$ is the electric field. The dipole energy term is again reproduced. This constitutes the first part of the work.

Obviously the Polyakov formulation of a reparameterization invariant theory is more transparent than the Nambu-Goto version involving a square root action, that we have studied so far. In this part we discuss the Polyakov formulation of the above model and ascertain how far it is justified to consider the (previous) Nambu-Goto and Polyakov forms as equivalent even in a classical scenario. (This means we are not considering the quantum anomalies.) The Polyakov form of the unconstrained model (1) is,

$$\mathcal{L}_{\mathcal{P}} = -\sqrt{-\gamma}\gamma^{ab}\partial_a X_\mu \partial_b N^\mu. \tag{29}$$

[4]In the special case of the target space being $2+1$-dimensional [11, 12], the spin sector can be removed altogether by exploiting the constraints. This leads to the NC algebra $\{x^\mu, x^\nu\} \approx \epsilon^{\mu\nu\lambda}p_\lambda$. In particular, this means that $\rightarrow \{x^1, x^2\} \approx p_0 \approx m \equiv constant$. Hence, this is compatible with the constant c-number x_μ-noncommutativity, at least in the non-relativistic limit. It is not clear whether similar phenomenon will occur in higher dimensions.

The non-dynamical metric γ^{ab} can be eliminated to reproduce the Nambu-Goto Lagrangian (1) in the conventional way. Diffeomorphism and Weyl invariances allow us to enforce locally the conformal gauge, $\gamma^{ab} = diag(-1, 1)$, which reduces (29) to a simple form,

$$\mathcal{L}_{\mathcal{P}} = -[-(\dot{X}\dot{N}) + (X'N')]. \tag{30}$$

This will lead to two decoupled free wave equations for X^μ and N^μ, as mentioned before. However the constraint χ_3 will change the dynamics of X^μ to,

$$\ddot{X}^\mu - X''^\mu - N_\nu(\ddot{X}^\nu - X''^\nu)N^\mu = 0. \tag{31}$$

The Hamiltonian is

$$\mathcal{H}_{\mathcal{P}} = (PP_N) + (X'N') \ ; \ P_\mu = \dot{N}_\mu \ , \ P_{N_\mu} = \dot{X}_\mu. \tag{32}$$

Since we are considering a classical theory, for the moment we ignore the fact that the Hamiltonian in (32) is not positive definite and invoke the constraint $\chi_3 = N^2 - 1$ on this model. Note that exploiting the conformal gauge *before* invoking the constraint χ_3 is justified from our previous experience of working in Nambu-Goto picture where we saw that χ_3 does not spoil the invariances. The conformal gauge removes the reparameterization invariance and we must check the stability of χ_3 under time translation. However, time persistence of the successive constraints now generates an infinite chain of constraints of the following form,

$$\chi_3 = N^2 - 1 \rightarrow (NP) \rightarrow P^2 - N'^2 \rightarrow (PN'') - (P'N') \rightarrow P'^2 - N''^2 \rightarrow ...$$
$$... \rightarrow (P'N'') - (P''N'') \rightarrow P''^2 - N'''^2 \rightarrow ... \ . \tag{33}$$

Notice that *all* the constraints in (33) are in involution, *i.e.* the constraints are FCC, the phase space being canonical (5). Quite clearly this (Polyakov) constrained system is very different from the finite number of FCC and SCCs present in the Nambu-Goto version given in (7). In fact one can choose a gauge in the Nambu-Goto form [6] which is equivalent to the conformal gauge in the Polyakov form. However, the structure of the SCCs (inducing the NC) is fixed in the Nambu-Goto form. This is precisely the disparity, even in the classical scenario, between Nambu-Goto and Polyakov formulations in the presence of constraints, that we had set out to establish.

A correspondence between the two formulations can be obtained if in the Polyakov form we choose a suitable gauge. In particular let us choose $\psi_2 = (PP_N) - (X'N')$ of (7) as the gauge condition. In the chain (33), it will keep the FC nature of $\chi_3 = N^2 - 1$ intact. The gauge condition ψ_2 together with $(NP) = \psi_1$ are rendered to an SCC pair. Subsequently rest of the FCCs will drop out of the picture once the Dirac brackets are exploited, which are precisely the ones derived before (9,10). Hence, we recover the constraint structure of the Nambu-Goto form (7) without the FCCs χ_1 and χ_2, since we have fixed the conformal gauge. Thus, in the Polyakov form, whether NC is induced or not, depends on our choice of gauge. Obviously any P^μ-independent gauge will fail to generate NC in X^μ. In this sense, we conclude that the Polyakov form is more general than the Nambu-Goto form.

Lastly, let us comment on the non-positive definite nature of the Hamiltonian in (32) which can hinder its quantization. A possible way out is to invoke the ideas of t'Hooft [20] where we rewrite $H_P = \int \mathcal{H}_{\mathcal{P}}$ in the following way,

$$H_P = H_+ - H_- \ , \ \{H_+, H_-\} = 0, \tag{34}$$

where both $H_\pm$ are positive definite. Subsequently one quantizes the system by taking H_+ as the Hamiltonian and employ H_- as a constraint such that $H_- \mid Physical\ State >= 0$. Thus we can express $\mathcal{H}_\mathcal{P}$ in (32) as,

$$\mathcal{H}_\mathcal{P} = \frac{1}{4}[((P+P_N)^2 + (X'+N')^2) - ((P-P_N)^2 + (X'-N')^2)] \equiv \mathcal{H}_+ - \mathcal{H}_-, \tag{35}$$

and proceed with the constraint analysis of $\mathcal{H}_\mathcal{P}$.

Finally, let us make a passing comment on the yet to be investigated quantum theory. Note that the classical conformal invariance of the model will be broken by quantum anomalies. However, since the model is new, the structure of the anomaly and also whether any non-trivial anomaly vanishing constraints emerge, are some of the topics of interest.

To conclude, we have constructed a *field theoretic* toy model, which yields a noncommutative space-time, without any external influence. The noncommutativity is induced by the constraints of the theory via Dirac brackets. We have also derived the symplectic structure for the zero momentum Fourier modes, which coincides with the algebra of a spinning particle model [11]. The algebra for the non-zero momentum sector is much more complicated.

To put our work in the proper perspective, it should be stressed that generation of the spinning particle model [11] is actually a by product of our construction and not the main issue involved. What we have achieved here is an explicit field theoretic construction of a noncommutative spacetime. This can serve as an alternative to the noncommutative spacetime where *commutative* (*i.e.* ordinary) spacetime coordinates can be used at the expense of working in an extended phase space [22] with extra auxiliary degrees of freedom. The noncommutativity was induced via the Dirac brackets [13], the latter being necessary since the system has Second Class constraints [13]. In the extended space [22], these constraints are modified to First Class constraints [13] and so the Dirac brackets are not needed. Incidentally, the noncanonical (and especially operator valued) Dirac brackets are a hindrance to the subsequent quantization of the model. It is very important to note that the extended space [22] is (by construction) completely canonical, which a prerequisite for carrying out the canonical quantization programme. The noncommutativity is reproduced through the auxiliary variables. The extended space formulation of the present model will proceed along the lines of [15, 22]. Although straightforward, this is a separate problem by itself and is postponed for a future publication.

In this model, various traits of a string induced noncommutative quantum theory on the D-brane, such as dipolar nature of the basic excitations [7] and momentum dependent shifts in the interaction point [8] appear naturally in our noncommutative spacetime. The observation [9] that noncommutative and ordinary field theories are alternative descriptions of the same underlying theory, is corroborated here by showing that they are gauge equivalent. This has been achieved by the introduction of an additional spin field which endows the point charges with a spin and subsequently a magnetic moment.

Also, treating the present model as an explicit example, we show that, in the presence of constraints, exact equivalence between the Nambu-Goto and Polyakov formulations can not be established in a naive way even classically.

Appendix: The target space consists of "position" vectors $x^i \equiv \{X^\mu, N^\nu\}$ with the metric,

$$g^{ij} = \begin{pmatrix} 0^\mu_\nu & \delta^\mu_\nu \\ \delta^\mu_\nu & 0^\mu_\nu \end{pmatrix}$$

where 0^μ_ν and δ^μ_ν represent null and unit matrices respectively. Hence, the distance ds in spacetime $(ds)^2 = g_{ij}dx^i dx^j = dX^\mu dN_\mu$ leads to the induced metric,

$$(ds)^2 = \gamma_{ab} d\xi^a d\xi^b \ ; \ \gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial N_\mu}{\partial \xi^b} \ ; \ \xi^a \equiv (\tau, \sigma).$$

Thus $\sqrt{-det\ \gamma}$ reproduces the action (1). Furthermore, restricting the x^i-space such that the induced metric γ_{ab} is symmetric, one obtains the interpretation of $\sqrt{-det\ \gamma}\ d\tau d\sigma$ as the infinitesimal area in the $\tau - \sigma$ plane trivially [21].

The off-diagonal nature of the target space metric requires some comments. The motivation was the construction of a reparameterization invariant geometric field theory to describe a noncommutative spacetime and the structure of the metric is geared for that purpose. It might be useful to note that prior to the introduction of the compactifying constraint on N_μ, the X_μ and N_μ sectors are treated in an identical way, which is reflected by their symmetrical appearence in the expressions. Unfortunately, at the present level of research concerning this model, it is difficult to ascribe a physical significance regarding the form of the metric.

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