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# Welfare indicators: A review and new perspectives. 1. Measurement of inequality

*Summary* - The purpose of this paper is to present significant results on welfare theoretic approaches to income distribution based measurement problems. The topics covered are related to the measurement of inequality. Alternative forms of indices have been analyzed. The problem of ranking income distributions in terms of welfare, graphical techniques, different forms of equalizing transfers, stochastic dominance and inverse stochastic dominance have been studied extensively. Formal connections between these notions of orderings and dispersive ordering studied by statisticians is also discussed.

*Key Words* - Inequality; Indices; Welfare; Dominance.

## 1. Introduction

The two dimensions of an income distribution that spring to a layman's mind are the total and spread. That is, the size of the cake and how is it divided? Given, the population size, we ask the questions what is the mean income and how unequally are incomes distributed around the mean? Most people believe that given other things, a reduction in inequality should lead to an increase in the well-being of the society. However, there exists wide disagreement of views about how to measure inequality in an accurate way. Promotion of higher equality is an important issue in welfare economics. But traditional welfare economics does not offer much help so far as distributional issue is concerned (Sen (1973)). This probably explains why in empirical works some statistical measure of the dispersion of incomes is taken as an indicator of inequality. Although Dalton (1920) pointed out that the degree of inequality cannot be measured without introducing social judgements, Atkinson (1970), Kolm (1969) and Sen (1973) initiated the modern social welfare approach to inequality measurement. In the Atkinson-Kolm-Sen approach social judgements

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concerning indices of inequality are made explicit through the use of social welfare functions, which simply ranks income distributions in terms of preferences of the society. Each index derived through this approach is based on a set of distributional value judgements. It therefore becomes clear what distributional objectives are being incorporated as a result of adopting a certain index of inequality.

This paper is a survey of the literature on welfare indicators based on distributions of income. We restrict attention to the most important topics in the recent literature on measurement of inequality. We start the discussion by assuming that welfare depends solely on income. But income as the sole attribute of welfare is often is regarded as inappropriate. Therefore we also discuss the multidimensional approaches suggested in the literature. Further, attempts have been made to relate welfare orderings to dispersive orderings investigated by statisticians. The relevance of failure rate (hazard rate) studied in reliability theory is also explained in this context. From these perspectives the survey seems to be quite exhaustive. None of the recent related contributions appears to be so much informative. For further treatments of some of these issues, the reader is referred to several recent books, surveys and unified approaches (see note 1).

The paper is organized as follows. The next section of the paper presents the notation, definitions and preliminaries. Section 3 sets out the different postulates by which indices of inequality can be selected and presents the Dalton and the Atkinson-Kolm-Sen approaches to the measurement of inequality and some related topics. In Section 4 we make a rigorous discussion on inequality and welfare dominance. Finally, Section 5 concludes.

## 2. Notation, definitions and preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $X$ be a non- negative random variable defined on it that has finite positive expectation $\mu(X)$ $(\mu)$. $\Omega$ may be seen as a population of individuals or households and $X(\omega)$ as the income of the individual (household) $\omega \in \Omega$. To facilitate the presentation we speak of individuals and their incomes, but do not exclude other interpretation. We may think the income distribution as a non- negative random variable $X$ with distribution function $F(x) = P(X \leq x)$ and finite positive expectation $\mu(X)$. If $\Omega = \{\omega_1, \dots, \omega_n\}$ and P gives equal mass $\frac{1}{n}$ to each $\omega_i$, we write $x_i = X(\omega_i)$ and $X = (x_1, x_2, \dots, x_n)$, for short. Let $\mathcal{F}$ be the class of distribution functions on $[0, \infty)$. Let $F \in \mathcal{F}$, we define:

$$H(y) = F^{-1}(y) = inf\{x : F(x) \geq y\}) \tag{1}$$

$0 \leq y \leq 1$. This is the left continuous version of the inverse of F.

We define, for $F \in \mathcal{F}$

$$F^{r+1}(t) = \int_0^t F^r(u)du \tag{2}$$

for all $t \in [0, \infty)$, where $r \geq 1$ is a positive integer. Obviously $F^1 = F$.

It is well known that:

$$F^r(t) = \frac{1}{(r-1)!}\int_0^t (t-y)^{r-1} dF(y)$$

for all $t \geq 0$. Analogously, we define:

$$H^{r+1}(t) = \int_0^t H^r(v)dv \tag{3}$$

for all $t \in [0, 1]$. Obviously $H^1(t) = H(t)$. $H^1(t)$ represents the income of the 100t-percent poorest individuals in the distribution $X$ and is referred to as the quantile function. Thus, the mean income $\mu(X)$ can now be calculated as

$$\mu(X) = \int_0^1 H(t)dt\,.$$

Similarly, $H^2(t)$ represents the aggregate income possessed by the 100t-percent poorest individuals in the distribution $X$. The sequences $F^{r+1}(t)$ and $H^{r+1}(t)$ will assure us to define in Section 4 a sequence of stochastic dominances.

In this context a fundamental order is the Lorenz order. To define the Lorenz order, consider the Lorenz function $L_X : [0, 1] \to [0, 1]$,

$$L_X(t) = \frac{1}{\mu(X)}\int_0^t H(s)ds \tag{4}$$

with $0 \leq t \leq 1$.

The graph of the Lorenz function is the *Lorenz curve*. An inequality measure is a functional that assigns a real number to every income distribution. One of the most common measures of inequality is the Gini index, which is defined as:

$$G(X) = 1 - 2\int_0^1 L_X(p)dp = 1 - \frac{2}{\mu(X)}\int_0^1\int_0^p H(t)\,dtdp\,. \tag{5}$$

As we pointed out in the Introduction it was suggested in the pioneering paper of Dalton (1920) that any measure of income inequality has an underlying social welfare function. Dalton's approach was developed further in Atkinson (1970). He defines the *equally distributed equivalent (EDE) income* of $X = (x_1, x_2, \ldots, x_n)$ to be that level of income which, if enjoyed by every individual, would make the total welfare exactly equal to the total welfare generated by $X$. This approach corresponds to the 'certainty equivalent' approach in decision theory under risk.

In this framework it seems to us that the natural way to characterize the EDE income is Chisini's functional approach regarding the mean (Chisini(1929)).

> "The search for a mean has the purpose of simplifying a given question by substituting a single summary variable for too many values and leaving the overall picture of the problem unchanged."

Consider a real-valued variable $X$ with distribution function $F$ and let $\mathcal{J}$ be a real-valued functional on the space of distribution functions. This functional represents the total social welfare. Let $I_\mu(x)$ be the indicator function of the set $[\mu, \infty]$, i.e. the distribution function of a mass concentrated at $\mu$. If there exists a unique real number $\mu$ solving

$$\mathcal{J}(F) = \mathcal{J}(I_\mu) \tag{6}$$

then $\mu$ is called the *mean of F for the evaluation of the welfare*, shortly, the *welfare mean of* $F$. To indicate its dependence on $\mathcal{J}$ and $F$, we denote it by: $\mu_{\mathcal{J}}(F)$ or $\mu_{\mathcal{J}}(X)$.

In other words, the mean is chosen so that one is indifferent, in order to evaluate the welfare, between the given cumulative distribution and a distribution concentrating all of its mass on $\mu$. For a characterization of $\mu_{\mathcal{J}}(X)$ we follow de Finetti (1931). In order to extend the Nagumo (1930)-Kolmogorov (1930) result to this context, de Finetti employs two postulates, monotonicity and associativity.

Let $\mathcal{P}_I$ denote the space of probability distributions with mass concentrated on some compact interval $A \subset R$.

**Strict monotonicity**

Let $F_1$ and $F_2$ be in $\mathcal{P}_A$. If $F_1(x) \leq F_2(x)$ for every $x$ (with strict inequality for at least one $x$) then

$$\mu_{\mathcal{J}}(F_1) > \mu_{\mathcal{J}}(F_2).$$

Consider now a distribution $F^*$ that results from a convex combination of two distributions $F_1$ and $F_2$,

$$F^* = \lambda F_1 + (1 - \lambda) F_2$$

with some $\lambda \in ]0, 1[$. The property of associativity requires that the mean of $F^*$ is unchanged if one of the two component distributions is replaced by another one with the same mean.

**Associativity**

If $F_1$, $F_2$ and $F_3$ in $\mathcal{P}_A$ are such that

$$\mu_{\mathcal{J}}(F_1) = \mu_{\mathcal{J}}(F_2)$$

then for every $F_3 \in \mathcal{P}_I$ and $\lambda \in ]0, 1[$

$$\mu_{\mathcal{J}}(\lambda F_1 + (1 - \lambda)F_3) = \mu_{\mathcal{J}}(\lambda F_2 + (1 - \lambda)F_3).$$

If, in the evaluation of welfare, we are indifferent between two distributions, this indifference is preserved if both distributions are mixed with a third distribution in the same proportions. In de Finetti's work, the characterisation of $\mu_{\mathcal{J}}(F)$ is the following:

**Theorem 2.1.** *Let A be a compact interval and let $\mu_{\mathcal{J}}$ be defined by* (6). *$\mu_{\mathcal{J}}$ satisfies strict monotonicity and associativity if and only if there exists a function u, continuous and strictly monotone, such that for every $F \in \mathcal{P}_A$*

$$\mu_{\mathcal{J}}(F) = u^{-1}\left(\int_A u(x)dF(x)\right) \tag{7}$$

*where u is unique up to a positive affine transformation.*

*If $u(x) = x$ is chosen in* (7), *we get the the arithmetic mean, i.e. the mathematical expectation, $\mu_F$ of F.*

The function $u(x)$ is interpreted as the individual utility function, and $u(\mu_{\mathcal{J}}(F))$ as the welfare index of $\mathcal{J}$. In a model of decision under risk, the quasi-linear mean in (7) corresponds to the expected utility index and $u$ to the von Neumann–Morgenstern utility function. In this framework, Ramsey (1926) and von Neumann–Morgenstern (1947) provided different axiomatizations. For a survey see Muliere and Parmigiani (1993).

$$W(F) = u(\mu_{\mathcal{J}}(F)) = \int_A u(x)dF(x) = E(u(X))$$

expresses the social welfare of a society with income distribution $F$. If $u(x)$ is strictly concave, the individual utility function $u$ increases at a decreasing rate, hence the social welfare increases when income is transfered from a richer to a poorer individual (for a review on welfare means, see Mosler and Muliere (1998)).

## 3. Measurement of inequality

In what follows an income distribution for a homogeneous population consisting of n persons ($n \geq 2$) ) is a random variable $X = (x_1, x_2, ..., x_n)$, taking values $x_i$ with probability $\frac{1}{n}$, where $x_i \geq 0$ is the income of individual i. The vector $X = (x_1, x_2, ..., x_n)$ is an element of $D^n$, the nonnegative orthant of the n-dimensional Euclidean space $R^n$ with the origin deleted. Deletion of origin from the domain ensures that there is at least one person with positive income. The set of all income distributions is $D = \cup_{n \in N} D^n$, where $N$ is the set of natural numbers. For any function $g : D \to R^1$, the restriction of g on $D^n$ will be denoted by $g^n$. For all $n \in N$ and $X \in D^n$, we will write $\mu(X)$ or $\mu$ for the mean of $X$ and $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ for the illfare ranked permutation of $X$, that is, $x_1^* \leq \cdots \leq x_n^*$. For all $n \in N$, $1^n$ will stand for the n-coordinated vector of ones. In view of our assumption, for any $n \in N$, $X \in D^n$, $\mu(X) > 0$. Sometimes we will use $D_+^n$, the strictly positive part of $D^n$, or $R_+^n$, the nonnegative part of $R^n$, as the set of income distributions in an $n$-person population. The corresponding sets of all income distributions will be denoted by $D_+$ and $R_+$ respectively.

### 3.1. *Properties for an index of inequality*

We begin this subsection with the widest possible definition of an inequality index, allowing both population size and total income to be variable. The term inequality index is used to indicate a continuous function $I : D \to R$ such that for any $m, n \in N$, $X \in D^n$, $Y \in D^n$

$$I^m(X) \leq I^n(Y) \tag{8}$$

will mean that the income distribution $X$ is no more unequal than distribution $Y$. Thus, each $I : D \to R$ is associated with a sequence $\{I^n : D^n \to R\}_{n \in N}$, one for each population size $n$.

Whether a specific type of change in incomes will keep inequality of income distribution unchanged is a matter of subjective evaluation. An inequality index $I : D \to R$ corresponds to the concept of relative inequality if proportional changes in all incomes do not change inequality, that is, for all $n \in N$ and for all $x \in D^n$,

$$I^n(X) = I^n(cX) \tag{9}$$

where $c > 0$ is any scalar. In contrast, an index $I : D \to R$ is an absolute index if it is invariant to equal absolute translation of incomes, that is, for all $n \in N$ and $X \in D^n$,

$$I^n(X) = I^n(X + c1^n) \tag{10}$$

where $c$ is a scalar such that $(X + c1^n) \in D^n$. Clearly, while in the former case income ratios are a source of envy, in the latter case people's feeling about deprivation due to higher incomes depends on absolute income differentials (see note 2). The classes of inequality indices satisfying invariance conditions (9) and (10) respectively may be rather large. Certain desirable properties can reduce the number of allowable indices. It has been argued in the literature that an inequality index $I : D \rightarrow R$, whether relative or absolute, should satisfy three postulates, namely, symmetry, the principle of population and the Pigou-Dalton transfers principle, which are stated below.

**Symmetry (SYM)**

For all $n \in N$, $X \in D^n$, $I^n(X) = I^n(Y)$, where $Y$ is any permutation of $X$.

Symmetry means that inequality remains unchanged under any reordering of incomes. Under SYM any two individuals can trade their positions. One implication of symmetry is that we can define an index of inequality directly on ordered distributions.

Quite often we become interested in cross population comparisons of inequality. The following postulate, suggested by Dalton (1920), enables us to compare inequality over different population sizes.

**Population Principle (POP)**

For all $n \in N$, $X \in D^n$, $I^n(X) = I^{mn}(Y)$ where $Y$ is the $m$-fold replication of $X$, that is, $Y = (x^{(1)}, x^{(2)}, \ldots, x^{(m)})$ with each $x^{(j)}$ being $X$.

According to POP, if a population is replicated several times, then the inequality levels of the original and the replicated populations are the same. In other words, POP views inequality as an average concept. Using replications, two distributions with different population sizes can be made to possess the same population size and POP keeps inequality unchanged under replications. It may be noted that POP is a property of all inequality indices that are defined on the continuum.

A third property which can be regarded as a central property of inequality indices is the Pigou (1912)-Dalton (1920) transfers principle.

The Pigou-Dalton transfers principle demands that a transfer of income from a person to anyone with a lower (higher) income should decrease (increase) inequality. We say that $X \in D^n$ is obtained from $Y \in D^n$ by a progressive transfer if there exist two persons $i$ and $j$ such that $x_k = y_k$ for all $k \neq i, j$; $x_i - y_i = y_j - x_j > 0$; $y_i < x_i < y_j$; and $y_i < x_j < y_j$. That is, $X$ and $Y$ are identical except for a positive transfer of income from person j to person i who has a lower income than $j$. Further, the transfer is such that it does not change the relative positions of the affected persons, that is, the donor of the transfer does not become poorer than the recipient. We can equivalently say that $Y$ has been obtained from $X$ by a regressive transfer.

**Pigou-Dalton Transfers Principle(PDT)**

For all $n \in N$, $Y \in D^n$, if $X$ is obtained from $Y$ by a progressive transfer, then $I^n(X) < I^n(Y)$

Thus, PDT means that a progressive transfer reduces inequality. Equivalently, a regressive transfer increases inequality. If I treats individuals symmetrically, that is, if the index of inequality I satisfies SYM then PDT allows only those transfers that do not alter rank orders of individuals. Properties SYM, POP and PDT are regarded as basic postulates of an index of inequality because of their consistency with the Lorenz/absolute Lorenz orderings (see Section 4).

It is often argued in the literature that higher weight should be attached to transfers lower down the income scale. This idea is captured by the principle of diminishing transfers that assigns more weight to a progressive transfer between individuals with a given income difference if the incomes are lower than when they are higher (Kolm (1976a)). More formally, $I : D \to R$ satisfies the principle of diminishing transfers if for all $n \in N$, $Y \in D^n$, $X$ is obtained from $Y$ by a progressive transfer of income from the person with income $x_i + h$ to the person with income $x_i$, then for a given $h > 0$, the magnitude of decrease in inequality $[I^n(Y) - I^n(X)]$ is higher the lower is $x_i$.

A stronger version of the principle of diminishing transfers is transfer sensitivity considered by Shorrocks and Foster (1987).

**Favorable composite transfer (FACT)**

For all $n \in N$, $Y \in D^n$, $X$ is obtained from $Y$ by a *favorable composite transfer* if there exist i, j, k and l ($i < j \leq k < l$) such that:

$$x_h = y_h \tag{11}$$

for all $h \neq i, j, k, l$

$$x_i - y_i = y_j - x_j;\, y_i < x_i < y_j;\, y_i < x_j < y_j \tag{12}$$

$$y_k - x_k = x_l - y_l;\, x_k < y_k < x_l;\, x_k < y_l < x_l \tag{13}$$

$$y_i < x_k \tag{14}$$

$$x_i^2 + x_j^2 + x_k^2 + x_l^2 = y_i^2 + y_j^2 + y_k^2 + y_l^2\,. \tag{15}$$

Conditions (11)-(15) say that a progressive transfer and a regressive transfer are jointly required to arrive at distribution $X$ from distribution $Y$ and that the progressive transfer involves lower incomes than the regressive transfer. Since means of the two distributions are the same, (11) along with (15) ensures that the variance of the original distribution does not get affected.

**Transfer Sensitivity (TRS)**

For all $n \in N$, $Y \in D^n$, $I^n(X) < I^n(Y)$, whenever $X$ is obtained from $Y$ by a FACT.

Transfer sensitivity requires inequality to decrease under a FACT, which is composed of a progressive transfer and a regressive transfer, the former taking place at lower incomes than the latter such that the variance of the distribution does not change.

A positional version of the diminishing transfers principle is the *principle of positional transfer sensitivity*, requiring that a transfer from any person to someone who has a lower income, given that there is a fixed proportion of population between them, should attach more weight at the lower end of the distribution (see Mehran (1976), Kakwani (1980a) and Zoli (1999)).

Let $\Delta I^n_{i+t,i}(Y^*(\delta))$ be the reduction in inequality in $Y^*$ due to a (rank preserving) progressive transfer of $\delta$ units of income from the person with rank $(i+t)$ to the person with rank $i$, where $t > 0$ is an integer.

**Principle of Positional Transfer Sensitivity (PPT)**

For all $n \in N$ and $Y^* \in D^n$ and for any pair of individuals i and j, $\Delta I^n_{i+t,i}(Y^*(\delta)) > \Delta I^n_{j+t,j}(Y^*(\delta))$, where $j > i$.

Note that for convenience PPT has been defined on ordered distributions. It implies that a combination of a (rank preserving) progressive transfer and a (rank preserving) regressive transfer of the same denomination, where the latter is taking place at higher incomes than the former reduces inequality.

It may be worthwhile to mention that recent experimental studies have not approved PDT unambiguously (see, for example, Amiel and Cowell (1992), Ballano and Ruiz-Castillo (1993) and Harrison and Seidl (1994)). This motivated several researchers to suggest weaker versions of PDT (see Eichhorn and Gehrig (1981), Castagnoli and Muliere (1990) and Mosler and Muliere (1996)). As weaker forms of PDT, Mosler and Muliere (1996) considered the principle of transfers about $\theta$ and star-shaped principle of transfers at $\theta$, where $\theta$ may be a given constant, a function of mean income or a quantile of the income distribution.

**Principle of transfers about $\theta$**

Given a fixed $\theta > 0$ and the non-identical ordered distributions $X^*, Y^* \in D^n$ with the same mean, we say that $X^*$ has been obtained from $Y^*$ by a sequence of transfers about $\theta$ if $x^*_i \leq \theta$ for $x^*_i - y^*_i \geq 0$, $x^*_i \geq \theta$ for $x^*_i - y^*_i \leq 0$.

That is, a transfer about $\theta$ is a rank preserving progressive transfer from a person with income above $\theta$ to someone who has a lower income than $\theta$. For instance, the distribution (100, 480, 490) results from (100, 470, 500) by a transfer about $\theta = 490$ but not about $\theta = 470$.

**Principle of transfers next to $\theta$**

Given a fixed $\theta > 0$ and non-identical ordered distributions $X^*, Y^* \in D^n$ with the same mean $\mu$, we say that $X^*$ is obtained from $Y^*$ by a sequence of transfers next to $\theta$ if there is some k with $y_k^* \leq \theta \leq y_{k+1}^*$, $x_i^* \geq y_i^*$ if $i \leq k$, $x_i^* \leq y_i^*$ if $i \geq k$, $y_k^* \leq x_i^* \leq y_{k+1}^*$, if $x_i^* \neq y_i^*$.

That is, a transfer next to $\theta$ is a rank preserving progressive transfer where only incomes next to $\theta$ get affected. Thus, we generate (100, 480, 490) from (100, 470, 500) by a transfer next to $\theta = 470$.

**Star-shaped principle of transfers**

A star-shaped transfer at $\theta$ is either a transfer about $\theta$ or a transfer next to $\theta$.

We will say that an inequality index satisfies the star-shaped principle of transfers at $\theta$ if it reduces under a star-shaped transfer at $\theta$.

We will explain the role of TRS, PPT and star-shaped transfers at $\theta$ in ranking alternative distributions of income in Section 4. We have stated all the properties in this section in terms of inequality. They all can be restated in terms of welfare. For instance, welfare counterparts to SYM and POP will require respectively that for any $n \in N$ and $X \in D^n$ the social welfare function $W^n : D^n \to R$ remains invariant under permutations of incomes and replications of populations. Similarly, PDT demands that welfare increases under a progressive transfer and PPT will mean that a progressive transfer between two individuals, with a given proportion of persons between them, is more welfare enhancing if it takes place at lower income levels.

*3.2. The Dalton approach*

Dalton (1920, p. 394) suggested to measure the inequality using:

> the ratio of total economic welfare attainable under an equal distribution to the total economic welfare attained under the given distribution.

Dalton chose the symmetric utilitarian form of social welfare function, that is, the welfare value of any income distribution $X \in D^n$ is given by $\sum_{i=1}^n u(x_i)$, where the identical individual utility function u is increasing and strictly concave, and $n \in N$ is arbitrary. Assuming positivity of the utility function, the Dalton index is defined by $I_{\mathrm{D}} : D \to R$, where for all $n \in N$ and $X \in D^n$,

$$I_{\mathrm{D}}^n(x) = 1 - \frac{\sum_{i=1}^n u(x_i)}{nu(\mu)} \tag{16}$$

$I_{\mathrm{D}}^{n}$ is bounded between zero and one, where the lower bound is achieved whenever the incomes are equal. This index tells us by how much (in relative terms) we can increase social welfare by distributing incomes equally. Since $u$ is cardinal, it is necessary that $I_{\mathrm{D}}^{n}$ should remain invariant under affine transformations of $u$. But $I_{\mathrm{D}}^{n}$ does not satisfy this property. For a discussion of Dalton's approach see also Ferreri (1978, 1980), Benedetti (1980), Giorgi (1984, 1985) and Muliere (1987).

### *3.3. The Atkinson-Kolm-Sen approach and related issues*

The form of social welfare function chosen by Dalton(1920) is quite restrictive. Therefore, following Sen (1973) we assume that ethical judgements on alternative distributions of income are summarized by the social welfare function $W : D \to R$, where W is ordinally significant. It is further assumed that for all $n \in N$, $W^n$ is continuous, increasing and strictly $S$-concave. Continuity ensures that minor observational errors on incomes does not give rise to abrupt jump in the value of the social welfare function. Increasing-ness means that if we increase any income, keeping the remaining fixed, social welfare increases. Increasing-ness is analogous to the strong Pareto preference condition. Strict $S$-concavity, as we will see, demands that a rank preserving transfer of income from a person to anybody who has a lower income increases social welfare (see note 3). Given any $X \in D^n$, the Atkinson (1970)-Kolm (1969)-Sen (1973) equally distributed equivalent (EDE) income is defined as that level of income which if given to everybody will make the existing distribution $X$ ethically indifferent (indifferent as measured by $W^n$). Thus, $x_e$ is implicitly defined by

$$W^n(x_e 1^n) = W^n(X)\,. \tag{17}$$

Given assumptions about $W^n$, we can solve (17) uniquely for $x_e$:

$$x_e = \mu_{\mathcal{J}}(X)\,. \tag{18}$$

By continuity of $W^n$, $\mu_{\mathcal{J}}(X)$ is a continuous function. Furthermore $\mu_{\mathcal{J}}(X)$ is a specific numerical representation of $W^n$, that is,

$$W^n(X) \geq W^n(Y) \iff \mu_{\mathcal{J}}(X) \geq \mu_{\mathcal{J}}(Y) \iff x_e \geq y_e\,. \tag{19}$$

Thus, one income distribution is socially better than another if and only if its EDE income is higher. The indifference surfaces of $\mu_{\mathcal{J}}(X)$ are numbered so that

$$\mu_{\mathcal{J}}(c1^n) = c \tag{20}$$

with $c > 0$.

The Atkinson-Kolm-Sen (AKS) index of inequality is defined by $I_{\text{AKS}} : D \to R$, where for all $n \in N$ and $X \in D^n$,

$$I^n_{\text{AKS}}(X) = 1 - \frac{\mu_{\mathcal{J}}(X)}{\mu(X)} . \tag{21}$$

$I_{\text{AKS}}$ is continuous, satisfies SYM and PDT (due to strict $S$-concavity of $W$) and bounded between zero and one, where the lower bound is achieved whenever incomes are equally distributed. It fulfils POP if $\mu_{\mathcal{J}}(X)$ fulfils the same. It gives the fraction of aggregate income that could be saved without any welfare loss if society distributed incomes equally. Since contours of $\mu_{\mathcal{J}}(X)$ are numbered, we can write the denominator of (21) as $\mu_{\mathcal{J}}(\mu(X)1^n)$. Hence $I_{\text{AKS}}$ can be interpreted as the proportional welfare loss that arises due to existence of inequality. Given a functional form for $I_{\text{AKS}}$, we have:

$$\mu_{\mathcal{J}}(X) = \mu(X)(1 - I^n_{\text{AKS}}(X)) . \tag{22}$$

Note that $\mu_{\mathcal{J}}(X)$ here corresponds to an ordinal social welfare function. $W^n$ can now be obtained by taking an increasing transformation of $\mu_{\mathcal{J}}(X)$ (since W is ordinal). We note that in (22) social welfare has been expressed as an increasing function of efficiency (the mean income) and a decreasing function of inequality. When efficiency considerations are absent (mean income is fixed), an increase in social welfare is equivalent to a reduction in inequality and vice-versa. Thus, the AKS inequality index is normatively significant or exact in the sense that it implies and is implied by a social welfare function.

Since in (21) two functions of incomes appear in a ratio form, it is intuitively reasonable to interpret $I_{\text{AKS}}$ as a relative index. However, in general it is not a relative index. $I_{\text{AKS}}$ is a relative index if and only if $\mu_{\mathcal{J}}(X)$ is linear homogeneous, which because of ordinal equivalence between $W^n$ and $\mu_{\mathcal{J}}(X)$ means homotheticity of $W^n$ (see Blackorby and Donaldson (1978) and Chakravarty (1990)) (see note 4).

In order to illustrate the formula in (21), suppose that the social welfare function is of symmetric utilitarian type. Assume that $X \in D^n_+$. It is well known (see de Finetti (1931), Pratt (1964), Atkinson (1970) and Daboni (1982)) that $\mu_{\mathcal{J}}(X)$ is linear homogeneous if and only if

$$u_r(x) = a\frac{x^r}{r} + b \tag{23}$$

with $a > 0$, $r \neq 0$ and $b \in R$.

$$u_r(x) = a log x + b \tag{24}$$

with $a > 0$ and r = 0.

Then

$$\mu^r_{\mathcal{J}}(X) = \left[\frac{1}{n}\sum_{i=1}^{n} x_j^r\right]^{\frac{1}{r}} \tag{25}$$

or

$$\mu_{\mathcal{J}}(X) = \left[\prod_{i=1}^{n} x_j\right]^{\frac{1}{n}} . \tag{26}$$

Therefore the only linear homogeneous means are the power-mean and the geometric mean. Consequently, the class of power-mean and the geometric mean is characterized by reflexivity, strict monotonicity, associativity and linear homogeneity. For $r = 0$, we obtain the geometric mean, whereas for $r = 1$ we get the arithmetic mean, for $r = -1$ it becomes the harmonic mean. The parameter $r$ determines the curvature of the social indifference surfaces. For any finite value of $r < 1$, the welfare contour becomes strictly convex to the origin and the degree of convexity increases as r decreases. As $r \to -\infty$, $\mu^r_{\mathcal{J}}(X) \to min_i(x_i)$ the Rawlsian maximin social welfare function (Rawls(1971)). On the other hand, as $r \to 1$, $\mu^r_{\mathcal{J}}(X) \to \mu(X)$, the mean income, which ignores distributional consideration and judges social welfare on the basis of size only. Therefore, if $r \leq 1$ (this means that u is concave) we obtain the inequality

$$x_e = \mu^r_{\mathcal{J}}(X) \leq \mu(X) .$$

The AKS index of inequality associated with the welfare function in (25) and (26) is the Atkinson (1970) index given by:

$$I^n_r(X) = 1 - \frac{1}{\mu(X)}\left[\frac{1}{n}\sum_{i=1}^{n} x_i^r\right]^{\frac{1}{r}} \tag{27}$$

or

$$I^n_r(X) = 1 - \frac{1}{\mu(X)}\left[\prod_{i=1}^{n} x_i\right]^{\frac{1}{n}} . \tag{28}$$

$I^n_r$ satisfies TRS for all values of $r < 1$. For a given $X$, $I^n_r$ is decreasing in $r$. As the value of $r$ decreases greater weight is attached to transfers at the lower end of the profile. As $r \to -\infty$, $I^n_r \to 1 - min_i(\frac{x_i}{\mu(X)})$, the relative maximin index, which corresponds to the maximin criterion.

An alternative of interest arises from the Gini social welfare function $\mu_G : D \to R$, where for all $n \in N$, $X \in D^n$,

$$\mu_G(X) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_i^* \tag{29}$$

which is a rank order weighted average of individual incomes, where the weight attached to the ith ranked income is independent of the income distribution. With a given rank order of incomes the Gini welfare function is linear and that is why the Gini index does not correspond to a strictly quasi-concave social welfare function (Newbery (1970), Kats (1972), Sheshinski (1972), Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973), Lambert (1985) and Chakravarty (1988, 1990)). The resulting AKS index of inequality is the well-known Gini index:

$$I_{\mathrm{G}}^{n}(X) = \frac{1}{2n^2\mu(X)} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \,. \tag{30}$$

As a third example we consider the Bonferroni social welfare function $\mu_B : D \to R$, where for all $n \in N$ and $X \in D^n$,

$$\mu_B(X) = \frac{1}{n} \sum_{i=1}^{n} \mu_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} x_j^* \tag{31}$$

where $\mu_i$ is the ith partial arithmetic mean, that is, $\mu_i = \frac{1}{i} \sum_{j=1}^{i} x_j^*$. The resulting AKS index becomes the Bonferroni index:

$$I_B^n(X) = 1 - \frac{1}{n\mu(X)} \sum_{i=1}^{n} \mu_i \,. \tag{32}$$

Strictly speaking, Bonferroni (1930) suggested the use of

$$I_{B*}^n = \frac{n-1}{n} I_B^n$$

as an index of inequality. Nygard and Sandstrom(1981) referred to $I_B^n$ as the Bonferroni index. One major limitation of $I_{B*}^n$ is that if we write it as an AKS index, then the underlying social welfare function becomes independent of the income of the richest person, however high or low it may be. Another important difference between the two versions of the Bonferroni index arises in the context of maximal inequality. Champernowne (1974) stipulated that in the limit as the number of incomes increases when one individual gets all the income, an inequality index should tend to the value one. This property is satisfied by $I_B^n$ but not by $I_{B*}^n$, because the maximal value for the former is $(1-\frac{1}{n})$ and for the latter it is 1. However, they both have same type of transfer sensitivity property. For a rank preserving income transfer in the distribution $X^*$ from a person with rank $j$ to another person with rank $i$ $(i < j)$ the reductions in $I_B^n$ and $I_{B*}^n$ become directly proportional to $\sum_{t=i}^{j-1} \frac{1}{t}$. Therefore, for a fixed number of persons $(j-i-1)$ between the donor $j$ and the recipient $i$, a progressive transfer is valued more by these indices if the transfer occurs at lower income levels. That is, they satisfy PPT. In contrast, for the Gini index the reduction due to the same progressive transfer depends on the difference $(j-i)$, which shows that given the difference $(j-i)$, the Gini index is sensitive to transfers in the same way whether they take place at the top of the income distribution or they concern low incomes and hence it fails to demonstrate positional transfer sensitivity. (For a comparison of Bonferroni index and Gini index in term of social welfare see Benedetti (1986).) However, one major shortcoming of the two Bonferroni indices is that they violate the principle of population and this makes them unsuitable for comparison of inequality across different-sized populations, but the Gini index is suitable in this context (see note 5).

It is therefore clear that to every homothetic social welfare function, there corresponds a different index of inequality and vice-versa. For instance, we can derive welfare functions associated with the Theil(1967) entropy index and the coefficient of variation. These indices will differ depending on the corresponding social welfare functions.

The concept of absolute inequality was introduced by Kolm (1976, 1976a). Blackorby and Donaldson (1980) made a detailed investigation on the properties of the social welfare functions associated with alternative absolute inequality indices. The Blackorby-Donaldson-Kolm (BDK) index of inequality is defined by $A_{\mathrm{BDK}} : R_+ \to R$, where for all $n \in N$ and $X \in R^n_+$,

$$A^n_{\mathrm{BDK}} = \mu(X) - \mu_J(X)\,. \tag{33}$$

$A_{\mathrm{BDK}}$ is continuous, strictly $S$-convex and bounded from below by zero, where this bound is achieved whenever incomes are equal. It satisfies POP if $\mu_{\mathcal{J}}(X)$ satisfies the same. It gives the per capita income that could be saved if society distributed incomes equally without any welfare loss. It also determines the size of absolute welfare loss associated with the existence of inequality.

Since in (33) two functions appear in a difference form, it is reasonable to regard $A^n_{\mathrm{BDK}}$ as an absolute index. $A^n_{\mathrm{BDK}}$ is an absolute index if and only if $\mu_{\mathcal{J}}$ is unit- translatable, that is

$$\mu_J(X + c1^n) = \mu_J(X) + c \tag{34}$$

for all $X$ and $c$, where $c$ is a scalar such that $(X + c1^n) \in R^n_+$. Since $\mu_J(X)$ and $W^n$ are ordinally equivalent, unit translatability of $\mu_{\mathcal{J}}(X)$ means that $W^n$ is translatable (see Blackorby and Donaldson (1980) and Chakravarty (1990)) (see note 6). From policy point of view the absolute index determines the total cost of per capita inequality in the sense that it tells us how much must be added in absolute terms to the income of every member in an $n$-person society to reach the same level of social welfare that would be achieved if everybody enjoyed the mean income of the current distribution.

The only quasi-linear unit-translatable means are obtained by setting

$$u(x) = \gamma x + \beta \tag{35}$$

with some $\gamma > 0$ and $\beta \in R$, or

$$u(x) = -\gamma \exp(-\alpha x) + \beta \tag{36}$$

with $\gamma > 0$, $\alpha > 0$ and $\beta \in R$. The first is the arithmetic mean and the second is the exponential mean.

The absolute index considered by Kolm (1976) is

$$A_\alpha^n(X) = \frac{1}{\alpha} \log \frac{1}{n} \sum_{i=1}^{n} e^{\alpha(\mu - x_i)} = \mu - \left[ -\frac{1}{\alpha} \log \frac{1}{n} \sum_{i=1}^{n} e^{-\alpha x_i} \right] \tag{37}$$

with $\alpha > 0$ where $n \in N$ and $X \in R_+^n$ are arbitrary. The third bracketed term in the second expression on the right hand side of (37), which we denote by $\mu_{\mathcal{J}}^\alpha$, is the equally distributed equivalent income associated with $A_\alpha^n$. Pollak (1971) suggested the use of

$$W_\alpha^n(X) = -\sum_{i=1}^{n} e^{-\alpha x_i}$$

with $\alpha > 0$, as a social welfare function. This function can be rewritten as

$$W_\alpha^n(X) = -n e^{(-\alpha \mu_{\mathcal{J}}^\alpha)}.$$

That is, the equally distributed equivalent income of the Kolm index is ordinally equivalent to the Pollak welfare function. Alternatively, we can start with the Pollak function and show that the corresponding BDK index is the Kolm index, which satisfies TRS for all $\alpha > 0$. The parameter $\alpha$ determines the curvature of the social indifference surfaces. As $\alpha \to \infty$, $\mu_{\mathcal{J}}^\alpha$ tends to the maximin criterion and $A_\alpha^n$ approaches $\mu(X) - min_i(x_i)$, the absolute maximin index.

The equally distributed equivalent (EDE) incomes $\mu_{\mathcal{J}}^r$ and $\mu_{\mathcal{J}}^\alpha$ satisfy a strict separability condition, which says that for any partitioning of the population into two or more subgroups, the aggregate EDE income can be calculated using subgroup EDE incomes. This property of these two functions enables the Atkinson and the Kolm-Pollak indices to decompose into between group and within group components, where the within group inequality is the population share weighted average of subgroup inequality levels and the between group component is defined as the inequality that would result if each person enjoys his subgroup's EDE income (see Blackorby, Donaldson and Auersperg (1981)) (see note 7).

Some welfare functions are both homothetic and translatable. Such welfare functions are called distributionally homothetic (Blackorby and Donaldson (1980)) (see note 8). Examples are the Gini and Bonferroni welfare functions. We can therefore generate both relative and absolute indices from such welfare functions. For instance, using the Gini and Bonferroni welfare functions in (33) we get the Gini and Bonferroni absolute indices which are given respectively by:

$$A^n_{\mathrm{G}}(X) = \mu(X) - \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x^*_i \tag{38}$$

and

$$A^n_{\mathrm{B}}(X) = \mu(X) - \frac{1}{n}\sum_{i=1}^{n}\mu_i\,. \tag{39}$$

An inequality index of this type is called a compromise index-when its relative form is multiplied by the mean income we get an absolute index and conversely, if the absolute version is divided by the mean income the resulting index becomes relative. (For further examples of such indices, see Ebert (1988b) and Chakravarty (1990)).

**Remark 4.1.** A comparison between utility based indices is possible using a comparison between $\mu_{\mathcal{J}}$. Let $\mu_{u_1}(F)$ and $\mu_{u_2}(F)$ be quasi-linear means with two different utility functions $u_1$ and $u_2$. If $u_1$ is increasing then

$$\mu_{u_1}(F) \geq \mu_{u_2}(F)$$

holds for every $F$ if and only if $u_1 \circ u_2$ is convex.

**Remark 4.2.** Chakravarty and Dutta (1987) proved that distributionally homothetic social welfare functions become useful for measuring economic distance between two income distributions. The economic distance between two populations is supposed to reflect the degree of affluence or well-being of one population relative to another. Hence this rules out a simple comparison of the inequality of incomes within respective populations, since this approach neglects the differences in mean incomes and so ignores an important factor which influences the relative well-being of two populations.

Assuming that $\mu_{\mathcal{J}}$ is population replication invariant, Chakravarty and Dutta (1987) characterized

$$|\mu_{\mathcal{J}}(X) - \mu_{\mathcal{J}}(Y))| \tag{40}$$

as an index of distance between the income distributions $X$ and $Y$, where $W$ is distributionally homogeneous (see also Dagum (1980), Shorrocks (1982), Ebert (1984) and Fields and Ok (1999, 1999a)).

**Remark 4.3.** As observed, $\mu_{\mathcal{J}}^{r}$ and $\mu_{\mathcal{J}}^{\alpha}$ are the only symmetric quasi-linear means that satisfy linear homogeneity and unit translatability respectively (see also Aczel (1987) and Muliere and Parmigiani (1993)).

In the preceding discussion we have assumed that the population is homogeneous with respect to all characteristics other than income. Individuals may differ because they may belong to households with different characteristics or they have different preferences. Therefore, for a heterogeneous population individuals will not be treated symmetrically with respect to factors other than income and hence appropriate generalizations of the properties of an inequality index are necessary (see Ebert (1995) and Shorrocks (1995)). In particular the perfectly equal distribution of income need not be the distribution with minimal inequality. The concept of EDE income is to be modified for defining the indices considered above. Weymark (1999) contains an elegant discussion on this. He starts from the observation that under standard assumptions about social welfare functions, $n\mu_{\mathcal{J}}(X)$ is the minimum amount of aggregate income necessary to arrive at an income distribution which is socially indifferent to $X$. Now let, $\lambda = (\lambda_1, \ldots, \lambda_n)$ and define the function $\Lambda : R_+^n \to R$ by setting

$$\Lambda(X) = Min_{\lambda \in R_+^n} \sum_{i=1}^{n} \lambda_i \tag{41}$$

such that

$$W^n(X) = W^n(\lambda)\,. \tag{42}$$

The heterogeneous population counterpart $I_H^n$ to the AKS index in (21) is obtained by replacing the EDE income by $\Lambda(X)$ and the mean income by the aggregate income. Formally,

$$I_H^n = 1 - \frac{\Lambda(X)}{n\mu(X)}\,. \tag{43}$$

The recent emphasis on basic needs and human development has put into focus the inadequacy of income as the sole attribute of well-being and argued that income should be supplemented by other attributes of welfare such as health and literacy. Composite indices of well-being have been developed for the purpose of interpersonal and international comparisons (see, for example, Kolm (1977), Atkinson and Bourguignon (1982), Maasoumi (1989), Slottje (1991), Mosler (1994), Sen (1987), UNDP (1991-2003) and Chakravarty (2003)).

A logical extension of this area of research is the construction of inequality indices which summarize inequalities with respect to different indicators of well-being. Contributions along this line have come from Maasoumi (1986, 1999), Tsui (1995,1999), Dardanoni (1995), Koshevoy and Mosler (1996, 1997), Bourguignon (1999) and others.

Suppose that the well-being of a person depends on $k$ attributes. Let $x_{ij}$ be the quantity of attribute $j$ possessed by person $i$. Assuming that there are only two attributes, Bourguignon (1999) considered the following CES type individual utility function:

$$U(x_{i1}, x_{i2}) = (\alpha_1 x_{i1}^{-\beta} + \alpha_2 x_{i2}^{-\beta})^{-\frac{1+\gamma}{\beta}} \tag{44}$$

where $-1 < \gamma < 0$ is the inequality sensitivity parameter and $\beta$ represents the degree of substitutability between the two attributes. A natural multidimensional extension to the Dalton index is then:

$$I_{\mathrm{D}}(C) = 1 - \frac{\sum_{j=1}^{n} (\alpha_1 x_{i1}^{-\beta} + \alpha_2 x_{i2}^{-\beta})^{-\frac{1+\gamma}{\beta}}}{n(\alpha_1 \mu_1^{-\beta} + \alpha_2 \mu_2^{-\beta})^{-\frac{1+\gamma}{\beta}}} \tag{45}$$

where $C$ is the matrix showing quantities of the two attributes possessed by different individuals and $\mu_1$ and $\mu_2$ are the means of attributes 1 and 2 respectively. We now consider an issue which is of very much practical importance in multidimensional measurement. Redistributing the two attributes so as to keep the marginal distributions constant and increase the correlation between them should increase or decrease inequality according as the attributes are substitutes or complements, that is, the cross derivative $u_{12}$ is negative or positive. In terms of the parameters of the utility function this condition becomes negativity or positivity of $\beta + 1 + \gamma$. By strict quasi-concavity of $u$, $\beta > -1$ and $\gamma < 0$. Values of these parameters can now be chosen appropriately to ensure increasing or decreasing inequality under a correlation increasing switch that keeps marginal distribution constant. It may be noted that the index in (45) is quite close to an index of multidimensional inequality suggested by Maasoumi (1986). Analogous extensions of the Atkinson and Kolm-Pollak indices to the multidimensional framework was developed by Tsui (1995).

The central idea underlying the inequality-welfare relationship is that social welfare should be an increasing function of mean income (efficiency) and a decreasing function of inequality. Evidently, alternatives to the formulations considered above are possible. For example, we can define social welfare function as

$$W^n(X) = \mu e^{-I^{(X)}}. \tag{46}$$

Another possible formulation is

$$W^n(X) = \frac{\mu}{(1 + I^n(X))}.$$

While these forms certainly capture the inequality-welfare relationship correctly, an attractive feature of the AKS formulation is its interpretation from different perspectives (see also Burk and Gehrig (1978)). One minor limitation of the AKS formulation is that although we start with an ordinal social welfare function, the inequality index derived is cardinal, it makes sense to speak of reducing inequality by 10% (see Blackorby and Donaldson (1978)). But often inequality is regarded as an ordinal concept-in common practice people frequently say that the distribution $X$ is more unequal than distribution $Y$. However, two ordinally equivalent inequality indices, for example, $I_G^n$ and $(I_G^n)^2$, imply social welfare functions which are not ordinal transformations of one another. Procedures which avoid this difficulty were discussed, among others, by Blackorby and Donaldson (1984), Ebert (1987) and Dutta and Esteban (1992).

## 4. Inequality and welfare dominance

The previous discussion suggests that it is possible to construct a wide range of inequality indices, where each index corresponds to a different social welfare function. Evidently, any index of inequality will completely order the set of all income distributions. But two different inequality indices may rank two alternative distributions of income in different directions. Therefore, this failure to make unambiguous judgements about inequality ranking of income distributions imposes a severe constraint on applications of theory to policy recommendations. The reason behind incomplete inequality ranking is that the set of properties embodied in different indices (hence in welfare functions) may come into conflict with each other. Actually, all these properties are essentially value judgements. A value judgement is a statement of ethics which cannot found to be true or false on the basis of factual evidence. One way to avoid this kind of contradictory rankings is to look at a dominance condition, that is, to identify a device using which we can say that one distribution can be regarded as more equitable than another by a certain class of inequality indices. However, the dominance conditions are generally incomplete, there may be cases in which it will not be possible to rank income distributions using them. Thus, the policy maker has to withhold judgements on inequality comparison between the concerned distributions. Nevertheless, a useful research agenda has been the attempt to identify the cases in which comparisons are conclusive. A very useful dominance condition is built upon the Lorenz curve, which indicates the share of total income enjoyed by the bottom t proportion $(0 \leq t \leq 1)$ of the population.

**Definition 4.1.** It will be said that $X$ Lorenz dominates $Y$, which we write as

$$X \geq_L Y$$

if

$$L_X(p) \geq L_Y(p)$$

for all $p \in [0, 1]$, with $>$ for some $p$.

That is, the Lorenz curve of $X$ is nowhere below that of $Y$ and strictly above at some places(at least). (An axiomatic characterization of the Lorenz ordering can be found in Aaberge (2001).) By scaling up the Lorenz curve of a distribution by its mean income, we get the generalized Lorenz curve of the distribution. Formally, the generalized Lorenz curve of $X$ is defined as

$$GL(X, p) = \mu(X)L_X(p)\,.$$

**Definition 4.2.** We say that $X$ generalized Lorenz dominates $Y$, $X \geq_{GL} Y$ for short, if

$$GL(X, p) \geq GL(Y, p)$$

for all $p \in [0, 1]$ with $>$ for some $p$.

If the means of the distributions are the same, the Lorenz and generalized Lorenz dominations coincide.

Atkinson (1970) made use of the formal similarity between the ranking of income distributions and the ranking of probability distributions in terms of expected utility. In particular, Atkinson used results from Rothschild and Stiglitz (1970) to demonstrate equivalence between Lorenz domination and second order stochastic dominance. To discuss these results formally, we first define stochastic dominance.

**Definition 4.3.** Given any two income distributions $X$ and $Y$ with distribution functions $F_X$ and $F_Y$, we say that $X$-rth order stochastic dominates $Y$, which we denote by $X \geq_r Y$, if

$$F_X^r(t) \leq F_Y^r(t) \tag{47}$$

for all $t \in [0, \infty]$ with$<$ for at least one $t$, where $r$ can be equal to any finite positive integer.

Thus, for first order stochastic dominance between $X$ and $Y$ we need inequality between the corresponding distribution functions.

Similarly, $X$ second order stochastic dominates $Y$ if we have

$$F_X^2(t) \leq F_Y^2(t)$$

for all $t \in [0, \infty]$ with $<$for some $t$.

The condition $X \geq_r Y$ is equivalent to the requirement that the expected utility under $F_X$ is greater than that under $F_Y$, where all odd order derivatives of the utility function $u$ through $r$ are positive and all even order derivatives are negative, that is,

$$\int_0^\infty u(t)dF_X(t) > \int_0^\infty u(t)dF_Y(t) \tag{48}$$

where $(-1)^{j+1}u^j > 0$, $u^j$ being the $j$-th order derivative of u, $j = 1, 2, \ldots, r$ (see Fishburn (1980) and Fishburn and Willig (1984)). Thus, efficiency preference or preference for higher incomes, ceteris paribus, is the main distinguishing characteristic for first order dominance. On the other hand, $X \geq_r Y$ holds for $r = 2$, that is, $X$ second order stochastic dominates $Y$ if and only if $X$ is preferred to $Y$ by all utilitarians who approve of both efficiency and equity.

Third order stochastic dominance means that all utilitarians have preference for efficiency, equity and diminishing transfers principle. Examples of utility functions identified in (48) are $u(t) = t^c$, $0 < c < 1$ and $u(t) = 1 - e^{-\omega t}$, where $\omega > 0$.

The preceding discussion sets the background for a set of seemingly unrelated equivalent conditions. Using results from Hardy, Littlewood and Polya (1934), Marshall and Olkin (1979), Kolm (1969), Atkinson (1970), Rothchild and Stiglitz (1970, 1973), Dasgupta, Sen and Starett (1973), Fields and Fei (1978), Foster (1985) and Chakravarty (1990), we can state the following theorem.

**Theorem 4.1.** *For arbitrary $n \in N$ and $X, Y \in D^n$, where $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, the following conditions are equivalent:*

(a) *$X^*$ can be obtained from $Y^*$ by a finite sequence of (rank preserving) progressive transfers.*

(b) *$X \geq_L Y$.*

(c) *$I^n(X) < I^n(Y)$ for all inequality indices $I^n$ that satisfy SYM and PDT.*

(d) *$I^n(X) < I^n(Y)$ for all inequality indices $I^n$ that satisfy strict $S$-convexity.*

(e) *$W^n(X) > W^n(Y)$ for all social welfare functions $W^n$ that satisfy strict $S$-concavity.*

(f) *$\sum_{i=1}^n u(x_i) > \sum_{i=1}^n u(y_i)$ for all utility functions $u : J \to R$ that are strictly concave, where $J$ is some interval in the non-negative part of the real line.*

(g) *$X \geq_2 Y$, that is, $X$ second order stochastic dominates $Y$.*

Theorem 4.1 shows that strict $S$-concavity of a social welfare function is sufficient to incorporate egalitarian bias into distributional judgements, that is, the value of a strictly $S$-concave welfare function increases under a rank preserving progressive transfer. It also shows the justification of using Lorenz domination for inequality ranking. More precisely, if $X$ and $Y$ are two income distributions of a given total over a given population size, and if $X$ Lorenz dominates Y, then $X$ is regarded as more equal than $Y$ by all inequality indices that fulfil symmetry and the Pigou-Dalton condition. The converse is true as well. However, if the two curves cross we can get two inequality indices satisfying PDT and SYM that disagree on the ranking of the two distributions.

Since Theorem 4.1 relies on constancy of mean income and the population size, its scope is quite limited, it is inapplicable to comparisons of inequality and welfare of distributions with variable means and population sizes. The following theorem due to Kolm (1969), Marshall and Olkin (1979) and Shorrocks (1983) shows that using the generalized Lorenz curve we can rank distributions with different means over a fixed population size.

**Theorem 4.2.** *Let $X, Y \in D^n$ be arbitrary. Then the following conditions are equivalent:*

(a) $X \geq_{GL} Y$.

(b) $X \geq_2 Y$.

(c) $\sum_{i=1}^{n} u(x_i) > \sum_{i=1}^{n} u(y_i)$ *for all utility functions $u : J \to R$ that are increasing and strictly concave, where $J$ is some interval in the non-negative part of the real line.*

(d) $W^n(X) > W^n(Y)$ *for all increasing and strictly S-concave social welfare functions $W^n$.*

Thus, Theorem 4.2 says that of two distributions $X$ and $Y$ over a given population, $X$ is regarded as better than $Y$ by all increasing and strictly $S$-concave social welfare functions if only if $X$ generalized Lorenz dominates $Y$. This in turn is equivalent to the condition that $X$ second order stochastic dominates $Y$. Theorem 4.2, however, does not tell us anything about inequality ranking of the concerned distributions. Inequality ranking here cannot be obtained by condition (a). To understand this, suppose that $X$ is obtained from $Y$ by increasing the income of the richest person. Then $X \geq_{GL} Y$. But in this case $X$ is also regarded as more unequal than $Y$ by all relative inequality indices that fulfil SYM and PDT (see Chakravarty (1990)). In fact, the following theorem of Foster (1985) (see also Fields and Fei (1978) and Chakravarty (1990)) shows that the appropriate technique here is the Lorenz criterion.

**Theorem 4.3.** *Let $X, Y \in D^n$ be arbitrary. Then the following conditions are equivalent:*

(a) $X \geq_L Y$.

(b) $I^n(X) < I^n(Y)$ *for all relative inequality indices $I^n$ that satisfy SYM and PDT.*

We can also focus our attention on fixed mean, arbitrary population size case. In this case the domain of definition of the inequality index is an appropriate subset $D_c$, where $D_c = \{X \in D | \mu(X) = c\}$. The following theorem shows that second degree stochastic dominance and Lorenz domination are the suitable methods for inequality and welfare rankings here (see Fields and Fei (1978), Foster (1985), Chakravarty (1990) and Moyes (1999)).

**Theorem 4.4.** *Let $X, Y \in D_c$ be arbitrary. Then the following conditions are equivalent:*

(a) *There exist replications $U$ and $V$ of $X$ and $Y$ respectively, such that $U$ and $V$ have the same population size and $U^*$ follows from $V^*$ by means of a finite sequence of rank preserving progressive transfers.*

(b) $X \geq_L Y$.

(c) $X \geq_2 Y$.

(d) $W(X) > W(Y)$ *for all strictly $S$-concave social welfare functions $W : D_c \to R$ that remain invariant under replications of the population.*

(e) $I(X) < I(Y)$ *for all inequality indices $I : D_c \to R$ that meet SYM, POP and PDT.*

An example of the social welfare functions identified in condition (d) in Theorem 4.4 is the Gini welfare function. Note that the Bonferroni welfare function cannot be used here.

When both mean income and population size are variable, we have Theorems 4.5 and 4.6 for welfare and inequality rankings respectively (see Dasgupta, Sen and Starrett (1973), Shorrocks (1983), Foster (1985), Chakravarty (1990) and Moyes (1999)).

**Theorem 4.5.** *Let $X, Y \in D$ be arbitrary. Then the following conditions are equivalent:*

(a) $X \geq_{GL} Y$.

(b) $W(X) > W(Y)$ *for all increasing, strictly $S$-concave $W : D \to R$ that remain invariant under replications of the population.*

**Theorem 4.6.** *Let $X, Y \in D$ be arbitrary. Then the following conditions are equivalent:*

(A) $X \geq_L Y$.

(b) $I(X) < I(Y)$ *for all relative inequality indices that meet SYM, POP and PDT.*

(c) *$X$ second order relative stochastic dominates $Y$, that is, the distribution $\frac{X}{\mu(X)}$ second order stochastic dominates the distribution $\frac{Y}{\mu(Y)}$.*

In Theorems 4.3 and 4.6 we have derived inequality ranking for relative indices. We can develop similar results for the absolute case. In this case the appropriate technique is the absolute Lorenz curve, which for any $X \in D^n$ is the generalized Lorenz curve of the centered distribution $[X - \mu 1^n]$ and we denote this by $AL(X, p)$ (see Moyes (1987)). $AL(X, p)$ represents the vertical distance evaluated at $p$ between the line of equality and $GL(X, p)$. It gives the average amount of income necessary to make everyone's (among bottom p proportion of population) income equal to the current mean income. Replacing $X \geq_L Y$ by $X \geq_{AL} Y$, the absolute Lorenz dominance, which we define in the same way as the Lorenz ordering, in part (a) of Theorems 4.3 and 4.6 and 'relative' by 'absolute' in part (b) of the theorems, we get absolute counterparts to Theorems 4.3 and 4.6. If we say that $X$ second order absolute stochastic dominates $Y$, whenever the distribution $[X - \mu(X)1^n]$ second order stochastic dominates the distribution $[Y - \mu(Y)1^n]$, then a condition analogous to (c) in Theorem 4.6 can be developed as well.

In practice, the Lorenz curves of income distributions are often found to intersect and hence the Lorenz ordering of the concerned distributions turns out to be inconclusive in such cases. Therefore, though the Lorenz domination and second degree stochastic dominance have appealing normative justifications, they have the serious problem of being inconclusive in many practical situations. Hence it may be necessary to appeal to the third degree stochastic dominance as a ranking criterion. Shorrocks and Foster (1987) proved the following analogue to Theorem 4.4 in this context.

**Theorem 4.7.** *Let $X, Y \in D_c$ be arbitrary. Then the following conditions are equivalent:*

(a) *There exist replications $U$ and $V$ of distributions $X$ and $Y$ respectively such that $U$ and $V$ have the same population size and $U^*$ can be obtained from $V^*$ by a finite sequence of rank preserving progressive transfers and/or FACT.*

(b) *$X \geq_3 Y$, that is, $X$ third order stochastic dominates $Y$.*

(c) *$I(X) < I(Y)$ for all inequality indices $I : D_c \to R$ that fulfil SYM, POP, PDT and TRP.*

Theorem 4.7 shows that third degree stochastic dominance is necessary and sufficient for unanimous ranking of two income distributions by all transfer sensitive inequality indices.

Although for intersecting Lorenz curves the inequality ranking of distributions by indices identified in condition (e) of Theorem 4.4 is not conclusive, it is possible to obtain an indisputable ordering for intersecting Lorenz curves under special circumstances when we restrict attention to transfer sensitive indices.

The variance of the distributions plays a crucial role here (Shorrocks and Foster (1987)). More precisely, when the Lorenz curve of $X$ intersects that of $Y$ *once from above*, then a sufficient condition for $X$ to be preferred to $Y$ by the third order stochastic dominance criterion is that the variance of $X$ is lower than that of $Y$ (Shorrocks and Foster (1987)).

Given $X, Y \in D$, $L_X(p)$ is said to intersect $L_Y(p)$ once and from above if there exists $p^* \in (0, 1)$ such that $L_X(p) > L_Y(p)$ for all $p \in (0, p^*)$ and

$L_X(p) < L_Y(p)$ for all $p \in (p^*, 1)$. That is, $L_X(p)$ lies above $L_Y(p)$ up to $p^*$ and thereafter $L_Y(p)$ lies above $L_X(p)$. Note that at the extreme points and at $p^*$ the two curves coincide (see note 9). Now, in practice two Lorenz curves may intersect more than once and it may be necessary to decide on unambiguous ranking here. Actually, the above procedure can be extended to any number of intersections.

Given $X, Y \in D$, we say that $L_X(p)$ intersects $L_Y(p)$ $T$ times and first from above if there exists $0 = p_0 < p_1 < \ldots < p_T = 1$ with $T \geq 1$ such that

$$L_X(p) > L_Y(p) \tag{49}$$

for all $p \in [p_{h-1}, p_h)$, if h is odd,

$$L_X(p) < L_Y(p) \tag{50}$$

for all $p \in [p_{h-1}, p_h)$, if h is even.

The following theorem due to Davies and Hoy (1995) shows that the variance condition, applied successively to each cumulated population proportion $p_h$, enables us to rank the two distributions in the case of multiple intersections.

**Theorem 4.8.** *Let $X, Y \in D_c$ be arbitrary. Then the following conditions are equivalent:*

(a) *$L_X(p)$ intersects $L_Y(p)$ $T$ times and first from above.*

(b) *$X \geq_3 Y$, that is, $X$ third order stochastic dominates $Y$.*

(c) *$I(X) < I(Y)$ for all inequality indices $I : D_C \to R$ that fulfil SYM, POP, PDT and TRP.*

(d) *The variance for any sub-population defined as the portion of the population below an intersection point is smaller for $X$ than for $Y$.*

There may be situations in which two Lorenz curves intersect, but generalized Lorenz curves do not. This is particularly true when the Lorenz curves intersect at a low income and the distribution corresponding to the curve that lies above beyond the point of intersection has a higher mean. The reason behind this is that the higher mean is sufficient to compensate for whatever differences may exist in the income distribution. However, intersections of generalized Lorenz curves may also occur in practical situations. Using the Yaari (1988) social welfare function, Zoli (1999) shows that in such a case the Gini index becomes decisive in determining welfare rankings of distributions. A Yaari social welfare function is additive and linear in income levels, but incomes are weighted according to positions of the individuals in the income ranking:

$$W_Y(F) = \int_0^1 H(p)v(p)dp \tag{51}$$

where $\upsilon(p) \geq 0$ is the weight attached to the income of the person with rank $p$. $W_Y(F)$ increases under a progressive transfer if and only if $\upsilon(p)$ is decreasing (Yaari (1988)). Similarly, for PPT to hold it is necessary and sufficient that $\upsilon(p)$ is strictly convex (Mehran (1976)). In fact, dominance in terms of the Yaari social welfare function corresponds to inverse stochastic dominance introduced by Muliere and Scarsini(1989).

**Definition 4.4.** Given two income distributions $X$ and $Y$ with distribution functions $F_X$ and $F_Y$ respectively, we say that $X$-rth order inverse stochastic dominates $Y$, which we write $X \geq_r^{-1} Y$, if

$$H_X^r(p) \geq H_Y^r(p) \tag{52}$$

for all $p \in [0, 1]$ with $>$ for some $p$, where $r$ is any arbitrary finite positive integer.

The orderings $\geq_r^{-1}$ form a sequence of progressively finer partial orderings:

$$X \geq_r^{-1} Y \rightarrow X \geq_s^{-1} Y \tag{53}$$

with $s \geq r$.

That is, $\geq_s^{-1}$ orders all pairs of distributions that are ordered by $\geq_r^{-1}$ and some more. Thus as we pass from $\geq_r^{-1}$to $\geq_{r+1}^{-1}$ each of the previously performed comparisons between pairs of distributions remains valid, and some more are included. It is easy to see that the direct first order stochastic dominance and the first order inverse stochastic dominance are equivalent. In fact, under equality of means equivalence holds for second order dominance as well. When $r \geq 3$ the equivalence does not hold anymore (Muliere and Scarsini (1989).)

We formally state this as

**Theorem 4.9.** *Let X and Y be two income distributions with the same mean $\mu$. Then the following conditions are equivalent:*

(a) $X \geq_2 Y$.

(b) $X \geq_2^{-1} Y$.

(c) $X \geq_L Y$, *that is, the Lorenz curve of X dominates that of Y.*

Theorem 4.9 gives a normative justification of the inverse second order dominance in terms of Lorenz ordering. But, as stated equivalence of the type given by (a) and (b) does not carry over beyond second order. However, Zoli (1999) established the following normative significance of the inverse third order dominance:

**Theorem 4.10.** *Suppose that the social welfare function W is of the type* (51). *Then for any two income distributions X and Y with distribution functions $F_X$ and $F_Y$ respectively, the following conditions are equivalent:*

(a $W(F_X) > W(F_Y)$, *where the social welfare function W satisfies PDT and PPT (that is, $v(p)$ is decreasing and strictly convex).*

(b) $X \geq_3^{-1} Y$ *and X has a higher mean than Y.*

Muliere and Scarsini (1989) showed that the Gini index is coherent with $\geq_3^{-1}$ when the distributions have the same mean. The Gini index is coherent with $\geq_2$ and (hence) with $\geq_2^{-1}$ (Yitzhaki (1982)). But the former result is stronger in that if $\geq_3^{-1}$ holds and the Lorenz curves intersect, the Gini coherence continues to exist. The following theorem due to Zoli (1999) shows the role of intersecting generalized Lorenz curves in the demonstration of the normative significance of the Gini index.

**Theorem 4.11.** *Suppose that the social welfare function W is of the type* (51). *Let X and Y be any two income distributions with the same mean and with the distribution functions $F_X$ and $F_Y$ respectively. Suppose that the generalized Lorenz curve of X intersects that of Y once from above. Then the following conditions are equivalent:*

(a) $W(F_X) > W(F_Y)$, *where the social welfare function satisfies PDT and PPT (that is, $v(p)$ is decreasing and strictly convex).*

(b) *The Gini index of X is less than that of Y.*

Zoli (1999) showed that this result can be extended to the unequal mean income case with a single crossing of generalized Lorenz curves provided that the least inequality averse Yaari social welfare functions are excluded (see note 10).

Using the idea that the social welfare should depend on individual incomes as well as their ranks, as is assumed in the Yaari form (51), we can define welfare function generally as a rank dependent quasilinear mean (see Chew, Karni and Safra (1987) and Chew (1990)).

**Definition 4.5.** A social welfare function $W$ is called a rank dependent quasi-linear mean if for all income distributions $X$ and corresponding distribution functions $F_X$, W(X) can be written as

$$W(X) = \psi^{-1}\left(\int_J \psi(t)dg(F_X(t))\right) \tag{54}$$

where $g : [0, 1] \rightarrow [0, 1]$ is continuous, non-decreasing, $g(0) = 0$, $g(1) = 1$ and $\psi : J \rightarrow R$ is continuous, increasing and bounded.

When $g$ is the identity function we get the symmetric quasi-linear mean. For $\psi(t) = t$ and $g(p) = 1 - (1 - p)^\delta$, where $\delta > 1$, we obtain the Donaldson-Weymark (1980, 1983) single parameter Gini social welfare function

$$W_\delta(X) = -\int_0^\infty t d[1 - F_X(t)]^\delta . \tag{55}$$

Therefore, if $X \geq_{r+1} Y$, then $W_\delta(F_X) \geq W_\delta(F_Y)$ for $\delta \geq r$. For $\delta = 2$, $W_\delta$ becomes the Gini welfare function. The higher is the value of the single parameter $\delta$, the closer are the implicit ethics to the maximin rule (see also Bossert (1990) and Aaberge (2000,2001)). Assuming that $\psi$ is the identity function and substituting $t = H(p)$ we note that $W(X)$ in (54) becomes W(F) in (51). Hence the Gini and Yaari welfare functions can be interpreted as rank dependent quasi-linear means.

Social welfare functions, which rely on the Lorenz divergence function and can be represented as rank dependent quasi-linear means, have also been suggested in the literature. An example is

$$W_g(X) = \mu(X)(1 - I_g(X)) \tag{56}$$

where $X$ is any arbitrary income distribution, $g : [0, 1] \to [0, 1]$ is continuous, increasing, $\int_0^1 g(p)dp = 0$ and

$$I_g(X) = \int_0^1 (p - L_X(p))dg(p) . \tag{57}$$

$I_g(X)$ is a weighted area between the diagonal line and the Lorenz curve, and can be regarded as an index of inequality. Boundedness of g ensures that $I_g(X)$ is also bounded between zero and one. Increasingness of $g$ is necessary and sufficient for PDT. The normalization $\int_0^1 g(p)dp = 0$ guarantees that $I_g(X)$ achieves its lower bound zero if everybody receives the same income.

If $g(p) = 2p - 1$, we get the Gini welfare function and Gini inequality index in (56) and (57) respectively. On the other hand, for $g(p) = 3p(2 - p)$, the corresponding welfare and inequality indices become the ones suggested by Mehran (1976) (see Nygard and Sandstrom (1981) and Mosler and Muliere (1998), for further discussion).

Although increasingness of $g$ ensures PDT for $I_g(X)$, there is no guarantee that TRS will hold. The following generalization of the Gini index suggested by Chakravarty (1988) avoids this shortcoming:

$$I_\phi(X) = 2\phi^{-1}\left[\int_0^1 \phi(p - L_X(p))dp\right] \tag{58}$$

where the real valued function $\phi$ defined on [0, 1] is increasing, strictly convex and $\phi(0) = 0$. Strict convexity of $\phi$ ensures satisfaction of TRS and consistency with third order stochastic dominance. Clearly, $I_\phi(X)$, which coincides with the Gini index when $\phi$ is affine, is a quasi-linear mean of the divergences between the line of equality and the Lorenz curve.

Next, we try to determine the restrictions that are imposed by the star-shaped principle of transfers in ranking alternative distributions. Inequality indices which are additive in functions that are star-shaped above at $\theta$ play important role here.

**Definition 4.6.** A function $g : J \to R$ is star-shaped above at $\theta$ if

$$\frac{(g(s) - g(\theta))}{(s - \theta)} \tag{59}$$

is increasing in $s \in J - \{\theta\}$ where $J$ is some interval in $R$.

The following theorem of Mosler and Muliere (1996) identifies a set of equivalent conditions in this context.

**Theorem 4.12.** *Suppose that the inequality index $I^n : D^n \to R$ is of the form*

$$I^n(X) = \sum_{i=1}^{n} g(x_i)$$

*where $g : J \to R$ with $J$ being some interval in $R$. Then the following conditions are equivalent:*

(a) *$X^*$ is obtained from $Y^*$ by a sequence of star-shaped transfers at $\theta$.*

(b) *$I^n(X) < I^n(Y)$ where $I^n$ satisfies SYM and the star-shaped principle of transfers at $\theta$.*

(c) *$g$ is star-shaped above at $\theta$.*

In the dominance theorems of this section we have not been very precise about income and income unit. It has been assumed that the individuals are the income receiving units and income is the only distinguishing characteristic among them. But population may consist of socially heterogeneous persons and households, e.g., households of different sizes and compositions. For welfare comparisons in the presence of heterogeneity, additional value judgements are necessary to take into account non-income factors such as family size, physical handicap, urban/rural location.

A device involving sequential dominance was suggested by Atkinson and Bourguignon (1987) for making welfare comparisons in the presence of social heterogeneity. Suppose that there are n types of households in the society. For instance, we may subdivide the society into three types of households

($n = 3$), couples with children, married couples without child and single person households. Assume that the households have been arranged in non-increasing order of needs. The income utility function for household of type i is denoted by $u_i$. Let $u_{\mathrm{AB}}$ be the class of all utility profiles $(u_1, u_2, \ldots, u_n)$ satisfying the following conditions:

(a) each $u_i$ is increasing and strictly concave.

(b) $(u_i^{'} - u_{i+1}^{'})$ is positive and decreasing in $i$.

The following theorem of Atkinson and Bourguignon (1987) can now be stated:

**Theorem 4.13.** *Let there be two societies with income distribution functions $F_X$ and $F_Y$ respectively. Suppose that the social welfare function W is additive across types, with utility function $u_i$ from profiles $u_{\mathrm{AB}}$ being applied to different types. Then the following conditions are equivalent:*

(a) $W(F_X) > W(F_Y)$ *for all utility profiles* $(u_1, u_2, \ldots, u_n) \in u_{\mathrm{AB}}$.

(b) *There is generalized Lorenz dominance of $F_X$ over $F_Y$ in each of the sub-populations comprising the $j$ most needy groups, $j = 1, 2, , \ldots, n$.*

The procedure is to take the neediest group first, then add the second neediest group, and so on until all groups are included, checking at each stage for generalized Lorenz domination. Obviously at the terminal stage of the sequential generalized Lorenz dominance we need conventional generalized Lorenz dominance. See also, Atkinson (1990), Jenkins and Lambert (1993), Ok and Lambert (1999) and Ebert (2000) (see note 11).

So far we have presented our discussion in terms income distributions. There exists formal connections between inequality ordering and dispersive ordering. A dispersive ordering is a partial ordering of distributions according to their degree of dispersion (see, Shaked (1982), Lynch, Mimmack and Proschan (1983)).

**Definition 4.7.** A distribution function $F_X$ is said to be less dispersed than another distribution function $F_Y$ if

$$H_X(\beta) - H_X(\alpha) \leq H_Y(\beta) - H_Y(\alpha) \tag{60}$$

for all $0 < \alpha < \beta < 1$ and we denote this by $F_X \leq_{disp} F_Y$.

That is, the income gap between $100\beta$percent poorest individuals and $100\alpha$percent poorest individuals is not higher under $F_X$ than that under $F_Y$.

The following theorem of Shaked (1982) and Lynch, Mimmack and Proschan (1983) shows that dispersive ordering is equivalent to the condition that some functions, determined by the pair of the underlying distributions, change sign at most once. This is also equivalent to first order stochastic dominance in the weak sense. Formally, we have:

**Theorem 4.14.** *Let X and Y be two non-negative random variables with a common support and their distribution functions are $F_X$ and $F_Y$ respectively. Then the following conditions are equivalent:*

(a) $F_X \leq_{\text{disp}} F_Y$.

(b) $F_{X-c} - F_Y$ *changes sign at most once, where* $F_{X-c}$ *is a translation of* $F_X$, *that is,* $F_{X-c}(t) = F_X(t-c)$, *c being any arbitrary real constant, and if the number of sign changes is one, then* $F_{X-c} - F_x$ *changes sign from minus to plus.*

(c) $F_X(t) - F_Y(t) \geq 0$ *for all t in the common support of the two distributions.*

Condition (c) of Theorem 4.14 states that $Y$ weakly first order stochastic dominates $X$. It also shows that $X$ cannot have a higher mean than $Y$. Another implication of Theorem 4.14 is that for any convex function $\phi$ the expectation of $\phi(X - \mu(X))$ cannot be higher than the corresponding expectation under $Y$ (see also Munoz-Perez and Sanchez-Gomez (1990)). This in particular shows that the variance of $Y$ cannot be lower than that of $X$. This is quite similar to the problem of looking at inequality preserving transformations of incomes (Moyes (1994)).

Attempts have also been made to study dispersive ordering in terms of failure rate (or hazard rate) studied in reliability theory (see, for example, Bartoszewicz (1987), Bhattacharjee (1991) and Mosler and Muliere (1998)).

If a random variable $X$ has distribution function $F_X$ and density function $f_X$, then the ratio

$$r_X(t) = \frac{f_X(t)}{1 - F_X(t)} \tag{61}$$

is called the hazard rate of $X$. In order to use this rate in ranking distributions $X$ and $Y$ by dispersive ordering, let us define $\phi(t) = H_Y F_X(t)$. We have:

**Definition 4.8.** $X$ is said to be new better than used in comparison with $Y$, which we write $X \leq_{NBU} Y$, if $\phi$ is superadditive, that is, $\phi(p+q) \leq \phi(p)+\phi(q)$ for all $p$ and $q$.

Similarly, $X$ is new worse than used in comparison with $Y$, which we write $X \leq_{NWU} Y$, if $\phi^{-1}$ is superadditive.

The following theorem of Bartoszewicz (1987) can now be stated:

**Theorem 4.15.** *Let X and Y be two non-negative random variables with a common support and their distribution functions are $F_X$ and $F_Y$ respectively. Suppose that $X \leq_{NBU} Y$ and $Y \leq_{NWU} X$. Assume further that the density functions of X and Y exist. Then the following conditions are equivalent:*

(a) $F_X \leq_{\text{disp}} F_Y$.

(b) $r_X(t) \geq r_Y(t)$ *for every* $t \geq 0$.

## 5. Concluding remarks

Ethical index number theory provides a way to link social indicators of inequality to the moral judgements required for policy decisions. As we have seen, the advantage of welfare indicators over descriptive ones is that the value judgements that are employed in both cases become explicit in the former. We also discuss a method for uncovering ethical judgements implicit in the application of descriptive indices to policy decisions. The literature on dominance which says how one distribution can be preferred to another on welfare ground is also surveyed extensively.

## 6. Notes

(1) See, for instance, Kakwani (1980), Ebert (1988), Chakravarty (1990, 1999), Cowell (1995, 2000), Foster and Sen (1997), Silber (1999), Blackorby, Bossert and Donaldson (1999), Lambert (2001), and Dutta (2002).

(2) Bossert and Pfingsten (1990) developed a more general notion of inequality equivalence using a convex mix of relative and absolute concepts (see also Zoli (2003)).

(3) Equivalently, $W^n : D^n \to R$ is called $S$-concave if

$$W^n(BY) \geq W^n(Y)$$

for all $Y$ and for all bistochastic matrices $B$ of order $n$. An $n \times n$ non-negative matrix $B$ is called a bistochastic matrix if each of its rows and columns sums to one. Strict $S$-concavity requires strict inequality whenever $BY$ is not a permutation of $Y$. A function $G^n : D^n \to R$ is called $S$-convex (strictly $S$-convex) if $-G^n$ is $S$-concave (strictly $S$-concave). All $S$-concave and $S$-convex functions are symmetric.

(4) Formally,$W^n$ is called homothetic if it can be written as $\psi(\tilde{W}^n(X))$, where $\psi$ is increasing in its argument and $\tilde{W}^n$ is linear homogenous.

(5) For further discussion on the Bonferroni index, see Tarsitano (1990), Giorgi and Mondani (1994,1995), Giorgi (1998), Giorgi and Crescenzi (2001a, 2001b, 2001c).

(6) Formally, $W'^n$ is called translatable if it can be written as $\psi(\widehat{W}^n(X))$,where $\psi$ is increasing in its argument and $\widehat{W}^n$ is unit translatable.

(7) On related matters, see Bhattacharya and Mahalanobis (1967), Bourguignon (1979), Cowell (1980, 1995), Cowell and Kuga (1981), Shorrocks (1980, 1984, 1988), Foster (1983), Ebert (1988a, 1999), Silber (1989), Lambert

and Aaronson (1993), Chakravarty and Tyagarupananda (1998), Foster and Shneyerov (2000, 2000a) and Chakravarty (2001).

(8) Formally, a social welfare function is called distributionally homothetic if it can be written as an increasing transformation of a distrbutionally homogeneous function, where distributional homogeneity of a function means that it is linear homogeneous and unit translatable.

(9) We can analogously define single crossing of two distribution functions. A sufficient condition for generalized Lorenz domination using single crossing property of distribution functions was developed by Ramos, Ollero and Sordo (1999).

(10) See Weymark (1981, 1995) and Ben Porath and Gilboa (1994) for characterizations of the Yaari social welfare function in the discrete case. Peragine (2002) used this welfare function for studying equality of opportunity.

(11) An alternative to sequential dominance for welfare comparisons in the presence of social heterogeneity is to use equivalence scales (see Glewwe (1991), Balckorby and Donaldson (1993), Ebert (1997, 2000), Ebert and Moyes (2003)).

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