On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems

M. Seetharama Gowda[*], Yoon Song[†]

Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21250

and

G. Ravindran[‡]

Indian Statistical Institute
8th Mile, Mysore Road
Bangalore 560 059, India

October 2001 (Revised January 2003)

ABSTRACT

In the setting of semidefinite linear complementarity problems on $\mathcal{S}^n$, the implications *strict monotonicity* $\Rightarrow \mathbf{P}_2 \Rightarrow \mathbf{GUS} \Rightarrow \mathbf{P}$ are known. Here, $\mathbf{P}$ and $\mathbf{P}_2$ properties for a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ are respectively defined by: $X \in \mathcal{S}^n$, $XL(X) = L(X)X \preceq 0 \Rightarrow X = 0$ and $X \succeq 0, Y \succeq 0$, $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0 \Rightarrow X = Y$; $\mathbf{GUS}$ refers to the global unique solvability in semidefinite linear complementarity problems corresponding to L. In this article, we show that the reverse implications hold for any self-adjoint linear transformation, and for normal Lyapunov and Stein transformations. By introducing the concept of a principal subtransformation of a linear transformation, we show that $L : \mathcal{S}^n \to \mathcal{S}^n$ has the $\mathbf{P}_2$-property if and only if for every $n \times n$ real invertible matrix Q, every principal subtransformation of $\widehat{L}$ has the $\mathbf{P}$-property where $\widehat{L}(X) := Q^T L(QXQ^T)Q$. Based on this, we show that $\mathbf{P}_2$, $\mathbf{GUS}$, and $\mathbf{P}$ properties coincide for the two-sided multiplication transformation.

[*]E-mail: gowda@math.umbc.edu; http://www.math.umbc.edu/~gowda
[†]E-mail: song@math.umbc.edu
[‡]E-mail: ravi@isibang.ac.in

1. INTRODUCTION

Given a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ and a matrix $Q \in \mathcal{S}^n$, the *Semidefinite Linear Complementarity Problem*, SDLCP(L, Q), is to find a matrix X such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0, \quad \text{and} \quad \langle X, Y \rangle := \text{trace}(XY) = 0$$

where $\mathcal{S}^n$ denotes the space of all real symmetric $n \times n$ matrices and $Z \succeq 0$ means that Z belongs to the cone $\mathcal{S}^n_+$ of all symmetric positive semidefinite matrices in $\mathcal{S}^n$.

This problem, studied in [3], [8], [9], [10], [11], [20], [21], [25], includes the (standard) LCP [5] and the geometric SDLCP of Kojima, Shindoh, and Hara [15]. Its applications include primal-dual semidefinite linear programs, control theory, linear and bilinear matrix inequalities [4], [18]. It is a special case of a cone complementarity problem, which in turn in a special case of a variational inequality problem [13]. Since the cone $\mathcal{S}^n_+$ is nonpolyhedral, (standard) LCP results/concepts cannot be routinely generalized to SDLCPs.

In connection with the semidefinite LCP, various concepts, such as the (strict) monotonicity, **GUS** and **P** properties were introduced and studied in the above cited works. We recall [9] that $L : \mathcal{S}^n \to \mathcal{S}^n$ is said to have the

(a) *monotonicity property (strict or strong monotonicity property)* if

$$\langle L(X), X \rangle \geq 0 \ (> 0) \quad \text{for } 0 \neq X \in \mathcal{S}^n;$$

(b) *Globally Uniquely Solvable (**GUS**)-property* if for all $Q \in \mathcal{S}^n$, SDLCP(L, Q) has a unique solution;

(c) **P**-*property* if

$$X \in \mathcal{S}^n, \ XL(X) = L(X)X \preceq 0 \Rightarrow X = 0;$$

(d) **P**$_2$-*property* if

$$X \succeq 0, \ Y \succeq 0, \ (X - Y)[L(X) - L(Y)](X + Y) \preceq 0 \Rightarrow X = Y.$$

One can define, see [8], a (stronger) noncommutative version of (c) : $X \in \mathcal{S}^n, \ XL(X) + L(X)X \preceq 0 \Rightarrow X = 0$. This property seems harder to analyze and will not be treated here. It has been shown, see [9], that under (c), SDLCP(L, Q) has a solution for all Q and the system $X \succ 0, \ L(X) \succ 0$ has a solution.

The properties $(b) - (d)$ (and the noncommutative version of (c)) are the semidefinite analogs of the following properties of a matrix $M \in R^{n \times n}$ in the (standard) LCP setting [5], [9]:

(b') For all $q \in R^n$, $\mathrm{LCP}(M, q)$ has a unique solution,

(c') $x \in R^n$, $\quad x * Mx \leq 0 \Rightarrow x = 0$,

(d') $x \geq 0$, $y \geq 0$, $(x - y) * [Mx - My] * [x + y] \leq 0 \Rightarrow x = y$,

where '$*$' denotes the Hadamard (i.e., componentwise) product. As is well known, these properties are equivalent to M being a **P**-matrix, that is, all principal minors of M are positive.

Motivated by the equivalence of properties $(b') - (d')$, we ask how properties $(a) - (d)$ are related to each other. Since the (generally irreversible) implications

$$\text{strict monotonicity} \Rightarrow \mathbf{P}_2 \Rightarrow \mathbf{GUS} \Rightarrow \mathbf{P} \tag{1}$$

are known (see [20] for the first implication and [9] for the last two implications), we are interested in the reverse implications. Before describing our findings, we briefly recall three important transformations studied in the SDLCP literature.

Given a matrix $A \in R^{n \times n}$, the *Lyapunov, Stein,* and *two-sided multiplication* transformations on $\mathcal{S}^n$ are defined (respectively) by

$$L_A(X) := AX + XA^T, \quad S_A(X) := X - AXA^T, \quad M_A(X) := AXA^T. \tag{2}$$

The Lyapunov transformation L_A has been extensively studied in the matrix/control/systems theory literature [1], [4], [6], [7]. A celebrated result of Lyapunov [17], [7] states that the continuous linear dynamical system $\frac{dx}{dt} = -Ax(t)$ is (globally) asymptotically stable if and only if all the eigenvalues of A lie in the positive right-half plane and that the latter condition holds if and only if the system $X \succ 0$, $L_A(X) \succ 0$ has a solution. In [9], it was shown that these conditions are equivalent to the **P**-property of L_A. In [11], the simultaneous stability of a finite number of matrices was studied based on the composition of several Lyapunov transformations and a fixed point map on $\mathcal{S}^n_+$. Recently, in [24], Sun and Sun express the differentiability properties of the projection map $X \mapsto \Pi_{\mathcal{S}^n_+}(X)$ in terms of Lyapunov transformations; these results were used in [19] to discuss the stability/regularity of a solution of a semidefinite (nonlinear) complementarity problem.

Along with the Lyapunov transformation, the Stein transformation S_A has also been extensively studied in the matrix/control/systems theory literature [1], [4], [6], [7]. It is well known that the discrete linear dynamical system $x(k+1) = Ax(k)$ is (globally) asymptotically stable if and only if all

the eigenvalues of A lie in the open unit disk and that the latter condition holds if and only if the system $X \succ 0$, $S_A(X) \succ 0$ has a solution. In [8], it was shown that these conditions are equivalent to the **P**-property of S_A. Since this **P**-property of S_A is equivalent to (spectral radius) $\rho(M_A) < 1$, see [8], we may view $S_A = I - M_A$ as a semidefinite analog of a nonsingular M-matrix (which is a matrix of the form $I - B$ where B is a nonnegative matrix with spectral radius is less than one) and expect interesting properties. [We may note that Theorem 2.3, [2] contains 50 equivalent properties of a nonsingular M-matrix.] Motivated by S_A, a general cone complementarity result was proved in [8] for transformations of the form $I - S$ where S leaves the (given) cone invariant.

The transformation M_A on $\mathcal{S}^n$ leaves the cone $\mathcal{S}^n_+$ invariant; it is analogous to a nonnegative matrix on R^n (which leaves the cone R^n_+ invariant). Similar to the nonnegative matrices in the standard LCP, one may expect interesting complementarity properties for M_A. Some of these properties have been explored in [3], [20].

Our findings in this paper are as follows:

First, by generalizing three recent results involving self-adjoint Lyapunov, Stein, and two-sided multiplication transformations [9], [10], [20], we show that *the reverse implications hold in (1) for any self-adjoint transformation and for normal Lyapunov and Stein transformations.* We conclude that $L : \mathcal{S}^n \to \mathcal{S}^n$ is strictly monotone if and only if $L + L^T$ has the **P**-property where L^T denotes the adjoint of L.

Second, by introducing the concept of a principal subtransformation of a linear transformation (see Section 5), we show that *L has the $\mathbf{P}_2$-property if and only if for each invertible real matrix Q, every principal subtransformation of $\widehat{L}$ has the **P**-property where $\widehat{L}(X) := Q^T L(Q X Q^T) Q$.* (Since linear automorphisms of the semidefinite cone $\mathcal{S}^n_+$ are given by $X \mapsto Q X Q^T$ for some real invertible Q [22], [16], one may view this result as the semidefinite analog of the following LCP/matrix theory result: A matrix $M \in R^{n \times n}$ is a **P**-matrix if and only if for each linear automorphism Λ of R^n_+ (this is of the form $x \mapsto ED x$ where E is a permutation matrix and D is a positive diagonal matrix), every principal submatrix of $\Lambda^T M \Lambda$ is a **P**-matrix.) As a byproduct of this characterization, we show that M_A has the $\mathbf{P}_2$-property if and only if A is either positive definite or negative definite, answering a question raised by Parthasarathy, Sampangi Raman, and Sriparna [21]. Combining this with a result of Bhimasankaram et al [3], *we deduce that $\mathbf{P}_2$, **GUS** and **P** properties are equivalent for M_A.*

2. PRELIMINARIES

For $x, y \in R^n$, the usual inner product is denoted either by $\langle x, y \rangle$ or by $x^T y$; In C^n, the inner product is denoted by $\langle \cdot, \cdot \rangle_C$.

For a matrix $A \in R^{n \times n}$, we recall the following definitions:

(a) The *trace* of A is the sum of all the diagonal elements of A, or equivalently, the sum of all the eigenvalues of A.

(b) A is *positive semidefinite (positive definite)* if the usual inner product $\langle Ax, x \rangle \geq 0$ (> 0) for all nonzero $x \in R^n$.

(c) A is *positive stable* if every eigenvalue of A has positive real part.

(d) A is *orthogonal* if $AA^T = I = A^T A$ where I is the $n \times n$ identity matrix.

(e) A is *normal* if $AA^T = A^T A$.

Recall that $\mathcal{S}^n_+$ is the cone of symmetric $n \times n$ positive semidefinite matrices in the space $\mathcal{S}^n$ of real symmetric $n \times n$ matrices. We use the notation

$$X \succeq (\succ) 0$$

to say that X is symmetric and positive semidefinite (positive definite); the notation $X \preceq 0$ means that $-X \succeq 0$. For $A, B \in R^{n \times n}$, we define $\langle A, B \rangle := \text{trace}(AB^T)$; In particular, for $X, Y \in \mathcal{S}^n$,

$$\langle X, Y \rangle = \text{trace}(XY).$$

For an $n \times n$ matrix A and for an index set $\alpha \subseteq \{1, 2, \cdots, n\}$, we write a_{ij} to denote the (i, j)-entry of A and write $A_{\alpha\alpha}$ to denote the submatrix of A consisting of entries a_{ij} with $i, j \in \alpha$.

For $x \in R^n$, $\|x\|$ denotes its Euclidean norm. For $A \in R^{n \times n}$,

$$\|A\|_{op} := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \|A\|_F := \left[\sum_{i,j} a_{ij}^2 \right]^{\frac{1}{2}}$$

denote, respectively, the operator norm and the Frobenius norm. The spectral radius of $A \in R^{n \times n}$ is denoted by $\rho(A)$.

Consider a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$. Then its transpose $L^T : \mathcal{S}^n \to \mathcal{S}^n$ is defined by

$$\langle L(X), Y \rangle = \langle X, L^T(Y) \rangle \quad (X, Y \in \mathcal{S}^n).$$

L is said to be self-adjoint on $\mathcal{S}^n$ if $L = L^T$, and normal if L commutes with L^T. For the Lypunov, Stein, and two-sided multiplication transformations, we note that $(L_A)^T = L_{A^T}, (S_A)^T = S_{A^T}, (M_A)^T = M_{A^T}$ and L_A, S_A, M_A are normal when A is normal.

Now let $[L] : R^k \to R^k$ (with $k = \frac{n(n+1)}{2}$) be a matrix representing L with respect to a complete orthonormal set in $\mathcal{S}^n$. Then L and $[L]$ have the same (real) eigenvalues and determinant; in particular, if every real eigenvalue of L is positive, then its determinant is positive. Moreover, L is self-adjoint if and only if $[L]$ is symmetric, and L is strictly monotone on $\mathcal{S}^n$ if and only if $[L]$ is positive definite. Hence, if L is self-adjoint and has only positive real eigenvalues, then it is strictly monotone. (This also follows from spectral theory.)

3. SELF-ADJOINT AND NORMAL **P**-TRANSFORMATIONS

We show below that self-adjoint **P** transformations are strictly monotone.

THEOREM 1. *Suppose a linear transformation* $L : \mathcal{S}^n \to \mathcal{S}^n$ *has the* **P***-property. Then all real eigenvalues of* L *are positive and hence* L *has a positive determinant. If, in addition,* L *is self-adjoint, then it is strictly monotone and the reverse implications hold in (1).*

Proof. Suppose λ is a nonpositive real eigenvalue of L. Then there exists a nonzero $X \in \mathcal{S}^n$ such that $L(X) = \lambda X$ and so $L(X)X = XL(X) = \lambda X^2 \preceq 0$. This contradicts the **P**-property of L. Thus all real eigenvalues of L are positive. It follows that the determinant is positive. Now suppose that L is also self-adjoint. Then all its eigenvalues are real and positive. It follows that L is strictly monotone. $\square$

Remark. The second part of the above theorem generalizes earlier results proved for L_A (when A is symmetric) [Theorem 9, [9]], for S_A (when A is symmetric or skew symmetric) [10], and for M_A (when A is symmetric) [Theorem 6, [20]].

Motivated by the above result, we ask if normal **P** transformations are strictly monotone. We answer this in the negative by the following example.

Example. Let
$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

This A is normal and positive definite. It can be shown (by direct algebraic manipulation) that M_A has the **P**-property. (This also follows from

Theorem 17, [3], and from Corollary 7 in Section 5.) Furthermore, it can be easily shown that M_A is normal. Yet, for

$$X = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix},$$

$\langle M_A(X), X \rangle = -12$, that is, M_A is not strictly monotone.

In spite of the above example, we show below that for normal Lyapunov and Stein transformations, **P**-property indeed implies the strict monotonicity property.

THEOREM 2. *If L_A is normal, then the reverse implications hold in (1).*

Proof. It is enough to show that the **P**-property implies strict monotonicity. To this end, let L_A have the **P**-property. Since $L_{A^T} = (L_A)^T$, it follows from $L_A \circ L_{A^T} = L_{A^T} \circ L_A$ that A is normal. Now the **P**-property of L_A implies that A is positive stable (cf. Theorem 5, [9]). We claim that A is positive definite. Let $x \in R^n$ be nonzero. Since A is normal, we have a unitary matrix U and a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ in $C^{n \times n}$ so that $A = U^* D U$ (cf. Theorem 2.5.4, [14]). Then

$$\alpha := \langle Ax, x \rangle = \langle Ax, x \rangle_C = \langle Dz, z \rangle_C = \sum_1^n d_i |z_i|^2 \tag{3}$$

where $z := Ux$ is nonzero in C^n. Since $\alpha \in R$, the last expression in (3) should be equal to its real part. But since A is positive stable, we have $\operatorname{Re}(d_i) > 0$, proving $\alpha > 0$.

Now, for any nonzero $X \in S^n$,

$$\langle X, L_A(X) \rangle = 2 \operatorname{trace}(XAX) = 2 \sum_{i=1}^n x_i^T A x_i \tag{4}$$

where x_i is the ith column in X. We see that $\langle X, L_A(X) \rangle > 0$ for all $X \neq 0$ proving the strict monotonicity of L_A. $\square$

THEOREM 3. *Let A be an $n \times n$ real matrix. For the Stein transformation S_A, consider the following statements.*

(i) $\|A\|_{op} < 1$.

(ii) S_A is strictly monotone.

*(iii) S_A has the **GUS**-property.*

8

*(iv) S_A has the **P**-property, i.e., $\rho(A) < 1$.*

(a) $\|A\|_{op} \le 1$.

(b) S_A is monotone.

(c) $\rho(A) \le 1$.

Then,

$$(i) \implies (ii) \implies (iii) \implies (iv),$$
$$(a) \implies (b) \implies (c) \quad \text{and} \quad (b) + (iv) \implies (iii).$$

Furthermore, when S_A is normal (i.e., when A is normal), reverse implications hold in the above statements.

Note that we have obvious implications: $(i) \implies (a)$, $(ii) \implies (b)$ and $(iv) \implies (c)$.

Proof. To show $(i) \implies (ii)$ and $(a) \implies (b)$, note that the Cauchy-Schwartz inequality in $R^{n \times n}$ gives, for $X \in \mathcal{S}^n$,

$$\text{trace}(AXA^TX) \quad = \quad \langle AX, XA \rangle \quad \le \quad \|AX\|_F \|XA\|_F.$$

By writing $X = [x_1, \cdots, x_n]$ (where x_i denotes the ith column of X), we get

$$\|AX\|_F \quad = \quad \left[\sum_1^n \|Ax_i\|^2 \right]^{\frac{1}{2}} \quad \le \quad \left[\|A\|_{op}^2 \sum_1^n \|x_i\|^2 \right]^{\frac{1}{2}} \quad = \quad \|A\|_{op} \|X\|_F.$$

Similarly,

$$\|XA\|_F \quad = \|A^TX\|_F \quad \le \|A^T\|_{op} \|X\|_F \quad = \quad \|A\|_{op} \|X\|_F.$$

Hence, $\text{trace}(AXA^TX) \le \|A\|_{op}^2 \|X\|_F^2$. From

$$\langle S_A(X), X \rangle = \|X\|_F^2 - \text{trace}(AXA^TX) \quad \ge \|X\|_F^2 (1 - \|A\|_{op}^2),$$

we have the implications $(i) \implies (ii)$ and $(a) \implies (b)$.
The implication $(ii) \implies (iii)$ follows from Cor.3.2, [13].
If (iii) holds, then S_A has the **P**-property (Theorem 7, [9]). That $\rho(A) < 1$ and the **P**-property of S_A are equivalent is given in Theorem 11, [8]. We thus have (iv).
Now suppose (b) holds. Then, for all $\epsilon > 0$, the transformation

$$\frac{1}{1+\epsilon}(S_A + \epsilon I) = S_{\frac{A}{\sqrt{1+\epsilon}}}$$

is strictly monotone. The implication $(ii) \implies (iv)$ gives $\rho(\frac{A}{\sqrt{1+\epsilon}}) < 1$. Taking the limit, we get $(b) \implies (c)$.

The statements (b) and (iv), respectively give, the convexity of the solution set for the SDLCP(S_A, Q) for all $Q \in \mathcal{S}^n$ (Theorem 6, [9]) and the **P**-property of S_A (Theorem 7, [9]). Hence the implication $(b) + (iv) \implies (iii)$. To see the last statement, suppose that S_A is normal. From $(S_A \circ S_{A^T})(I) = (S_{A^T} \circ S_A)(I)$, we get $(AA^T)^2 = (A^TA)^2$. From the uniqueness of square root (Theorem 6.4, [27]), we get $AA^T = A^TA$ proving that A is normal. Since $\|A\|_{op}$ is also the norm of A computed over C^n, and $\rho(A) = \|A\|_{op}$ (because of normality) we get the last statement of the theorem. $\qquad\square$

4. PRINCIPAL SUBTRANSFORMATIONS

Analogous to the definition of a principal submatrix of a matrix, we now formulate the concept of a principal subtransformation of a linear transformation.

DEFINITION 4. Let $L : S^n \to S^n$ be a linear transformation. For any $\alpha \subseteq \{1, 2, \cdots, n\}$, we define a linear transformation $L_{\alpha\alpha} : S^{|\alpha|} \to S^{|\alpha|}$ by

$$L_{\alpha\alpha}(Z) = [L(X)]_{\alpha\alpha} \quad (Z \in S^{|\alpha|})$$

where, corresponding to $Z \in S^{|\alpha|}$, $X \in S^n$ is the unique matrix such that $X_{\alpha\alpha} = Z$ and $x_{ij} = 0$ for all $(i, j) \notin \alpha \times \alpha$. We call $L_{\alpha\alpha}$ *the principal subtransformation of L corresponding to α.*

It can be easily verified that for the Lyapunov transformation L_A, we have $(L_A)_{\alpha\alpha} = L_{A_{\alpha\alpha}}$. Since the positive stable property of a matrix is not inherited by its principal submatrices, we may infer that the **P**-property of a transformation is not inherited by its principal subtransformations. We make this statement precise in the following example.

Example 6. Consider the Lyapunov transformation $L = L_A : S^2 \to S^2$ corresponding to

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

so that for any

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix},$$

$$L(X) = AX + XA^T = \begin{bmatrix} -2y & x+y-z \\ x+y-z & 2(y+z) \end{bmatrix}.$$

Then the principal subtransformation of L_A corresponding to $\alpha = \{1\}$ is given by

$$L_{11}(x) = \left[L\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) \right]_{11} = 0 \quad (x \in S^1).$$

We note that A is positive stable and positive semidefinite. Hence L_A has the **GUS** and **P** properties. Yet L_{11} does not have the **GUS** and **P** properties.

Remarks. Monotonicity and strict monotonicity properties are inherited by principle subtransformations. Only under certain conditions, principal subtransformations inherit the **GUS** and **P** properties. For example, suppose $\alpha = \{1, 2, \ldots, .m\}$, and

$$L\left(\begin{bmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{bmatrix} \right) = L(X) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

for any $X_{\alpha\alpha}$. Then it can be easily shown (see [12]) that $L_{\alpha\alpha}$ has the **P**-property (**GUS**-property) whenever L has the **P**-property (respectively, **GUS**-property).

$\square$

5. THE $\mathbf{P}_2$-PROPERTY

The $\mathbf{P}_2$-property of a linear transformation $L : \mathcal{S}^n \to \mathcal{S}^n$ is introduced in [9] as a sufficient condition for the **GUS**-property. Answering a question raised in [9], namely, whether strict monotonicity is related to the $\mathbf{P}_2$-property, Parthasarathy, Sampangi Raman, and Sriparna [20] show that

$$strict\ monotonicity \Rightarrow \mathbf{P}_2$$

and that the converse holds for the Lyapunov transformation L_A and (when A is symmetric) for M_A. In a subsequent article, the same authors [21] show that for a 2×2 or a 3×3-matrix A, M_A has the $\mathbf{P}_2$-property if and only if either A is positive definite or negative definite, and provide an example of M_A for which the $\mathbf{P}_2$-property holds but not strict monotonicity. They also raise the question whether their result holds for any $n \times n$ matrix A.

Below, we characterize the $\mathbf{P}_2$-property, and, as a byproduct, show that for any A, M_A has the $\mathbf{P}_2$-property if and only if A is either positive definite or negative definite.

In what follows, for a given invertible $Q \in R^{n \times n}$, we define a linear transformation $\widehat{L}$ on $\mathcal{S}^n$ by

$$\widehat{L}(X) = Q^T L(QXQ^T)Q. \qquad (5)$$

THEOREM 5. *For any linear transformation L on $\mathcal{S}^n$, the following are equivalent:*

(a) *L has the $\mathbf{P}_2$-property.*

(b) *For every invertible $Q \in R^{n \times n}$, every principal subtransformation of $\widehat{L}$ has the $\mathbf{GUS}$-property.*

(c) *For every invertible $Q \in R^{n \times n}$, every principal subtransformation of $\widehat{L}$ has the $\mathbf{P}$-property.*

Proof. $(a) \Rightarrow (b)$: Assume that L has the $\mathbf{P}_2$-property and fix an invertible Q and an index set α. First we claim that $\widehat{L}$ defined in (5) has the $\mathbf{P}_2$-property. To see this, suppose

$$V \succeq 0, \ W \succeq 0, \ \text{and} \ (V - W)[\widehat{L}(V) - \widehat{L}(W)](V + W) \preceq 0.$$

Then $Q(V-W)[\widehat{L}(V)-\widehat{L}(W)](V+W)Q^T \preceq 0$. Upon writing $X := QVQ^T$, $Y := QWQ^T$ and using the definition of $\widehat{L}$, we see that $X \succeq 0$, $Y \succeq 0$, with $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$. From the $\mathbf{P}_2$-property of L, we get $X = Y$ and (from the invertibility of Q) $V = W$. This proves the $\mathbf{P}_2$-property of $\widehat{L}$.

Now we show that $(\widehat{L})_{\alpha\alpha}$ has the $\mathbf{P}_2$-property. To simplify the notation, let $\Gamma := \widehat{L}$ and let without loss of generality, $\alpha = \{1, 2, \ldots, m\}$, $m \leq n$. Suppose $X_m \succeq 0$, $Y_m \succeq 0$ with

$$(X_m - Y_m)[\Gamma_{\alpha\alpha}(X_m) - \Gamma_{\alpha\alpha}(Y_m)](X_m + Y_m) \preceq 0$$

where X_m and Y_m belong to $\mathcal{S}^m$. Then letting

$$X = \begin{bmatrix} X_m & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_m & 0 \\ 0 & 0 \end{bmatrix}$$

and writing

$$\Gamma(X) - \Gamma(Y) = \begin{bmatrix} \Gamma_{\alpha\alpha}(X_m - Y_m) & * \\ * & * \end{bmatrix},$$

we see that

$$(X - Y)[\Gamma(X) - \Gamma(Y)](X + Y) = \begin{bmatrix} E_m & 0 \\ 0 & 0 \end{bmatrix} \preceq 0.$$

where $E_m := (X_m - Y_m)[\Gamma_{\alpha\alpha}(X_m) - \Gamma_{\alpha\alpha}(Y_m)](X_m + Y_m)$. Now from the $\mathbf{P}_2$-property of Γ, we have $X = Y$ and hence $X_m = Y_m$. This shows that $\Gamma_{\alpha\alpha}$, that is, $(\widehat{L})_{\alpha\alpha}$ has the $\mathbf{P}_2$-property. Since $\mathbf{P}_2$-property implies the $\mathbf{GUS}$-property (Remark 7, [9]), we see that for all invertible Q and all α, $(\widehat{L})_{\alpha\alpha}$ has the $\mathbf{GUS}$-property. Thus (b) holds.

$(b) \Rightarrow (c)$ This is immediate since the $\mathbf{GUS}$-property implies the $\mathbf{P}$-property always.

$(c) \Rightarrow (a)$: To some extent, our proof follows that of Theorem 4 in [20] where it is shown that strict monotonicity implies the $\mathbf{P}_2$-property. Assume that (c) holds and suppose that $X \succeq 0, Y \succeq 0, X \neq Y$ with $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$. Since $X + Y$ is symmetric, positive semidefinite and nonzero, there exist a (real) invertible matrix Q such that

$$X + Y = Q \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^T$$

where I_r is the identity matrix of size $r \times r$ and $1 \leq r \leq n$ (see Theorem 6.3, [27] or the proof of Theorem 4 in [20]). Put $A := Q^{-1}X(Q^{-1})^T$ and $B := Q^{-1}Y(Q^{-1})^T$. Then A and B are symmetric positive semidefinite matrices with

$$A + B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix}$$

where A_r and B_r are $r \times r$-matrices. Now $(X - Y)[L(X) - L(Y)](X + Y) \preceq 0$ yields

$$Q^{-1}(X - Y)(Q^{-1})^T Q^T [L(QAQ^T) - L(QBQ^T)]QQ^{-1}(X + Y)(Q^{-1})^T \preceq 0.$$

This gives

$$(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0$$

where $\widehat{L}(Z) := Q^T L(QZQ^T) Q$. Writing

$$\widehat{L}(A) - \widehat{L}(B) = \begin{bmatrix} P & N \\ N^T & R \end{bmatrix},$$

we get from $(A - B)[\widehat{L}(A) - \widehat{L}(B)](A + B) \preceq 0$,

$$\begin{bmatrix} A_r - B_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & N \\ N^T & R \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_r - B_r)P & 0 \\ 0 & 0 \end{bmatrix} \preceq 0.$$

This implies that $(A_r - B_r)P \preceq 0$ (which means that $(A_r - B_r)P$ is symmetric and negative semidefinite). By the block form of $A - B$, we note that $P = (\widehat{L})_{\alpha\alpha}(A_r - B_r)$ where $\alpha = \{1, 2, \ldots, r\}$. Thus

$$(A_r - B_r)(\widehat{L})_{\alpha\alpha}(A_r - B_r) \preceq 0.$$

By our assumption, $(\widehat{L})_{\alpha\alpha}$ has the **P**-property. Hence $A_r = B_r$ proving $A = B$ and $X = Y$. This is a contradiction. Therefore L has the $\mathbf{P}_2$-property. This completes the proof.
□

Remarks. Suppose L has the $\mathbf{P}_2$-property. Then, because of Lemma 1, for every invertible Q, each principal subtransformation of $\widehat{L}$ has positive determinant. It is not clear if the converse holds.

COROLLARY 6. *For any $A \in R^{n \times n}$, consider M_A defined by (2). Then the following statements are equivalent:*

(a) *A is positive definite or negative definite.*

(b) *M_A has the $\mathbf{P}_2$-property.*

Remark. Bhimasankaram et al [3] have shown that for M_A, **GUS**, **P**, and $\mathbf{R}_0$-properties are equivalent to (a). Hence for M_A, $\mathbf{P}_2$, **GUS**, and **P** properties are equivalent.

Proof. $(a) \Rightarrow (b)$: Assume without loss of generality that A is positive definite. We verify condition (c) of the previous theorem. For any invertible Q,

$$\widehat{M_A}(X) = Q^T M_A(QXQ^T)Q = Q^T AQXQ^T A^T Q = BXB^T = M_B(X)$$

where $B := Q^T AQ$. Note that B is also positive definite as Q is invertible. We need to show that every principal subtransformation of M_B has the **P**-property. Without loss of generality, let $\alpha = \{1, 2, \ldots, m\}$. Let $0 \neq Z \in S^m$ be such that
$$Z(M_B)_{\alpha\alpha}(Z) = (M_B)_{\alpha\alpha}(Z)Z \preceq 0.$$
Writing B in the block form

$$B = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

where E conforms with Z, we get $(M_B)_{\alpha\alpha}(Z) = EZE^T$. Writing $Z = V^T DV$ where V is orthogonal and D is diagonal and nonzero, we get

$DM_C(D) \preceq 0$ with $C := VEV^T$ (which is positive definite). By using an appropriate permutation matrix, we may assume that

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where D_1 is diagonal and invertible. Writing C in a block form that conforms with D and letting C_1 be the submatrix that conforms with D_1, we see that $DM_C(D) \preceq 0$ yields $D_1 C_1 D_1 C_1^T \preceq 0$. Note that C_1 is positive definite. For ease of notation, we assume $D_1 = D$ and $C_1 = C$ so that D is invertible and C is positive definite. Let λ be a negative eigenvalue of $DCDC^T$ with a corresponding eigenvector u. From $DCDC^T u = \lambda u$, we get $CDC^T u = \lambda D^{-1} u$. Hence

$$\langle CDC^T u, DC^T u \rangle = \lambda \langle D^{-1} u, DC^T u \rangle = \lambda \langle u, C^T u \rangle$$

where we compute the inner product in an appropriate Euclidean space. Since C is positive definite, the left-hand side of the above expression is positive while the right-hand side is negative. Thus we reach a contradiction.

The implication $(b) \Rightarrow (a)$ can be proved, as in [21], as follows: Suppose $x^T A x = 0$ with $x \neq 0$. With $X = xx^T$, $Y = 0$, we have $(X - Y)M_A(X - Y)[X + Y] = 0$, contradicting the $\mathbf{P}_2$-property. □

6. CONCLUDING REMARKS

In this paper, we studied the reverse implications in the statement

$$\text{strict monotonicity} \Rightarrow \mathbf{P}_2 \Rightarrow \mathbf{GUS} \Rightarrow \mathbf{P}.$$

We showed that the reverse implications hold for self-adjoint transformations and for normal Lyapunov and Stein transformations. We also gave a characterization of the $\mathbf{P}_2$-property in terms of principal subtransformations $\widehat{L}$ defined with respect to invertible Qs. We do not know if the same characterization holds when Q is restricted to orthogonal matrices. More open questions can be found in [12] where one also finds an elaborated discussion on topics covered in this article.

REFERENCES

1   R. Bellman, Introduction to Matrix Analysis, SIAM, Philadelphia, 1995.

2   A. Berman and R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.

3   P. Bhimasankaram, A.L.N. Murthy, G.S.R. Murthy, and T. Parthasarathy, Complementarity problems and positive definite matrices, Research Report, Indian Statistical Institute, Street No. 8, Habshiguda, Hyderabad 500 007, India, 2000. (Revised June 27, 2001).

4   S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.

5   R.W. Cottle, J.-S. Pang, and R. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.

6   B.N. Datta, Stability and inertia, Linear Alg. and Its Appl. 302-303 (1999) 563-600.

7   F.R. Gantmacher, Theory of Matrices, Vol. II, Chelsea Publishing Company, New York, 1959.

8   M.S. Gowda, T. Parthasarathy, Complementarity forms of the theorems of Lyapunov and Stein, and related results, Linear Alg. and Its Appl. 320 (2000) 131-144.

9   M.S. Gowda, Y. Song, On semidefinite linear complementarity problems, Math. Prog. Series A 88 (2000) 575-587.

10 M.S. Gowda, Y. Song, Errata: On semidefinite linear complementarity problems, Math. Prog. Series A 91 (2001) 199-200.

11 M.S. Gowda, Y. Song, Some new results for the semidefinite linear complementarity problem, SIAM Jour. Matrix Analysis, 24 (2002) 25-39.

12 M.S. Gowda, Y. Song, G. Ravindran, Some interconnections between strict monotonicity, GUS, and P properties in semidefinite linear complementarity problems, Research Report TR01-11, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, October 2001.

13 P.T. Harker, J.-S. Pang, Finite dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms, and applications, Math. Prog. 48 (1990) 161-220.

14 R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

15 M. Kojima, S. Shindoh, S. Hara, Interior-point methods for the monotone semidefinite linear complementarity problems, SIAM Optim. 7 (1997) 86-125.

16   R. Loewy, On ranges of real Lyapunov transformations, Linear Alg. and Its Appl. 13 (1976) 79-89.

17   M.A. Lyapunov, Probléme général de la stabilité des mouvement, Ann. Fac. Sci. Toulouse 9 (1907) 203-474 (French, translation of the original paper published in 1893 in Comm. Soc. Math. Kharkow; reprinted as Vol. 17 in Annals of Mathematics Study, Princeton University Press, Princeton, NJ, 1949).

18   M. Mesbahi, G.P. Papavassilopoulos, A cone programming approach to the bilinear matrix inequality problem and its geometry, Math. Prog. 77 (1997) 247-272.

19   J.-S. Pang, D. Sun, and J. Sun, Semismooth homeomorphims and strong stability of semidefinite and Lorentz complementarity problems, Research Report, Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21228, U.S.A, (revised) September 2002.

20   T. Parthasarathy, D. Sampangi Raman, B. Sriparna, Relationship between strong monotonicity, $\mathbf{P}_2$-property, and the GUS property in semidefinite LCPs, Math. Operations Research, 27 (2002) 326-331.

21   T. Parthasarathy, D. Sampangi Raman, B. Sriparna, $\mathbf{P}_2$-property need not imply strong monotonicity in the semidefinite LCPs, Research Report, Indian Statistical Institute, Chennai, June 2001.

22   H. Schneider, Positive operators and an inertia theorem, Numerische Mathematik, (1965) 11-17.

23   P. Stein, Some general theorems on iterants, J. Res. Natl. Bur. Standards 48 (1952) 82-83.

24   D. Sun and J. Sun, Semismooth matrix valued functions, Math. of Operations Research, 27 (2002) 150-169.

25   P. Tseng, Merit functions for semi-definite complementarity problems, Math. Prog. 83 (1998) 159-185.

26   N. Yamashita, M. Fukushima, A new merit function for SDCP, in reformulation: nonsmooth, piecewise smooth, semismooth and smoothing methods, Kluwer Academic Publishers, Boston, 1999.

27   F. Zhang, Matrix Theory, Springer-Verlag, New York, 1999.