LIMITING SPECTRAL DISTRIBUTION OF CIRCULANT MATRIX WITH DEPENDENT ENTRIES

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Abstract. In this article, we derive the limiting spectral distribution of the *circulant matrix* when the input sequence is a stationary infinite order two sided moving average process.

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1. Introduction and Main result

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are all the eigenvalues of a square matrix A_n of order n. Then the *empirical spectral distribution function (ESDF)* of A_n is defined as

$$F_n(x,y) = n^{-1} \sum_{i=1}^{n} I\{Re\lambda_i \le x,\ Im\lambda_i \le y\}.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices with the corresponding ESDF $\{F_n\}_{n=1}^{\infty}$. The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence $\{F_n\}_{n=1}^{\infty}$, if it exists.

If $\{A_n\}$ are random, the limit is understood to be in some probabilistic sense, such as "almost surely" or "in probability". Suppose elements of $\{A_n\}$ are defined on some probability space $(\Omega, \mathcal{F}, P)$, that is $\{A_n\}$ are random. Let F be a nonrandom distribution function. We say the ESD of A_n converges to the *limiting spectral distribution* (LSD) F in L_2 if at all continuty points (x,y) of F,

$$\int_{\omega} \big(F_n(x,y) - F(x,y)\big)^2 dP(\omega) \to 0 \text{ as } n \to \infty$$

and converges in probability to F if for every $\epsilon > 0$ and at all continuty points (x,y) of F,

$$P\big(|F_n(x,y) - F(x,y)| > \epsilon\big) \to 0 \text{ as } n \to \infty.$$

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For detailed information on limiting spectral distributions of large dimensional random matrices see [Bai(1999)] and also [Bose and Sen(2008)].

In this article we focus on obtaining the LSD of the *circulant matrix* (C_n) given by

$$C_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \ldots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_0 \end{bmatrix}.$$

So, the $(i,j)th$ element of the matrix is $x_{(j-i+n) mod\ n}$. The eigenvalues are given by (see for example [Brockwell and Davis(2002)]),

$$\lambda_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_k l} = b_k + ic_k \quad \forall\ k = 1, 2, \cdots, n,$$

where

$$\omega_k = \frac{2\pi k}{n}, \quad b_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \cos(\omega_k l), \quad c_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \sin(\omega_k l).$$

The existence of the LSD of C_n is given by the following theorem of [Bose and Mitra(2002)].

Theorem 1.1. Let $\{x_i\}$ be a sequence of independent random variables with mean 0 and variance 1 and $\sup_i E \mid x_i \mid^3 < \infty$. Then the ESD of C_n converges in L_2 to the two-dimensional normal distribution given by $N_2(0, D)$ where D is a diagonal matrix with diagonal entries 1/2.

We investigate the existence of LSD of C_n under a dependent situation. Let $\{x_n; n \geq 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are iid random variables with mean zero and variance one. We show that the LSD of C_n continues to exist in this dependent situation. Define $\gamma_h = Cov\,(x_{t+h}, x_t)$. Then it is easy to see that $\sum_{j\in\mathbb{Z}} |\gamma_j| < \infty$ and the *spectral density function* of $\{x_n\}$ is given by

$$f(\omega) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi}\big[\gamma_0 + 2\sum_{k\geq 1} \gamma_k \cos(k\omega)\big] \text{ for } \ \omega \in [0, 2\pi].$$

Let $f^* = \inf_{\omega\in[0,2\pi]} f(\omega)$ and $C_0 = \{\omega \in [0, 2\pi]; f(\omega) = 0\}$. For $k = 1, 2, \cdots, n$, define

$$\xi_{2k-1} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \sin(\omega_k t).$$

Define

$$B(\omega)=\begin{pmatrix} a_1(e^{i\omega}) & -a_2(e^{i\omega}) \\ a_2(e^{i\omega}) & a_1(e^{i\omega}) \end{pmatrix},$$

where $a_1(e^{i\omega})=\mathcal{R}[a(e^{i\omega})],\ a_2(e^{i\omega})=\mathcal{I}[a(e^{i\omega})]$, $a(e^{i\omega})$ is same as defined in Lemma 1.3 and for $z\in\mathbb{C}$, $\mathcal{R}(z),\mathcal{I}(z)$ denote the real and imaginary part of z respectively. It is easy to see that

$$|a(e^{i\omega})|^2=a_1(e^{i\omega})^2+a_2(e^{i\omega})^2=2\pi f(\omega).$$

Define for $(x,y)\in\mathbb{R}^2$ and $\omega\in[0,2\pi]$,

$$H(\omega,x,y)=\begin{cases} P\big(B(\omega)(N_1,N_2)'\le\sqrt{2}(x,y)'\big) & \text{if} \quad f(\omega)\neq 0,\\ \mathbb{I}(x\ge 0,y\ge 0) & \text{if} \quad f(\omega)=0.\end{cases}$$

Since $a(e^{i\omega})$ is continuous on $[0,2\pi]$, it is easy to verify that for fixed (x,y), H is bounded continuous function in ω. Hence we may define

$$F(x,y)=\int_0^1 H(2\pi s,x,y)ds.$$

F is a proper distribution function.

For any Borel set B, let $\lambda(B)$ denote the corresponding Lebesgue measure. It is easy to see that

(i) if $\lambda(C_0)=0$ then F is continuous everywhere and,

(ii) if $\lambda(C_0)\neq 0$ then F is discontinuous *only* on $D_1=\{(x,y):xy=0\}$.

Theorem 1.2. Suppose $\{\epsilon_i\}$ are iid with $E|\epsilon_i|^{(2+\delta)}<\infty$. Then the ESD of C_n converges in L_2 to the LSD

$$F(x,y)=\int_0^1 H(2\pi s,x,y)ds,$$

and if $\lambda(C_0)=0$ we have

$$F(x,y)=\iint \mathbb{I}_{\{(v_1,v_2)\le(x,y)\}}\Big[\int_0^1 \mathbb{I}_{\{f(2\pi s)\neq 0\}}\frac{1}{2\pi^2 f(2\pi s)}e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}}ds\Big]dv_1dv_2.$$

Remark 1.1. If $\inf_{\omega\in[0,2\pi]} f(\omega)>0$, we can write $\mathcal{F}_g$ in the following form

$$F(x,y)=\iint \mathbb{I}_{\{(v_1,v_2)\le(x,y)\}}\Big[\int_0^1 \frac{1}{2\pi^2 f(2\pi s)}e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}}ds\Big]dv_1dv_2.$$

Remark 1.2. If $\{x_i\}$ are i.i.d, then $f(\omega)=1/2\pi$ for all $\omega\in[0,2\pi]$ and the LSD is standard complex normal distribution. This agrees with Theorem 1.1.

Proof of the theorem mainly depends on following two lemmas. Lemma 1.3 follows from [Fan and Yao(2003)] (Theorem 2.14(ii), page 63). For completeness, we have provided a proof. The proof of Lemma 1.4 follows easily from [Bhattacharya and Ranga Rao(1976)] (Corollary 18.3, page 184). We omit the details.

Lemma 1.3. Let $x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j}$ for $t \geq 0$, where $\{\epsilon_t\}$ are i.i.d random variables with mean 0, variance 1 and $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then for $k = 1, 2, \cdots, n$,

$$\lambda_k = a(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k),$$

where $a(e^{i\omega_k}) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j}$ and $\max_{0\leq k<n} E|Y_n(\omega_k)|^2 \to 0$ as $n \to \infty$.

Proof.

$$\begin{aligned}
\lambda_k &= \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t e^{i\omega_k t} \\
&= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \sum_{t=0}^{n-1} \epsilon_{t-j} e^{i\omega_k (t-j)} \\
&= \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \left(\sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t} + U_{nj} \right) \\
&= a(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}] + Y_n(\omega_k),
\end{aligned}$$

where

$$a(e^{i\omega_k}) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j}, \quad U_{nj} = \sum_{t=-j}^{n-1-j} \epsilon_t e^{i\omega_k t} - \sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t}, \quad Y_n(\omega_k) = n^{-1/2} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} U_{nj}.$$

Note that if $|j| < n$, U_{nj} is a sum of $2|j|$ independent random variables, whereas if $|j| \geq n$, U_{nj} is a sum of $2n$ independent random variables. Thus $E|U_{nj}|^2 \leq 2\min(|j|, n)$. Therefore, for any fixed positive integer l and $n > l$,

$$\begin{aligned}
E|Y_n(\omega_k)|^2 &\leq \frac{1}{n} \left(\sum_{j=-\infty}^{\infty} |a_j| (EU_{nj}^2)^{1/2} \right)^2 \quad \left(\because \sum_{-\infty}^{\infty} |a_j| < \infty \right) \\
&\leq \frac{2}{n} \left(\sum_{j=-\infty}^{\infty} |a_j| \{\min(|j|, n)\}^{1/2} \right)^2 \\
&\leq 2 \left(\frac{1}{\sqrt{n}} \sum_{|j|\leq l} |a_j| |j|^{1/2} + \sum_{|j|>l} |a_j| \right)^2.
\end{aligned}$$

Note that the right-hand side of the above expression is independent of k and as $n \to \infty$, it can be made smaller than any given positive constant by choosing l large enough. Hence, $\max_{1\leq k\leq n} E|Y_n(\omega_k)|^2 \to 0$. □

Lemma 1.4. Let $X_1, \ldots, X_k$ be independent random vectors with values in $\mathbb{R}^d$, having zero means and an average positive-definite covariance matrix $V_k = k^{-1} \sum_{j=1}^{k} CovX_j$. Let G_k

denote the distribution of $k^{-1/2}T_k(X_1+\ldots+X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \geq 1$. If for some $\delta > 0$, $E \parallel X_j \parallel^{(2+\delta)} < \infty$, then

$$\begin{aligned}\sup_{C\in\mathcal{C}} |G_k(C) - \Phi_{0,I}(C)| &\leq ck^{-\delta/2}\big[k^{-1}\sum_{j=1}^{k} E \parallel T_k X_j \parallel^{(2+\delta)}\big] \\ &\leq ck^{-\delta/2}(\lambda_{\min}(V_k))^{-(2+\delta)}\big[k^{-1}\sum_{j=1}^{k} E \parallel X_j \parallel^{(2+\delta)}\big]\end{aligned}$$

where $\Phi_{0,I}$ is the normal probability function with mean zero and identity covariance matrix, $\mathcal{C}$, the class of all Borel-measurable convex subsets of $\mathbb{R}^d$ and c is a constant, depending only on d.

Proof of Theorem 1.2: We first assume $\lambda(C_0) = 0$. To prove the theorem it suffices to show that for each $x, y \in \mathbb{R}$,

$$(1.1) \qquad E(F_n(x,y)) \to F(x,y) \text{ and } V(F_n(x,y)) \to 0.$$

Note that we may ignore the eigenvalue λ_n and also $\lambda_{n/2}$ whenever n is even since they contribute atmost $2/n$ to the ESD $F_n(x,y)$. So for $x, y \in \mathbb{R}$,

$$E[F_n(x,y)] \sim n^{-1}\sum_{k=1,(k\neq n/2)}^{n-1} P(b_k \leq x, c_k \leq y).$$

Define for $k = 1, 2, \cdots, n$,

$$\eta_k = (\xi_{2k-1}, \xi_{2k})', \quad Y_{1n}(\omega_k) = \mathcal{R}[Y_n(\omega_k)], \quad Y_{2n}(\omega_k) = \mathcal{I}[Y_n(\omega_k)],$$

$$A_k = \begin{pmatrix} a_1(e^{i\omega_k}) & -a_2(e^{i\omega_k}) \\ a_2(e^{i\omega_k}) & a_1(e^{i\omega_k}) \end{pmatrix},$$

where $a(e^{i\omega_k}), Y_n(\omega_k)$ are same as defined in Lemma 1.3. Then $(b_k, c_k)' = A_k\eta_k + (Y_{1n}(\omega_k), Y_{2n}(\omega_k))'$. From Lemma 1.3, it is intutively clear that for large n, $\lambda_k \sim a(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}]$. So first we show that for large n

$$\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(b_k \leq x, c_k \leq y) \sim \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k \leq (x,y)').$$

Note

$$\begin{aligned}
& \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(b_k\le x, c_k\le y) - \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k\le (x,y)')\Big| \\
= \quad & \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k + (Y_{1n}(\omega_k), Y_{2n}(\omega_k))' \le (x,y)') - P(A_k\eta_k\le (x,y)')\Big| \\
\le \quad & \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P((|Y_{1n}(\omega_k)|, |Y_{2n}(\omega_k)|) > (\epsilon,\epsilon)) \\
+ \quad & \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k\le (x,y)', (|Y_{1n}(\omega_k)|, |Y_{2n}(\omega_k)|) \le (\epsilon,\epsilon)) - P(A_k\eta_k\le (x,y)')\Big| \\
= \quad & T_1 + T_2, \text{ say.}
\end{aligned}$$

Now using Lemma 1.3, as $n\to\infty$

$$T_1 \le \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(|Y_n(\omega_k)|^2 > 2\epsilon^2) \le \frac{1}{2\epsilon^2}\sup_k E|Y_n(\omega_k)|^2 \to 0.$$

$$\begin{aligned}
T_2 \quad \le \quad & \max\Big\{\Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k\le (x+\epsilon, y+\epsilon)' - P(A_k\eta_k\le (x,y)')\Big|, \\
& \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k\le (x-\epsilon, y-\epsilon)' - P(A_k\eta_k\le (x,y)')\Big|\Big\}
\end{aligned}$$

and

$$\Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k\le (x+\epsilon, y+\epsilon)' - P(A_k\eta_k\le (x,y)')\Big| \le T_3 + T_4 + T_5.$$

where

$$T_3 = \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\xi_k\le (x,y)') - P(A_k(N_1\ N_2)'\le (\sqrt{2}x, \sqrt{2}y)')\Big|,$$

$$T_4 = \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\xi_k\le (x+\epsilon, y+\epsilon)') - P(A_k(N_1\ N_2)'\le (\sqrt{2}x+\sqrt{2}\epsilon, \sqrt{2}y+\sqrt{2}\epsilon)')\Big|,$$

$$T_5 = \Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k(N_1\ N_2)'\le (\sqrt{2}x+\sqrt{2}\epsilon, \sqrt{2}y+\sqrt{2}\epsilon)') - P(A_k(N_1\ N_2)'\le (\sqrt{2}x, \sqrt{2}y)')\Big|.$$

To show $T_3, T_4\to 0$ define for $k=1,2,\cdots,n-1$, (except for $k=n/2$) and $l=0,1,2,\cdots,n-1$,

$$X_{l,k} = (\sqrt{2}\epsilon_l\cos(\omega_k l), \quad \sqrt{2}\epsilon_l\sin(\omega_k l))'.$$

Note that

$$E(X_{l,k}) = 0 \ \ \forall\ \ l,\ k,\ n. \tag{1.2}$$

(1.3) $$n^{-1}\sum_{l=0}^{n-1} Cov(X_{l,k}) = I \ \ \forall \ \ k, \ n.$$

Note that for $k \neq n/2$

$$\{A_k\eta_k \leq (x,y)'\} = \{A_k(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\}.$$

Since $\{(r,s) : A_k(r,s)' \leq (\sqrt{2}x, \sqrt{2}y)'\}$ is a convex set in $\mathbb{R}^2$ and $\{X_{l,k}, \ l = 0, 1, \ldots (n-1)\}$ satisfies (1.2) and (1.3), we can apply Lemma 1.4 for $k \neq n/2$ to get

$$\big|P(A_k(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)') - P(A_k(N_1, \ N_2)' \leq (\sqrt{2}x, \sqrt{2}y)')\big| \leq cn^{-\delta/2}[n^{-1}\sum_{l=0}^{n-1} E \parallel X_{lk} \parallel^{(2+\delta)}],$$

where N_1, N_2 are independent standard normal variates. Note that

$$\sup_{1\leq k\leq n} [n^{-1}\sum_{l=0}^{n-1} E \parallel X_{lk} \parallel^{(2+\delta)}] \leq M < \infty$$

and, as $n \to \infty$

$$\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} \big|P(A_k(n^{-1/2}\sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)') - P(A_k(N_1, \ N_2)' \leq (\sqrt{2}x, \sqrt{2}y)')\big| \leq cMn^{-\delta/2} \to 0.$$

Hence $T_3 \to 0$ and similarly $T_4 \to 0$. and also

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k \leq (x,y)') &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} H(\frac{2\pi k}{n}, x, y) \\ &= \int_0^1 H(2\pi s, x, y)ds. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} T_5 &= \Big| \int_0^1 H(2\pi s, x+\epsilon, y+\epsilon)ds - \int_0^1 H(2\pi s, x, y)ds \Big| \\ &\leq \int_0^1 \big|H(2\pi s, x+\epsilon, y+\epsilon)ds - H(2\pi s, x, y)\big|ds. \end{aligned}$$

Note that

$$\big|H(2\pi s, x+\epsilon, y+\epsilon)ds - H(2\pi s, x, y)\big| \leq 2$$

and for fixed $(x,y) \in \mathbb{R}^2$ as $\epsilon \to 0$,

(1.4) $$\big|H(2\pi s, x+\epsilon, y+\epsilon)ds - H(2\pi s, x, y)\big| \to 0.$$

Hence by DCT $\lim_{\epsilon\to 0}\lim_{n\to\infty} T_5 = 0$ and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k \leq (x+\epsilon, y+\epsilon)' - P(A_k\eta_k \leq (x,y)')\Big| = 0.$$

Also note that for fixed (x, y) as $\epsilon \to 0$,

$$\big|H(2\pi s, x-\epsilon, y-\epsilon)ds - H(2\pi s, x, y)\big| \to 0, \tag{1.5}$$

outside the measure zero set C_0. Using this fact, proceeding as above we can show that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big|\frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k \leq (x-\epsilon, y-\epsilon)' - P(A_k\eta_k \leq (x,y)')\Big| = 0,$$

and hence $\lim_{\epsilon\to 0}\lim_{n\to\infty} T_2 = 0$. Therefore as $n \to \infty$,

$$E[F_n(x,y)] \sim \frac{1}{n}\sum_{k=1,(k\neq n/2)}^{n-1} P(A_k\eta_k \leq (x,y)') \to \int_0^1 H(2\pi s, x, y)ds,$$

and since $\lambda(C_0) = 0$, we have

$$\begin{aligned}
\int_0^1 H(2\pi s, x, y)ds &= \int_0^1 \mathbb{I}_{\{f(2\pi s)\neq 0\}} H(2\pi s, x, y)ds \\
&= \int_0^1 \mathbb{I}_{\{f(2\pi s)\neq 0\}}\Big[\iint \mathbb{I}_{\{B(2\pi s)(u_1,u_2)'\leq (x,y)'\}}\frac{1}{2\pi}e^{-\frac{u_1^2+u_2^2}{2}}du_1du_2\Big]ds \\
&= \int_0^1 \mathbb{I}_{\{f(2\pi s)\neq 0\}}\Big[\iint \mathbb{I}_{\{(v_1,v_2)\leq (x,y)\}}\frac{1}{2\pi^2 f(2\pi s)}e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}}dv_1dv_2\Big]ds \\
&= \iint \mathbb{I}_{\{(v_1,v_2)\leq (x,y)\}}\Big[\int_0^1 \mathbb{I}_{\{f(2\pi s)\neq 0\}}\frac{1}{2\pi^2 f(2\pi s)}e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}}ds\Big]dv_1dv_2 \\
&= F(x,y).
\end{aligned}$$

Now, to show $V[F_n(x,y)] \to 0$, it is enough to show that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} Cov(J_k, J_{k'}) \to 0. \tag{1.6}$$

where for $1 \leq k \leq n$, J_k is the indicator that $\{b_k \leq x, c_k \leq y\}$. Observe that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} Cov(J_k, J_{k'}) = \frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n}\left[E(J_k, J_{k'}) - E(J_k)E(J_{k'})\right].$$

Now as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} E(J_k)E(J_{k'}) = \Big(\frac{1}{n}\sum_{k=1}^{n}E(J_k)\Big)^2 - \frac{1}{n^2}\sum_{k=1}^{n}(E(J_k))^2 \to H(x,y)^2.$$

So to show (1.6), it is enough to show as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{n} E(J_k, J_{k'}) \to H(x,y)^2.$$

Along the lines of the proof used to show $\frac{1}{n}\sum_{k=1}^{n} P(A_k(N_1\ N_2)' \leq (\sqrt{2}x, \sqrt{2}y)') \to F(x,y)$, one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details.

When $\lambda(C_0) \neq 0$, we have to show (1.1) only at continuty points of F and it is continuous on complement of D_2. All the above steps except (1.4),(1.5) in the proof will go through for all (x, y), but on complement of D (1.4),(1.5) also holds. Hence if $\lambda(C_0) \neq 0$, we have our required LSD. This proves the Theorem. □

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