Coherent States and Squeezed States in Interacting Fock Space

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In this paper we study coherent states and squeezed states in one mode interacting Fock space.

KEY WORDS: coherent states; squeezed states; Fock space.

1. INTRODUCTION

Interacting Fock space, studied by Accardi and Bozejko (1998) and Accardi and Nhani (in press), grew out from the stochastic limit of QED. It is expected that the idea of interacting Fock space can provide a natural frame work to describe a large class of self-interacting quantum field theories (nonlinear) analogous to that played by usual Fock space for noninteracting fields (linear theories). In this paper, we study coherent states and squeezed states in one mode interacting Fock space and their Hilbert space properties. In subsequent papers we use coherent states to investigate problems of nonlinear interactions in quantum optics.

2. PRELIMINARIES AND NOTATIONS

As a vector space one mode interacting Fock space $\Gamma(\mathbb{C})$ is defined by

$$\Gamma(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle \tag{1}$$

where $\mathbb{C}|n\rangle$ is called the n-particle subspace. The different n-particle subspaces are orthogonal, that is, the sum in Eq. (1) is orthogonal. The norm of the vector $|n\rangle$ is given by

$$\langle n \mid n\rangle = \lambda_n \tag{2}$$

where $\{\lambda_n\} > 0$. The norm introduced in Eq. (2) makes $\Gamma(\mathbb{C})$ a Hilbert space.

An arbitrary vector f in $\Gamma(\mathbb{C})$ is given by

$$f \equiv c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + \cdots + c_n|n\rangle + \cdots \tag{3}$$

with $\|f\| = (\sum_{n=0}^{\infty} |c_n|^2 \lambda_n)^{1/2} < \infty$.

We now consider the following actions on $\Gamma(\mathbb{C})$:

$$\begin{aligned} a^+|n\rangle &= |n+1\rangle \\ a|n+1\rangle &= \frac{\lambda_{n+1}}{\lambda_n}|n\rangle \end{aligned} \tag{4}$$

a^+ is called the *creation operator* and its adjoint a is called the *annihilation operator*. To define the annihilation operator we have taken the convention $0/0 = 0$.

We observe that

$$\langle n \mid n\rangle = \langle a^+(n-1), n\rangle = \langle (n-1), an\rangle = \frac{\lambda_n}{\lambda_{n-1}}\langle n-1, n-1\rangle = \cdots \tag{5}$$

and

$$\||n\rangle\|^2 = \frac{\lambda_n}{\lambda_{n-1}} \cdot \frac{\lambda_{n-1}}{\lambda_{n-2}} \cdots \frac{\lambda_1}{\lambda_0} = \frac{\lambda_n}{\lambda_0} \tag{6}$$

From Eqs. (2) and (6) we observe that $\lambda_0 = 1$.

The commutation relation takes the form

$$[a, a^+] = \frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}} \tag{7}$$

where N is the number operator defined by $N|n\rangle = n|n\rangle$.

3. ORTHONORMAL SET

Proposition. *The set* $\{\frac{|n\rangle}{\sqrt{\lambda_n}}, n = \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$ *forms a complete orthonormal set.*

Proof: If $f_n = \frac{|n\rangle}{\sqrt{\lambda_n}}, n = 0, 1, 2, 3, \ldots$, then

$$\|f_n\| = (f_n, f_n)^{1/2} = 1$$

and $(f_n, f_m) = 0$. Hence $\{f_n\}$ forms an orthonormal set.

Also, it is complete. For, if $f = \sum a_n|n\rangle \in \Gamma(\mathbb{C})$, then

$$(f_n, f) = \lambda_n \cdot \frac{1}{\sqrt{\lambda_n}} \cdot a_n$$

Hence

$$\sum |(f_n, f)|^2 = \sum |a_n|^2 \lambda_n = \|f\|^2$$

By Parseval's theorem $\{f_n\}$ is complete. □

4. COHERENT STATES

We define the coherent state as an eigenstate of the annihilation operator

$$af_\alpha = \alpha f_\alpha \tag{8}$$

where α is a complex number and f_α is given by

$$f_\alpha \equiv \alpha_0|0\rangle + \alpha_1|\mathbf{1}\rangle + \alpha_2|\mathbf{2}\rangle + \cdots + \alpha_n|n\rangle + \cdots.$$

Now

$$\begin{aligned} af_\alpha &= a[\alpha_0|0\rangle + \alpha_1|\mathbf{1}\rangle + \alpha_2|\mathbf{2}\rangle + \cdots + \alpha_n|n\rangle + \cdots] \\ &= \alpha_1\frac{\lambda_1}{\lambda_0}|0\rangle + \alpha_2\frac{\lambda_2}{\lambda_1}|\mathbf{1}\rangle + \alpha_3\frac{\lambda_3}{\lambda_2}|\mathbf{2}\rangle + \cdots + \alpha_n\frac{\lambda_n}{\lambda_{n-1}}|n-1\rangle \\ &\quad + \alpha_{n+1}\frac{\lambda_{n+1}}{\lambda_n}|n\rangle + \cdots \end{aligned} \tag{9}$$

and

$$\alpha f_\alpha = \alpha\alpha_0|0\rangle + \alpha\alpha_1|\mathbf{1}\rangle + \alpha\alpha_2|\mathbf{2}\rangle + \cdots + \alpha\alpha_n|n\rangle + \cdots \tag{10}$$

From Eqs. (8), (9), and (10) we observe that a_n satisfies the following difference equation:

$$\alpha_{n+1}\frac{\lambda_{n+1}}{\lambda_n} = \alpha\alpha_n \tag{11}$$

That is,

$$\alpha_{n+1} = \alpha \cdot \frac{\lambda_n}{\lambda_{n+1}} \cdot \alpha_n \tag{12}$$

Hence

$$\alpha_1 = \alpha \cdot \frac{\lambda_0}{\lambda_1} \cdot \alpha_0,\ \alpha_2 = \alpha\frac{\lambda_1}{\lambda_2} \cdot \alpha_1 = \alpha^2\frac{\lambda_0}{\lambda_2} \cdot \alpha_0,\ \alpha_3 = \alpha \cdot \frac{\lambda_2}{\lambda_3} \cdot \alpha_2 = \alpha^3\frac{\lambda_0}{\lambda_3} \cdot \alpha_0, \ldots$$

Thus

$$\alpha_n = \alpha^n\frac{\lambda_0}{\lambda_n}\alpha_0$$

Hence

$$f_\alpha = \sum \alpha_n |n\rangle = \alpha_0 \sum \frac{\alpha^n}{\lambda_n} |n\rangle$$

We choose α_0 so that f_α is normalized:

$$1 = (f_\alpha, f_\alpha) = |\alpha_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\lambda_n}$$

We denote $\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\lambda_n}$ by $\psi(|\alpha|^2)$. Thus we have

$$|\alpha_0| = \psi(|\alpha|^2)^{-1/2}$$

Thus aside from some trivial phase factor the eigenvector of a is

$$f_\alpha = \psi(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\lambda_n} |n\rangle \tag{13}$$

We call f_α a *coherent vector* in $\Gamma(\mathbb{C})$.

5. HILBERT SPACE PROPERTIES OF COHERENT STATE

5.1. Completeness of Coherent States

Theorem 1.

$$I = \int_{\alpha\in\mathbb{C}} d\mu(\alpha) |f_\alpha\rangle\langle f_\alpha| \tag{14}$$

where

$$d\mu(\alpha) = \psi(|\alpha|^2)\sigma(|\alpha|^2) r\, dr\, d\theta \tag{15}$$

where $\alpha = re^{i\theta}$ and $\sigma(x)$ is some weight function to be determined.

To prove the theorem we need the following

Lemma. *To find a weight function $\sigma(x)$ such that*

$$\int_0^\infty \sigma(x) x^n\, dx = \frac{\lambda_n}{\pi} \tag{16}$$

Proof: In order to find $\sigma(x)$ we multiply both sides of (16) by $\frac{(iy)^n}{n!}$ to get

$$\int_0^\infty \sigma(x) x^n \frac{(iy)^n}{n!}\, dx = \frac{\lambda_n}{\pi} \frac{(iy)^n}{n!}$$

On summation over n we get

$$\int_0^\infty \sigma(x)\left(\sum_{n=0}^{\infty}\frac{(ixy)^n}{n!}\right)dx = 1/\pi\sum_{n=0}^{\infty}\lambda_n\frac{(iy)^n}{n!}$$

or

$$\int_0^\infty \sigma(x)e^{ixy}\,dx = \phi(y) \tag{17}$$

where $\phi(y) = 1/\pi\sum_{n=0}^{\infty}\frac{(iy)^n\lambda_n}{n!}$ is absolutely convergent for $|y| < 1$ and for $|y| > 1$ may be defined by analytic continuation. We can now determine the weight function by taking an inverse Fourier transformation of Eq. (17). The result is

$$\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(y)\,e^{-ixy}\,dy,\ x > 0 \tag{18}$$

□

Proof of the Theorem: We define the operator

$$|f_\alpha\rangle\langle f_\alpha| : \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C}) \tag{19}$$

by

$$|f_\alpha\rangle\langle f_\alpha|f = (f_\alpha, f)f_\alpha \tag{20}$$

with $f = \sum_{n=0}^{\infty} b_n|n\rangle$. Now,

$$(f_\alpha, f) = \psi(|\alpha|^2)^{-1/2}\sum_{n=0}^{\infty}\bar{\alpha}^n b_n$$

Then,

$$(f_\alpha, f)f_\alpha = \psi(|\alpha|^2)^{-1}\sum_{m,n=0}^{\infty}\frac{\alpha^m}{\lambda_m}\bar{\alpha}^n b_n|m\rangle$$

Hence,

$$\begin{aligned}\int_{\alpha\in\mathbb{C}} d\mu(\alpha)|f_\alpha\rangle\langle f_\alpha|f &= \sum_{m,n=0}^{\infty}\frac{|m\rangle}{\lambda_m}b_n\int_0^\infty r\,dr\cdot\sigma(r^2)r^{m+n} \\ &\quad\times\int_0^\pi d\theta\cdot e^{i(m-n)\theta} \\ &= \sum_{n=0}^{\infty}\frac{|n\rangle}{\lambda_n}b_n\pi\int_0^\infty dr^2\cdot\sigma(r^2)\cdot r^{2n}\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{|n\rangle}{\lambda_n} b_n \pi \int_0^{\infty} dx \cdot \sigma(x) \cdot x^n$$

$$= \sum_{n=0}^{\infty} b_n |n\rangle$$

$$= f \tag{21}$$

where we have taken $x = r^2$ and utilized the fact $\int_0^{\infty} dx \cdot \sigma(x) \cdot x^n = \lambda_n/\pi$. □

5.2. Nonorthogonality of Coherent States

Coherent states are not orthogonal, for

$$(f_\alpha, f_{\alpha'}) = \psi(|\alpha|^2)^{-1/2} \psi(|\alpha'|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{(\bar{\alpha})^n (\alpha')^n}{\lambda_n} \neq 0$$

5.3. Hilbert Space Expansion

Any arbitrary state f of norm one in $\Gamma(\mathbb{C})$ can be expressed in terms of the orthonormal states $\{f_n\}$ in the form

$$f = \sum_{n=0}^{\infty} a_n f_n \tag{22}$$

with

$$1 = \sum_{n=0}^{\infty} |a_n|^2$$

Using the overcompleteness of coherent states $\{f_\alpha\}$ we can also expand f in terms of coherent states:

$$\begin{aligned} f &= \int_{\alpha \in \mathbb{C}} d\mu(\alpha) | f_\alpha\rangle\langle f_\alpha | f \\ &= \int_{\alpha \in \mathbb{C}} d\mu(\alpha) (f_\alpha, f) f_\alpha \\ &= \int_{\alpha \in \mathbb{C}} d\mu(\alpha) \phi(\bar{\alpha}) f_\alpha \end{aligned} \tag{23}$$

where $\phi(\bar{\alpha})$ is the coherent state representation of f:

$$\theta(\bar{\alpha}) = \psi(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} (\bar{\alpha})^n b_n$$

and

$$d\mu(\alpha) = \psi(|\alpha|^2)\sigma(|\alpha|^2) r\, dr\, d\theta.$$

In particular, if $f \equiv f_n$, the corresponding coherent state representation is

$$(f_\alpha, f_n) = \psi(|\alpha|^2)^{-1/2} \frac{(\bar{\alpha})^n}{\sqrt{\lambda_n}}. \tag{24}$$

6. SQUEEZED STATES

Squeezed state is generated by the action of $a - \alpha a^+$ on an arbitrary state f in $\Gamma(\mathbb{C})$ (Solomon and Katriel, 1990) and satisfying the following equation:

$$(a - \alpha a^+) f = 0 \tag{25}$$

where $f = \sum_{n=0}^{\infty} \alpha_n |n\rangle$.

Now

$$\begin{aligned} af &= a[\alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_n|n\rangle + \cdots] \\ &= \alpha_1 \frac{\lambda_1}{\lambda_0}|0\rangle + \cdots + \alpha_n \frac{\lambda_n}{\lambda_{n-1}}|n-1\rangle + \alpha_{n+1}\frac{\lambda_{n+1}}{\lambda_n}|n\rangle + \cdots \end{aligned} \tag{26}$$

and

$$\begin{aligned} a^+ f &= a^+[\alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_n|n\rangle + \cdots] \\ &= \alpha_0|1\rangle + \alpha_1|2\rangle + \cdots + \alpha_{n-1}|n\rangle + \alpha_n|n+1\rangle + \cdots \end{aligned} \tag{27}$$

From Eqs. (25), (26), and (27) we observe that α_n satisfies the following difference equation:

$$\alpha_{n+1}\frac{\lambda_{n+1}}{\lambda_n} = \alpha\alpha_{n-1} \tag{28}$$

That is,

$$\alpha_{n+2} = \alpha \frac{\lambda_{n+1}}{\lambda_{n+2}} \alpha_n \tag{29}$$

and

$$\alpha_1 = 0 \tag{30}$$

Thus we see that

$$\alpha_{2n} = \alpha^n \frac{\lambda_1\lambda_3\lambda_5\cdots\lambda_{2n-1}}{\lambda_2\lambda_4\lambda_6\cdots\lambda_{2n}}\alpha_0$$

and

$$\alpha_1 = \alpha_3 = \alpha_5 = \cdots = \alpha_{2n-1} = 0$$

Hence f takes the form

$$f = \alpha_0 \left[|0\rangle + \alpha \frac{\lambda_1}{\lambda_2} |2\rangle + \cdots + \alpha^n \frac{\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1}}{\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n}} |2n\rangle + \cdots \right]$$
$$= \alpha_0 \sum_{n=0}^{\infty} \alpha^n \frac{\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1}}{\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n}} |2n\rangle \tag{31}$$

To normalize we have

$$1 = (f, f) = |\alpha_0|^2 \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{(\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1})^2}{(\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n-1})^2 \lambda_{2n}} \tag{32}$$

Thus aside from a trivial phase we have

$$\alpha_0 = \left\{ \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{(\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1})^2}{(\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n-1})^2 \lambda_{2n}} \right\}^{-1/2} \tag{33}$$

and the squeezed state f takes the form

$$f = \left[\sum_{n=0}^{\infty} |\alpha|^{2n} \frac{(\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1})^2}{(\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n-1})^2 \lambda_{2n}} \right]^{-1/2} \sum_{n=0}^{\infty} \alpha^n \frac{\lambda_1 \lambda_3 \lambda_5 \cdots \lambda_{2n-1}}{\lambda_2 \lambda_4 \lambda_6 \cdots \lambda_{2n}} |2n\rangle \tag{34}$$

7. CONCLUSION

In conclusion, we showed the existence of an orthonormal basis in interacting Fock space, representation of coherent states, and their overcompleteness as a basis. Finally, we have formulated the form of squeezed states in this space.

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