CERTAIN INEQUALITY RELATIONSHIPS AMONG THE COMBINATORIAL PARAMETERS OF INCOMPLETE BLOCK DESIGNS

By K. R. NAIR,
Statistical Laboratory, Calcutta.

1. In agricultural field experiments where the accuracy of the final results depends on the successful elimination of the effects of soil heterogeneity, certain types of designs involving blocks with fewer number (k) of plots than the number (v) of treatments have been introduced, in order to improve the general level of accuracy or efficiency of the experiment. These designs have come to be known as *Incomplete Block* Designs on account of the fact that $v>k$.

2. One outstanding class of incomplete block designs which was discovered by Yates (1936*b*) has the following description :—

In any design there are b blocks of k plots in which v varieties (or treatments) are replicated r times. Every pair of varieties repeats in λ blocks. Hence, we have

$$bk = rv\ ;\ \lambda = r(k-1)/(v-1) \qquad \ldots \quad (1)$$

3. We know by definition that $v>k$, from which it follows that $b>r>\lambda$.

But there is one other important inequality relation, namely,

$$b \geqslant v \text{ or } r \geqslant k \qquad \ldots \quad (2)$$

a proof for which has been provided by Fisher (1940) using an ingenious method.

4. When Yates' designs are resolvable, that is, when the b blocks can be grouped together in r sets of n blocks, each set constituting a complete replication of the v varieties, we will have $b = nr$ and $v = nk$. Bose (1942) has recently shown that a more stringent inequality than (2), namely,

$$b \geqslant v + r - 1 \qquad \ldots \quad (3)$$

should hold in the case of such resolvable designs.

5. A much simpler alternative proof for the inequality relations (2) and (3) is given later, in the present paper.

6. The special importance of the 'method of standard deviation' used by Fisher and Bose lies in another direction which has not been noticed by them. Thus it can easily be proved by their method that a still closer lower limit for b can be obtained as shown below.

Fisher proved that if x_i is the number of treatments common to a given block and any other (ith) block

$$\left.\begin{aligned} \sum_{1}^{b-1} x_i &= k(r-1) \\ \sum_{1}^{b-1} x_i^2 &= k[(r-k) + \lambda(k-1)] \end{aligned}\right\} \qquad \ldots \quad (4)$$

From the definition of standard deviation of x, we get

$$b - 1 \geqslant (\Sigma x)^2/\Sigma(x^2) \qquad \ldots \quad (5)$$

$$\text{or} \quad b \geqslant 1 + k(r-1)^2/[(r-k) + \lambda(k-1)] \qquad \ldots \quad (6)$$

7. For resolvable designs considered by R. C. Bose, Σx_i and Σx^2_i will have the same value as in (4) but since we definitely know that x will be 0 in $n-1$ out of the $b-1$ blocks, i varies from 1 to $b-n$ or $b(1-1/r)$. Therefore we have

$$b(1-1/r) \geqslant (\Sigma x)^2/\Sigma(x^2) \qquad \text{.. (7)}$$

or $$b \geqslant rk(r-1)/[(r-k)+\lambda(k-1)] \qquad \text{.. (8)}$$

Knowing that $b \geqslant v$, it can easily be shown that the r.h.s. of (6) is $\geqslant v$. Similarly knowing that $b \geqslant v+r-1$, it is easy to see that the r. h. s. of (8) is $\geqslant v+r-1$. It therefore follows that (6) and (8) give closer lower limits for b than those obtained by Fisher and Bose.

8. A more general class of incomplete block designs than Yates' designs has been introduced by Bose and Nair (1939). Here also the parameters b, k, r and v have the same definition as in Yates' design. But instead of a single parameter λ, we will have several λ-parameters. In these latter designs, with respect to any variety the remaining varieties fall into m associate-classes of $n_1, n_2, \ldots n_m$ varieties so that the first variety occurs with each of the n_i varieties of the ith group in λ_i blocks. The λ's need not all be unequal (Nair and Rao, 1942), nor should all be equal in which case we get Yates' design. The n_i varieties will be called the ith associates of the first variety. These designs are known as Partially Balanced Incomplete Block Designs and include as special cases many of the confounded designs for factorial and quasi-factorial experiments (Nair and Rao, 1942). All these special cases give resolvable designs.

9. There is an additional set of parameters in Partially Balanced Incomplete Block Designs which is not present in Balanced Incomplete Block Designs. Any parameter of this set is denoted by p^i_{jk} and stands for the number of varieties common to the jth associates of any variety (say V_1) and the kth associates of another variety which belongs to the group of ith associates of V_1. The following relationships between all the combinatorial parameters have been established by Bose and Nair (1939).

$$bk = rv\;;\; \sum_1^m n_i = v-1\;;\; \sum_1^m n_i\lambda_i = r(k-1) \qquad (9)$$

$$\left.\begin{aligned} \sum_{k=1}^m p^i_{jk} &= n_j - 1 \quad \text{if } i = j \\ &= n_j \qquad \text{if } i \neq j \end{aligned}\right\} \qquad (10)$$

$$n_i\, p^i_{jk} = n_j\, p^j_{ik} = n_k\, p^k_{ij} \qquad (11)$$

10. As before, $b > r$, since $v > k$. But in the place of $r > \lambda$, we have $r \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m$, where the λ's are not all equal. As regards the relation between b and v, actual designs constructed in the previous paper include not only the cases $b \geqslant v$ but also the case $b < v$. Similarly there are many resolvable partially balanced designs with $b < v+r-1$ and even with $b < v$.

It may be mentioned that incomplete block designs with $b < v$, or, in other words, $r < k$ are of special appeal for experimental work as such designs require comparatively less number of replications of the treatments. Balanced incomplete block designs do not possess this advantage.

11. We shall presently introduce a certain function involving r, λ_i and p^i_{jk} which will be (i) zero when $b < v$ and $< v + r - 1$ depending on the resolvability of the design, and which will be (ii) positive only when $b \geqslant v$ or $\geqslant v + r - 1$. For balanced design this function reduces to $r - \lambda$, which being positive and never 0 leads to the conclusion that in those designs b cannot be less than v or $v + r - 1$, thus providing us an alternative proof for the inequalities obtained by Fisher and by Bose respectively.

12. Let us suppose that the individual plot values of the character observed (say, yield) are not available but have got mixed up in such a way that total value for each block is known (Nair, 1940). Let us denote these totals by $B_1, B_2, \ldots B_b$. In such a situation we cannot separate block effects from treatment effects and so shall ignore the former and try to estimate the latter from the block totals. These estimates of treatment effects will be called interblock (between block) estimates to distinguish them from intra-block (within block) estimates which could have been obtained if individual plot values were known. We shall estimate a general mean, m; and v treatment effects, $v_1, v_2, \ldots v_v$, (where $\sum_1^v v_i = 0$), from the b known quantities $B_1, B_2, \ldots B_b$ by the method of least squares, that is, by minimising the quantity

$$\sum_{j=1}^{b} (B_j - km - \sum_{i=1}^{k} v_{ij})^2 \qquad (12)$$

It is evident that a necessary condition for the solution to exist is $b \geqslant v$ as there are v independent unknowns and b knowns.

If the partially balanced incomplete block design is resolvable we could set about estimating r replication effects: $r_1, r_2, \ldots r_r$ (where $\Sigma r = 0$) besides m and $v_i (i = 1$ to $v)$. We will then have to estimate $v + r - 1$ independent unknowns from the b known block totals, by minimising

$$\sum_{i=1}^{n} \sum_{j=1}^{r} (B_{ij} - km - kr_j - \sum_{1}^{k} v)^2 \quad .. \quad (13)$$

It is evident therefore that for resolvable designs, inter-block estimates can be obtained for varietal effects only if $b \geqslant v + r - 1$. In fact if the design is resolvable only in sets of $c (\geqslant 2)$ replications, where c is a factor of r, the corresponding inequality relation will be $b \geqslant v + r/c - 1$.

If we denote by T_i the sum of the totals of the r blocks in which variety V_i occurs, and by T_o the sum of the totals of all the blocks, we have for both (12) and (13)

$$P_i = T_i - (k/v)\, T_o = r(1 - k/v)\, v_i + \sum_{j=1}^{m} \{(\lambda_j - rk/v)\, (\sum_{(n_j)} v_{ij})\} \quad .. \quad (14)$$

where $\sum_{(n_j)} v_{ij}$ = sum of the effects of the n_j, jth associates of V_i. If we denote by $\sum_{(n_j)} P_{ij}$, the sum of the P's corresponding to those jth associates of V_i, we have

$$\sum_{(n_j)} P_{ij} = n_j\, (\lambda_j - rk/v)\, v_i + r \sum_{(n_j)} v_{ij} + \sum_{l=1}^{m} [\{ \sum_{k=1}^{m} (\lambda_k\, p^l_{jk}) - (rk/v) n_j \} (\sum_{(n_l)} v_{il})] \quad .. \quad (15)$$

The m equations of the type (15) for a given value of i, along with the condition $\Sigma v = 0$, will be enough to solve $v_i, \Sigma v_{i1}, \Sigma v_{i2}, \ldots \Sigma v_{im}$ and we get

$$v_i = \Delta_i / \Delta \qquad (16)$$

where

$$\Delta_i = \begin{vmatrix} P_i & (1 - rk/v + \lambda_1) & . & . & (1 - rk/v + \lambda_m) \\ \Sigma\, P_{1i} & \{r + \sum_1^m \lambda_k\, p^1_{1k} - (rk/v)\, n_1\} & . & . & \{\sum_1^m \lambda_k\, p^m_{1k} - (rk/v) n_1\} \\ . & . & . & . & . \\ . & . & . & . & . \\ \Sigma\, P_{im} & \{\sum_1^m \lambda_k\, p^1_{mk} - (rk/v)\, n_m\} & . & . & \{r + \sum_1^m \lambda_k\, p^m_{mk} - (rk/v) n_m\} \end{vmatrix} \quad .. \quad (17)$$

and

$$\Delta = \begin{vmatrix} r + \sum_1^m \lambda_k\, p^1_{1k} - n_1\, \lambda_1 & . & . & \sum_1^m \lambda_k\, p^m_{1k} - n_1\, \lambda_1 \\ . & . & . & . \\ . & . & . & . \\ \sum_1^m \lambda_k\, p^1_{mk} - n_m\, \lambda_m & . & . & r + \sum_1^m \lambda_k\, p^m_{mk} - n_m\, \lambda_m \end{vmatrix} \quad .. \quad (18)$$

13. In the general case where the estimates are based on (12) a solution for v_i can exist only if $b \geqslant v$, although this is not a sufficient condition. We therefore conclude that if $\Delta > 0$, then $b \geqslant v$; and if $b < v$, then $\Delta = 0$. But Δ may also vanish when $b \geqslant v$.

In the special case of resolvable designs, where the estimates are based on (13) a solution for v_i can exist only if $b \geqslant v + r - 1$, although this is not a sufficient condition. We therefore conclude that if $\Delta > 0$, then $b \geqslant v + r - 1$; and if $b < v + r - 1$, then $\Delta = 0$. But Δ may also vanish when $b \geqslant v + r - 1$.

14. Now, Δ is an mth degree polynomial in r. When $m = 2$ it simplifies to

$$\Delta = (r - \lambda_1)\,(r - \lambda_2) + (\lambda_1 - \lambda_2)\,[p^2_{12}(r - \lambda_1) - p^1_{12}(r - \lambda_2)] \quad .. \quad (19)$$

The equation $\Delta = 0$ if expressed as a quadratic equation in $r - (\lambda_1 + \lambda_2)/2$ will have two real roots of opposite sign. But since $r \geqslant \lambda_1 > \lambda_2$, $r - (\lambda_1 + \lambda_2)/2$ is always positive. Therefore the inequality $\Delta \geqslant 0$ can, when $m = 2$, be replaced by

$$r - (\lambda_1 + \lambda_2)/2 \geqslant \tfrac{1}{2}\,(\lambda_1 - \lambda_2)\,[(p^1_{12} - p^2_{12}) + \sqrt{\{(p^1_{12} - p^2_{12})^2 + 2(p^1_{12} + p^2_{12}) + 1\}}] \quad .. \quad (20)$$

15. When $\Delta = 0$ the quantity within the square root in (20) must become a perfect square, as r cannot have a fractional value.

When $p^1_{12} = 0$ or $p^2_{12} = 0$ this quantity automatically becomes a perfect square and in these cases (20) reduces either to $r \geqslant \lambda_1$ or $(r - \lambda_1) \geqslant n_2(\lambda_1 - \lambda_2) > 0$ respectively.

16. For Yates' balanced incomplete block designs, $\lambda_1 = \lambda_2 = \lambda$ and (20) reduces to $(r - \lambda) \geqslant 0$. But by definition $r > \lambda$ so that Δ never vanishes, or b can never be less than v in this type of design; and in the resolvable cases of it, b can never be less than $v + r - 1$. Inter-block estimate of varietal effects can therefore be always obtained, and is given by $v_i = P_i/(r - \lambda)$.

17. As shown by Nair and Rao (1942) many of the confounded designs for factorial and quasi-factorial experiments are special cases of partially balanced incomplete block designs, and all of them are resolvable. For example, in the two-dimensional quasi-factorial experiments in two groups, [Yates (1936a)] where $b < v + r - 1$, the interaction cannot be

estimated from block totals as it has been confounded *within* blocks. And unless the main effects as well as the interaction can be estimated from block totals we cannot get inter-block estimate for each individual treatment effect.

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