Magnus force on quantum Hall skyrmions and vortices

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Abstract

We have discussed here the Magnus force acting on the vortices and skyrmions in the quantum Hall systems. We have found that it is generated by the chirality of the system which is associated with the Berry phase and is same for both the cases.

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In recent times, the topological excitations near the filling factor $\nu = 1$ in quantum Hall effect have drawn much attention [1–3]. There are two kinds of topological excitations in single layer quantum Hall systems. When the system is fully polarized the relevant charged quasiparticles are *topological vortices*. At $\nu = \frac{1}{2n+1}$ the charge and spin of such a vortex is $\frac{e}{2n+1}$ and $\frac{1}{2(2n+1)}$, respectively. In a remarkable paper Sondhi et al. [1] have argued that for a weak Zeeman coupling the lowest energy charged quasiparticle is a *topological soliton* or *skyrmion*. The charge is still $\frac{e}{2n+1}$ but the total spin can be substantially larger. This large spin with moderate charge explains the observed depolarization when the filling factor slightly deviates from $\nu = \frac{1}{2n+1}$. Indeed, the difference between a vortex and a skyrmion is that, a vortex is fully polarized while inside the skyrmion core there is some number of electrons with reversed spins. It may be noted that there is an interplay of two factors, namely the Zeeman and the Coulomb energy. Vortex solutions exists for any value of the Zeeman coupling and is well known from the studies on the vortices in the Ginzburg–Landau model of the fully polarized quantum Hall effect [4]. For weak Zeeman coupling the relevant quasiparticles are the skyrmions. In this Letter, we have studied the Magnus force acting on these skyrmions and vortices from their topological properties when it is studied in terms of $(3+1)$-dimensional nonlinear sigma model and found that they are the same.

To investigate the dynamics of vortices and skyrmions we begin with the Landau–Ginzburg theory of the Hall effect introduced by Zhang, Hansson and Kivelson [5] and modified by Lee and Kane [6] to incorporate the effect of the spin. For spin 1/2 particles if we set $\hbar = e = c = 1$, the Lagrangian can be written as

$$L = \overline{\Psi}_\sigma \left[\partial_0 - i(a_0 + eA_0)\right]\Psi_\sigma - \frac{1}{2m^*}\left|\left[\partial_i - i(a_i + A_i)\right]\Psi_\sigma\right|^2 - \frac{\lambda}{2}(\overline{\Psi}_\sigma \Psi_\sigma - \rho_0)^2 + \frac{1}{4\theta}\epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha. \tag{1}$$

Here Ψ_σ is a two-component Bose field with effective mass m^* and short-range repulsive interaction λ, which couples to the external and statistical gauge fields A_i and a_i ($i = 0, 1, 2$). Here ρ_0 is the uniform density $|\phi_1|^2 + |\phi_2|^2$ at filling factor $\nu = \frac{1}{2n+1}$. In order to separate the charge and spin degree of freedom, we explicitly separate the magnitude and $U(1)$ and $SU(2)$ phases of Ψ_σ: $\Psi_\sigma \to \sqrt{\rho}\,\phi z_\sigma$ with $\bar{\phi}\phi = \bar{z}_\sigma z_\sigma = 1$. By direct substitution and keeping the leading-order gradient terms we obtain

$$L = \rho\left[\bar{\phi}\partial_0\phi + \bar{z}_\sigma \partial_0 z_\sigma - i(a_0 + eA_0)\right] - \frac{\rho}{2m^*}\left|\bar{\phi}\partial_i\phi + \bar{z}_\sigma \partial_i z_\sigma - i(a_i + A_i)\right|^2 - \frac{\rho}{2m^*}\left[|\partial_i z_\sigma|^2 + (\bar{z}_\sigma \partial_i z_\sigma)^2\right] - \frac{\lambda}{2}(\rho - \rho_0)^2 + \frac{1}{4\theta}\epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha. \tag{2}$$

We now note that we have the identity

$$\frac{\rho}{2m^*}\left[|\partial_i z_\sigma|^2 + (\bar{z}_\sigma \partial_i z_\sigma)^2\right] = \frac{\rho}{8m^*}(\nabla \mathbf{n})^2, \tag{3}$$

where $\nabla\mathbf{n} = (\partial_1\mathbf{n}, \partial_2\mathbf{n})$ with $\mathbf{n} = \bar{z}_\sigma \sigma_{\alpha\beta} z_\sigma$, $\sigma_{\alpha\beta}$ being Pauli matrices. We now introduce Hubbard–Stratonovich fields $\mathbf{J}$ to decouple the second term of Eq. (2) to obtain

$$L = i\rho\left[\bar{\phi}\partial_0\phi + \bar{z}_\sigma \partial_0 z_\sigma - i(a_0 + eA_0)\right] + i\left[\bar{\phi}\partial_i\phi + \bar{z}_\sigma \partial_i z_\sigma - i(a_i + eA_i)\right]J_i + \frac{m^*}{2\rho}|\mathbf{J}|^2 - \frac{\rho}{8m^*}(\nabla\mathbf{n})^2 - \frac{\lambda}{2}(J_0 - \rho_0)^2 + \frac{1}{4\theta}\epsilon^{\mu\nu\alpha} a_\mu \partial_\nu a_\alpha, \tag{4}$$

where $\mathbf{J}$ is the three vector $\rho = (J_0, J_1, J_2)$. After integrating out the longitudinal fluctuations in ψ, we have the conserved current relation $J_\mu a_\mu = 0$, which can be satisfied by taking J_μ as the curl of a three-dimensional vector field. We can now set $J_\mu^{[0]} =$ $(\rho_0, 0, 0)$ equal to $\epsilon_{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda^{[0]}$ and

$$J_\mu - J_\mu^{[0]} = \epsilon_{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda. \tag{5}$$

Now, following Stone [7], we integrate out the Chern–Simons field a_μ and using the relation $2\theta\rho_0 = eB_z$, we can write

$$L = 2\pi\left[\mathcal{J}_\mu^V\left(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]}\right) + \mathcal{J}_\mu^S\left(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]}\right)\right] - \theta|\mathbf{J}|^\mu \mathcal{A}_\mu + \frac{m^*}{2\rho}\mathbf{J}^2 - \frac{\rho}{8m^*}(\nabla\mathbf{n})^2 - \frac{\lambda}{2}\left(J^0 - J^0_{[0]}\right)^2, \tag{6}$$

where

$$\mathcal{J}_\mu^V = \frac{1}{2\pi i}\epsilon_{\mu\nu\lambda}\partial_\nu\bar{\phi}\partial_\lambda\phi, \tag{7}$$

$$\mathcal{J}_\mu^S = \frac{1}{2\pi i}\epsilon_{\mu\nu\lambda}\partial_\nu\bar{z}_\sigma\partial_\lambda z_\sigma \tag{8}$$

are the skyrmion and vortex three currents, respectively. It is observed that the skyrmion current can be written in the familiar form

$$\mathcal{J}_\mu^S = \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\mathbf{n}.(\partial_\nu\mathbf{n} \times \partial_\lambda\mathbf{n}). \tag{9}$$

Now setting $\rho = \rho_0$ in the kinetic energy term and adjusting the units of length and time such that $c = \sqrt{\lambda\rho_0/m^*}$, the velocity of density wave in the absence of the magnetic field becomes unity and defining the field strength tensor $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$, we can write

$$L = 2\pi\left[\mathcal{J}_\mu^S\left(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]}\right) + \mathcal{J}_\mu^V\left(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]}\right)\right] - \frac{1}{2}\theta\,\epsilon_{\mu\nu\sigma}\mathcal{A}_\mu\mathcal{F}_{\nu\sigma} - \frac{\lambda}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{8\lambda}(\nabla\mathbf{n})^2. \tag{10}$$

It is observed that here $\mathcal{A}_\mu$ is a topologically massive gauge field and $\mathcal{A}_\mu^{[0]}$ just represents the background field.

To study the Magnus force on these vortices and skyrmions, we now take resort to the spherical geometry. In a recent paper [8], we have studied quantum Hall skyrmions in terms of the $(3+1)$-dimensional nonlinear sigma model where we have taken a system of 2D electron gas residing on the surface of a 3D sphere with a monopole at the centre. We note that taking the spin variable $\mathbf{z} = U\mathbf{z}_0$ where $\mathbf{z}_0 = \binom{1}{0}$ and $U \in SU(2)$, we may write the nonlinear sigma model Lagrangian in terms of U. The $(3+1)$-dimensional generalization of the skyrmion current can now be defined as

$$\mathcal{J}_\mu^S = \frac{1}{24\pi^2}\epsilon_{\mu\nu\lambda\sigma} \times \mathrm{Tr}\big[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\sigma U)\big], \quad (11)$$

where it is associated with an $O(4)$ nonlinear sigma model with U defined as

$$U = \pi_0 + i\vec{\pi}.\vec{\sigma}, \quad U \in SU(2), \quad (12)$$

where $\vec{\sigma}$ are Pauli matrices and $\vec{\pi}$ are chiral boson fields, satisfying the constraint

$$\pi_0^2 + \vec{\pi}^2 = 1. \quad (13)$$

The $(3+1)$-dimensional generalization of the vortex current can be written as

$$\mathcal{J}_\mu^V = \epsilon_{\mu\nu\lambda\sigma}\,\partial_\nu\phi\,\partial_\lambda\phi\,\partial_\sigma\phi. \quad (14)$$

In a pure gauge, we can take a gauge field B_μ such that

$$B_\mu = \partial_\mu\phi. \quad (15)$$

Now noting that vortex–antivortex pair can be taken as an $SU(2)$ doublet, we may consider B_μ as an $SU(2)$ gauge field and write

$$B_\mu = \widetilde{U}^{-1}\partial_\mu\widetilde{U}, \quad \widetilde{U} \in SU(2). \quad (16)$$

In view of this, we can write the vortex current taking into account proper normalization

$$\mathcal{J}_\mu^V = \frac{1}{24\pi^2}\epsilon_{\mu\nu\lambda\sigma} \times \mathrm{Tr}\big[(\widetilde{U}^{-1}\partial_\nu\widetilde{U})(\widetilde{U}^{-1}\partial_\lambda\widetilde{U})(\widetilde{U}^{-1}\partial_\sigma\widetilde{U})\big]. \quad (17)$$

Thus comparing Eq. (11) and Eq. (17) we note that the skyrmion current and vortex current can be written in a similar form which is also shown by Duan et al. [9].

In $3+1$ dimensions we can generalize the Lagrangian (10) with non-Abelian gauge field $\mathcal{A}_\mu[\mathcal{A}_\mu^{[0]}] \in SU(2)$ and the θ-term in the form

$$L = 2\pi\big[\mathcal{J}_\mu^S(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]}) + \mathcal{J}_\mu^V(\mathcal{A}_\mu + \mathcal{A}_\mu^{[0]})\big] - \frac{M^2}{16}\mathrm{Tr}(\partial_\mu U^\dagger\partial_\nu U) - \frac{\theta}{16\pi^2}\mathrm{Tr}\,{}^*\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{4}\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \quad (18)$$

Here $\theta = g/c^2$ with $g = \nu e^2/h$ as Hall conductivity and ${}^*\mathcal{F}_{\mu\nu}$ is a Hodge dual given by

$${}^*\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\mathcal{F}_{\lambda\sigma}. \quad (19)$$

The third term in Eq. (18) is the θ-term which is the $(3+1)$-dimensional relative to the $(2+1)$-dimensional Chern–Simons term. It is noted that the θ-term is related to chiral anomaly and Berry phase [10]. Indeed, it is a four divergence and we can write

$$-\frac{1}{16\pi^2}\mathrm{Tr}\,{}^*\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} = \partial_\mu\Omega_\mu, \quad (20)$$

where Ω_μ is the Chern–Simons characteristic class given by

$$\Omega_\mu = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\lambda\sigma}\,\mathrm{Tr}\left(\mathcal{A}_\nu\mathcal{F}_{\lambda\sigma} + \frac{2}{3}\mathcal{A}_\nu\mathcal{A}_\lambda\mathcal{A}_\sigma\right). \quad (21)$$

When a fermionic chiral current interacts with a gauge field, we may define

$$\tilde{J}_\mu^5 = J_\mu^5 + 2\Omega_\mu, \quad (22)$$

where J_μ^5 is the axial vector current so that $\overline{\Psi}\gamma_\mu\gamma_5\Psi$, we have $\partial_\mu\tilde{J}_\mu^5 = 0$ whereas $\partial_\mu J_\mu^5 \neq 0$. Indeed, we have the chiral anomaly given by

$$\partial_\mu J_\mu^5 = -2\partial_\mu\Omega_\mu = \frac{1}{8\pi^2}\mathrm{Tr}\,{}^*\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}. \quad (23)$$

The Pontryagin index is given by

$$q = 2\mu = -\frac{1}{16\pi^2}\int \mathrm{Tr}\,{}^*\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\,d^4x = \int d^4x\,\partial_\mu\Omega_\mu = -\frac{1}{2}\partial_\mu J_\mu^5\,d^4x. \quad (24)$$

Here μ represents a magnetic charge. The Berry phase $e^{i\Phi_B}$ is associated with the chiral anomaly [10,11] through the relation

$$\Phi_B = 2\pi\mu. \quad (25)$$

In Euclidean space–time if we demand $\mathcal{F}_{\mu\nu} = 0$, then the gauge potential tends to a pure gauge in the limiting case towards the boundary, i.e., we can take

$$\mathcal{A}_\mu = U^{-1}\partial_\mu U, \quad U \in SU(2). \quad (26)$$

This gives

$$\Omega_\mu = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta} \times \mathrm{Tr}\big[(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)\big]. \quad (27)$$

Comparing this with Eq. (11) we note that Ω_μ can be related to the skyrmion current. It is noted that on the boundary S^3 where $\mathcal{F}_{\mu\nu}=0$, we have $\partial_\mu \Omega_\mu = 0$ as is evident from Eq. (23). However, inside the volume V^4 where $\mathcal{F}_{\mu\nu} \neq 0$, Ω_μ is associated with Berry phase as follows from Eq. (24). On the boundary S^3 of a volume V^4, the term $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ in the Lagrangian (18) gives rise to the Skyrme term $[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}]^2$ in this limiting case. This ensures the stability of the skyrmion so that it does not shrink to zero size. Here also $\mathcal{A}_\mu$ appears as a topologically massive gauge field as it has been shown elsewhere that when a gauge field interacts with a chiral current it acquires mass topologically through chiral anomaly [12].

If we take the compactified 3-sphere we have the winding number associated with the homotopy $\Pi_3(S^3) = Z$

$$q = 2\mu = \frac{1}{24\pi^2} \int_{S^3} dS_\mu \, \epsilon_{\mu\nu\alpha\beta} \times \mathrm{Tr}\big[\big(U^{-1}\partial_\nu U\big)\big(U^{-1}\partial_\alpha U\big)\big(U^{-1}\partial_\beta U\big)\big]. \tag{28}$$

This effectively represents the geometric phase associated with the θ-term in the nonlinear sigma model action.

The configuration of a skyrmion is such that spins wrap an unit sphere with a Dirac flux quanta within it and the resultant spin arising out of spin reversals will give rise to a specific chirality. When a skyrmion of charge α ($\alpha = \frac{e}{2n+1}$) moves around a closed path, the Berry phase is given by $2\pi\alpha N$ where N is the number of skyrmions enclosed by the path as is evident from Eq. (25). The Magnus force [13] is now given by

$$F = 2\pi\alpha N \hat{c} \times \mathbf{V}, \tag{29}$$

where $\hat{c}$ represents the axis of resultant chirality and $\mathbf{V}$ is the velocity of the skyrmion with respect to the quantum Hall fluid.

When the Zeeman energy is large, i.e., the vortices are the dominant excitations in the system, these vortices will also experience a similar Magnus force. Since the two currents are of similar form, the vortex current can also be associated with the chiral anomaly giving rise to the Berry phase. We know that topologically a vortex is equivalent to a magnetic flux, so when a vortex moves in a closed path, the Berry phase is given by $2\pi\alpha N$ where $\alpha = \frac{e}{2n+1}$ is the charge of the vortex and N is the number of vortices enclosed by the loop. The Magnus force is given by the vector product of the vorticity and the velocity relative to the quantum Hall fluid

$$F = \pm 2\pi\alpha N \hat{z} \times \mathbf{V}_{\text{vortex}}, \tag{30}$$

where $\pm$ corresponds to a vortex parallel (antiparallel) to the z-direction and $\mathbf{V}_{\text{vortex}}$ is the vortex velocity. Thus a comparison between Eq. (29) and Eq. (30) suggests that the Magnus force experienced by a vortex and a skyrmion in a quantum Hall fluid is the same. This is consistent with the result obtained by Dziarmaga [14].

This analysis supports the Ao–Thouless theory [13] of Magnus force which associates the Magnus force with the Berry phase. In this scenario, the skyrmion current as well as the vortex current effectively represents a chiral current which is associated with the chiral anomaly when it interacts with a topologically massive gauge field. The Magnus force is generated by the background field associated with the chirality of the system. Finally, we may mention that in a recent paper [15] we have analyzed the Magnus force acting on vortices in high temperature superconductivity and is also found to be generated by the background field associated with the chirality of the system. We observe here that if we study the Magnus force from the topological properties of the excitations concerned the Magnus force acting on vortices and skyrmions in quantum Hall fluid as well as on vortices in high T_c superconductivity is of similar origin.

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