Linear Algebra to Quantum Cohomology: The Story of Alfred Horn's Inequalities

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We want first an overview of the aim and of the road; we want to understand the *idea* of the proof, the deeper context. A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.

Hermann Weyl

A long-standing problem in linear algebra—Alfred Horn's conjecture on eigenvalues of sums of Hermitian matrices—has been solved recently. The solution appeared in two papers, one by Alexander Klyachko [20] in 1998 and the other by Allen Knutson and Terence Tao [23] in 1999. This has been followed by a flurry of activity that has brought to the mathematical centre-stage what for many years had been somewhat of a side-show. The aim of this article is to describe the problem, its origins, some of the early work on it, and some ideas that have gone into its solution.

A substantial part of this article should be accessible to anyone who has had a second course on linear algebra. The reader who wants to know more will find it rewarding to read the comprehensive and advanced account [11] by William Fulton.

1. LINEARITY, QUASILINEARITY, AND CONVEXITY The principal characters in our story are $n \times n$ Hermitian matrices A and B, their sum $C = A + B$, and the eigenvalues of A, B, and C enumerated as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, respectively. Sometimes we would like to emphasize the dependence of the eigenvalues on the matrix. We then use the notation $\lambda_j^\downarrow(A)$ for the jth eigenvalue of A when the eigenvalues are arranged in a (weakly) decreasing order. Thus $\alpha_j = \lambda_j^\downarrow(A)$. This n-tuple of eigenvalues of A as a whole is denoted by α or $\lambda^\downarrow(A)$.

The story begins with a simple question: what are the relationships between α, β, and γ?

Now, the eigenvalues are *not* linear functions of A and no simple relation between α, β, and γ is apparent, except one. The *trace* of A, denoted $\operatorname{tr} A$ is the sum of the diagonal entries of A and also of the eigenvalues of A. So, $\operatorname{tr} C = \operatorname{tr} A + \operatorname{tr} B$ and hence

$$\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_j. \tag{1}$$

We can think of A as a linear operator on the complex Euclidean space $\mathbb{C}^n$ equipped with its usual inner product $\langle x, y\rangle$, written also as x^*y and the associated norm $\|x\| = (x^*x)^{1/2}$. The Spectral Theorem tells us that every Hermitian operator A can be diagonalised in some orthonormal basis; or equivalently, there exists a unitary matrix U such that $UAU^* = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, a diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. If u_j are the orthonormal eigenvectors corresponding to its eigenvalues α_j, we write $A = \sum \alpha_j u_j u_j^*$, and call this the *spectral resolution* of A. Using this, it is easy to see that the set $\{\langle x, Ax\rangle : \|x\| = 1\}$ (called the *numerical range* of A) is equal to the interval $[\alpha_n, \alpha_1]$. In particular, we have

$$\alpha_1 = \max_{\|x\|=1} \langle x, Ax\rangle, \tag{2}$$

$$\alpha_n = \min_{\|x\|=1} \langle x, Ax\rangle. \tag{3}$$

For each fixed vector x, the quantity $\langle x, Ax\rangle$ depends linearly on A. Equations (2) and (3) express α_1, α_n as a maximum or minimum over these linear functions. Such expressions are called *quasilinear*. Very often, they lead to interesting inequalities. Thus, from (2) and (3) we have

$$\gamma_1 \leq \alpha_1 + \beta_1, \tag{4}$$

$$\gamma_n \geq \alpha_n + \beta_n. \tag{5}$$

In this way, we begin to get *linear inequalities* between α, β, and γ. There is another way of looking at (4). The set of $n \times n$ Hermitian matrices is a real vector space. The inequality (4) says $\lambda_1^{\downarrow}(A)$ is a *convex* function on this space; the inequality (5) says that $\lambda_n^{\downarrow}(A)$ is *concave*.

The inequalities (4) and (5) are not independent. Note that the eigenvalues of $-A$ are the same as the negatives of the eigenvalues of A. But taking negatives reverses order; so for $1 \leq j \leq n$,

$$\lambda_j^{\downarrow}(-A) = -\lambda_{n-j+1}^{\downarrow}(A) = -\lambda_j^{\uparrow}(A), \tag{6}$$

where the notation $\lambda_j^{\uparrow}(A)$ indicates that we are now enumerating the eigenvalues of A in increasing order. Using this observation we can see that (2) and (3) are equivalent, as are (4) and (5). Many of the inequalities stated below lead to complementary inequalities by this argument.

2. THE MINMAX PRINCIPLE AND WEYL'S INEQUALITIES The relations (2) and (3) are subsumed in a variational principle called the *minmax principle*. It says that for all $1 \leq j \leq n$

$$\alpha_j = \max_{\substack{V \subset \mathbb{C}^n \\ \dim V = j}} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax\rangle = \min_{\substack{V \subset \mathbb{C}^n \\ \dim V = n-j+1}} \max_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax\rangle \tag{7}$$

Here dim V stands for the dimension of a linear space V contained in $\mathbb{C}^n$. This principle was first mentioned in a 1905 paper of E. Fischer. Its proof is easy. Use the spectral resolution $A = \sum \alpha_j u_j u_j^*$. Let W be the space spanned by the vectors $u_j, \ldots, u_n$. Then dim $W = n - j + 1$. So, if V is any j-dimensional subspace of $\mathbb{C}^n$, then V and W have a nonzero intersection. If x is a unit vector in this intersection, then $\langle x, Ax\rangle$ lies in the interval $[\alpha_n, \alpha_j]$. This shows that

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax\rangle \leq \alpha_j.$$

If we choose V to be the subspace spanned by $u_1, \ldots, u_j$, we obtain equality here. This proves the first relation in (7). The second has a very similar proof.

This principle has several very interesting consequences. Hermitian matrices can be ordered in a natural way. We say that $A \leq B$ if $\langle x, Ax\rangle \leq \langle x, Bx\rangle$ for all x. One sees at once from (7) that if $A \leq B$, then $\lambda_j^{\downarrow}(A) \leq \lambda_j^{\downarrow}(B)$ for all j. This is called *Weyl's monotonicity principle*. (The applied mathematics classic by Courant and Hilbert [**8**] is full of applications of eigenvalue problems in physics. The Weyl monotonicity principle has the following physical interpretation: *If the system stiffens, the pitch of the fundamental tone and every overtone increases* [**8**, p. 286, Theorem IV]. This indeed is the experience of anyone tuning the wires of a musical instrument.)

Weyl's monotonicity principle, and several other relations between eigenvalues of A, B, and $A+B$ were derived by H. Weyl in a famous paper in 1912 [**33**]. Particularly important for our story is the family of inequalities

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad \text{for } i+j-1 \leq n. \tag{8}$$

These can be proved using the same idea as the one that gave us the min-max principle. Let A, B, and $A+B$ have spectral resolutions $A = \sum \alpha_j u_j u_j^*$, $B = \sum \beta_j v_j v_j^*$, $A+B = \sum \gamma_j w_j w_j^*$. Consider the three subspaces spanned by $\{u_i, \ldots, u_n\}$, $\{v_j, \ldots, v_n\}$, and $\{w_1, \ldots, w_k\}$. These spaces have dimensions $n-i+1$, $n-j+1$, and k respectively. If $k = i+j-1$, these numbers add up to $2n+1$. This implies that these three subspaces of $\mathbb{C}^n$ have a nontrivial intersection. Let x be a unit vector in this intersection. Then $\langle x, Ax\rangle$ is in the interval $[\alpha_n, \alpha_i]$, $\langle x, Bx\rangle$ in $\left[\beta_n, \beta_j\right]$, and $\langle x, (A+B)x\rangle$ in $[\gamma_k, \gamma_1]$. Hence

$$\gamma_k \leq \langle x, (A+B)x\rangle = \langle x, Ax\rangle + \langle x, Bx\rangle \leq \alpha_i + \beta_j.$$

This proves (8).

Note that the inequality (4) is a very special case of (8). Another special consequence of (8) is the inequality

$$\alpha_i + \beta_n \leq \gamma_i \leq \alpha_i + \beta_1 \quad \text{for } 1 \leq i \leq n. \tag{9}$$

The second inequality is derived from (8) simply by putting $j = 1$; the first by the sort of argument indicated at the end of Section 1.

As an aside, let us mention the interest such results have for numerical analysts. For any operator A on $\mathbb{C}^n$ define

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \tag{10}$$

If A is Hermitian, then it is easy to see that

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle| = \max\left(|\alpha_1|, |\alpha_n|\right). \tag{11}$$

Using this, one can see from (9) that

$$\alpha_i - \|B\| \leq \gamma_i \leq \alpha_i + \|B\|. \tag{12}$$

By a change of labels (replace B by $B-A$) this leads to the *Weyl perturbation theorem*

$$\max_j |\alpha_j - \beta_j| \leq \|A - B\|. \tag{13}$$

In numerical analysis one often replaces a matrix A by a nearby matrix B whose eigenvalues might be easier to calculate. Inequalities like (13) then provide useful information on the error caused by such approximations.

Some of the inequalities in the following sections provide finer information of interest to the numerical analyst. We do not discuss this further in this article; see [5].

Convexity properties of eigenvalues and intersection properties of eigenspaces are closely related, as we have already seen. This is the *leitmotif* of our story.

3. THE CASE $n = 2$ When $n = 2$, the statement (8) contains three inequalities

$$\gamma_1 \le \alpha_1 + \beta_1, \quad \gamma_2 \le \alpha_1 + \beta_2, \quad \gamma_2 \le \alpha_2 + \beta_1. \tag{14}$$

It turns out that, together with the trace equality (1), these three inequalities are sufficient to characterise the possible eigenvalues of A, B, and C; i.e., if three pairs of real numbers $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$, each ordered decreasingly ($\alpha_1 \ge \alpha_2$, etc.), satisfy the relations (1) and (14), then there exist 2×2 Hermitian matrices A and B such that these pairs are the eigenvalues of A, B, and $A + B$.

Let us indicate why this is so. Choose two pairs α, β, say

$$\alpha_1 = 4, \quad \alpha_2 = 1, \quad \beta_1 = 3, \quad \beta_2 = -2.$$

What are the γ that satisfy (1) and (14)? The condition (1) says

$$\gamma_1 + \gamma_2 = 6.$$

This gives a line in the plane $\mathbb{R}^2$. The restriction $\gamma_1 \ge \gamma_2$ gives half of this line—its part in the half-plane $\gamma_1 \ge \gamma_2$. One of the three inequalities in (14) is redundant; the other two are

$$\gamma_1 \le 7, \quad \gamma_2 \le 2.$$

So, the set of γ that satisfy (1) and (14) constitutes the line segment with end points $(4, 2)$ and $(7, -1)$; see Figure 1. We want to show that each point on this segment corresponds to the two eigenvalues of a Hermitian matrix $C = A + B$, where A has eigenvalues $(4, 1)$ and B has eigenvalues $(3, -2)$.

Start with the diagonal matrices

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Let U_θ be the 2×2 rotation matrix

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and let

$$B_\theta = U_\theta B_0 U_\theta^*, \quad C_\theta = A + B_\theta.$$

This gives a family of Hermitian matrices parametrised by the real number θ. Note that

$$C_0 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}.$$

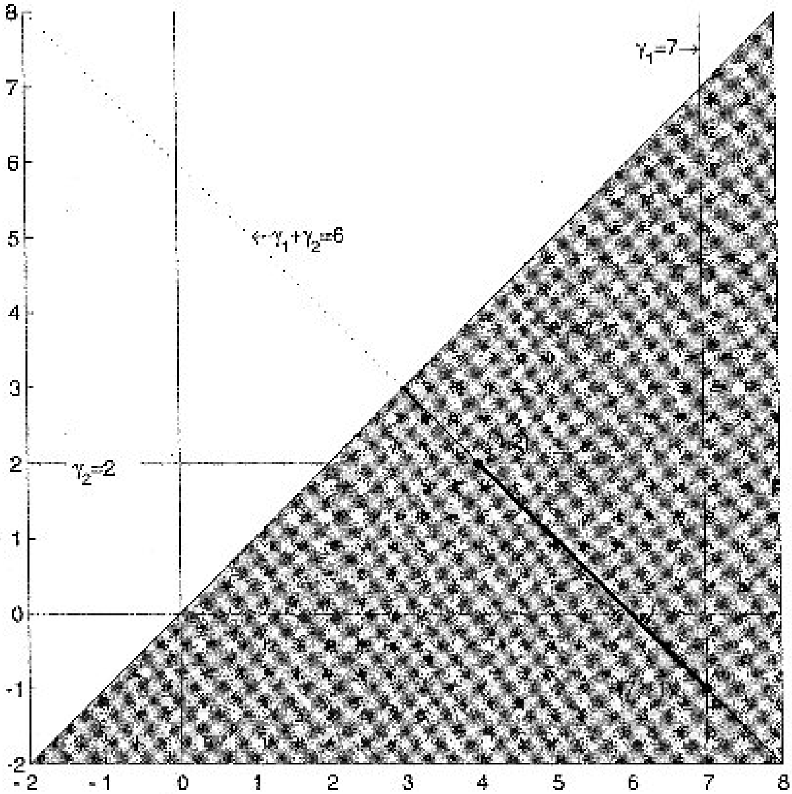

Figure 1. The line segment given by Weyl's inequalities.

$$C_{\pi/2} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus the two end points of our line segment correspond to $\left(\lambda_1^{\downarrow}(C_\theta), \lambda_2^{\downarrow}(C_\theta)\right)$ for the values $\theta = 0$ and $\theta = \pi/2$. It is a fact that $\lambda_j^{\downarrow}(C_\theta)$ is a continuous functions of θ; see [**5**, p. 154]).

Condition (1) tells us that the eigenvalues of C_θ must lie on the line $\gamma_1 + \gamma_2 = 6$. So, by the intermediate value theorem each point of the line segment between $(7, -1)$ and $(4, 2)$ must be the pair of eigenvalues of C_θ for some $0 \le \theta \le \pi/2$.

Figure 2 shows a plot of the two eigenvalues $\lambda_1^{\downarrow}(C_\theta)$ and $\lambda_2^{\downarrow}(C_\theta)$, $0 \le \theta \le \pi/2$. The two curves are symmetric about the line $y = 3$ because of the trace condition (1).

Some comments are in order here. We chose numerical values for α, β for concrete illustrations. The same argument would work for any pairs. The matrices A and B we got are not just Hermitian; they are real symmetric. The condition (1) brought us down from the plane onto a line, the condition $\gamma_1 \ge \gamma_2$ to a part of this line, and the inequalities (14) to a closed interval on it. We have proved the following theorem.

Theorem 1. *Let A, B be two real symmetric 2×2 matrices with eigenvalues $\alpha_1 \ge \alpha_2$ and $\beta_1 \ge \beta_2$, respectively. Then the set of (decreasingly ordered) eigenvalues of the family $A + UBU^*$, where U varies over rotation matrices, is a convex set (actually a line segment). This convex set is described by Weyl's inequalities (14).*

This is also a good opportunity to comment on two features of Figure 2. Neither the

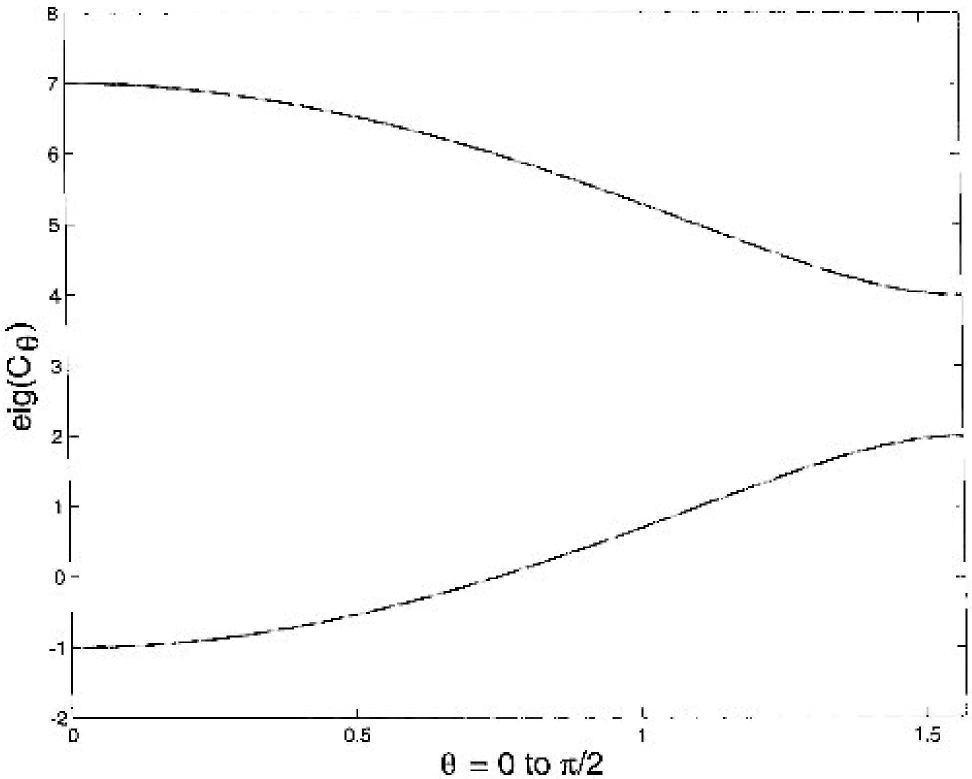

Figure 2. The two eigenvalues of the family C_θ

smoothness of the two curves, nor their avoidance of crossing each other is fortuitous. See the book **[27**, p. 113] (and the picture on its cover) for a discussion and explanation of these phenomena.

4. MAJORISATION Before proceeding further, it would be helpful to introduce the concept of majorisation of vectors. The theorems of Ky Fan, Lidskii–Wielandt, and Schur are best understood in the language of majorisation.

Let $x = (x_1, x_2, \ldots, x_n)$ be an element of $\mathbb{R}^n$. We write $x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \ldots, x_n^\downarrow)$ for the vector whose coordinates are obtained by rearranging the x_j in decreasing order $x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow$.

Let x, y be two elements of $\mathbb{R}^n$. If

$$\sum_{j=1}^{k} x_j^\downarrow \leq \sum_{j=1}^{k} y_j^\downarrow \quad \text{for } 1 \leq k \leq n, \tag{15}$$

then we say x is *weakly majorised* by y, and write $x \prec_w y$. If, in addition to the inequalities (15), we have

$$\sum_{j=1}^{n} x_j^\downarrow = \sum_{j=1}^{n} y_j^\downarrow, \tag{16}$$

then we say x is *majorised* by y and write $x \prec y$.

As an example, let $p = (p_1, \ldots, p_n)$ be any probability vector; i.e., $p_j \geq 0$ and $\sum p_j = 1$. Then

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (p_1, \ldots, p_n) \prec (1, 0, \ldots, 0).$$

The notion of majorisation is important. A good part of the classic [18] and all of the more recent book [29] are concerned with majorisation. See also [5].

Among the several characterisations of majorisation the following two are especially useful and interesting; see [5, p. 33].

1. Let σ be a permutation on n symbols. Given $y \in \mathbb{R}^n$, let $y_\sigma = \left(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right)$. Then $x \prec y$ if and only if x is in the convex hull of the $n!$ points y_σ.
2. $x \prec y$ if and only if $x = Sy$ for a doubly stochastic matrix S.

Recall that a matrix $S = [s_{ij}]$ is *doubly stochastic* if $s_{ij} \geq 0$, $\sum_j s_{ij} = 1$ for all i, and $\sum_i s_{ij} = 1$ for all j.

Let us write $x^\uparrow = (x_1^\uparrow, \ldots, x_n^\uparrow)$ for the vector whose coordinates are obtained by rearranging x_j in increasing order: $x_1^\uparrow \leq \cdots \leq x_n^\uparrow$. Note that $x_j^\uparrow = x_{n-j+1}^\downarrow$. Then x is majorised by y if and only if

$$\sum_{j=1}^{k} x_j^\uparrow \geq \sum_{j=1}^{k} y_j^\uparrow \qquad 1 \leq k \leq n \tag{17}$$

and the equality (16) holds.

One of the basic theorems about majorisation says that for any x, y in $\mathbb{R}^n$

$$x^\downarrow + y^\uparrow \prec x + y \prec x^\downarrow + y^\downarrow; \tag{18}$$

see [5, p. 49]. This relation describes the effect of rearrangement on addition of vectors. Some of the inequalities in the following sections have this form; the vectors involved are n-tuples of eigenvalues of Hermitian matrices.

5. THE THEOREMS OF SCHUR AND FAN Return now to the Hermitian matrix A with eigenvalues α. Let $d = (a_{11}, \ldots, a_{nn})$ be the vector whose coordinates are the diagonal entries of A. Since $a_{jj} = \langle e_j, Ae_j \rangle$, the inequality

$$d_1^\downarrow \leq \alpha_1 \tag{19}$$

follows from (2). A famous theorem of Schur (1923), closely related to our main story, extends this inequality. This theorem says that we have the majorisation

$$d \prec \alpha. \tag{20}$$

Here is an easy proof. By the spectral theorem, there exists a unitary matrix U such that $A = UDU^*$, where $D = \text{diag}(\alpha_1, \ldots, \alpha_n)$. From this one sees that

$$a_{ii} = \sum_{j=1}^{n} |u_{ij}|^2 \alpha_j \qquad 1 \leq i \leq n.$$

This can be rewritten as $d = S\alpha$, where S is the matrix with entries $s_{ij} = |u_{ij}|^2$. This matrix is doubly stochastic since U is unitary. Hence, by one of the characterisations in Section 4, we have the majorisation (20).

The eigenvalues of A do not change under a change of orthonormal basis. So, from the relation (20) we get the following extremal representation called *Fan's Maximum Principle*

$$\sum_{j=1}^{k} \alpha_j = \max_{\text{orthonormal } \{x_j\}} \sum_{j=1}^{k} \langle x_j, Ax_j \rangle, \quad 1 \leq k \leq n. \tag{21}$$

Here the maximum is taken over all orthonormal k-tuples $x_1, \ldots, x_k$. The summands on the right hand side of (21) are diagonal entries of a matrix representation of A. So their sum is always less than or equal to $\sum_{j=1}^{k} \alpha_j$ by (20). For the special choice when x_j are eigenvectors of A with $Ax_j = \alpha_j x_j$, we have equality here.

When $k = 1$, (21) reduces to (2), and when $k = n$ both sides are equal to $\operatorname{tr} A$. This expression gives a quasilinear representation of the sum $\sum \alpha_j$. Among other things it tells us that for each k between 1 and n, $\sum_{j=1}^{k} \lambda_j^{\downarrow}(A)$ is a convex function of A. Thus each $\lambda_j^{\downarrow}(A)$ is a difference of two convex functions. Generalising (4) we now have inequalities

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \alpha_j + \sum_{j=1}^{k} \beta_j, \quad 1 \leq k \leq n, \tag{22}$$

proved by Ky Fan in 1949. Again, note that when $k = 1$, the inequality (22) reduces to (4) and when $k = n$, this is just the equality (1). In terms of majorisation we can express the family of inequalities (22) as

$$\lambda(A + B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B). \tag{23}$$

This is a matrix analogue of the right hand side of (18).

6. INEQUALITIES OF LIDSKII AND WIELANDT The next event in our story is quite dramatic. In 1950, V. B. Lidskii announced the following result: the vector γ lies in the convex hull of the $n!$ points $\alpha + \beta_\sigma$, where σ runs over all permutations σ of n indices. Lidskii, it would seem, was providing an elementary proof of this theorem that F. A. Berezin and I. M. Gel'fand had discovered in connection with their work on Lie groups. The paper of Berezin and Gel'fand appeared in 1956 and alludes to this. Lidskii's elementary proof may have been clear to the members of Gel'fand's famous Moscow seminar. However, the published version did not give all the details and it could not be understood by many others. H. Wielandt saw the connection between Lidskii's theorem and Fan's inequalities (22) and provided another proof, very different in method from the one sketched by Lidskii.

Let $1 \leq k \leq n$ and let $1 \leq i_1 < \cdots < i_k \leq n$. Then the assertion of Lidskii's theorem is equivalent to saying that for all such choices

$$\sum_{j=1}^{k} \gamma_{i_j} \leq \sum_{j=1}^{k} \alpha_{i_j} + \sum_{j=1}^{k} \beta_j. \tag{24}$$

The equivalence is readily seen using the characterisations of majorisation given in Section 4.

Note that Fan's inequalities (22) are included in (24). To derive these inequalities Wielandt proved a minmax principle that is far more general than (21). We return to this later.

Now several proofs of Lidskii's theorem are known. Some of them are fairly easy and are given in [**5**]. The *easiest* proof, however, is the following one due to C.-K. Li and R. Mathias [**28**]:

Fix k and the indices $1 \le i_1 < \cdots < i_k \le n$. We want to prove that

$$\sum_{j=1}^{k} \left[\lambda_{i_j}^{\downarrow}(A+B) - \lambda_{i_j}^{\downarrow}(A)\right] \le \sum_{j=1}^{k} \lambda_j^{\downarrow}(B). \tag{25}$$

We can replace B by $B - \lambda_k^{\downarrow}(B)I$, and thus may assume that $\lambda_k^{\downarrow}(B) = 0$. Let $B = B_+ - B_-$ be the decomposition of B into its positive and negative parts (if B has the spectral resolution $\sum \beta_j u_j u_j^*$, then $B_+ = \sum \beta_j^+ u_j u_j^*$ where $\beta_j^+ = \max(\beta_j, 0)$). Since $B \le B_+$, by Weyl's monotonicity principle $\lambda_{i_j}^{\downarrow}(A+B) \le \lambda_{i_j}^{\downarrow}(A+B_+)$. So, the left hand side of (25) is not bigger than

$$\sum_{j=1}^{k} \left[\lambda_{i_j}^{\downarrow}(A+B_+) - \lambda_{i_j}^{\downarrow}(A)\right].$$

By the same principle, this is not bigger than

$$\sum_{j=1}^{n} \left[\lambda_j^{\downarrow}(A+B_+) - \lambda_j^{\downarrow}(A)\right].$$

(All of the summands are nonnegative.) This sum is $\operatorname{tr} B_+$, and since we assumed $\lambda_k^{\downarrow}(B) = 0$, it is equal to $\sum_{j=1}^{k} \lambda_j^{\downarrow}(B)$. This proves (25).

Using the observation (6), it is not difficult to obtain from the Lidskii-Wielandt inequalities (24) the relation

$$\lambda^{\downarrow}(A) + \lambda^{\uparrow}(B) \prec \lambda(A+B). \tag{26}$$

Together with (23), this gives a *noncommutative analogue* of (18): if A, B were commuting Hermitian matrices the relations (23) and (26) would reduce to (18).

7. THE CASE $n = 3$ Let us see what we have obtained so far when $n = 3$. We get six relations from Weyl's inequalities (8):

$$\begin{gathered}\gamma_1 \le \alpha_1 + \beta_1, \quad \gamma_2 \le \alpha_1 + \beta_2, \quad \gamma_2 \le \alpha_2 + \beta_1 \\ \gamma_3 \le \alpha_1 + \beta_3, \quad \gamma_3 \le \alpha_3 + \beta_1, \quad \gamma_3 \le \alpha_2 + \beta_2.\end{gathered} \tag{27}$$

One more follows from Fan's inequalities (22):

$$\gamma_1 + \gamma_2 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_2. \tag{28}$$

Four more relations can be read off from the Lidskii–Wielandt inequalities (24):

$$\begin{aligned}&\gamma_1 + \gamma_3 \le \alpha_1 + \alpha_3 + \beta_1 + \beta_2,\\ &\gamma_2 + \gamma_3 \le \alpha_2 + \alpha_3 + \beta_1 + \beta_2,\\ &\gamma_1 + \gamma_3 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_3, \text{ and}\\ &\gamma_2 + \gamma_3 \le \alpha_1 + \alpha_2 + \beta_2 + \beta_3.\end{aligned} \tag{29}$$

(Use the symmetry in A and B.)

It was shown by Horn [16] that one more inequality

$$\gamma_2 + \gamma_3 \le \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \tag{30}$$

is valid, and further, together with the trace equality (1), the twelve inequalities (27)–(30) are sufficient to characterise all triples α, β, γ that can be eigenvalues of A, B, and $A + B$. The proof of this assertion is not as simple as the one we gave for the case $n = 2$ in Section 3.

Where does the inequality (30) come from? Horn derived all inequalities that sums like $\gamma_i + \gamma_j$ satisfy for any dimension n; the inequality (30) is one of them. For the special case $n = 3$, one can derive this inequality from the majorisation (26), which is a consequence of the Lidskii–Wielandt theorem. For $n = 3$, this says

$$(\alpha_1 + \beta_3,\ \ \alpha_2 + \beta_2,\ \ \alpha_3 + \beta_1) \prec (\gamma_1,\ \gamma_2,\ \gamma_3).$$

Now using (17) one sees that the last three inequalities in (27) are hidden in this assertion. (Only the first five inequalities in (27) can be derived from the Lidskii–Wielandt inequalities in their raw form (24).) The inequality (30) too follows from this majorisation: if $\alpha_2 + \beta_2$ is larger than $\alpha_1 + \beta_3$ and $\alpha_3 + \beta_1$, this is clear from (17); if it is smaller than one of them, this follows from (29).

Let us consider a simple example. Let

$$\alpha = \{4, 3, -2\}, \quad \beta = \{2, -1, -6\}.$$

Then the condition (1) says

$$\gamma_1 + \gamma_2 + \gamma_3 = 0.$$

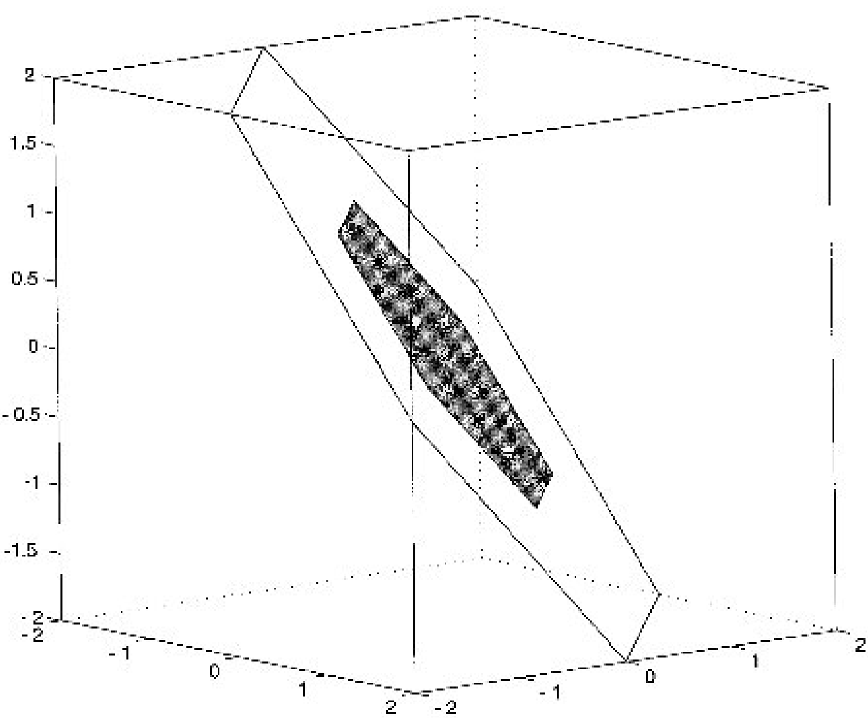

Figure 3. Part of the plane $\{\gamma_1 + \gamma_2 + \gamma_3 = 0\}$; small hexagon $= \{|\gamma_i| \le 1\}$

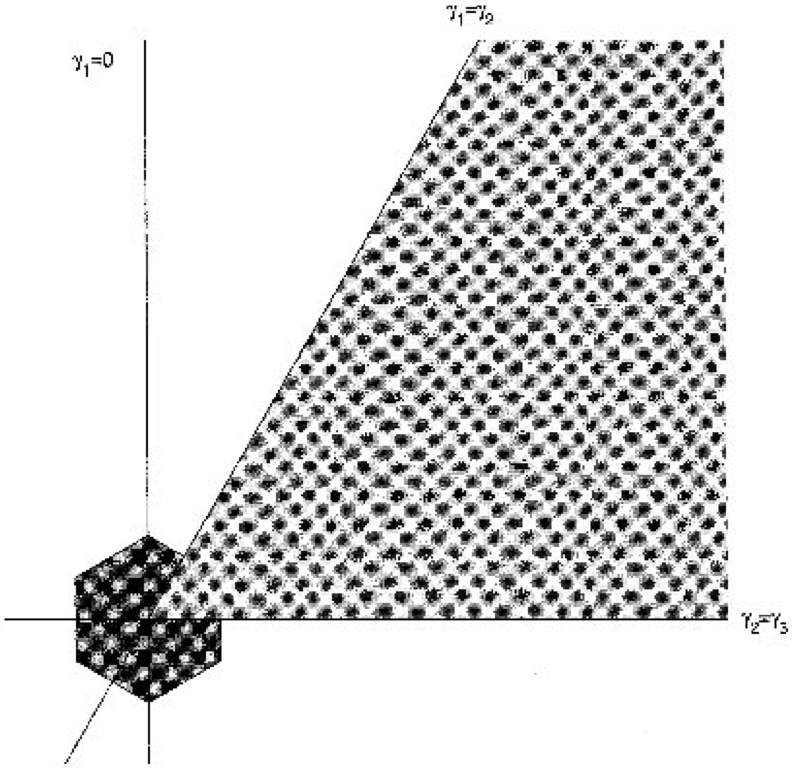

Figure 4. $\gamma_1 \geq \gamma_2 \geq \gamma_3$; the small hexagon $= \{|\gamma_2| \leq 1\}$

This is a plane in $\mathbb{R}^3$; see Figure 3. For convenience rotate it to the x-y plane. The condition $\gamma_1 \geq \gamma_2 \geq \gamma_3$ gives the part of the plane shown in Figure 4. The six inequalities of Weyl in (27) give three restrictions

$$\gamma_1 \leq 6, \quad \gamma_2 \leq 3, \quad \gamma_3 \leq -2.$$

This restricts γ further to the pentagon shown in Figure 5. A new restriction is imposed by Fan's inequality (28):

$$\gamma_1 + \gamma_2 \leq 8,$$

and this constrains γ to be in the hexagon in Figure 6. Of the four inequalities (29) of Lidskii–Wielandt, two are redundant. The remaining two are

$$\gamma_1 + \gamma_3 \leq 3, \quad \gamma_2 + \gamma_3 \leq 0.$$

However, they do not impose any new constraint; see Figure 7. We have a new inequality from Horn's condition (30). This says

$$\gamma_2 + \gamma_3 \leq -2,$$

and cuts down the set of permissible γ to the heptagon shown in Figure 8.

Horn's theorem says that each point γ in this set is the eigenvalue triple of a matrix $C = A + B$, where A, B are Hermitian matrices with eigenvalues α, β.

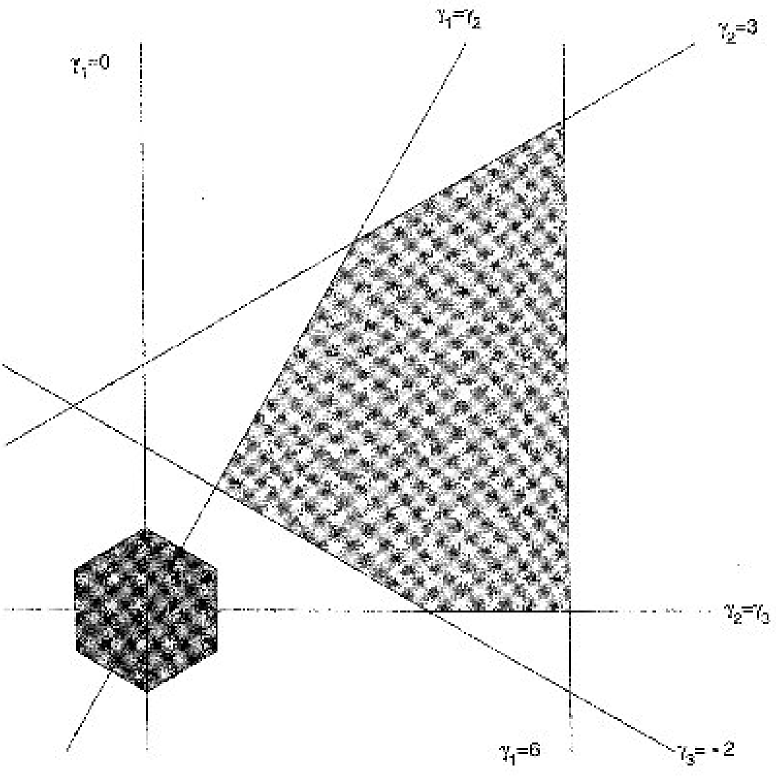

Figure 5. In the plane $\{\gamma_1+\gamma_2+\gamma_3=0\}$; the Weyl pentagon

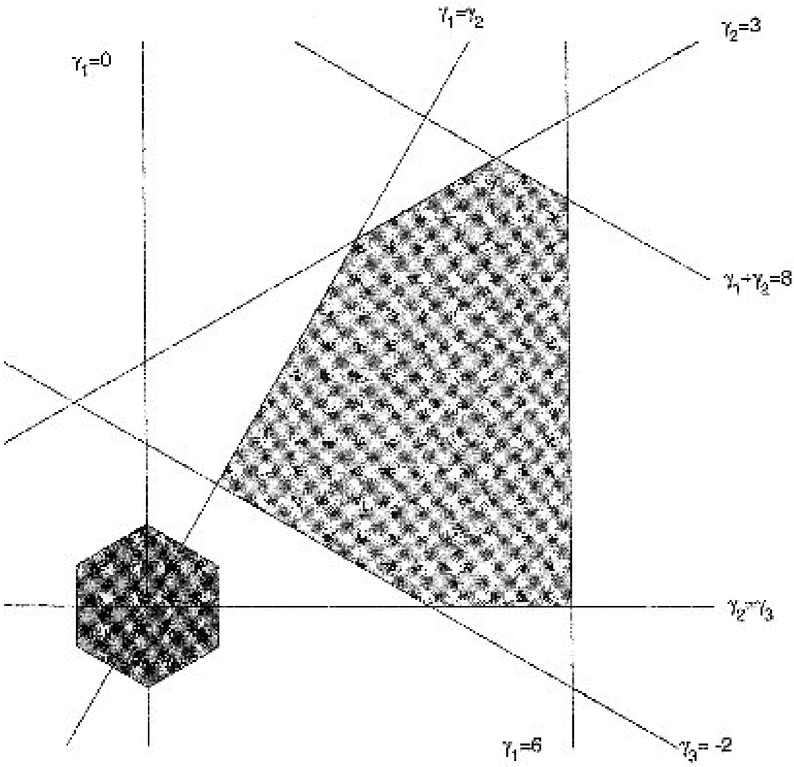

Figure 6. In the plane $\{\gamma_1+\gamma_2+\gamma_3=0\}$; the Ky Fan hexagon

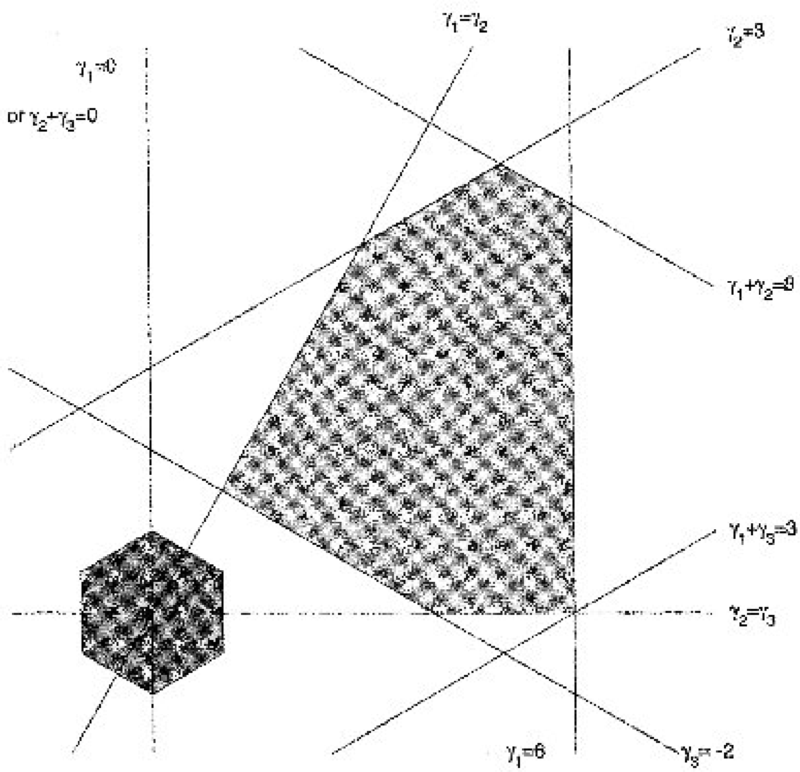

Figure 7. Lidskii–Wielandt inequalities have no effect in this example

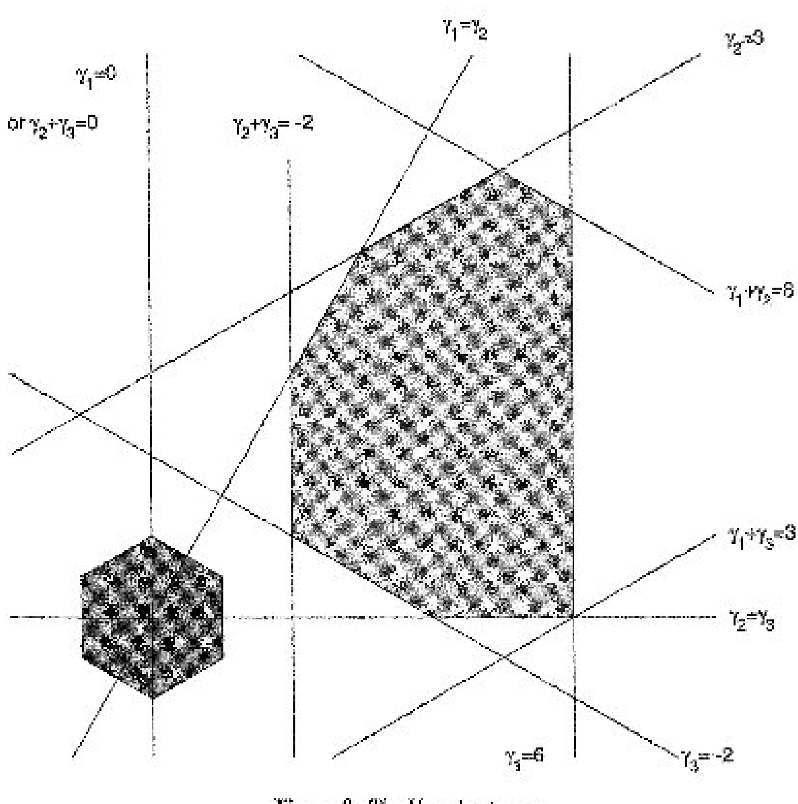

Figure 8. The Horn heptagon

The majorisations (23) and (26) give in this example

$$(2, 0, -2) \prec \gamma \prec (6, 2, -8).$$

In the plane $\gamma_1 + \gamma_2 + \gamma_3 = 0$, the set of γ satisfying $\gamma \prec (6, 2, -8)$ is shown in Figure 9; the set of γ satisfying $(2, 0, -2) \prec \gamma$ is shown in Figure 10. The intersection of these two sets is a hexagon. The Weyl inequality $\gamma_2 \leq 3$ imposes a constraint not included in these two majorisations. This additional constraint gives us the heptagon of Figure 8.

8. THE HORN CONJECTURE The Lidskii–Wielandt Theorem aroused a lot of interest, and more inequalities connecting α, β, γ were discovered. Some of these looked very complicated. A particularly attractive one proved by R. C. Thompson and L. Freede in 1971 says

$$\sum_{j=1}^{k} \gamma_{i_j+p_j-j} \leq \sum_{j=1}^{k} \alpha_{i_j} + \sum_{j=1}^{k} \beta_{p_j} \tag{31}$$

for any choice of indices $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq p_1 < \cdots < p_k \leq n$ satisfying $i_k + p_k - k \leq n$. This includes the Lidskii–Wielandt inequalities (24) (choose $p_j = j$) and treats α, β more symmetrically.

But where does the story end? Can one go on finding more and more inequalities like this? This question was considered, and an answer to it suggested, by A. Horn in a remarkable paper in 1962 [16]. This paper followed the ideas of Lidskii's original approach to the problem.

The inequalities (8), (22), (24), (31) all have a special form:

$$\sum_{k\in K} \gamma_k \leq \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j, \tag{32}$$

where I, J, K are certain subsets of $\{1, 2, \ldots, n\}$ having the same cardinality. One may raise here two questions:

- (i) What are all triples (I, J, K) of subsets of $\{1, 2, \ldots, n\}$ for which inequalities (32) are true? Let us call such triples *admissible*.
- (ii) Are these inequalities, together with (1), sufficient to characterise the α, β, γ that can be eigenvalues of Hermitian matrices A, B, and $A + B$?

Horn conjectured that the answer to the second question is in the affirmative and that the set T_r^n of admissible triples (I, J, K) of cardinality r can be described by induction on r as follows.

Let us write $I = \{i_1 < i_2 < \cdots < i_r\}$ and likewise for J and K. Then for $r = 1$, (I, J, K) is in T_1^n if $k_1 = i_1 + j_1 - 1$. For $r > 1$, (I, J, K) is in T_r^n if

$$\sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \binom{r+1}{2}, \tag{33}$$

and, for all $1 \leq p \leq r - 1$ and all $(U, V, W) \in T_p^r$,

$$\sum_{u\in U} i_u + \sum_{v\in V} j_v \leq \sum_{w\in W} k_w + \binom{p+1}{2}. \tag{34}$$

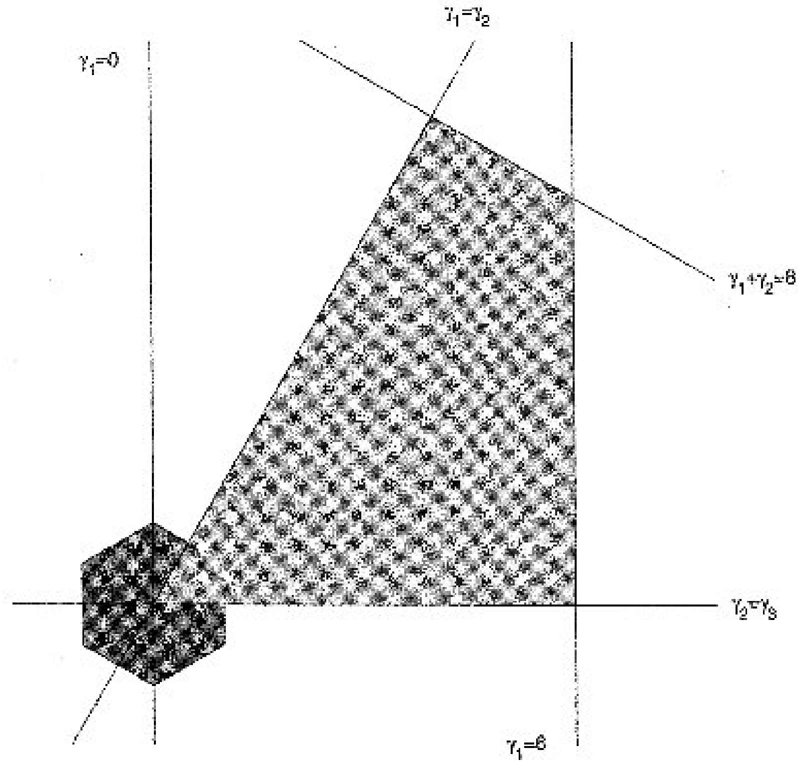

Figure 9. The quadrilateral containing all γ majorised by $(6, 2, -8)$

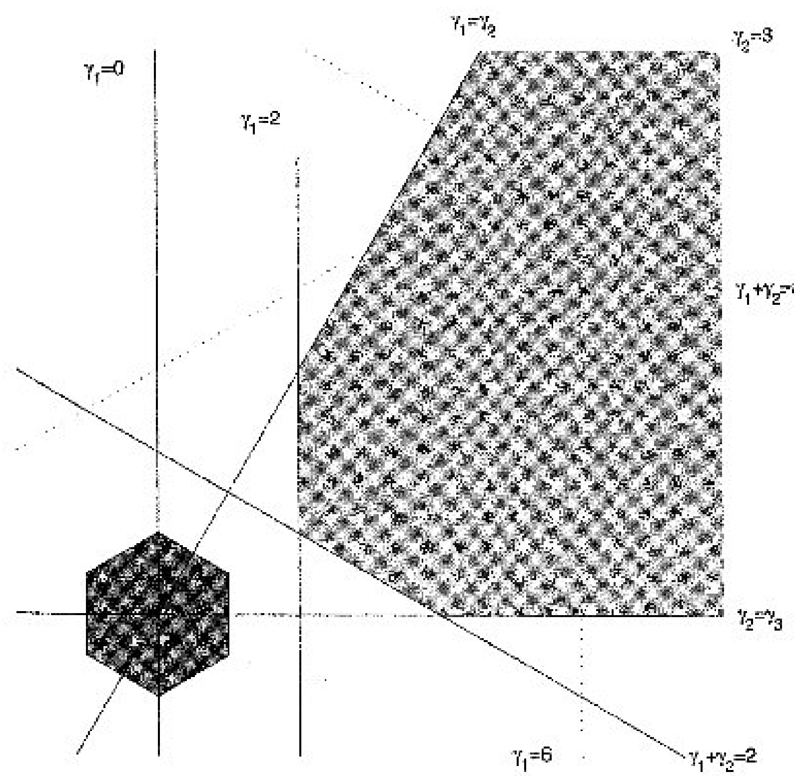

Figure 10. A part of the region containing γ that majorise $(2, 0, -2)$

Horn proved his conjecture for $n = 3$ and 4. When $n = 2$, these conditions just reduce to the three Weyl inequalities (14). When $n = 3$, they reduce to the twelve inequalities (27)–(30). When $n = 7$, there are 2062 inequalities given by these conditions, not all of which may be independent.

There is not much to explain about the conditions (33) and (34) themselves. The striking features of the conjecture—now a theorem—are the following. It says three things:

(i) Fix α, β and choose two Hermitian matrices A, B with eigenvalues α, β. Then the set of γ that are eigenvalues of $A + UBU^*$, as U varies over unitary matrices is a convex polyhedron in $\mathbb{R}^n$.

(ii) This convex polyhedron is described by Horn's inequalities.

(iii) These inequalities can be obtained by an inductive procedure.

We should emphasize that none of these is a statement of an obvious fact, and while each of them has now been proved the deeper reasons for their being true are still to be understood.

9. THE SCHUR-HORN THEOREM AND CONVEXITY A simple theorem like (20) is often an impetus for the development of several subjects. The theory of majorisation, a good part of matrix theory, and some important work in Lie groups and geometry, were inspired by this simple inequality.

In 1954 A. Horn [15] proved a converse to this theorem of Schur. Namely, if x and y are two real n-vectors such that $x \prec y$, then there exists a Hermitian matrix A such that the entries of x are the diagonal of A and the entries of y are the eigenvalues of A.

Using the properties of majorisation given in Section 4, we can state the theorem of Schur and its converse due to Horn as follows.

Theorem 2. *Let α be an n-tuple of real numbers and let $\mathcal{O}_\alpha$ be the set of Hermitian matrices with eigenvalues α. Let $\Phi : \mathcal{O}_\alpha \to \mathbb{R}^n$ be the map that takes a matrix to its diagonal. Then the image of Φ is a convex polyhedron, whose vertices are the $n!$ permutations of α.*

Now, the set of skew-Hermitian matrices $\mathcal{U}(n)$ is the Lie algebra associated with the compact Lie group $U(n)$ consisting of $n \times n$ unitary matrices. The set of Hermitian matrices is $i\mathcal{U}(n)$. The set $\mathcal{O}_\alpha$ is the *orbit* of the diagonal matrix with diagonal α under the action of $U(n)$: it consists of all matrices $U \operatorname{diag}(\alpha)U^*$ as U varies over $U(n)$. This led B. Kostant in 1970 to interpret Theorem 2 as a special case of a general theorem for compact Lie groups. (The role of diagonal matrices is now played by a maximal compact abelian subgroup, that of the permutation group by the Weyl group.) This in turn led to a much wider generalisation in 1982 by M. Atiyah, and independently by V. Guillemin and S. Sternberg. An explanation of these ideas is beyond our scope. However, let us state the theorem of Atiyah et al. to give a flavour of the subject.

Theorem 3. *Let M be a compact connected symplectic manifold, with an action of a torus T. Let $\Phi : M \to \mathfrak{t}^*$ be a moment map for this action. Then the image of Φ is a convex polytope, whose vertices are the images of the T-fixed points on M.*

The curious reader should see the article [22] by A. Knutson (from where we have borrowed this formulation) for an explanation of the terms and the ideas. Another informative article is one by Atiyah [2].

For the present, we emphasize that the *moment map* and its convexity properties are now a major theme in geometry. Especially interesting for our story is the fact that the first part of Horn's conjecture stated at the end of Section 8 was proved in 1993 by A. H. Dooley, J. Repka, and N. J. Wildberger [9], using convexity properties of the moment map.

10. SCHUBERT CALCULUS AND THE HEART OF THE MATTER

R. C. Thompson seems to have been the first one to realise that there are deep connections between the spectral inequalities we have been talking about and a topic in algebraic geometry called Schubert Calculus. Let us indicate these ideas briefly.

Start with the minmax principle [7]. For convenience we rewrite it as

$$\alpha_j = \max_{\dim V = j} \min_{x \in V,\, \|x\|=1} \operatorname{tr} Axx^*. \tag{35}$$

Note that xx^*, the orthogonal projection operator onto the 1-dimensional space spanned by x, depends not on the *vector* x but on the *space* spanned by it.

The set of all 1-dimensional subspaces of $\mathbb{C}^{m+1}$ is known as the *complex projective space* $\mathbb{CP}_m$ *of dimension* m. These spaces are the basic objects studied by classical algebraic geometers and it is perhaps worth explaining briefly the geometers' notation of homogeneous coordinates in projective spaces. Any non-zero vector of $\mathbb{C}^{m+1}$ determines a point in $\mathbb{CP}_m$; two points $(z_0, \ldots, z_m)$, $(z'_0, \ldots, z'_m)$ determine the same 1-dimensional subspace (i.e., point of $\mathbb{CP}_m$) if and only if there is a non-zero $c \in \mathbb{C}$ such that $z'_i = cz_i$ for each $i = 0, \ldots, m$. (The practice of using $\{0, \ldots, m\}$ to index the coordinates of $\mathbb{C}^{m+1}$ ensures that in m-dimensional projective space the last coordinate has index m rather than $m + 1$.) In view of this the point ℓ of $\mathbb{CP}_m$ determined by $(z_0, \ldots, z_m)$ is denoted by $[z_0 : \cdots : z_m]$ and these are called the *homogeneous coordinates* of ℓ. Note that the homogenous coordinates of a point in $\mathbb{CP}_m$ are not uniquely determined; they are defined only up to multiplication by non-zero complex numbers.

Now, if f is a nonconstant homogenous polynomial in $z_0, \ldots, z_m$, then there is a well-defined zero locus of f:

$$Z_f = \{[z_0 : \cdots : z_m] \in \mathbb{CP}_m : f(z_0, \ldots, z_m) = 0\}$$

This is known as the *projective hypersurface* defined by f. If f is a linear polynomial, Z_f is called a *hyperplane*, if f is quadratic, Z_f is called a *quadric hypersurface* and so on. *Projective varieties* are intersections of a finite number of projective hypersurfaces.

These spaces enjoy interesting symmetry properties since it is easy to see that $\mathbb{CP}_m$ is homeomorphic to $U(m+1)/(U(1) \times U(m))$, where $U(1) \times U(m)$ is the subgroup of unitary matrices whose first row is $(1, 0, \ldots, 0)$.

A generalization of the notion of projective space is the *Grassmannian*, $G_k(\mathbb{C}^n)$, the set of k-dimensional subspaces of $\mathbb{C}^n$. From our perspective of matrices it is easy to get a model of these spaces. Associate with any k-dimensional subspace V of $\mathbb{C}^n$ the unitary operator $P_V - P_{V^\perp}$, where P_W is the orthogonal projection onto the subspace W. This sets up a bijective correspondence between $G_k(\mathbb{C}^n)$ and the set of $n \times n$ unitary matrices having trace equal to $2k - n$.

These Grassmannians can be embedded in projective spaces as subvarieties in the following way. Given a subspace $V \subset \mathbb{C}^n$ of dimension k, choose a basis

$$u_1 = \begin{bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_n^1 \end{bmatrix}, \ldots, u_k = \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_n^k \end{bmatrix}$$

for V. Then the *Plücker coordinates* of V are the $\binom{n}{k}$ numbers

$$p_{i_1 \ldots i_k}(V) = \det \begin{vmatrix} u_{i_1}^1 & u_{i_2}^1 & \cdots & u_{i_k}^1 \\ \vdots & \vdots & \ldots & \vdots \\ u_{i_1}^k & u_{i_2}^k & \cdots & u_{i_k}^k \end{vmatrix} \quad \text{for } 1 \le i_1 < i_2 < \cdots < i_k \le n.$$

If we choose a different basis $u_1', \ldots, u_k'$ for V, then the Plücker coordinates are all multiplied by the same non-zero scalar factor (the determinant of the unitary transformation that takes each u_i to u_i' and is the identity map when restricted to $V^{\perp}$). So once an ordering has been chosen for the k-tuples $1 \le i_1 < \cdots < i_k \le n$,

$$V \mapsto [\cdots : p_{i_1 \ldots i_k}(V) : \cdots]$$

yields an embedding of $G_k(\mathbb{C}^n)$ in $\mathbb{CP}_{\binom{n}{k}-1}$. In fact, the image of this embedding is a projective variety and its defining equations are well known.

Let us now return to matrix inequalities.

Given any Hermitian operator A on $\mathbb{C}^n$, and a subspace L of $\mathbb{C}^n$ (which we think of as a point in $G_k(\mathbb{C}^n)$), let $A_L = P_L A P_L$. Note that $\operatorname{tr} A_L = \operatorname{tr} P_L A P_L = \operatorname{tr} A P_L$.

To prove the inequality (24), Wielandt invented a most remarkable minmax principle. This says that whenever $1 \le i_1 < \cdots < i_k \le n$, then

$$\sum_{j=1}^{k} \alpha_{i_j} = \max_{\substack{V_1 \subset \cdots \subset V_k \\ \dim V_j = i_j}} \ \min_{\substack{L \in G_k(\mathbb{C}^n) \\ \dim(L \cap V_j) \ge j}} \operatorname{tr} A_L. \tag{36}$$

When $k = 1$, this reduces to (12).

Another such principle was found by Hersch and Zwahlen. Let A have the spectral resolution $A = \sum \alpha_j v_j v_j^*$. For $1 \le m \le n$ let V_m be the linear span of $v_1, \ldots, v_m$. Then

$$\sum_{j=1}^{k} \alpha_{i_j} = \min_{L \in G_k(\mathbb{C}^n)} \left\{ \operatorname{tr} A_L : \dim \left(L \bigcap V_{i_j} \right) \ge j, \quad j = 1, \ldots, k \right\}. \tag{37}$$

This can be proved using ideas familiar to us from Section 2. Let L be any k-dimensional subspace of $\mathbb{C}^n$ such that $\dim \left(L \bigcap V_{i_j} \right) \ge j$. Since $\dim \left(L \bigcap V_{i_1} \right) \ge 1$, we can find a unit vector x_1 in $L \bigcap V_{i_1}$. Since V_{i_1} is spanned by $\{v_1, \ldots, v_{i_1}\}$ we have the inequality $\alpha_{i_1} \le \langle x_1, A x_1 \rangle$. Since $\dim \left(L \bigcap V_{i_2} \right) \ge 2$, we can find a unit vector x_2 in $L \bigcap V_{i_2}$ that is orthogonal to x_1. Then $\alpha_{i_2} \le \langle x_2, A x_2 \rangle$. Continuing in this way, we obtain an orthonormal basis $x_1, \ldots, x_k$ for L such that $\alpha_{i_j} \le \langle x_j, A x_j \rangle$ for $1 \le j \le k$. Thus

$$\sum_{j=1}^{k} \alpha_{i_j} \le \sum_{j=1}^{k} \langle x_j, A x_j \rangle = \operatorname{tr} A_L.$$

For the special choice $L = \text{span}\{v_{i_1}, \dots, v_{i_k}\}$, we have equality here. This proves the Hersch–Zwahlen principle (37).

The minimum in (37) is taken over a special kind of subset of $G_k(\mathbb{C}^n)$ studied by geometers and topologists for many years.

A sequence of nested subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n,$$

where $\dim V_j = j$, is called a *complete flag*. Given such a flag $\mathcal{F}$, for each multiindex $I = \{i_1 < \cdots < i_k\}$ the subset

$$S(I; \mathcal{F}) = \{W \in G_k(\mathbb{C}^n) : \dim(W \bigcap V_{i_j}) \geq j, \quad 1 \leq j \leq n\}$$

of the Grassmannian is called a *Schubert variety*.

The Hersch–Zwahlen principle says that the sum $\sum_{i \in I} \alpha_i$ is the minimal value of $\text{tr}\, A_L$ evaluated on the Schubert variety $S(I; \mathcal{F})$ corresponding to the flag constructed from the eigenvectors of A.

Hersch and Zwahlen developed a technique for obtaining inequalities like (32) using the principle (37). The essence of this technique can be described as follows. Consider the spectral resolutions

$$A = \sum \alpha_j u_j u_j^*, \quad B = \sum \beta_j v_j v_j^*, \quad C = A + B = \sum \gamma_j w_j w_j^*.$$

We find it convenient to write

$$-A - B + C = 0. \tag{38}$$

Recall that $\lambda_j^\downarrow(-A) = -\lambda_{n-j+1}^\downarrow(A)$. Given an index set $I = \{1 \leq i_1 < \cdots < i_k \leq n\}$ let $I' = \{i : n - i + 1 \in I\}$ and arrange the elements of I' in increasing order. For $1 \leq j \leq n$ consider the three families of subspaces

$$\begin{aligned} U_j &= \text{span}\{u_n, \dots, u_{n-j+1}\}, \\ V_j &= \text{span}\{v_n, \dots, v_{n-j+1}\}, \\ W_j &= \text{span}\{w_1, \dots, w_j\}. \end{aligned}$$

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be the complete flags formed by these three families. Now suppose our index sets I, J, K (of the same cardinality) are such that the Schubert varieties $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$ have a nonempty intersection. Choose a point L in this intersection. Then using (38) and (37) gives the inequality

$$\begin{aligned} 0 = &\ \text{tr}\,(-A_L - B_L + C_L) \\ &\geq \sum_{i \in I'} \lambda_i^\downarrow(-A) + \sum_{j \in J'} \lambda_j^\downarrow(-B) + \sum_{k \in K} \lambda_k^\downarrow(C). \end{aligned}$$

In other words,

$$\sum_{k \in K} \lambda_k^\downarrow(C) \leq -\sum_{i \in I'} \lambda_i^\downarrow(-A) - \sum_{j \in J'} \lambda_j^\downarrow(-B) = \sum_{i \in I} \lambda_i^\downarrow(A) + \sum_{j \in J} \lambda_j^\downarrow(B).$$

This is the kind of equality (32) we are looking for, and we have now touched the heart of the matter. Whenever the Schubert varieties $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$

have a nontrivial intersection, the triple (I, J, K) is admissible. The simplest instance of this idea at work is the proof of Weyl's inequalities (8) that we gave in Section 2. The triple $I = \{i\}$, $J = \{j\}$, $K = \{k\}$ is admissible if $k = i + j - 1 \leq n$.

The full significance of these ideas was grasped by R. C. Thompson; see especially the Ph.D. thesis of his student S. Johnson [19], and his unpublished lecture notes [32]; see also [14]. Among other things, Thompson asked whether the admissibility of three triples (I, J, K) in Horn's inequalities was *equivalent* to the condition that Schubert varieties $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$ corresponding to *any* three complete flags $\mathcal{F}, \mathcal{G}, \mathcal{H}$ (not necessarily constructed from eigenvectors of A, B, and $A + B$) have a nontrivial intersection. This equivalence has now been proved by Klyachko [20].

Theorem 4. *The triple (I, J, K) is admissible if and only if for any three complete flags $\mathcal{F}, \mathcal{G}, \mathcal{H}$, the intersection of the Schubert varieties $S(I'; \mathcal{F})$, $S(J'; \mathcal{G})$, and $S(K; \mathcal{H})$ is nonempty.*

The study of intersection properties of Schubert varieties is the subject of *Schubert calculus*. It reduces geometric questions about intersection of Schubert varieties to algebraic questions about multiplication in a ring called the *integral cohomology ring* $H^*(G_k(\mathbb{C}^n))$ associated with the Grassmannian. *Schubert cycles* S_I are equivalence classes of Schubert varieties (the dependence on $\mathcal{F}$ is removed). They form a basis for the ring $H^*(G_k(\mathbb{C}^n))$. Given triples I, J, K, consider the product $S_I \cdot S_J$ in this ring and expand it as

$$S_I \cdot S_J = \sum c_{I,J}^L S_L, \tag{39}$$

where $c_{I,J}^L$ are nonnegative integers. It turns out that the triple (I, J, K) is admissible if and only if the coefficient $c_{I,J}^K$ in (39) is nonzero (i.e., S_K occurs in the expansion of the product $S_I \cdot S_J$.)

It can now be said that the proof of Weyl's inequalities given in Section 2, and some others such as Wielandt's proof of (24) and the Thompson–Freede proof of (31), really amount to showing *using ideas from linear algebra alone* that certain Schubert varieties always intersect.

We raised two questions in Section 8. Theorem 4 answers the first of these questions by reformulating the problem of admissible triples in terms of Schubert calculus. Other equivalent formulations have been found. For example, the problem is related also to some important questions in the representation theory of the group $GL(n)$. We explain this connection briefly in Section 13. The answer to the second question—and the full proof of Horn's Conjecture—came partly from the work of Klyachko [20] on this connection. The last crucial step was the solution by Knutson and Tao [23] of a related problem in representation theory called the Saturation Conjecture. An exposition of this may be found in [7].

The proofs need advanced facts from algebraic geometry and representation theory. However, to quote from [22], "In fact the details of the proofs are not actually very different from the hands-on techniques used e.g. by Horn himself."

Other parts of the picture have been filled in since the appearance of the papers by Klyachko [20] and Knutson and Tao [23]. Belkale [3] has shown that if $c_{I,J}^K > 1$, then the inequalities (32) that correspond to the triple (I, J, K) are redundant, that is, they can be derived from other inequalities in the list. On the other hand, Knutson, Tao, and Woodward [25] have shown that the inequalities in the list (32) that correspond to those (I, J, K) for which $c_{I,J}^K = 1$ are independent.

Together, these results give the smallest set of inequalities needed to completely characterise the convex polyhedron whose points are eigenvalues of $A + UBU^*$, where A, B are given Hermitian matrices and U varies over unitaries.

11. SINGULAR VALUES OF PRODUCTS OF MATRICES In this section A, B, etc. are arbitrary $n \times n$ matrices, not necessarily Hermitian any more.

The *singular values* of A are the nonnegative numbers $s_1(A) \geq \cdots \geq s_n(A)$ that are the square roots of the eigenvalues of A^*A. It is easy to see that $s_1(A) = \|A\|$, and that

$$s_1(AB) \leq s_1(A)s_1(B). \tag{40}$$

Compare this with (4) and a natural problem stares at us: are there counterparts of inequalities for eigenvalues of sums of Hermitian matrices that are valid for products of singular values of arbitrary matrices? This question too has been of great interest and importance in linear algebra.

The k-fold antisymmetric tensor product $\bigwedge^k A$ has singular values $s_{i_1}(A) \cdots s_{i_k}(A)$, where $1 \leq i_1 < \cdots < i_k \leq n$. Since $\bigwedge^k(AB) = \bigwedge^k(A) \bigwedge^k(B)$, we get from (8) the inequality

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A) \prod_{j=1}^{k} s_j(B). \tag{41}$$

This is the singular value analogue of (22). (Incidentally, there is a perfect analogy here. We have derived (41) by applying (40) to a tensor object. We can derive (22) from (4) by a quite similar argument [**4**, p. 23].) The analogue of (24) is the following inequality proved by Gel'fand and Naimark

$$\prod_{j=1}^{k} s_{i_j}(AB) \leq \prod_{j=1}^{k} s_{i_j}(A) \prod_{j=1}^{k} s_j(B). \tag{42}$$

Once again, the theorem was proved in connection with questions about Lie groups, a matrix-theoretic proof was given by V. B. Lidskii, the inequality was discussed and proved in [**5**], and the simplest proof was found by Li and Mathias [**28**] soon afterwards. More inequalities of this type had been discovered by others, notably by R. C. Thompson and his students. The conjecture parallel to that of Horn was discussed by Thompson. Now it has been proved:

Theorem 5. *Let $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $c_1 \geq \cdots \geq c_n$, be three triples of nonnegative real numbers. Then there exist matrices A, B with singular values $s_j(A) = a_j$, $s_j(B) = b_j$, $s_j(AB) = c_j$, if and only if*

$$\prod_{k \in K} c_k \leq \prod_{i \in I} a_i \prod_{j \in J} b_j$$

for all admissible triples (I, J, K).

This is stated as Theorem 16 in [**11**]. The reason why it is true and the connection with Horn's problem are provided by the following theorem [**21**].

Theorem 6. *Let a, b, c be three n-tuples of decreasingly ordered real numbers. Then the following statements are equivalent:*

(i) *There exist nonsingular matrices A, B with $s_j(A) = a_j$, $s_j(B) = b_j$, $s_j(AB) = c_j$.*

(ii) *There exist Hermitian matrices X, Y, with $\lambda_j^\downarrow(X) = \log a_j$, $\lambda_j^\downarrow(Y) = \log b_j$, $\lambda_j^\downarrow(X+Y) = \log c_j$.*

12. EIGENVALUES OF PRODUCTS OF UNITARY MATRICES Eigenvalues of two unitary matrices and their product are the next objects we consider. Here the formulation of the problem is much more delicate and needs more advanced machinery. We can indicate only somewhat vaguely what it involves.

To get rid of ambiguities arising from multiplication on the unit circle we restrict ourselves to the set $SU(n)$ of $n \times n$ unitary matrices with determinant one. For $A \in SU(n)$ let $\mathrm{Eig}^\downarrow(A)$ be the set of its eigenvalues $\exp(2\pi i\lambda_j)$, labelled so that $\lambda_1 \geq \cdots \geq \lambda_n$. Since $\det A = 1$, we must have $\lambda_1 + \cdots + \lambda_n \equiv 0 (\mathrm{mod}\, 1)$. Choose a normalisation that has $\lambda_1 + \cdots + \lambda_n = 0$ and $\lambda_1 - \lambda_n < 1$. With this normalisation, call the numbers λ_j occuring here $\lambda_j^\downarrow(A)$.

Our problem is to find relations between $\lambda_j^\downarrow(A)$, $\lambda_j^\downarrow(B)$, and $\lambda_j^\downarrow(AB)$ for two elements A, B of $SU(n)$.

The analogue of the Lidskii–Wielandt inequalities (24) in this context was discovered in 1958 by A. Nudel'man and P. Svarcman. This has exactly the form (24). However, the analogue of Horn's conjecture in this context involves some objects that arise in the study of vector bundles, and are related to *quantum Schubert calculus*, a subject of very recent origin.

In Section 10 we alluded to the cohomology ring $H^*(G_K(\mathbb{C}^n))$ and how multiplication in this ring gives us information about intersection of Schubert cycles. Quantum cohomology associates with the Grassmannian the object

$$qH^*(G_k(\mathbb{C}^n)) = H^*(G_k(\mathbb{C}^n)) \otimes \mathbb{C}[[q]],$$

where $\mathbb{C}[[q]]$ is the ring of formal power series. Multiplication $S_I * S_J$ in this ring is more complicated. Instead of (39) we have an expansion that looks like

$$S_I * S_J = \sum_L \sum_{d \geq 0} \left(c_{I,J}^L\right)_d q^d S_L. \tag{43}$$

The new result on eigenvalues of unitary matrices is the following:

Let (I, J, K) be triples such that the coefficient $\left(c_{I,J}^K\right)_d$ in the expansion (43) is nonzero. Then for all A, B in $SU(n)$

$$\sum_{i \in I} \lambda_i^\downarrow(A) + \sum_{j \in J} \lambda_j^\downarrow(B) \leq d + \sum_{k \in K} \lambda_k^\downarrow(AB). \tag{44}$$

Further, these inequalities give a *complete* set of restrictions (in the same sense as in Horn's problem).

This theorem has been proved by S. Agnihotri and C. Woodward [1] and by P. Belkale [3], with earlier contributions by I. Biswas [6]. A crucial component of the proof is a 1980 theorem of V. B. Mehta and C. S. Seshadri [30] on vector bundles on the projective space $\mathbb{CP}_1$. Let us explain, in bare outline, this theorem, and the fascinating connection it has with our problem.

For brevity let $\mathbb{P}_1$ denote the projective space $\mathbb{CP}_1$ introduced in Section 10. This space can be identified with the two-dimensional sphere S^2. This, in turn, can be thought of as the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane. The point ∞ is thought of as the north pole of the sphere and the point 0 as the south pole. To points in the open set $\mathbb{P}_1\setminus\{\infty\}$ we assign the usual complex coordinate z while on the open set $\mathbb{P}_1\setminus\{0\}$ we define the complex coordinate w by putting $w = 1/z$.

This space is simply connected: its fundamental group $\pi_1(\mathbb{P}_1)$ is trivial. $\mathbb{P}_1$ with one puncture (i.e., one of its points removed) can be identified with $\mathbb{C}$. This too is simply connected, and its fundamental group is trivial. $\mathbb{P}_1$ with two punctures is isomorphic to the punctured plane $\mathbb{C}\setminus\{0\}$. The fundamental group of this space is $\mathbb{Z}$, a group generated by one element. Carry out this construction further. Let $S = \{p_1, \dots, p_k\}$ be any finite subset of $\mathbb{P}_1$. Without loss of generality, think of p_k as the point at ∞. To identify the fundamental group of this space, choose a base point p in $\mathbb{P}_1\setminus S$. Loops, with fixed base point p, can be composed in the usual way. With this law of composition the product of the loops going counterclockwise around the points p_j, $1 \le j \le k-1$, is the loop going clockwise around $p_k = \infty$; see Figure 11.

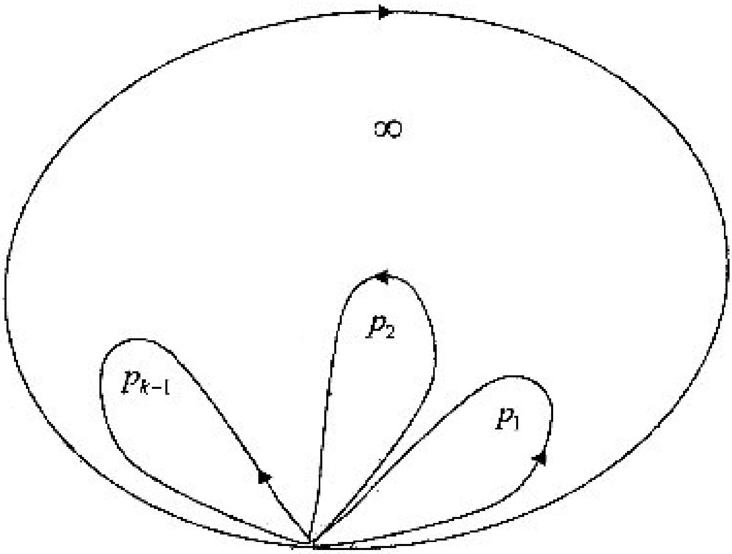

Figure 11. Computing the fundamental group of $\mathbb{P}_1\setminus S$

Thus the fundamental group $\pi_1(\mathbb{P}_1\setminus S)$ is the free group with generators $g_1, \dots, g_k$ with one relation $g_k = (g_1 \cdots g_{k-1})^{-1}$.

A homomorphism of a group G into another group H is called a *representation* of G in H.

Let ρ be a representation of the fundamental group $\pi_1(\mathbb{P}_1\setminus S)$ in the group $U(n)$ or $SU(n)$. If $A_j = \rho(g_j)$, this gives unitary matrices $A_1, \dots A_k$, with their product $A_1 A_2 \cdots A_k = I$.

In our original problem we are given three n-tuples of numbers and we want to know when they can be the eigenvalues of matrices A, B, and AB in $SU(n)$. Prescribing eigenvalues means fixing the conjugacy class of A under unitary conjugations $A \to UAU^*$. Thus our problem is to find conditions for the existence of three elements A, B, C of $SU(n)$ with prescribed conjugacy classes such that $ABC = I$. Instead of three matrices, we can equally well consider the same question for k matrices $A_1, \dots, A_k$. In the preceding paragraph we saw how this problem is connected with representations of the fundamental group of $\mathbb{P}_1$ with k punctures.

Next we recall the notion of a vector bundle. For simplicity, we make some restrictions in our definitions; see [31] for a splendid introduction. Let B be a compact connected Hausdorff topological space. A *vector bundle* over (the *base space*) B consists of the following:

(i) a topological space $\mathcal{E}$ called the *total space*,

(ii) a continuous map $\Pi : \mathcal{E} \to B$ called the *projection map*,

(iii) on each set $E_b = \Pi^{-1}(b)$, $b \in B$, the structure of an n-dimensional real or complex vector space. (The bundle is accordingly called real or complex.) The vector space E_b is called the *fibre* over b.

These objects are required to satisfy a restriction called *local triviality*: for each $b \in B$ there exists a neighbourhood U, and a homeomorphism $h : U \times \mathbb{K}^n \to \Pi^{-1}(U)$, such that for each $a \in U$ the map $x \mapsto h(a, x)$ from $\mathbb{K}^n$ to E_a is an isomorphism of vector spaces. Here, $\mathbb{K}^n$ is the space $\mathbb{R}^n$ or $\mathbb{C}^n$ depending on whether the bundle is real or complex. The pair (U, h) is called a *local trivialisation* about b. If it is possible to choose U equal to the entire base space B, then the bundle $\mathcal{E}$ is called a *trivial bundle*. In this case $\mathcal{E} = B \times \mathbb{K}^n$.

The number n is called the *rank* of the bundle $\mathcal{E}$. If $n = 1$, the bundle is called a *line bundle*.

If B is a contractible space, every vector bundle on it is trivial. On the base space S^1 (the unit circle) the cylinder is a trivial line bundle while the Moebius strip is a nontrivial line bundle.

Let U be any open set in B. A *section* over U is a continuous map $s : U \to \mathcal{E}$ such that $s(b) \in E_b$ for all $b \in U$.

Let (U, h) be a local trivialisation. Let $\{x_j\}$ be the standard basis for $\mathbb{K}^n$ and let $e_j^U(a) = h(a, x_j)$, $a \in U$. Then $\{e_j^U(a)\}$ is a basis for the vector space E_a. The maps e_j^U are sections over U. The family $\{e_j^U\}$ is called a *local basis* for $\mathcal{E}$ over U. Let $\{e_j^U\}$ and $\{e_j^V\}$ be two local bases for $\mathcal{E}$ over open sets U and V. Then for each point $a \in U \cap V$, we can find an invertible matrix $g_{V,U}(a)$ that carries the basis $\{e_j^U(a)\}$ onto the basis $\{e_j^V(a)\}$ of E_a. This is called a *transition function*. Note that $a \to g_{V,U}(a)$ is a continuous map from $U \cap V$ into $GL(n)$.

If the spaces involved have more structure, we could define *smooth* bundles or *holomorphic* bundles by putting the appropriate conditions on the maps involved.

Let $\mathcal{E}$ and $\mathcal{F}$ be two bundles over the same base space B such that (the total space) $\mathcal{F}$ is contained in (the total space) $\mathcal{E}$ and each fibre F_b in the bundle $\mathcal{F}$ is a vector subspace of the corresponding fibre E_b. Then we say that $\mathcal{F}$ is a *subbundle* of $\mathcal{E}$. A trivial bundle may have nontrivial subbundles.

We are interested in complex vector bundles on the base space $\mathbb{P}_1$. One more notion that we need is the *degree* of a vector bundle on $\mathbb{P}_1$. We have identified $\mathbb{P}_1$ with the sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The complement of the north pole ∞ is an open set U that can be identified with the complex plane $\mathbb{C}$ with its coordinate z. This is a contractible space; so any vector bundle $\mathcal{E}$ on $\mathbb{P}_1$ admits a local trivialisation on U. Let $\{e_j^U\}$ be the corresponding local basis over U. Similarly, the set $V = \mathbb{P}_1 \backslash \{0\}$—the complement of the south pole—is identified with the complex plane with coordinate $w = 1/z$. So $\mathcal{E}$ admits a local trivialisation over V and a local basis $\{e_j^V\}$. The equator $|z| = 1$ lies in $U \cap V$. Let $g_{V,U}(z)$ be the transition function between the two bases. Identifying the equator with S^1 with coordinate z, we get a map $g_{V,U}(z)$ from S^1 into $GL(n)$. Then $\psi(z) = \det g_{V,U}(z)$ is a map from S^1 into nonzero complex numbers. The winding number of this map around 0 is called the *degree* of the vector bundle.

For example, consider the *tautological line bundle* on $\mathbb{P}_1$. This associates with each point of $\mathbb{P}_1$ the complex line through that point. In the open set $U = \mathbb{P}_1 \backslash \{\infty\}$ we associate with the point $[z : 1]$ the line $\mathbb{C}(z, 1)$ in $\mathbb{C}^2$. In the open set $V = \mathbb{P}_1 \backslash \{0\}$, we associate with the point $[1 : 1/z]$ the line $\mathbb{C}(1, 1/z)$ in $\mathbb{C}^2$. The total space for this bundle is a subset of $\mathbb{P}_1 \times \mathbb{C}^2$. In the intersection $U \cap V$ the transition from the basis $(z, 1)$ to $(1, 1/z)$ is given by multiplication by $g(z) = 1/z$. This function on S^1 has winding number -1 around the origin. So this bundle has degree -1.

The *slope* of a vector bundle $\mathcal{E}$ is defined as

$$\text{slope}(\mathcal{E}) = \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})}. \tag{45}$$

The bundle $\mathcal{E}$ is said to be *stable* if

$$\text{slope}(\mathcal{F}) < \text{slope}(\mathcal{E}) \tag{46}$$

for every subbundle $\mathcal{F}$ of $\mathcal{E}$, and *semistable* if

$$\text{slope}(\mathcal{F}) \leq \text{slope}(\mathcal{E}). \tag{47}$$

The bundle $\mathcal{E}$ is said to be *polystable* if it is isomorphic to a direct sum of stable bundles of the same slope. Polystable bundles are semistable. Each semistable bundle is equivalent to a canonical polystable bundle (under an equivalence relation that we do not define here).

Let $\mathcal{E}$ be a vector bundle of rank n on the space $\mathbb{P}_1$. Let $S = \{p_1, \ldots, p_k\}$ be a given finite subset of $\mathbb{P}_1$. A *parabolic structure* on $\mathcal{E}$ consists of the following objects given at each point $p \in S$:

(i) in each fibre E_p, a complete flag

$$\{0\} = V_0^p \subset V_1^p \subset \cdots \subset V_n^p = \mathbb{C}^n, \tag{48}$$

(ii) an n-tuple of real numbers α_j^p, $1 \leq j \leq n$ satisfying

$$\alpha_1^p \geq \alpha_2^p \geq \cdots \geq \alpha_n^p > \alpha_1^p - 1. \tag{49}$$

The flag (48) is also called a *filtration*, and the sequence (49) is called a *weight sequence*. We should remark that in the original definition due to C. S. Seshadri, flags in (i) were not required to be complete, and the weights were restricted to be in the interval $[0, 1)$.

Let $\mathcal{F}$ be a subbundle of $\mathcal{E}$ with rank $(\mathcal{F}) = r$. At each point p, the fibre F_p is an r-dimensional subspace of the n-dimensional space E_p. For $p \in S$, consider the intersections $F_p \bigcap V_j^p$, $\quad 0 \leq j \leq n$, where the V_j form the flag (48). If $r < n$, some of these spaces coincide. Retain only the distinct members of this sequence and label them as W_i^p, $\quad 0 \leq i \leq r$. Assign to the space W_i^p the highest possible weight allowed by this intersection; i.e., the weight $\beta_i^p = \alpha_j^p$, where j is the smallest number satisfying $W_i^p = F_p \bigcap V_j^p$. Then the subbundle $\mathcal{F}$ with parabolic structure given by the filtration

$$\{0\} = W_0^p \subset W_1^p \subset \cdots \subset W_r^p = \mathbb{C}^r \tag{50}$$

and weights

$$\beta_1^p \geq \beta_2^p \geq \cdots \geq \beta_r^p \tag{51}$$

is called a *parabolic subbundle* of $\mathcal{E}$. For brevity $\mathcal{E}$ is called a *parabolic bundle*.

The *parabolic degree* of $\mathcal{E}$ is defined as

$$\text{par degree}(\mathcal{E}) = \text{degree}(\mathcal{E}) + \sum_{p\in S}\sum_{j=1}^{n} \alpha_j^p, \tag{52}$$

and its *parabolic slope* as

$$\text{par slope}(\mathcal{E}) = \frac{\text{par degree}(\mathcal{E})}{\text{rank}(\mathcal{E})}. \tag{53}$$

The notions of stability, semistability, and polystability of a parabolic bundle are defined by replacing the quantity "slope" in the inequalities (46) or (47) by "parabolic slope".

Now we have all the pieces needed to describe the theorem of Mehta and Seshadri (as modified by Belkale and others to suit our needs).

We began by looking at $SU(n)$ representations of the fundamental group $\pi_1(\mathbb{P}_1\backslash S)$, where $S = \{p_1, \ldots, p_k\}$. We saw that this amounts to finding matrices $A_1, \ldots, A_k$ in $SU(n)$ whose product is I. For $i = 1, 2, \ldots, k$, let $\alpha_j^i = \lambda_j^\downarrow(A_i)$, where $\lambda_j^\downarrow(A)$ is as defined at the beginning of this section. For each $i = 1, 2, \ldots, k$, let V_m^i be the m-dimensional space spanned by the eigenvectors $u_1^i, \ldots, u_m^i$ corresponding to the eigenvalues $\lambda_1^\downarrow(A_i), \ldots, \lambda_m^\downarrow(A_i)$. Use these data to give a parabolic structure to the trivial rank n bundle on $\mathbb{P}_1$ as follows:

(i) in each fibre E_{p_i} a filtration is given by $\{0\} = V_0^i \subset V_1^i \subset \cdots \subset V_n^i = \mathbb{C}^n$

(ii) the numbers $\alpha_1^i \geq \cdots \geq \alpha_n^i$ give a weight sequence.

The theorem of Mehta and Seshadri says that *the parabolic bundle obtained in this way is polystable*, and conversely *every polystable bundle arises in this way*.

Now to the denouement: families of inequalities such as (43) are used by Agnihotri–Woodward, Belkale, and Biswas to prove that certain vector bundles on $\mathbb{P}_1\backslash\{p_1, p_2, p_3\}$ are semistable. (Semistability is defined by a family of inequalities.) To each semistable parabolic bundle there corresponds a unique polystable parabolic bundle. The Mehta–Seshadri Theorem then leads to the existence of unitary matrices whose eigenvalues are the given n-tuples.

The proof of Klyachko for the original Horn problem uses ideas similar to these, but it involves bundles on $\mathbb{P}_2$ and a theorem of Donaldson.

13. REPRESENTATIONS OF $GL(n)$ We began this story with Weyl's inequalities. It is befitting to end it with another subject in which Weyl was a pioneer—the theory of representations of groups. A fascinating connection between the two subjects has been discovered in recent years.

Let GL_n be the group consisting of $n \times n$ complex invertible matrices. By the *standard representation* of GL_n we mean the homomorphism from GL_n into the space $GL(V)$ of all linear operators on the space $V = \mathbb{C}^n$. If W is any m-dimensional complex vector space, a homomorphism $\rho : GL_n \to GL(W)$ is called a *representation* of GL_n in W. Such a representation is called an m-dimensional representation. For example, the map det gives a 1-dimensional representation. For brevity we denote a representation in W by W.

For simplicity, let us consider only *polynomial representations*, ones in which the entries of $\rho(A)$ are polynomials in the entries of A. The determinant representation is an example of such a representation. Another example is the tensor product, in which $W = \otimes^k V = V \otimes \cdots \otimes V$ (k times), and $\rho(A) = \otimes^k A$.

The space $\otimes^k V$ has several subspaces that are invariant under all the operators $\otimes^k A$, $A \in GL(V)$. Two examples are the spaces $\bigwedge^k V$ and $\mathrm{Sym}^k V$ of antisymmetric and symmetric tensors, respectively. The restrictions of $\otimes^k A$ to these spaces are written as $\bigwedge^k A$ and $\mathrm{Sym}^k A$. The spaces $\bigwedge^k V$ and $\mathrm{Sym}^k V$ are examples of *irreducible* representations of GL_n; they have no proper subspace invariant under all operators $\bigwedge^k A$ or $\mathrm{Sym}^k A$. These are subrepresentations of $\otimes^k V$. All polynomial representations are subrepresentations of $\otimes^k V$ for some k.

Let $\mathbf{N}_+$ be the set of all upper triangular matrices with diagonal entries 1, $\mathbf{N}_-$ the set of all lower triangular matrices with diagonal entries 1, and $\mathbf{D}$ the set of all nonsingular diagonal matrices. Each of these sets is a subgroup of GL_n. A matrix A is called *strongly nonsingular* if all its leading principal minors are nonzero. (These are the minors of the top left $k \times k$ blocks of A, $1 \le k \le n$.) It is a basic fact that every such matrix can be factored as

$$A = LDR, \tag{54}$$

where L, D, and R belong to $\mathbf{N}_-$, $\mathbf{D}$, and $\mathbf{N}_+$, respectively [**17**, pp. 158–165]. This is used in the Gaussian elimination method in solving linear equations, and (54) is called the *Gauss decomposition* of A. For representation theory, its significance lies in the consequence that every irreducible representation of GL_n is induced by a one-dimensional unitary representation (character) of $\mathbf{D}$. The set $\mathbf{B}$ consisting of all nonsingular upper triangular matrices (or, equivalently, all products LR with $L \in \mathbf{D}$, $R \in \mathbf{N}_+$) is another subgroup of GL_n. This is a solvable group. It is known that every irreducible representation of such a group is 1-dimensional.

Let ρ be a representation of GL_n in W. A vector v in W is called a *weight vector* if it is a simultaneous eigenvector for $\rho(D)$ for all $D \in \mathbf{D}$. If v is such a vector let

$$\rho(D)v = \lambda(D)v, \quad D \in \mathbf{D}.$$

Then λ is a complex-valued function on $\mathbf{D}$ such that

$$\lambda(DD') = \lambda(D)\lambda(D').$$

So, if $D = \mathrm{diag}(d_1, \ldots, d_n)$, then

$$\lambda(D) = d_1^{m_1} \cdots d_n^{m_n}$$

for some nonnegative integers $m_1, \ldots, m_n$, called the *associated weights*. For example, if V is the standard representation, then the only weight vectors are the basis vectors e_i, and the associated weights are $(0, 0, \ldots, 1, 0, \ldots, 0)$, $\quad 1 \le i \le n$. If $W = \bigwedge^k (\mathbb{C}^n)$, then $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a weight vector with weight $(1, 1, \ldots, 1, 0, \ldots, 0)$ where 1 occurs k times. If $W = \mathrm{Sym}^k \mathbb{C}^n$, $e_1 \vee e_1 \vee \cdots \vee e_1$ is a weight vector with weight $(k, 0, \ldots, 0)$.

A weight vector is called a *maximal weight vector* if it is left fixed by all elements of $\rho(\mathbf{N}_-)$, or equivalently, if it is a simultaneous eigenvector for all elements of $\rho(\mathbf{B})$. Thus, for the standard representation the only such vector is e_1. The associated weights in this case are called *highest weights*.

A fundamental theorem of representation theory says that an irreducible representation ρ of GL_n is determined completely by a unique maximal weight vector and associated weights $m_1 \ge \cdots \ge m_n$.

This is a bare-bones summary of a vast area; see [**12**] or [**13**] for details.

Decomposing representations into their irreducible components is a central problem in the theory of representations. In particular, the tensor product of two irreducible representations is not always irreducible and one wants to find its irreducible components. This is an intricate business. One important outcome of the recent work of Klyachko, Knutson–Tao, and others is the following theorem:

Theorem 7. *Let $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$ be three n-tuples of nonnegative integers. Let V_α, V_β, V_γ be the irreducible representations of $GL(V)$ with highest weights α, β, γ. Then V_γ is a component of $V_\alpha \otimes V_\beta$ if and only if there exist Hermitian matrices A and B such that $\alpha = \lambda(A)$, $\beta = \lambda(B)$, $\gamma = \lambda(A+B)$.*

The motivation for Gel'fand and Berezin in their study that led to the Lidskii–Wielandt inequalities was to unravel properties of tensor products of representations. This, in turn, led to Horn's conjecture. So, the connection between these problems is not new.

Let us show Theorem 7 in action in a simple example.

Consider irreducible representations of GL_2 with highest weights $\alpha = (4, 2)$ and $\beta = (3, 1)$. By results in Section 3, the admissible γ (that can occur as eigenvalues of $C = A + B$, where A, B are 2×2 Hermitian matrices with eigenvalues α, β) are the ones that satisfy the condition

$$(5,5) \prec \gamma \prec (7,3).$$

If we restrict γ to have integral entries, there are three possibilities

$$\gamma = (5,5), \quad (6,4), \quad (7,3).$$

By the rules for calculations with highest weights, we write

$$\alpha = (4,2) = (2,2) + (2,0) = 2(1,1) + (2,0),$$
$$\beta = (3,1) = (1,1) + (2,0).$$

The weights $(1, 1)$ correspond to the representation $\bigwedge^2 V$; $2(1, 1)$ to two copies of this; $(2, 0)$ to $\mathrm{Sym}^2 V$. So,

$$V_\alpha = (\textstyle\bigwedge^2 V)^{\otimes 2} \otimes \mathrm{Sym}^2 V,$$
$$V_\beta = \textstyle\bigwedge^2 V \otimes \mathrm{Sym}^2 V,$$
$$V_\alpha \otimes V_\beta = (\textstyle\bigwedge^2 V)^{\otimes 3} \otimes \left(\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V\right).$$

The last factor can be decomposed by using the *Clebsch–Gordan formula* [**13**, p. 306], which gives in our particular situation

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V = \mathrm{Sym}^4 V \oplus \left[\textstyle\bigwedge^2 V \otimes \mathrm{Sym}^2 V\right] \oplus \left(\textstyle\bigwedge^2 V\right)^{\otimes 2}.$$

Thus, we have the direct sum decomposition

$$V_\alpha \otimes V_\beta = \left[\left(\textstyle\bigwedge^2 V\right)^{\otimes 3} \otimes \mathrm{Sym}^4 V\right] \oplus \left[\left(\textstyle\bigwedge^2 V\right)^{\otimes 4} \otimes \mathrm{Sym}^2 V\right] \oplus \left(\textstyle\bigwedge^2 V\right)^{\otimes 5}.$$

The three direct summands here are irreducible representations corresponding to highest weights

$$\begin{aligned} 3(1,1)+(4,0) &= (7,3) \\ 4(1,1)+(2,0) &= (6,4) \\ 5(1,1) &= (5,5) \end{aligned}$$

respectively. This is what Theorem 7 predicted.

It is not easy to write down irreducible components of representations; intricate calculations with Young tableaux enter the picture. Theorem 7 gives another way of making a list of such representations. Thus from results in Section 7 we know that representations with highest weights (3, 2, 2), (3, 3, 1), and (4, 2, 1) are the irreducible components of the two representations of GL_3 with weights (2, 1, 0) and (2, 1, 1). It is an interesting exercise to write this decomposition explicitly.

The general problem of finding irreducible components of tensor products of irreducible representations of Lie groups (including GL_n) has been studied under the name "PRV Conjecture" and solved [26]. Several proofs of this conjecture have been given, and one more has come out of the recent work on Horn's inequalities.

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