Injectivity Sets for Spherical Means on the Heisenberg Group

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In this paper we prove that cylinders of the form $\Gamma_R = S_R \times \mathbb{R}$, where S_R is the sphere $\{z \in \mathbb{C}^n : |z| = R\}$, are injectivity sets for the spherical mean value operator on the Heisenberg group H^n in L^p spaces. We prove this result as a consequence of a uniqueness theorem for the heat equation associated to the sub-Laplacian. A Hecke–Bochner type identity for the Weyl transform proved by D. Geller and spherical harmonic expansions are the main tools used.

Key Words: Fourier transform; Heisenberg group; heat equation; spherical means; Laguerre functions; unitary group; spherical harmonics; sub-Laplacian; unitary representations; Weyl transform.

1. INTRODUCTION AND THE MAIN RESULTS

On $\mathbb{R}^n$ consider the spherical means of a continuous function f defined by

$$f * \sigma_r(x) = \int_{|y|=r} f(x-y)\, d\sigma_r(y), \tag{1.1}$$

where σ_r is the normalized surface measure on the sphere $S_r = \{y \in \mathbb{R}^n : |y| = r\}$. Let V be a class of functions on $\mathbb{R}^n$. We say that a subset Γ of $\mathbb{R}^n$ is a set of injectivity for the spherical mean value operator in V if $f * \sigma_r(x) = 0$ for all $r > 0$ and $x \in \Gamma$ implies $f = 0$ for every $f \in V$. Determining sets of injectivity for the spherical means in various classes of functions is an interesting and important problem which has received considerable attention in recent times (see [2–4]).

For each $\lambda > 0$ consider the function φ_λ defined by

$$\varphi_\lambda(x) = 2^{n/2-1}\Gamma\left(\frac{n}{2}\right)(\lambda|x|)^{-n/2+1}J_{n/2-1}(\lambda|x|),$$

where $J_{n/2-1}$ is the Bessel function of order $(\frac{n}{2}-1)$. Then it is well known that

$$\varphi_\lambda * \sigma_r(x) = \varphi_\lambda(r)\varphi_\lambda(x).$$

Therefore, if λR is a zero of the Bessel function $J_{n/2-1}(t)$ then $\varphi_\lambda * \sigma_r$ vanishes on the sphere S_R for all $r > 0$. Since $\varphi_\lambda \in L^p(\mathbb{R}^n)$ for all $p > \frac{2n}{n-1}$ the above shows that spheres are not sets of injectivity for the spherical means in $L^p(\mathbb{R}^n)$ with $p > \frac{2n}{n-1}$. On the other hand, when $p \leq \frac{2n}{n-1}$, spheres and, more generally, boundaries of bounded regions in $\mathbb{R}^n$ are sets of injectivity for the spherical means in $L^p(\mathbb{R}^n)$. This interesting result has been proved recently by Agranovsky et al. [2].

It is natural to ask what happens if the ordinary spherical means are replaced by twisted spherical means on $\mathbb{C}^n$. Let μ_r be the normalized surface measure on the sphere S_r in $\mathbb{C}^n$. The twisted spherical means of a function f on $\mathbb{C}^n$ is then defined by

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w)e^{(i/2)\mathrm{Im}(z.\bar{w})}\, d\mu_r. \tag{1.2}$$

These twisted spherical means are related to the spherical means on the Heisenberg group, and they play an important role in some problems on the Heisenberg group.

Let $\varphi_k(z)$, $k = 0, 1, 2, \ldots$ be the Laguerre functions defined by

$$\varphi_k(z) = L_k^{n-1}(\tfrac{1}{2}|z|^2)e^{-(1/4)|z|^2}, \tag{1.3}$$

where L_k^{n-1} are the Laguerre polynomials of type $(n-1)$. Then it is well known that [17]

$$\varphi_k \times \mu_r(z) = \frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r)\varphi_k(z). \tag{1.4}$$

Therefore, if $\frac{1}{2}R^2$ is a zero of $L_k^{n-1}(t)$ then it follows that $\varphi_k \times \mu_r(z) = 0$ on the sphere S_R for all $r > 0$. Since φ_k are Schwartz functions it follows that spheres are not sets of injectivity for the twisted spherical means on any $L^p(\mathbb{C}^n)$.

Injectivity sets for the twisted spherical means have recently been studied by Agranovsky and Rawat [4]. For $\epsilon > 0$ let $V_{p,\epsilon}$ be the class of functions f such that $f(z)e^{(1/4+\epsilon)|z|^2}$ is in $L^p(\mathbb{C}^n)$. They showed that the boundary Γ of any bounded domain in $\mathbb{C}^n$ is a set of injectivity for the twisted spherical means in $V_{p,\epsilon}$. In view of the examples we have, it is natural to expect that the same result is true with $\epsilon = 0$ as well. Unfortunately, their method does not give this result, and it is still an open problem. In this paper we show that when Γ is a sphere we can take $\epsilon = 0$.

THEOREM 1.1. *For $1 \le p \le \infty$, let V_p be the space of functions satisfying $f(z)e^{(1/4)|z|^2} \in L^p(\mathbb{C}^n)$. Then spheres are sets of injectivity for the twisted spherical mean value operator in V_p.*

This theorem was proved in [13] for $n = 1$. Here we treat the higher dimensional case, using spherical harmonic expansions and a Hecke–Bochner-type identity for the Weyl transform which is due to Geller [8].

As we said earlier, twisted spherical means are related to the spherical means on the Heisenberg group H^n. Recall that $H^n = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t)(w, s) = (z + w, s + t + \tfrac{1}{2}\operatorname{Im}(z.\bar{w})).$$

Let μ_r be as above, but now it is considered as a measure on H^n supported on $S_r \times \{0\}$. Then the spherical means of a function f on the Heisenberg group is defined by

$$f * \mu_r(z, t) = \int_{H^n} f(z - w, t - s - \tfrac{1}{2}\operatorname{Im}(z.\bar{w}))\, d\mu_r. \tag{1.5}$$

Injectivity sets for the spherical mean value operator on H^n have been studied by Agranovsky and Rawat [4]. A different problem, namely the injectivity of the spherical mean value operator, was studied earlier by Agranovsky et al. [1] and Thangavelu [15].

On the Heisenberg group consider the functions e_k^λ given by

$$e_k^\lambda(z, t) = e^{i\lambda t}\varphi_k^\lambda(z),$$

where $\varphi_k^\lambda(z) = \varphi_k(|\lambda|^{1/2}z)$. These are "elementary spherical functions" on H^n, and they satisfy the equation

$$e_k^\lambda * \mu_r(z, t) = \frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(r)e_k^\lambda(z, t), \tag{1.6}$$

where $\varphi_k^\lambda(r) = L_k^{n-1}(\frac{1}{2}|\lambda|r^2)e^{-(1/4)|\lambda|r^2}$. Thus $e_k^\lambda * \mu_r(z, t) = 0$ for all $r > 0$ if (z, t) belongs to the cylinder $\Gamma_R = S_R \times \mathbb{R}$, where $\frac{1}{2}|\lambda|R^2$ is a zero of the Laguerre polynomial $L_k^{n-1}(t)$.

Note that $e_k^\lambda \in L^p(H^n)$ only for $p = \infty$. Therefore, it is natural to expect that cylinders Γ_R are all sets of injectivity for the spherical means in $L^p(H^n)$, $1 \le p < \infty$. We prove that this is indeed the case.

THEOREM 1.2. *Let $1 \le p \le 2$ and let $\Gamma_R = S_R \times \mathbb{R}$. Then Γ_R is a set of injectivity for the spherical means on H^n in $L^p(H^n)$.*

The restriction $p \leq 2$ is imposed for technical reasons. We believe that the theorem is true for all $p, 1 \leq p < \infty$. Our method of proof requires that we should be able to take the partial Fourier transform of $f(z,t)$ in the t variable. The theorem can be proved, for example, for all functions in the mixed norm space $L^{p,q}(H^n), 1 \leq p < \infty, 1 \leq q \leq 2$, consisting of functions for which

$$\|f\|_{p,q}^p = \int_{\mathbb{C}^n} \left(\int_{-\infty}^{\infty} |f(z,t)|^q dt \right)^{p/q} dz < \infty.$$

See Agranovsky and Rawat [4] for a version of the above theorem under a different assumption on f.

In proving the main result in [2] the authors have used the wave equation associated to the Laplacian on $\mathbb{R}^n$. In this paper we make use of the heat equations associated to the sub-laplacian and the twisted Laplacian. If $u(z,t)$ is the solution of the heat equation

$$\partial_t u(z,t) = -Lu(z,t), \qquad u(z,0) = f(z), \tag{1.7}$$

given by $u(z,t) = f \times p_t(z)$, where $p_t(z)$ is the explicitly known heat kernel for the operator L, then it can be shown that $u(z,t) = 0$ for all $t > 0$ on S_R whenever $f \times \mu_r(z) = 0$ for all $r > 0$ on S_R. Therefore, Theorem 1.1 will follow from the following result.

THEOREM 1.3. *Let $f \in V_p, 1 \leq p \leq \infty$, where V_p is as before. If $u(z,t) = f \times p_t(z) = 0$ for all $t > 0$ on a sphere S_R then $f = 0$.*

Similarly, Theorem 1.2 will follow once we prove the following uniqueness theorem for the heat equation associated to the sub-Laplacian on H^n. Let $q_t(z,s)$ be the heat kernel associated to $\mathcal{L}$, whose Fourier transform in the s variable is explicitly known.

THEOREM 1.4. *Let $f \in L^p(H^n), 1 \leq p \leq 2$. If $u(z,s;t) = f * q_t(z,s) = 0$ for all $t > 0$ on a cylinder Γ_R, then $f = 0$.*

There is an analogue of Theorem 1.3 also for the Hermite operator $H = -\Delta + |x|^2$. Let e^{-tH} be the Hermite semigroup generated by this operator, so that $u(x,t) = e^{-tH} f(x)$ solves the heat equation associated to H with initial condition f. For $1 \leq p \leq \infty$ let B_p be the space of continuous functions f on $\mathbb{R}^n$ for which $f(x) e^{(1/2)|x|^2} \in L^p(\mathbb{R}^n)$.

THEOREM 1.5. *Let $1 \leq p \leq \infty$ and $f \in B_p$. If $u(x,t) = e^{-tH} f(x) = 0$ for all $t > 0$ on a sphere, then $f = 0$.*

This theorem can be restated in terms of the Ornstein–Uhlenbeck semigroup U_t. This semigroup is given by the kernel

$$K_t(x,y) = \pi^{-n/2}(1 - e^{-2t})^{-n/2} e^{-|e^{-t}x - y|^2/(1-e^{-2t})} \tag{1.8}$$

in the sense that $U_t f(x) = \int_{\mathbb{R}^n} K_t(x, y) f(y) d\gamma(y)$ for $f \in L^p(\gamma)$. Here $L^p(\gamma)$ are the L^p spaces taken with respect to the Gaussian measure $d\gamma(x) = e^{-|x|^2} dx$. It is easily checked that

$$e^{-nt} U_{2t} f(x) = e^{(1/2)|x|^2} e^{-tH} F(x),$$

where $F(y) = f(y) e^{-(1/2)|y|^2}$. Thus if $f \in L^p(\mathbb{R}^n)$ then the vanishing of $U_t f(x)$ on a sphere for all $t > 0$ implies that $f = 0$.

There is another notion of spherical means for functions on $\mathbb{R}^n$ which is related to the Hermite operator. This is defined by

$$M_r f(x) = \int_{|w|=r} e^{i(x.u + (1/2)u.v)} f(x+v)\, d\mu_r(w), \tag{1.9}$$

where $w = u + iv \in \mathbb{C}^n$. This spherical means is nothing but $W(\mu_r)f$ and has the expansion

$$M_r f(x) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) P_k f(x), \tag{1.10}$$

where $P_k f(x)$ are the spectral projections of f associated to the Hermite expansions. As shown by Ratnakumar and Thangavelu [11], these means can be expressed in terms of the spherical means on the (reduced) Heisenberg group.

It can be shown that (see Section 3) the functions $\psi_k(x) = L_k^{n/2-1} \times (|x|^2) e^{-(1/2)|x|^2}$ satisfy the relation

$$M_r \psi_k(x) = \frac{(2k)!(n-1)!}{(2k+n-1)!} \varphi_{2k}(r) \psi_k(x), \tag{1.11}$$

so that if R^2 is a zero of the polynomial $L_k^{n/2-1}$ then $M_r \psi_k(x)$ vanishes on S_R for all $r > 0$. Therefore, the analogue of Theorem 1.1 for these means is the following.

THEOREM 1.6. *For $1 \leq p \leq \infty$ let B_p be as in the previous theorem. Then spheres are sets of injectivity for the spherical mean value operator M_r in B_p.*

We will show that the vanishing of $M_r f(x)$ for all $r > 0$ is equivalent to the vanishing of $e^{-tH} f(x)$ for all $t > 0$. Thus Theorems 1.5 and 1.6 are equivalent.

Theorems 1.3, 1.4, and 1.5 will be proved in Section 3. In the next section we collect all of the relevant material needed to prove our main results. We closely follow the notations used in [16] and [17].

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2. WEYL TRANSFORM AND SPHERICAL HARMONICS

To prove Theorem 1.5 we need some properties of spherical harmonics and a Hecke–Bochner-type identity for the Hermite projection operators. Let $\mathscr{H}_m$ be the space of solid harmonics of degree m. That is, the elements of $\mathscr{H}_m$ are harmonic polynomials on $\mathbb{R}^n$ which are homogeneous of degree m. The restrictions of elements of $\mathscr{H}_m$ to the unit sphere S^{n-1} are spherical harmonics. We let $\mathscr{S}_m$ stand for the space of spherical harmonics of degree m. Then $L^2(S^{n-1})$ is the orthogonal direct sum of $\mathscr{S}_m, m \geq 0$. Given a continuous function f on $\mathbb{R}^n$, we have the spherical harmonic expansion $f(rx') = \sum_{m=0}^{\infty} f_m(rx')$ where $x = rx'$ with $x' \in S^{n-1}$. We can express f_m in terms of certain representations of $O(n)$.

Let $O(n)$ be the orthogonal group whose natural action on S^{n-1} defines a unitary representation on $L^2(S^{n-1})$. The restriction of this to each $\mathscr{S}_m$ defines an irreducible, unitary representation of $O(n)$, denoted by δ_m. Let χ_m be the character and let d_m be the degree of δ_m. The following proposition has been proved by Helgason [9].

PROPOSITION 2.1. *Let $f(x) = \sum_{m=0}^{\infty} f_m(x)$ be the spherical harmonic expansion of a continuous function f on $\mathbb{R}^n$. Then*

$$f_m(x) = d_m \int_{O(n)} \chi_m(\sigma) f(\sigma.x)\, d\sigma,$$

where $d\sigma$ is the normalized Haar measure on $O(n)$.

Applying the Peter–Weyl theorem to the function $F(\sigma) = f(\sigma.x)$ on $O(n)$, we obtain the decomposition

$$f(x) = \sum_{\delta \in O(n)} d(\delta) \int_{O(n)} \chi_\delta(\sigma) f(\sigma.x)\, d\sigma.$$

In Proposition 2.7 in Helgason [9] it is shown that the integrals in the above decomposition survive only when $\delta = \delta_m$ for some m.

We need explicit formulas for $P_k f_m$, where f_m are the components of f appearing in the above decomposition. Let Φ_α, where α are multi-indices, be the normalized Hermite functions on $\mathbb{R}^n$ so that the projections $P_k f(x)$ are given by

$$P_k f(x) = \sum_{|\alpha|=k} \left(\int_{\mathbb{R}^n} f(y) \Phi_\alpha(y)\, dy \right) \Phi_\alpha(x).$$

The following Hecke–Bochner-type identity has been proved in [16] (see Theorem 3.4.1). For a function g defined on $\mathbb{R}^+$ define

$$R_k^\delta(g) = \frac{2\Gamma(k+1)}{\Gamma(k+\delta+1)} \int_0^\infty g(r) L_k^\delta(r^2) e^{-(1/2)r^2} r^{2\delta+1}\, dr,$$

where L_k^δ are Laguerre polynomials of type δ.

THEOREM 2.2. *Assume that $f(x) = g(|x|)P(x)$, where P is a solid harmonic of degree m. Then, if $k = 2j + m$ we have*

$$P_k f_m(x) = R_j^\delta(g)P(x)L_j^\delta(|x|^2)e^{-(1/2)}|x|^2,$$

where $\delta = \frac{n}{2} + j + m$ and $P_k f_m = 0$ for all other values of k.

To prove Theorems 1.3 and 1.4 we need a Hecke–Bochner-type identity for the Weyl transform which is related to the group Fourier transform on the Heisenberg group. Recall that the irreducible, infinite-dimensional, unitary representations of H^n are given by π_λ, where λ is a non-zero real number. All of these representations are realized on the same Hilbert space $L^2(\mathbb{R}^n)$ and given by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x.\xi+(1/2)x.y)}\varphi(\xi + y)$$

for $\varphi \in L^2(\mathbb{R}^n)$. By suppressing the central variable, we let $\pi_\lambda(z) = \pi_\lambda(z, 0)$, which defines a projective representation of the abelian group $\mathbb{C}^n$.

Given a function $f \in L^1(\mathbb{C}^n)$, we define a family of operators $G_\lambda(f)$ all acting on $L^2(\mathbb{R}^n)$ by the prescription

$$G_\lambda(f) = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z)\,dz. \tag{2.1}$$

When $\lambda = 1$ the usual notation is $W(f)$, which is called the Weyl transform of f. Here we are using the notation employed by Geller in his paper [8]. Let $\hat{f}(\lambda)$ be the group Fourier transform of $f \in L^1(H^n)$ given by

$$\hat{f}(\lambda) = \int_{H^n} f(z, t)\pi_\lambda(z, t)\,dz\,dt.$$

Then from the definition of G_λ it follows that $G_\lambda(f^\lambda) = \hat{f}(\lambda)$, where

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z, t)e^{i\lambda t}\,dt \tag{2.2}$$

is the partial inverse Fourier transform of f in the t variable.

For each λ not equal to zero, define the λ-twisted convolution of two functions on $\mathbb{C}^n$ by

$$f *_\lambda g(z) = \int_{\mathbb{C}^n} f(z - w)g(w)e^{(i/2)\lambda \operatorname{Im}(z.\bar{w})}\,dw.$$

When $\lambda = 1$ we simply call this the twisted convolution of f and g and denote it by $f \times g$. If f and g are functions on H^n their group convolution is defined by

$$f * g(z, t) = \int_{H^n} f((z, t)(w, s)^{-1})g(w, s)\,dw\,ds.$$

It is then easily checked that $(f * g)^\lambda(z) = f^\lambda *_\lambda g^\lambda(z)$. We make use of this relation in transforming $f * \mu_r(z, t) = 0$ into the family of equations $f^\lambda *_\lambda \mu_r(z) = 0$. We also note that $G_\lambda(f *_\lambda g) = G_\lambda(f)G_\lambda(g)$ for any two functions on $\mathbb{C}^n$.

The λ-twisted convolution of f with the constant function $g = 1$ is called the λ-symplectic Fourier transform. Explicitly,

$$\mathcal{F}_\lambda f(z) = \int_{\mathbb{C}^n} f(z - w)e^{(i/2)\lambda \operatorname{Im}(z.\bar{w})}\, dw.$$

When $\lambda = 1$ we omit the subscript and call it just the symplectic Fourier transform. We then define the Weyl correspondence W_λ by setting $W_\lambda(f) = G_\lambda(\mathcal{F}_\lambda^{-1} f)$. The Hecke–Bochner-type identity which we need is the one which gives a formula for the Weyl transform of certain functions defined in terms of spherical harmonics. We will now recall relevant definitions before stating this important formula.

For each pair of non-negative integers p and q let $\mathcal{P}_{pq}$ be the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Let $\mathcal{H}_{pq} = \{P \in \mathcal{P}_{pq} : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{C}^n$. Elements of $\mathcal{H}_{pq}$ are called bigraded, solid harmonics on $\mathbb{C}^n$. For non-zero real λ we let $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$. By $L_k^\delta(t)$ we denote the kth Laguerre polynomial of type $\delta > -1$. See Szego [14] for various properties of these polynomials.

With the above notations we are now in a position to state Geller's result concerning the Weyl transform [8].

THEOREM 2.3. *Suppose $gP \in L^1(\mathbb{C}^n)$ or $L^2(\mathbb{C}^n)$, where g is a radial function and $P \in \mathcal{H}_{pq}$. For $\lambda > 0$, $G_\lambda(gP) = (-1)^q W_\lambda(P)S$, where the action of S on Φ_α^λ is given as follows: $S\Phi_\alpha^\lambda = c_{|\alpha|}$, where $c_k = 0$ for $k < p$, and for $k \geq p$, it is given by*

$$c_k = \frac{(k-p)!(n-1)!}{(k+q+n-1)!} \int_{\mathbb{C}^n} g(|z|) L_{k-p}^{n-1+p+q}\left(\frac{\lambda}{2}|z|^2\right) e^{-(\lambda/4)|z|^2} |z|^{2(p+q)}\, dz.$$

When $\lambda < 0$ the roles of p and q are reversed in the definition of c_k.

The Hermite functions Φ_α^λ are eigenfunctions of the operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ with eigenvalues $(2|\alpha| + n)|\lambda|$. If we let $P_k(\lambda)$ stand for the orthogonal projection onto the eigenspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = k\}$ then it is known that $P_k(\lambda) = |\lambda|^n G_\lambda(\varphi_k^\lambda)$ (see [17]). Using this relation in the previous theorem, we obtain

$$G_\lambda(gP *_\lambda \varphi_k^\lambda) = (-1)^q|\lambda|^{-n} W_\lambda(P)SP_k(\lambda) = (-1)^q|\lambda|^{-n} c_k W_\lambda(P)P_k(\lambda).$$

On the other hand, if we take

$$g(z) = \psi_k(z) = L_{k-p}^{n-1+p+q}\left(\frac{\lambda}{2}|z|^2\right)e^{-(\lambda/4)|z|^2}$$

in the above theorem, then it follows that $c_j = 0$ for all $j \neq k$ and $c_k = 1$, which simply means that

$$G_\lambda(\psi_k P) = (-1)^q W_\lambda(P) P_k(\lambda).$$

Thus we have the following corollary.

COROLLARY 2.4. *Suppose $gP \in L^1(\mathbb{C}^n)$ or $L^2(\mathbb{C}^n)$, where g is radial and $P \in \mathscr{H}_{pq}$. Then we have the formula*

$$(gP) *_\lambda \varphi_k^\lambda(z) = c_k P(z) L_{k-p}^{n-1+p+q}(\tfrac{1}{2}|\lambda||z|^2)e^{-(1/4)|\lambda||z|^2},$$

where c_k is as in the theorem.

We will now look at the spaces $\mathscr{H}_{pq}$ more closely. Let $\mathscr{S}_{pq}$ be the space of functions which are restrictions to the unit sphere S^{2n-1} of functions from $\mathscr{H}_{pq}$. The elements of $\mathscr{S}_{pq}$ are called bigraded spherical harmonics. The space $L^2(S^{2n-1})$ is the orthogonal direct sum of $\mathscr{S}_{pq}$ as p and q range over all non-negative integers. Given a continuous function f on $\mathbb{C}^n$, we can expand the function $f_r(\omega) = f(r\omega)$, where $r > 0$ and $\omega \in S^{2n-1}$ in terms of spherical harmonics. Thus there is an expansion

$$f(r\omega) = \sum_{k=0}^{\infty} \sum_{p+q=k} f_{pq}(r\omega), \tag{2.3}$$

with $f_{pq}(r\omega)$ coming from $\mathscr{S}_{pq}$. We would like to express f_{pq} in terms of certain representations of the unitary group $U(n)$.

The natural action of $U(n)$ on the unit sphere S^{2n-1} defines a unitary representation of $U(n)$ on the Hilbert space $L^2(S^{2n-1})$. When restricted to $\mathscr{S}_{pq}$ it defines an irreducible representation, which we denote by δ_{pq}. Let $d(p, q)$ be the dimension of $\mathscr{S}_{pq}$ and let χ_{pq} be the character associated to δ_{pq}.

PROPOSITION 2.5. *Given a continuous function f on $\mathbb{C}^n$ the projections f_{pq} appearing in the spherical harmonic expansion* (2.3) *are given by*

$$f_{pq}(z) = d(p, q) \int_{U(n)} \chi_{pq}(\sigma) f(\sigma.z)\, d\sigma,$$

where $d\sigma$ is the normalized Haar measure on the group $U(n)$.

The proof of this proposition is similar to that of Proposition 2.7 in Helgason [9], where the ordinary spherical harmonic expansion has been considered. Let $G = U(n)$ and let $\widehat{G}$ stand for the unitary dual of G. Then the Peter–Weyl theorem applied to the function $F(\sigma) = f(\sigma.z)$ leads to the expansion

$$f(z) = \sum_{\delta \in \widehat{G}} d(\delta) \int_G \chi_\delta(\sigma) f(\sigma.z)\, d\sigma. \tag{2.4}$$

Let $K = U(n-1)$ be considered as a subgroup of G. As in Helgason [9], we can show that the integral $\int_G \chi_\delta(\sigma) f(\sigma.z)\, d\sigma$ is nonzero only if the group $\delta(K)$ has a non-zero fixed vector. For each pair (p, q) the representation δ_{pq} has a unique K-fixed vector in $\mathscr{H}_{pq}$. Moreover, all such representations are accounted for by these δ_{pq}. Thus the above expansion reduces to

$$f(z) = \sum_{p,q} d(p, q) \int_G \chi_{p,q}(\sigma) f(\sigma.z)\, d\sigma. \tag{2.5}$$

This completes the proof of the proposition.

3. SPHERICAL MEANS AND HEAT EQUATIONS

In this section we prove our main results. We start by collecting some information regarding the heat kernels associated to the sub-Laplacian and the twisted Laplacian. The sub-Laplacian $\mathscr{L}$ which plays the role of the Laplacian for H^n is the second-order differential operator explicitly given by

$$\mathscr{L} = -\Delta_z - \tfrac{1}{4}|z|^2 \partial_s^2 - N\partial_s,$$

where Δ_z is the Laplacian on $\mathbb{C}^n$ and N is the rotation operator

$$N = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

The twisted Laplacian L and the sublaplacian $\mathscr{L}$ are related by $\mathscr{L}(e^{is} f \times (z)) = e^{is} Lf(z)$; or, explicitly, $L = -\Delta_z + \frac{1}{4}|z|^2 - iN$.

The spectral decomposition of L is explicitly given by the special Hermite expansion

$$Lf(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k+n) f \times \varphi_k(z),$$

where φ_k are the Laguerre functions defined in (1.3). The heat kernel associated to L is given by

$$p_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} \varphi_k(z).$$

The series can be summed, and we get the formula (see [16])

$$p_t(z) = (2\pi)^{-n} (\sin ht)^{-n} e^{-\frac{1}{4}(\cot ht)|z|^2}. \tag{3.1}$$

Thus $p_t(z)$ is a radial function, and hence the solution $u(z,t) = f \times p_t(z)$ of the heat equation for L satisfies $u(z,t) = 0$ for all $t > 0$ whenever $f \times \mu_r(z) = 0$ for all $r > 0$. In view of this, once we prove Theorem 1.2, Theorem 1.1 will follow.

The sublaplacian $\mathcal{L}$ generates a contraction semigroup $e^{-t\mathcal{L}}$, which is given by a nice kernel $q_t(z,s)$ (see Folland [6]). The function $u(z,s;t) = f * q_t(z,s)$ solves the equation

$$\partial_t u(z,s;t) = -\mathcal{L} u(z,s;t), \qquad u(z,s;0) = f(z,s).$$

Though q_t itself is not explicitly known, its Fourier transform in the s variable is known (see Gaveau [7] and Hulanicki [10]). Indeed,

$$q_t^\lambda(z) = c_n \left(\frac{\lambda}{\sin h\lambda t} \right)^n e^{-(1/4)(\lambda \cot h\lambda t)|z|^2}. \tag{3.2}$$

Note that $q_t^\lambda(z) = b_n \lambda^{-n} p_{\lambda t}(\sqrt{\lambda} z)$, and q_t is a Schwartz class function. Since q_t is radial in z it follows that $f * q_t(z,s)$ can be expressed in terms of $f * \mu_r(z,s)$, and hence $f * q_t(z,s) = 0$ for all $t > 0$ whenever $f * \mu_r(z,s) = 0$ for all $r > 0$. Therefore, once we prove Theorem 1.4, Theorem 1.3 will follow immediately.

We begin by proving Theorem 1.2. For the proof we require the following result.

PROPOSITION 3.1. *Let $f \in L^p(\mathbb{C}^n)$, $1 \le p \le \infty$. If $f \times p_t(z)$ vanishes on a sphere for all $t > 0$ then so does $f \times \varphi_k(z)$ for all $k \ge 0$.*

Proof. Note that for a fixed z the heat kernel $p_t(z)$ can be extended to the right half-plane $\mathrm{Re}(t) > 0$ as a holomorphic function. A simple calculation shows that $|p_t(z)|$ has exponential decay in z in that region. Therefore, $f \times p_t(z)$ is well defined and holomorphic in $\mathrm{Re}(t) > 0$. Since $f \times p_t(z) = 0$ on a sphere for $t > 0$ it follows that $f \times p_{t+is}(z) = 0$ on the same sphere for all s if $t > 0$. Observe that $f \times p_{t+is}(z)$ is given by the expansion

$$f \times p_{t+is}(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} e^{-(2k+n)is} f \times \varphi_k(z),$$

which vanishes for all s on a sphere for $t > 0$ fixed. By calculating the Fourier coefficients of $f \times p_{t+is}(z)$ as a function of s we obtain $f \times \varphi_k(z) = 0$ on the same sphere. This proves the proposition.

We remark that this proposition can be used to prove the equivalence of Theorems 1.1 and 1.2. Indeed, as is well known (see [17]),

$$f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) f \times \varphi_k(z).$$

If $f \times \varphi_k(z) = 0$ for all k, then by the uniqueness theorem for the Laguerre expansions it follows that $f \times \mu_r(z) = 0$ for all $r > 0$. Thus $f \times p_t(z) = 0$ for all $t > 0$ is equivalent to $f \times \mu_r(z) = 0$ for all $r > 0$.

If f were a radial function, then $f \times \varphi_k(z)$ would reduce to

$$f \times \varphi_k(z) = C_k \left(\int_{\mathbb{C}^n} f(w) \varphi_k(w)\, dw \right) \varphi_k(z).$$

Suppose now $f \times \varphi_k(z)$ vanishes on the sphere S_R. Since the Laguerre polynomials L_k^{n-1} have distinct zeros (see Szego [14]), $L_k^{n-1}(\frac{1}{2}R^2)$ can vanish for at most one value, say $k = K$ of k, and hence the above integral vanishes for all $k \neq K$. This means that $f(z) = A_K \varphi_K(z)$. But then $f(z)e^{(1/4)|z|^2}$ cannot be in any $L^p(\mathbb{C}^n)$. This proves Theorem 1.2 if f were radial.

To deal with the general case we assume, without loss of generality, that f is continuous and expand f in terms of bigraded spherical harmonics, getting $f(z) = \sum_{p,q} f_{pq}(z)$, where

$$f_{pq}(z) = \sum_{m=1}^{d(p,q)} f_{pq}^m(|z|) P_{pq}^m(z), \tag{3.3}$$

with $P_{pq}^m \in \mathscr{H}_{pq}$. Consider

$$f_{pq} \times \varphi_k(z) = \int_{\mathbb{C}^n} \varphi_k(z-w) e^{(-i/2)\mathrm{Im}(z.\bar{w})} f_{pq}(w)\, dw,$$

which in view of Proposition 2.5 is equal to the integral

$$\int_{\mathbb{C}^n} \int_{U(n)} \varphi_k(z-w) e^{(-i/2)\mathrm{Im}(z.\bar{w})} \chi_{pq}(\sigma) f(\sigma.w)\, d\sigma\, dw.$$

Since $\varphi_k(z)$ is radial and any $\sigma \in U(n)$ preserves the symplectic form $\mathrm{Im}(z.\bar{w})$, we obtain the formula

$$f_{pq} \times \varphi_k(z) = \int_{U(n)} f \times \varphi_k(\sigma.z) \chi_{pq}(\sigma)\, d\sigma.$$

Therefore, if $f \times \varphi_k$ vanishes on a sphere S_R, then so does $f_{pq} \times \varphi_k$ for any pair p, q of non-negative integers.

As f_{pq} is given by the expression (3.3), using Corollary 2.4, we obtain

$$f_{pq} \times \varphi_k(z) = \left(\sum_{m=1}^{d(p,q)} c_{pq}^m(k) P_{pq}^m(z) \right) L_{k-p}^{n-1+p+q}(\tfrac{1}{2}|z|^2) e^{-(1/4)|z|^2}, \tag{3.4}$$

where the constants $c^m_{pq}(k)$ are given by the integral

$$\frac{(k-p)!(n-1)!}{(k+q+n-1)!}\int_{\mathbb{C}^n} f^m_{pq}(|z|)L^{n-1+p+q}_{k-p}\left(\frac{1}{2}|z|^2\right)e^{-(1/4)|z|^2}|z|^{2(p+q)}\,dz.$$

If $f_{pq}\times\varphi_k(z)$ vanishes on S_R, then, as before, since the zeros of $L^{n-1+p+q}_k$ are distinct, we get

$$\sum_{m=1}^{d(p,q)} c^m_{pq}(k)P^m_{pq}(z)=0$$

for all values of k, except possibly for one value, say, $k=K$. The restrictions of P^m_{pq} to the unit sphere are orthonormal, and so the above implies $c^m_{pq}(k)=0$ for all m when $k\neq K$. This means that $f_{pq}\times\varphi_k=0$ for all $k\neq K$, and hence

$$f_{pq}(z)=\left(\sum_{m=0}^{d(p,q)} c^m_{pq}(K)P^m_{pq}(z)\right)L^{n-1+p+q}_{K-p}(\tfrac{1}{2}|z|^2)e^{-(1/4)|z|^2}.$$

The condition $f(z)e^{(1/4)|z|^2}\in L^p(\mathbb{C}^n)$ holds good for f_{pq} as well, and hence the above is possible only when $f_{pq}=0$. As this is true for all p and q we conclude that $f=0$. This completes the proof of Theorem 1.2.

The proof of Theorem 1.4 is similar. Again we can assume that f is a continuous function. Since $f*q_t(z,s)$ is assumed to be zero on $\Gamma_R=S_R\times\mathbb{R}$, by taking the Fourier transform in the s variable we get $f^\lambda *_\lambda q^\lambda_t(z)=0$ for all $\lambda\neq 0$, $t>0$, and $z\in S_R$. As $q^\lambda_t(z)$ is proportional to $p_{\lambda t}(\sqrt{\lambda}z)$ we obtain the equations $f^\lambda *_\lambda \varphi^\lambda_k(z)=0$ on S_R for all $\lambda\neq 0$ and $k\geq 0$. Now we expand $f^\lambda(z)$ in terms of spherical harmonics, getting $f^\lambda=\sum_{p,q} f^\lambda_{pq}(z)$, and as before this leads to the equations $f^\lambda_{pq} *_\lambda \varphi^\lambda_k(z)=0$ on S_R.

Let us write $f^\lambda_{pq}(z)=\sum_{m=1}^{d(p,q)} f^{\lambda,m}_{pq}(|z|)P^m_{pq}(z)$, where $P^m_{pq}\in\mathscr{H}_{pq}$, so that Corollary 2.4 leads to the formula

$$f^\lambda_{pq} *_\lambda \varphi^\lambda_k(z)=\left(\sum_{m=1}^{d(p,q)} c^{\lambda,m}_{pq}(k)P^m_{pq}(z)\right)L^{n-1+p+q}_{k-p}(\tfrac{1}{2}|\lambda||z|^2)e^{-(1/4)|\lambda||z|^2}.$$

Consider now the conditions $f^\lambda_{pq} *_\lambda \varphi^\lambda_k(z)=0$ on S_R. For each k, $L^{n-1+p+q}_{k-p}\times(\frac{1}{2}|\lambda|R^2)=0$ only for finitely many values of λ. Thus there is a countable set $\mathscr{D}$ such that when λ is not in $\mathscr{D}$,

$$\sum_{m=1}^{d(p,q)} c^{\lambda,m}_{pq}(k)P^m_{pq}(z)=0$$

for all k. This in turn leads to the vanishing of $c^{\lambda,m}_{pq}(k)$ for all m and k.

Consequently, $f_{pq}^{\lambda} = 0$ for all p and q, which means that $f^{\lambda}(z) = 0$ for all λ not in $\mathcal{D}$ and $z \in \mathbb{C}^n$. But then the Fourier transform of $f(z, s)$ in the s variable is supported on the countable set $\mathcal{D}$, which contradicts the assumption that $f \in L^p(H^n)$ unless $f = 0$. This completes the proof of Theorem 1.4, and, as we mentioned earlier, Theorem 1.3 follows from this.

Finally, we take up Theorems 1.5 and 1.6. As in Proposition 3.1 we can show that $e^{-tH}f(x) = 0$ for all $t > 0$ implies that $P_k f(x) = 0$ for all $k \geq 0$. Also, since $M_r f(x)$ is given by (1.10), by the uniqueness theorem for Laguerre expansions we can conclude that $P_k f(x) = 0$ for all $k \geq 0$ if $M_r f(x) = 0$ for all $r > 0$. Thus it is enough to show that $P_k f(x) = 0$ on a sphere S_R for all $k \geq 0$ is not possible for functions $f \in B_p$ unless $f = 0$.

Let $\Phi_k(x, y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y)$ be the kernel of P_k. These are given by the generating function

$$\sum_{k=0}^{\infty} r^k \Phi_k(x, y) = \pi^{-n/2}(1 - r^2)^{-n/2} e^{-((1+r^2)/2(1-r^2))(|x|^2+|y|^2)+(2r/(1-r^2))x.y}. \tag{3.5}$$

By taking $y = 0$ we observe that the right-hand side of the above reduces to the generating function for the Laguerre functions $L_k^{n/2-1}(|x|^2)e^{-(1/2)|x|^2}$ (see [16]). Therefore,

$$\sum_{k=0}^{\infty} r^{2k} L_k^{n/2-1}(|x|^2)e^{-(1/2)|x|^2} = \sum_{k=0}^{\infty} r^k \Phi_k(x, 0),$$

and hence we get $\Phi_{2k+1}(x, 0) = 0$ and $\Phi_{2k}(x, 0) = L_k^{n/2-1}(|x|^2)e^{-(1/2)|x|^2}$. Since

$$M_r\Phi_{2k}(x, 0) = \sum_{|\alpha|=2k} M_r\Phi_\alpha(x)\Phi_\alpha(0) = \frac{(2k)!(n-1)!}{(2k+n-1)!}\varphi_{2k}(r)\Phi_{2k}(x, 0),$$

we see that our assertion (1.11) is justified.

Let $f = \sum_{m=0}^{\infty} f_m(x)$ be the spherical harmonic expansion of f where the components are given by Proposition 2.1. From (3.5) it is clear that $\Phi_k(\sigma.x, \sigma.y) = \Phi_k(x, y)$ for all $\sigma \in O(n)$. Therefore,

$$P_k f_m(x) = d_m \int_{O(n)} P_k f(\sigma.x)\chi_m(\sigma)\, d\sigma,$$

which shows that whenever $P_k f(x)$ vanishes on S_R so does $P_k f_m(x)$ on the same sphere. Now

$$f_m(x) = \sum_{j=1}^{d_m} f_m^j(|x|)P_m^j(x),$$

where P_m^j are solid harmonics of degree m. The rest of the proof proceeds as in the twisted spherical means case. We make use of Theorem 2.2 to conclude that $f_m = 0$ for all m. This completes the proofs of Theorems 1.5 and 1.6.

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