MINIMAL ISOMETRIC DILATIONS OF OPERATOR COCYCLES*

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We obtain existence, uniqueness results for minimal isometric dilations of contractive cocycles of semigroups of unital $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$. This generalizes the result of Sz. Nagy on minimal isometric dilations of semigroups of contractive operators on a Hilbert space. In a similar fashion we explore results analogus to Sarason's characterization that subspaces to which compressions of semigroups are again semigroups are semi-invariant subspaces, in the context of cocycles and quantum dynamical semigroups.

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded operators on $\mathcal{H}$. An e_0-semigroup θ on $\mathcal{B}(\mathcal{H})$ is a one parameter semigroup $\theta = \{\theta_t : 0 \leq t < \infty\}$ of normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$, such that for each $X \in \mathcal{B}(\mathcal{H})$ the map $t \mapsto \theta_t(X)$ is continuous in strong (equivalently weak) operator topology. It is said to be an E_0-semigroup if each θ_t is unital. A family $R = \{R_t : 0 \leq t < \infty\}$ of bounded operators on $\mathcal{H}$ is said to be a *(left) cocycle* with respect to an e_0-semigroup θ if $R_0 = I$, $R_{s+t} = R_s\theta_s(R_t)$ for all s, t and the map $t \mapsto R_t$ is continuous in strong operator topology. The cocycle R is said to be contractive, isometric, unitary, positive, or projection cocycle if each R_t has that property. It is said to be *local* if for every t, $\{R_t\}$ commutes with $\theta_t(X)$ for all X.

Cocycles adapted to Fock filtration is a recurring theme in the theory of Quantum Stochastic Processes ([AM], [LW], [Pa]). Here usually there are two kinds of cocycles namely cocycles made up of $*$-homomorphisms and those made up of operators. We will be talking about operator cocycles only. For us cocycles are important as they give us new semigroups. Indeed if R is an isometric left cocycle of an e_0-semigroup θ, then $\tau = \{\tau_t : 0 \leq t < \infty\}$, defined by

$$\tau_t(X) = R_t\theta_t(X)R_t^* \quad \text{for } X \in \mathcal{B}(\mathcal{H}) \tag{1.1}$$

is another e_0-semigroup. In general whenever R is a left cocycle of θ, τ defined by (1.1) is a semigroup of completely positive maps. Semigroups of completely positive maps play an important role in quantum theory as they describe irreversible dynamics and are known

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as quantum dynamical semigroups. We will restrict our attention to quantum dynamical semigroups consisting of normal maps and which are continuous in strong operator topology.

If θ is the identity semigroup ($\theta_t(X) \equiv X$) then its cocycles are nothing but semigroups of bounded operators. Thanks to Sz. Nagy we know that semigroups of contractions on a Hilbert space can be dilated to a semigroup of isometries and under a minimality condition the dilation is unique up to unitary equivalence. Here we have a similar result on dilation of contractive left cocycles of general E_0-semigroups to isometric cocycles.

We recall that every contractive quantum dynamical semigroup dilates to an e_0-semigroup (see [Bh2], [Bh3]). Here too the dilation is unique (up to conjugation by a unitary) under a minimality assumption. Now if τ, θ are as in (1.1) it is natural to ask as to whether the minimal dilation $\hat{\tau}$ of τ can be drastically different from θ. This question can be better formulated as follows. Powers [Po4] has divided E_0-semigroups into three types. The type I E_0-semigroups are well understood. There are a host of type II, III examples (see [Po2], [Po3], [Ts1], [Ts2]). We have partial understanding of type II E_0-semigroups. But except for their existence we know very little about type III E_0-semigroups. With this in mind we can ask as to whether types of $\hat{\tau}$ and θ can be different. Plenty of cocycles of type I E_0-semigroups have been constructed by many authors using quantum stochastic calculus and other means (see [Jo], [Fa], [LW], [MS], [Pa]). It would have been nice if we could have got type II or type III E_0-semigroups with the help of these cocycles. But in Section 4 we will see that this is not possible because if θ is of type I then so is $\hat{\tau}$. We also show that in general index $(\hat{\tau}) \leq$ index (θ), here by index we mean the numerical index Arveson attaches as an invariant for e_0-semigroups.

Practically all the results we prove have discrete time versions where we consider powers of a single normal $*$-endomorphism (or completely positive map) as the semigroup. It is not clear as to how far the results can be pushed to cocycles of semigroups of endomorphisms of general C^*-algebras to study product systems of Hilbert modules (see [BS]).

In this article our Hilbert spaces will be complex, separable with an inner product $\langle \cdot, \cdot \rangle$, which is antilinear in the first variable. For any two vectors u, v in the Hilbert space by $|u\rangle\langle v|$, we will denote the operator which sends any vector w to $\langle v, w\rangle u$.

2 Sarason's Theorem

In this Section $\mathcal{H}$ is a closed subspace of a Hilbert space $\mathcal{K}$ and we denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ by P. Note that here and throughout this paper we identify operators X on $\mathcal{B}(\mathcal{H})$ with operators PXP on $\mathcal{B}(\mathcal{K})$, and this we do for any two Hilbert space $\mathcal{H}, \mathcal{K}$ with $\mathcal{H} \subseteq \mathcal{K}$ without warning. For instance, the identity operator $1_{\mathcal{H}}$ is also the projection of $\mathcal{K}$ onto $\mathcal{H}$.

Suppose $V = \{V_t : 0 \leq t < \infty\}$ is a semigroup of bounded operators on $\mathcal{K}$ and $R = \{R_t : 0 \leq t < \infty\}$ is the compression of V onto $\mathcal{H}$, defined by,

$$R_t = PV_tP, \quad 0 \leq t < \infty.$$

Then a famous result of Sarason ([Sa], see also [Pi] for various versions of this theorem) states that R is a semigroup in its own right if and only if $\mathcal{H}$ is semi-invariant for V, that is, $P = P_1 - P_2$, where $P_1 \geq P_2$ are projections on $\mathcal{K}$ such that their ranges are invariant under all V_t. We wish to prove a similar result for left cocycles of E_0-semigroups. Note that when V_t is a contraction the condition that P_1 is invariant for V_t is equivalent to saying $V_t P_1 V_t^* \leq P_1$.

THEOREM 2.1 *Let ψ be an E_0-semigroup on $B(\mathcal{K})$ such that $\psi_t(P) = P$ for every t, where P is the orthogonal projection to a subspace $\mathcal{H}$. Let θ be the E_0-semigroup of $B(\mathcal{H})$ defined by $\theta_t(X) = \psi_t(X)$ for every $X \in B(\mathcal{H})$. Now suppose $V = \{V_t\}$ is a contractive left cocycle of ψ. Its compression $R = \{R_t : 0 \leq t < \infty\}$ defined by $R_t = PV_tP$, is a left cocycle of θ if and only if $P = P_1 - P_2$ for some projections $P_1 \geq P_2$ such that $\tilde{\psi}_t(P_1) \leq P_1, \tilde{\psi}_t(P_2) \leq P_2$, where $\tilde{\psi}$ is the quantum dynamical semigroup defined by $\tilde{\psi}_t(Z) = V_t\psi_t(Z)V_t^*$, $Z \in B(\mathcal{K})$.*

Proof: Suppose $P = P_1 - P_2$ as above. First we claim that for every t, V_t maps $\mathcal{H}$ = range (P) into range (P_1), that is, $V_tP = P_1V_tP$ or $(1 - P_1)V_tP = 0$. This is clear as,

$$\begin{aligned}
(1 - P_1)V_tP[(1 - P_1)V_tP]^* &= (1 - P_1)V_tPV_t^*(1 - P_1) \\
&= (1 - P_1)V_t\psi_t(P)V_t^*(1 - P_1) \\
&= (1 - P_1)\tilde{\psi}_t(P)(1 - P_1) \\
&\leq (1 - P_1)\tilde{\psi}_t(P_1)(1 - P_1) \\
&\leq (1 - P_1)P_1(1 - P_1) \\
&= 0.
\end{aligned}$$

We also claim, $PV_s\psi_s(P_2V_tP) = 0$ for all s, t. This is a similar computation. Indeed,

$$\begin{aligned}
PV_s\psi_s(P_2V_tP)(PV_s\psi_s(P_2V_tP))^* &= PV_s\psi_s(P_2V_tPV_t^*P_2)V_s^*P \\
&\leq \|V_tPV_t^*\|PV_s\psi_s(P_2)V_s^*P \\
&\leq \|V_tPV_t^*\|P\tilde{\psi}_s(P_2)P \\
&\leq \|V_tPV_t^*\|PP_2P \\
&= 0.
\end{aligned}$$

Now for $0 \leq s, t < \infty$,

$$\begin{aligned}
R_{s+t} &= PV_{s+t}P = PV_s\psi_s(V_t)P = PV_s\psi_s(V_t)\psi_s(P) \\
&= PV_s\psi_s(V_tP) = PV_s\psi_s(P_1V_tP) \\
&= PV_s\psi_s(PV_tP) + PV_s\psi_s(P_2V_tP) \\
&= PV_s\psi_s(PV_tP) = PV_s\theta_s(R_t) \\
&= PV_sP\theta_s(R_t) = R_s\theta_s(R_t).
\end{aligned}$$

The continuity of $t \mapsto R_t$ in strong operator topology is obvious. Hence R is a left cocycle of θ.

Conversely suppose R is a left cocycle of θ. Let P_1 be the projection onto $\mathcal{H}_1 = \overline{\text{span}}\{V_tu : u \in \mathcal{H}, 0 \leq t < \infty\}$ and $P_2 = P_1 - P$. Then for $0 \leq r, s < \infty$, and $u, v \in \mathcal{H}$,

$$\tilde{\psi}_t(|V_ru\rangle\langle V_sv|) = V_t\psi_t(V_r|u\rangle\langle v|V_s^*)V_t^* = V_{r+t}\psi_t(|u\rangle\langle v|)V_{s+t}^* = V_{r+t}\theta_t(|u\rangle\langle v|)V_{s+t}^*.$$

As $\theta_t(|u\rangle\langle v|) \in \mathcal{B}(\mathcal{H})$, the operator $V_{r+t}\theta_t(|u\rangle\langle v|)V_{s+t}^* \in \mathcal{B}(\mathcal{H}_1)$. Then it is clear that $\tilde{\psi}_t(Y) \in \mathcal{B}(\mathcal{H}_1)$ for every $Y \in \mathcal{B}(\mathcal{H}_1)$. Therefore $\tilde{\psi}_t(P_1) \leq P_1$. Now range $(P_2) = \overline{\text{span}}\{V_t u - R_t u : u \in \mathcal{H}, 0 \leq t < \infty\}$, and for $0 \leq r, s, t < \infty, u, v \in \mathcal{H}$,

$$\begin{aligned}
& P\tilde{\psi}_t(|V_r u - R_r u\rangle\langle V_s v - R_s v|)P \\
= \ & PV_t\psi_t(|V_r u - R_r u\rangle\langle V_s v - R_s v|)V_t^* P \\
= \ & PV_t\psi_t(|V_r u\rangle\langle v|V_s^*)V_t^* P - PV_t\psi_t(|R_r u\rangle\langle v|V_s^*)V_t^* P \\
& -PV_t\psi_t(V_r|u\rangle\langle R_s v|)V_t^* P + PV_t\psi_t(|R_r u\rangle\langle R_s v|)V_t^* P \\
= \ & R_{r+t}\theta_t(|u\rangle\langle v|)R_{s+t}^* - R_{r+t}\theta_t(|u\rangle\langle v|)R_{s+t}^* - R_{r+t}\theta_t(|u\rangle\langle v|)R_{s+t}^* + R_{r+t}\theta_t(|u\rangle\langle v|)R_{s+t}^* \\
= \ & 0.
\end{aligned}$$

Therefore $\tilde{\psi}_t(Y) \in \mathcal{B}(\mathcal{H}_1 \ominus \mathcal{H})$ for all $Y \in \mathcal{B}(\mathcal{H}_1 \ominus \mathcal{H})$, and hence $\tilde{\psi}_t(P_2) \leq P_2$. ∎

We may ask as to whether a similar result holds for quantum dynamical semigroups. One may appeal to vector space or Banach space versions of Sarason's Theorem directly. For this we need to consider general subspaces of the Banach space $\mathcal{B}(\mathcal{K})$ and not just 'corners' $P\mathcal{B}(\mathcal{K})P$ defined by projections. We do not wish to do it here, instead we have a somewhat weaker, but perhaps more useful statement.

THEOREM 3.2 *Let β be a contractive quantum dynamical semigroup on $\mathcal{B}(\mathcal{K})$. Let P be a projection to a subspace $\mathcal{H}$ of $\mathcal{K}$ and let $\alpha = \{\alpha_t : 0 \leq t < \infty\}$ be the compression of β, $\alpha_t(X) = P\beta_t(X)P$, for $X \in \mathcal{B}(\mathcal{H})$. Then (i) α is a unital quantum dynamical semigroup if and only if $\beta_t(P) \geq P$; (ii) $P = P_1 - P_2$, where $P_1 \geq P_2$ are projections such that $\beta_t(P_1) \leq P_1, \beta_t(P_2) \leq P_2$, if and only if α is a quantum dynamical semigroup and $\beta_s(Q\beta_t(P)Q) \leq Q$ for all s, t, where $Q = 1 - P$.*

Proof: The 'only if' part of (i) is obvious. The 'if' part follows from (ii) by taking $P_1 = 1$ and $P_2 = 1 - P$. To prove (ii) suppose P has the decomposition as described. Consider $0 \leq X \leq P \leq P_1$. Then $0 \leq \beta_t(X) \leq \beta_t(P) \leq \beta_t(P_1) \leq P_1$, and therefore

$$\begin{aligned}
\alpha_{s+t}(X) &= P\beta_{s+t}(X)P = P\beta_s(\beta_t(X))P = P\beta_s(P_1\beta_t(X)P_1)P \\
&= P\beta_s((P + P_2)\beta_t(X)(P + P_2))P.
\end{aligned}$$

Clearly $P\beta_s(P_2\beta_t(X)P_2)P \leq P\beta_s(P_2)P \leq PP_2P = 0$. Now Kadison's inequality $\beta_s(Y^*)\beta_s(Y) \leq \beta_s(Y^*Y)$ applied to the contractive completely positive map β_s with $Y = P\beta_t(X^*)P_2$ yields

$$P\beta_s(P_2\beta_t(X)P)\beta_s(P\beta_t(X^*)P_2)P \leq P\beta_s(P_2\beta_t(X)PP\beta_t(X^*)P_2)P \leq P\beta_s(P_2)P \leq PP_2P = 0.$$

Hence $P\beta_s(P_2\beta_t(X)P) = 0$. Similarly $\beta_s(P\beta_t(X)P_2)P = 0$. Therefore we obtain $\alpha_{s+t}(X) = P\beta_s(P\beta_t(X)P)P = \alpha_s(\alpha_t(X))$. Also $\beta_s(Q\beta_t(P)Q) \leq \beta_s(Q\beta_t(P_1)Q) \leq \beta_s(QP_1Q) = \beta_s(P_2) \leq P_2 \leq Q$.

In the converse direction, take P_1 as the projection onto $\mathcal{H}_1 := \overline{\text{span}}\{\beta_t(X)x : 0 \leq t < \infty, X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{K}\}$. (It is important to note that here we have taken all $x \in \mathcal{K}$ and not just $x \in \mathcal{H}$.) For $0 \leq r, s < \infty, X, Y \in \mathcal{B}(\mathcal{H}), x, y \in \mathcal{K}$

$$|\beta_r(X)x\rangle\langle\beta_s(Y)y| = \beta_r(X)|x\rangle\langle y|\beta_s(Y^*).$$

Then it is clear that any operator of the form $\beta_r(X)Z\beta_s(Y)$, $Z \in \mathcal{B}(\mathcal{K})$ is in $\mathcal{B}(\mathcal{H}_1)$. We wish to show that this implies that $\beta_r(X) \in \mathcal{B}(\mathcal{H}_1)$ for $X \in \mathcal{B}(\mathcal{H})$. For $\epsilon > 0$ take $Z_\epsilon = (\beta_r(P)+\epsilon)^{-1}$. Then as ϵ decreases to zero $Z_\epsilon\beta_r(P)$ converges to the support projection G of $\beta_r(P)$ in norm and as $G\beta_r(X)G = \beta_r(X)$ it follows that $\beta_r(X)Z_\epsilon\beta_r(P)$ converges to $\beta_r(X)$. Therefore $\beta_r(X) \in \mathcal{B}(\mathcal{H}_1)$. Now for any $Z \in \mathcal{B}(\mathcal{K})$ and $0 \leq t < \infty$ by repeated application of Kadison's inequality

$$\begin{aligned}\beta_t(\beta_r(X)Z)\beta_t(Z^*\beta_r(X^*)) &\leq \beta_t(\beta_r(X)ZZ^*\beta_r(X^*))\\ &\leq \|Z\|^2\beta_t(\beta_r(X)\beta_r(X^*))\\ &\leq \|Z\|^2\beta_t(\beta_r(XX^*))\\ &= \|Z\|^2\beta_{t+r}(XX^*) \in \mathcal{B}(\mathcal{H}_1).\end{aligned}$$

Consequently $(1-P_1)\beta_t(\beta_r(X)Z) = 0$. In a similar way $\beta_t(Z\beta_r(X))(1-P_1) = 0$. It follows that for any $0 \leq r, s, t < \infty, X, Y \in \mathcal{B}(\mathcal{H}), x, y \in \mathcal{K}$,

$$\beta_t(|\beta_r(X)x\rangle\langle\beta_s(Y)y|) \in \mathcal{B}(\mathcal{H}_1)$$

and hence $\beta_t(Z) \in \mathcal{B}(\mathcal{H}_1)$ for $Z \in \mathcal{B}(\mathcal{H}_1)$ and in particular $\beta_t(P_1) \leq P_1$. Now take $P_2 = P_1 - P$. We wish to prove that $\beta_t(P_2) \leq P_2$. We already have $\beta_t(P_2) \leq \beta_t(P_1) \leq P_1$. Therefore it is enough to show that $P\beta_t(P_2)P = 0$. Now as range $(P_2) = \overline{\text{span}}\ \{Q\beta_r(X)x : 0 \leq r < \infty, X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{K}\}$, $P\beta_t(P_2)P = 0$ follows from

$$P\beta_t(|Q\beta_r(X)x\rangle\langle Q\beta_s(Y)y|)P = 0,$$

for all r, s, t, X, Y, x, y as above. This follows by yet another application of Kadison's inequality. Take $Z = |x\rangle\langle Q\beta_s(Y)y|$. Then

$$\begin{aligned}P\beta_t(Q\beta_r(X)Z)P(P\beta_t(Q\beta_r(X)Z)P)^* &\leq P\beta_t(Q\beta_r(X)ZZ^*\beta_r(X^*)Q)P\\ &\leq \|Z\|^2P\beta_t(Q\beta_r(XX^*)Q)P\\ &\leq \|Z\|^2\|X\|^2P\beta_t(Q\beta_r(P)Q)P\\ &= 0.\end{aligned}$$

■

COROLLARY 2.3 *Let β be an e_0-semigroup of $\mathcal{B}(\mathcal{K})$ and suppose α is a compression of β by a projection P (with range $P = \mathcal{H}$) as in THEOREM 2.2. Then α is an e_0-semigroup if and only if for every $X \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$, $\beta_t(X)$ leaves $\mathcal{H}$ invariant and $P = P_1 - P_2$, where $P_1 \geq P_2$ are projections such that $\beta_t(P_1) \leq P_1$ and $\beta_t(P_2) \leq P_2$.*

Proof: Taking $Q = 1 - P$, we have

$$\begin{aligned}\alpha_t(X^*X) &= P\beta_t(X^*)(P+Q)\beta_t(X)P\\ &= \alpha_t(X^*)\alpha_t(X) + P\beta_t(X^*)QQ\beta_t(X)P.\end{aligned}$$

From this relation it is clear that α_t is a $*$-endomorphism if and only if $Q\beta_t(X)P = 0$, that is, $\beta_t(X)$ leaves $\mathcal{H}$ invariant for all X.

Now if α is also an e_0-semigroup, we have

$$\begin{aligned}\alpha_{s+t}(P) &= P\beta_s(\beta_t(P))P = P\beta_s((P+Q)\beta_t(P)(P+Q))P \\ &= P\beta_s(P\beta_t(P)P)P + P\beta_s(Q\beta_t(P)Q)P \\ &= \alpha_s(\alpha_t(P)) + P\beta_s(Q\beta_t(P)Q)P.\end{aligned}$$

Therefore $P\beta_s(Q\beta_t(P)Q)P = 0$, or $\beta_s(Q\beta_t(P)Q) \leq Q$. So we obtain the decomposition of P from THEOREM 2.2 (ii). The converse statement is another simple application of the same result. ∎

There do exist quantum dynamical semigroups α, β, with projections $P, Q = 1 - P$ such that α is a compression of β by P, but $P\beta_t(Q\beta_s(P)Q)P$ is non-zero for some s, t, so that (α, β) do not come under the purview of THEOREM 2.2 (ii).

EXAMPLE 2.4 *First consider a discrete time example. Take $\mathcal{K} = \mathbb{C}^2$, with standard orthonormal basis $\{e_1, e_2\}$. Take $\mathcal{H} = \mathbb{C}e_1$. Define β by*

$$\beta(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \frac{1}{N}\begin{bmatrix} 2(a+d)-(b+c) & a \\ a & a \end{bmatrix} \tag{2.1}$$

where N is a suitable positive scalar to make β contractive. It is easily verified that β is completely positive on $\mathcal{B}(\mathcal{K})$ and

$$\beta^n(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \frac{x}{4}\left(\frac{2}{N}\right)^n\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.2}$$

with $x = 2(a+d) - (b+c)$, for $n \geq 2$. Let α be the compression of β to $\mathcal{B}(\mathcal{H})$:

$$\alpha(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = \frac{2}{N}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that α^n is the compression of β^n for all n. Clearly if P is the projection onto $\mathcal{H}$, and $Q = 1 - P$, $\beta(Q\beta(P)Q) \leq Q$ does not hold. A continuous time example can be got by considering contractive completely positive semigroups

$$\tau_t = e^{t(\alpha - M)}, \gamma_t = e^{t(\beta - M)} \quad for \quad t \geq 0,$$

where M is a suitable positive scalar to make τ, γ contractive. Clearly τ is a compression of γ. Making use of (2.1), (2.2) we have

$$\gamma_t(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = e^{-tM}\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \frac{t}{N}\begin{bmatrix} x & a \\ a & a \end{bmatrix} + \frac{x}{4}(e^{\frac{2t}{N}} - 1 - \frac{2t}{N})\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\},$$

and then it is immediate that $\gamma_s(Q\gamma_t(P)Q) \leq Q$ does not hold, in general.

This example also reveals a subtle point in the dilation theory of contractive quantum dynamical semigroups. Let θ be an e_0-semigroup of $\mathcal{B}(\mathcal{K})$. Suppose that it is a dilation of a contractive quantum dynamical semigroup τ of $\mathcal{B}(\mathcal{H})$ ($\mathcal{H} \subseteq \mathcal{K}$). Consider

$$\hat{\mathcal{H}} = \overline{\text{span}}\{\theta_{r_1}(Y_1)\cdots\theta_{r_n}(Y_n)u : r_1 \geq \cdots \geq r_n \geq 0, Y_1, \ldots, Y_n \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}, n \geq 0\}.$$

and the compression η of θ to $B(\hat{\mathcal{H}})$ defined by

$$\eta_t(X) = 1_{\hat{\mathcal{H}}}\theta_t(X)1_{\hat{\mathcal{H}}}, \quad X \in B(\hat{\mathcal{H}}).$$

The question we want to ask is whether η is the minimal dilation of τ.

First suppose τ is unital. Then from the relation $P = P\theta_s(P)P$, we have $P \leq \theta_s(P) \leq \theta_t(P)$ for $0 \leq s \leq t$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. By first considering vectors ξ_1, ξ_2, ξ_3 of the form $\theta_{r_1}(Y_1)\cdots\theta_{r_n}(Y_n)u$, with $r_1 \geq \cdots \geq r_n \geq 0$, it is easily seen that $\theta_t(|\xi_1\rangle\langle\xi_2|)\xi_3 \in \hat{\mathcal{H}}$ for $\xi_1, \xi_2, \xi_3 \in \hat{\mathcal{H}}$. It follows that $\theta_t(X)$ leaves $\hat{\mathcal{H}}$ invariant for every $X \in B(\hat{\mathcal{H}})$. Furthermore considering an orthonormal basis $\{e_k : 1 \leq k < \infty\}$ of $\mathcal{H}$, for $r_1 \geq \cdots \geq r_m \geq t \geq r_{m+1} \geq \cdots \geq r_n \geq 0$,

$$\begin{aligned}\theta_{r_1}(Y_1)\cdots\theta_{r_n}(Y_n)u &= \sum_k \theta_{r_1}(Y_1)\cdots\theta_{r_m}(Y_m)\theta(|e_k\rangle\langle e_k|)\theta_{r_{m+1}}(Y_{m+1})\cdots\theta_{r_n}(Y_n)u \\ &= \sum_k \theta_t(Z_k)\xi_k,\end{aligned}$$

where $Z_k = |\theta_{r_1-t}(Y_1)\cdots\theta_{r_m-t}(Y_m)e_k\rangle\langle e_k|$, $\xi_k = \theta_{r_{m+1}}(Y_{m+1})\cdots\theta_{r_n}(Y_n)u$. Therefore $\overline{\text{span}}$ $\{\theta_t(Z)\xi : Z \in B(\hat{\mathcal{H}}), \xi \in \hat{\mathcal{H}}\} = \hat{\mathcal{H}}$. Hence $1_{\hat{\mathcal{H}}}\theta_t(1_{\hat{\mathcal{H}}})1_{\hat{\mathcal{H}}} = 1_{\hat{\mathcal{H}}}$ and $\theta_t(1_{\hat{\mathcal{H}}}) \geq 1_{\hat{\mathcal{H}}}$. So in this case η is an e_0-semigroup (make use of COROLLARY 2.3) and it is the minimal dilation of τ (See [Ar3] for an alternative argument).

If τ is contractive but not unital it is possible that $\theta_t(X), X \in B(\mathcal{H})$ does not leave $\hat{\mathcal{H}}$ invariant. In such cases by COROLLARY 2.3 η is not an e_0-semigroup and hence not the minimal dilation of τ. An example of this kind is got by taking θ as the minimal dilation of the quantum dynamical semigroup γ of EXAMPLE 2.4. Note that θ is then a dilation of τ as well. We wish to show that η defined for the pair (τ, θ) is not the minimal dilation of τ. From [Bh3] we know that if η is the minimal dilation of τ then

$$\eta_s(X)\eta_t(Y)P = \eta_s(X\tau_{t-s}(Y))P$$

for $0 \leq s \leq t < \infty, X, Y \in B(\mathcal{H})$. In particular, $\eta_s(P)\eta_t(P)P = \eta_s(P\tau_{t-s}(P))P$. From COROLLARY 2.3, if η is an e_0-semigroup $\theta_t(X)$ must leave $\hat{\mathcal{H}}$ invariant for all $X \in B(\hat{\mathcal{H}}), t \geq 0$. Now recalling the definition of η we obtain, $\theta_s(P)\theta_t(P)P = \theta_s(P\tau_{t-s}(P))P$. We arrive at a contradiction by showing that this equality is not true for some $0 < s < t < \infty$. Consider

$$\begin{aligned}&[\theta_s(P)\theta_t(P)P - \theta_s(P\tau_{t-s}(P))P]^*[\theta_s(P)\theta_t(P)P - \theta_s(P\tau_{t-s}(P))P] \\ &= P\theta_t(P)\theta_s(P)\theta_t(P)P - P\theta_t(P)\theta_s(P\theta_{t-s}(P)P)P \\ &-P\theta_s(P\theta_{t-s}(P)P)\theta_t(P)P + P\theta_s(P\theta_{t-s}(P)P\theta_{t-s}(P)P)P\end{aligned}$$

Now this is computable in terms of γ as θ is the minimal dilation of γ, and $P \in B(\mathcal{H}) \subsetneq B(\mathcal{K})$, and we obtain

$$P\gamma_s[\gamma_{t-s}(P)P\gamma_{t-s}(P) - \gamma_{t-s}(P)P\gamma_{t-s}(P)P - P\gamma_{t-s}(P)P\gamma_{t-s}(P) + P\gamma_{t-s}(P)P\gamma_{t-s}(P)P]P,$$

which can be seen to be non-zero.

3 Isometric Dilation

Definition 3.1 *Let $R = \{R_t\}_{t\geq 0}$ be a left cocycle with respect to an E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{H})$. Suppose $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ and $\psi = \{\psi_t\}_{t\geq 0}$ is an E_0-semigroup of $\mathcal{B}(\mathcal{K})$ such that $\psi_t(X) = \theta_t(X), X \in \mathcal{B}(\mathcal{H}), t \geq 0$ (ψ is an extension of θ). A left cocycle $V = \{V_t\}_{t\geq 0}$ of ψ is said to be a* dilation *of R if $\langle u, V_t v\rangle = \langle u, R_t v\rangle, u, v \in \mathcal{H}, t \geq 0$ or equivalently*

$$R_t = PV_tP, \tag{3.1}$$

(P being the projection of $\mathcal{K}$ onto $\mathcal{H}$). It is said to be isometric *if each V_t is isometric and is said to be* minimal *if $\overline{span}\{V_t u : u \in \mathcal{H}, 0 \leq t < \infty\} = \mathcal{K}$.*

Throughout this section we assume that we have a contractive left cocycle R with respect to an E_0-semigroup θ on $\mathcal{B}(\mathcal{H})$. The aim is to study existence, uniqueness properties of minimal isometric dilations of R. The first step in this direction is the following lemma.

Lemma 3.2 *Let $\mathcal{M}$ be the cartesian product $\mathbb{R}_+ \times \mathcal{H}$. Define $K : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$, by*

$$K((s,u),(t,v)) = \begin{cases} \langle u, v\rangle & \text{if } s = t, \\ \langle u, \theta_s(R_{t-s})v\rangle & \text{if } s < t, \\ \langle u, \theta_t(R^*_{s-t})v\rangle & \text{if } s > t. \end{cases}$$

Then K is a positive definite kernel.

Proof: For $c_1, c_2, \ldots, c_n \in \mathbb{C}, (s_1, u_1), \ldots, (s_n, u_n) \in \mathcal{M}$ we want to show that

$$\sum \bar{c}_i c_j K((s_i, u_i), (s_j, u_j)) \geq 0.$$

Without loss of generality we can assume $0 \leq s_1 \leq \cdots \leq s_n < \infty$. Then positive definiteness of K follows easily if we show that block matrix operators $A = [A_{ij}]$, defined by

$$A_{ij} = \begin{cases} I & \text{if } i = j, \\ \theta_{s_i}(R_{s_j - s_i}) & \text{if } i < j, \\ \theta_{s_j}(R^*_{s_i - s_j}) & \text{if } i > j. \end{cases}$$

are positive for all $0 \leq s_1 \leq \cdots \leq s_n < \infty$. We prove this by induction on n. The result is trivial for $n = 1$. For $n > 1$, we decompose A as

$$A = DBD^* + E \tag{3.2}$$

where

$$D_{ij} = \begin{cases} \theta_{s_1}(R_{s_2 - s_1}) & \text{if } i = 1, j = 2, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

$$B_{ij} = \begin{cases} 0 & \text{if } i = 1, \text{ or } j = 1, \\ A_{ij} & \text{otherwise.} \end{cases}$$

$$E_{ij} = \begin{cases} 1 - \theta_{s_1}(R_{s_2 - s_1}R^*_{s_2 - s_1}) & \text{if } i = 1, j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here B is positive by induction hypothesis and E is positive as $R_{s_2-s_1}$ is a contraction. Finally (3.2) can be verified as follows:

$$(DBD^*)_{ij} = \sum_{k,l} D_{ik}B_{kl}(D^*)_{lj} = \sum_{k,l\geq 2} D_{ik}A_{kl}D^*_{jl}.$$

If $i > 1, j > 1$, $(DBD^*)_{ij} = \sum_{k,l\geq 2}\delta_{ik}A_{kl}\delta_{jl} = A_{ij}$. If $i = 1, j = 1$, $(DBD^*)_{ij} = D_{12}A_{22}D^*_{12} = \theta_{s_1}(R_{s_2-s_1}R^*_{s_2-s_1})$. If $i = 1, j > 1$,

$$\begin{aligned}(DBD^*)_{ij} &= D_{12}A_{2j}\delta_{jl} = \theta_{s_1}(R_{s_2-s_1})\theta_{s_2}(R_{s_j-s_2}) \\ &= \theta_{s_1}(R_{s_2-s_1}\theta_{s_2-s_1}(R_{s_j-s_2})) = \theta_{s_1}(R_{s_j-s_1}) = A_{ij}.\end{aligned}$$

Similarly if $i > 1, j = 1$, $(DBD^*)_{ij} = A_{ij}$. ∎

Let $\tilde{\mathcal{H}}$ be the GNS Hilbert space of positive definite kernel (see Section 15 of [Pa]) K of LEMMA 3.2 , with associated embedding $\lambda : \mathcal{M} \to \tilde{\mathcal{H}}$ satisfying

$$\langle\lambda((s,u)),\lambda((t,v))\rangle = K((s,u),(t,v)),$$

for $(s,u),(t,v) \in \mathcal{M}$, and $\overline{\text{span}}\ \{\lambda((s,u)) : 0 \leq s < \infty, u \in \mathcal{H}\} = \tilde{\mathcal{H}}$. As

$$\|\lambda(s,u) - \lambda(t,v)\|^2 = \|u\|^2 + \|v\|^2 - \langle u, \theta_s(R_{t-s})v\rangle - \langle v, \theta_s(R^*_{t-s})u\rangle$$

for $s \leq t$, by making use of the joint continuity of the map $(s, X) \mapsto \theta_s(X)$ on $\mathbb{R}_+ \times \mathcal{B}(\mathcal{H})$, with usual topology on $\mathbb{R}_+$ and σ-weak topology on $\mathcal{B}(\mathcal{H})$ (see Lemma 3.2 of [AM]) we can see that the mapping $\lambda : \mathbb{R}_+ \times \mathcal{H} \to \tilde{\mathcal{H}}$ is continuous. Then it is clear that the Hilbert space $\tilde{\mathcal{H}}$ is separable. For $t \geq 0$, define $\tilde{R}_t : \mathcal{H} \to \tilde{\mathcal{H}}$ by $\tilde{R}_t u = \lambda(t,u)$, $u \in \mathcal{H}$. Then it follows that $\tilde{R}_t$ are isometries , satisfying

$$\tilde{R}^*_s\tilde{R}_t = \begin{cases} I & \text{if } s = t, \\ \theta_s(R_{t-s}) & \text{if } s < t, \\ \theta_t(R^*_{s-t}) & \text{if } s > t. \end{cases}$$

Moreover $\overline{\text{span}}\ \{\tilde{R}_s u : 0 \leq s < \infty, u \in \mathcal{H}\} = \tilde{\mathcal{H}}$ and the map $t \to \tilde{R}_t$ is continuous in strong operator topology.

Making use of the family of operators $\{\tilde{R}_t\}$ we will construct an e_0-semigroup on $\mathcal{B}(\tilde{\mathcal{H}})$. For this purpose we need to recall some constructions from [Bh2] and [Bh4]. Fix $a \in \mathcal{H}$ with $\|a\| = 1$. Take $\mathcal{P}_t = \text{range}\ \theta_t(|a\rangle\langle a|)$. Define $W_t : \mathcal{H} \otimes \mathcal{P}_t \to \mathcal{H}$ by setting

$$W_t(u \otimes \theta_t(|a\rangle\langle a|)v) = \theta_t(|u\rangle\langle a|)v, \quad \text{for } u, v \in \mathcal{H},$$

and extending linearly. Then W_t is a unitary operator. Moreover

$$\theta_t(X) = W_t(X \otimes 1_{\mathcal{P}_t})W^*_t, \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

Thus θ_t is expressed as the identity representation with multiplicity. We will call W_t as *canonical implementing unitary* of θ_t with ground vector a.

Further if $U_{s,t}$ denotes the restriction of W_t to domain $\mathcal{P}_s \otimes \mathcal{P}_t$ and range $\mathcal{P}_{s+t}$, then it is a unitary operator. Moreover for $r, s, t \geq 0$,

$$(i) \quad W_s(W_t \otimes 1_{\mathcal{P}_s}) = W_{s+t}(1_{\mathcal{H}} \otimes U_{t,s}),$$

$$(ii) \quad U_{r,s+t}(1_{\mathcal{P}_r} \otimes U_{s,t}) = U_{r+s,t}(U_{r,s} \otimes 1_{\mathcal{P}_t}).$$

The Hilbert spaces $\{\mathcal{P}_t\}$ with unitaries $U_{s,t}$ form a continuous tensor product system of Hilbert spaces. This is anti-isomorphic to the product system Arveson associates with the E_0-semigroup θ. This construction of maps $\{W_t\}$, $\{U_{s,t}\}$ and product system $\{\mathcal{P}_t\}$ also works for e_0-semigroups, the only difference being that maps $\{W_t\}$ are no longer unitaries but just isometries.

THEOREM 3.3 *There exists unique e_0-semigroup $\tilde{\theta}$ of $\mathcal{B}(\tilde{\mathcal{H}})$ satisfying*

$$\tilde{\theta}_t(\tilde{R}_r X \tilde{R}_s^*) = \tilde{R}_{r+t}\theta_t(X)\tilde{R}_{s+t}^*, \quad \forall X \in \mathcal{B}(\mathcal{H}), r, s, t \geq 0. \tag{3.3}$$

Proof: Define a linear map $\tilde{W}_t$ from $\tilde{\mathcal{H}} \otimes \mathcal{P}_t$ to $\tilde{\mathcal{H}}$ by,

$$\tilde{W}_t(\tilde{R}_s u \otimes \theta_t(|a\rangle\langle a|)v) = \tilde{R}_{s+t}\theta_t(|u\rangle\langle a|)v$$

for $0 \leq s < \infty, u, v \in \mathcal{H}$. It is easily verified that $\tilde{W}_t$ as defined is isometric and as $\overline{\text{span}}\{\tilde{R}_s u : 0 \leq s < \infty, u \in \mathcal{H}\} = \tilde{\mathcal{H}}$, $\tilde{W}_t$ extends uniquely to an isometry from $\tilde{\mathcal{H}} \otimes \mathcal{P}_t$ to $\tilde{\mathcal{H}}$. Define $\tilde{\theta}_t$ by

$$\tilde{\theta}_t(Z) = \tilde{W}_t(Z \otimes 1_{\mathcal{P}_t})\tilde{W}_t^*, \quad Z \in \mathcal{B}(\tilde{\mathcal{H}}).$$

Clearly each $\tilde{\theta}_t$ is a normal $*$-endomorphism. We claim that they form a semigroup. It is easy to see that

$$\tilde{W}_t(\tilde{W}_s \otimes 1_{\mathcal{P}_t}) = \tilde{W}_{s+t}(1_{\tilde{\mathcal{H}}} \otimes U_{s+t})$$

on $\mathcal{H} \otimes \mathcal{P}_s \otimes \mathcal{P}_t$, and so

$$\begin{aligned}
\tilde{\theta}_{s+t}(Z) &= \tilde{W}_{s+t}(Z \otimes 1_{\mathcal{P}_{s+t}})\tilde{W}_{s+t}^* \\
&= \tilde{W}_{s+t}(1_{\tilde{\mathcal{H}}} \otimes U_{s,t})(Z \otimes 1_{\mathcal{P}_s} \otimes 1_{\mathcal{P}_t})(1_{\tilde{\mathcal{H}}} \otimes U_{s,t}^*)\tilde{W}_{s+t}^* \\
&= \tilde{W}_t(\tilde{W}_s \otimes 1_{\mathcal{P}_t})(Z \otimes 1_{\mathcal{P}_s} \otimes 1_{\mathcal{P}_t})(\tilde{W}_s^* \otimes 1_{\mathcal{P}_t})\tilde{W}_t^* \\
&= \tilde{\theta}_t(\tilde{\theta}_s(Z)).
\end{aligned}$$

Continuity in weak operator topology of maps $t \to \tilde{\theta}_t(Z)$, follows from measurability of maps $t \to \langle x, \tilde{\theta}_t(Z)y\rangle$ for $x, y \in \tilde{\mathcal{H}}, Z \in \mathcal{B}(\mathcal{H})$ and measurability can be proved directly, but this approach seems to be rather tedious. Instead one can also prove this by making use of continuity of $t \mapsto V_t$ in THEOREM 3.5.

We also note that

$$\tilde{W}_t(\tilde{R}_s u \otimes \theta_t(|a\rangle\langle a|)v) = \tilde{R}_{s+t}\theta_t(|u\rangle\langle a|)v = \tilde{R}_{s+t}W_t(u \otimes \theta_t(|a\rangle\langle a|)v).$$

It follows that on $\mathcal{H} \otimes \mathcal{P}_t$

$$\tilde{W}_t(\tilde{R}_s \otimes 1_{\mathcal{P}_t}) = \tilde{R}_{s+t}W_t.$$

and therefore (3.3) is satisfied. The uniqueness is clear on rank one operators $X = |u\rangle\langle v|$, $u, v \in \mathcal{H}$, as $\overline{\text{span}}\{\tilde{R}_s u : 0 \leq s < \infty, u \in \mathcal{H}\} = \tilde{\mathcal{H}}$, and due to normality assumption on $\tilde{\theta}_t$, it is determined by its action on rank one operators. ■

Note that $\{\tilde{R}_t\}$ do not form a cocycle because they are maps from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ and not from $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{H}}$. So we must extend their definition and also to talk of cocycles we need E_0-semigroups on $\mathcal{B}(\tilde{\mathcal{H}})$. This is quite unlike the situation of semigroups of operators where we are having the trivial (identity) semigroup as our E_0-semigroup and we naturally extend it to the identity semigroup. An immediate consequence of next lemma is that $\mathcal{H}^\perp$ in $\tilde{\mathcal{H}}$ is infinite dimensional (unless all $\{R_t\}$ are isometric and in which case we need not dilate them). So there are plenty of E_0-semigroups on $\mathcal{B}(\tilde{\mathcal{H}})$ which extend θ. For example identify $\tilde{\mathcal{H}}$ with $\mathcal{H} \otimes \mathcal{K}$ for some infinite dimensional Hilbert space $\mathcal{K}$ and identify $\mathcal{H}$ with $\mathcal{H} \otimes e$ for some unit vector e of $\mathcal{K}$, so as to have $\mathcal{H} \subseteq \mathcal{K}$. Then taking $\psi_t = \theta_t \otimes id$, we have ψ as an E_0-semigroup extension of θ.

Lemma 3.4 *Let Q be the projection onto $\mathcal{H}^\perp$. Then $Q = 0$ if and only if $\{R_t\}$ is isometric for every t. If $Q \neq 0$ then $Q > \tilde{\theta}_s(Q) > \tilde{\theta}_t(Q) > 0$, for $0 < s < t < \infty$.*

Proof: If $Q = 0$ then $R_t = \tilde{R}_t$, and hence R_t is isometric. On the other hand if R_t is isometric for every t, we see that

$$\langle R_s u, R_t v\rangle = K((s,u),(t,v)) \quad \text{for} \quad 0 \leq s,t < \infty,$$

where K is the positive definite kernel of LEMMA 3.2. Then by uniqueness of GNS construction the Hilbert space $\tilde{\mathcal{H}}$ is same as $\mathcal{H}$.

Now assume $Q \neq 0$. First we claim that $\tilde{\theta}_t(Q) \leq Q$. This follows by arguments similar to the last part in the proof of THEOREM 2.1, by first showing:

$$P\tilde{\theta}_t(|\tilde{R}_r u - R_r u\rangle\langle\tilde{R}_s v - R_s v|)P = 0,$$

for $0 \leq r, s < \infty$, $u, v \in \mathcal{H}$, $P = 1 - Q$. Hence for $0 < s < t < \infty$,

$$Q \geq \tilde{\theta}_s(Q) \geq \tilde{\theta}_t(Q) \geq 0. \tag{3.4}$$

If $Q = \tilde{\theta}_{t_0}(Q)$ for some $t_0 > 0$ we have $Q = \tilde{\theta}_t(Q) = \tilde{\theta}_{t_0}(Q)$, for $0 \leq t \leq t_0$, and hence $Q = \tilde{\theta}_t(Q)$ for all t. We claim that this is not possible. Indeed if $Q = \tilde{\theta}_t(Q)$ then $P \geq \tilde{\theta}_t(P) = \tilde{R}_t P \tilde{R}_t^*$, and therefore $Q\tilde{R}_t P \tilde{R}_t Q = 0$ or $Q\tilde{R}_t P = 0$. That is $\tilde{R}_t$ leaves $\mathcal{H}$ invariant for every t. So we obtain $\tilde{\mathcal{H}} = \mathcal{H}$, contradicting the assumption $Q \neq 0$. Now recalling that every non-trivial endomorphism of $\mathcal{B}(\tilde{\mathcal{H}})$ is injective we obtain strict inequality in (3.4). ■

Here is our main theorem.

Theorem 3.5 *Let $R = \{R_t : 0 \leq t < \infty\}$ be a left cocycle with respect to an E_0-semigroup θ of $\mathcal{B}(\mathcal{H})$ and let $\tilde{\mathcal{H}}$ be the Hilbert space constructed before with associated operators $\{\tilde{R}_t : 0 \leq t < \infty\}$. Let ψ be an E_0-semigroup of $\mathcal{B}(\tilde{\mathcal{H}})$, such that $\psi_t(X) = \theta_t(X)$ for $X \in \mathcal{B}(\mathcal{H}), t \geq 0$. Then there exists a unique isometric left cocycle $V = \{V_t : 0 \leq t < \infty\}$ of ψ such that*

$$V_t u = \tilde{R}_t u, \quad u \in \mathcal{H}, 0 \leq t < \infty.$$

(In particular V is a minimal isometric dilation of R.) Furthermore, irrespective of the choice of ψ,

$$V_t\psi_t(Z)V_t^* = \tilde{\theta}_t(Z), \quad \forall Z \in \mathcal{B}(\tilde{\mathcal{H}}),$$

where $\tilde{\theta}$ is the e_0-semigroup of THEOREM 2.3.

Proof: Fix $a \in \mathcal{H}$, with $\|a\| = 1$, and consider $\mathcal{P}_t, W_t, \tilde{W}_t$ as before. Define linear maps $\bar{W}_t : \tilde{\mathcal{H}} \otimes \mathcal{P}_t \to \tilde{\mathcal{H}}$ by

$$\bar{W}_t(x \otimes \theta_t(|a\rangle\langle a|)v = \psi_t(|x\rangle\langle a|)v, \quad x \in \tilde{\mathcal{H}}, v \in \mathcal{H}.$$

Clearly $\bar{W}_t$ are implementing unitaries for the E_0-semigroup ψ, and therefore

$$\psi_t(Z) = \bar{W}_t(Z \otimes 1_{\mathcal{P}_t})\bar{W}_t^*, \quad \forall Z \in \mathcal{B}(\tilde{H}).$$

Take $V_t = \tilde{W}_t\bar{W}_t^*$, for $0 \leq t < \infty$. Obviously $\bar{W}_t$ is same as W_t on $\mathcal{H} \otimes \mathcal{P}_t$. Also recall that W_t is a unitary from $\mathcal{H} \otimes \mathcal{P}_t$ to $\mathcal{H}$. Hence for $u \in \mathcal{H}$, there exists $\xi \in \mathcal{H} \otimes \mathcal{P}_t$ such that $\bar{W}_t\xi = W_t\xi = u$. Therefore for $u \in \mathcal{H}$,

$$V_tu = \tilde{W}_t\bar{W}_t^*\bar{W}_t\xi = \tilde{W}_t\xi = \tilde{W}_t(\tilde{R}_0 \otimes 1_{\mathcal{P}_t})\xi = \tilde{R}_tW_t\xi = \tilde{R}_tu.$$

$(\tilde{R}_0 = 1_{\mathcal{H}})$. Evidently $\tilde{\theta}_t(Z) = V_t\psi_t(Z)V_t^*, \quad \forall Z \in \mathcal{B}(\tilde{\mathcal{H}})$. We need to show that $\{V_t\}$ is a cocycle for ψ. We have

$$\begin{aligned}
V_s\psi_s(V_t) &= \tilde{W}_s\bar{W}_s^*\bar{W}_s(V_t \otimes 1_{\mathcal{P}_s})\bar{W}_s^* \\
&= \tilde{W}_s(\tilde{W}_t\bar{W}_t^* \otimes 1_{\mathcal{P}_s})\bar{W}_s^* \\
&= \tilde{W}_s(\tilde{W}_t \otimes 1_{\mathcal{P}_s})(\bar{W}_t^* \otimes 1_{\mathcal{P}_s})\bar{W}_s^* \\
&= \tilde{W}_{s+t}(1_{\tilde{\mathcal{H}}} \otimes U_{t,s})(1_{\tilde{\mathcal{H}}} \otimes U_{t,s}^*)\bar{W}_{s+t}^* \\
&= \tilde{W}_{s+t}\bar{W}_{s+t}^* = V_{s+t}.
\end{aligned}$$

It is also easily verified that $V_tV_t^* = \tilde{\theta}_t(1)$, and $V_t^*V_t = \psi_t(1) = 1$. In view of [Arl], Proposition 2.5, (also see the Appendix A of [Bh4]) to show the continuity of the map $t \mapsto V_t$, it suffices to show weak measurability, i.e, the measurability of the map $t \mapsto \langle y, V_tz\rangle$, for all $y, z \in \tilde{\mathcal{H}}$. Clearly it suffices to consider y, z in a total set. If $z = \bar{W}_t(\tilde{R}_sx \otimes \theta_t(|a\rangle\langle a|)v)$, for some $s \geq 0, x \in \mathcal{H}, v \in \mathcal{H}$, then

$$V_tz = \tilde{W}_t\bar{W}_t^*\bar{W}_t(\tilde{R}_sx \otimes \theta_t(|a\rangle\langle a|)v) = \tilde{W}_t(\tilde{R}_sx \otimes \theta_t(|a\rangle\langle a|)v) = \tilde{R}_{s+t}\theta_t(|x\rangle\langle a|)v$$

and the measurability of $t \mapsto \langle y, V_tz\rangle$ is clear.

Finally we need to prove the uniqueness of V. Let $A = \{A_t : 0 \leq t < \infty\}$ be another cocycle of ψ such that $A_tu = \tilde{R}_tu$ for $u \in \mathcal{H}$. Now for $r, s \geq 0, X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}
A_t\psi_t(\tilde{R}_rX\tilde{R}_s^*)A_t^* &= A_t\psi_t(A_rXA_s^*)A_t^* \\
&= A_{r+t}\psi_t(X)A_{s+t}^* \\
&= A_{r+t}\theta_t(X)A_{s+t}^* \\
&= \tilde{R}_{r+t}\theta_t(X)\tilde{R}_{s+t}^*.
\end{aligned}$$

Hence by uniqueness of $\tilde{\theta}$,

$$\tilde{\theta}_t(Y) = A_t\psi_t(Y)A_t^* = V_t\psi_t(Y)V_t^*, \tag{3.5}$$

where $\{A_t\}, \{V_t\}$ are isometric cocycles of ψ. This implies $A_t = V_tU_t$, for a local unitary cocycle $U = \{U_t\}$ of ψ_t. Indeed take $U_t = V_t^*A_t$, then elementary computations using (3.5) and the observation $A_tA_t^* = V_tV_t^* = \tilde{\theta}_t(1)$, show this fact. In the present context we also have $A_t1_{\mathcal{H}} = V_t1_{\mathcal{H}}$, and we wish to show that this forces $U_t = V_t^*A_t$ to be the identity operator for every t.

Now U is local for ψ and hence $U_t = \tilde{W}_t(1_{\tilde{\mathcal{H}}} \otimes \tilde{U}_t)\tilde{W}_t^*$, for some unitary $\tilde{U}_t \in \mathcal{B}(\mathcal{P}_t)$. Hence

$$\begin{aligned} V_t1_{\mathcal{H}} = A_t1_{\mathcal{H}} &= A_t\tilde{W}_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})\tilde{W}_t^* \\ &= V_tU_t\tilde{W}_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})\tilde{W}_t^* \\ &= V_t\tilde{W}_t(1_{\mathcal{H}} \otimes 1_{\tilde{U}_t})\tilde{W}_t^*\tilde{W}_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})\tilde{W}_t^* \\ &= V_t\tilde{W}_t(1_{\mathcal{H}} \otimes \tilde{U}_t)\tilde{W}_t^* \\ &= V_tW_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})W_t^*. \end{aligned}$$

Also, $V_t1_{\mathcal{H}} = V_tW_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})W_t^*$. Hence

$$V_tW_t(1_{\mathcal{H}} \otimes U_t)W_t^* = V_tW_t(1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t})W_t^*,$$

or $(1_{\mathcal{H}} \otimes U_t) = 1_{\mathcal{H}} \otimes 1_{\mathcal{P}_t}$, and hence $\tilde{U}_t = 1_{\mathcal{P}_t}$, consequently $A_t = V_t$, for all t. ∎

An interesting consequence of LEMMA 3.2 is the following result which says that as block matrix operator with respect to the decomposition $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, V_t has the form

$$V_t = \begin{bmatrix} R_t & 0 \\ * & * \end{bmatrix},$$

just as it happens for the usual Sz. Nagy dilation of contraction semigroups.

REMARK 3.6 *Let V_t be the minimal isometric dilation as in THEOREM 2.9. Then V_t^* leaves $\mathcal{H}$ invariant for every t.*

Proof: For $0 \leq t < \infty$, we have $\tilde{\theta}_t(Q) \leq Q$. Hence

$$\begin{aligned} V_t\psi_t(Q)V_t^* &\leq Q, \\ V_tQV_t^* &\leq Q, \\ PV_tQV_t^*P &\leq 0. \end{aligned}$$

Therefore $QV_t^*P = 0$. ∎

4 Application

The most basic goal of the study of e_0-semigroups is their classification. The first step in this direction is to define cocycle conjugacy for E_0-semigroups (Earlier this concept used to be called as outer conjugacy, see [Ar1], [Po1], [Po4]). We may extend this notion to include e_0-semigroups as follows.

DEFINITION 4.1 *Let α, β be two e_0-semigroups on $B(\mathcal{H})$. Then α, β are said to be* cocycle conjugate *if there exists a left cocycle $V = \{V_t : 0 \le t < \infty\}$ of β such that*

$$\alpha_t(X) = V_t\beta_t(X)V_t^* \quad \forall X \in B(\mathcal{H}), t \ge 0$$

and $V_tV_t^* = \alpha_t(1), V_t^*V_t = \beta_t(1)$ for all $t \ge 0$. This is denoted by $\alpha \sim \beta$.

Note that if $\alpha \sim \beta$ are E_0-semigroups then they are cocycle conjugate through unitaries but in general we are having partial isometries. For example in THEOREM 3.3 θ and $\tilde{\theta}$ are cocycle conjugate through isometries $\{V_t\}$. It is easy to see that cocycle conjugacy is an equivalence relation. Extending Arveson's result [Ar1] we see that two e_0-semigroups on $B(\mathcal{H})$ are cocycle conjugate if and only if they have isomorphic product systems.

At times two e_0-semigroups one wishes to compare could be acting on operators of two different Hilbert spaces. If the two Hilbert spaces are of same dimension then we can still talk of cocycle conjugacy by first identifying two Hilbert spaces through a unitary isomorphism or equivalently by first replacing one of the e_0-semigroups by a unitary conjugate to have both the e_0-semigroups acting on a common space. Of course in either approach it doesn't matter which unitary we choose. Two e_0-semigroups are cocycle conjugate if and only if basic Hilbert spaces are of same dimension and the product systems are isomorphic. (In finite dimensions only e_0-semigroups we have are semigroups of automorphisms and they have trivial product systems.)

Through out this Section we consider the set up of Section 3 with τ as the contractive quantum dynamical semigroup, $\tau_t(X) = R_t\theta_t(X)R_t^*$ on $B(\mathcal{H})$, where R is a left cocycle of an E_0-semigroup θ of $B(\mathcal{H})$. Then we see that $\tilde{\theta}$ of THEOREM 3.3 is a dilation of τ. One may ask as to whether $\tilde{\theta}$ is the minimal dilation of τ. In general this is not the case, in fact as we will see, usually the minimal dilation $\hat{\tau}$ of τ and $\tilde{\theta}$ have non-isomorphic product systems, whereas θ and $\tilde{\theta}$ have isomorphic product systems. The product system of $\hat{\tau}$ is a 'sub-system' of that of θ.

The following proposition is a surprise in view of the 'non-minimal' example towards the end of Section 2.

PROPOSITION 4.2 *Suppose*

$$\hat{\mathcal{H}} = \overline{span}\{\tilde{\theta}_{r_1}(Y_1)\cdots\tilde{\theta}_{r_n}(Y_n)u : r_1 \ge \cdots \ge r_n \ge 0, Y_1, \ldots, Y_n \in B(\mathcal{H}), u \in \mathcal{H}, n \ge 0\}.$$

Let η be the compression of $\tilde{\theta}$ to $B(\hat{\mathcal{H}})$. Then η is the minimal dilation $\hat{\tau}$ of τ.

Proof: We observe that for $a, b, c \ge 0$, and $X, Y, Z \in B(\mathcal{H})$

$$\tilde{\theta}_a(X)\tilde{\theta}_b(Y)\tilde{\theta}_c(Z) = \begin{cases} \tilde{\theta}_a(X\tau_{b-a}(Y))\tilde{\theta}_c(Z) & \text{if } b \ge a \ge c \\ \tilde{\theta}_a(X)\tilde{\theta}_c(\tau_{b-c}(Y)Z) & \text{if } b \ge c \ge a \\ \tilde{\theta}_a(X\tau_{b-a}(Y)Z) & \text{if } b \ge a = c \end{cases}$$

For instance if $b \ge a \ge c$,

$$\tilde{\theta}_a(X)\tilde{\theta}_b(Y)\tilde{\theta}_c(Z) = V_a\theta_a(X)V_a^*V_b\theta_b(Y)V_b^*V_c\theta_c(Z)V_c^*$$

$$
\begin{aligned}
&= V_a\theta_a(X)\theta_a(R_{b-a})\theta_b(Y)\theta_c(R^*_{b-c})\theta_c(Z)V_c^* \\
&= V_a\theta_a(X)\theta_a(R_{b-a})\theta_b(Y)\theta_c([\theta_{a-c}(R^*_{b-a})R^*_{a-c}])\theta_c(Z)V_c^* \\
&= V_a\theta_a(X\tau_{b-a}(Y))\theta_c(R^*_{a-c})\theta_c(Z)V_c^* \\
&= \tilde{\theta}_a(X\tau_{b-a}(Y))\tilde{\theta}_c(Z).
\end{aligned}
$$

This shows that $\tilde{\theta}_t(X)$ leaves $\hat{\mathcal{H}}$ invariant for $t \geq 0, X \in \mathcal{B}(\mathcal{H})$, and hence on $\hat{\mathcal{H}}$ the action of $\eta_t(X)$ is same as that of $\tilde{\theta}_t(X)$. Moreover we are able to compute (see [Bh1]) inner products between vectors of the form $\tilde{\theta}_{r_1}(Y_1)\cdots\tilde{\theta}_{r_n}(Y_n)u : r_1 \geq \cdots \geq r_n \geq 0, Y_1, \ldots, Y_n \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}, n \geq 0$ and we see that

$$\langle\tilde{\theta}_{r_1}(Y_1)\cdots\tilde{\theta}_{r_m}(Y_m)u, \tilde{\theta}_{s_1}(Z_1)\cdots\tilde{\theta}_{s_n}(Z_n)v\rangle = \langle\hat{\tau}_{r_1}(Y_1)\cdots\hat{\tau}_{r_m}(Y_m)u, \hat{\tau}_{s_1}(Z_1)\cdots\hat{\tau}_{s_n}(Z_n)v\rangle.$$

Then the result is clear due to uniqueness of the minimal dilation. ■

Now we claim that the product systems of θ and $\tilde{\theta}$ are isomorphic. Fix $a \in \mathcal{H}$, with $\|a\| = 1$, and take

$$\mathcal{P}_t = \text{ range } \theta_t(|a\rangle\langle a|), \quad \mathcal{Q}_t = \text{ range } \tilde{\theta}_t(|a\rangle\langle a|).$$

Take $A_t = V_t\theta_t(|a\rangle\langle a|)$. Then as $V_t\theta_t(|a\rangle\langle a|) = \tilde{\theta}_t(|a\rangle\langle a|)V_t$, A_t maps $\mathcal{P}_t$ to $\mathcal{Q}_t$. (A_t is nothing but the restriction of V_t to domain $\mathcal{P}_t$ and range $\mathcal{Q}_t$). It is elementary to see that A_t defines an isomorphism between product systems $\{\mathcal{P}_t\}, \{\mathcal{Q}_t\}$ of $\theta, \tilde{\theta}$. In particular if $\mathcal{H}$ is infinite dimensional, so that dim $\mathcal{H} =$ dim $\tilde{\mathcal{H}}$, then $\theta, \tilde{\theta}$ are cocycle conjugate.

THEOREM 4.3 *Suppose that $\mathcal{H}$ is infinite dimensional. Then the product system of $\hat{\tau}$ is isomorphic to a subsystem of that of θ and $\hat{\tau}$ is cocycle conjugate to $\theta_t(\cdot)E_t$ where E_t is a local projection cocycle of θ. If θ is of type I then so is $\hat{\tau}$. If θ is of type II then $\hat{\tau}$ is of type I or II. If θ is of type III then so is $\hat{\tau}$. In general index $\hat{\tau} \leq$ index θ.*

Proof: In view of PROPOSITION 4.2 we may take $\hat{\tau} = \eta$. Now if $\tilde{W}_t$ are implementing unitaries for $\tilde{\theta}$ with ground vector a. Taking $\tilde{\mathcal{P}}_t =$ range $\hat{\tau}_t(|a\rangle\langle a|)$, we have

$$\hat{\tau}_t(Y) = \tilde{W}_t(Y \otimes 1_{\tilde{\mathcal{P}}_t})\tilde{W}_t^*$$

for $Y \in \mathcal{B}(\hat{\mathcal{H}})$. Modifying the domain of definition we take

$$\tilde{\tau}_t(Z) = \tilde{W}_t(Z \otimes 1_{\tilde{\mathcal{P}}_t})\tilde{W}_t^*$$

for $Z \in \mathcal{B}(\tilde{\mathcal{H}})$. Then we see that $\tilde{\tau}$ is another e_0-semigroup (See the notion of induced semigroup in [Bh4]) and it is cocycle conjugate to $\hat{\tau}$ as they have identical product systems. Clearly,

$$\tilde{\tau}_t(Z) = \tilde{\theta}_t(Z)F_t$$

where F_t is the local projection cocycle given by $F_t = \tilde{\tau}_t(1)$. Now the first claim follows easily from cocycle conjugacy of θ and $\tilde{\theta}$. Projection local cocycles of type I E_0-semigroups have been described explicitly in [Bh4], combining it with the results here, it follows that $\hat{\tau}$ also must be of type I if θ is of type I . As the product system of $\hat{\tau}$ is isomorphic to a subsystem of that θ every unit of $\hat{\tau}$ gives us a unit of θ. Therefore if θ is of type III then so is $\tilde{\tau}$. Finally we show that θ or equivalently $\tilde{\theta}$ has a unit then $\tilde{\tau}$ also must have atleast one unit. This would settle the type **II** case, and we also get the inequality for indices as the product system of $\tilde{\tau}$ is isomorphic to a subsystem of that of θ. Here it is convenient to view units as one parameter semigroups of intertwining operators (See [Ar1], Section 6 of [Bh4]). Suppose $U = \{U_t\}$ is a unit of $\tilde{\theta}$, that is, U is a one parameter semigroup and $\tilde{\theta}_t(Z)U_t = U_tZ$ for all t, Z. Then taking $V = \{V_t\}$, where $V_t = F_tU_t$ it is easily seen that V is a semigroup as $F_sU_sF_tU_t = F_s\tilde{\theta}_t(F_t)U_sU_t = F_{s+t}U_{s+t}$, and each V_t intertwines with $\tilde{\tau}_t$ as $\tilde{\tau}_t(Z)V_t = \tilde{\theta}_t(Z)F_tU_t = F_t\tilde{\theta}_t(Z)U_t = F_tU_tZ$. ∎

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