Points of Weak-Norm Continuity in the Unit Ball of Banach Spaces

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Using the M-structure theory, we show that several classical function spaces and spaces of operators on them fail to have points of weak-norm continuity for the identity map on the unit ball. This gives a unified approach to several results in the literature that establish the failure of strong geometric structure in the unit ball of classical function spaces. Spaces covered by our result include the Bloch spaces, dual of the Bergman space L_a^1 and spaces of operators on them, as well as the space $C(T)/A$, where A is the disc algebra on the unit circle T. For any unit vector f in an infinite-dimensional function algebra A we explicitly construct a sequence $\{f_n\}$ in the unit ball of A that converges weakly to f but not in the norm.

Key Words: points of weak-norm continuity; function spaces; M-ideals.

INTRODUCTION

Let X be a Banach space and let X_1 denote its closed unit ball. The geometry and structure of the unit ball of function spaces and spaces of operators have received a lot of attention in the literature (see, e.g., [1, 3, 5, 10, 11]). In this paper we are interested in demonstrating the failure of strong extremal structures like strongly exposed and denting points (see [4] for the definitions) in the unit ball of several classical function spaces and spaces of operators on them. To do this we study the points of continuity of the identity map $i: (X_1, weak) \to (X_1, \| \ \|)$. We refer to these points as "points of weak-norm continuity." We note that in an infinite-dimensional space such a point is a unit vector. A well-known result of Lin *et al*. [9] asserts that $x_0 \in X_1$ is a denting point if and only if it is an extreme point and a point of weak-norm continuity (note that any strongly exposed point is a denting point). Thus by exhibiting the lack of points of weak-norm continuity we get a strong negative result.

A closed subspace $M \subset X$ is said to be an M-ideal if there is a linear projection P on X^* such that $\ker(P) = M^{\perp}$ and $\|P(f)\| + \|f - P(f)\| = \|f\|$ for all $f \in X^*$. See Chapters I, III, and VI of [6] for several examples of M-ideals from among function spaces and spaces of operators on them.

The main result of this paper shows that if $M \subset X$ is an M-ideal of infinite dimension such that $P(X^*)_1$ is weak* dense in X_1^* and M is of infinite codimension, then there are no points of weak-norm continuity in M_1 or in X_1. As an application we get that the unit balls of the little and big Bloch spaces and the predual and the dual of the Bergman space L_a^1 fail to have points of weak-norm continuity. We also show the lack of these points in the unit ball of certain injective tensor product spaces like $l^2 \otimes_\varepsilon l^2 \otimes_\varepsilon l^2$.

Let A be an infinite-dimensional function algebra. Beneker and Wiegerinck [1] have proved that there are no strongly exposed points in the unit ball of A. We recall from [4, Theorem V.10] that the existence of strongly exposed points implies the dentability of the unit ball and the existence of points of weak-norm continuity. Nygaard and Werner [10] have recently proved that the unit ball is not dentable, and it does not have any points of weak-norm continuity.

For an infinite compact Hausdorff space K and for any Banach space X this author has recently studied the structure of the unit ball of the space $WC(K, X)$ of X-valued functions on K that are continuous w.r.t weak topology, equipped with the supremum norm. It was shown in [11] that for any $f \in WC(K, X)$, $\|f\| = 1$, there exists a sequence $\{f_n\}$ in the unit ball of $WC(K, X)$ such that $f_n \to f$, in the weak topology but not in the norm. Thus there can be no strongly exposed or denting points in the unit ball of such a space.

In the concluding part of this note we indicate how to modify the arguments of [1] and [11] to show that for any unit vector $f \in A$ there exists a sequence $\{f_n\}$ in A_1 such that $f_n \to f$ weakly, but not in the norm. Thus the unit ball has no points of weak-norm sequential continuity.

We refer to the monographs [4] and [6] for the definitions of various geometric notions that we will be using in this paper. Since reflexive spaces always have denting and hence points of weak-norm continuity in the unit ball, we only consider these questions for non-reflexive spaces.

MAIN RESULTS

Our main result shows that a Banach space fails to have points of weak-norm continuity in the unit ball, depending on the position of an

M-ideal. We always consider a Banach space X as canonically embedded in its bidual X^{**}. We will be using several times a result of Hu and Lin ([7, Corollary 2.10]), which states that a point of weak-norm continuity of X_1 is a point of weak*-norm continuity of X_1^{**}. We also note that any point of weak*-norm continuity of X_1^{**} belongs to X_1 and is a point of weak-norm continuity of X_1.

When X is an M-ideal in its bidual (i.e., X is an M-embedded space in the terminology of [6, Chap. III]), then X and X^{**} satisfy the hypothesis of the following theorem (see Corollary III.3.7 in [6]).

THEOREM. *Let $M \subset X$ be an infinite-dimensional M-ideal such that the corresponding L-projection P: $X^* \to X^*$ satisfies the condition that $P(X^*)_1$ is weak* dense in X_1^*. Assume further that M is not of finite codimension in X. Then there are no points of weak-norm continuity in M_1 or in X_1.*

Proof. Since M is an M-ideal, any $m^* \in M^*$ has a unique norm-preserving extension to X, still denoted by m^* (see [6, Chap. I]). Define J: $X \to M^{**}$ by $J(x)(m^*) = P(m^*)(x)$ for $m^* \in M^*$ and $x \in X$. Since $\ker P = M^{\perp}$, this is a well-defined linear map. By our assumption of weak* density of the ball we get that J is an isometry. It is easy to see that $J \mid M$ is the canonical embedding of M in its bidual. Thus if $m_0 \in M_1$ is a point of weak-norm continuity, then it follows from the result of Hu and Lin quoted above that m_0 is a point of weak*-norm continuity and hence a point of weak-norm continuity of M_1^{**}. Therefore m_0 is a point of weak-norm continuity in X_1. Applying the result of Hu and Lin once more, we get that it is a point of weak*-norm continuity in X_1^{**}. Now since M is an M-ideal, $X^{**} = M^{**} \oplus (M^*)^{\perp}$ (l^∞ direct sum). Since $P^*(m_0) = m_0$, by the non-triviality and infinite-dimensionality of $(M^*)^{\perp}$ we can choose a net of unit vectors in $(M^*)^{\perp}$ that converges to zero weakly. Since P^* is an M-projection, by adding m_0 to this net we get a net in X_1^{**} that converges weakly to m_0, but not in the norm. Hence there are no points of weak-norm continuity in M_1.

We now show that there are no points of weak-norm continuity in X_1. Let $x_0 \in X_1$ be a point of weak-norm continuity. It again follows from the result of Hu and Lin quoted before that x_0 is a point of weak*-norm continuity of X_1^{**}.

We get the required contradiction by observing that $P^*(x_0)$ is a point of weak*-norm continuity of M_1^{**} and hence is a point of weak-norm continuity of M_1. To see our claim, let $\{m_\alpha^{**}\}$ be a net in the unit ball of M^{**} converging to $P^*(x_0)$, in the weak* topology. Since $M^{\perp\perp} = M^{**}$ is a weak* closed subspace we have the weak* convergence in X^{**} as well. Thus $m_\alpha^{**} + x_0 - P^*(x_0) \to x_0$ in the weak* topology. Since x_0 is a point of weak*-norm continuity of X_1^{**}, we have the norm convergence and hence the claim.

Remark 1. See [6, Example III.1.4] for several examples of M-embedded spaces. In particular we get that there are no strongly exposed or denting points in the unit balls of $C(T)/A$, the Bloch spaces B_0 and B (see also [3]). We also have from the remarks in [6, p. 150] that both the predual and the dual of the Bergman space L_a^1 (or B_1) fail to have points of weak-norm continuity in the unit ball. For $1 < p < \infty$, $\mathscr{K}(l^p)$ is an M-ideal in its bidual $\mathscr{L}(l^p)$ ([6, Chap. VI, Example 4.1]); thus there are no points of weak-norm continuity in their unit ball. This answers a question raised by the author in [12].

One major motivation for considering the above abstract formulation is its applicability to spaces of compact operators. Let X and Y be Banach spaces such that $\mathscr{K}(X,Y)$ is an M-ideal in the space $\mathscr{L}(X,Y)$ (see [6, Chap. VI] and [8] for several important examples of this situation). Let P be the L-projection in the dual such that $\ker(P) = \mathscr{K}(X,Y)^{\perp}$. For $x \in X_1$ and $y^* \in Y_1^*$, let $x \otimes y^*$ denote the functional defined by $(x \otimes y^*)(T) = y^*(T(x))$. These functionals are defined on either space and are of the same norm. Since for any $\Lambda \in \mathscr{L}(X,Y)^*$, $P(\Lambda)$ is the unique norm preserving extension of its restriction to the compact operators (see [6, Chap. I]), we get that $P(x \otimes y^*) = x \otimes y^*$. Since $\{x \otimes y^* : x \in X_1, y^* \in Y_1^*\}$ is a norming set, we conclude that the projection P satisfies the condition mentioned in the hypothesis of the above theorem. We apply this idea in the following corollary, which extends several of the results from [12].

COROLLARY 1. *Let X and Y be such that $\mathscr{K}(X,Y)$ is a proper subspace and an M-ideal in $\mathscr{L}(X,Y)$. Then there are no points of weak-norm continuity in $\mathscr{K}(X,Y)_1$ or $\mathscr{L}(X,Y)_1$.*

Proof. It follows from the remarks in [6, p. 335] that under the above hypothesis $\mathscr{K}(X,Y)$ is not complemented in $\mathscr{L}(X,Y)$. Thus the conclusion follows from our theorem and the above remark.

Remark 2. For $1 < p < \infty$, for the Bergman space B_p, it follows from [8, Corollary 4.8] that $\mathscr{K}(B_p)$ is an M-ideal in $\mathscr{L}(B_p)$; thus there are no points of weak-norm continuity in the unit balls of these spaces. When X is separable and $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ it would be interesting to construct, for any operator of norm one, an explicit sequence in $\mathscr{L}(X)_1$ that converges to it weakly but not in the norm. Such a construction when $X = l^p$ was done in [12].

Our next corollary deals with points of weak-norm continuity in the unit ball of injective tensor product spaces.

COROLLARY 2. *Let X be a non-reflexive Banach space which is an M-ideal in its bidual and let Y be any Banach space. Then there are no points of weak-norm continuity in $(X \otimes_\varepsilon Y)_1$ or in $(X^{**} \otimes_\varepsilon Y)_1$.*

Proof. Since X is an M-ideal in its bidual it follows from [6, Proposition VI.3.1] that $X \otimes_\varepsilon Y$ is an M-ideal in $X^{**} \otimes_\varepsilon Y$. It follows from the proof of Proposition VI.3.1 in [6] that the L-projection in the dual is obtained by composing with the canonical L-projection Q in X^{***}. It is easy to see that the unit ball of the range of this projection is weak* dense in $(X^{**} \otimes_\varepsilon Y)_1^*$. Thus the inclusion $X \otimes_\varepsilon Y \subset X^{**} \otimes_\varepsilon Y$ satisfies the first part of the hypothesis of our theorem. Also we have from [6, Corollary III.4.7] that X has a complemented copy of c_0. Since $c_0 \otimes_\varepsilon Y$ is not of finite codimension in $l^\infty \otimes_\varepsilon Y$, it is easy to see that $X \otimes_\varepsilon Y$ is not of finite codimension in $X^{**} \otimes_\varepsilon Y$. Therefore the conclusion follows from our theorem.

Remark 3. It was shown in [8, Proposition 6.7] that $\mathscr{K}(l^2, \mathscr{K}(l^2)) = l^2 \otimes_\varepsilon l^2 \otimes_\varepsilon l^2$ is not an M-ideal in $\mathscr{L}(l^2, \mathscr{K}(l^2))$ (in fact since l^2 is the range of a norm one projection in $\mathscr{K}(l^2)$, it follows from [6, Proposition VI.4.2] that $\mathscr{K}(\mathscr{K}(l^2))$ is not an M-ideal in $\mathscr{L}(\mathscr{K}(l^2))$). However, since $\mathscr{K}(l^2)$ is an M-ideal in its bidual $\mathscr{L}(l^2)$, we get that there are no points of weak-norm continuity in the unit balls of $l^2 \otimes_\varepsilon l^2 \otimes_\varepsilon l^2$ and $\mathscr{L}(l^2) \otimes_\varepsilon l^2$.

We conclude this note by showing that no infinite-dimensional function algebra has points of weak-norm sequential continuity in the unit ball. We follow the standard notation and terminology of [2].

PROPOSITION. *Let A be an infinite-dimensional function algebra. Let $f \in A$, $\|f\| = 1$. Then there exists a sequence $\{f_n\}$ in A_1 such that $f_n \to f$ weakly but not in the norm.*

Proof. Since our conclusion is isometric in nature we may assume w.l.o.g that A is a function algebra on its Shilov boundary X. Let ∂A denote the Choquet boundary of A. Since A is infinite-dimensional, ∂A is an infinite set. It is well known (see [2, Corollary 2.2.10]) that ∂A is dense in X. Let $f \in A$, $\|f\| = 1$. Following the arguments in [11], we divide the proof into several cases.

Case i. Suppose there exists a sequence $\{x_n\} \subset \partial A$ of distinct terms such that $f(x_n) \to \alpha \neq 0$. Let $\{V_n\}$ be a pairwise disjoint sequence of open sets such that $x_n \in V_n$. Since the points x_n are in the Choquet boundary, by applying Theorem 2.3.4 in [2] we can choose $f_n \in A$ with $f_n(x_n) = 1 = \|f_n\|$ and $|f_n| < 1$ outside V_n.

Let ϕ denote a Riemann mapping of the unit disc to the interior of the Jordan curve $\{z \in C : |z| \leq 1, |\arg z| = (1 - |z|)^2\} \cup \{0\}$ such that $\phi(1) = 1$. Then as remarked during the proof of the main result in [1], $(\phi \circ f_n)^n \in A$ and $(\phi \circ f_n)^n \to 0$ pointwise. Since $\{(\phi \circ f_n)^n\}_{n \geq 1}$ is also a bounded sequence, we have $(\phi \circ f_n)^n \to 0$. Also since $f(1 - (\phi \circ f_n)^n) \in A$ and $\|1 - (\phi \circ f_n)^n\| \to 1$ we have a sequence in A_1 converging weakly to f. However, since $\|f(\phi \circ f_n)^n\| \geq |f(x_n)|$, we see that the above sequence does not converge to f in the norm.

Case ii. Suppose $f(X)$ is an infinite countable set with 0 as the only accumulation point. Let $f(X) = \{t_n\}_{n \geq 1} \cup \{0\}$. Let $U_n = f^{-1}(t_n)$. Then $\chi_{U_n} \in A$. Choose $0 < \delta < 1$ and N such that $|t_n| \leq \delta$ for $n \geq N$. For $n \geq N$ let $g_n = (1 - \delta)\chi_{U_n}$. Then $g_n \in A$ and $g_n \to 0$ weakly. Also $\|f + g_n\| \leq 1$. Thus $\{f + g_n\}_{n \geq 1}$ is a sequence in the unit ball that converges weakly to f but not in the norm.

Case iii. Suppose $f(X)$ is a finite set. Suppose zero is the only value assumed on an infinite subset of the Choquet boundary by f. Let $U = f^{-1}(0)$. Then since $\chi_U \in A$, making use of the sequence that converges to zero weakly but not in the norm that we have constructed in Case i, we get the desired conclusion.

Arguments identical to the ones given in Case i can be used to deal with the case where f takes a non-zero value on an infinite subset of the Choquet boundary.

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