A PT-symmetric QES partner to the Khare–Mandal potential with real eigenvalues

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Abstract

We consider a PT-symmetric partner to Khare–Mandal's recently proposed non-Hermitian potential with complex eigenvalues. Our potential, which is quasi-exactly solvable, is shown to possess only real eigenvalues.

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Exploration of non-Hermitian Hamiltonians, in particular the PT-symmetric ones, is currently a topic of active research interest (see, for example, [1–14]). As is well known, PT-symmetric Hamiltonians are conjectured [15] to preserve the reality of their bound state eigenvalues except possibly for situations when PT may be spontaneously broken. It should be noted that PT-invariance in itself is not a sufficient condition for the Hamiltonian to possess an entirely real spectrum [1,2].

Recently, Khare and Mandal (KM) have inquired [9] into the invariance of a non-Hermitian Hamiltonian under the combined operations of a complex shift ($x \to a - x$, $a = i(\pi/2)$) and time reversal ($p \to -p$, $i \to -i$) symmetries and have argued, by considering a specific model potential, that the quasi-exactly solvable eigenvalues can emerge as complex conjugate pairs if one of the potential parameters is an even integer or if it is an odd integer and, in addition, the other potential parameter is large enough. They also identify their complex shift with that of parity and as such look upon their potential as the one enjoying PT-symmetry:

$$V(x) = -(\zeta \cosh 2x - iM)^2, \tag{1}$$

where the parameter ζ is real and M is restricted to integer values only.

We wish to point out in this Letter that although the above potential is invariant under the aforesaid transformations $x \to a - x$ and $i \to -i$ as rightly claimed by KM, it is non-PT-invariant because with a imaginary these transformations *do not* commute between themselves. We also observe that potential (1)

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does admit of a PT-symmetric partner, namely

$$V(x) = -(\zeta \sinh 2x - iM)^2 \tag{2}$$

(where, as in (1), ζ is real and M an integer) which, as can be easily checked, is invariant under the joint action of parity ($x \to -x$) and time reversal ($i \to -i$). Further, it is quasi-exactly solvable as we demonstrate below.

We begin by considering simultaneously the KM potential (1) and its modified version (2). The corresponding Hamiltonians are ($\hbar = 2m = 1$)

$$H^{(+)} = -\frac{d^2}{dx^2} - (\zeta \cosh 2x - iM)^2, \tag{3}$$

$$H^{(-)} = -\frac{d^2}{dx^2} - (\zeta \sinh 2x - iM)^2, \tag{4}$$

which can also be expressed together as

$$H^{(\pm)} = -\frac{d^2}{dx^2} - \left[\frac{\zeta}{2}\left(e^{2x} \pm e^{-2x}\right) - iM\right]^2. \tag{5}$$

Now, under a change of variable $x = (1/2)\log z$, $H^{(\pm)}$ become

$$H^{(\pm)} = -4z^2\frac{d^2}{dz^2} - 4z\frac{d}{dz} - \left[\frac{\zeta}{2}\left(z \pm \frac{1}{z}\right) - iM\right]^2. \tag{6}$$

As such if we set

$$\mu^{(\pm)}(z) = z^{(1-M)/2} e^{i(\zeta/4)(z \pm 1/z)}, \tag{7}$$

Hamiltonians (6) can be mapped to their gauge-transformed forms

$$H_g^{(\pm)} = \left[\mu^{(\pm)}(z)\right]^{-1} H^{(\pm)} \left[\mu^{(\pm)}(z)\right]. \tag{8}$$

From the relations

$$\mu^{-1}\frac{d}{dz}\mu = \frac{d}{dz} + \frac{\mu'}{\mu}, \tag{9}$$

$$\mu^{-1}\frac{d^2}{dz^2}\mu = \frac{d^2}{dz^2} + 2\frac{\mu'}{\mu}\frac{d}{dz} + \frac{\mu''}{\mu}, \tag{10}$$

where primes denote differentiations with respect to z, we easily obtain

$$H_g^{(\pm)} = -4z^2\frac{d^2}{dz^2} - 4z\left(2\frac{\mu'}{\mu}z + 1\right)\frac{d}{dz} - 4z^2\frac{\mu''}{\mu} - 4z\frac{\mu'}{\mu} - \left[\frac{\zeta}{2}\left(z \pm \frac{1}{z}\right) - iM\right]^2. \tag{11}$$

Taking now into account the expressions

$$\frac{\mu'}{\mu} = \frac{1-M}{2z} + i\frac{\zeta}{4}\left(1 \mp \frac{1}{z^2}\right), \tag{12}$$

$$\frac{\mu''}{\mu} = \left(\frac{\mu'}{\mu}\right)^2 - \frac{1-M}{2z^2} \pm i\frac{\zeta}{2}\frac{1}{z^3}, \tag{13}$$

we arrive at the following representations of $H_g^{(\pm)}$:

$$H_g^{(\pm)} = -4z^2\frac{d^2}{dz^2} - \left[2i\zeta z^2 - 4(M-2)z \mp 2i\zeta\right]\frac{d}{dz} + 2i\zeta(M-1)z + 2M - 1 \mp \zeta^2. \tag{14}$$

We thus find that the Schrödinger equation

$$H^{(\pm)}\psi^{(\pm)}(x) = E^{(\pm)}\psi^{(\pm)}(x) \tag{15}$$

is equivalent to

$$H_g^{(\pm)}\phi^{(\pm)}(z) = E^{(\pm)}\phi^{(\pm)}(z), \tag{16}$$

where $\psi^{(\pm)}(x) = \mu^{(\pm)}(z)\phi^{(\pm)}(z)$.

It should be noted here that in terms of the $s\ell(2, R)$ generators [16]

$$J_+ = z^2\frac{d}{dz} - 2jz, \qquad J_0 = z\frac{d}{dz} - j, \qquad J_- = \frac{d}{dz} \tag{17}$$

the gauged Hamiltonians $H_g^{(\pm)}$ can be rewritten as

$$H_g^{(\pm)} = -4J_0^2 - 2i\zeta J_+ \pm 2i\zeta J_- + M^2 \mp \zeta^2 \tag{18}$$

(provided $j = (M-1)/2$) in the true spirit of quasi-solvability [16].

We now turn to some specific cases of $\phi^{(\pm)}(z)$ by focusing on the following choices:

(i) $\phi^{(\pm)}(z) = c_0^{(\pm)}$,
(ii) $\phi^{(\pm)}(z) = c_0^{(\pm)} + c_1^{(\pm)}z$ ($c_1^{(\pm)} \neq 0$),
(iii) $\phi^{(\pm)}(z) = c_0^{(\pm)} + c_1^{(\pm)}z + c_2^{(\pm)}z^2$ ($c_2^{(\pm)} \neq 0$),
(iv) $\phi^{(\pm)}(z) = c_0^{(\pm)} + c_1^{(\pm)}z + c_2^{(\pm)}z^2 + c_3^{(\pm)}z^3$ ($c_3^{(\pm)} \neq 0$),

where $c_i^{(\pm)}$ ($i = 0, 1, 2, 3$) are constants. It is obvious that we can generalize $\phi^{(\pm)}(z)$ to higher degrees of z apart from the ones chosen here.

First consider $\phi^{(\pm)}(z) = c_0^{(\pm)}$. For this case, Eq. (16) becomes

$$2i\zeta(M-1)z + 2M - 1 \mp \zeta^2 - E^{(\pm)} = 0 \tag{19}$$

leading to

$$2\zeta(M-1)=0, \tag{20}$$

$$2M-1\mp\zeta^2-E^{(\pm)}=0. \tag{21}$$

Hence $M=1$ ($j=0$) and, as a result,

$$E^{(\pm)}=1\mp\zeta^2. \tag{22}$$

The accompanying wave functions read

$$\psi^{(+)}\propto e^{(i\zeta/2)\cosh 2x}, \tag{23}$$

$$\psi^{(-)}\propto e^{(i\zeta/2)\sinh 2x}. \tag{24}$$

We therefore see that for $M=1$ the energy eigenvalues corresponding to (1) as well as its modified PT-symmetric version (2) are real.

Next consider $\phi^{(\pm)}(z)=c_0^{(\pm)}+c_1^{(\pm)}z$. Eq. (16) gives

$$2i\zeta(M-2)\,c_1^{(\pm)}=0, \tag{25}$$

$$2i\zeta(M-1)c_0^{(\pm)}+\left(6M-9\mp\zeta^2-E^{(\pm)}\right)c_1^{(\pm)}=0, \tag{26}$$

$$\pm 2i\zeta c_1^{(\pm)}+\left(2M-1\mp\zeta^2-E^{(\pm)}\right)c_0^{(\pm)}=0. \tag{27}$$

Eq. (25) implies $M=2$ while Eqs. (26) and (27) give

$$2i\zeta c_0^{(\pm)}-\epsilon^{(\pm)}c_1^{(\pm)}=0, \tag{28}$$

$$-\epsilon^{(\pm)}c_0^{(\pm)}\pm 2i\zeta c_1^{(\pm)}=0, \tag{29}$$

where $\epsilon^{(\pm)}=E^{(\pm)}-3\pm\zeta^2$.

Solving (28) and (29) we get $\epsilon_\pm^{(+)}=\pm 2i\zeta$ (complex) and $\epsilon_\pm^{(-)}=\pm 2\zeta$ (real). We thus find for $M=2$ ($j=1/2$) the results

$$E_\pm^{(+)}=3\pm 2i\zeta-\zeta^2, \tag{30}$$

$$\psi_\pm^{(+)}\propto e^{(i\zeta/2)\cosh 2x}\left(e^{-x}\pm e^{x}\right) \tag{31}$$

for the KM potential and

$$E_\pm^{(-)}=3\pm 2\zeta+\zeta^2, \tag{32}$$

$$\psi_\pm^{(-)}\propto e^{(i\zeta/2)\sinh 2x}\left(e^{-x}\pm ie^{x}\right) \tag{33}$$

for potential (2). Expectedly the eigenvalues for the KM case turn out to be a complex conjugate pair, $M=2$ being an even integer. However, those for the PT-symmetric potential (2) emerge real as borne out by (32).

Proceeding now to the case $\phi^{(\pm)}(z)=c_0^{(\pm)}+c_1^{(\pm)}z+c_2^{(\pm)}z^2$, we find from Eq. (16)

$$2i\zeta(M-3)c_2^{(\pm)}=0, \tag{34}$$

$$2i\zeta(M-2)c_1^{(\pm)}+\left(10M-25\mp\zeta^2-E^{(\pm)}\right)c_2^{(\pm)}=0, \tag{35}$$

$$2i\zeta(M-1)c_0^{(\pm)}+\left(6M-9\mp\zeta^2-E^{(\pm)}\right)c_1^{(\pm)}\pm 4i\zeta c_2^{(\pm)}=0, \tag{36}$$

$$\pm 2i\zeta c_1^{(\pm)}+\left(2M-1\mp\zeta^2-E^{(\pm)}\right)c_0^{(\pm)}=0. \tag{37}$$

Here we have $M=3$ and, defining $\epsilon^{(\pm)}=E^{(\pm)}-9\pm\zeta^2$, we get

$$2i\zeta c_1^{(\pm)}-\left(\epsilon^{(\pm)}+4\right)c_2^{(\pm)}=0, \tag{38}$$

$$4i\zeta c_0^{(\pm)}-\epsilon^{(\pm)}c_1^{(\pm)}\pm 4i\zeta c_2^{(\pm)}=0, \tag{39}$$

$$-\left(\epsilon^{(\pm)}+4\right)c_0^{(\pm)}\pm 2i\zeta c_1^{(\pm)}=0. \tag{40}$$

Solving (38)–(40) we obtain for $M=3$ ($j=1$) the solutions

$$E_0^{(+)}=5-\zeta^2,\qquad E_\pm^{(+)}=7-\zeta^2\pm 2\sqrt{1-4\zeta^2}, \tag{41}$$

$$\psi_0^{(+)}\propto e^{(i\zeta/2)\cosh 2x}\sinh 2x, \tag{42}$$

$$\psi_\pm^{(+)}\propto e^{(i\zeta/2)\cosh 2x}\left[2\cosh 2x-\frac{i}{\zeta}\left(1\pm\sqrt{1-4\zeta^2}\right)\right] \tag{43}$$

for the KM potential and

$$E_0^{(-)}=5+\zeta^2,\qquad E_\pm^{(-)}=7+\zeta^2\pm 2\sqrt{1+4\zeta^2}, \tag{44}$$

$$\psi_0^{(-)}\propto e^{(i\zeta/2)\sinh 2x}\cosh 2x, \tag{45}$$

$$\psi_\pm^{(-)}\propto e^{(i\zeta/2)\sinh 2x}\left[2\sinh 2x-\frac{i}{\zeta}\left(1\pm\sqrt{1+4\zeta^2}\right)\right] \tag{46}$$

for potential (2). Contrary to the Khare–Mandal potential (1) for which two of the eigenvalues (namely $E_\pm^{(+)}$) become complex if $|\zeta|$ is larger than the critical value $\zeta_c=1/2$, all three eigenvalues of the PT-symmetric potential (2) remain real for all values of ζ.

We now take up the case (iv), namely $\phi^{(\pm)}(z)=c_0^{(\pm)}+c_1^{(\pm)}z+c_2^{(\pm)}z^2+c_3^{(\pm)}z^3$, for which we obtain from Eq. (16) the relations

$$2i\zeta(M-4)c_3^{(\pm)}=0, \tag{47}$$

$$2i\zeta(M-3)c_2^{(\pm)}+\left(14M-49\mp\zeta^2-E^{(\pm)}\right)c_3^{(\pm)}=0, \tag{48}$$

$$2i\zeta(M-2)c_1^{(\pm)}+\left(10M-25\mp\zeta^2-E^{(\pm)}\right)c_2^{(\pm)}\pm 6i\zeta c_3^{(\pm)}=0, \tag{49}$$

$$2i\zeta(M-1)c_0^{(\pm)}+\left(6M-9\mp\zeta^2-E^{(\pm)}\right)c_1^{(\pm)}\pm 4i\zeta c_2^{(\pm)}=0, \tag{50}$$

$$\pm 2i\zeta c_1^{(\pm)}+\left(2M-1\mp\zeta^2-E^{(\pm)}\right)c_0^{(\pm)}=0. \tag{51}$$

We are thus led to $M=4$ and, defining $\epsilon^{(\pm)}=E^{(\pm)}-15\pm\zeta^2$, we get

$$2i\zeta c_2^{(\pm)}-\left(\epsilon^{(\pm)}+8\right)c_3^{(\pm)}=0, \tag{52}$$

$$4i\zeta c_1^{(\pm)}-\epsilon^{(\pm)}c_2^{(\pm)}\pm 6i\zeta c_3^{(\pm)}=0, \tag{53}$$

$$6i\zeta c_0^{(\pm)}-\epsilon^{(\pm)}c_1^{(\pm)}\pm 4i\zeta c_2^{(\pm)}=0, \tag{54}$$

$$-\left(\epsilon^{(\pm)}+8\right)c_0^{(\pm)}\pm 2i\zeta c_1^{(\pm)}=0. \tag{55}$$

An analysis of Eqs. (52)–(55) reveals that for consistency of the equations $\epsilon^{(\pm)}$ have to fulfill the condition

$$\left(\epsilon^{(\pm)}+8\right)\left[\left(\epsilon^{(\pm)}+8\right)\left(\epsilon^{(\pm)2}\pm 16\zeta^2\right)\pm 24\zeta^2\epsilon^{(\pm)}\right]+144\zeta^4=0. \tag{56}$$

This fourth-degree equation can be factorized into two quadratic ones, namely

$$\left(\epsilon^{(+)}+8\right)\left(\epsilon^{(+)}\pm 4i\zeta\right)+12\zeta^2=0, \tag{57}$$

$$\left(\epsilon^{(-)}+8\right)\left(\epsilon^{(-)}\pm 4\zeta\right)-12\zeta^2=0. \tag{58}$$

Let us now consider the solutions of (57) and (58). It is obvious that in the former case, all the four solutions are complex. This corresponds to the KM scenario. However, in the latter case (which corresponds to the PT-symmetric model (2)), we get quadratic equations in $\epsilon^{(-)}$ with real coefficients:

$$\epsilon^{(-)2}+4(2\pm\zeta)\epsilon^{(-)}+4\zeta(\pm 8-3\zeta)=0. \tag{59}$$

For the upper signs we obtain two real solutions for any ζ:

$$\epsilon_{+,\pm}^{(-)}=-2(2+\zeta)\pm 4\sqrt{1-\zeta+\zeta^2}, \tag{60}$$

and the same is true for the lower signs:

$$\epsilon_{-,\pm}^{(-)}=-2(2-\zeta)\pm 4\sqrt{1+\zeta+\zeta^2}. \tag{61}$$

Note that we can combine the four real solutions given by (60) and (61) in the manner

$$\epsilon_{\sigma,\tau}^{(-)}=-2(2+\sigma\zeta)+4\tau\sqrt{1-\sigma\zeta+\zeta^2}, \tag{62}$$

where $\sigma,\tau=+,-$.

Thus corresponding to the PT-symmetric potential (2) our findings for $M=4$ $(j=3/2)$ are

$$E_{\sigma,\tau}^{(-)}=11-2\sigma\zeta+\zeta^2+4\tau\sqrt{1-\sigma\zeta+\zeta^2}, \tag{63}$$

$$\psi_{\sigma,\tau}^{(-)}\propto e^{(i\zeta/2)\sinh 2x}\left(e^{-x}-\sigma i e^{x}\right)\times\left[\sinh 2x-\frac{i}{\zeta}\left(1+\tau\sqrt{1-\sigma\zeta+\zeta^2}\right)\right]. \tag{64}$$

To summarize, we observe from the foregoing treatment of the cases $M=1,2,3,4$ that unlike the case of KM potential in which QES eigenvalues occur in complex conjugate pairs for M an even integer but may be real for M an odd integer and $|\zeta|$ smaller than or equal to some critical value ζ_c, our PT-symmetric potential (2) exhibits real energy eigenvalues both for even and odd integer values of M and any value of ζ. It should be remarked that although we restricted our discussion up to $M=4$ which corresponds to keeping a fourth-degree wave function, it is clear that we can deal with, in an identical way, higher degree contributions in $\phi^{(\pm)}(z)$. We conjecture that all of them will lead to real eigenvalues.

Finally, we can write down recurrence relations for polynomials by substituting

$$\phi^{(\pm)}(z)=\sum_{n=0}^{\infty}\frac{R_n^{(\pm)}(E^{(\pm)})}{n!}t^n, \quad t=\pm\frac{z}{2i\zeta},$$

in the Schrödinger equation (16). It can be readily seen that the coefficients $R_n^{(\pm)}(E^{(\pm)})$ satisfy the three-term recursion relation

$$R_{n+1}^{(\pm)}\left(E^{(\pm)}\right)=\left(E^{(\pm)}-b_n^{(\pm)}\right)R_n^{(\pm)}\left(E^{(\pm)}\right)-a_n^{(\pm)}R_{n-1}^{(\pm)}\left(E^{(\pm)}\right), \tag{65}$$

where

$$a_n^{(\pm)}=\mp 4n(M-n)\zeta^2, \tag{66}$$

$$b_n^{(\pm)}=4n(M-1-n)+2M-1\mp\zeta^2. \tag{67}$$

If $M = k$, a positive integer, then $a_k^{(\pm)} = 0$ and, for $n = k$, Eq. (65) reduces to a two-term recursion relation. As a consequence, $R_{k+1}^{(\pm)}$, and more generally, $R_{k+n}^{(\pm)}$, is proportional to $R_k^{(\pm)}$:

$$R_{k+n}^{(\pm)} = R_k^{(\pm)} \bar{R}_n^{(\pm)}, \tag{68}$$

where $\bar{R}_n^{(\pm)}$ satisfies the three-term recursion relation

$$\bar{R}_{n+1}^{(\pm)}\big(E^{(\pm)}\big) = \big(E^{(\pm)} - b_{M+n}^{(\pm)}\big)\bar{R}_n^{(\pm)}\big(E^{(\pm)}\big) - a_{M+n}^{(\pm)}\bar{R}_{n-1}^{(\pm)}\big(E^{(\pm)}\big). \tag{69}$$

QES eigenvalues are obtained as solutions of the kth-degree equation $R_k^{(\pm)}(E^{(\pm)}) = 0$.

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References

[1] C.M. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.
[2] C.M. Bender, S. Boettcher, J. Phys. A 31 (1998) L273.
[3] E. Delabaere, F. Pham, Phys. Lett. A 250 (1998) 25, 29.
[4] M. Znojil, Phys. Lett. A 259 (1999) 220.
[5] B. Bagchi, R. Roychoudhury, J. Phys. A 33 (2000) L1.
[6] M. Znojil, J. Phys. A 33 (2000) L61.
[7] M. Znojil, J. Phys. A 33 (2000) 4561.
[8] B. Bagchi, F. Cannata, C. Quesne, Phys. Lett. A 269 (2000) 79.
[9] A. Khare, B.P. Mandal, Phys. Lett. A 272 (2000) 53.
[10] B. Bagchi, C. Quesne, Phys. Lett. A 273 (2000) 285.
[11] F. Cannata, M. Ioffe, R. Roychoudhury, P. Roy, Phys. Lett. A 281 (2001) 305.
[12] G. Lévai, F. Cannata, A. Ventura, J. Phys. A 34 (2001) 839.
[13] Z. Ahmed, Phys. Lett. A 282 (2001) 343.
[14] P. Dorey, C. Dunning, R. Tateo, J. Phys. A 34 (2001) L391.
[15] D. Bessis, unpublished (1992).
[16] A. Ushveridze, Quasi-Exactly Solvable Models in Quantum Mechanics, Institute of Physics Publishing, Bristol, 1994.