# Measuring multidimensional deprivation

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Received 30 September 1999; received in revised form 31 December 2000; accepted 31 January 2001

**Abstract**

This paper examines the problem of measuring deprivation in an economy with more than one attribute. The income-based methods falls short in situations where for some attribute market do not exist. In this case, we explore suitable methods of measuring commodity specific and aggregate personal deprivation directly based on the distribution of attributes. This paper also characterizes simple class of deprivation measures for different possible situations. These measures would be helpful in practice, as they are simply parametrized. We also define deprivation orderings, both commodity specific and aggregate. Finally we provide a numerical illustration of our measures and orderings.

*Keywords:* Multidimensional distribution; Deprivation measures; Deprivation ordering; Characterization; Substitution and complementation

*JEL classification:* D31; D63

## 1. Introduction

A person's feeling of deprivation in a society arises out of comparing his situation with those of better off persons. Using the example of promotion, Runciman (1966) illustrated the feeling of deprivation of an individual and argued that the extent of deprivation felt by an individual who is not promoted increases with the number of promoted people. Sen (1976) was the first person to introduce this notion of deprivation in the income distribution literature. He posited that an individual's level of deprivation is an increasing function of the number of persons who are better off than the person in question. Yitzhaki (1979) considered the feeling of deprivation in terms of income or lack thereof and showed that the product of mean income and the Gini coefficient could be one plausible index of deprivation in a society.[1] Later on, Hey and Lambert (1980) provided an alternative characterization of the Yitzhaki index. Their demonstration hinges on Runciman's (1966) remark "The magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it" (op. cit. p10). Temkin (1986) also argued that aggregate inequality in a society should be measured in terms of such differences. The size of someone's complaint depends on "how she fares relative to all of the other people who are better off than she" (op. cit. p105) and such complaints can be used to formulate inequality. (Amiel and Cowell (1994) and Chakravarty (1990, 1999) provide further discussions along this line.) A discussion on the different relative and absolute measures of deprivation using social satisfaction functions and their relationships is provided in Chakravarty and Mukherjee (1999a).[2]

Although discussion and measurement of deprivation has received fair attention in the literature, most of it has been only in terms of the income distribution. The feeling of deprivation, however, is essentially a multifaceted phenomenon. Income being only one of the many attributes in terms of which a person may feel deprived. In fact, income may not always be sufficient in reflecting a persons aggregate feeling of deprivation. While it is true that an increase in the purchasing power enables a person to assuage his feeling of deprivation in other commodities also, but this may not always be possible. Specially when a market for certain attributes do not exist. This is true for the public goods in particular. An example is *access to the malaria prevention programme* in many underdeveloped countries.[3]

In this context, a very important issue is how to choose the relevant basket of attributes. This should include all basic need items (e.g., food stuff, clothing items, fuel and light, characteristics of dwelling, health facilities, sanitation, etc.) and some non-basic commodities, the choice of which would depend on the society or culture of the economy under consideration (e.g., whisky may be relevant for the Scots but not for Indians, for whom pan and tobacco may be relevant).

Another moot question in this regard is whether or not to include income as one of the attributes under consideration. If income is included, then all the marketable attributes in the basket should be assigned lower weightage to mitigate double counting. We should not drop them altogether, because, recent empirical studies (Behrman and Deolalikar, 1988) have shown that an increase in income do not necessarily result in increased consumption of even the basic need items even though they are available in the market. Hence, the inclusion of income, which seems reasonable, should be dealt with particular care.

Measurement of such multidimensional nature has been attempted in the literature in terms of inequality. See Sen (1992) for a discussion on this. These issues are ably

[1] Sen (1973, 1997) also interpreted the Gini coefficient from a quite similar perspective.

[2] See also the references cited therein.

[3] See Streeten (1981) for an extensive discussion on related issues.

surveyed in Tsui (1995, 1999). Multidimensional poverty measurement is discussed in Tsui (1996a, 1998), Zheng (1997) and Chakravarty et al. (1998). Tsui (1996) and Chakravarty and Mukherjee (1999b) take up the issue of multidimensional measurement of well-being.[4]

In this paper, it is our aim to address the measurement of deprivation in terms of many attributes and formulate plausible measures for this purpose in terms of ethical judgment.[5] To this end, we initiate our discussion on how to measure a person's feeling of deprivation in terms of a particular commodity in Section 2. We characterize and use a simple variant of the usual income-based measure of relative deprivation for this purpose. In Section 3 we go on to devise suitable methods of aggregating such commodity specific deprivation into aggregate personal deprivation index. This section considers the various possibilities of the attributes being substitutes or complements. Evidently, such considerations affect a person's aggregate feeling of deprivation to a large extent. This section also provides simple characterization results of such aggregate personal deprivation measures.

In order to develop an ordering among income distributions in terms of deprivation, Kakwani (1984) plotted the sum of income share shortfalls of different individuals from richer individuals against the cumulative proportion of persons to generate the *relative deprivation curve* (RDC). He also showed that the area under the curve is the Gini coefficient for the society.[6] In Sections 2 and 3, we also define the RDC in terms of each commodity and in aggregate and define an ordering between two populations in terms of these curves.

Following Chakravarty (1999) and Chakravarty et al. (1998), in this paper, we discuss the problem of aggregating the measures of deprivation for each person in the society to arrive at a social measure of deprivation. We briefly recount a few such measures in Section 4. In this context, we also consider an egalitarian principle in terms of the correlation among the attribute holdings considered by Atkinson and Bourguignon (1982).

Our mode of discussion is essentially a three-step approach that moves from the disaggregated to the aggregate value of deprivation. Alternatively, one can start with the social deprivation function and do a disaggregation analysis by adopting suitable decomposability axioms. This method can also lead to interesting results, but we do not explore this possibility in the present paper.

Section 5 provides a numerical illustration for the commodity specific and aggregate personal deprivation values using a sample data on households from two districts in the state of West Bengal, India. We also calculate the social deprivation and plot all relevant RDCs for the purpose of demonstrating our definitions. Finally, Section 6 concludes the paper.

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[4] Sen (1985, 1987); Kakwani (1993) discusses the single dimensional issues.

[5] We encourage the reader to look at Dutta and Pattanaik (1999), a recent and highly interesting empirical study in a similar vein.

[6] For a detailed discussion of these issues, see Chakravarty and Mukherjee (1999).

## 2. Preliminaries and commoditywise personal deprivation

The purpose of this section is to lay down the preliminary definitions and postulates for the measurement of a person's feeling of deprivation with respect to a particular attribute. To this end, we consider an economy with $n$ persons and $k$ commodities ($n$ and $k$ are integers, $\geq 2$). Let $R_+^m$ stand for the nonnegative orthant of the Euclidean $m$-space $R^m$. For the set of n persons, the $i$th one possesses a $k$-vector $\mathbf{x_i} = (x_{i1}, \ldots, x_{ik}) \in R_+^k$ of attributes. $M^n$ be the set of all $n \times k$ matrices whose entries $\in R_+$. A typical $X \in M^n$ shows an arrangement of values of $k$ attributes possessed by $n$ persons in a matrix form. The $i$th row of $X$ is $\mathbf{x_i}$. The $j$th column of $X$, $\mathbf{x^j}$, shows the distribution of attribute $j$ among the $n$ persons in the society. The $(i,j)$th entry of $X$ is $x_{ij}$, the quantity of the $j$th attribute possessed by the person $i$. Let $M = \cup_{n \in N} M^n$, where $N$ is the set of natural numbers. Define the mean of the vector $\mathbf{x^j}$, or the average $j$th commodity holding in the economy, by $\mu_j = (1/n)\Sigma_i x_{ij}$.

We define the $i$th person's deprivation in the $j$th commodity in general as

$$d_{ij} = d_{ij}(X)$$

where $d_{ij}: M^n \to R_+$. To pindown a specific class of indices, we assume the following.

**(A1).** The deprivation $d_{ij}$ felt by person $i$ in commodity $j$ is a function of the population size $n$ and the normalized shortfall of person $i$'s $j$th commodity holding from that of persons having higher holding of the $j$th attribute. Or, we can say

$$d_{ij} = d_{ij}(\mathrm{Max}\{\frac{x_{1j} - x_{ij}}{\mu_j}, 0\}, \ldots, \mathrm{Max}\{\frac{x_{nj} - x_{ij}}{\mu_j}, 0\}; n) \tag{1}$$

So, we are assuming that the extent of deprivation in the $j$th commodity is independent of the distribution of other commodities. This assumption is similar to the assumptions like factor decomposability discussed in the inequality and poverty literature (see Chakravarty et al., 1998). (A1) is strong, but here we want to focus on strongly decomposable measures which are very useful in practice for ease of comparison. The division by $\mu_j$ is to make the measure of deprivation scale free. This is done for two reasons. First, we do not want comparison to be distorted by a change of units (e.g., weight in kg rather than in lb). Second, and more important, we need the measure to be unit free so that we can aggregate across commodities in a cardinally meaningful way to arrive at the aggregate personal deprivation to be discussed in the next section.

We are also ignoring the interrelation of the deprivation of different commodities at this stage, it would make our analysis intractable. This will be considered when we aggregate across commodities.

**(A2).** Rank preserving transfers of $j$th commodity between two persons who has higher $j$th commodity holding than person $i$ do not effect $d_{ij}$. (Due to (A1), this also holds trivially for two persons having less than person $i$.)

This assumption imposes a transfer insensitivity on the class of admissible indices.

The consequence of (A2) is a generalized linearity (see Apostol, 1969, for details) on the $d_{ij}$ function (more on this later). To establish the connection, consider the following lemmata.

**Lemma 1.** *If $d_{ij}$ satisfies (A2) then it satisfies the following condition.*

$$\frac{\partial d_{ij}}{\partial x_{l_1 j}} - \frac{\partial d_{ij}}{\partial x_{l_2 j}} = 0,$$

*if both $x_{l_1 j}, x_{l_2 j} > x_{ij}$.*

**Proof.** The axiom (A2) implies that the function $d_{ij}$ satisfy the following condition:

$$d_{ij}\left(\text{Max}\left\{\frac{x_{1j} - x_{ij}}{\mu_j}, 0\right\}, \ldots, \text{Max}\left\{\frac{x_{nj} - x_{ij}}{\mu_j}, 0\right\}; n\right)$$
$$= d_{ij}\left(\text{Max}\left\{\frac{y_{1j} - y_{ij}}{\mu_j}, 0\right\}, \ldots, \text{Max}\left\{\frac{y_{nj} - y_{ij}}{\mu_j}, 0\right\}; n\right),$$

where $y_{lj} = x_{lj}$ for all $l = 1, \ldots, n$ except for $l_1$ and $l_2$ such that $x_{l_1 j}, x_{l_2 j} > x_{ij}$, $y_{l_1 j} = x_{l_1 j} + h$ and $y_{l_2 j} = x_{l_2 j} - h$ with $h > 0$ being so that the transfer preserves ranks. Now considering the difference between the two $d_{ij}$ values in the above equation divided by $h$ and letting $h \to 0$ we get the desired result. □

We now take recourse to the following result assuming that $d_{ij}$ is differentiable with respect to each of its arguments.

**Lemma 2.** (*Theorem* 9.1, *p.* 285, *Apostol*, 1969) *Let g be a differentiable function on R and let $f: R^2 \to R$ be defined by the equation*

$$f(x, y) = g(bx - ay), \tag{2}$$

*a,b constant, not both zero. Then f satisfies the first order partial differential equation*

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = 0 \tag{3}$$

*everywhere in $R^2$. Conversely, every differentiable solution of* (3) *necessarily has the form* (2) *for some g.* □

Now considering $d_{ij}$ in place of $f$ as a function of two arguments $x_{l_1 j}, x_{l_2 j}$ and taking $a = -1$ and $b = 1$, we have the desired result. More specifically,

$$d_{ij} = G_{ij}\left(\sum_{l=1}^{n} \text{Max}\left\{\frac{x_{lj} - x_{ij}}{\mu_j}, 0\right\}; n\right),$$

where $G_{ij}: R_+ \times N \to R_+$.

We further assume that

**(A3).** $d_{ij}$ is population replication invariant. If we replicate the distribution $X \in M^n$ $p$ times to get $X^{(p)} \in M^{np}$, the $d_{ij}$ values do not change. This forces $d_{ij}$ to be of the form

$$d_{ij} = g_{ij}\left(\frac{1}{n}\sum_{l=1}^{n} \text{Max}\left\{\frac{x_{lj} - x_{ij}}{\mu_j}, 0\right\}\right). \tag{4}$$

Here $g_{ij}: R_+ \to R_+$.[7]

We now look at the effect of transfer between a person having more than person $i$ and someone having less than person $i$. We postulate

**(A4).** Rank preserving transfers of $j$th commodity from a person having more than person $i$ to someone having less decreases $d_{ij}$. This assumption forces $g_{ij}$ to be increasing in its argument.

(A3) and (A4) are standard in the literature of deprivation. (A3) allows one to extend this discussion for an $n$-person economy to one with a continuum of individuals, which is important for empirical purposes. At this stage, we would like to note that replication invariance is at odds with Runciman's original idea of relative deprivation. The way he visualized it, deprivation of an individual depends upon the group the person is compared with. This works best within some small community or group one belongs to and knows about. For a large group, say a country like India, there would be severe informational constraints.[8] (See also Yitzhaki, 1982.)

The somewhat restrictive assumption (A2) is postulated to simplify the measurement issue. Allowing for transfer sensitivity would lead to a non-linear aggregation rule to arrive at $d_{ij}$. For example, if we think that the feeling of deprivation becomes less severe per unit distance as we move up the amount of attribute holding ladder, then one might postulate $d_{ij}$ to be of the form

$$d_{ij} = g_{ij}\left(\frac{1}{n}\sum_{l=1}^{n}\left[\text{Max}\left\{\frac{x_{lj} - x_{ij}}{\mu_j}, 0\right\}\right]^{\gamma}\right), \quad 0 < \gamma < 1.$$

We do not go into this complication in the present paper.

The measure in (4) with $g'_{ij} > 0$, in addition to (A1)–(A4) also satisfies the following.

**(A5).** $d_{ij}$ is symmetric in the normalized shortfalls. Any permutation of the vector $\mathbf{x}^{\mathbf{j}}$ do not effect $d_{ij}$. this amounts to anonymity in the economy.

**(A6).** $d_{ij}$ satisfies scale invariance. If all the $j$th attribute holdings are measured in a new unit, the value of $d_{ij}$ remains unchanged (e.g., weight is measured in lb instead of kg, etc.).

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[7] The proof of this statement follows that of Theorem 4 in Shorrocks (1980).

[8] We thank an anonymous referee for this point.

We can easily check that the deprivation function $d_{ij}$ given by (4) satisfies the postulates (A5) and (A6) also. The simplest example of such a $d_{ij}$ function is

$$d_{ij} = \frac{1}{n\mu_j}\sum_{l=1}^{n}\mathrm{Max}\{(x_{lj} - x_{ij}),0\}. \tag{5}$$

From the above discussion we have the following proposition.

**Proposition 1.** (*a*) *the deprivation function* $d_{ij}$ *satisfies the axioms* (A1)–(A4) *if and only if it is ordinally equivalent to the form of* $d_{ij}$ *given by* (5). (*b*) *the deprivation function given by* (5) *satisfies the axioms* (A5) and (A6).

In the following discussion, for the sake of simplicity, we will only consider deprivation function $d_{ij}$ of the form given by (5). Note that, when we have only one attribute, i.e., $k=1$, the form of $d_{ij}$ in Eq. (5) reduces to the usual individual deprivation function discussed in Chakravarty and Mukherjee (1999a).

Before we conclude this section, we briefly define the *relative deprivation curves* (RDCs) for the $k$ commodities/attributes under consideration. For the $j$th commodity, we first order the $d_{ij}$ values non-increasingly, say $d_{(1)j} \geq d_{(2)j} \geq \ldots \geq d_{(n)j}$. The RDC associated with the $j$th commodity is defined as the plot of $d_{(i)j} = d_{(i)j}(\mathbf{x}^{\mathbf{j}})$ against the cumulative proportion of population $i/n$, where $i=1,\ldots,n$.

Hence, we can now define the deprivation ordering, in commodity $j$, between two distributions $X$ and $Y \in M$ in the following way:

$X$ dominates $Y$ by the deprivation criterion in the $j$th commodity, $X \geq_{D_j} Y$ for short, if $d^X_{(i)j} \geq d^Y_{(i)j}$ for $i=1,\ldots,n$. This amounts to saying that the RDC of $X$ according to the $j$th commodity lies nowhere below and somewhere strictly above that of $Y$.

## 3. Aggregate personal deprivation

Once we have settled the question of the form a commodity specific personal deprivation function, the next logical step would be to devise suitable methods to aggregate these deprivation values across commodities and across individuals in a meaningful and consistent way. To this end, we first consider the problem of aggregating the deprivation values across commodities. Otherwise, if we aggregate across individuals first, we would be throwing away the information we have, in $X$, on individual substitution/complementation effects across commodities. Needless to say, this is necessary only if $k \geq 2$, i.e., we have at least two attributes. We take up the issue of aggregating personal deprivation values across individuals in the next section.

We define the $i$th person's deprivation function $d_i$ as

$$d_i = d_i(d_{i1}, d_{i2}, \ldots, d_{ik}), \tag{6}$$

where $d_i : R^k_+ \to R_+$. To impose some structure on the function $d_i$, the main task now would be to propose suitable properties that the $d_i$ function should satisfy. So, we propose the following:

**(B1).** Minimum value of $d_i$ will be achieved when $d_{ij}=0$ for all $j$, that is, When person $i$ has no feeling of deprivation for any commodity (he has maximum amount of each). We propose the minimum value to be 0. So we will have $d_i = d_i(0,0, \ldots ,0) = 0$.

Similarly, when person $j$ has zero amount of each of the commodities, or $d_{ij}=1$ for all $j$, then $d_i$ is maximum (=1, say). So, then $d_i = d_i(1,1, \ldots ,1) = 1$.

So, we will always have $0 \le d_i \le 1$.

The following assumptions on marginal effects of $d_{ij}$ values on $d_i$ are reasonable:

**(B2).** $\partial d_i / \partial x_{ij} \le 0$ for all $j$. That is, if the holding of the $j$th attribute by person $i$ increases, keeping all other things constant, in aggregate, deprivation must not increase.

**(B3).** $\partial d_i / \partial x_{lj} \ge 0$ for $l \ne i$ and for all $j$ ($>$ if $x_{lj} > x_{ij}$; $=0$, otherwise). That is, if the holding of the $j$th attribute by some other person $l$ increases, keeping all else constant, person $i$'s aggregate deprivation must not decrease. If the holding of person $l$ is above that of person $i$, then $d_i$ must increase.

(B2) and (B3) are very reasonable in the sense that they match with our intuitive notion of deprivation. If any good is a 'good', then more of it should always decrease our deprivation and if someone else has more of it, then deprivation cannot decrease. We have,

$$\frac{\partial d_i}{\partial d_{ij}} = n\mu_j \frac{\partial d_i}{\partial x_{lj}} \ge 0$$

for $l \ne i$, $x_{lj} > x_{ij}$ (=0 otherwise) and

$$0 \ge \frac{\partial d_i}{\partial x_{ij}} = -\frac{n_i^j}{n\ \mu_j}\frac{\partial d_i}{\partial d_{ij}},$$

where $n_i^j$=number of $\{1 \le l \le n | x_{lj} > x_{ij}\}$. So, again $\partial d_i / \partial d_{ij} \ge 0$ for all $j$. Therefore, (B2) and (B3) force $\partial d_i / \partial d_{ij}$ to be$>0$ for all $j$. That is, we must have function $d_i$ increasing in each argument.

We now propose alternative postulates on substitution effects among commodities. Consider the following.

**(B4).** $\partial^2 d_i / \partial x_{ij} \partial x_{ik} \ge (\le) 0$ if commodity $k$ and $j$ ($k \ne j$) are substitutes (complements).

While considering commodity specific deprivation, we ignored the interrelation of deprivation between different commodities. This enabled us to do our analysis in a simple manner. We now attempt to capture this in (B4). This postulate is akin to the law of DMRS. If $j$ and $k$ are substitutes, then rate of change in deprivation due to changes in $x_{ik}$ (rate of change $<0$) will become slower (increase) with increase in $x_{ij}$. This is because substitution will make the marginal valuation of $x_{ik}$ smaller as $x_{ij}$ increases. Now, we have

$$\frac{\partial^2 d_i}{\partial x_{Ij}\,\partial x_{Ik}} = \frac{1}{n^2\;\mu_k\;\mu_j}\,\frac{\partial^2 d_i}{\partial d_{ij}\,\partial d_{ik}}$$

for $x_{Ik} > x_{ik}$ and $x_{Ij} > x_{ij}$ (=0 otherwise) and

$$\frac{\partial^2 d_i}{\partial x_{ij}\,\partial x_{ik}} = \frac{n_i^j\,n_i^k}{n^2\;\mu_k\;\mu_j}\,\frac{\partial^2 d_i}{\partial d_{ij}\,\partial d_{ik}}.$$

It follows that $\partial^2 d_i/\partial d_{ij}\partial d_{ik}$ must be $\geq (\leq)$ 0, if commodities $k$ and $j$ $(k \neq j)$ are substitutes (complements).

Under (B1)–(B4) we have some resulting restrictions on the form of the deprivation function $d_i$. For instance, we are not allowing an additive deprivation function. We now consider a few examples of such deprivation functions.

**Example 1.** We consider the two commodity case ($k=2$).

(a) Suppose the two commodities are gross substitutes with constant substitution effect. Then the functional form of $d_i$ might be $d_i = d_{i1}d_{i2}$.
(b) Conversely, if we assume that the commodities are complements then one example of a $d_i$ function might be $d_i = 4/3[1 - 1/(1+d_{i1})(1+d_{i2})]$.

**Example 2.** Now let us consider the three commodity case ($k=3$). Here one might consider one of the following functional forms.

(a) $d_i = 1/2[d_{i1}d_{i2} + d_{i3}]$, here goods 1 and 2 are substitutes with constant substitution effect.
(b) $d_i = 1/2[d_{i1} + d_{i2}]d_{i3}$, pair of attributes (1,3) and (2,3) are substitutes.
(c) $d_i = 4/7[d_{i1} - 1/(1+d_{i2})(1+d_{i3})]$, etc.

**Note** that we have

$$\frac{\partial^2 d_i}{\partial x_{ij}^2} = \frac{\partial^2 d_i}{\partial d_{ij}^2}\left(\frac{n_i^j}{n\;\mu_j}\right)^2, \quad \text{and} \quad \frac{\partial^2 d_i}{\partial x_{Ij}^2} = \frac{1}{(n\;\mu_j)^2}\,\frac{\partial^2 d_i}{\partial d_{ij}^2}.$$

So, these two own second derivatives also has the same sign (<0 ?). Structure imposed on the deprivation function through these might also help us in getting some insight into the possible forms it might take.

But, even with all these assumptions we do not get enough restrictions on the form of the function $d_i$ that give us a particular functional form or a small class of functions that is easily parametrizable. For the purpose of comparing two distributions $X, Y \in M$, we have the necessary apparatus in the form of the above assumptions and consequent restrictions. Even then, for the sake of analytical completeness, below we suggest a few stronger assumptions that could be imposed on the functional form of $d_i$ and demonstrate the consequence of these assumptions in the form of resulting specific functional form for $d_i$.

First we propose the following postulate.

**(B5).** $\partial^3 d_i / \partial d_{ij_1} \partial d_{ij_2} \partial d_{ij_3} = 0$ for all distinct $j_1, j_2, j_3 \in \{1, \dots, k\}$.

This very awkward looking assumption is actually quite simple when we look at it in terms of substitution effect. This assumption says that the amount held of any third commodity do not affect the substitution/complementation effect between two other commodities. Thus, we are essentially neglecting third order interactions which are most often negligible and difficult to interpret in empirical literature anyway. We incorporate this feature into our deprivation function mainly for analytical simplicity, but this assumption does not impose too much on the form of the deprivation function. We still retain a fairly rich class of admissible functions that contains all the simple functions we have considered and which are most likely to be useful in practice. For instance, this assumption is satisfied by a class of functions similar to the popular class of utility functions known as the CES (Constant Elasticity of Substitution) utility function

$$(\sum_{i=1}^{n} x_i^r)^{1/r}$$

where elasticity of substitution between any two commodities is a constant. This class is given by

$$\sum_{i,j} \sum_{=1}^{k} (x_i^r + x_j^r)^{1/r}.$$

As a consequence of (B5) we can easily obtain the following lemma.

**Lemma 3.** If the deprivation function defined in (6) satisfies (B1)–(B5), then it must be of the form

$$d_i(d_{i1}, d_{i2}, \dots, d_{ik}) = \sum_{j_1 \neq j_2 \in \{1, \dots, k\}} \alpha_{i(j_1 j_2)} d_{i(j_1 j_2)}(d_{ij_1}, d_{ij_2}), \tag{7}$$

where $\alpha_{i(j_1 j_2)}$ are arbitrary positive constants satisfying (B1). The derivatives of $d_{i(j_1 j_2)}$ has the signs postulated by (B2)–(B4).

**Proof.** [9]

To simplify notation, consider the function $y = f(x_1, x_2, \dots, x_k)$. In terms of this function, (B5) implies $\partial^3 f / \partial x_i \partial x_j \partial x_l = 0$ for distinct $i, j, l \in \{1, \dots, k\}$.

Now consider $j=1$, $l=2$ and $i \geq 3$. Then we have $\partial / \partial x_i (\partial^2 f / \partial x_1 \partial x_2) = 0$ for $i = 3, \dots, k$. By repeated integration we get,

$$f(x_1, x_2, \dots, x_k) = f_{12}(x_1, x_2) + k\ g(x_3, \dots, x_k),$$

where $k$ is an arbitrary constant of integration. Note that $(x_1, x_2)$ plays no special role,

[9] The theory of integration applied here is explained in Apostol (1969, Chapter 11, pp. 353–416).

this relation is true for any pair of variables. Generalizing to all possible pairs we have the solution as given by (7). □

That is, we can see that the consequence of the assumption (B5) is that the function $d_i$ becomes pairwise strictly separable in the $d_{ij}$ variables. Now that we have broken down the form of the function $d_i$ with any arbitrary number of commodities into a number of two variable functions, we can focus on the pairwise interaction terms or the $d_{i(j_1 j_2)}$ functions and look for refinements of (B4) in order to realize some specific forms for these functions which we call the *pairwise deprivation functions*. On these, we might impose the following stricter version of (B4).

**(B4′).**

(a) $\partial^2 d_{i(j_1 j_2)} / \partial d_{ij_1} \partial d_{ij_2} = \gamma > 0$. That is, we are imposing a constant substitution effect.
(b) $\partial^2 d_{i(j_1 j_2)} / \partial d_{ij_1} \partial d_{ij_2} = -\gamma < 0$. That is, constant complementation effect.
(c) $\partial^2 d_{i(j_1 j_2)} / \partial d_{ij_1} \partial d_{ij_2} = -(d_{i(j_1 j_2)})^2 < 0$. This postulate says that the complementation effect increases with the level of pairwise deprivations. This assumption is consonant with a very simple form of deprivation function, which is quite intuitive, as we will see below.

Note that (B4′)(a) and (b) ⇒ (B5). These alternative assumptions in (B4′) results in several choices of pairwise deprivation functions.

**Lemma 4.**

(a) If $d_{i(j_1 j_2)}$ functions as defined in (7) also satisfy (B4′) (a) then it must be of the form

$$d_{i(j_1 j_2)} = A(d_{ij_1}) + B(d_{ij_2}) + \gamma d_{ij_1} d_{ij_2},$$

where $A', B' > 0$, and $A, B$ satisfies some boundary conditions at 0 and 1 imposed by (B1), e.g., $A(0) + B(0) = 0$, $A(1) + B(1) = 1 - \gamma (> 0 \Rightarrow \gamma < 1)$.
(b) If $d_{i(j_1 j_2)}$ satisfies (B4′) (b) then it must be of the form

$$d_{i(j_1 j_2)} = A(d_{ij_1}) + B(d_{ij_2}) + \gamma(d_{ij_1} + d_{ij_2}) - \gamma d_{ij_1} d_{ij_2},$$

where $A', B' > 0$. Similarly, here we also need $A(0) + B(0) = 0$, $A(1) + B(1) = 1 - \gamma (> 0 \Rightarrow \gamma < 1)$.
(c) (B4′) (c) is satisfied by

$$d_{i(j_1 j_2)} = -\frac{1}{(\beta_{j_1} + d_{ij_1})(\beta_{j_2} + d_{ij_2})},$$

where $\beta_{j_1}$ and $\beta_{j_2}$ are such that the measure satisfies (B1). This measure is analogous to the one discussed in Example 1.

**Proof.** The proof of this lemma uses techniques similar to that used in Lemma 3. Here also, repeated integration gives us the desired results. For illustration, we prove the case (a).

We have, again using simplified notation, $\partial^2 f(x_1, x_2)/\partial x_1 \partial x_2 = \gamma$. Integration with respect to $x_1$ gives

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \gamma x_1 + k_1(x_2)$$

and, integrating again

$$f(x_1, x_2) = \gamma x_1\, x_2 + K_1(x_2) + p_1.$$

Now, reversing the order of integration we get

$$f(x_1, x_2) = \gamma x_1\, x_2 + K_2(x_1) + p_2,$$

where $p_1, p_2$ are arbitrary constants of integration. From (B1) we have $p_1 = p_2 = 0$. Since the two integrals have to be identical, we have the desired result

$$f(x_1, x_2) = \gamma x_1\, x_2 + K_2(x_1) + K_1(x_2)$$

where $K_1, K_2$ are strictly increasing due to (B2) and (B3). □

From Lemmata 3 and 4 we can conclude the following simple characterization.

**Proposition 2.** If the deprivation function $d_i$ is as given in (7), that is, it satisfies (B1)–(B5), and

(a) if all commodity pairs satisfy the axiom (B4′) (a), then $d_i$ must be of the form

$$d_i = \sum_{j=1}^{k} A_j(d_{ij}) + \sum_{l \neq} \sum_{j=1}^{k} \gamma_{lj} d_{il} d_{ij}. \tag{8}$$

(b) if all commodity pairs satisfy (B4′) (b) then $d_i$ must be of the form

$$d_i = \sum_{j=1}^{k} A_j(d_{ij}) + \sum_{j=1}^{k} \delta_j d_{ij} - \sum_{l \neq} \sum_{j=1}^{k} \gamma_{lj} d_{il} d_{ij}, \tag{9}$$

where $\delta_j = \delta(\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{kj})$.

(c) Consider the set $C = \{(j, j') | 1 \leq j, j' \leq k$ and attributes $j$ and $j'$ are complements$\}$. This is the set that contains all the pair of attributes that are complements to each other. Call the corresponding set of substitute as $S$. Also define $C_1 = \{j | (j, j') \in C$ for some $1 \leq j' \leq k\}$. If all the pair of attributes in $C$ satisfy (B4′) (b) and those in $S$ satisfy (B4′) (a), then $d_i$ must be of the form

$$d_i = \sum_{j=1}^{k} A_j(d_{ij}) + \sum_{(j_1, j_2) \in S} \gamma_{j_1 j_2} d_{ij_1} d_{ij_2} - \sum_{(j_1, j_2) \in C} \gamma_{j_1 j_2} d_{ij_1} d_{ij_2} + \sum_{j \in C_1} \delta_j d_{ij}. \tag{10}$$

**Proof.** The proof of this result follows from combining the Lemmata 3 and 4. The particular form of the function $d_i$ follows from combining the forms of the pairwise deprivation functions obtained from Lemma 4. □

The above proposition essentially gives a simple class of deprivation functions as a consequence of the axioms postulated. The functional forms $A_j(.)$ and the constants $\delta_j$ in the Eqs. (8)–(10) determine the relative importance of the commodities under consideration. The $\gamma$ parameters show the degree of substitution or complementation that is present between the elements of any pair of attributes. The particular choices that are exercised in practice would often indeed be one of these as they are simply parametrized and hence easy to implement.

Precise choice of any form is essentially a matter of value judgment. But, this is eminently a non-trivial task. One has to decide upon the extent of substitution/complementation between each pair of chosen attributes. Also, some of the attributes are not perfectly divisible (e.g., access to malaria prevention programme is either 0 for denied or 1 for accessed). They must be measured on a suitably defined scale.

Note that, though we have suppressed the '$i$' suffix in the statement of (B4′) and Proposition 2, the form of $d_i$ in each of Eqs. (8)–(10) are actually individual specific. So we are not restricting all individuals to have identical preferences.

Here also, one can define the RDC in terms of aggregate personal deprivation ($d_i$) values and deprivation ordering in terms of this curve in a manner similar to the attribute wise RDCs defined in Section 2.

## 4. Social deprivation

After we have pinned down a specific form for the personal deprivation function $d_i$, the next logical step is to aggregate these deprivation values across individuals. So, in this section, it is our aim to devise suitable methods for measuring social deprivation as some aggregate function of the personal deprivation values. Note that, once we have aggregated across commodities, the form of these functions are identical to their single-attribute counterparts. In the literature, some indices has been proposed. Chakravarty and Mukherjee (1999a) provides a survey on this topic. We will now discuss a few of these here to illustrate the social deprivation function. We define a social deprivation function $d$ as

$$d = d(d_1, d_2, \ldots, d_n): R^n_+ \rightarrow R_+ \qquad (11)$$

where $d$ is (C1) continuous, (C2) increasing along the ray of equality and each deprivation contour crosses the ray of equality, (C3) symmetric and (C4) quasi-concave.

The continuity and symmetry assumptions are quite standard in the deprivation literature. The second assumption ensures that as the individual deprivation values increase, the social deprivation will also increase. Quasi-concavity is requiring that the deprivation contours are convex to the origin. This assumption is reasonable in that it demands the social deprivation function to be more sensitive to the deprivation of poorer persons (in terms of aggregate deprivation). In this section, we will only consider those social deprivation functions that satisfy (C1)–(C4).

To illustrate the general formula in (11) let us consider the following examples:

(a) symmetric mean of order $r$

$$\begin{aligned} d_{(r)}(d_1,\ldots,d_n) &= \left(\frac{1}{n}\sum d_i^r\right)^{1/r},\ r\leq 1, r\neq 0 \\ &= (\Pi d_i)^{1/n},\ r=0. \end{aligned} \tag{12}$$

The parameter $r$ in (12) determines the curvature of the social deprivation contours. As $r$ decreases, $d$ in (12) becomes more sensitive to the deprivation of the poorer persons. For $r=1$, $d$ becomes the Gini deprivation function (Kakwani, 1984)

$$d_{(1)}(d_1,\ldots,d_n)=\frac{1}{n}\sum d_i.$$

(b) Kolm-Pollak type

$$d_\theta(d_1,\ldots,d_n) = -\frac{1}{\theta}\log\frac{\sum e^{-\theta d_i}}{n},\quad \theta\geq 0. \tag{13}$$

Again $\theta$ determines the curvature of deprivation contours. As $\theta\to 0$, $d_\theta\to$ Gini deprivation function (Yitzhaki, 1979).

Many other measures of social deprivation can be derived along this line. Here our aim was to give only a few for the purpose of illustration. The choice of a particular social deprivation function is a subjective issue.

We now consider an important aspect of multidimensional measurement of deprivation, namely, the correlation among attributes. the property we now introduce generalizes an egalitarian principle considered by Atkinson and Bourguignon (1982). The Atkinson–Bourguignon (AB) study focuses on the welfare ranking of matrices $\in M^n$ using the additively separable social welfare function:

$$W(X)=U(\boldsymbol{x}_1)+U(\boldsymbol{x}_2)+\cdots+U(\boldsymbol{x}_n).$$

This property explores the possibility of the correlation among the $\mathbf{x}^j$ vectors affecting deprivation. Let us consider the following simple example of switching some amount of one attribute from one person to another. We consider $n=3$ and $k=2$.

$$A=\begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 7 & 2 \end{bmatrix}\to\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 7 & 5 \end{bmatrix}=B.$$

The switch is in attribute 2 from person 2 to person 3 resulting in person 3 having more of both attributes than person 2. This results in an increase in correlation among the two columns of the distribution matrix. We call this a *correlation increasing switch*. *AB* explored the restriction on $U$ so that we have $W(A)\geq W(B)$, or, in general, $W(X)\geq W(Y)$ whenever $Y$ is obtained from $X$ by a correlation increasing switch.

In the context of deprivation measurement, it seems intuitively reasonable to argue that the aggregate deprivation $d_i$ should not be higher in $A$ than in $B$. the following axiom captures this property.

**(C5).** Let $X$ and $Y$ be two matrices $\in M^n$ such that $x_{11} \leq x_{21} \leq \cdots \leq x_{n1}$ and $y_{11} \leq y_{21} \leq \cdots \leq y_{n1}$. Then for any deprivation function $d$, $d(X) \leq d(Y)$ whenever $Y$ is obtained from $X$ by a sequence of correlation increasing switches in the attribute distribution vectors.

Suppose a correlation increasing switch in the $j$th commodity (in concordance with the $l$th commodity) takes place between persons $i_1$ and $i_2$. That is, $x_{i_1 j} > x_{i_2 j}$, $x_{i_1 l} < x_{i_2 l}$ and the amounts after switch is $y_{i_1 j} = x_{i_2 j} < x_{i_1 j} = y_{i_2 j}$. Now one can easily check that only person $i_1$ and $i_2$'s deprivations are affected by this switch. So the net change in social deprivation is (ignoring things that remain constant)

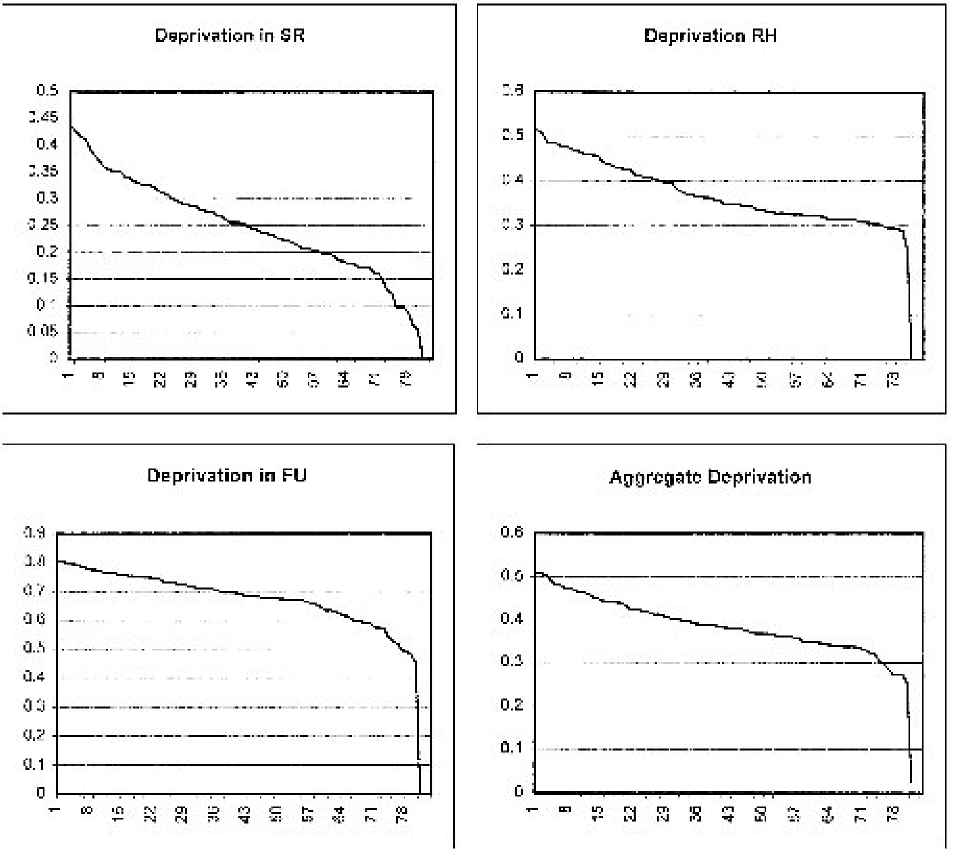
$$d(d_{i_1}(x_{i_1 j},x_{i_1 l}),d_{i_2}(x_{i_2 j},x_{i_2 l})) - d(d_{i_1}(x_{i_2 j},x_{i_1 l}),d_{i_2}(x_{i_1 j},x_{i_2 l})). \tag{14}$$

$AB$ condition stipulates that the expression in (14) be $\leq 0$. Now using a little algebraic manipulation and the fact that $\partial d_{ij}/\partial x_{ij} \leq 0$, one can show that, in the case when the commodities $j$ and $l$ are substitutes, this is true if the deprivation contours are convex to the origin. When the two commodities are complements, actually the reverse phenomenon occurs. Person $i_1$ being more deprived in both commodities becomes socially better than the two persons $i_1$ and $i_2$ feeling deprived in one commodity each. This is due to the fact that the feeling of deprivation, in two commodities that are complements, has an overall mitigating effect. Hence, we have the following proposition.

**Proposition 3.** *For the class of deprivation measures we consider here, the AB condition is automatically satisfied when the correlation increasing switch occurs involving two commodities that are substitutes. The reverse is seen when they are complements.*

## 5. Numerical illustration

The illustration provided here is based on survey data on basic needs collected by Rudra et al. (1995). This survey was conducted in December 1990–May 1991 covering five districts of the state of West Bengal, India. A total of 2598 households was surveyed and data collected on 17 basic need attributes. For our purpose, we chose three attributes: (i) number of 'saris' per adult female in the household, where sari is the chief garment for an Indian woman (SR), (ii) the roof height of the dwelling (RH) and (iii) number of months for which the household members had two square meals a day throughout the last year (365 days) (FU). We chose data from two of the districts, namely Darjeeling and Jalpaiguri. A small sample of 81 (99) households for the district of Darjeeling (Jalpaiguri) were used. We plotted the RDC according to the three attributes for the two districts in Figs. 1 and 2.




Number of households = 81, Aggregate social deprivation = 0.3794464

Fig. 1. District Darjeeling.

To aggregate the commodity specific deprivations to arrive at the personal deprivation value, we make use of the formula [Example 2(a)]

$$d_i = \frac{1}{2}\,[d_{i1}d_{i2} + d_{i3}],$$

where 1,2 and 3 denote SR, RH and FU, respectively. We also plot the RDC according to aggregate deprivation for the two districts in the same figures.

To aggregate the personal deprivation values to get the aggregate social deprivation, we use the *symmetric mean of order r* considered in (12) where $r = 1/2$. More specifically, to compute $d$, we use the following:

$$d(d_1, d_2, \ldots, d_n) = \left(\frac{1}{n}\sum d_i^{1/2}\right)^2.$$

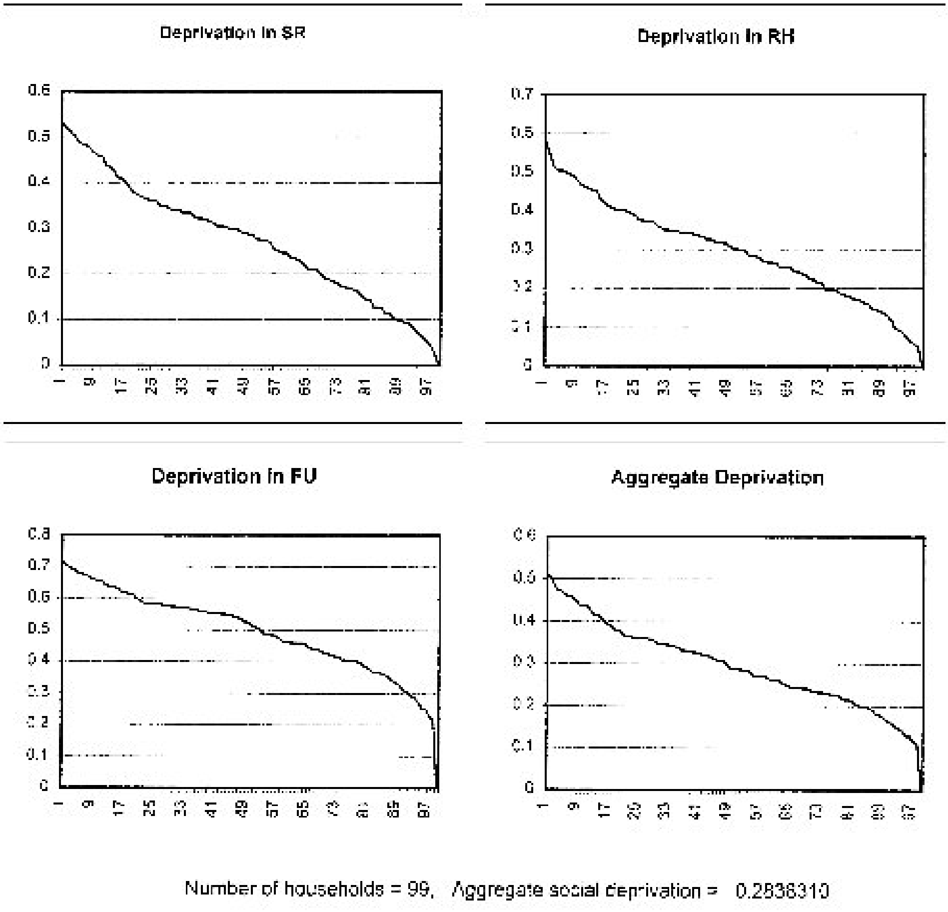



Fig. 2. District Jalpaiguri.

These values are also reported in the two figures.

From the two figures, we can see that the RDCs according to the attribute SR are quite similar for the two districts. On the other hand, comparing the RDCs according to RH, it is easily observable that the proportion of population with high deprivation in RH is much more in Darjeeling than in Jalpaiguri. A somewhat similar picture is also seen for FU. Consequently, the aggregate RDC for Darjeeling also shows a more sudden drop at the end (very low proportion of people with low deprivation) than that for Jalpaiguri.

We do not make any general statements about the comparative deprivation situation of the two districts because our calculations are sample and choice of measure specific. Any conclusions based on these sort of computations are highly susceptible to such choices. Our calculations and consequent observations no way lead to any sort of policy statements. These should only be considered in the light of helping to make our concepts and definitions easier to understand.

## 6. Conclusion

A person feels deprived if someone in the society has more of an attribute than this person. This feeling of deprivation has been modeled in the literature as a function of the difference of holdings between these two persons. Most authors have focussed on measuring deprivation in terms of income only. But, the feeling of deprivation is essentially a multifaceted phenomenon. So, in this paper, we have tried to devise measures of a person's and society's feeling of deprivation in terms of many attributes which are considered necessary in the society. We have first modeled a person's feeling of deprivation in terms of a single commodity only and provided a simple characterization of such measure. This measure is quite similar to the income based measure of deprivation for a single person. The paper goes on to propose suitable measures of a person's aggregate deprivation combining the commodity wise deprivation values. We also characterize a few alternative measures using intuitive and plausible axioms. Finally, we define a benchmark by which two multi-attribute distribution matrices can be compared in terms of a partial order. We call this the deprivation order, both in terms of a single attribute and also in aggregate.

The next problem that we tackle is how to aggregate the personal deprivation of all persons in the society to arrive at a social deprivation function. We discuss the desirable properties a social deprivation function should have and give a few illustrations. In this context, we also discuss an egalitarian principle referred to as the *AB* condition and relate this to our social deprivation measures. The empirical illustration helps to clarify our proposed definitions and orderings.

## Acknowledgements

I am extremely grateful to Dr. Peter J. Lambert for starting me on this problem. I would also like to thank Dr. Satya R. Chakravarty and Mr. Bratisankar Chakraborty for helpful discussions. My thanks go to seminar participants at the Fifth Meeting of the Society for Social Choice and Welfare held in Alicante, Spain, Jawaharlal Nehru University, New Delhi and Jadavpur University, Kolkata for their interesting suggestions and two anonymous referees for their constructive criticism and suggestions. The general disclaimer applies.

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