More on positive subdefinite matrices and the linear complementarity problem

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Abstract

In this article, we consider positive subdefinite matrices (PSBD) recently studied by J.-P. Crouzeix et al. [SIAM J. Matrix Anal. Appl. 22 (2000) 66] and show that linear complementarity problems with PSBD matrices of rank $\geqslant 2$ are processable by Lemke's algorithm and that a PSBD matrix of rank $\geqslant 2$ belongs to the class of sufficient matrices introduced by R.W. Cottle et al. [Linear Algebra Appl. 114/115 (1989) 231]. We also show that if a matrix A is a sum of a merely positive subdefinite copositive plus matrix and a copositive matrix, and a feasibility condition is satisfied, then Lemke's algorithm solves LCP(q, A). This supplements the results of Jones and Evers.

Keywords: Sufficient matrix; Pseudomonotone; Copositive star; Lemke's algorithm

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1. Introduction

We say that a real square matrix A of order n is *positive subdefinite* (PSBD) if for all $x \in \mathbb{R}^n$

$$x^{t}Ax < 0 \quad \text{implies} \quad \text{either } A^{t}x \leqslant 0 \text{ or } A^{t}x \geqslant 0.$$

The class of PSBD matrices is a generalization of the class of positive semidefinite matrices and is useful in the study of quadratic programming problem. The class of

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symmetric PSBD matrices has been introduced by Martos [8] while characterizing a pseudo-convex quadratic function. Cottle and Ferland [3] also studied the class of PSBD matrices nearly at the same time in connection with the class of quadratic pseudo-convex functions. Recently nonsymmetric PSBD matrices have been studied by Crouzeix et al. [4], in the context of generalized monotonicity and the linear complementarity problem.

Given a real square matrix A of order n and a vector $q \in \mathbb{R}^n$, the linear complementarity problem is to find $w \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ such that

$$w - Az = q, \quad w \geqslant 0,\ z \geqslant 0, \tag{1.1}$$

$$w^{\mathrm{t}} z = 0. \tag{1.2}$$

This problem is denoted as LCP(q, A). It is well studied in the literature on Mathematical Programming and arises in a number of applications in Operations Research, Mathematical Economics and Engineering. In particular, the problem of computing a Karush–Kuhn–Tucker point of a convex quadratic programming problem with linear constraints on the variables can be formulated as an LCP. For recent books on this problem, see [1,9].

In this paper we use the following convention. Suppose a class of matrices $\mathscr{C} \subseteq \mathbb{R}^{n\times n}$ is defined by specifying a property which is satisfied by each square matrix of order n in $\mathscr{C}$. We then say that A is a $\mathscr{C}$ matrix. Thus the symbol $\mathscr{C}$ is used for the class of matrices satisfying the specified property as well for the property itself. For the definition of various classes of matrices see Section 2.

In this paper, we study PSBD matrices and related classes. In Section 2, we present the required definitions, introduce the notations and state the relevant results used in this paper. In Section 3, we prove our main results.

2. Preliminaries

We consider matrices and vectors with real entries. Any vector $x \in \mathbb{R}^n$ is a column vector unless otherwise specified, and x^{t} denotes the row transpose of x. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$. For any vector $x \in \mathbb{R}^n$, x^+ and x^- are the vectors whose components are $x_i^+ (= \max\{x_i, 0\})$ and $x_i^- (= \max\{-x_i, 0\})$, respectively, for all i. We say that a vector $x \in \mathbb{R}^n$ is *unisigned* if either $x \in \mathbb{R}^n_+$ or $-x \in \mathbb{R}^n_+$. For any matrix $A \in \mathbb{R}^{m\times n}$, a_{ij} denotes its ith row and jth column entry. For any matrix $A \in \mathbb{R}^{m\times n}$, let $A_{i.}$ denote its ith row and $A_{.j}$ denote its jth column. For any positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. For any set $\alpha \subseteq \{1, 2, \ldots, n\}$, $\bar{\alpha}$ denotes its complement in $\{1, 2, \ldots, n\}$. If A is a matrix of order $n \times n$, $\alpha \subseteq \{1, 2, \ldots, n\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$, then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β, respectively. Given a symmetric matrix $S \in \mathbb{R}^{n\times n}$, let $\nu_+(S)$, $\nu_-(S)$, $\nu_0(S)$ denote the number of positive, negative and zero eigenvalues of S, respectively. Let A be a given $m \times n$ matrix, not necessarily symmetric. We say that A is *positive semidefinite* (PSD) if $x^{\mathrm{t}}Ax \geqslant 0\, \forall x \in \mathbb{R}^n$ and

A is *positive definite* (PD) if $x^t Ax > 0\ \forall 0 \neq x \in \mathbb{R}^n$. A is said to be *merely positive subdefinite* (MPSBD) if A is a PSBD matrix but not a PSD matrix. A is said to be a $P(P_0)$ *matrix* if all its principal minors are positive (nonnegative). A is said to be *column sufficient* if for all $x \in \mathbb{R}^n$ the following implication holds:

$$x_i(Ax)_i \leqslant 0\ \forall i \quad \text{implies} \quad x_i(Ax)_i = 0\ \forall i.$$

A is said to be *row sufficient* if A^t is column sufficient. A is *sufficient* if A and A^t are both column sufficient. For details see [1,2,11].

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ we define the feasible set $F(q, A) = \{z \geqslant 0 \mid Az + q \geqslant 0\}$ and the solution set of LCP(q, A) by $S(q, A) = \{z \in F(q, A) \mid z^t(Az + q) = 0\}$. We say that A is a Q_0 *matrix* if $F(q, A) \neq \emptyset$ implies $S(q, A) \neq \emptyset$. Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, an affine map $\mathscr{F}(x) = Ax + q$, where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, is said to be *pseudomonotone* on $\mathbb{R}^n_+$ if

$$(y - z)^t(Az + q) \geqslant 0,\ \ y \geqslant 0,\ \ z \geqslant 0 \quad \Rightarrow \quad (y - z)^t(Ay + q) \geqslant 0.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be pseudomonotone if $\mathscr{F}(x) = Ax$ is pseudomonotone on the nonnegative orthant. Crouzeix et al. [4] proved that an affine map $\mathscr{F}(x) = Ax + q$, where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, is pseudomonotone if and only if

$$z \in \mathbb{R}^n, \quad z^t Az < 0 \quad \Rightarrow \quad \begin{cases} A^t z \geqslant 0 \ \text{ and } \ z^t q \geqslant 0 \ \text{ or} \\ A^t z \leqslant 0,\ \ z^t q \leqslant 0 \ \text{ and } \ z^t(Az^- + q) < 0. \end{cases}$$

$A \in \mathbb{R}^{n\times n}$ is said to be *copositive* (C_0) if $x^t Ax \geqslant 0\ \forall x \geqslant 0$ and *conegative* if $x^t Ax \leq 0\ \forall x \geqslant 0$. We say that $A \in \mathbb{R}^{n\times n} \cap C_0$ is *copositive plus* (C_0^+) if

$$\left[x^t Ax = 0,\ x \geqslant 0\right] \ \Rightarrow\ (A + A^t)x = 0$$

and *copositive star* (C_0^*) if

$$\left[x^t Ax = 0, Ax \geqslant 0,\ x \geqslant 0\right] \ \Rightarrow\ A^t x \leqslant 0.$$

We require the following theorems in the next section. For proof of these results see Crouzeix et al. [4].

Theorem 2.1 [4, Proposition 2.1]. *Let $A = ab^t$, where $a \neq b$, $a, b \in \mathbb{R}^n$. A is PSBD if and only if one of the following holds*:

(i) $\exists$ *a* $t > 0$ *such that* $b = ta$,

(ii) *for all* $t > 0$, $b \neq ta$ *and either* $b \geqslant 0$ *or* $b \leqslant 0$.

Further suppose that $A \in$ MPSBD. Then $A \in C_0$ if and only if either ($a \geqslant 0$ and $b \geqslant 0$) or ($a \leqslant 0$ and $b \leqslant 0$) and $A \in C_0^*$ if and only if A is copositive and $a_i = 0$ whenever $b_i = 0$.

Combining Theorem 2.1 and Proposition 2.5 in [4], we get:

Theorem 2.2 [4, Theorem 2.1]. *Suppose $A \in \mathbb{R}^{n\times n}$ is PSBD and* rank$(A) \geqslant 2$. *Then A^t is PSBD and at least one of the following conditions holds*:

(i) *A is PSD*,

(ii) $(A + A^t) \leqslant 0$,
(iii) A *is* C_0^*.

Theorem 2.3 [4, Proposition 2.2]. *Assume that* $A \in \mathbb{R}^{n\times n}$ *is MPSBD and* $\operatorname{rank}(A) \geqslant 2$. *Then*
(a) $\nu_-(A + A^t) = 1$,
(b) $(A + A^t)z = 0 \Leftrightarrow Az = A^t z = 0$.

Theorem 2.4 [4, Theorem 3.3]. *A matrix* $A \in \mathbb{R}^{n\times n}$ *is pseudomonotone if and only if A is PSBD and copositive with the additional condition that in case* $A = ab^t$, $b_i = 0 \Rightarrow a_i = 0$.

Theorem 2.5 [5, Corollary 4]. *If A is pseudomonotone, then A is a row sufficient matrix.*

Murthy and Parthasarathy [10] have proved the following result on nonnegative square matrices.

Theorem 2.6 [10, Theorem 2.5]. *Let* $A \geqslant 0$ *be an* $n \times n$ *matrix. A is a* Q_0*-matrix if and only if for any* i, $A_{i\cdot} \neq 0 \Rightarrow a_{ii} > 0$.

3. PSBD and MPSBD matrices

Since a PSBD matrix is a natural generalization of a PSD matrix, it is of interest to determine which of the properties of a PSD matrix also holds for a PSBD matrix. In particular we may ask whether
(i) A is PSBD if and only if $(A + A^t)$ is PSBD and
(ii) any PPT (Principal Pivot Transform [1, p. 79]) of a PSBD matrix is a PSBD matrix.

The following examples show that these statements are false.

Example 3.1. Let

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x^t Ax = x_1 x_2 < 0$ implies x_1 and x_2 are of opposite sign. Clearly $A \in$ PSBD since $x^t Ax < 0$ and $A^t x = \begin{bmatrix} -x_2 \\ 2x_1 \end{bmatrix}$ imply either $A^t x \leqslant 0$ or $A^t x \geqslant 0$.

Also it is easy to see that

$$A + A^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is not a PSBD matrix.

Similarly let

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \quad \text{so that} \quad A + A^{\mathrm{t}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

It is easy to verify that $A + A^{\mathrm{t}}$ is PSBD but A is not a PSBD matrix.

Example 3.2. Let us consider the matrix

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

in Example 3.1. Note that $A \in$ PSBD but it is easy to see that

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ 0.5 & 0 \end{bmatrix}$$

is not a PSBD matrix.

Since A^{-1} is a PPT of A therefore any PPT of a PSBD matrix is not a PSBD matrix.

Theorem 3.1. *Suppose $A \in \mathbb{R}^{n\times n}$ is a PSBD matrix. Then $A_{\alpha\alpha} \in$ PSBD, where $\alpha \subseteq \{1, \ldots, n\}$.*

Proof. Let $A \in$ PSBD and $\alpha \subseteq \{1, \ldots, n\}$. Let $x_\alpha \in \mathbb{R}^{|\alpha|}$ and

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

Suppose that $x_\alpha^{\mathrm{t}} A_{\alpha\alpha} x_\alpha < 0$. Now define $z \in \mathbb{R}^n$ by taking $z_\alpha = x_\alpha$ and $z_{\bar{\alpha}} = 0$. Then $z^{\mathrm{t}} A z = x_\alpha^{\mathrm{t}} A_{\alpha\alpha} x_\alpha$. Since A is a PSBD matrix $z^{\mathrm{t}} A z = x_\alpha^{\mathrm{t}} A_{\alpha\alpha} x_\alpha < 0 \Rightarrow$ either $A^{\mathrm{t}} z \geqslant 0$ which implies that $A_{\alpha\alpha}^{\mathrm{t}} x_\alpha \geqslant 0$ or $A^{\mathrm{t}} z \leqslant 0$ (which implies $A_{\alpha\alpha}^{\mathrm{t}} x_\alpha \leqslant 0$). Hence $A_{\alpha\alpha} \in$ PSBD. As α was arbitrary, it follows that every principal submatrix of A is a PSBD matrix. □

Theorem 3.2. *Suppose $A \in \mathbb{R}^{n\times n}$ is a PSBD matrix. Let $D \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix. Then $A \in$ PSBD if and only if $DAD^{\mathrm{t}} \in$ PSBD.*

Proof. Let $A \in$ PSBD. For any $x \in \mathbb{R}^n$ let $y = D^{\mathrm{t}} x$. Note that $x^{\mathrm{t}} DAD^{\mathrm{t}} x = y^{\mathrm{t}} A y < 0 \Rightarrow A^{\mathrm{t}} y = A^{\mathrm{t}} D^{\mathrm{t}} x \leqslant 0$ or $A^{\mathrm{t}} y = A^{\mathrm{t}} D^{\mathrm{t}} x \geqslant 0$. This implies that either $DA^{\mathrm{t}} D^{\mathrm{t}} x \leqslant 0$ or $DA^{\mathrm{t}} D^{\mathrm{t}} x \geqslant 0$, since D is a positive diagonal matrix. Thus $DAD^{\mathrm{t}} \in$ PSBD. The converse follows from the fact that D^{-1} is a positive diagonal matrix and $A = D^{-1}(DAD^{\mathrm{t}})(D^{-1})^{\mathrm{t}}$. □

Theorem 3.3. *PSBD matrices are invariant under principal rearrangement, i.e., if $A \in \mathbb{R}^{n\times n}$ is a PSBD matrix and $P \in \mathbb{R}^{n\times n}$ is any permutation matrix, then $PAP^{\mathrm{t}} \in$ PSBD.*

Proof. Let $A \in$ PSBD and let $P \in \mathbb{R}^{n\times n}$ be any permutation matrix. For any $x \in \mathbb{R}^n$, let $y = P^{\mathrm{t}}x$. Note that $x^{\mathrm{t}}PAP^{\mathrm{t}}x = y^{\mathrm{t}}Ay < 0 \Rightarrow A^{\mathrm{t}}y = A^{\mathrm{t}}P^{\mathrm{t}}x \leqslant 0$ or $A^{\mathrm{t}}y = A^{\mathrm{t}}P^{\mathrm{t}}x \geqslant 0$. This implies that either $PA^{\mathrm{t}}P^{\mathrm{t}}x \leqslant 0$ or $PA^{\mathrm{t}}P^{\mathrm{t}}x \geqslant 0$, since P is just a permutation matrix. It follows that PAP^{t} is a PSBD matrix. The converse follows from the fact that $P^{\mathrm{t}}P = I$ and $A = P^{\mathrm{t}}(PAP^{\mathrm{t}})(P^{\mathrm{t}})^{\mathrm{t}}$. □

We now settle the question *whether PSBD $\subseteq Q_0$ and Lemke's algorithm possesses PSBD matrices.* In this connection we rewrite Theorem 2.1 as follows.

Theorem 3.4. *Let $A = ab^{\mathrm{t}} \in \mathbb{R}^{n\times n}$, $a, b \in \mathbb{R}^n$, $a, b \neq 0$, be a PSBD matrix. Suppose either $a \geqslant 0$ or $a \leqslant 0$ when $b \neq ta$ for any $t > 0$. Then $A \in Q_0$ if and only if one or more of the following conditions hold:*
(i) *A is PSD,*
(ii) *a and b have opposite signs,*
(iii) *a and b has the same sign and*

$$a_i = 0 \quad \textit{whenever} \;\; b_i = 0 \;\; \forall i = \{1, 2, \ldots, n\}. \tag{3.1}$$

Proof.
Case 1. There exists a $t > 0$ so that $b = ta$. It is easy to see that A is PSD and hence $A \in Q_0$.

Case 2. For all $t > 0$, $b \neq ta$. In this case it follows from Theorem 2.1 that either $b \geqslant 0$ or $b \leqslant 0$. Under our hypothesis about a, either $A \leqslant 0$ or $A \geqslant 0$. If $A \leqslant 0$, then $A \in Q_0$. But if $A \geqslant 0$, then from Theorem 2.6, it is easy to see that $A \in Q_0$ if and only if

$$a_i = 0 \quad \text{whenever} \;\; b_i = 0 \;\; \forall i = \{1, 2, \ldots, n\}. \qquad \square$$

Remark 3.1. Note that any PSBD matrix $A = ab^{\mathrm{t}} \in \mathbb{R}^{n\times n}$, $a, b \in \mathbb{R}^n$, $a, b \neq 0$ is a sufficient matrix if $a_i = b_i = 0$ or $a_i b_i > 0$. See [11, Corollary 4.2].

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a PSBD matrix with* rank$(A) \geqslant 2$ *and let $A + A^{\mathrm{t}} \leqslant 0$. We have:*
(i) *If $a_{ii} < 0$, then the column/row containing a_{ii} is nonpositive.*
(ii) *If A has a principal submatrix of the form*

$$\begin{bmatrix} 0 & a_{ks} \\ a_{sk} & 0 \end{bmatrix}$$

with $(a_{ks} + a_{sk}) < 0$, *then the* sth *and* kth *rows as well as* sth *and* kth *columns of* A *are nonpositive.*

Proof. By Theorem 2.2, A^t is a PSBD matrix. By Theorem 3.1 every principal submatrix of A as well as A^t are also PSBD matrices. To prove (i) we proceed as follows. Suppose the diagonal entry $a_{ii} < 0$. Let (assuming $i < k$) $\alpha = \{i, k\}$. Consider the 2×2 submatrix

$$A_{\alpha\alpha} = \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix},$$

which is a PSBD matrix. Now for any $x = \binom{x_1}{x_2} \in \mathbb{R}^2$,

$$x^t A_{\alpha\alpha} x = a_{ii} x_1^2 + x_1 x_2 (a_{ik} + a_{ki}) + a_{kk} x_2^2 < 0$$

if x is nonnegative with $x_1 > 0$, since by hypothesis, $a_{kk} \leqslant 0$ and $a_{ik} + a_{ki} \leqslant 0$. Thus $(A_{\alpha\alpha})^t x$ is unisigned for any nonnegative x with $x_1 > 0$. Now by taking $x_2 = 0$, $x_1 > 0$ we conclude that $a_{ik} \leqslant 0$. Applying the same argument for A^t and $(A^t_{\alpha\alpha}) = (A_{\alpha\alpha})^t$ we conclude that $A_{\alpha\alpha} x$ is also unisigned and hence $a_{ki} \leqslant 0$. This completes the proof of (i).

To prove (ii) we proceed as follows: Note that for any $y \in \mathbb{R}^n$,

$$y^t A y = \sum_{i=1}^{n} a_{ii} y_i^2 + \sum_{i<j} (a_{ij} + a_{ji}) y_i y_j .$$

By our hypothesis a_{ii} and $a_{ij} + a_{ji}$ are nonpositive for all i and j. Suppose now $a_{kk} = a_{ss} = 0$ and $(a_{ks} + a_{sk}) < 0$. In this case note that if $z \in \mathbb{R}^n$ is any vector such that $z_i = 0, i \neq k, s,\ \ z_k > 0$ and $z_s > 0$, then $z^t A z = z_s z_k (a_{ks} + a_{sk}) < 0$. Therefore it follows that for such a z, $A^t z$ is unisigned. Suppose now for some r, $r \neq s, k,\ a_{kr} > 0$. Choose $z_k = 1$. Let δ be a positive number such that $a_{kr} + a_{sr}\delta > 0$. It is easy to see that such a δ exists. Define the vector $\bar{z}$ by taking $\bar{z_i} = 0, i \neq k, s,\ \ \bar{z_k} = 1, \bar{z_s} = \delta$. Note that $A^t \bar{z}$ is not unisigned, a contradiction. This contradiction shows that $a_{kr} \leqslant 0\ \forall r$. In a similar manner it can be shown that a_{sr} is nonpositive for all r. From the fact that A^t is also a PSBD matrix, by a similar argument it follows that a_{rk} and a_{rs} are also nonpositive for all r.

This completes the proof. □

Lemma 3.2. *Suppose* $A \in \mathbb{R}^{n \times n}$ *is a PSBD matrix with* $\operatorname{rank}(A) \geqslant 2$ *and* $A + A^t \leqslant 0$. *If* A *is not a skew-symmetric matrix, then* $A \leqslant 0$.

Proof. Let the index sets L_1, L_2 and L be defined as follows:

$$L_1 = \{i \mid a_{ii} < 0\}; \quad L_2 = \{i \mid a_{ii} = 0,\ \exists k,\ \text{with } a_{kk} = 0,\ a_{ik} + a_{ki} < 0\}.$$

Note that if $i \in L_2$, then L_2 will also contain the index k that satisfies the defining conditions of L_2 for i. Let $L = L_1 \cup L_2$. By the hypothesis of the lemma L is nonempty, for otherwise, A is skew-symmetric. Consider the following partitioned form of A induced by the index set L:

$$PAP^{\mathrm{t}} = \begin{bmatrix} A_{LL} & A_{L\bar{L}} \\ A_{\bar{L}L} & A_{\bar{L}\bar{L}} \end{bmatrix},$$

where $\bar{L}$ denotes the set of indices $\{1, 2, \ldots, n\}\backslash L$ and P is the appropriate permutation matrix. (In what follows we will simply use the symbol A to denote PAP^{t}.) By the earlier lemmas, $A_{LL} \leqslant 0$, $A_{L\bar{L}} \leqslant 0$ and $A_{\bar{L}L} \leqslant 0$. Also note that by definition, $A_{\bar{L}\bar{L}}$ is a skew-symmetric matrix. For any $y \in \mathbb{R}^n$, let $y = \begin{pmatrix} y_L \\ y_{\bar{L}} \end{pmatrix}$ denote the corresponding partition of y. Note that

$$y^{\mathrm{t}}Ay = y_L^{\mathrm{t}} A_{LL} y_L + y_{\bar{L}}^{\mathrm{t}} A_{\bar{L}\bar{L}} y_{\bar{L}} + y_{\bar{L}}^{\mathrm{t}} (A_{\bar{L}L} + A_{L\bar{L}}^{\mathrm{t}}) y_L.$$

Since $A_{\bar{L}\bar{L}}$ is skew-symmetric it follows that for all $y \in \mathbb{R}^n$, $y_{\bar{L}}^{\mathrm{t}} A_{\bar{L}\bar{L}} y_{\bar{L}} = 0$. It follows that for all vectors y such that y_L is positive, $y^{\mathrm{t}}Ay$ is negative and hence both Ay and $A^{\mathrm{t}}y$ are unisigned. To complete the proof we need to show that none of the entries of $A_{\bar{L}\bar{L}}$ is positive. Suppose to the contrary that for some $s \in \bar{L}, r \in \bar{L}, a_{sr} > 0$. Choose ϵ such that

$$\epsilon \sum_{i \in L} a_{ir} + a_{sr} > 0.$$

Define the vector $\bar{y}$ by taking $y_i = \epsilon\ \forall i \in L$ and $y_i = 0\ \forall i \neq r \in \bar{L}$ and $y_r = 1$. Note that since each row and column of A_{LL} contains at least one negative entry and all the entries of A_{LL}, and $A_{\bar{L}L}$ are nonpositive it follows that $(A^{\mathrm{t}}y)_i < 0\ \forall i \in L$. Also by construction $(A^{\mathrm{t}}y)_r > 0$. This is a contradiction! Hence $A_{\bar{L}\bar{L}} \leqslant 0$ and the lemma follows. □

Theorem 3.5. *Suppose $A \in \mathbb{R}^{n\times n}$ is a PSBD matrix with* rank$(A) \geqslant 2$. *Then A is a Q_0 matrix.*

Proof. By Theorem 2.2, A^{t} is a PSBD matrix. Also by the same theorem, either $A \in$ PSD or $(A + A^{\mathrm{t}}) \leqslant 0$ or $A \in C_0^*$. If $A \in C_0^*$, then $A \in Q_0$ (see [1]). Now if $(A + A^{\mathrm{t}}) \leqslant 0$, and A is not skew-symmetric, then by Lemma 3.2 it follows that $A \leqslant 0$. In this case $A \in Q_0$ [1]. However if A is skew-symmetric, then $A \in$ PSD. Therefore $A \in Q_0$. □

Corollary 3.1. *Suppose A is a PSBD matrix with* rank$(A) \geqslant 2$. *Then LCP(q, A) is processable by Lemke's algorithm. If* rank$(A) = 1$, *(i.e., $A = ab^{\mathrm{t}}$, $a, b \neq 0$) and $A \in C_0$, then LCP(q, A) is processable by Lemke's algorithm whenever $b_i = 0 \Rightarrow a_i = 0$.*

Proof. Suppose rank$(A) \geqslant 2$. From Theorem 2.2 and the proof of Theorem 3.5, it follows that A is either a PSD matrix or $A \leqslant 0$ or $A \in C_0^*$. Hence LCP(q, A) is processable by Lemke's algorithm (see [1]). For PSBD $\cap\, C_0$ matrices of rank$(A) = 1$, i.e., for $A = ab^{\mathrm{t}}, a, b \neq 0$, such that $b_i = 0 \Rightarrow a_i = 0$. Note that $A \in C_0^*$ by Theorem 2.1. Hence LCP(q, A) with such matrices are processable by Lemke's algorithm. □

Theorem 3.6. *Suppose A is a PSBD* $\cap C_0$ *matrix with* $\text{rank}(A) \geqslant 2$. *Then* $A \in \mathbb{R}^{n\times n}$ *is a sufficient matrix.*

Proof. Note that by Theorem 2.2 A^{t} is a PSBD $\cap\, C_0$ matrix with $\text{rank}(A^{\text{t}}) \geqslant 2$. Now by Theorem 2.4, A and A^{t} are pseudomonotone. Hence A and A^{t} are row sufficient by Theorem 2.5. Therefore A is sufficient. □

The following example shows that in general PSBD matrices need not be a P_0 matrix.

Example 3.3. Let

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$x^{\text{t}}Ax = -2x_1x_2 < 0$$

implies x_1 and x_2 are of same sign. $A \in \text{PSBD}$, since $A^{\text{t}}x = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$ implies either $A^{\text{t}}x \leqslant 0$ or $A^{\text{t}}x \geqslant 0$ but $A \notin P_0$.

The following example shows that PSBD matrices need not be a Q_0 matrix in general.

Example 3.4. Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$A^{\text{t}}x = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$$

implies either $A^{\text{t}}x \leqslant 0$ or $A^{\text{t}}x \geqslant 0$. Hence $A \in \text{PSBD}$. Taking $q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ we note that $\text{LCP}(q, A)$ is feasible but has no solution. Therefore A is not a Q_0 matrix.

The following theorem provides a new sufficient condition to solve $\text{LCP}(q, A)$ by Lemke's algorithm. (See [1] for a detailed discussion on Lemke's algorithm.)

Theorem 3.7. *Suppose* $A \in \mathbb{R}^{n\times n}$ *can be written as* $M + N$, *where* $M \in MPSBD \cap C_0^+$, $\text{rank}(M) \geqslant 2$ *and* $N \in C_0$. *If the system* $q + Mx - N^{\text{t}}y \geqslant 0$, $y \geqslant 0$, *is feasible, then Lemke's algorithm for LCP*(q, A) *with covering vector* $d > 0$ *terminates with a solution.*

Proof. Assume that the feasibility condition of the theorem holds so that there exists an $x^0 \in \mathbb{R}^n$ and a $y^0 \in \mathbb{R}^n_+$ such that $q + Mx^0 - N^t y^0 \geqslant 0$. First we shall show that for any $x \in \mathbb{R}^n_+$, if $Ax \geqslant 0$ and $x^t Ax = 0$, then $x^t q \geqslant 0$. Note that for given $x \geqslant 0$, $x^t Ax = 0 \Rightarrow x^t(M+N)x = 0$ and since $M, N \in C_0$, this implies that $x^t Mx = 0$. As M is a MPSBD matrix $x^t Mx = 0 \Leftrightarrow x^t(M + M^t)x = 0 \Leftrightarrow (M + M^t)x = 0 \Leftrightarrow M^t x = 0 \Leftrightarrow Mx = 0$. See Theorem 2.3. Also since $Ax \geqslant 0$, it follows that $Nx \geqslant 0$ and hence $x^t N^t y^0 \geqslant 0$. Further since $q + Mx^0 - N^t y^0 \geqslant 0$ and $x \geqslant 0$, it follows that $x^t(q + Mx^0 - N^t y^0) \geqslant 0$. This implies that $x^t q \geqslant x^t N^t y^0 \geqslant 0$.

Now from Corollary 4.4.12 and Theorem 4.4.13 of [1, p. 277] it follows that Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ terminates with a solution. □

The following example shows that the class MPSBD $\cap\ C_0^+$ is nonempty.

Example 3.5. Let

$$M = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $x^t Mx = 2(x_1 + x_2)(x_1 + 2x_2)$. Using this expression it is easy to verify that $x^t Mx < 0 \Rightarrow$ either $M^t x \leqslant 0$ or $M^t x \geqslant 0$. Also it is easy to see that $M \in C_0^+$.

Remark 3.2. The above theorem cannot be extended to a PSBD matrix. Note that the class PSBD matrices includes PSD matrices. In the example below, we consider a matrix A which may be written as $M + N$, where $M \in$ nonsymmetric PSD and $N \in C_0$ and show that Theorem 3.7 does not hold.

Example 3.6. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Taking $q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ we note that LCP(q, A) is feasible but the problem has no solution. Therefore A is not a Q_0 matrix.

Let

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Note that M is a nonsymmetric PSD matrix of rank 2 and $N \in C_0$ and it is easy to check that the system $q + Mx - N^t y \geqslant 0,\ y \geqslant 0$, is feasible. Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ (for example $d = e$, where e is a n dimensional column vector of all 1s) terminates with a secondary ray for this q, as LCP(q, A) has no solution. Thus if M is a nonsymmetric PSD matrix, Theorem 3.7 does not hold. (See also [6] and [7].)

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