Hilbert Modules and Stochastic Dilation of a Quantum Dynamical Semigroup on a von Neumann Algebra

Debashish Goswami[1,⋆], **Kalyan B. Sinha**[2]

[1] Indian Statistical Institute, Delhi Centre, 7, SJSS Marg, New Delhi - 110016, India.
E-mail: debashis@isid.ac.in

[2] Indian Statistical Institute, Delhi Centre and Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore, India

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Abstract: A general theory for constructing a weak Markov dilation of a uniformly continuous quantum dynamical semigroup T_t on a von Neumann algebra $\mathcal{A}$ with respect to the Fock filtration is developed with the aid of a coordinate-free quantum stochastic calculus. Starting with the structure of the generator of T_t, existence of canonical structure maps (in the sense of Evans and Hudson) is deduced and a quantum stochastic dilation of T_t is obtained through solving a canonical flow equation for maps on the right Fock module $\mathcal{A} \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$, where k_0 is some Hilbert space arising from a representation of $\mathcal{A}'$. This gives rise to a $*$-homomorphism j_t of $\mathcal{A}$. Moreover, it is shown that every such flow is implemented by a partial isometry-valued process. This leads to a natural construction of a weak Markov process (in the sense of [B-P]) with respect to Fock filtration.

1. Introduction

Given a uniformly continuous quantum dynamical semigroup T_t on a von Neumann algebra $\mathcal{A}$, a general theory for constructing a (weak) Markov dilation of T_t with respect to the Fock-filtration is developed. While doing this, we introduce in a natural way (in Sect. 2) a coordinate-free stochastic calculus and quantum Ito formula which combines the initial space and the Fock space. The Sect. 3 is devoted to the solution of a class of quantum stochastic differential equations, both of the Hudson–Parthasarathy as well as the Evans–Hudson types. Here we find that the language of Hilbert right $\mathcal{A}$-modules is very useful to describe a quantum stochastic flow equation which is now a differential equation for maps on the module $\mathcal{A} \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$, where k_0 is a certain Hilbert space associated with a representation of $\mathcal{A}'$. The proof of the $*$-homomorphism property of the solution $j_t(x)$ of the flow equation (see [Mo-S]) becomes particularly transparent in this language needing no extra assumptions as in [Mo-S]. We also prove that every such

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flow can be implemented by a partial isometry-valued process, and that $\{j_t, j_t(1)\mathbb{E}_t\}_{t\geq 0}$ is an example of a weak Markov process (see [B-P]), where $\mathbb{E}_t$ denotes the conditional expectation in Fock space . We would like to add that there is some overlap in the study of Evans–Hudson flows in Sect. 3 with the work of Lindsay and Wills ([L-W]).

Let us consider a unital von Neumann algebra $\mathcal{A}$ in $\mathcal{B}(h)$, where h is a (not necessarily separable) Hilbert space. Let $(T_t)_{t\geq 0}$ be a uniformly continuous quantum dynamical semigroup (that is, a contractive, normal, completely positive semigroup) with the bounded generator $\mathcal{L} : \mathcal{A} \to \mathcal{A}$. It is known due to Christensen and Evans ([C-E]) that there exist a Hilbert space $\mathcal{K}$, bounded operator $R : h \to \mathcal{K}$, normal $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and $\ell \in \mathcal{A}$ such that,

$$\mathcal{L}(x) = R^*\pi(x)R + \ell^*x + x\ell. \tag{1.1}$$

We say that $(T_t)_{t\geq 0}$ is conservative if $\mathcal{L}(1) = 0$, which is equivalent to saying that $T_t(1) = 1$ for $t \geq 0$. It is simple to note that in case when $\pi(1) = 1$ and $\mathcal{L}(1) = 0$,

$$\mathcal{L}(x) = R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x], \tag{1.2}$$

where $H = i(l + \frac{1}{2}R^*R)$, a self-adjoint element of $\mathcal{A}$, and $R^*\pi(x)R \in \mathcal{A}$ for all $x \in \mathcal{A}$. By replacing $\mathcal{K}$ by $\hat{\mathcal{K}} \equiv \pi(1)\mathcal{K}$; and R by $\hat{R} \equiv \pi(1)R$, it is clear that

$$\hat{R}^*\pi(x)\hat{R} + \ell^*x + x\ell = R^*\pi(1)\pi(x)\pi(1)R + \ell^*x + x\ell = \mathcal{L}(x).$$

Since the range of $\pi(x)$ is contained in $\hat{\mathcal{K}}$ for all $x \in \mathcal{A}$, π may be thought of as a $*$-representation from $\mathcal{A}$ to $\mathcal{B}(\hat{\mathcal{K}})$. In view of this, we may assume that $\pi(1) = 1$. That we can also put $\mathcal{L}(1) = 0$ follows by a slight modification of the reasoning in Theorem 2.13 of [B-P].

It has been observed elsewhere ([S, A-L]) that the symmetric or bosonic Fock space often acts as a model for a heat-bath or reservoir. While the evolution of the state of the combined system, consisting of the observed physical object and the reservoir to which it is coupled, is given by a quantum stochastic differential equation (or equivalently in the dual picture, by a quantum stochastic flow equation of the observables), that of the observed subsystem is given by some kind of averaging or expectation with respect to the Fock variables. Thus though the total evolution is not given by a group, the evolution of the observed subsystem is given by a quantum dynamical semigroup. However, in most cases of physical interest, the semigroup is expected to be only strongly continuous and not uniformly continuous as has been assumed here. In this study, we are interested only in the structural aspects of the theory and the more realistic cases of a strongly continuous dynamical semigroup can often be discussed as a suitable limit of a sequence of uniformly continuous ones and will be treated elsewhere.

2. A Coordinate-Free Quantum Stochastic Calculus

Thus we shall assume that $\pi(1) = 1_{\mathcal{K}}$, $\mathcal{L}(1) = 0$ for the rest of the article. Our present aim is to develop a coordinate-free theory of quantum stochastic calculus, which will be needed for constructing a dilation of $(T_t)_{t\geq 0}$.

2.1. Basic processes. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and A be a (possibly unbounded) linear operator from $\mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}$. For each $f \in \mathcal{H}_2$, we define a linear operator $\langle f, A\rangle$ with domain $\mathcal{D}$ and taking value in $\mathcal{H}_1$ such that,

$$\langle\langle f, A\rangle u, v\rangle = \langle Au, v \otimes f\rangle \tag{2.1}$$

for $u \in \mathcal{D},\ v \in \mathcal{H}_1$. This definition makes sense because we have, $|\langle Au, v \otimes f\rangle| \leq \|Au\|\ \|f\|\ \|v\|$, and thus $\mathcal{H}_1 \ni v \to \langle Au, v \otimes f\rangle$ is a bounded linear functional. Moreover, $\|\langle f, A\rangle u\| \leq \|Au\|\ \|f\|$, for all $u \in \mathcal{D},\ f \in \mathcal{H}_2$. Similarly, for each fixed $u \in \mathcal{D}, v \in \mathcal{H}_1,\ \mathcal{H}_2 \ni f \to \langle Au, v \otimes f\rangle$ is a bounded linear functional, and hence there exists a unique element of $\mathcal{H}_2$, to be denoted by $A_{v,u}$, satisfying

$$\langle A_{v,u}, f\rangle = \langle Au, v \otimes f\rangle = \langle\langle f, A\rangle u, v\rangle. \tag{2.2}$$

We shall denote by $\langle A, f\rangle$ the adjoint of $\langle f, A\rangle$, whenever it exists. Clearly, if A is bounded, then so is $\langle f, A\rangle$ and $\|\langle f, A\rangle\| \leq \|A\|\ \|f\|$. Similarly, for any $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $f \in \mathcal{H}_2$, one can define $T_f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ by setting $T_f u = T(u \otimes f)$. For any Hilbert space $\mathcal{H}$, we denote by $\Gamma(\mathcal{H})$ and $\Gamma^f(\mathcal{H})$ the symmetric Fock space and the full Fock space of $\mathcal{H}$. For a systematic discussion of such spaces, the reader may be referred to [Par], from which we shall borrow all the standard notations and results. Now, we define a map $S : \Gamma^f(\mathcal{H}_2) \to \Gamma(\mathcal{H}_2)$ by setting,

$$S(g_1 \otimes g_2 \otimes \cdots \otimes g_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)}, \tag{2.3}$$

and linearly extending it to $\mathcal{H}_2^{\otimes^n}$, where S_n is the group of permutations of n objects. Clearly, $\|S|_{\mathcal{H}_2^{\otimes^n}}\| \leq n$. We denote by $\tilde{S}$ the operator $1_{\mathcal{H}_1} \otimes S$.

Let us now define the creation operator $a^\dagger(A)$ abstractly which will act on the linear span of vectors of the form $vg^{\otimes^n}$ and $ve(g)$ (where $g^{\otimes^n}$ denotes $\underbrace{g \otimes \cdots \otimes g}_{n \text{ times}}$), $n \geq 0$, with $v \in \mathcal{D},\ g \in \mathcal{H}_2$. It is to be noted that we shall often omit the tensor product symbol $\otimes$ between two or more vectors when there is no confusion. We define,

$$a^\dagger(A)(vg^{\otimes^n}) = \frac{1}{\sqrt{n+1}}\, \tilde{S}((Av) \otimes g^{\otimes^n}). \tag{2.4}$$

It is easy to observe that $\sum_{n \geq 0} \frac{1}{n!} \|a^\dagger(A)(vg^{\otimes^n})\|^2 < \infty$, which allows us to define $a^\dagger(A)(ve(g))$ as the direct sum $\bigoplus_{n \geq 0} \frac{1}{(n!)^{\frac{1}{2}}}\, a^\dagger(A)(vg^{\otimes^n})$. We have the following simple but useful observation, the proof of which is straightforward and hence omitted.

Lemma 2.1.1. *For $v \in \mathcal{D},\ u \in \mathcal{H}_1, g, h \in \mathcal{H}_2$,*

$$\langle a^\dagger(A)(ve(g)),\ ue(h)\rangle = \langle A_{u,v}, h\rangle\langle e(g),\ e(h)\rangle = \frac{d}{d\varepsilon}\langle e(g + \varepsilon A_{u,v}), e(h)\rangle|_{\varepsilon=0}. \tag{2.5}$$

In the same way, one can define annihilation and number operators in $\mathcal{H}_1 \otimes \Gamma(\mathcal{H}_2)$ for $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as:

$$a(A)ue(h) = \langle A, h \rangle\, ue(h),$$

$$\Lambda(T)ue(h) = a^\dagger(T_h)ue(h).$$

One can also verify that in this case $a^\dagger(A)$ is the adjoint of $a(A)$ on $\mathcal{H}_1 \otimes \mathcal{E}(\mathcal{H}_2)$, where $\mathcal{E}(\mathcal{H}_2)$ is the linear span of exponential vectors $e(g)$, $g \in \mathcal{H}_2$. Next, to define the basic processes, we need some more notations. Let k_0 be a Hilbert space, $k = L^2(\mathbb{R}_+, k_0)$, $k_t = L^2([0,t]) \otimes k_0$, $k^t = L^2((t,\infty)) \otimes k_0$, $\Gamma_t = \Gamma(k_t)$, $\Gamma^t = \Gamma(k^t)$, $\Gamma = \Gamma(k)$. We assume that $R \in \mathcal{B}(h, h \otimes k_0)$ and define $R_t^\Delta : h \otimes \Gamma_t \to h \otimes \Gamma_t \otimes k^t$ for $t \geq 0$ and a bounded interval Δ in (t, ∞) by,

$$R_t^\Delta(u\psi) = P((1_h \otimes \chi_\Delta)(Ru) \otimes \psi),$$

where $\chi_\Delta : k_0 \to k^t$ is the operator which takes α to $\chi_\Delta(\cdot)\alpha$ for $\alpha \in k_0$, and P is the canonical unitary isomorphism from $h \otimes k \otimes \Gamma$ to $h \otimes \Gamma \otimes k$. We define the creation field $a_R^\dagger(\Delta)$ on either of the domains consisting of the finite linear combinations of vectors of the form $u_t \otimes f^{t \otimes^n}$ or of $u_t \otimes e(f^t)$ for $u_t \in h \otimes \Gamma_t$, $f^t \in \Gamma^t$, $n \geq 0$, as:

$$a_R^\dagger(\Delta) = a^\dagger(R_t^\Delta), \tag{2.6}$$

where $a^\dagger(R_t^\Delta)$ carries the meaning discussed before Lemma 2.1.1, with $\mathcal{D} = \mathcal{E}(\mathcal{H}_2)$, $\mathcal{H}_1 = h \otimes \Gamma_t$, $\mathcal{H}_2 = k^t$. Similarly the two fields $a_R(\Delta)$ and $\Lambda_T(\Delta)$ can be defined as:

$$a_R(\Delta)(u_t e(f^t)) = ((\int_\Delta \langle R, f(s) \rangle ds)u_t)e(f^t), \tag{2.7}$$

and for $T \in \mathcal{B}(h \otimes k_0)$,

$$\Lambda_T(\Delta)(u_t e(f^t)) = a^\dagger(T_{f^t}^\Delta)(u_t e(f^t)). \tag{2.8}$$

In the above, $T_{f^t}^\Delta : h \otimes \Gamma_t \to h \otimes \Gamma_t \otimes k^t$ is defined as,

$$T_{f^t}^\Delta(u\alpha_t) = P(1 \otimes \hat{\chi}_\Delta)(\hat{T}(uf^t) \otimes \alpha_t), \tag{2.9}$$

and $\hat{T} \in \mathcal{B}(h \otimes L^2((t,\infty), k_0))$ is given by, $\hat{T}(u\varphi)(s) = T(u\varphi(s))$, $s > t$, and $\hat{\chi}_\Delta$ is the multiplication by $\chi_\Delta(\cdot)$ on $L^2((t,\infty), k_0)$. Clearly, $\|\hat{T}\| \leq \|T\|$, which makes $T_{f^t}^\Delta$ bounded. We note here that objects similar to $a_R(.)$, $a_R^\dagger(.)$ and $\Lambda_T(.)$ were used in [H-P2], however in a coordinatized form. In what follows, we shall assume that $(H_t)_{t \geq 0}$ and $(H_t')_{t \geq 0}$ are two operator-valued Fock-adapted processes (in the sense of [Par]), having all vectors of the form $ve(f_t)\psi^t$ in their domains, where $v \in h$, $f_t \in k_t$, $\psi^t \in \Gamma^t$. We also assume that there exist constants $c(t, f)$ and $c'(t, f)$ such that for $t \geq 0$,

$$\sup_{0 \leq s \leq t} \|H_s(ue(f))\| \leq c(t,f)\|u\|, \quad \sup_{0 \leq s \leq t} \|H_s'(ue(f))\| \leq c'(t,f)\|u\|. \tag{2.10}$$

We shall often denote an operator B and its trivial extension $B \bigotimes I$ to some bigger space by the same notation, unless there is any confusion in doing so. We also denote the unitary isomorphism from $h \otimes k_0 \otimes \Gamma(k)$ onto $h \otimes \Gamma(k) \otimes k_0$ and that from $h \otimes k \otimes \Gamma(k)$

onto $h \otimes \Gamma(k) \otimes k$ by the same letter P. Clearly, $H_t P$ acts on any vector of the form $w \otimes e(g)$, where $w \in h \otimes_{alg} k_0$, $g \in k$ and $\sup_{0 \le s \le t} \|H_s P(we(g))\| \le c(t,g)\|w\|$. This allows one to extend $H_t P$ on the whole of the domain containing vectors of the form $\bar{w}e(g)$, $\bar{w} \in h \otimes k_0$, $g \in k$. We denote this extension again by $H_t P$. Similarly we define $H_t' P$. When P is taken to be the isomorphism from $h \otimes k \otimes \Gamma(k)$ onto $h \otimes \Gamma(k) \otimes k$, we define $H_t P$ and $H_t' P$ in an exactly parallel manner.

Next we prove a few preliminary results which will be needed for establishing the quantum Ito formula in the next subsection.

Lemma 2.1.2. *Let $\Delta, \Delta' \subseteq (t, \infty)$ be intervals of finite length, $R, S \in \mathcal{B}(h, h \otimes k_0)$; $u, v \in h$; $g, f \in k$. Then we have,*

$$\begin{aligned}
&\langle H_t a_R^\dagger(\Delta)(ve(g)),\ H_t' a_S^\dagger(\Delta')(ue(f))\rangle \\
&= e^{\langle g^t, f^t\rangle}\{\langle H_t R_t^{\Delta}(ve(g_t)),\ H_t' S_t^{\Delta'}(ue(f_t))\rangle \\
&\quad + \langle\langle f^t, H_t R_t^{\Delta}\rangle ve(g_t),\ \langle g^t, H_t' S_t^{\Delta'}\rangle ue(f_t)\rangle\} \\
&= \int_{\Delta\cap\Delta'} \langle (H_t P R)(ve(g)),\ (H_t' P S)(ue(f))\rangle ds \\
&\quad + \int_{\Delta}\int_{\Delta'} \langle\langle f(s), H_t P R\rangle(ve(g)), \langle g(s'),\ H_t' P S\rangle(ue(f))\rangle ds\, ds'.
\end{aligned} \tag{2.11}$$

Proof. For the present proof, we make the convention of writing $\frac{df(\varepsilon)}{d\varepsilon}|_{\varepsilon=\varepsilon_0}$ for the limit $\lim_{n\to\infty} n(f(\varepsilon_0 + \frac{1}{n}) - f(\varepsilon_0))$ whenever it exists, and R_Δ will denote $(1 \otimes \chi_\Delta)R \in \mathcal{B}(h, h \otimes k)$ for $R \in \mathcal{B}(h, h \otimes k_0)$. Let us now choose and fix orthonormal bases $\{e_\nu\}_{\nu\in J}$ and $\{k_\alpha\}_{\alpha\in I}$ of $h \otimes \Gamma_t$ and Γ^t respectively ($t \ge 0$). We also choose subsets J_0 and I_0, which are at most countable, of J and I respectively as follows. Let J_0 be such that $\langle H_t P R_\Delta(ve(g)),\ e_\nu \otimes k_\alpha\rangle = 0 = \langle e_\nu \otimes k_\alpha,\ H_t' P S_{\Delta'}(ue(f))\rangle$ for all $\alpha \in I$ whenever $\nu \notin J_0$. Fixing this J_0, we choose I_0 to be the union of $I_{\nu,n}$, $\nu \in J_0$, $n = 1, 2, \ldots, \infty$, such that

$$\langle e(g^t + \frac{1}{n}(H_t P R_\Delta)_{e_\nu, ve(g_t)}),\ k_\alpha\rangle = 0 = \langle k_\alpha,\ e(f^t + \frac{1}{n}(H_t' P S_{\Delta'})_{e_\nu, ue(f_t)})\rangle$$

for all $\alpha \ne I_{\nu,n}$ when $n < \infty$, and

$$\langle e(g^t),\ k_\alpha\rangle = 0 = \langle k_\alpha, e(f^t)\rangle \ \text{ for } \alpha \ne I_{\nu,\infty}.$$

We have now,

$$\begin{aligned}
&\langle H_t a_R^\dagger(\Delta)(ve(g)),\ H_t' a_S^\dagger(\Delta')(ue(f))\rangle \\
&= \sum_{\substack{\nu\in J_0\\ \alpha\in I_0}} \langle H_t a_R^\dagger(\Delta)(ve(g)),\ e_\nu \otimes k_\alpha\rangle\langle e_\nu \otimes k_\alpha,\ H_t' a_S^\dagger(\Delta')(ue(f))\rangle \\
&= \sum_{\substack{\nu\in J_0\\ \alpha\in I_0}} (\frac{d}{d\varepsilon}\langle e(g^t + \varepsilon(H_t P R_\Delta)_{e_\nu, ve(g_t)}), k_\alpha\rangle|_{\varepsilon=0}) \times (\frac{d}{d\eta}\overline{\langle e(f^t + \eta(H_t' P S_{\Delta'})_{e_\nu, ue(f_t)}), k_\alpha\rangle}|_{\eta=0}) \\
&= \sum_{\nu\in J_0} \frac{\partial^2}{\partial\varepsilon\partial\eta}(\sum_{\alpha\in I_0} \langle e(g^t + \varepsilon(H_t P R_\Delta)_{e_\nu, ve(g_t)}),\ k_\alpha\rangle\langle k_\alpha, e(f^t + \eta(H_t' P S_{\Delta'})_{e_\nu, ue(f_t)})\rangle|_{\varepsilon=0=\eta})
\end{aligned}$$

$$= \sum_{\nu \in J_0} \frac{\partial^2}{\partial\varepsilon\partial\eta}(\langle e(g^t + \varepsilon(H_t P R_\Delta)_{e_\nu, \nu e(g_t)}), e(f^t + \eta(H_t' P S_{\Delta'})_{(e_\nu, u e(f_t))})\rangle|_{\varepsilon=0=\eta})$$

$$= \sum_{\nu \in J_0} e^{\langle g^t, f^t\rangle}(\langle (H_t P R_\Delta)_{e_\nu, \nu e(g_t)}, (H_t' P S_{\Delta'})_{e_\nu, u e(f_t)}\rangle$$

$$+ \langle (H_t P R_\Delta)_{e_\nu, \nu e(g_t)}, f^t\rangle\langle g^t, (H_t' P S_{\Delta'})_{e_\nu, u e(f_t)}\rangle).$$

Before proceeding further, let us justify the intermediate step in the above calculations, which involves an interchange of summation and limit, by appealing to the dominated convergence theorem. Indeed, for any fixed $\alpha \in I_0, \psi, \ \psi' \in k^t$, if we write $k_\alpha^{(n)}$ for the projection of k_α on $k^{t^{\otimes^n}}$ $(n \geq 0)$, then $\langle e(g^t + \varepsilon\psi), k_\alpha\rangle$ can be expressed as $\sum_{i \geq 0} c_i^{(\alpha)}\varepsilon^i$, where $c_i^{(\alpha)} = \sum_{n \geq i} \frac{1}{\sqrt{n!}}\binom{n}{i}\langle g^{t\otimes^{(i)}} \otimes \psi^{\otimes^{(n-i)}}, k_\alpha^{(n)}\rangle$, where $g^{t\otimes^{(i)}} \equiv \underbrace{g^t \otimes \cdots \otimes g^t}_{i-\text{times}}$, and $\psi^{\otimes^{(n-i)}} \equiv \underbrace{\psi \otimes \cdots \otimes \psi}_{(n-i)-\text{times}}$. It can be easily verified that the above is an absolutely summable power series in ε, converging uniformly for $\varepsilon \in [0, M]$, say, for any fixed $M > 0$. Similar analysis can be done for $\langle k_\alpha, e(f^t + \eta\psi')\rangle$. By Mean Value Theorem and some straightforward estimate, we have that for $\varepsilon, \eta, \varepsilon', \eta'$ in $[0, M]$,

$$\frac{1}{|(\varepsilon - \varepsilon')(\eta - \eta')|}\sum_{\alpha \in I_0}|(\langle e(g^t + \varepsilon\psi), k_\alpha\rangle - \langle e(g^t + \varepsilon'\psi), k_\alpha\rangle)$$

$$\times(\langle k_\alpha, e(f^t + \eta\psi')\rangle - \langle k_\alpha, e(f^t + \eta'\psi')\rangle)|$$

$$\leq \sum_{\substack{n \geq 0, m \geq 0,\\ 0 \leq i \leq n, 0 \leq j \leq m}} \frac{i.j.M^{i+j-2}}{\sqrt{n!m!}}\binom{n}{i}\binom{m}{j}$$

$$\times \sum_{\alpha \in I_0}|\langle g^{t\otimes^{(i)}} \otimes \psi^{\otimes^{(n-i)}}, k_\alpha^{(n)}\rangle\,\langle k_\alpha^{(n)}, f^{t\otimes^{(j)}} \otimes \psi'^{\otimes^{(m-j)}}\rangle|$$

$$\leq \sum_{n,m,i,j} \frac{ijM^{i+j-2}}{\sqrt{n!m!}}\binom{n}{i}\binom{m}{j}\|g^{t\otimes^{(i)}} \otimes \psi^{\otimes^{(n-i)}}\|\,\|f^{t\otimes^{(j)}} \otimes \psi'^{\otimes^{(m-j)}}\|$$

[since$\{k_\alpha^{(n)}\}_{\alpha \in I_0}$ are mutually orthogonal for any fixed n,

with $\|k_\alpha^{(n)}\| \leq 1 \ \forall\alpha$]

$$\leq \sum_{\substack{n \geq 0\\ m \geq 0}} \frac{mn\|g^t\|\,\|f^t\|\,(M\|g^t\| + \|\psi\|)^{n-1}(M\|f^t\| + \|\psi'\|)^{m-1}}{\sqrt{m!n!}} < \infty.$$

This allows us to apply dominated convergence theorem.

Let us now choose a countable subset I_0' of I so that $0 = \langle (H_t P R_\Delta)_{e_\nu, \nu e(g_t)}, k_\alpha\rangle = \langle k_\alpha, (H_t' P S_{\Delta'})_{e_\nu, u e(f_t)}\rangle$ for α not in I_0', for all $\nu \in J_0$.

Clearly, we have

$$\begin{aligned}
&\sum\nolimits_{\nu\in J_0}\langle (H_t P R_\Delta)_{e_\nu, ve(g_t)}, (H_t' P S_{\Delta'})_{e_\nu, ue(f_t)}\rangle \\
&= \sum_{\nu\in J_0,\alpha\in I_0'} \langle (H_t PR_\Delta)(ve(g_t)), e_\nu\otimes k_\alpha\rangle\ \langle e_\nu\otimes k_\alpha, (H_t' P S_{\Delta'})(ue(f_t))\rangle \\
&= \langle (H_t PR_\Delta)(ve(g_t)), (H_t' PS_{\Delta'})(ue(f_t))\rangle.
\end{aligned}$$

We choose sequences $\omega^{(n)},\ \omega'^{(n)}$ of vectors which can be written as finite sums of the form, $\omega^{(n)} = \sum v_i^{(n)}\otimes\beta_i^{(n)},\ \omega'^{(n)} = \sum_i u_i^{(n)}\otimes\alpha_i^{(n)}$, where $u_i^{(n)}, v_i^{(n)}\in h,\ \beta_i^{(n)},\alpha_i^{(n)}\in k_0$, and $\omega^{(n)}\to Rv,\ \omega'^{(n)}\to Su$ as $n\to\infty$. Then we have,

$$\begin{aligned}
&\| H_t P(1\otimes\chi_\Delta)\ (\omega^{(n)}\otimes e(g_t)) - H_t P R_\Delta(ve(g_t))\| \\
&\quad\le c(t,g)\|\omega^{(n)} - (Rv)\|\ |\Delta|\ \to 0 \text{ as } n\to\infty,
\end{aligned}$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. Similarly, $\| H_t' P(1\otimes\chi_{\Delta'})\ (\omega'^{(n)}\otimes e(f_t)) - (H_t' PS_{\Delta'})(ue(f_t))\|\to 0$ as $n\to\infty$. Hence we obtain

$$\begin{aligned}
&\langle (H_t PR_\Delta)(ve(g_t)), (H_t' PS_{\Delta'}(ue(f_t))\rangle \\
&= \lim_{n\to\infty}\langle H_t P(1\otimes\chi_\Delta)(\omega^{(n)}e(g_t)), H_t' P(1\otimes\chi_{\Delta'})(\omega'^{(n)}e(f_t))\rangle \\
&= \lim_{n\to\infty}\int\langle H_t(\sum_i v_i^{(n)}\otimes e(g_t)\otimes\beta_i^{(n)}), H_t'(\sum_i u_i^{(n)}\otimes e(f_t)\otimes\alpha_i^{(n)})\rangle\chi_{\Delta\cap\Delta'}(s)ds \\
&= \lim_{n\to\infty}|\Delta\cap\Delta'|\langle (H_t P)(\omega^{(n)}e(g_t)), (H_t' P)(\omega'^{(n)}e(f_t))\rangle \\
&= |\Delta\cap\Delta'|\langle (H_t PR)(ve(g_t)), (H_t' PS)(ue(f_t))\rangle \\
&= \int_{\Delta\cap\Delta'}\langle H_t PR(ve(g_t)), H_t' PS(ue(f_t))\rangle ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{\nu\in J_0}\langle (H_t PR_\Delta)_{e_\nu, ve(g_t)}, f^t\rangle\ \langle g^t, (H_t' PS_{\Delta'})_{e_\nu, ue(f_t)}\rangle \\
&= \sum_{\nu\in J_0}\langle\langle f^t, H_t PR_\Delta\rangle(ve(g_t)), e_\nu\rangle\ \langle e_\nu, \langle g^t, H_t' PS_{\Delta'}\rangle\ (ue(f_t))\rangle \\
&= \langle\langle f^t, H_t PR_\Delta\rangle(ve(g_t)), \langle g^t, (H_t' PS_{\Delta'})\rangle(ue(f_t))\rangle,
\end{aligned}$$

where the last step follows by Parseval's identity, noting the fact that for $\nu\notin J_0$, $\langle\langle f^t, H_t PR_\Delta\rangle(ve(g_t)), e_\nu\rangle = 0$ because for such ν, $\langle (H_t PR_\Delta)(ve(g_t)), e_\nu\otimes k_\alpha\rangle = 0$ for all $\alpha\in I$; and similarly $\langle e_\nu, \langle g^t, (H_t' PS_{\Delta'})\rangle(ue(f_t))\rangle = 0\ \forall\nu\notin J_0$.

We complete the proof by observing that

$$\begin{aligned}
\langle f^t, H_t R_t^\Delta\rangle &= \int_\Delta\langle f(s), H_t PR\rangle ds, \text{ and} \\
\langle g^t, H_t' S_t^{\Delta'}\rangle &= \int_{\Delta'}\langle g(s'), H_t' PS\rangle ds'.
\end{aligned}$$

To see this, it is enough to note that for $\omega \in h$, $h_t \in k_t$, we have,

$$\langle\langle f^t, H_t R_t^{\Delta}\rangle(ve(g_t)), \omega e(h_t)\rangle = \int\limits_{\Delta} \langle (H_t PR)(ve(g_t)), \omega e(h_t) \otimes f^t(s)\rangle ds,$$

which can be justified by considering $\omega^{(n)}$ as before and applying dominated convergence theorem.
Since $\Delta \subseteq (t, \infty)$ and hence $f^t(s) = f(s)$ for $s \in \Delta$, the above expression can now be written as

$$\int\limits_{\Delta} \langle (H_t PR)(ve(g_t)), we(h_t) f(s)\rangle ds = \int\limits_{\Delta} \langle\langle f(s), H_t PR\rangle(ve(g_t)), we(h_t)\rangle ds.$$

This completes the proof. □

Remark 2.1.3. If H_t and H_t' are bounded, then (2.11) of Lemma 2.1.2 holds with u, v replaced by arbitrary vectors in $h \otimes \Gamma_t$ and f, g by the same in k^t.

Lemma 2.1.4. *Let $T, T' \in \mathcal{B}(h \otimes k_0)$. Then we have,*

$$\begin{aligned} &\langle (H_t T_{g^t}^{\Delta})(ve(g)), (H_t' T_{f^t}^{\Delta'})(ue(f))\rangle \\ &= \int\limits_{\Delta \cap \Delta'} \langle H_t P T P^*(ve(g)g(s)), H_t' P T' P^*(ue(f)f(s))\rangle ds, \end{aligned}$$

and

$$\langle g^t, H_t' T_{f^t}^{\Delta'}\rangle = \int\limits_{\Delta'} \langle g(s), H_t' T'_{f(s)}\rangle ds,$$

where $T'_{f(s)} \in \mathcal{B}(h \otimes \Gamma(k), h \otimes \Gamma(k) \otimes k_0)$ is defined in (2.9).

The proof is omitted since it is very similar to that of Lemma 2.1.2.

Lemma 2.1.5. *For $\eta \in k_0$, $\langle \eta, H_t PR\rangle ve(g) = H_t((\langle \eta, R\rangle v)e(g))$, where $v \in h$, $g \in k$.*

Proof. It is easy to see that by virtue of (2.10), for every fixed g, $f \in k$, $\eta \in k_0$, $t \geq 0$, $\langle H_t(ve(g)), ue(f)\rangle = \langle v, M_t u\rangle$ defines an operator $M_t \in \mathcal{B}(h)$. Let $\tilde{M}_t = M_t \otimes 1_{k_0}$. Then we have, for $w = v \otimes \alpha$, $w' = u \otimes \beta$; $\alpha, \beta \in k_0$, $u, v \in h$,

$$\langle H_t P(we(g)),\ P(w'e(f))\rangle = \langle w, \tilde{M}_t w'\rangle.$$

By the density of $h \otimes_{alg} k_0$ in $h \otimes k_0$, we have that $\langle H_t P(we(g)),\ P(w'e(f))\rangle = \langle w, \tilde{M}_t w'\rangle$ for all $w, w' \in h \otimes k_0$. Thus

$$\begin{aligned} &\langle\langle \eta, H_t PR\rangle ve(g)\, ue(f)\rangle = \langle H_t P((Rv)e(g)),\ ue(f)\eta\rangle \\ &= \langle H_t P((Rv)e(g)),\ P(u\eta e(f))\rangle = \langle Rv,\ \tilde{M}_t(u\eta)\rangle = \langle Rv,\ (M_t u) \otimes \eta\rangle \\ &= \langle\langle \eta, R\rangle v,\ M_t u\rangle = \langle H_t((\langle \eta, R\rangle v)e(g)), ue(f)\rangle. \end{aligned}$$

This completes the proof, since the vectors of the form $ue(f)$ are total in $h \otimes \Gamma(k)$. □

2.2. Stochastic integrals and Left Quantum Ito formulae. Following [H-P1] and [Par], we call an adapted process $(H_t)_{t\geq 0}$ satisfying $\sup_{0\leq s\leq t} \|H_s v e(g)\| \leq c(t,g)\|v\|$ (for all $v \in h,\ f \in k$), to be *simple* if H_t is of the form,

$$H_t = \sum_{i=0}^{m} H_{t_i} \chi_{[t_i, t_{i+1})}(t),$$

where m is an integer (≥ 1), and $0 \equiv t_0 < t_1 < \cdots < t_m < t_{m+1} \equiv \infty$. If M denotes one of the four basic processes a_R, $a_R^\dagger$ and Λ_T and tI, and if (H_t) is simple, then we define the left and right integrals $\int_0^t H_s M(ds)$ and $\int_0^t M(ds) H_s$ respectively in the natural manner:

$$\int_0^t H_s M(ds) = \sum_{i=0}^{m} H_{t_i} M([t_i, t_{i+1}) \cap [0,t]),$$

$$\int_0^t M(ds) H_s = \sum_{i=0}^{m} M([t_i, t_{i+1}) \cap [0,t]) H_{t_i}.$$

We call H_t to be *regular* if $t \mapsto H_t(ue(f))$ is continuous for all fixed $u \in h$ and $f \in k$. Also note that if H_t is regular, then so is the extension $H_t P$. The next proposition gives the quantum Ito formulae for simple integrands.

Proposition 2.2.1. *Let $u, v \in h$; $f, g \in L^2(\mathbb{R}_+, k_0)$; $R, S, R', S' \in \mathcal{B}(h, h \otimes k_0)$ and let $T, T' \in \mathcal{B}(h \otimes k_0)$. Furthermore, assume that E, F, G, H and E', F', G', H' are adapted simple processes satisfying the bound given at the beginning of this subsection, and that*

$$X_t = \int_o^t \left(E_s \Lambda_T(ds) + F_s a_R(ds) + G_s a_S^\dagger(ds) + H_s ds \right),$$

$$X_t' = \int_0^t \left(E_s' \Lambda_{T'}(ds) + F_s' a_{R'}(ds) + G_s' a_{S'}^\dagger(ds) + H' ds \right).$$

Then we have,

(i) *(first fundamental formula)*

$$\langle X_t ve(g), ue(f)\rangle \tag{2.12}$$
$$= \int_0^t ds \big\langle \{\langle f(s), E_s PT_{g(s)}\rangle + F_s \langle R, g(s)\rangle + G_s \langle f(s), S\rangle + H_s\}(ve(g)), ue(f)\big\rangle.$$

(ii) *(second fundamental formula or Quantum Ito formula). For this part suppose that $f, g \in k \cap L^\infty(\mathbb{R}_+, k_0)$. Then*

$$\langle X_t ve(g), X_t' ue(f)\rangle$$
$$= \int_0^t ds \left[\left\langle X_s ve(g), \{\left\langle g(s), E_s' PT'_{f(s)}\right\rangle + F_s'\langle R', f(s)\rangle + G_s'\langle g(s), S'\rangle + H_s'\}(ue(f))\right\rangle\right]$$
$$+ \int_0^t ds \left[\langle \{\langle f(s), E_s PT_{g(s)}\rangle + F_s\langle R, g(s)\rangle + G_s\langle f(s), S\rangle + H_s\}(ve(g)), X_s' ue(f)\rangle\right] +$$

$$+\int_0^t ds\Big[\Big\langle E_sPT_{g(s)}(ve(g)),\, E_s'PT'_{f(s)}(ue(f))\Big\rangle + \langle E_sPT_{g(s)}(ve(g)),\, G_s'PS'(ue(f))\rangle$$
$$+\Big\langle G_sPS(ve(g)),\, E_s'PT'_{f(s)}(ue(f))\Big\rangle + \langle G_sPS(ve(g)),\, G_s'PS'(ue(f))\rangle\Big]. \quad (2.13)$$

Since the proof is very similar in spirit to the proof in [H-P1, Par], it is omitted. However, a comment with regard to the notation used above is in order. For example, for almost all $s \in \mathbb{R}_+$, the expression $E_sPT_{g(s)}(ve(g))$ is to be understood as $(E_s \otimes I_{k_0})P(T_{g(s)}v \otimes e(g)) = (E_s \otimes I_{k_0})P(T(v \otimes g(s)) \otimes e(g)) \in h \otimes \Gamma \otimes k_0$. Thus the operator $E_sPT_{g(s)}$ maps $h \otimes \Gamma$ into $h \otimes \Gamma \otimes k_0$ and therefore by the discussion in Subsect. 2.1, $\langle f(s), E_sPT_{g(s)}\rangle$ maps $h \otimes \Gamma$ into $h \otimes \Gamma$.

For a simple integrand H_t, one can easily derive the following estimate by Gronwall's Lemma as in [Par].

Lemma 2.2.2. *Let v, g, X_t be as in Proposition 2.2.1 (ii). Then one has*

$$||X_tve(g)||^2 \le e^t \int_0^t ds\big[||\{E_sPT_{g(s)} + G_sPS\}(ve(g))||^2$$
$$+||\{\langle g(s), E_sPT_{g(s)}\rangle + F_s\langle R, g(s)\rangle + \langle g(s), G_sPS\rangle + H_s\}(ve(g))||^2\big]. \quad (2.14)$$

The extension of the definition of X_t to the case when the coefficients (E, F, G, H) are regular is now fairly standard and we have the following result:

Proposition 2.2.3. *The integral X_t with regular coefficients (E, F, G, H) exists as a regular process and the first and second fundamental formulae as well as the estimate (2.14) remain valid in such a case.*

Corollary 2.2.4. *(i) Assume that in the above proposition E, F, G, H satisfy $C = \sup_{0\le t\le t_0}(||E_s|| + ||F_s|| + ||G_s|| + ||H_s||) < \infty$. Suppose furthermore that R, S, T are functions of t such that $t \mapsto R(t)u, S(t)u, T(t)\psi$ are strongly continuous for u belonging to a dense subspace $\mathcal{D} \subseteq \mathrm{Dom}(R(t)) \cap \mathrm{Dom}(S(t)) \subseteq h$ and $\psi \equiv u \otimes f(t) \in \mathrm{Dom}(T(t)) \subseteq h \otimes k_0$ for all $t \in [0, t_0]$ with $f \in \mathcal{C}$, the set of all bounded continuous functions in $L^2(\mathbb{R}_+, k_0)$. Then the integral:*

$$X(t) = \int_0^t (E_s\Lambda_T(ds) + F_sa_R(ds) + G_sa_S^\dagger(ds) + H_sds)$$

defines an adapted regular process satisfying the estimate (2.14) with the constant coefficients T, R, S replaced by $T(s)$, $R(s)$ and $S(s)$ respectively.

(ii) In the first part of the corollary, if we replace $T(t), R(t), S(t)$ by adapted processes denoted by the same symbols respectively but with $\mathcal{D}$ replaced by $\mathcal{D} \otimes_{\mathrm{alg}} \mathcal{E}(\mathcal{C}_t)$, where $\mathcal{C}_t = \mathcal{C} \cap k_t$, then the conclusions as in (i) remain valid.

Proof. (i) Clearly we can choose sequences $T^{(n)}(t), R^{(n)}(t), S^{(n)}(t)$ of simple coefficients such that $T^{(n)}(t)\psi$, $R^{(n)}(t)u$ and $S^{(n)}(t)u$ converge to $T(t)\psi$, $R(t)u$ and $S(t)u$ respectively for u and ψ as mentioned in the statement of the corollary. With these, we can define the integral $X^{(n)}(t)$ on $u \otimes e(f)$ in a natural way using Proposition 2.2.3. The hypotheses of continuity of the coefficients will allow one to pass to the limit in the integral as well by using the estimate (2.14). For example, $|| \int_0^t E_s(\Lambda_{T^{(n)}}(ds) - \Lambda_{T^{(m)}}(ds))ue(f)||^2 \le Ce^t||e(f)||^2 \int_0^t(1+||f(s)||^2)||(T^{(n)}(s) - T^{(m)}(s))(uf(s))||^2ds \to 0$ as $m, n \to \infty$. The estimate for $||X(t)ue(f)||$ will also follow by continuity.

(ii) This part follows easily from (i) with obvious adaptations. For instance, in the estimate above we shall have instead $\| \int_0^t E_s[\Lambda_{T^{(n)}}(ds) - \Lambda_{T^{(m)}}(ds)]ue(f)\|^2 \leq$ $Ce^t \int_0^t ds(1 + \|f(s)\|^2)\|(T^{(n)}(s) - T^{(m)}(s))(u \otimes e(f_s) \otimes f(s))\|^2 \|e(f^s)\|^2$. □

Remark 2.2.5. Instead of the left integral, one could as well have dealt with the right integral $\int_0^t M(ds)H_s$ and obtained formulae similar to those in Propositions 2.2.1 and 2.2.3.

Remark 2.2.6. (i) The Ito formulae derived in Proposition 2.2.3 can be put in a convenient symbolic form. Let $\tilde{\pi_0}(x)$ denote $x \otimes 1_{\Gamma(k)}$ and $\pi_0(x)$ denote $x \otimes 1_{k_0}$. Then the Ito formulae are:
$a_R(dt)\tilde{\pi_0}(x)a_S^\dagger(dt) = R^*\pi_0(x)Sdt$, $\Lambda_T(dt)\tilde{\pi_0}(x)\Lambda_{T'}(dt) = \Lambda_{T\pi_0(x)T'}(dt)$, $\Lambda_T(dt)\tilde{\pi_0}(x)a_S^\dagger(dt) = a^\dagger_{T\pi_0(x)S}(dt)$, $a_S(dt)\tilde{\pi_0}(x)\Lambda_T(dt) = a_{T^*\pi_0(x)S}(dt)$, and the products of all other types are 0.

(ii) The coordinate-free approach of quantum stochastic calculus developed here includes the old coordinatized version as presented in [Par]. Let us consider for example, for $f \in L^2(\mathbb{R}_+, k_0)$, the operator R_f defined by $R_f u = u \otimes f$, for $u \in h$. It is easy to see that the creation and annihilation operators $a^\dagger(R_f)$, $a(R_f)$ coincide with the creation and annihilation operators $a^\dagger(f)$ and $a(f)$ (respectively) defined in [Par] associated with f. Indeed, it is easy to see that $(R_f)_{u,v} = \langle u, v\rangle f$ for $u, v \in h$. Thus, for $g, l \in k$, $\langle a^\dagger(R_f)ve(g), ue(l)\rangle = \frac{d}{d\varepsilon}(\langle e(g + \varepsilon\langle u, v\rangle f), e(l)\rangle)|_{\varepsilon=0} = e^{\langle g,l\rangle}\langle v, u\rangle\langle f, l\rangle$ $= \langle v, u\rangle \frac{d}{d\varepsilon}\langle e(g + \varepsilon f), e(l)\rangle|_{\varepsilon=0} = \langle v, u\rangle\langle a^\dagger(f)e(g), e(l)\rangle$. It is also clear that $\langle R_f, g\rangle$ $= \langle f, g\rangle$ and hence $a(R_f)(ve(g)) = \langle f, g\rangle ve(g) = v(a(f)e(g))$. Finally, the number operator $\Lambda(T)$ in the sense of [Par] for $T \in \mathcal{B}(k)$ can be identified with $\Lambda_{1_h \otimes T}$.

3. Quantum Stochastic Differential Equations

3.1. Equations of Hudson–Parthasarethy (H-P) type. We consider the quantum stochastic differential equations (q.s.d.e.) of the form,

$$dX_t = X_t(a_R(dt) + a_S^\dagger(dt) + \Lambda_T(dt) + Adt), \tag{3.1}$$

$$dY_t = (a_R(dt) + a_S^\dagger(dt) + \Lambda_T(dt) + Adt)Y_t, \tag{3.2}$$

with prescribed initial values $\tilde{X_0} \otimes 1$ and $\tilde{Y_0} \otimes 1$ respectively, with $\tilde{X_0}, \tilde{Y_0} \in \mathcal{B}(h)$, where $R, S \in \mathcal{B}(h, h \otimes k_0)$, $T \in \mathcal{B}(h \otimes k_0)$, $A \in \mathcal{B}(h)$.

Proposition 3.1.1. *The q.s.d.e.'s (3.1) and (3.2) admit unique solutions as regular processes.*

Proof. The standard proofs of existence and uniqueness of solutions along the lines of that given in [Par] (Sect. 26 for the left equation and Sect. 27 for the right equation) work here also. For the iteration process in the case of the right equation to make sense, one has to take into account Corollary 2.2.4(ii) while interpreting the right integrals involved. The estimates in the same corollary also prove the regularity of the solutions as well as the estimates : $\sup_{0 \leq s \leq t}\{\|X_s ue(f)\| + \|Y_s ue(f)\|\} \leq c(t, f)\|u\|$, for $u \in h$, $f \in \mathcal{C}$ and some constant $c(t, f)$. □

We now consider a pair of special q.s.d.e.'s:

$$dU_t = U_t(a_R^\dagger(dt) + \Lambda_{T-1}(dt) - a_{T^*R}(dt) + (iH - \frac{1}{2}R^*R)dt), \quad U_0 = I, \quad (3.3)$$

$$dW_t = (a_R(dt) + \Lambda_{T^*-I}(dt) - a_{T^*R}^\dagger(dt) - (iH + \frac{1}{2}R^*R)dt)W_t, \quad W_0 = I; \quad (3.4)$$

where T is a contraction in $\mathcal{B}(h \otimes k_0)$, $R \in \mathcal{B}(h, h \otimes k_0)$ and H is a selfadjoint element of $\mathcal{B}(h)$. Then we have:

Proposition 3.1.2 (see [Mo] also)**.**

(i) *The solutions of both Eq. (3.3) and (3.4) exist as regular contraction-valued processes and $W_t = U_t^*$.*

(ii) *If furthermore T is a co-isometry, then W_t is an isometry, or equivalently U_t is a co-isometry.*

(iii) *If T is unitary, then U_t is a unitary process.*

Proof. (i) We have already seen the existence and uniqueness of the solutions U_t and W_t in the previous proposition. A simple calculation using the second fundamental formula in Proposition 2.2.3 and the right equation (3.4) give for $u, v \in h$ and $f, g \in \mathcal{C}$:

$$\langle W_t ve(g), W_t ue(f)\rangle - \langle ve(g), ue(f)\rangle = \int_0^t \big\langle W_s ve(g), \big\langle g(s), (TT^* - I)_{f(s)}\big\rangle W_s ue(f)\big\rangle, \quad (3.5)$$

which implies that W_t is a contraction for all t. Since $W_t \in \mathcal{B}(h \otimes \Gamma)$, an application of the first fundamental formula in Proposition 2.2.3 shows that U_t admits a bounded extension (which we denote also by U_t) to the whole of $h \otimes \Gamma$ and that $U_t^* = W_t$.

(ii) The relation (3.5) shows clearly that W_t is an isometry if and only if T is a co-isometry.

(iii) We note the following simple facts:

(a) For fixed $g, f \in L^2(\mathbb{R}_+, k_0) \cap L^\infty(\mathbb{R}_+, k_0)$ and $t \geq 0$, there exists a unique operator $M_t^{f,g} \in \mathcal{B}(h)$ such that $\langle v, M_t^{f,g} u\rangle = \langle U_t(ve(g)), \ U_t(ue(f))\rangle$.

(b) Setting $\tilde{M}_t^{f,g} = M_t^{f,g} \otimes 1_{k_0}$, we have for all $w, w' \in h \otimes k_0$,

$$\langle U_t P(we(g)), U_t P(w'e(f))\rangle = \langle w, \tilde{M}_t^{f,g} w'\rangle.$$

It is an easy computation using the Quantum Ito formulae (Proposition 2.2.3) to verify that,

$$\begin{aligned}
\langle v, M_t^{f,g} u\rangle - \langle ve(g), ue(f)\rangle = \int_0^t ds[&-\langle v, M_s^{f,g}\langle T^*R, f(s)\rangle u\rangle \\
- \langle u, \langle g(s), T^*R\rangle M_s^{f,g} u\rangle &+ \langle v, M_s^{f,g}\langle g(s), R\rangle u\rangle + \langle vg(s), \tilde{M}_s^{f,g}((T-1)(uf(s)))\rangle \\
+ \langle v, M_s^{f,g}(iH - \frac{1}{2}R^*R)u\rangle &+ \langle v, \langle R, f(s)\rangle M_s^{f,g} u\rangle \\
+ \langle vg(s), (T^* - 1)\tilde{M}_s^{f,g}(uf(s))\rangle &+ \langle v, (-iH - \frac{1}{2}R^*R)M_s^{f,g} u\rangle
\end{aligned}$$

$$+ \langle Rv, \tilde{M}_s^{f,g}(Ru)\rangle + \langle Rv, \tilde{M}_s^{f,g}(T-1)(uf(s))\rangle$$
$$+ \langle vg(s), (T^*-1)\tilde{M}_s^{f,g}(Ru)\rangle + \langle vg(s), (T^*-1)\tilde{M}_s^{f,g}(T-1)(uf(s))\rangle].$$

Let us consider maps $Y_i, i = 1, \ldots, 5$ from $[0,\infty) \times \mathcal{B}(h)$ to $\mathcal{B}(h)$ given by: $Y_1(s,A) = -A\langle T^*R, f(s)\rangle - \langle g(s), T^*R\rangle A + A\langle g(s), R\rangle - \frac{1}{2}(AR^*R + R^*RA) + i[A,H] + \langle R, f(s)\rangle A$, $Y_2(s,A) = R^*\tilde{A}R$, where $\tilde{A} = A \otimes 1_{k_0}$, $Y_3(s,A) = \langle g(s), \{(T^*-1)\tilde{A} + \tilde{A}(T-1) + (T^*-1)\tilde{A}(T-1)\}_{f(s)}\rangle$, $Y_4(s,A) = \langle (T^*-I)\tilde{A}^*R, f(s)\rangle$, $Y_5(s,A) = \langle (T^*-I)\tilde{A}R, g(s)\rangle^*$. Then it follows that

$$\langle v, M_t^{f,g}u\rangle - \langle ve(g), ue(f)\rangle = \int_s^t \langle v, \sum_{i=1}^{5} Y_i(s, M_s^{f,g})u\rangle ds,$$

i.e.,

$$\frac{dM_t^{f,g}}{dt} = \sum_{i=1}^{5} Y_i(t, M_t^{f,g}).$$

We also have that $M_0^{f,g} \equiv \langle e(g), e(f)\rangle I$ is a solution since the isometry property of T implies that $Y_i(t,I) = 0 \;\forall\; i$. Moreover, Y_i's are linear and bounded, hence by the uniqueness of the solution of the Banach space valued initial value problem, we conclude that $M_t^{f,g} = M_0^{f,g}$ for all t, or equivalently that U_t is an isometry. □

3.2. Fock modules. For any Hilbert space $\mathcal{H}$, we denote by $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ the subspace of $\mathcal{B}(h, h \otimes \mathcal{H})$ consisting of finite sums of the form $\sum x_i \otimes \alpha_i$, where $x_i \in \mathcal{A},\ \alpha_i \in \mathcal{H}$. We also equip $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{A} \otimes_{alg} \mathcal{H}$ by defining $\langle x \otimes \alpha,\ y \otimes \beta\rangle = x^*y\langle\alpha,\beta\rangle$ and extending this linearly. One denotes by $\mathcal{A} \otimes_{c^*} \mathcal{H}$ the completion of $\mathcal{A} \otimes_{alg} \mathcal{H}$ under the operator norm inherited from $\mathcal{B}(h, h \otimes \mathcal{H})$. It is easy to see that $\|X\| = \|X^*X\| = \|\langle X, X\rangle\|^{\frac{1}{2}}$. This is a usual Hilbert C^* module with $\mathcal{A}$ being the underlying algebra. For a comprehensive study of such structures, the reader may be referred to [Lan]. However, instead of the norm topology, we need to topologize $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ by the inherited strong operator topology from $\mathcal{B}(h, h \otimes \mathcal{H})$ and the closure of $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ under this topology will be denoted by $\mathcal{A} \otimes \mathcal{H}$. Note that $\mathcal{A} \otimes \mathcal{H} \ni X_n \to X \in \mathcal{A} \otimes \mathcal{H}$ if and only if $X_n u \to Xu \ \ \forall u \in h$. It is clear that $\langle\cdot,\cdot\rangle$ extends naturally to both $\mathcal{A} \otimes_{c*} \mathcal{H}$ and $\mathcal{A} \otimes \mathcal{H}$; and they also have a natural right $\mathcal{A}$-module action, namely, $(Xa)u := X(au)$ for $a \in \mathcal{A},\ X \in \mathcal{A} \otimes_{c^*} \mathcal{H}$ or $\mathcal{A} \otimes \mathcal{H}$. For other applications of Hilbert modules in quantum stochastic processes, the reader is referred to [A-L] and [Sk].

Lemma 3.2.1. *Any element X of $\mathcal{A} \otimes \mathcal{H}$ can be written as, $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$, where $\{\gamma_\alpha\}_{\alpha \in J}$ is an orthonormal basis of $\mathcal{H}$ and $x_\alpha \in \mathcal{A}$. The above sum over a possibly uncountable index set J makes sense in the usual way: it is strongly convergent and $\forall u \in h$, there exists an at most countable subset J_u of J such that $Xu = \sum_{\alpha \in J_u} (x_\alpha u) \otimes \gamma_\alpha$. Moreover, once $\{\gamma_\alpha\}$ is fixed, x_α's are uniquely determined by X.*

Proof. Set $x_\alpha = \langle\gamma_\alpha, X\rangle$ as per notation of Sect. 2.1. Clearly, if $X \in \mathcal{A} \otimes_{alg} \mathcal{H}, x_\alpha \in \mathcal{A}$ for all α. Since any element of $\mathcal{A} \otimes \mathcal{H}$ is a strong limit of elements from $\mathcal{A} \otimes_{alg} \mathcal{H}$; and since $\mathcal{A}$ is strongly closed, if follows that $x_\alpha \in \mathcal{A}$ for an arbitrary $X \in \mathcal{A} \otimes \mathcal{H}$. Now, for a fixed $u \in h$, let J_u be the (at most countable) set of indices such that $\forall \alpha \in J_u$, $\exists\, v_\alpha \in h$ with $\langle Xu, v_\alpha \otimes \gamma_\alpha \rangle \neq 0$. Then for any $v \in h$ and $\gamma \in \mathcal{H}$, we have with $c_\alpha^\gamma = \langle \gamma_\alpha, \gamma \rangle$,

$$\begin{aligned}\langle Xu, v \otimes \gamma \rangle &= \sum_{\alpha \in J_u} c_\alpha^\gamma \langle Xu, v \otimes \gamma_\alpha \rangle = \sum_{\alpha \in J_u} c_\alpha^\gamma \langle \langle \gamma_\alpha, X \rangle u, v \rangle \\ &= \sum_{\alpha \in J_u} \langle x_\alpha u, v \rangle \langle \gamma_\alpha, \gamma \rangle = \langle \sum_{\alpha \in J_u} (x_\alpha \otimes \gamma_\alpha) u,\ v \otimes \gamma \rangle;\end{aligned}$$

that is, $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$ in the sense described in the statment of the lemma. Given $\{\gamma_\alpha\}$, the choice of x_α's is unique, because for any fixed α_0, $\langle \gamma_{\alpha_0}, X \rangle = x_{\alpha_0}$, which follows from the previous computation if we take γ to be γ_{α_0}. □

Corollary 3.2.2. *Let $X, Y \in \mathcal{A} \otimes \mathcal{H}$ be given by $X = \sum_{\alpha \in J} x_\alpha \otimes \gamma_\alpha$ and $Y = \sum_{\alpha \in J} y_\alpha \otimes \gamma_\alpha$ as in the lemma above. For any finite subset I of J, if we denote by X_I and Y_I the elements $\sum_{\alpha \in I} x_\alpha \otimes \gamma_\alpha$ and $\sum_{\alpha \in I} y_\alpha \otimes \gamma_\alpha$ respectively, then $\lim_I \langle X_I, Y_I \rangle = \langle X, Y \rangle$, where the limit is taken over the directed family of finite subsets of J with usual partial ordering by inclusion.*

Proof. The proof is an easy adaptation of Lemma 27.7 in [Par]. □

We give below a convenient necessary and sufficient criterion for verifying whether an element of $\mathcal{B}(h, h \otimes \mathcal{H})$ belongs to $\mathcal{A} \otimes \mathcal{H}$.

Lemma 3.2.3. *Let $X \in \mathcal{B}(h, h \otimes \mathcal{H})$. Then X belongs to $\mathcal{A} \otimes \mathcal{H}$ if and only if $\langle \gamma, X \rangle \in \mathcal{A}$ for all γ in some dense subset $\mathcal{E}$ of $\mathcal{H}$.*

Proof. That $X \in \mathcal{A} \otimes \mathcal{H}$ implies $\langle \gamma, X \rangle \in \mathcal{A}\ \forall\, \gamma \in \mathcal{H}$ has already been observed in the proof of the previous lemma. For the converse, first we claim that $\langle \gamma, X \rangle \in \mathcal{A}$ for all γ in $\mathcal{E}$ (where $\mathcal{E}$ is dense in $\mathcal{H}$) will imply $\langle \gamma, X \rangle \in \mathcal{A}$ for all $\gamma \in \mathcal{H}$. Indeed, for any $\gamma \in \mathcal{H}$ there exists a net $\gamma_\alpha \in \mathcal{E}$ such that $\gamma_\alpha \to \gamma$, and hence $\| \langle \gamma, X \rangle - \langle \gamma_\alpha, X \rangle \| \le \| \gamma_\alpha - \gamma \| \| X \| \to 0$. Now let us fix an orthonormal basis $\{\gamma_\alpha\}_{\alpha \in J}$ of $\mathcal{H}$ and write $X = \sum_{\alpha \in J} \langle \gamma_\alpha, X \rangle \otimes \gamma_\alpha$ by Lemma 3.2.1. Clearly, the net $X_{\mathcal{I}}$ indexed by finite subsets $\mathcal{I}$ of J (partially ordered by inclusion) converges strongly to X. Since $X_{\mathcal{I}} \in \mathcal{A} \otimes_{alg} \mathcal{H}$ for any such finite subset $\mathcal{I}$ (as $\langle \gamma_\alpha, X \rangle \in \mathcal{A}\ \forall\, \alpha$), the proof follows by noting that $\mathcal{A}$ is strongly closed. □

In case $\mathcal{H} = \Gamma(k)$, we call the module $\mathcal{A} \otimes \Gamma(k)$ as the right Fock $\mathcal{A}$-module over $\Gamma(k)$, for short the *Fock module*, and denote it by $\mathcal{A} \otimes \Gamma$.

3.3. Solution of Evans–Hudson type q.s.d.e. In the previous subsections, we have considered q.s.d.e.'s on the Hilbert space $h \otimes \Gamma$. Now we shall study an associated class of q.s.d.e.'s, but on the Fock module $\mathcal{A} \otimes \Gamma$. This is closely related to the Evans–Hudson type of q.s.d.e.'s ([Ev, Par]).

For this part of the theory, we assume that we are given the *structure maps*, that is, the triple of normal maps $(\mathcal{L}, \delta, \sigma)$, where $\mathcal{L} \in \mathcal{B}(\mathcal{A})$, $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0)$ and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$ satisfying:

(S1) $\sigma(x) = \pi(x) - x \otimes I_{k_0} \equiv \Sigma^*(x \otimes I_{k_0})\Sigma - x \otimes I_{k_0}$, where Σ is a partial isometry in $h \otimes k_0$ such that π is a $*$-representation on $\mathcal{A}$.

(S2) $\delta(x) = Rx - \pi(x)R$, where $R \in \mathcal{B}(h, h \otimes k_0)$ so that δ is a π-derivation, i.e. $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$.

(S3) $\mathcal{L}(x) = R^*\pi(x)R + lx + xl^*$, where $l \in \mathcal{A}$ with the condition $\mathcal{L}(1) = 0$ so that $\mathcal{L}$ satisfies the second order cocycle relation with δ as coboundary, i.e.

$$\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x)^*y = \delta(x)^*\delta(y)\ \forall x, y \in \mathcal{A}.$$

Given the generator $\mathcal{L}$ of a q.d.s., that one can choose k_0 and Σ such that the hypotheses (S1)–(S3) are satisfied will be established in the next section.
To describe Evans–Hudson flow in this language, it is convenient to introduce a map Θ encompassing the triple $(\mathcal{L}, \delta, \sigma)$ as follows:

$$\Theta(x) = \begin{pmatrix} \mathcal{L}(x) & \delta^\dagger(x) \\ \delta(x) & \sigma(x) \end{pmatrix}, \tag{3.6}$$

where $x \in \mathcal{A}$, $\delta^\dagger(x) = \delta(x^*)^* : h \otimes k_0 \to h$, so that $\Theta(x)$ can be looked upon as a bounded linear normal map from $h \otimes \hat{k_0} \equiv h \otimes (\mathbb{C} \oplus k_0)$ into itself. It is also clear from (S1)-(S3) that Θ maps $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\hat{k_0})$. The next lemma sums up important properties of Θ.

Lemma 3.3.1. *Let Θ be as above. Then one has:*

(i)
$$\Theta(x) = \Psi(x) + K(x \otimes 1_{\hat{k_0}}) + (x \otimes 1_{\hat{k_0}})K^*, \tag{3.6}$$
where $\Psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{k_0})$ is a completely positive map and $K \in \mathcal{B}(h \otimes \hat{k_0})$.

(ii) *Θ is conditionally completely positive and satisfies the structure relation:*

$$\Theta(xy) = \Theta(x)(y \otimes 1_{\hat{k_0}}) + (x \otimes 1_{\hat{k_0}})\Theta(y) + \Theta(x)\hat{Q}\Theta(y), \tag{3.8}$$

where $\hat{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{h\otimes k_0} \end{pmatrix}$.

(iii) *There exists $D \in \mathcal{B}(h \otimes \hat{k_0})$ such that $||\Theta(x)\zeta|| \le ||(x \otimes 1_{\hat{k_0}})D\zeta||\ \forall \zeta \in h \otimes \hat{k_0}$.*

Proof. Define the following maps with respect to the direct sum decomposition $h \otimes \hat{k_0} = h \oplus (h \otimes k_0)$:
$\tilde{R} = \begin{pmatrix} 0 & 0 \\ R & -I \end{pmatrix}$, $\tilde{\pi}(x) = \begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix}$, $K = \begin{pmatrix} l & 0 \\ R & -\frac{1}{2}1_{h\otimes k_0} \end{pmatrix}$, $\tilde{\Sigma} = \begin{pmatrix} 1_h & 0 \\ 0 & \Sigma \end{pmatrix}$.
Then it is easy to see that (i) is verified with $\Psi(x) = \tilde{R}^*\tilde{\Sigma}^*(x \otimes 1_{\hat{k_0}})\tilde{\Sigma}\tilde{R} = \tilde{R}^*\tilde{\pi}(x)\tilde{R}$. Clearly, Ψ is completely positive. That Θ is conditionally completely positive and satisfies the structure relation (3.8) is also an easy consequence of (i) and (S1)-(S3). The estimate in (iii) follows from the structure of Ψ given above with the choice of D as

$$D = ||\tilde{\Sigma}\tilde{R}||\ \tilde{\Sigma}\tilde{R} + ||K||\ 1_{h\otimes\hat{k_0}} + K^*. \qquad \square$$

We now introduce the basic processes. Fix $t \ge 0$, a bounded interval $\Delta \subseteq (t, \infty)$, elements $x_1, x_2, \ldots, x_n \in \mathcal{A}$ and vectors $f_1, f_2, \ldots, f_n \in k$; $u \in h$. We define the following:

$$\left(a_\delta(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right)u = \sum_{i=1}^n a_{\delta(x_i^*)}(\Delta)(ue(f_i)),$$
$$\left(a_\delta^\dagger(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right)u = \sum_{i=1}^n a_{\delta(x_i)}^\dagger(\Delta)(ue(f_i)),$$

$$\left(\Lambda_\sigma(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right)u = \sum_{i=1}^n \Lambda_{\sigma(x_i)}(\Delta)(ue(f_i)),$$
$$\left(\mathcal{I}_{\mathcal{L}}(\Delta)(\sum_{i=1}^n x_i \otimes e(f_i))\right)u = \sum_{i=1}^n |\Delta|(\mathcal{L}(x_i)u) \otimes e(f_i)),$$

where $|\Delta|$ denotes the length of Δ.

Lemma 3.3.2. *The above processes are well defined on $\mathcal{A}\otimes_{\rm alg}\mathcal{E}(k)$ and they take values in $\mathcal{A} \otimes \Gamma(k)$.*

Proof. First note that $e(f_1), \ldots, e(f_n)$ are linearly independent whenever $f_1, \ldots, f_n$ are distinct from which it is easy to see that $\sum_{i=1}^n x_i \otimes e(f_i) = 0$ implies $x_i = 0\, \forall i$, whenever f_i's are distinct. This will establish that the processes are well defined. The second part of the lemma will follow from Lemma 3.2.3 with the choice of the dense set $\mathcal{E}$ to be $\mathcal{E}(k)$ and $\mathcal{H} = \Gamma(k)$ and by some simple computation, noting the fact that $\mathcal{L}$, δ, σ are structure maps. For example, $\langle e(g), a_\delta^\dagger(\Delta)(x \otimes e(f))\rangle = \langle e(g), e(f)\rangle \int_\Delta \langle g(s), \delta(x)\rangle ds \in \mathcal{A}$, which shows that the range of $a_\Delta^\dagger$ is in $\mathcal{A} \otimes \Gamma(k)$. Similarly, one verifies that $\langle e(g), \Lambda_\sigma(\Delta)(x \otimes e(f))\rangle = \langle e(g), e(f)\rangle \int_\Delta \langle g(s), \sigma(x)_{f(s)}\rangle ds \in \mathcal{A}$, since $\sigma(x) \in \mathcal{A} \otimes \mathcal{B}(k_0)$. □

Next, we want to consider the solution of an equation of the Evans–Hudson type which in our notation can be written as:

$$J_t = id_{\mathcal{A}\otimes\Gamma} + \int_0^t J_s \circ (a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_{\mathcal{L}})(dt), \;\; 0 \le t \le t_0, \tag{3.9}$$

where the solution is looked for as a map from $\mathcal{A} \otimes \Gamma$ into itself. For this, we first need an abstract lemma which allows us to interpret the above integral on the right-hand side and to get an appropriate bound for such integrals.

Lemma 3.3.3 (*The Lifting Lemma*). *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}$ be a vector space. Let $\beta : \mathcal{A} \otimes_{\rm alg} \mathcal{V} \to \mathcal{A} \otimes \mathcal{H}$ be a linear map satisfying the estimate*

$$||\beta(x \otimes \eta)u|| \le c_\eta ||(x \otimes 1_{\mathcal{H}''})ru|| \tag{3.10}$$

for some Hilbert space $\mathcal{H}''$ and $r \in \mathcal{B}(h, h \otimes \mathcal{H}'')$ (both independent of η) and for some constant c_η depending on η. Then, for any Hilbert space $\mathcal{H}'$, we can define a map $\tilde{\beta} : (\mathcal{A} \otimes \mathcal{H}') \otimes_{\rm alg} \mathcal{V} \to \mathcal{A} \otimes (\mathcal{H} \otimes \mathcal{H}')$ by $\tilde{\beta}(x \otimes f \otimes \eta) = \beta(x \otimes \eta) \otimes f$ for $x \in \mathcal{A}, \eta \in \mathcal{V}, f \in \mathcal{H}'$. Moreover, $\tilde{\beta}$ admits the estimate

$$||\tilde{\beta}(X \otimes \eta)u|| \le c_\eta ||(X \otimes 1_{\mathcal{H}''})ru||, \tag{3.11}$$

where $X \in \mathcal{A} \otimes \mathcal{H}'$.

Proof. Let $X \in \mathcal{A}\otimes\mathcal{H}'$ be given by the strongly convergent sum $X = \sum x_\alpha \otimes e_\alpha$, where $x_\alpha \in \mathcal{A}$ and $\{e_\alpha\}$ is an orthonormal basis of $\mathcal{H}'$. It is easy to verify that $||\tilde{\beta}(\sum x_\alpha \otimes e_\alpha \otimes \eta)u||^2 = \sum ||\beta(x_\alpha \otimes \eta)u||^2 \le c_\eta^2 \sum_\alpha ||(x_\alpha \otimes 1_{\mathcal{H}''})ru||^2 = c_\eta^2 ||(X \otimes 1_{\mathcal{H}''})ru||^2$, and thus $\tilde{\beta}$ is well defined and admits the required estimate. □

Corollary 3.3.4. *If we take $\mathcal{V} = \mathbb{C}$ and identify $\mathcal{A}\otimes_{\text{alg}}\mathcal{V}$ with $\mathcal{A}$, then $\beta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{H}$ is a bounded normal map and $\tilde{\beta}$ is also a bounded normal map from $\mathcal{A}\otimes\mathcal{H}'$ to $\mathcal{A}\otimes\mathcal{H}\otimes\mathcal{H}'$.*

The proof of this corollary is a simple consequence of the estimates.

We now want to define $\int_0^t Y(s)\circ(a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_\mathcal{L})(ds)$, where $Y(s) : \mathcal{A}\otimes_{\text{alg}}\mathcal{E}(k) \to \mathcal{A}\otimes\Gamma(k)$ is an adapted strongly continuous process satisfying the estimate

$$\sup_{o\le t\le t_0} \|Y(t)(x\otimes e(f))u\| \le \|(x\otimes 1_{\mathcal{H}''})ru\|, \tag{3.12}$$

for $x \in \mathcal{A}$, $f \in \mathcal{C}$ and where $\mathcal{H}''$ is a Hilbert space and $r \in \mathcal{B}(h, h\otimes\mathcal{H}'')$. In this the integrals corresponding to a_δ and $\mathcal{I}_\mathcal{L}$ belong to one class while the other two belong to another. In fact, we define $\int_0^t Y(s)\circ(a_\delta + \mathcal{I}_\mathcal{L})(ds)(x\otimes e(f))$ by setting it to be equal to $\int_0^t Y(s)((\mathcal{L}(x) + \langle\delta(x^*), f(s)\rangle)\otimes e(f))ds$. For the integral involving the other two processes, we need to consider $\widetilde{Y(s)} : \mathcal{A}\otimes k_0\otimes\mathcal{E}(k_s) \to \mathcal{A}\otimes\Gamma_s\otimes k_0$ as is given by the previous lemma and fix $x \in \mathcal{A}$ and $g \in \mathcal{C}$ (see Corollary 2.2.4). Define two maps $S(s) : h\otimes_{\text{alg}}\mathcal{E}(\mathcal{C}_s) \to h\otimes\Gamma_s\otimes k_0$ and $T(s) : h\otimes_{\text{alg}}\mathcal{E}(\mathcal{C}_s)\otimes k_0 \to h\otimes\Gamma_s\otimes k_0$ by

$$S(s)(ue(f_s)) = \widetilde{Y(s)}(\delta(x)\otimes e(f_s))u,$$

and

$$T(s)(ue(g_s)\otimes f(s)) = \widetilde{Y(s)}(\sigma(x)_{f(s)}\otimes e(g_s))u.$$

By virtue of the hypotheses on $Y(s)$, the lifting lemma and the fact that $s \mapsto e(g_s)$ is strongly continuous, the families S and T satisfy the hypotheses of Corollary 2.2.4(ii). Therefore we can define the integral $\int_0^t Y(s)\circ(\Lambda_\sigma(ds) + a_\delta^\dagger(ds)(x\otimes e(f))u$ by setting it to be equal to $\left(\int_0^t \Lambda_T(ds) + a_S^\dagger(ds)\right) ue(f)$. Thus we have:

Proposition 3.3.5. *The integral $Z(t) \equiv \int_0^t Y(s)\circ(a_\delta^\dagger + a_\delta + \Lambda_\sigma + \mathcal{I}_\mathcal{L})(ds)$, where $Y(s)$ satisfies (3.12) is well defined on $\mathcal{A}\otimes_{\text{alg}}\mathcal{E}(\mathcal{C})$ as a regular process. Moreover, the integral satisfies an estimate:*

$$\begin{aligned}\|\{Z(t)(x\otimes e(f)\}u\|^2 \le 2e^t \int_0^t exp(\|f^s\|^2)\{\|\hat{Y}(s)(\Theta(x)_{\hat{f}(s)}\otimes e(f_s))u\|^2 + \\ \|\langle f(s), \hat{Y}(s)(\Theta(x)_{f(s)}\otimes e(f_s))u\|^2\}ds,\end{aligned} \tag{3.13}$$

where Θ was as defined earlier, $\hat{Y}(s) = Y(s)\oplus\widetilde{Y(s)} : \mathcal{A}\otimes\hat{k}_0\otimes_{\text{alg}}\mathcal{E}(\mathcal{C}_s) \to \mathcal{A}\otimes\Gamma_s\otimes\hat{k}_0$, $\hat{f}(s) = 1\oplus f(s)$ and $f(s)$ is identified with $0\oplus f(s)$ in $\hat{k}_0$.

Proof. We have already seen that the integral is well defined. The estimate (3.13) follows from the estimate (2.14) in Corollary 2.2.4 by setting $E = F = G = H = I$ and recalling the definition of Θ. □

Now we are ready to prove the main result of this section .

Theorem 3.3.6. (i) *There exists a unique solution J_t of equation (3.9), which is an adapted regular process mapping $\mathcal{A}\otimes\mathcal{E}(\mathcal{C})$ into $\mathcal{A}\otimes\Gamma$. Furthermore, one has an estimate*

$$\sup_{0\le t\le t_0} \|J_t(x\otimes e(g))u\| \le C'(g)\|(x\otimes 1_{\Gamma^f(\hat{k})})E_{t_0}u\|,$$

where $g \in \mathcal{C}$, $\hat{k} = L^2([0, t_0], \hat{k}_0)$, $E_t \in \mathcal{B}(h, h\otimes\Gamma^f(\hat{k}))$ and $\Gamma^f(\hat{k})$ is the full Fock space over $\hat{k}$.

(ii) *Setting* $j_t(x)(ue(g)) = J_t(x \otimes e(g))u$, *we have*
  (a) $\langle j_t(x)ue(g), j_t(y)ve(f)\rangle = \langle ue(g), j_t(x^*y)ve(f)\rangle \ \forall g, f \in \mathcal{C}$, *and*
  (b) j_t *extends uniquely to a normal* $*$*-homomorphism from* $\mathcal{A}$ *into* $\mathcal{A} \otimes \mathcal{B}(\Gamma)$,

(iii) *If* $\mathcal{A}$ *is commutative, then the algebra generated by* $\{j_t(x) | x \in \mathcal{A}, 0 \le t \le t_0\}$ *is commutative.*

(iv) $j_t(1) = 1 \ \forall t \in [0, t_0]$ *if and only if* $\Sigma^*\Sigma = 1_{h \otimes k_0}$.

Proof. (i) We write for $\Delta \subseteq [0, \infty)$, $M(\Delta) \equiv a_\delta(\Delta) + a_\delta^\dagger(\Delta) + \Lambda_\sigma(\Delta) + \mathcal{I}_\mathcal{L}(\Delta)$, and set up an iteration by

$$J_t^{(n+1)}(x \otimes e(f)) = \int_0^t J_s^{(n)} \circ M(ds)(x \otimes e(f)),\ J_t^{(0)}(x \otimes e(f)) = x \otimes e(f),$$

with $x \in \mathcal{A}$ and $f \in \mathcal{C}$ fixed. Since $J_t^{(1)} = M([0,t])$, $J_t^{(1)}$ is adapted regular and has the estimate (by the definition of $M(\Delta)$, estimate (2.14) and Lemma 3.3.1(iii)): $||J_t^{(1)}(x \otimes e(f))u||^2 \le 4e^{t_0}||e(f)||^2 \int_0^t ds \|\Theta(x)(u \otimes \hat{f}(s))\|^2 ||\hat{f}(s)||^2 \le 4||e(f)||^2 e^{t_0} \int_0^t ds ||\hat{f}(s)||^2 ||(x \otimes 1_{\hat{k_0}})D(u \otimes \hat{f}(s))||^2$. For the given f, define $E_t^{(1)} : h \to h \otimes \hat{k}$ by $(E_t^{(1)}u)(s) = D(u \otimes \hat{f}(s) || \hat{f}_t(s)||)$, where $\hat{f}_t(s) = \chi_{[0,t]}(s)\hat{f}(s)$. Then the above estimate reduces to

$$||J_t^{(1)}(x \otimes e(f))u||^2 \le 4||e(f)||^2 e^{t_0} ||(x \otimes 1_{\hat{k}})E_t^{(1)}u||^2. \tag{3.14}$$

Now, an application of the lifting lemma leads to

$$||\widehat{J_t^{(1)}}(X \otimes e(f))u||^2 \le 4||e(f)||^2 e^{t_0} ||(X \otimes 1_{\hat{k}})E_t^{(1)}u||^2,$$

for $X \in \mathcal{A} \otimes \hat{k_0}$, where as in the previous proposition, $\widehat{J_t^{(1)}} = J_t^{(1)} \oplus \widetilde{J_t^{(1)}}$. As an induction hypothesis, assume that $J_t^{(n)}$ is a regular adapted process having an estimate $||J_t^{(n)}(x \otimes e(f))u||^2 \le C^n ||e(f)||^2 ||(x \otimes 1_{\hat{k}^{\otimes n}})E_t^{(n)}u||^2$, where $C = 4e^{t_0}$, $E_t^{(n)} : h \to h \otimes \hat{k}^{\otimes^n}$ defined as:

$$(E_t^{(n)}u)(s_1, s_2, \dots s_n) = (D \otimes 1_{\hat{k}^{\otimes^{n-1}}})P_n\{(E_{s_1}^{(n-1)}u)(s_2, \dots s_n) \otimes \hat{f}(s_1) \| \hat{f}_t(s_1) \|\}.$$

Furthermore, $P_n : h \otimes \hat{k}^{\otimes^{(n-1)}} \otimes \hat{k_0} \to h \otimes \hat{k_0} \otimes \hat{k}^{\otimes^{(n-1)}}$ is the operator which interchanges the second and third tensor components and $E_t^{(0)} = 1_h$. Then by an application of Proposition 3.3.5 one can verify that $J_t^{(n+1)}$ also satisfies a similar estimate. Thus, if we put $J_t = \sum_{n=0}^\infty J_t^{(n)}$, then

$$||J_t(x \otimes e(f))u|| \le \sum_{n=0}^\infty ||J_t^{(n)}(x \otimes e(f))u||$$

$$\le ||e(f)|| \sum_{n=0}^\infty C^{\frac{n}{2}} (n!)^{-\frac{1}{4}} ||(x \otimes 1_{\hat{k}^{\otimes^n}})(n!)^{\frac{1}{4}} E_t^{(n)}u||$$

$$\le ||e(f)|| \left(\sum_{n=0}^\infty \frac{C^n}{\sqrt{n!}} \right)^{\frac{1}{2}} ||(x \otimes 1_{\Gamma^f(\hat{k})})E_t u||, \tag{3.15}$$

where we have set $E_t : h \to h \otimes \Gamma^f(\hat{k})$ by $E_t u = \bigoplus_{n=0}^{\infty} (n!)^{\frac{1}{4}} E_t^{(n)} u$. It is easy to see that

$$\begin{aligned} ||E_t u||^2 &= \sum_{n=0}^{\infty} (n!)^{\frac{1}{2}} ||E_t^{(n)} u||^2 \\ &\le ||u||^2 \sum_{n=0}^{\infty} (n!)^{\frac{1}{2}} ||D||^{2n} \left\{ \int_{0<s_n<s_{n-1}<\ldots s_1<t} ds_n \ldots ds_1 ||\hat{f}(s_n)||^4 \ldots ||\hat{f}(s_1)||^4 \right\} \\ &= ||u||^2 \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} ||D||^{2n} \mu_f(t)^n, \end{aligned}$$

where $\mu_f(t) = \int_0^t ||\hat{f}(s)||^4 ds$. The estimate (3.15) proves the existence of the solution of Eq. (3.9), as well as its continuity relative to the strong operator topology in $\mathcal{B}(h)$. The uniqueness of the solution follows along standard lines of reasoning.

(ii) First we prove the following identity:

$$\langle J_t(x \otimes e(f))u, J_t(y \otimes e(g))v \rangle = \langle ue(f), J_t(x^* y \otimes e(g))v \rangle. \tag{3.16}$$

For this it is convenient to lift the maps J_t to the module $\mathcal{A} \otimes \Gamma^f(\hat{k_0}) \otimes_{\text{alg}} \mathcal{E}(\mathcal{C})$, that is, replace $\mathcal{A}$ by $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$. We define $\hat{J}_t : \mathcal{A} \otimes \Gamma^f(\hat{k_0}) \otimes_{\text{alg}} \mathcal{E}(\mathcal{C}) \to \mathcal{A} \otimes \Gamma \otimes \Gamma^f(\hat{k_0})$ by $\hat{J}_t = (J_t \otimes id)P$, where P interchanges the second and third tensor components. Recalling from Sect. 2 that $\Theta(x)_\zeta \in \mathcal{B}(h, h \otimes \hat{k_0})$ for $x \in \mathcal{A}, \zeta \in \hat{k_0}$, we can define $\Theta_\zeta : \mathcal{A} \to \mathcal{A} \otimes \hat{k_0}$ by $\Theta_\zeta(x) = \Theta(x)_\zeta$, and extend as above to $\widehat{\Theta_\zeta} : \mathcal{A} \otimes \Gamma^f(\hat{k_0}) \to \mathcal{A} \otimes \Gamma^f(\hat{k_0})$ by setting $\widehat{\Theta_\zeta}|_{\mathcal{A} \otimes \hat{k_0}^{\otimes n}} = \Theta_\zeta \otimes id_{\hat{k_0}^{\otimes n}}$. By the lifting lemma, both $\hat{J}_t$ and $\widehat{\Theta_\zeta}$ are well defined and enjoy the estimates as in Theorem 3.3.6 and Lemma 3.3.1(iii) respectively.

Next, note that for fixed $f, g \in \mathcal{C}$ and $x, y \in \mathcal{A}$, one has using Eq. (3.9) for J_t and quantum Ito formula (Proposition 2.2.4) and the structure relation in Lemma 3.3.1(ii):

$$\begin{aligned} &\langle J_t(x \otimes e(f))u, J_t(y \otimes e(g))v \rangle \langle xu \otimes e(f), yv \otimes e(g) \rangle \\ &+ \int_0^t ds \{ \langle \hat{J}_s(\Theta_{\hat{f}(s)}(x) \otimes e(f))u, \hat{J}_s(y \otimes \hat{g}(s) \otimes e(g))v \rangle \\ &\quad + \langle \hat{J}_s(x \otimes \hat{f}(s) \otimes e(f))u, \hat{J}_s(\Theta_{\hat{g}(s)}(y) \otimes e(g))v \rangle \\ &\quad + \langle \hat{J}_s(\Theta_{f(s)}(x) \otimes e(f))u, \hat{J}_s(\Theta_{g(s)}(y) \otimes e(g))v \rangle \}, \end{aligned} \tag{3.17}$$

where $f(s)$ and $g(s)$ in k_0 are identified with $0 \oplus f(s)$ and $0 \oplus g(s)$ in $\hat{k_0}$ respectively. We claim that the identity above remains valid even when we replace x, y by $X, Y \in \mathcal{A} \otimes \Gamma^f(\hat{k_0})$ and $\Theta_\zeta(x), \Theta_\zeta(y)$ by $\widehat{\Theta}_\zeta(X), \widehat{\Theta}_\zeta(Y)$ respectively, where ζ is one of the vectors $\hat{f}(s), \hat{g(s)}, f(s)$ and $g(s)$. To see this, it suffices to observe that in the resulting identity, both left and right-hand sides vanish if $X \in \mathcal{A} \otimes \hat{k_0}^{\otimes^n}$ and $Y \in \mathcal{A} \otimes \hat{k_0}^{\otimes^m}$ with $m \neq n$, and then use the definition of $\hat{J}_t$ and $\widehat{\Theta}_\zeta$ to prove the identity for $X, Y \in \mathcal{A} \otimes_{\text{alg}} \hat{k_0}^{\otimes^m}$. Finally, use Corollary 3.2.2 and strong continuity of $\hat{J}_t$ obtained from the estimate in (i) to extend the identity from $X = \sum x_\alpha \otimes e_\alpha$, $Y = \sum y_\alpha \otimes e_\alpha$ (finite sums) to arbitrary X and Y. Thus one has upon setting

$$\Phi_t(X, Y) \equiv \langle \hat{J}_t(X \otimes e(f))u, \hat{J}_t(Y \otimes e(g))v\rangle - \langle ue(f), \hat{J}_t(\langle X, Y\rangle \otimes e(g))v\rangle$$

the equation:

$$\begin{aligned}\Phi_t(X, Y) = \int_0^t ds\{&\Phi_s(\widehat{\Theta}_{\hat{f}(s)}(X), \mathcal{J}_{\hat{g}(s)}(Y)) \\ &+ \Phi_s(\mathcal{J}_{\hat{f}(s)}(X), \widehat{\Theta}_{\hat{g}(s)}(Y)) + \Phi_s(\widehat{\Theta}_{f(s)}(X), \widehat{\Theta}_{g(s)}(Y))\},\end{aligned} \tag{3.18}$$

where $\langle X, Y\rangle$ is the module inner product in $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$ and we have set for $\zeta, \eta_1, \dots \eta_n \in \hat{k}_0$ the map $\mathcal{J}_\zeta(x \otimes \eta_1 \dots \otimes \eta_n) = x \otimes \eta_1 \dots \otimes \eta_n \otimes \zeta$, and extend it naturally as a map from $\mathcal{A} \otimes \Gamma^f(\hat{k_0})$ to itself. It is clear that the estimates in Lemma 3.3.1(iii) and Theorem 3.3.6(i) extend to

$$\|\widehat{\Theta}_\zeta(X)u\| \le \|(X \otimes 1_{\hat{k_0}})D(u \otimes \zeta)\|$$

and

$$\sup_{0 \le t \le t_0} \|\hat{J}_t(X \otimes e(f))u\| \le C'(f)\|(X \otimes 1_{\Gamma^f(\hat{k})})E_{t_0}u\|.$$

From the above estimates and definition of Φ_t, it is clear that $|\Phi_t(X, Y)| \le \|u\|\|v\|\|X\|\|Y\|\|E_{t_0}\|C'(g)\{\|E_{t_0}\|C'(f) + \|e(f)\|\}$. This implies, by iterating the expression (3.18) sufficient number of times, that $\Phi_t(X, Y) = 0$ which leads to $\Phi_t(x, y) = 0$ for all $x, y \in \mathcal{A}$. Since $\langle ve(g), j_t(x)(\sum_{i=1}^n u_i e(f_i))\rangle = \langle J_t(x^* \otimes e(g))v, \sum u_i e(f_i)\rangle$ by the above identity, it follows that $j_t(x)$ is well defined on $h \otimes_{\mathrm{alg}} \mathcal{E}(\mathcal{C})$, and thus (ii)(a) is proven. The proof of (ii)(b) and (iii) are as in [Ev] and [Par] respectively. For (iv), we note that $j_t(1) = 1$ for all t if and only if $dJ_t(1 \otimes e(f))u = 0$ $\forall\, u \in h,\ f \in \mathcal{C}$; and from Eq. (3.9) and (S1) it is clear that this can happen if and only if $0 = \int_0^t ds\langle ue(f), J_s \circ (\Lambda_{\Sigma^*\Sigma - I}(ds)(ve(f))\rangle = -\int_0^t ds\langle ue(f), J_s(\langle f(s), (\Sigma^*\Sigma - I)_{f(s)}\rangle \otimes e(f))v\rangle$ for all t, since $\pi(1)\delta = \delta$. But this is possible if and only if $\langle \zeta, (\Sigma^*\Sigma - I)_\zeta\rangle = 0\ \forall\, \zeta \in k_0$ which is same as $\Sigma^*\Sigma = I$. □

4. Dilation of a Quantum Dynamical Semigroup

In this section, first a unitary evolution U_t is constructed in $h \otimes \Gamma$ such that the vaccum expectation of $j_t^0(x) \equiv U_t(x \otimes 1_\Gamma)U_t^*$ gives back the q.d.s. T_t that we started with in Sect. 1. However, $j_t^0(x)$ in general will not satisfy a flow equation of the Evans–Hudson type. Here it is also shown that there exists a suitable choice of a partial isometry in $h \otimes k_0$ such that the above kind of flow equation can be implemented by a partial isometry-valued process in $h \otimes \Gamma$. It is to be noted here that in [H-S] an Evans–Hudson type dilation was achieved with $\mathcal{A} = \mathcal{B}(h)$ for a countably infinite dimensional h only.

Before proceeding further, we note the following two theorems whose proofs can be found in their respective references .

Theorem 4.0.1 ([C-E]). *Let $(T_t)_{t \ge 0}$ be a conservative uniformly continuous q.d.s. on $\mathcal{A}$ with $\mathcal{L}$ as its generator. Then*

(i) *There is a unital normal $*$-representation ρ of $\mathcal{A}$ in a Hilbert space $\mathcal{K}$ and a ρ-derivation $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that the set $\mathcal{D} \equiv \{\alpha(x)u | x \in \mathcal{A}, u \in h\}$ is total in $\mathcal{K}$.*

(ii) *Furthermore, there exists* $R \in \mathcal{B}(h, \mathcal{K})$ *such that* $\alpha(x) = Rx - \rho(x)R$, *and* $\mathcal{L}(x) = R^*\rho(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x]$ *with* $R^*\rho(x)R \in \mathcal{A}\ \forall x \in \mathcal{A}$ *and* H *a selfadjoint element in* $\mathcal{A}$.

The property of ρ-derivation has already been introduced in (S2) of Sect. 3 and note that the above pair $(\mathcal{L}, \alpha)$ satisfy the cocycle relation (S3) in Sect. 3.

Theorem 4.0.2 **([Dix]).** *Every normal* $*$*-representation* ρ *of a von Neumann algebra* $\mathcal{A}$ *in a Hilbert space* $\mathcal{K}$ *is of the form* $\rho(x) = \Sigma_1^*(x \otimes 1_{k_1})\Sigma_1$, *where* k_1 *is a Hilbert space and* Σ_1 *is a partial isometry with initial set* $\mathcal{K}$ *and final set* $h \otimes k_1$ *such that the projection* $P_1 \equiv \Sigma_1\Sigma_1^*$ *commutes with* $x \otimes 1_{k_1}$ *for all* $x \in \mathcal{A}$. *If* ρ *is unital, then* $\Sigma_1^*\Sigma_1 = 1_{\mathcal{K}}$. *Moreover, in case* h *is separable, one can choose* k_1 *to be separable also.*

4.1. Hudson-Parthasarathy (H-P) dilation. Let ρ, α, R be as in Theorem 4.1.1 and Σ_1 as in Theorem 4.1.2. Then set $\tilde{R} = \Sigma_1 R$, $\tilde{R} \in \mathcal{B}(h, h \otimes k_1)$ so that $\tilde{R}^* = R^*\Sigma_1^*$ and we have

$$\begin{aligned}\tilde{R}^*(x \otimes 1_{k_1})\tilde{R} &= R^*\Sigma_1^*(x \otimes 1_{k_1})\Sigma_1 R\\ &= R^*\rho(x)R.\end{aligned}$$

Also,

$$\tilde{R}^*\tilde{R} = R^*\Sigma_1^*\Sigma_1 R = R^*R, \text{ as } \Sigma_1^*\Sigma_1 = 1_{\mathcal{K}}.$$

Now, we take the unitary process U_t which satisfies the following q.s.d.e. (as in Sect. 3.1)

$$dU_t = U_t(a^{\dagger}_{\tilde{R}}(dt) - a_{\tilde{R}}(dt) + (iH - \frac{1}{2}\tilde{R}^*\tilde{R})dt), \quad U_0 = I. \tag{4.1}$$

Let $\tilde{\Gamma}$ denote $\Gamma(L^2(\mathbb{R}_+, k_1))$. Taking $j_t^0(x) = U_t(x \otimes 1_{\tilde{\Gamma}})U_t^*$, we see that for each t, $j_t^0(\cdot)$ is a $*$-homomorphism. We now claim that $\langle ve(0), j_t^0(x)ue(0)\rangle = \langle v, T_t(x)u\rangle$. To prove this, it is enough to show that $\langle ve(0), \frac{d}{dt} j_t^0(x)(ue(0))\rangle = \langle v, T_t(\mathcal{L}(x))u\rangle$, and this follows from the quantum Ito formula for right integrals as mentioned in Remark 2.2.5. Indeed we have,

$$\langle U_t^*(ve(0)), (x \otimes 1_{\tilde{\Gamma}})U_t^*(ue(0))\rangle = \int_0^t ds\langle ve(0), j_s^0(\mathcal{L}(x))(ue(0))\rangle,$$

where

$$\begin{aligned}\mathcal{L}(x) &= R^*\rho(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x]\\ &= \tilde{R}^*(x \otimes 1_{k_1})\tilde{R} - \frac{1}{2}\tilde{R}^*\tilde{R}x - \frac{1}{2}x\tilde{R}^*\tilde{R} + i[H, x].\end{aligned}$$

Thus, if we denote by $\mathbb{E}_0$ the vacuum expectation map which takes an element G of $\mathcal{B}(h \otimes \tilde{\Gamma})$ to an element $\mathbb{E}_0 G$ in $\mathcal{B}(h)$ satisfying $\langle v, (\mathbb{E}_0 G)u\rangle = \langle ve(0), G(ue(0))\rangle$ (for $u, v \in h$), then

$$\frac{d}{dt}\mathbb{E}_0 j_t^0(x) = \mathbb{E}_0 j_t^0(\mathcal{L}(x)),$$

which implies (since $\mathcal{L}$ is bounded), that $\mathbb{E}_0 j_t^0(x) = T_t(x)$.

A simple calculation using the quantum Ito formula and Eq. (4.1) shows that

$$dj_t^0(x) = U_t[a^{\dagger}_{\alpha(x)}(dt) - a_{\alpha(x)}(dt) + \mathcal{L}(x)dt]U_t^*, \tag{4.2}$$

where $\alpha(x) = \tilde{R}x - (x \otimes 1_{k_1})\tilde{R} = \Sigma_1[Rx - \rho(x)R]$, for $x \in \mathcal{A}$. In general, $\alpha(x)$ may not be in $\mathcal{A} \otimes k_1$ and therefore Eq. (4.2) is not a flow equation of the Evans–Hudson type. However, in case $\mathcal{A} = \mathcal{B}(h)$, it is trivially a flow equation.

4.2. Existence of structure maps and Evans–Hudson dilation of T_t. In the context of Theorems 4.1.1 and 4.1.2, it should be noted that in general $\mathcal{K}$ need not be of the form $h \otimes k'$ and neither ρ or α be structure maps as defined in Sect. 3, that is, ρ need not be in $\mathcal{A} \otimes \mathcal{B}(k')$ nor $\alpha(x)$ be in $\mathcal{A} \otimes k'$. However, the following theorem asserts that one can "rotate" the whole structure appropriately so that the "rotated" ρ and α (denoted π and δ respectively) become structure maps without changing $\mathcal{L}$ (see also [P-S]).

Theorem 4.2.1. *Let T_t be a conservative norm-continuous q.d.s. with generator $\mathcal{L}$. Then there exist a Hilbert space k_0, a normal $*$-representation $\pi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(k_0)$ and a π-derivation δ of $\mathcal{A}$ into $\mathcal{A} \otimes k_0$ such that the hypotheses* (S1)–(S3) *in Sect. 3 are satisfied.*

Proof. (i) Let $(\rho, \alpha, \mathcal{D})$ be as in theorem 4.1.1. We define a map $\rho' : \mathcal{A}' \to \mathcal{B}(\mathcal{K})$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$ in $\mathcal{B}(h)$, by

$$\rho'(a)(\alpha(x)u) = \alpha(x)\, au, x \in \mathcal{A}, \quad u \in h, \quad a \in \mathcal{A}', \tag{4.3}$$

and extend it linearly to the algebraic span of $\mathcal{D}$.

To show that it is well defined, we need to show that whenever $\sum_{i=1}^m \alpha(x_i)u_i = 0$ for $x_i \in \mathcal{A}$, $u_i \in h$, one has $\rho'(a)(\sum_{i=1}^m \alpha(x_i)u_i) = 0$. Since $\alpha(x_i)^*\alpha(y) = \mathcal{L}(x_i^* y) - \mathcal{L}(x_i^*)y - x_i^*\mathcal{L}(y) \in \mathcal{A}$ for $y \in \mathcal{A}$ by the comments at the end of Theorem 4.1.1, we have for $a \in \mathcal{A}'$,

$$\begin{aligned}\langle \rho'(a)(\sum_{i=1}^m \alpha(x_i)u_i),\ \alpha(y)v\rangle &= \sum_{i=1}^m \langle \alpha(x_i)au_i, \alpha(y)v\rangle \\ &= \sum_{i=1}^m \langle u_i, a^*\alpha(x_i)^*\alpha(y)v\rangle \\ &= \sum_{i=1}^m \langle u_i, \alpha(x_i)^*\alpha(y)a^*v\rangle \\ &= \langle \sum_{i=1}^m \alpha(x_i)u_i,\ \alpha(y)a^*v\rangle,\end{aligned} \tag{4.4}$$

thereby proving that ρ' is well defined. A similar computation gives,

$$\|\rho'(a)(\sum_{i=1}^m \alpha(x_i)u_i)\|^2 = \sum_{i=1}^m \sum_{j=1}^m \langle u_i, \alpha(x_i)^*\alpha(x_j)a^*au_j\rangle. \tag{4.5}$$

Denoting the operator $\alpha(x_i)^*\alpha(x_j)$ by A_{ij}, and noting that $A \equiv ((A_{ij}))_{i,j=1,\dots m}$ acts as a positive operator on $\underbrace{h \oplus \dots \oplus h}_{\text{m copies}}$, which commutes with the positive operator $C \otimes I_m$, where $C = \|a\|^2.1 - a^*a$ and I_m denotes the identity matrix of order m, we observe that $A(C \otimes I_m)$ is also a positive operator. Thus, considering $u_1 \oplus u_2 \oplus \ldots \oplus u_m \in \underbrace{h \oplus \ldots \oplus h}_{\text{m copies}}$, the right-hand side of (4.5) can be estimated as:

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(x_j)a^*au_j\rangle \\ &\leq \|a\|^2 \sum_{i=1}^{m}\sum_{j=1}^{m}\langle u_i, \alpha(x_i)^*\alpha(x_j)u_j\rangle \\ &= \|\sum_{i=1}^{m}\alpha(x_i)u_i\|^2\|a\|^2,\end{aligned}$$

proving that $||\rho'(a)|| \leq ||a||$ since $\mathcal{D}$ is total in $\mathcal{K}$. It is also easy to see from the definition of ρ' and (4.4) that it is a unital $*$-representation of $\mathcal{A}'$ in $\mathcal{K}$. Next we show that ρ' is normal. For this, take a net $\{a_\alpha\}$ such that $0 \leq a_\alpha \uparrow a$, where $a_\alpha, a \in \mathcal{A}'$. It is clear from the definition of ρ' that $\rho'(a_\alpha)\alpha(x)u \rightarrow \rho'(a)\alpha(x)u$ for all $x \in \mathcal{A}$, $u \in h$, and thus, $\rho'(a_\alpha) \overset{s}{\rightarrow} \rho'(a)$ on $\mathcal{K}$ by totality of $\mathcal{D}$ in $\mathcal{K}$ and since $\|\rho'(a_\alpha)\| \leq \|a_\alpha\| \leq ||a|| \; \forall \alpha$.

(ii) By (i), $\rho' : \mathcal{A}' \rightarrow \mathcal{B}(\mathcal{K})$ is a unital normal *- representation. By Theorem 4.1.2, there exist a Hilbert space k_2, an isometry $\Sigma_2 : \mathcal{K} \rightarrow h \otimes k_2$ with $\mathcal{K}_2 = \mathrm{Ran}\Sigma_2 \overset{\sim}{=} \mathcal{K}$, satisfying,

$$\rho'(a) = \Sigma_2^*(a \otimes 1_{k_2})\Sigma_2, \tag{4.6}$$

and for all $a \in \mathcal{A}'$, $a \otimes 1_{k_2}$ commutes with $P_2 \equiv \Sigma_2\Sigma_2^*$. Let us now take $\tilde{\delta}(x) = \Sigma_2\alpha(x)$, $\tilde{\pi}(x) = \Sigma_2\rho(x)\Sigma_2^*$. It is clear that $\tilde{\delta}$ is a $\tilde{\pi}$-derivation. Moreover, $\tilde{\delta}(x^*)^*\tilde{\delta}(y) = \alpha(x^*)^*\Sigma_2^*\Sigma_2\alpha(y) = \alpha(x^*)^*\alpha(y)$ and hence $\mathcal{L}(xy) - x\mathcal{L}(y) - \mathcal{L}(x)y = \tilde{\delta}(x^*)^*\tilde{\delta}(y)$ holds. Taking $\tilde{R} = \Sigma_2 R \in \mathcal{B}(h, h \otimes k_2)$, we observe that $\tilde{\delta}(x) = \Sigma_2\alpha(x) = \Sigma_2(Rx - \rho(x)R) = \tilde{R}x - \tilde{\pi}(x)\tilde{R}$. It is also clear that $\mathcal{L}(x) = \tilde{R}^*\tilde{\pi}(x)\tilde{R} - \frac{1}{2}\tilde{R}^*\tilde{R}x - \frac{1}{2}x\tilde{R}^*\tilde{R} + i[H, x] = R^*\rho(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x]$.

To show that $\tilde{\delta}(x) \in \mathcal{A} \otimes k_2$ for all $x \in \mathcal{A}$, it is enough (by Llemma 3.2.3) to verify that for any $f \in k_2$, $\langle f, \tilde{\delta}(x)\rangle \in \mathcal{A}$, or equivalently that $\langle f, \tilde{\delta}(x)\rangle$ commutes with all $a \in \mathcal{A}'$. For $f \in k_2, a \in \mathcal{A}', u, v \in h, x \in \mathcal{A}$, since P_2 and $(a \otimes 1_{k_2})$ commute, we have,

$$\begin{aligned}\langle\langle f, \tilde{\delta}(x)\rangle au, v\rangle &= \langle\tilde{\delta}(x)au, v \otimes f\rangle \\ &= \langle\Sigma_2\alpha(x)au, v \otimes f\rangle = \langle\Sigma_2\rho'(a)(\alpha(x)u), v \otimes f\rangle \\ &= \langle\Sigma_2\Sigma_2^*(a \otimes 1_{k_2})\Sigma_2\alpha(x)u, v \otimes f\rangle \\ &= \langle P_2(a \otimes 1_{k_2})\Sigma_2\alpha(x)u, v \otimes f\rangle \\ &= \langle(a \otimes 1_{k_2})\Sigma_2\Sigma_2^*\Sigma_2\alpha(x)u, v \otimes f\rangle \\ &= \langle\Sigma_2\alpha(x)u, (a^*v) \otimes f\rangle = \langle\langle f, \tilde{\delta}(x)\rangle u, a^*v\rangle \\ &= \langle a\langle f, \tilde{\delta}(x)\rangle u, v\rangle.\end{aligned}$$

Next, we want to show that $\tilde{\pi}(x) \in \mathcal{A} \otimes \mathcal{B}(k_2)$ for $x \in \mathcal{A}$; and for this it is enough to verify $\tilde{\pi}(x)(a \otimes 1_{k_2}) = (a \otimes 1_{k_2})\tilde{\pi}(x)$ for all $a \in \mathcal{A}'$. Since $\Sigma_2^*P_2^{\perp} = 0$, and since

$\Sigma_2\mathcal{D}$ is total in $\mathcal{K}_2$, it suffices to verify that $\tilde{\pi}(x)(a \otimes 1_{k_2})P_2 = (a \otimes 1_{k_2})\tilde{\pi}(x)P_2$, or equivalently that $\tilde{\pi}(x)(a \otimes 1_{k_2})\Sigma_2\alpha(y)u = (a \otimes 1_{k_2})\tilde{\pi}(x)\Sigma_2\alpha(y)u$, for all $y \in \mathcal{A}$, $u \in h$. For this, observe that,

$$\begin{aligned}
&\tilde{\pi}(x)(a \otimes 1_{k_2})\Sigma_2\alpha(y)u = \Sigma_2\rho(x)\Sigma_2^*(a \otimes 1_{k_2})\Sigma_2\alpha(y)u = \Sigma_2\rho(x)\rho'(a)\alpha(y)u \\
&= \Sigma_2\rho(x)\alpha(y)au = \Sigma_2\alpha(xy)au - \Sigma_2\alpha(x)yau = \Sigma_2\rho'(a)(\alpha(xy) - \alpha(x)y)u \\
&= \Sigma_2\rho'(a)\rho(x)\alpha(y)u = \Sigma_2\Sigma_2^*(a \otimes 1_{k_2})\Sigma_2\rho(x)\Sigma_2^*(\Sigma_2\alpha(y)u) \\
&= P_2(a \otimes 1_{k_2})\tilde{\pi}(x)(\Sigma_2\alpha(y)u) = (a \otimes 1_{k_2})\tilde{\pi}(x)(\Sigma_2\alpha(y)u).
\end{aligned}$$

(iii) It follows from the above and Theorem 4.1.2 that $\tilde{\pi}(x) = \Sigma_2\Sigma_1^*(x \otimes 1_{k_1})\Sigma_1\Sigma_2^* \equiv \tilde{\Sigma}^*(x \otimes 1_{k_1})\tilde{\Sigma}$ on $h \otimes k_2$ so that $\tilde{\Sigma}$ is a partial isometry with initial set $P_2(h \otimes k_2)$ and final set $P_1(h \otimes k_1)$. Now set $k_0 = k_1 \oplus k_2$ and $\Sigma = \tilde{\Sigma} \oplus 0 : h \otimes k_0 \to h \otimes k_0$ with initial set $(0 \oplus P_2)(h \otimes k_0)$ and final set $(P_1 \oplus 0)(h \otimes k_0)$ and $\pi(x) = \tilde{\pi}(x) \oplus 0$, $\delta(x)u = \tilde{\delta}(x)u$ for $x \in \mathcal{A}$, $u \in h$. It is clear that $\delta(x) \in \mathcal{A} \otimes k_0$, $\pi(x) \in \mathcal{A} \otimes \mathcal{B}(k_0)$ and (S1)–(S3) are satisfied. □

Remark 4.2.2. Although ρ was assumed to be unital, π chosen by us is not unital. However, in some cases it may be possible to choose Σ, k_0 in such a manner that π is unital.

We summarise the main result of this section in form of the following theorem:

Theorem 4.2.3. *Let $(T_t)_{t\geq 0}$ be a conservative norm-continuous q.d.s. with $\mathcal{L}$ as its generator. Then there is a flow $J_t : \mathcal{A} \otimes \mathcal{E}(\mathcal{C}) \to \mathcal{A} \otimes \Gamma$ satisfying an Evans–Hudson type q.s.d.e. (3.9) with structure maps $(\mathcal{L}, \delta, \sigma)$ satisfying (S1)-(S3), where $\Gamma \equiv \Gamma(L^2(\mathbb{R}_+, k_0))$ and $\mathcal{C}$ consists of bounded continuous functions in $L^2(\mathbb{R}_+, k_0)$, such that $j_t(x)$ defined in Theorem* 3.3.6 *is a (not necessarily unital) $*$-homomorphism of $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(k_0)$ and $\mathbb{E}_0 j_t(x) = T_t(x)\ \forall x \in \mathcal{A}$.*

Proof. The proof is immediate by (i) observing the existence of structure maps δ and σ satisfying (S1),(S2) from Theorem 4.2.1, (ii) observing that $\mathcal{L}$ satisfies (S3), and finally (iii) constructing the solution J_t of Eq. (3.9) with structure maps $(\mathcal{L}, \delta, \sigma)$ as in Theorem 3.3.6. That $\mathbb{E}_0 j_t(x) = T_t(x)\ \forall x \in \mathcal{A}$ follows from the q.s.d.e (3.9). □

Remark 4.2.4. With reference to the last sentence in the statement of Theorem 4.1.2, it may be noted that both the Hilbert spaces k_1 and k_2 and hence k_0 can be chosen to be separable if the initial Hilbert space h is separable. In such a case, if we choose an orthonormal basis $\{e_j\}$ in k_0, then the estimate for δ in (S2) is precisely the coordinate-free form of the condition

$$\sum_{i=0} ||\mu_0^i(x)u||^2 \leq \sum_{i\in\mathcal{I}_0} ||xD_0^i u||^2$$

with $\sum_{i\in\mathcal{I}_0} ||D_0^i u||^2 \leq \alpha_0||u||^2$ in [Mo-S]. The similar conditions on $\mu_j^i (j \neq 0)$ as in [Mo-S] is trivially satisfied by $((\mu_j^k))_{k,j=1}^{\infty} \equiv \sigma$ and for $\mu_0^0 \equiv \mathcal{L}$ as can be seen easily from (S1) and (S3). It may also be noted that j_t satisfies the E-H equation $dj_t(x) = \sum_{i,j} j_t(\mu_j^i(x))d\Lambda_i^j(t)$, with $j_0 = id$, in the coordinatized form with the appropriate choices of μ_j^i's in terms of $\mathcal{L}, \delta$ and σ as above. The flow equation (3.9) is in fact a coordinate-free modification of the old coordinatized E-H equation given above.

4.3. Implementation of $E - H$ flow. Combining Theorems 4.1.1 and 4.2.3, we see that for $x \in \mathcal{A}$, $\pi(x) = \Sigma(x \otimes 1_{k_0})\Sigma^*$
$\in \mathcal{A} \otimes \mathcal{B}(k_0)$, $\delta(x) = Rx - \pi(x)R \in \mathcal{A} \otimes k_0$ for a suitable Hilbert space k_0, $R \in \mathcal{B}(h, h \otimes k_0)$, Σ a partial isometry in $h \otimes k_0$. Now let us consider the H-P type q.s.d.e.:

$$dV_t = V_t(a_R^\dagger(dt) + \Lambda_{\Sigma - I}(dt) - a_{\Sigma^* R}(dt) + (iH - \frac{1}{2}R^*R)dt),\ V_0 = I. \qquad (4.7)$$

Then by Proposition 3.1.2, there is a contraction valued unique solution V_t as a regular process on $h \otimes \Gamma$. The following theorem shows that every Evans–Hudson type flow J_t satisfying Eq. (3.9) is actually implemented by a process V_t satisfying Eq. (4.7).

Theorem 4.3.1. *Every flow J_t satisfying Eq. (3.9) is implemented by a partial isometry valued process V_t satisfying (4.7), that is, $J_t(x \otimes e(f))u = V_t(x \otimes 1_\Gamma)V_t^*(ue(f))$. Furthermore, the projection-valued processes $P_t \equiv V_tV_t^*$ and $Q_t \equiv V_t^*V_t$ belong to $\mathcal{A} \otimes \mathcal{B}(\Gamma)$ and $\mathcal{A}' \otimes \mathcal{B}(\Gamma)$ respectively.*

We need a lemma for the proof of this theorem.

Lemma 4.3.2. *If $\mathcal{B}$ is a von Neumann algebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and p is a projection in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{B} \ni x \mapsto pxp$ is a $*$-homomorphism of $\mathcal{A}$, then $p \in \mathcal{B}'$.*

Proof of the lemma. Let q be any projection in $\mathcal{B}$. We have by the hypothesis that,

$$pqp = pq^np = (pqp)^n \quad \forall n \geq 1.$$

But

$$(pqp)^n = \underbrace{pqp \cdot pqp \ldots}_{\text{n times}} = (pq)^n p \xrightarrow{s} (p \wedge q)p = p \wedge q,$$

by von Neumann's Theorem, where $p \wedge q$ denotes the projection onto $\text{Ran}(p) \bigcap \text{Ran}(q)$. Thus we have, $(qp)^*qp = pqp = p \wedge q$, which implies that qp is a partial isometry with the initial space $\text{Ran}(p \wedge q)$ and hence $qp.p \wedge q = qp$. But $qp.p \wedge q = p \wedge q$, and thus $qp = p \wedge q = (p \wedge q)^*$ (since $p \wedge q$ is a projection) $= pq$. This completes the proof because $\mathcal{B}$ is generated by its projections. □

Proof of the theorem. Setting $J_t'(x \otimes e(f))u = V_t(x \otimes 1_\Gamma)V_t^*(ue(f))$ for $u \in h$, $f \in \mathcal{C}$, and using Eq. (4.7) we verify easily that $J_0' = id$ and J_t' satisfies the same flow equation (3.9) as does J_t. By the uniqueness of the solution of the initial value problem (3.9) we conclude that $J_t = J_t'$. Now, as in Theorem 3.3.6, if we set $j_t(x)ue(f) = J_t(x \otimes e(f))u$, it follows that $j_t(x) = V_t(x \otimes 1_\Gamma)V_t^*$ and that $j_t(.)$ is a $*$-homomorphism of $\mathcal{A}$. Therefore, $V_t(xy \otimes 1_\Gamma)V_t^* = V_t(x \otimes 1_\Gamma)Q_t(y \otimes 1_\Gamma)V_t^*$ for $x, y \in \mathcal{A}$. In particular $P_t = j_t(1) = V_tV_t^*$ is a projection, that is, V_t is a partial isometry valued regular process. It also follows from the same identity that $Q_t(xy \otimes 1_\Gamma)Q_t = Q_t(x \otimes 1_\Gamma)Q_t(y \otimes 1_\Gamma)Q_t$, that is, $x \otimes 1_\Gamma \mapsto Q_t(x \otimes 1_\Gamma)Q_t$ is a $*$-homomorphism of $\mathcal{A} \otimes 1_\Gamma$. Therefore $Q_t \in (\mathcal{A} \otimes 1_\Gamma)' = \mathcal{A}' \otimes \mathcal{B}(\Gamma)$, by the Lemma 4.3.2. □

5. Weak Markov Process Associated with the $E - H$ Type Flow

Here we consider the solution J_t of Eq. (3.9) or the associated j_t and construct a weak Markov process (see [B-P]) with respect to the Fock filtration. The next theorem summarizes the results:

Theorem 5.0.3. (i) *Let j_t be as in Theorem 4.2.3. Set $F_t \equiv j_t(1)\mathbb{E}_t$, where $\mathbb{E}_t$ is the conditional expectation operator given by $\mathbb{E}_t(ue(f)) = ue(f_t)$. Then there exists a nonzero projection $j_\infty(1)$ such that the family of projections $\{j_t(1)\}$ and $\{F_t\}$ decreases and increases to $j_\infty(1)$ respectively.*

(ii) *The triple $\{j_t, h \otimes \Gamma, F_t\}$ is a weak Markov process as defined in [B-P], that is, $\mathbb{E}_0^F j_0(x) = xF_0$, $j_t(x)F_t = F_t j_t(x) = F_t j_t(x)F_t$, $\mathbb{E}_s^F j_t(x) = j_s(T_{t-s}(x))F_s$ for $0 \le s \le t < \infty$, $x \in \mathcal{A}$, where $\mathbb{E}_s^F(X) = F_s X F_s$ for $X \in \mathcal{B}(h \otimes \Gamma)$.*

(iii) If we set $k_t(x) = j_t(x)F_t = j_t(x)\mathbb{E}_t = \mathbb{E}_t j_t(x)$, then the triple $\{k_t, h \otimes \Gamma, F\}$ is a conservative weak Markov flow subordinate to $\{F_t\}$ (see [B-P]), that is, k_t satisfies the properties listed in (ii) above and also $k_t(1) = F_t$.

Proof. (i) In the notation of Theorem 4.2.3, $j_t(1) = V_t V_t^*$ and therefore taking $W_t = V_t^*$ and using the relation (3.5) with T replaced by Σ we see that $\{j_t(1)\}$ is a decreasing family of projections. On the other hand, a simple computation shows that for $t > s > 0$,

$$\langle ve(g), (F_t - F_s)(ue(f)\rangle =$$
$$\langle W_t ve(g), W_t ue(f)\rangle e^{-\int_t^\infty \langle g(\tau), f(\tau)\rangle d\tau} - \langle W_s ve(g), W_s ue(f)\rangle e^{-\int_s^\infty \langle g(\tau), f(\tau)\rangle d\tau} =$$
$$\int_s^t exp\{-\int_\tau^\infty \langle g(\tau'), f(\tau')\rangle d\tau'\}\langle W_\tau ve(g), \langle g(\tau), \Sigma\Sigma^*_{f(\tau)}\rangle W_\tau ue(f)\rangle d\tau,$$

from which it follows that $\{F_t\}$ is an increasing family of projections. Since $\mathbb{E}_t$ increases to I on $h \otimes \Gamma$, and since $j_t(1)$ converges strongly to say $j_\infty(1)$, we have that F_t increases to $j_\infty(1)$. Therefore $j_\infty(1)$ cannot be the zero projection.

(ii) Let $u, v \in h, f, g \in k, x \in \mathcal{A}$. Since $j_s(1)j_t(1) = j_t(1)$ for $s \le t$, we have,

$$\begin{aligned}
&\langle F_s j_t(x)F_s(ve(g)), ue(f)\rangle = \langle j_s(1)j_t(x)j_s(1)ve(g_s), ue(f_s)\rangle \\
&= \langle j_t(x)ve(g_s), ue(f_s)\rangle = \langle J_t(x \otimes e(g_s))v, ue(f_s)\rangle \\
&= \langle J_s(x \otimes e(g_s))v, ue(f_s)\rangle + \int_s^t \langle J_\tau(\mathcal{L}(x)e(g_s))v, ue(f_s)\rangle d\tau \\
&= \langle j_s(x)F_s(ve(g)), ue(f)\rangle + \int_s^t \langle F_s j_\tau(\mathcal{L}(x))F_s(ve(g)), ue(f)\rangle d\tau,
\end{aligned}$$

because $f_s(\tau) = 0$, $g_s(\tau) = 0$ for $\tau > s$. Thus, if we denote by Ξ_t the map $\mathcal{A} \ni x \mapsto F_s j_t(x)F_s$, then $\frac{d\Xi}{dt} = \Xi_t \circ \mathcal{L}$, for $s \ge t$. On the other hand, denoting by Π_t the map given by, $\Pi_t(x) = j_s(T_{t-s}(x))F_s$, we can easily verify that $\frac{d\Pi}{dt} = \Pi_t \circ \mathcal{L}$. Since $\Xi_s(x) = \mathbb{E}_s j_s(x) = j_s(x)F_s = \Pi_s(x)$, the initial values of Π and Ξ agree. Thus by the standard uniqueness result of differential equations, we conclude that $\Pi_t \equiv \Xi_t$ for all $t \ge s$.

(iii) The proof of this part is obvious from the definitions. □

We have no result with regards to minimality of the above process in the sense of [B-P]. However, if we denote the closed linear span of $\{j_{t_1}(x_1) j_{t_2}(x_2) \ldots j_{t_n}(x_n) ue(0) | x_j \in \mathcal{A},\ t > t_1 \geq t_2 \geq \ldots \geq t_n > 0\}$ by $\mathcal{K}'_t$ for $0 \leq t \leq \infty$, then it is an easy observation that $\mathcal{K}'_t$ is contained in the range of F_t for each $t < \infty$, and thus $\mathcal{K}'_\infty$ is contained in $j_\infty(1)(h \otimes \Gamma)$. We suspect that $\mathcal{K}'_\infty = j_\infty(1)(h \otimes \Gamma)$, which we have not been able to prove. If this turns out to be true, then the above provides a complete general theory of stochastic dilation for a uniformly continuous quantum dynamical semigroup on a von Neumann algebra. It should also be noted that the final weak Markov process (j_s, F_s) is actually living in $h \otimes \Gamma(k)$ and its filtration is subordinate to that in the Fock space.

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