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ON THE DISTRIBUTION OF SQUARENESS

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SUMMARY. Squareness of a job with circular face is defined to be the amount of projection of the face on the axis of the job. In ideal situation the plane of the face is perpendicular to the axis and the squareness equals zero. Under simple assumptions it is shown that the distribution of squareness is of the form $K|X+\mu|$ where $K > 0$, $-\infty < \mu < \infty$ and X is a normal deviate. Some data sets are analysed by the model.

1. Introduction

In many industrial jobs, metal made cap or covers are required to fit face to face to another cap or plane sheet of metal so as to enclose instrumental components inside it.

The cap or cover usually posses an axis passing through the centre of the face and the outer circular face of the cover is required to be perpendicular with the axis. If the plane of the face is not so with the axis of the cover to a great extent, then the fitting of the cap will be improper which will affect the function of the job.

As for example in the case of back cover of a table fan, the rotar of the fan is placed along the axis of the back cover. Now, if the squareness, a measure of deviation of the face of the cover from ideal perpendicular position, is high then this will hamper the smooth running of the rotar inside it, affecting the function of the fan.

The squareness is defied to be the amount of projection of the outer plane of the face on the axis of the job. This measure is non-negative ; zero being attained when the outer face is exactly perpendicular to the axis.

Let r be the radius of the outer circular face which is tilted at an angle θ with the perpendicular position of the axis. The measure of sequareness is then $|2r \text{ Sin } \theta|$, $-\pi/2 \leqslant \theta < \pi/2$.

2. Description of the operation and model

The back cover of a table fan consist of a thick outer circular ring which is connected to another small hollow base where the bush has to be placed.

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The rotar of the fan is adjusted along the axis of the cover and an end of the rotar moves inside the bush.

The processing of the cover is done as follows. The outer ring of the face is given an approximate radius $r = r_n$ by the metal cutting operation described in Dasgupta, Ghosh and Rao (1981). Next, to process the outer plane of the face, the cover is gripped along its axis, possibly with an inclination θ with the axis of rotation, by a jaw chuck in a lathe machine. The cover is then rotated by the machine and the metal is removed from the face to give it a smooth shape and the angle of inclination $\theta = \theta_t$ varies over m rotations. We study the limiting behavior of the squareness as $n \to \infty$, $m \to \infty$.

There are several factors which affects squareness. For processing the face in a lathe machine, in order that there is no squareness operator intends to fix the cover along the axis of rotation, before processing starts. The target is to make $\theta = 0$, but for error in setting θ is in a neighbourhood of zero. Like any other error component one may assume θ to be normally distributed with zero mean. The variance will of course depend on the efficiency of the operator in setting the job. A systematic factor responsible for squareness is the machine defect while setting the job. If the jaw chuck by which the cover is gripped for processing is itself tilted at an angle θ_0 with the perpendicular position of the axis then one may take $\theta \sim N(\theta_0, .)$. In perfect machine, for an operator without any bias in setting $\theta_0 = 0$. For an experienced operator the variance of the distribution of θ will be small and we shall assume $K^{1/2}(\theta - \theta_0) \sim N(0, 1)$ where K is large compared to θ and θ_0 in radians.

Usually outer face of the cover is also thick and before processing the face, the outer ring is given an approximate radius r by a different process of metal cutting. Assume for the moment that r is not a random variable but a constant. Also let $\theta_0 = 0$. Since the magnitude of θ is small one may write $\sin \theta = \theta(1+o(1))$. Hence $2r|\sin\theta| \simeq 2r|\theta|(1+o(1)) \simeq 2rK^{-1/2}|X| \simeq K_1|X|$ for some $K_1 > 0$, where X is a normal deviate. Next for $\theta_0 \neq 0$, $\sin\theta \simeq \sin\theta_0 + (\theta - \theta_0)\cos\theta_0$.

Therefore

$$|2r \sin\theta| \simeq K^{-1/2}|2rK^{1/2}\sin\theta_0 + 2rK^{1/2}(\theta-\theta_0)\cos\theta_0|$$

$$\simeq K_1|X+\mu| \text{ for some } K_1 > 0 \text{ and } -\infty < \mu < \infty.$$

Hence the distribution of squareness is of the form $K_1|X+\mu|$ where $\mu = 0$ if $\theta_0 = 0$.

Now consider the case where r is random. It follows from the cutting model considered in Dasgupta *et al.* (1981, 186-187) that the cut upto n-th stage T_n is of the form

$$T_n = T_{n-1}+(\epsilon_n-T_{n-1})^+c_n,\ T_1 = c_1\epsilon_1^+,\ 0 < c_n \leqslant 1,\ n \geqslant 1,\ \epsilon^+ = \max(\epsilon, 0).$$

ϵ_i is the random shift of the center at i-th stage towards the cutting tool while shaping the outer ring, c_i is the fraction of the cut to the shift of the center at i-th rotation. c_i depends on hardness of the cutting tool and the job. e.g. $c_i \equiv 1$ if the tool is too hard and job is made of relatively soft material. One may assume that max ϵ_i has a limiting distribution after suitable standardisation i.e., $(\max \epsilon_i - a_n)/b_n \sim$ an extreme value distribution. It is also assumed $c_i = 1-o(|a_i^{-1} b_i|)$; $c_i \uparrow 1$ and $\overline{\lim}_{i\to\infty} f(ik)/f(i) < \infty\ \forall\ k$ where $f(i) = |a_i b_i^{-1}|$ is nondecreasing in i. The assumed conditions imply that $T_n = \max \epsilon_i + o_p(b_n)$; then $r = r_n = d - T_n$ where d is some specified value, ϵ_i's are i,i.d random variable and $b_n \downarrow 0$ as $n \to \infty$ e.g. $b_n = n^{-1}$ if ϵ_i's are i.i.d uniform on (0, 1). Hence $r_n = d - T_n = d-(a_n+b_nz(1+o_p(1)))$ where z is a random variable with extreme value distribution. If the r.v's ϵ_i's are bounded above, one may take $a_n = a$ and $r_n = d-a-b_n z(1+o_p(1)) = d^*-b_nz(1+o_p(1))$ where $d^* = d-a$. Therefore, since $b_n \downarrow 0$

$$\begin{aligned} |2r_n \sin\theta| &= |2(d^*-b_nz(1+o_p(1)))\sin\theta| \\ &= |2d^*\sin\theta(1+o_p(b_n))| \simeq K_1|X+\mu|. \end{aligned}$$

So, a distribution of the type $K_1|X+\mu|$ is a good approximation to the distribution of squareness.

Next assume that the number of revolutions to which the face of the cover is exposed for processing be m. It is natural to assume that the inclination θ_i at the i-th revolution is a random quantity with a systematic part θ present in it. We shall assume $\theta_i = \theta+\eta_i$ where η_i's are i.i.d r.v's centred at origin. The calculations and arguments in the above paragraph hold for the i-th revolution with θ replaced by θ_i. It is also clear that in this representation, the squareness over m revolutions is $\max_{i=1\ldots m} |2r_n \sin\theta_i|$. From A1 of Dasgupta *et al.* (1981) with $c > 0$ it follows that for large m the squareness is approximatele equal to $|2r_n \text{ Sin } (\theta+\delta)|$ where $\delta = \sup\ \{\alpha : p(\eta > \alpha) > 0\}$, letting $\theta > 0$ and $(\theta+\delta) < \pi/2$.

The earlier calculations remain valid with θ replaced by $(\theta+\delta)$ and the form of the distribution of squareness remains the same.

The contribution to μ come from two different components in this representation. A part is from θ_0, the mean of θ, which may be attributed to the bias of operator/machine in setting the job. The other part comes from the fluctuation of the angle of inclination over m rotations. This part may be attributed to the machine efficiency. It may not be possible to separate these two components except when one of these is zero.

3. Fitting the model and analysis of data

If $|\mu|$ is quite large then virtually speaking $K_1|X+\mu| = K_1(X+|\mu|) \sim N(.,.)$. For moderate values of $|\mu|$, the distribution $K_1|X+\mu|$ folds the part of the normal distribution on the negative side of the axis to its absolute value on the positive side.

Estimate of μ may be obtained by trimming the observations equidistant from the model value and then taking the mean of the trimmed observations. The variance of the distribution may be obtained by distributing, the half of the frequencies of the classes which are far of the model value in positive side, to the negative side. Hence reconstructing the original normal distribution one may proceed to estimate σ^2.

We analyse two sets of data. The following are the squareness of back covers of 80 table fans.

squareness (in 10^{-3} inch)	1	1.5	2	2.5	3	3.5	4	4.5	5	6	6.6	7	8
frequency	1	3	3	5	10	3	21	3	16	11	1	1	2

Grouped data in class intervals are shown below :

class interval	frequency
(0—1.5]	4
(1.5—2.5]	8
(2.5—3.5]	13
(3.5—4.5]	24
(4.5—5.5]	16
(5.5—6.5]	12
> 6.5	3

The model class is (3.5–4.5]. From row data we observe that only two observations corresponding to the squareness 8 fall outside the equidistant range of both sides of model class. Therefore we may carry out usual normal fit to the data since the distribution is not perturbed by the absolute sign.

class interval	mid point	frequency	expected frequency
(0—1.5]	.75	4	3.27
(1.5—2.5]	2	8	8.10
(2.5—3.5]	3	13	16.36
(3.5—4.5]	4	24	21.00
(4.5—5.5]	5	16	17.55
(5.5—6.5]	6	12	9.43
> 6.5	7	3	4.29
total	—	80	80

$\mu = 4.088$, $\sigma = 1.49$, calculated $\chi^2 = 2.5$, $\chi_{05,4} = 9.49$

Calculated χ^2 is insignificant at 5% level and the fit is satisfactory. 95% confidence interval for squareness is $(\mu \pm 1.96\sigma) = (1.17, 7.01)$. Only two observation fall outside the upper limit and the process seems to be under control.

However $\mu = 4.09$ and there is a possibility of reducing this value (the ideal being zero) by improvement of the setting of the machine.

The second set of data relates to the squareness of the back cover of 48 table fans after pressing the bush inside it. Following are the row of squareness in 10^{-3} inch.

squareness	1.5	2	3	3.5	4	4.5	5	6	7	8	9	10	11	12	13	14	16	20
frequency	2	1	2	2	3	1	6	4	4	1	4	5	3	2	3	2	2	1

Grouped data in class intervals are as follows :

class interval	frequency	mid point
(0—3]	5	1.5
(3—4]	5	3.5
(4—7]	15	5.5
(7—10]	10	8.5
(10—13]	8	11.5
> 13	5	
total	48	—

It appears from the above table that the mean μ estimated from the trimmed observations excluding the last class is quite large and the distributon is not much perturbed by the absolute sign. Therefore usual normal fit is carried out. $\mu = 7.33$, $\sigma = 4.029$

class interval	expected frequency
$(-\infty, 0]$	1.65
(0—3]	5.13
(3—4]	3.03
(4—7]	12.62
(7—10]	13.39
(10—13]	8.35
> 13	3.83

Since we consider only the absolute value of the variable the frequency of the class $(-\infty, 0]$ should be added to the last class. So the expected frequencies of the distribution of squareness are as follows.

class interval	observed frequency	expected frequency
(0—3]	5	5.13
(3—4]	5	3.03
(4—7]	15	12.62
(7—10]	10	13.39
(10—13]	8	8.35
> 13	5	5.48
Total	48	48

Calculated $\chi^2 = 2.65$, $\chi^2_{.05,3} = 7.815$, χ^2 is insignificant and the fit is satisfactory. 95% confidence interval for squareness is $(\mu \pm 1.96\sigma) = (0, 15.23)$. From row data we see that 3 observations fall outside the upper limit, caution should be taken for those points.

Like data set I, here also mean value may be reduced by proper setting of the machine.

Next we compare this model with another competitor $|\mu + Y\sigma|$ where $Y \sim \frac{1}{2} e^{-|y|}$, $-\infty < y < \infty$. The m.l.e. μ and σ are median and mean deviation when absolute sign is ignored, which seems to be appropriate if μ is large.

The fit for two sets of data are shown in the following tables. The fit seems to be worse as compared to corresponding normal fit.

Data set I

Fit for $|\mu+Y\sigma|$, $Y \sim \frac{1}{2} e^{-|y|}$

class interval	observed frequency	expected frequency
(0—1.5]	4	3.046
(1.5—2.5]	8	5.715
(2.5—3.5]	13	13.480
(3.5—4.5]	24	27.609
(4.5—5.5]	16	16.697
(5.5—6.5]	12	7.079
> 6.5	3	6.374
total	80	80

$\mu = 4.125$, $\sigma = 1.1656$, $\chi^2 = 6.94$, corresponding χ^2 for normal fit $= 2.5$.

Data set II

class interval	observed frequency	expected frequency
(0—3]	5	4.49
(3—4]	5	2.69
(4—7)	15	15.32
(7—10]	10	13.62
(10—13]	8	5.40
> 13	5	6.48
total	48	48

$\mu = 6.8$, $\sigma = 3.24$, $\chi^2 = 4.60$, corresponding χ^2 for normal fit $= 2.65$.

Discussion. The empirical data conform quite satisfactorily with the theoretical model of the distribution of squareness as $k|X+\mu|$ where $k > 0$, $-\infty < \mu < \infty$, $X \sim N(0, 1)$.

In the first data set two observations fall outside the upper confidence limit and the process seems to be under control.

In the second data set three observations corresponding to squareness 16 and 20 fall outside the upper limit. As the magnitude of squareness is too high, caution should be taken for these observations.

In both data sets I and II there is a possibility of reducing the value of μ, 4.09 for 1st set, 7.33 for 2nd set, towards the ideal value zero by improvement of the setting of the machine.

Both the data sets fit well to the folded normal distribution relative to folded exponential distribution suggesting that the data can distinguish between two distributions and the folded normal model is satisfactory.

Reference

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