Spectral Triples and Metric Aspects of Geometry on Some Noncommutative Spaces

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To

my parents

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Contents

Notations

N	Natural numbers
N_0	$N \cup \{0\}$
N	number operator
$\mathbb{T}$	complex numbers of unit modulus
$\mathcal{H}, \mathfrak{h}$ etc.	Hilbert spaces
$\mathcal{K}$	The ideal of compact operators in a separable Hilbert space
$\mathcal{B}(\mathcal{H})$	The algebra of bounded operators on a Hilbert space
ℓ	annihilation operator on $l^2(\mathbb{N}_0)$ or $l^2(\mathbb{Z})$
$\mathcal{A}$	involutive algebra
$Dom(T)$	domain of the operator T
tr_ω	Dixmier trace associated with the state ω
$\mathcal{A}_\theta$	noncommutative torus
$\mathcal{A}_\hbar$	quantum Heisenberg manifold
$\Omega^\bullet(\cdot)$	The universal differential graded algebra
$\Omega^\bullet_D(\cdot)$	The complex of Connes-deRham forms
$\widetilde{\Omega}^\bullet(\cdot)$	The complex of square integrable forms.

Introduction

Quantization of mathematical theories is now more than half a century old idea in mathematics. It goes back to Gelfand-Naimark's seminal paper [37] in 1943. As the name suggests noncommutative geometry is the "quantization" of differential geometry. It is the study of noncommutative algebras as if they were algebras of functions on spaces like the commutative algebras associated to affine algebraic varieties, smooth manifolds, topological spaces. One can trace its roots in the Gelfand-Naimark theorems (1943, [37]). In modern terminology their theorem says there is an antiequivalence between the category of (locally) compact Hausdorff spaces and (proper, vanishing at infinity) continuous maps and the category of (not necessarily) unital C^*-algebras and $*$-homomorphisms. In other words the entire topological information of a locally compact Hausdorff space is encoded in the commutative C^*-algebra of continuous functions vanishing at infinity. This observation suggests an immediate extension of the notion of topological spaces by considering a not necessarily commutative C^*-algebra as the algebra of "functions on some noncommutative space".

This idea of extending classical notions to the domain of noncommutative algebras was exploited by Karoubi in the early 70's. He showed that topological K-theory can be extended to Banach algebras. Next major breakthrough towards extending algebraic topological ideas in the noncommutative arena were the works of Brown, Douglas and Fillmore and Kasparov. Kasparov gave a unified approach towards extending the notion of analytical K-homology and topological K-theory.

Strictly speaking all these developments were taking part in the realm of noncommutative topology. Noncommutative geometry took off in the hands of Connes with the introduction of cyclic (co)homology. It was introduced as an extension of the deRham cohomology of differentiable manifolds to the noncommutative setting, and serves as a natural target for the Chern character homomorphism from K-theory. At this point we should also mention that Boris Tsygan also independently arrived at the notion of cyclic homology. Novelty of the various constructions of Connes lies in the explicit nature of the pairing between cyclic cohomology and K-theory. He achieves this by lifting the notion of Dirac operators to the noncommutative arena. The essential features of geometry of spin manifolds are extended by the notion of spectral triples, which consists of a separable Hilbert space $\mathcal{H}$, an involutive subalgebra $\mathcal{A}$ of the algebra of bounded operators, and D, a selfadjoint operator with compact resolvents deriving $\mathcal{A}$ in the sense that the commutator of D with every element of $\mathcal{A}$ is densely defined and admits a bounded extension. This operator D contains almost all the 'geometric' information. With any closed Riemannian spin manifold M there is associated a canonical spectral triple with $\mathcal{A} = C^\infty(M)$, the algebra of complex valued smooth functions on M, $\mathcal{H} = L^2(M, S)$, the Hilbert space of square integrable sections of the irreducible spinor bundle over M and D, the Dirac operator associated with the Levi-Civita connection. For this spectral triple Connes has a recipe for getting back the usual differential calculus of forms on M. In fact the prescription given in the last chapter of his book [24] works for any spectral triple. In the general context we will call the calculus associated with a spectral triple the Connes-de Rham calculus. Connes extended metric notions like volume measure, connection, curvature etc. in the general noncommutative set up. His ideas were further extended by Frohlich, Grandjean and Recknagel [36]. They extended the metric notions like scalar curvature, Ricci curvature etc. On the other hand results of Baaj and Julg [4] imply spectral triples on $\mathcal{A}$ are enough to describe the elements of K-homology of the norm closure of $\mathcal{A}$. One says a spectral triple is nontrivial or equivalently, has nontrivial Chern character if the associated element in K-homology is non trivial. Another property of the canonical spectral triple described above is its nontriviality. A natural question in this context is, given a concrete C^*-algebra can we construct spectral triples with nontrivial Chern character for which the associated algebra is a dense subalgebra of the given C^*-algebra? Can we explicitly describe the associated differential calculus? This takes us to one of the objectives of the thesis, namely construction of spectral triples with nontrivial Chern character and its associated calculus.

In **chapter 1** we recall some preliminaries. We begin with the most basic object namely a C^*-algebra. Then we briefly recall the K-theory/K-homology of C^*-algebras and the fundamental pairing between them. Then comes the notion of entire cyclic (co)homology. This was introduced by Connes in [21]. Our discussion follows [38],[39]. Connes has shown us how to recover the volume form from the canonical spectral triple for a compact spin manifold. This is done by using Dixmier trace. We introduce this concept along the lines of the appendix of [27]. In chapter 6 of [24], he has described ways to capture de Rham cohomology using the canonical spectral triple. These ideas have further been extended by Frohlich, Grandjean and Recknagel [36]. We briefly recall some of their notions like scalar curvature, torsion etc. In the last section we introduce the relatively new notion of compact quantum metric spaces due to Rieffel [78].

In **chapter 2** we look into the noncommutative torus (NCT), an example studied in great depth by Connes and Rieffel. Rieffel's study of this example mostly deals with C^*-algebraic aspects, whereas Connes dealt with this in the spirit of noncommutative geometry. He studied one particular spectral triple. At this point one can ask: are their other spectral triples on NCT? Are they distinguishable by their associated volume forms, scalar curvature etc.? By a result of Bratteli, Eliott & Jorgensen ([9]) one can list down all spectral triples satisfying a mild condition. For this class of spectral triples we show that the volume form remains invariant ([16]). Scalar curvature as introduced by Frohlich et. al ([36]) also does not change. For some specific cases we show that the Connes-De Rham cohomology changes, thereby showing that these spectral triples are not unitarily equivalent to the one studied by Connes. Another approach to study geometry in the classical case is via the heat semigroup. One may also like to use completely positive semigroups with 'local' generators to investigate these 'noncommutative spaces'. In particular using the near zero asymptotics of the trace of heat kernel one can introduce concepts like volume form, integrated scalar curvature etc. We show that although the volume form for a perturbed family of Laplacians remains invariant, the integrated scalar curvature may vary.

We introduce in [16] quantum 2d-dimensional spaces as quantization of Euclidean 2d-dimensional space. These are examples of 'locally compact quantum spaces'. In this case also one can introduce the idea of volume form and show it remains invariant under quantization.

In **chapter 3** we deal with Quantum Heisenberg manifolds (QHM), introduced by Rieffel in [73]. He attached concrete meaning to the concept of deformation quantization along a Poisson bracket in the C^*-algebraic framework and constructed quantum Heisenberg manifolds as example of "noncommutative manifold" obtained by strict deformation quantization of Heisenberg manifolds along a given Poisson bracket. These are C^*-algebras $\mathcal{A}^{c,\hbar}_{\mu,\nu}$ parametrized by four nonzero parameters μ, ν, $\hbar \in \mathbb{R}$, $\hbar > 0$ and $c \in \mathbb{N}$ with an ergodic action of the Heisenberg group, and an invariant trace. Now, whenever there is a smooth C^* dynamical system $(\mathcal{A}, G, \alpha)$ with an n dimensional Lie group G and an invariant trace τ, one can adopt the following principle to construct spectral triples on a subalgebra of smooth vectors. Let $\mathcal{H} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N, N = 2^{\lfloor \frac{n}{2} \rfloor}$, and $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$ of the Lie algebra of G. Since G acts as a strongly continuous unitary group on $L^2(\mathcal{A}, \tau)$ we can form self adjoint operators d_{X_i} on $L^2(\mathcal{A}, \tau)$. Let us define $D : \mathcal{H} \to \mathcal{H}$ by $D = \sum_i d_{X_i} \otimes \gamma_i$, where $\gamma_1, \ldots \gamma_n$ are self adjoint matrices in $M_N(\mathbb{C})$ such that $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$. We call this the Dirac operator associated with the basis $X_1, \cdots, X_n$. The operator D along with $\mathcal{A}^\infty$ and $\mathcal{H}$ should be a candidate for a spectral triple. For such a D, clearly one has $[D, \mathcal{A}^\infty] \subseteq \mathcal{A}^\infty \otimes M_N(\mathbb{C})$. Therefore, the only thing that remains to verify is whether D is selfadjoint with compact resolvents. We carry through this programme in the context of Quantum Heisenberg Manifolds (QHM). In the context of QHM, we identify the Lie algebra of Heisenberg group as the algebra of 3×3 strictly upper triangular matrices. Let

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha > 1$. Then we show (see [15])

1. The Dirac operator associated with the basis X_1, X_2, X_3 is a self adjoint operator with compact resolvent, provided $\alpha > 1$, and gives rise to an odd spectral triple on $\mathcal{A}_\hbar^\infty$, the algebra of smooth elements in the QHM.

2. This spectral triple has dimension 3, in the sense that $|D|^{-3}$ is Dixmier traceable and has nonzero Dixmier trace.

3. The operator D depends on the real parameter α, but they induce same element in $K^1(\mathcal{A}_{\mu,\nu}^{c,\hbar})$.

4. The Chern character in Entire cyclic cohomology associated with the different D's are cohomologous.

5. We explicitly compute the space of Connes-deRham forms as defined in the last chapter of [24], and show that forms of degree higher than four vanish. There are not too many instances where one explicitly knows these spaces.

6. Space of square integrable forms as defined in [36] are easily seen to coincide with the Connes-de Rham forms and using this we characterize unitary and torsionless connections and show that a connection can not be simultaneously torsionless and unitary.

7. We exhibit connections for which the associated scalar curvature is nontrivial and an element of the smooth algebra. Some of these metric notions like scalar curvature, Ricci curvature were introduced in [36], but we have not seen any concrete example other than the noncommutative torus where actual computations have been done.

Another area of mathematics where the program of quantization has been successfuly carried through is the theory of compact topological groups. It started with the search for a selfdual category to be called quantum groups containing the category of locally compact topological groups. The solution was provided by the category of Kac algebras obtained independently by Enock and Schwartz and by Kac and Vainerman. But the example of q-deformation of the $SU(2)$ studied by Woronowicz showed the inadequacy of the category of Kac algebras. He gave a satisfactory definition of compact quantum groups and studied their representation theory. Later Podles constructed quantum spheres S^2_{qc} as homogeneous space for $SU_q(2)$. Now in the context of Lie groups and their homogeneous spaces they have their own geometry and quite naturally given a concrete quantum group or quantum homogeneous space one would like to implement programs of noncommutative geometry on them. In **chapter 4** we take up the issue of construction of spectral triples and associated calculus in the context of $SU_q(2)$ and S^2_{qc}. Here to construct explicit spectral triple we begin with computation of K-groups, and then from explicit generators we construct spectral triples which induce generating elements in K-homology. We also compute a modified version of the space of Connes deRham forms and the associated calculus. The space of L^2 forms have also been described explicitly.

In the computations of chapter 4, the fact that $SU_q(2)$ is a compact quantum group does not play any role. But in the classical context of a compact Lie group G its tangent bundle is the trivial product bundle $G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Let $X_1, \cdots, X_n$ be an orthonormal basis of the $\mathfrak{g}$, then the Dirac operator is given by $\sum d_{X_i} \otimes \gamma_i$, where the γ_i's are the Clifford gamma matrices. Note that this is a selfadjoint operator on $L^2(G) \otimes \mathbb{C}^N$ and commutes with the left regular representation of G. In general a left invariant differential operator commutes with the left regular representation of G. Now in the case of abelian G, the C^*-algebra generated by the left regular representation is nothing but $C(\hat{G})$. Therefore we can rephrase the left invariance condition as a commutation condition with $C(\hat{G})$. For $C(SU_q(2))$, Woronowicz has explicitly described the generators for $C(\hat{G})$. Therefore, a proper analog of a left invariant Dirac operator would be a Dirac operator commuting with these generators. With this formulation of left invariance in mind, in **chapter 5**

1. We show the existence of equivariant Dirac operators with nontrivial Chern character,

2. characterize equivariant nontrivial Dirac operators.

3. It is shown that an equivariant nontrivial Dirac operator must have dimension at least three and there is one with dimension three. This observation is in agreement with the dimension of its classical counterpart namely $SU(2)$.

4. We also show that equivariant Dirac operators are universal in the sense that given any odd spectral triple there is an equivariant Dirac operator D inducing the same element in odd K-homology.

In noncommutative geometry, the natural way to specify a metric is by means of a suitable "Lipschitz seminorm". This idea was first suggested by Connes ([22]), and developed further in [24]. Connes pointed out ([22], [24]) that from a Lipschitz seminorm one obtains in a simple way an ordinary metric on the state space of a C^*-algebra. A natural question in this context is when does this metric topology coincides with the weak* topology. In his search for an answer to this question, Rieffel ([76], [77], [78]) has identified a larger class of spaces, namely order unit spaces on which one can answer these questions. He has introduced the concept of Compact Quantum Metric Spaces (CQMS) as a generalization of compact metric spaces, and used ([78]) this new concept for rigorous study of convergence questions of algebras much in the spirit of Gromov-Hausdorff convergence. In **chapter 6** we take up the problem of construction of examples of compact quantum metric spaces. Here we basically discuss two classes.

1. Let (X, d) be a compact metric space. Suppose we have a faithful representation $C(X) \subseteq \mathcal{B}(\mathcal{H})$. Let $\widetilde{A_0} = \{((a_{ij})) \in \mathcal{K}(l^2(\mathbb{N})) \otimes C(X) |$ (i) a_{ij} are selfadjoint elements in $C(X)$, (ii) $a_{ij} = a_{ji}$ (iii) a_{ij} is actually a Lipschitz function $\}$. Suppose we have a short exact sequence of C^*-algebras
$$0 \longrightarrow \mathcal{K} \otimes C(X) \xrightarrow{i} \widetilde{A_1} \xrightarrow{\pi} \widetilde{A_2} \longrightarrow 0$$
with $\widetilde{A_1}$, $\widetilde{A_2}$ unital and a positive linear splitting $\sigma : \widetilde{A_2} \to \widetilde{A_1}$. Let (A_2, L_2) be a compact quantum metric space with A_2 a dense subspace of selfadjoint elements of $\widetilde{A_2}$. Then we put Lip norm L_1 on $A_1 = i(\widetilde{A_0}) \oplus \sigma(A_2)$, so that (A_1, L_1) becomes a compact quantum metric space. As concrete example of this phenomena we produce compact quantum metric spaces out of $C(SU_q^2)$ and $C(S_{qc}^2)$.

2. Rieffel has constructed compact quantum metric spaces using ergodic action of compact Lie group on a C^*-algebra. Although Heisenberg group acts ergodically on QHM, Rieffel's result does not apply because of non-compact nature of the Heisenberg group. Weaver had partial success in construction of CQMS out of QHM. Essentially modifying Rieffel's argument we produce examples of CQMS on QHM (see [12]).

Chapter 1

Preliminaries

One standard way to obtain noncommutative theories is by applying a two step algorithm on classical theories. These steps are (i) algebraization, i.e, state the classical theory in algebraic terms; (ii) quantization, i.e, remove certain commutativity hypothesis. In this chapter we will see how that is carried out in the context of geometry. We begin with quantized version of topological spaces. Then we discuss the noncommutative version of topological K-theory. This lies at the heart of Connes' noncommutative geometry. Entire cyclic cohomology/homology is discussed following Getzler-Szenes ([39]), and Getzler ([38]). Then the metric notions like Riemannian volume measure, curvature etc. are introduced. Some of these concepts are from [36]. Recently Marc Rieffel has introduced the notion of compact quantum metric spaces. We end with a discussion of this notion essentially following [76],[77], and [78].

1.1 C^*-algebras

In point set topology one studies various categories of topological spaces. One of the most well studied category is that of locally compact spaces and proper continuous maps. Similarly in noncommutative topology a subcategory of Banach algebras plays central role called C^*-algebras.

Definition 1.1.1 A C^*-algebra is a Banach algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a conjugate linear isometric involution $* : \mathcal{A} \to \mathcal{A}$ such that

$$(xy)^* = y^*x^*, \quad \|x^*x\| = \|x\|^2$$

for all x, y in $\mathcal{A}$.

It is a basic fact of the theory of C^*-algebras that the norm on a C^*-algebra is uniquely determined by the algebra structure. The spectral radius $\rho(x)$ of an element x of $\mathcal{A}$ is defined by,

$$\rho(x) = sup\{|\lambda|, \lambda \in \mathbb{C} : \lambda \in \sigma(x)\},$$

where $\sigma(x) = \{\lambda \in \mathbb{C} : (x - \lambda)\text{is not invertible}\}$ is the spectrum of x. Then the square of the norm of any element x of a C^*-algebra is the same as the spectral radius of x^*x:

$$\|x\|^2 = \rho(x^*x).$$

The above definition introduces the so called abstract C^*-algebras. Concrete C^*-algebras are defined in terms of their representations in a Hilbert space $\mathcal{H}$. In fact an involutive algebra is a C^*-algebra iff it admits a $*$-representation π on a Hilbert space $\mathcal{H}$ such that $\pi(x) = 0$ implies $x = 0$, and the image $\pi(\mathcal{A})$ of $\mathcal{A}$ is closed in norm in the algebra of bounded operators on $\mathcal{H}$.

Example 1.1.2 Let X be a compact Hausdorff space. Denote by $C(X)$ the collection of all complex valued continuous functions on X, which is a C^*-algebra if we define,

$$\|f\| = sup_{x\in X}|f(x)|, \; f^*(x) = \bar{f}(x) \text{ for } f \in C(X).$$

Let μ be a measure on X then $C(X)$ acts by multiplication operators on $\mathcal{H} = L^2(X, \mu)$, and this representation gives a $*$-homomorphism from $C(X)$ to $\mathcal{B}(\mathcal{H})$, which is isometric provided that μ assigns nonzero measure to each nonempty open set. Thus $C(X)$ is a C^*-algebra. If X is only locally compact, then an analogous construction shows that the algebra $C_0(X)$ of continuous functions which tend to zero at infinity is a C^*-algebra. Notice that $C_0(X)$ does not have a unit if X is not compact.

All these are examples of commutative C^*-algebras. A celebrated theorem of Gelfand and Naimark ([37]) says that these are all. To state their result more precisely we need the notion of the Gelfand transform.

Let $\mathcal{A}$ be a commutative Banach algebra and $\hat{\mathcal{A}}$ be the space of nonzero continuous algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. It is a locally compact Hausdorff space in the topology of pointwise convergence. The homomorphism,

$$\pi : \mathcal{A} \to C_0(\hat{\mathcal{A}})$$

which maps $a \in \mathcal{A}$ to the function $\pi(\alpha)$ given by $\pi(\alpha)(\hat{a}) = \alpha(a)$ is called the Gelfand transform.

Theorem 1.1.3 (Gelfand, Naimark) *If $\mathcal{A}$ is a commutative C^*-algebra then the Gelfand transform is an isometric $*$-isomorphism from $\mathcal{A}$ onto $C_0(\mathcal{A})$.*

If $\alpha : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism between commutative C^*-algebras then there exists an induced proper continuous function $\hat{\alpha} : \mathcal{B} \to \mathcal{A}$ given by $\hat{\alpha}(\phi) = \phi \circ \alpha$ for $\phi \in \hat{\mathcal{B}}$. Conversely, a proper continuous map from $\mathcal{B}$ to $\mathcal{A}$ induces a $*$-homomorphism from $C_0(\hat{\mathcal{A}})$ to $C_0(\hat{\mathcal{B}})$. In other words we have the following equivalence of categories.

Theorem 1.1.4 *The category of all commutative C^*-algebras and $*$-homomorphisms is equivalent to the opposite category of locally compact Hausdorff spaces and proper maps.*

This theorem says that the category of commutative C^*-algebras gives a good algebraization of the theory of locally compact Hausdorff spaces and one should quantize the situation by considering not necessarily commutative C^*-algebras as continuous function algebras of spaces non-existent.

1.2 K-theory

Among the various cohomology/homology theories studied in algebraic topology one cohomology theory namely topological K-theory admits extensions to the noncommutative situation. In this section we recall the basic definitions following [8], [10], [84].

Definition 1.2.1 Let $\mathbf{C}^*$ be the category of C^*-algebras and $\mathbf{Ab}$ be the category of abelian groups. A covariant functor $\mathbf{F} : \mathbf{C}^* \to \mathbf{Ab}$ is called a homology functor if it satisfies:
(i) **Half exactness:** Every short exact sequence of C^*-algebras $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is carried to an exact sequence $\mathbf{F}\mathcal{A} \to \mathbf{F}\mathcal{B} \to \mathbf{F}\mathcal{C}$.
(ii) **Homotopy invariance:** Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. We say that two $*$-homomorphisms $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ are homotopic if there exists a family of point-norm continuous $*$-homomorphisms $\gamma_t : \mathcal{A} \to \mathcal{B}$, for $t \in [0, 1]$ such that $\gamma_0 = \alpha, \gamma_1 = \beta$. In such a situation homotopy invariance means $\mathbf{F}(\alpha) = \mathbf{F}(\beta)$.

Definition 1.2.2 A homology theory on a subcategory $\mathfrak{X}$ of the category $\mathbf{C}^*$ is a sequence (H_n) of homology functors from $\mathfrak{X}$ to $\mathbf{Ab}$ so that for every short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ there are group homomorphisms

$$\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$$

connecting all the exact sequences $H_n(\mathcal{A}) \to H_n(\mathcal{B}) \to H_n(\mathcal{C})$ to a long exact sequence

$$\cdots \to H_n(\mathcal{A}) \to H_n(\mathcal{B}) \to H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A}) \to H_{n-1}(\mathcal{B}) \to \cdots$$

Let us briefly recall the central definition of topological K-theory. If X is a compact Hausdorff space then $K^0(X)$ is the abelian group generated by the isomorphism classes of complex vector bundles over X subject to the relations

$$[E] + [F] = [E \oplus F]$$

for vector bundles E and F. Now, a complex vector bundle can alternatively be described by a continuous function $p : X \to M_n(\mathbb{C})$ such that $p(x)$ is a projection for each $x \in X$. The ranges of the projections then fit together to form a vector bundle over X. This gives a bijective correspondence between the isomorphisms classes of vector bundles over X and homotopy classes of projection valued functions. Therefore we have the alternative description of $K^0(X)$ as abelian group generated by homotopy classes of maps from X to the space of projections in $M_n(\mathbb{C})$ as n runs over the natural numbers. Since a projection valued function from X to $M_n(\mathbb{C})$ is the same thing as a projection in the C^*-algebra $M_n(C(X))$ of $n \times n$ matrices over $C(X)$ one extends the notion of K^0 to C^*-algebras as follows:

Definition 1.2.3 Let $\mathcal{A}$ be a unital C^*-algebra. Denote by $K_0(\mathcal{A})$ the abelian group with one generator, $[p]$, for each projection p in each matrix algebra $M_n(\mathcal{A})$, and the following relations:
(i) if both p and q are projections in $M_n(\mathcal{A})$, for some n, and if p and q are joined by a continuous path of projections in $M_n(\mathcal{A})$, then $[p] = [q]$;
(ii) $[0] = 0$, for any size of square zero matrix; and
(iii) $[p] + [q] = [p \oplus q]$ for any sizes of projection matrices p and q.

If $p \in M_m(\mathcal{A})$ and $q \in M_n(\mathcal{A})$ then the notation $p \oplus q$ refers to the projection $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $M_{m+n}(\mathcal{A})$.

Because of relations (ii) and (iii) every element of $K_0(\mathcal{A})$ is in fact a formal difference $[p] - [q]$ of projections in some $M_n(\mathcal{A})$. Two such formal differences $[p] - [q]$ and $[p'] - [q']$ define the same K-theory class iff they are stably homotopic, which is to say that there exists a third projection r such that $p \oplus q' \oplus r$ can be joined by a continuous path of projections to $p' \oplus q \oplus r$.

Note that K_0 is a functor, i.e., if $\alpha : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism, and p is a projection in $M_n(\mathcal{A})$, then $\alpha(p)$ defined by applying α elementwise to the matrix p is a projection in $M_n(\mathcal{B})$. So, α induces a homomorphism $K_0(\alpha) : K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

Remark 1.2.4 Because of condition (i), K_0 is a homotopy invariant functor.

Example 1.2.5 Let $\mathcal{A} = \mathbb{C}$. Two projections in $M_n(\mathbb{C})$ are connected by a path of projections iff they have the same rank. The map $[p] \mapsto Rank(p)$ induces an isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

Example 1.2.6 Similarly $K_0(\mathcal{A}) \cong \mathbb{Z}$, if $\mathcal{A}$ is $M_m(\mathbb{C})$. In fact for a unital algebra $\mathcal{A}$, $K_0(\mathcal{A})$ is canonically isomorphic with $K_0(M_m(\mathcal{A}))$.

Example 1.2.7 If $\mathcal{A} = C(X)$, where X is a compact Hausdorff space then $K_0(\mathcal{A})$ is the topological K-theory group $K^0(X)$. Here the placement of subscripts and superscripts reflects the fact that $K^0(X)$ is contravariant in X, whereas $K_0(\mathcal{A})$ is covariant in $\mathcal{A}$.

Example 1.2.8 $K_0(\mathcal{B}(\mathcal{H}))$ is trivial because given any two projection $p, q \in M_n(\mathcal{B}(\mathcal{H}))$ there is a continuous path of projections joining $p \oplus 1 \oplus 0$ with $q \oplus 1 \oplus 0$.

For a nonunital C^*-algebra $\mathcal{A}$ let $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ be the C^*-algebra with coordinatewise addition, multiplication and the norm $\|(a, \lambda)\| := max\{\|a\|, |\lambda|\}$. Then we have a short exact sequence $0 \longrightarrow \mathcal{A} \xrightarrow{\imath} \mathcal{A}^+ \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$. $K_0(\mathcal{A})$ is defined as:

$$K_0(\mathcal{A}) := \ker\{K_0(\pi) : K_0(\mathcal{A}^+) \to K_0(\mathbb{C}) = \mathbb{Z}\}$$

Definition 1.2.9 For a C^*-algebra we define the suspension $S\mathcal{A}$ of the algebra $\mathcal{A}$, by

$$S\mathcal{A} = \{f : \mathbb{T} \to \mathcal{A} : f(1) = 0\}.$$

Then we define $K_1(\mathcal{A}) = K_0(S\mathcal{A})$; in general $K_n(\mathcal{A}) = K_0(S^n\mathcal{A})$.

Theorem 1.2.10 *(i) $\{K_n\}$ determines a homology theory on* $\mathbf{C}^*$.
(ii) **Bott Periodicity:** *This theory is periodic in the sense that there are natural isomorphisms $K_i(\mathcal{A}) \cong K_{i+2}(\mathcal{A})$.*

Remark 1.2.11 For a short exact sequence ,

$$0 \longrightarrow \mathcal{A} \xrightarrow{\imath} \mathcal{B} \xrightarrow{\pi} \mathcal{C} \longrightarrow 0$$

Bott periodicity combined with the long exact sequence produces the six term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \xrightarrow{i_0} & K_0(\mathcal{B}) & \xrightarrow{\pi_0} & K_0(\mathcal{C}) \\ \uparrow \text{index} & & & & \downarrow \exp \\ K_1(\mathcal{C}) & \xleftarrow{\pi_1} & K_1(\mathcal{B}) & \xleftarrow{i_1} & K_1(\mathcal{A}) \end{array}$$

Here the vertical maps are appropriate connecting maps.

Remark 1.2.12 All these can be done in the context of Banach algebras provided in the definition of K_0 we replace projections by idempotents. However, for C^*-algebras both the theories coincide.

We have a more concrete description of K_1.

Definition 1.2.13 Let $\mathcal{A}$ be a unital C^*-algebra. Then $K_1^u(\mathcal{A})$ denotes the abelian group with one generator $[u]$ for each unitary matrix in each $M_n(\mathcal{A})$, and the following relations:
(i) if u and v lie in the same $M_n(\mathcal{A})$, and if u and v can be joined by a continuous path of unitaries in $M_n(\mathcal{A})$, then $[u] = [v]$;
(ii) $[1] = 0$; and
(iii) $[u] + [v] = [u \oplus v]$, for any sizes of unitary matrices u and v;

Let us temporarily denote path connectedness through unitaries by the symbol $\sim$. If u and v are unitaries in $\mathcal{A}$ then the following relations hold in $M_2(\mathcal{A})$:

$$u \oplus 1 \sim 1 \oplus u,\ u \oplus v \sim uv \oplus 1 \sim vu \oplus 1,\ u \oplus u^* \sim 1 \oplus 1.$$

The first relation is implemented by the path $R_t \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} R_t^*$ where, R_t is the rotation matrix

$$R_t = \begin{pmatrix} cos(\frac{\pi}{2}t) & sin(\frac{\pi}{2}t) \\ -sin(\frac{\pi}{2}t) & cos(\frac{\pi}{2}t) \end{pmatrix}$$

The other relations follow easily from this.

Theorem 1.2.14 *Let $\mathcal{A}$ be a unital C^*-algebra. The group $K_1^u(\mathcal{A})$ is naturally isomorphic to the group $K_1(\mathcal{A}) = K_0(S\mathcal{A})$.*

Remark 1.2.15 In the description of K_1^u if we use invertibles instead of unitaries then this theorem holds even for Banach algebras.

Now with this description of K_1 we can describe the vertical maps. we will do this in the general context of Banach algebras. We need a simple lemma on lifting invertibles.

Lemma 1.2.16 *(i) Let $\mathcal{A}$ be a unital Banach algebra. Then the identity component of the group of invertible elements is the group generated by $exp(x)$ with $x \in \mathcal{A}$.*
(ii) Let ϕ be a unital surjective homomorphism from a Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$. Then every element in the identity component of invertibles in $\mathcal{B}$ can be lifted to the identity component of invertibles in $\mathcal{A}$.

Proof: (i) Suppose there is a path of invertibles starting at 1 and ending at u. An easy compactness argument implies that there is a sequence of invertibles $1 = u_0, u_1, \cdots, u_r = u$ such that $\|1 - u_{i-1}^{-1}u_i\| < 1$. Taking $x_i = log(u_{i-1}^{-1}u_i)$ we get $u = exp(x_1)exp(x_2)\cdots exp(x_r)$.
(ii)An application of (i) yields (ii). □

Let $0 \longrightarrow \mathcal{A} \xrightarrow{\imath} \mathcal{B} \xrightarrow{\pi} \mathcal{C} \longrightarrow 0$ be a short exact sequence of Banach algebras. Let $[u] \in M_n(\mathcal{C})$ be an invertible matrix. Then the discussions following definition 1.2.13 implies $u \oplus u \in M_{2n}(\mathcal{C})$ can be joined with identity by a continuous path of invertibles and hence by the above lemma admits a lift v in the identity component of $M_{2n}(\mathcal{B})$. Then the vertical map from $K_1(\mathcal{C})$ to $K_0(\mathcal{A})$ is given by $Index([u]) = [vp_nv^{-1}] - [p_n]$, where, $p_n = I_n \oplus 0_n$. For the other vertical map,let p be an idempotent in $M_n(\mathcal{C})$. Then $exp([p]) = [e^{2\pi ix}]$, where $x \in M_n(\mathcal{B})$ is any lift of p.

Example 1.2.17 $K_1(\mathbb{C}) = 0$, because every unitary matrix u is path connected to the identity matrix through unitary matrices, as can be seen by diagonalizing u, then rotating each eigenvalue continuously to one.

Example 1.2.18 Let $\mathcal{A}$ be any unital C^*-algebra. We consider a unitary $u \in \mathcal{A}$ such that spectrum of u is not the whole circle. Then there is a continuous branch of the function $z \mapsto \log(z)$ defined on spectrum of u, and so by the function calculus we may define a homotopy $u_t = exp(t \log u)$ from the identity to u. Thus $[u] = 0$ in $K_1(\mathcal{A})$.

Example 1.2.19 Each unitary in $\mathcal{B}(\mathcal{H})$ admits a logarithm by the Borel function calculus, hence $K_1(\mathcal{B}(\mathcal{H})) = 0$.

Example 1.2.20 Using the six term cyclic sequence for the extension

$$0 \to \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) \to 0, \tag{1.2.1}$$

where $\mathcal{Q}(\mathcal{H})$ is the Calkin algebra along with the K-groups of $\mathcal{K}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ we get $K_1(\mathcal{Q}(\mathcal{H})) = 0$.

1.3 K-homology

K-homology arose from the study of elliptic operators on manifolds and in good cases it is dual to K-theory. The initial ideas are due to Atiyah ([3]), Which were later developed by Brown-Douglas-Fillmore, Kasparov, Connes.

Definition 1.3.1 Let $\mathcal{A}$ be an involutive algebra over $\mathbb{C}$. Then a **Fredholm module** $(\mathcal{H}, F)$ over $\mathcal{A}$ is given by: (i) an involutive representation π of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$;
(ii) an operator $F = F^*, F^2 = I$; on $\mathcal{H}$ such that $[F, \pi(a)]$ is compact for any $a \in \mathcal{A}$.
Such a Fredholm module will be called **odd**. An **even** Fredholm module is given by a Fredholm module as above together with a $\mathbb{Z}/2$ grading $\gamma, \gamma = \gamma^*, \gamma^2 = I \in \mathcal{B}(\mathcal{H})$ such that: (a) $\gamma\pi(a) = \pi(a)\gamma$, $\forall a \in \mathcal{A}$, (b) $F\gamma = -\gamma F$.
In the context of $\mathbb{Z}/2$ graded algebras (a) becomes $\gamma\pi(a) = (-1)^{deg(a)}\pi(a)\gamma$.

The above definition is upto trivial changes, the same as Atiyah's definition of abstract elliptic operators ([3]).

Example 1.3.2 Let M be a smooth manifold, $\mathcal{A} = C(M)$, the involutive algebra of continuous functions on M. Let $E^{\pm}$ be smooth Hermitian vector bundles over M and $P : C^{\infty}(M, E^+) \to C^{\infty}(M, E^-)$ an elliptic pseudodifferential operator of order 0. Then being of order 0 it extends to a bounded operator

$$P : L^2(M, E^+) \to L^2(M, E^-),$$

and the existence of a parametrix Q for P such that both $QP - I$ and $PQ - I$ are compact shows that P almost intertwines the representations of $C(M)$ by multiplication operators in $L^2(M, E^{\pm})$,

$$\pi^{\pm}(f)\xi = f\xi, \ \forall \xi \in L^2(M, E^{\pm}), \ f \in C(M).$$

Indeed one has

$$P\pi^+(f) - \pi^-(f)P \in \mathcal{K}, \text{ the ideal of compact operators}, \forall f \in C(M).$$

Taking $\mathcal{H} = L^2(M, E^+) \oplus L^2(M, E^-) = \mathcal{H}^+ \oplus \mathcal{H}^-, \pi = \pi^+ \oplus \pi^-, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$ one has $[F, \pi(f)] \in \mathcal{K}, \forall f \in C(M)$, and $F^2 - 1 \in \mathcal{K}$, so that upto an easy modification we get an even fredholm module over $C(M)$.

Example 1.3.3 Let Γ be a discrete group. Consider the Hilbert space $l^2(\Gamma)$. Let $g \mapsto \lambda_g$ be the left regular representation of Γ. The algebra $C^*_r(\Gamma)$ is the C^*-algebra generated by $\{\lambda_g : g \in \Gamma\}$.

Consider now a tree T, i.e., a one dimensional connected, simply connected, simplicial complex, on which Γ acts freely and transitively. Take T^i to be the set of vertices when $i = 0$ and edges when $i = 1$. For $p \in T^0$ put $\varphi : T^0 - \{p\} \to T^1$ to the map which sends a vertex q to the unique 1-simplex containing q which is a subset of the path $[p,q]$. Let us define $\mathcal{H}^+ = l^2(T^0), \mathcal{H}^- = L^2(T^1)$ and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Furthermore let us define an operator P from $\mathcal{H}^+$ to $\mathcal{H}^-$ by $P(\xi_p) = 0$ and $P(\xi_q) = \xi_{\varphi(q)}$, where ξ_q's are the indicator functions in respective l^2 spaces. Then there is the following involution $F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$ on $\mathcal{H}$ anticommuting with the involution γ whose eigenspaces are the Hilbert spaces $\mathcal{H}^{\pm}$. As in the previous examples after an easy modification this defines an even Fredholm module over $C^*_r(\Gamma)$.

A useful method for constructing examples is by spectral triples which we now define.

Definition 1.3.4 A **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is a triple where
(i) $\mathcal{H}$ is a separable Hilbert space;
(ii) $\mathcal{A}$ is an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ closed under holomorphic function calculus;
(iii) D is selfadjoint with compact resolvents such that $[D, \mathcal{A}] \subset \mathcal{B}(\mathcal{H})$.
A spectral triple is called **even**, if there is an involution $\gamma, \gamma = \gamma^*, \gamma^2 = I \in \mathcal{B}(\mathcal{H})$, that commutes with $\mathcal{A}$ and anticommutes with D. Otherwise the spectral triple is called **odd**.

A spectral triple gives rise to Fredholm modules in the following way:
Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Suppose this is odd then define $F = sign(D)$ on $\ker(D)^{\perp}$ and $F \equiv 1$ on $\ker(D)$. Then $(\mathcal{H}, F)$ defines an odd Fredholm module over the norm closure of $\mathcal{A}$. One has to be little more careful while dealing with even spectral triples. In that case take $\mathcal{H}' = \mathcal{H} \oplus \ker(D)$. Orient the extra copy of $\ker(D)$ by $-\gamma$, and F switches the two copies of $\ker(D)$. In fact results of Baaj and Julg ([4]) imply that in some sense to construct Fredholm modules it is enough to construct spectral triples. These are basic objects of study in noncommutative geometry. In fact they define noncommutative geometric spaces and one often says noncommutative space defined by a spectral triple. Since one of our objectives is construction of spectral triple with certain nontriviality condition to be made precise in the next section, we abstain from discussing any noncommutative example. Instead, we content ourselves with the classical example coming from a compact Riemannian spin manifold.

Example 1.3.5 Let M be a compact Riemannian spin manifold. Let $\mathcal{A}$ be the algebra of complex valued smooth functions with its natural involutive algebra structure. $\mathcal{A}$ is faithfully represented in the Hilbert space of square integrable sections of the associated spinor bundle. These along with the Dirac operator forms a spectral triple often referred as the canonical spectral triple associated with a compact Riemannian spin manifold.

Homotopy classes of Fredholm modules constitute classes in K-homology. For the duality between K-homology and K-theory we need the notion of Fredholm operators. A bounded linear operator T on a separable Hilbert space $\mathcal{H}$ is called a Fredholm operator if T has finite dimensional kernel and finite dimensional cokernel. Then the index of T is defined as,

$$Index(T) = dim\, ker(T) - dim\, coker(T)$$

The index of Fredholm operator is invariant under compact perturbations:

$$Index(T + S) = Index(T),$$

for any compact operator S. Fredholm operators are precisely lifts of invertible elements of Calkin algebra in $\mathcal{B}(\mathcal{H})$. If we identify $\mathbb{Z}$ with $K_0(\mathcal{K}(\mathcal{H}))$ then $Index$ is nothing but the vertical upward arrow associated with the short exact sequence (1.2.1). This explains the name $Index$ in the six term exact sequence.
The duality between K-homology and K-theory is given in the following theorem. In the context of example 1.3.2 this yields the index of elliptic operators with coefficients in an auxiliary vector bundle.

Theorem 1.3.6 (Atiyah, Kasparov) *Let $\mathcal{A}$ be a Banach *-algebra, $(\mathcal{H}, F)$ a Fredholm module over $\mathcal{A}$, and for $q \in \mathbb{N}$, let $(\mathcal{H}_q, F_q)$ be the Fredholm module over $M_q(\mathcal{A}) = \mathcal{A} \otimes M_q(\mathbb{C})$ given by*

$$\mathcal{H}_q = \mathcal{H} \otimes \mathbb{C}^q,\ F_q = F \otimes I,\ \pi_q = \pi \times I.$$

(a) Let $(\mathcal{H}, F)$ be even, with $\mathbb{Z}/2$ grading γ, and let p be a projection in $M_q(\mathcal{A})$. Then,

$$\pi_q^-(p) F_q \pi_q^+(p) : \pi_q^+(p)\mathcal{H}_q^+ \to \pi_q^-(p)\mathcal{H}_q^-$$

is a Fredholm operator. An additive map ϕ of $K_0(\mathcal{A})$ to $\mathbb{Z}$ is determined by

$$\begin{aligned} \phi([p]) &= Index(\pi_q^-(p) F_q \pi_q^+(p)) \\ &=: \langle [(\mathcal{H}, F)], [p] \rangle. \end{aligned} \tag{1.3.1}$$

(b) Let $(\mathcal{H}, F)$ be odd and let $E = \left(\frac{1+F}{2}\right)$. Let u be an invertible element in $M_q(\mathcal{A})$. Then

$$E_q \pi_q(u) E_q : E_q \mathcal{H}_q \to E_q \mathcal{H}_q$$

is a Fredholm operator. An additive map of $K_1(\mathcal{A})$ to $\mathbb{Z}$ is determined by

$$\begin{aligned} \phi([p]) &= Index(E_q \pi_q(u) E_q) \\ &=: \langle [(\mathcal{H}, F)], [u] \rangle. \end{aligned} \tag{1.3.2}$$

Remark 1.3.7 The pairing described in the above theorem will often be described as the index pairing.

1.4 Entire Cyclic Cohomology

Cyclic cohomology was discovered by Connes ([20]). It was introduced as an extension of the deRham cohomology of differentiable manifolds to the noncommutative setting, and serves as a natural target for the Chern character from K-theory. Later on he introduced entire cyclic cohomology ([21], [23]). There are Chern character homomorphisms from K-theory/homology to entire cyclic homology/cohomology and one can obtain alternative expressions for the index pairing. In this section we closely follow [39] and [38]. Let $\mathcal{A}$ be a Banach algebra with identity, and $\overline{\mathcal{A}}$ is the Banach space $\mathcal{A}/\mathbb{C}$, let $C_*(\mathcal{A})$ be the $\mathbb{Z}$ graded locally Frechet space obtained by taking the union of the completions of the space $\sum_n \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$ with respect to the collection of seminorms

$$\|\sum_n A_n\|_z = \sup_n z^n \frac{\|A_n\|_\pi}{\Gamma(n/2)}, \qquad z > 0. \tag{1.4.1}$$

Here, $\|\cdot\|_\pi$ is the projective tensor product norm on $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$. The projective norm is characterized as follows: the continuous dual $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n'}$ is isomorphic to the space of continuous multilinear maps $f : \mathcal{A} \times \overline{\mathcal{A}}^{\times n} \to \mathbb{C}$. The completion of $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$ in this topology will be denoted by $C_n(\mathcal{A})$. The elementary tensor $a_0 \otimes \ldots \otimes a_n$ of $C_n(\mathcal{A})$ will be denoted by $(a_0, \ldots, a_n)_n$, where $a_i \in \mathcal{A}$. Here the normalization in defining $\|\cdot\|_z$ is as in [38], it is slightly different from Connes. Consider the two bounded operators on $C_*(\mathcal{A})$.

$$b(a_0, \ldots, a_n)_n = \sum_0^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots a_n)_{n-1} + (-1)^n (a_n a_0, \ldots, a_{n-1})_{n-1} \tag{1.4.2}$$

and

$$B(a_0, \ldots, a_n)_n = \sum_0^{n} (-1)^{in} (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1})_{n+1}. \tag{1.4.3}$$

Note that $b^2 = B^2 = bB + Bb = 0$. The entire cyclic homology $HE_*(\mathcal{A})$ is defined to be the homology of the complex $(C_*(\mathcal{A}), b + B)$. The operator $b + B$ is inhomogeneous, so that $HE_*(\mathcal{A})$ is only $\mathbb{Z}/2$ graded; the even subspace is denoted HE_0 and the odd subspace by HE_1.

The cobar complex $C^*(\mathcal{A})$ is the topological dual $C_*(\mathcal{A})'$, of the bar complex; this is the same thing as the space of the continuous multilinear forms on $\mathcal{A} \times \overline{\mathcal{A}}^{\times n}$. This space carries two boundary operators obtained by forming adjoints of b and B acting on $C_*(\mathcal{A})$, denoted by the same symbols. The cohomology of the complex $(C^*(\mathcal{A}), b + B)$ is called the entire cyclic cohomology of $\mathcal{A}$, and is denoted $HE^*(\mathcal{A})$. The pairing between $C^*(\mathcal{A})$ and $C_*(\mathcal{A})$ induces a pairing between $HE^*(\mathcal{A})$ and $HE_*(\mathcal{A})$, which is denoted by $(\cdot,\cdot) : HE^*(\mathcal{A}) \times HE_*(\mathcal{A}) \to \mathbb{C}$. Chern character defines a homomorphism from K-theory/K-homology to entire cyclic theories and one can obtain the index pairing in terms of $(\cdot,\cdot)$

1.4.1 Chern character in Entire Cyclic Theory

Chern character defines homomorphism from K-homology, K-theory to cyclic theories. For the description of this homomorphism on K-theory we need the following trace map. Let $\mathcal{A}$ be a Banach algebra. Let $A^{(0)} = ((a_{ij}^{(0)})), A^{(1)} = ((a_{ij}^{(1)})), A^{(2)} = ((a_{ij}^{(2)})), \cdots, A^{(l)} = ((a_{ij}^{(l)}))$ be $k \times k$ matrices with entries from $\mathcal{A}$.

$$Tr(A^{(0)}, A^{(1)}, \cdots, A^{(l)})_l = \sum_{i_0, i_1, \cdots, i_n} (a^{(0)}_{i_0,i_1}, a^{(1)}_{i_1 i_2}, \cdots, a^{(l)}_{i_l i_0})_l \in C_l(\mathcal{A}).$$

Theorem 1.4.1 *(i) Let $p \in M_r(\mathcal{A})$ be an idempotent. Define the Chern character in $C_*(\mathcal{A})$ of p by the formula*

$$Ch_*(p) = Tr(p)_0 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} Tr(p - \frac{1}{2}, p, \cdots, p)_{2k}$$

then, $(b+B)Ch_*(p) = 0$ and $[p] \mapsto Ch_*(p)$ defines a homomorphism from $K_0(\mathcal{A})$ to $HE_0(\mathcal{A})$.
(ii) If $g \in GL_N(\mathcal{A})$, then

$$Ch_*(g) = \sum_{k=0}^{\infty} k! Tr(g^{-1}, g, \cdots, g^{-1}, g)_{2k+1}$$

is a closed element of $C^*(\mathcal{A})$ and hence defines an element in $HE_1(\mathcal{A})$.

Chern character on K-homology is defined by defining them on the generators, i.e., theta summable unbounded Fredholm modules. Connes introduced the notion of these modules in [21]. We insert the word unbounded to distinguish it from the already defined Fredholm modules.

A **theta summable** unbounded Fredholm module $(\mathcal{H}, D)$ over an involutive Banach algebra $\mathcal{A}$ with identity consists of a Hilbert space $\mathcal{H}$ carrying a continuous involutive representation of $\mathcal{A}$ and a self adjoint operator $D : \mathcal{H} \to \mathcal{H}$ with the following properties:
(i) if $a \in \mathcal{A}$, the operator $[D, a] \in \mathcal{B}(\mathcal{H})$, and $\|a\| + \|[D, a]\| \leq N(D)\|a\|_{\mathcal{A}}$ for some constant $N(D)$;
(ii) D has compact resolvents and $tre^{-tD^2} < \infty, \ \forall t > 0$.
It is called **even** if there is an involution $\gamma, \gamma = \gamma^*, \gamma^2 = I$ that commutes with $\mathcal{A}$ and anti-commutes with D. Otherwise the Fredholm module is called **odd**.

Note that the way one constructs a Fredholm module from a spectral triple, one can construct a Fredholm module from an unbounded Fredholm module too.
Let $(\mathcal{H}, D)$ be an unbounded Fredholm module. In case it is even the **Chern character** is defined as

$$Ch^n((a_0, \cdots, a_n)_n) = \int_{\Delta^n} Str(a_0 e^{-s_0 D^2}[D, a_1]e^{-s_1 D^2} \cdots [D, a_n]e^{-s_n D^2})d^n s$$

where Δ^n is the n-simplex and $StrA = trA|_{\mathcal{H}^+} - trA|_{\mathcal{H}^-}$ and $\mathcal{H}^{\pm}$ are the eigenspaces of the grading operator γ corresponding to eigenvalues ± 1 respectively.

Theorem 1.4.2 *Let $(\mathcal{H}, D)$ be an even unbounded theta summable Fredholm module. $Ch^\bullet(D)$ defines an entire cyclic cocycle, often referred as JLO cocycle after Jaffe, Lesniewski, Osterwalder. Let $(\mathcal{H}, F)$ be the associated Fredholm module. Then for an idempotent $p \in M_q(\mathcal{A})$ one has $\langle[(\mathcal{H}, F)], [p]\rangle = (Ch^\bullet(D), Ch_\bullet(p))$.*

In the odd case one has to be little more careful. In that case define $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2$ with the grading operator $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\widetilde{\pi} = \pi \otimes I$ and $\widetilde{D} = D \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Then the **Chern character** is given by

$$Ch^n(D)((a_0, a_1, \cdots, a_n)_n) = \int_{\Delta^n} Str(c_1 \widetilde{\pi}(a_0)e^{-s_0 \widetilde{D}^2}[\widetilde{D}, \widetilde{\pi}(a_1)]e^{-s_1 \widetilde{D}^2} \cdots [\widetilde{D}, \widetilde{\pi}(a_n)]e^{-s_n \widetilde{D}^2})d^n s$$

where Str is as earlier and $c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem 1.4.3 *(i) $Ch^\bullet(D)$ defines an odd entire cocycle. (ii) Let $(\mathcal{H}, F)$ be the canonically associated Fredholm module, then for an invertible $u \in GL_N(\mathcal{A})$ one has $\langle[(\mathcal{H}, F)], [u]\rangle = (Ch^\bullet(D), Ch_\bullet(u))$.*

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple such that $tre^{-tD^2} < \infty$, $\forall t > 0$. Let $\bar{\mathcal{A}}$ be the norm closure of $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Define $\mathcal{A}_{(1)} = \{a \in \bar{\mathcal{A}} : [D, a] \in \mathcal{B}(\mathcal{H})\}$. Consider the new norm $\|\cdot\|_1$ on $\mathcal{A}_{(1)}$, given by $\|a\|_1 := \|a\| + \|[D, a]\|$. It is a matter of straightforward verification that $\mathcal{A}_{(1)}$ is a Banach algebra with isometric involution and $(\mathcal{H}, D)$ is a theta summable unbounded Fredholm module on $\mathcal{A}_{(1)}$ with associated Chern character $Ch^\bullet(D) \in HE^*(\mathcal{A}_{(1)}$. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has nontrivial Chern character if $Ch^\bullet$ gives a nontrivial homomorphism from $HE_*(\mathcal{A}_{(1)})$ to $\mathbb{Z}$. We are going to construct spectral triples with nontrivial Chern characters.

1.5 Dixmier Trace

Dixmier introduced a class of tracial functionals to show existence of nonnormal traces on the algebra of bounded operators on a separable Hilbert space. Later on by clever use of Weyl asymptotics Connes has shown us how to use these singular traces to capture various metric notions from geometry. Here our exposition closely follows the appendix of [25].
Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{L}^1 \subset \mathcal{K}$ the ideal of trace class operators:

$$\mathcal{L}^1 = \{T \in \mathcal{K} : \sum_0^\infty \mu_n(T) < \infty\}$$

where $\mu_n(T) = inf\{\|TE^\perp\| \ : \ dimE = n\}$ is the $(n+1)^{\text{th}}$ singular value of T. One defines trT for $T \in \mathcal{L}^1$ as $trT = \sum \langle T\xi_n, \xi_n\rangle$. This converges because T is an l^1 sum of rank one operators and is independent of the choice of the basis. Trace norm for a compact operator is defined by $\|T\|_1 = tr|T| = \sum_0^\infty \mu_n(T)$.

Definition 1.5.1 For each integer $N > 1, T \in \mathcal{K}$ let

$$\sigma_N(T) = sup\{\|TE\|_1 : E \text{ is a subspace of } \mathcal{H},\ dimE = N\},$$

$\sigma_N(T)$ can also be described as the sum of N largest singular values of T. By construction this is a norm. For $\lambda > 0$, define $\sigma_\lambda(T)$ to be the piecewise linear function that agrees with $\sigma_N(T)$ for $N \in \mathbb{N}$ and equals zero at zero. Then each $\sigma_\lambda(T)$ is a norm and for fixed T, $\lambda \mapsto \sigma_\lambda(T)$ is a concave function. Moreover for $T_1, T_2 \geq 0$ and $\lambda_1, \lambda_2 \geq 0$ we have,

$$\sigma_{\lambda_1+\lambda_2}(T_1 + T_2) \geq \sigma_{\lambda_1}(T_1) + \sigma_{\lambda_2}(T_2).$$

Definition 1.5.2 The Dixmier ideal of compact operators is defined by

$$\mathcal{L}^{1+} := \{T \in \mathcal{K} : \ \|T\|_{1+} := \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{log\lambda} < \infty\}.$$

A compact operator T is in the Dixmier ideal iff $\mu_n(T) = O(\frac{1}{n})$. Clearly $\|\cdot\|_{1+}$ is a norm, satisfying $\sigma_\lambda(T) \leq \|T\|_{1+} log\lambda$ for $\lambda > 0$. Since $\lambda \mapsto \sigma_\lambda(T)$ is bounded for $T \in \mathcal{L}^1$ it is clear that $\mathcal{L}^1 \subseteq \mathcal{L}^{1+}$. In fact $\mathcal{L}^{1+} \subseteq \mathcal{L}^p$ for any $p > 1$, whence the notation for the Dixmier ideal. Consider the following Cesaro mean of the function $\frac{\sigma_\lambda(T)}{log\lambda}$:

$$\tau_\lambda(T) := \frac{1}{log\lambda} \int_a^\lambda \frac{\sigma_u(T)}{log\, u} \frac{du}{u} \quad \text{for } \lambda \geq a \geq e.$$

Subadditivity of $\sigma_u(\cdot)$ descends to that of $\tau_\lambda(\cdot)$. It is asymptotically additive in the following sense:

Proposition 1.5.3 *If $T, S \in \mathcal{L}^{1+}$ are positive operators, then*

$$|\tau_\lambda(T+S) - \tau_\lambda(T) - \tau_\lambda(S)| = O\left(\frac{log\, log\lambda}{log\lambda}\right) \text{ as } \lambda \to \infty.$$

In the quotient C^*-algebra $B_\infty = C_b([a,\infty))/C_0([a,\infty))$, let $\tau(T) \in B_\infty$ be the class of $\lambda \mapsto \tau_\lambda(T)$, for T, a positive element of $\mathcal{L}^{1+}$. Then τ is additive and positive homogeneous, i.e., $\tau(T+S) = \tau(T) + \tau(S), \tau(cT) = c\tau(T)$ for $c \geq 0$.

Proposition 1.5.4 *Extending the definition of τ by linearity we get a positive linear map from $\mathcal{L}^{1+}$ to B_∞ such that for any bounded operator S in $\mathcal{B}(\mathcal{H})$: $\tau(ST) = \tau(TS)$ $\forall T \in \mathcal{L}^{1+}$.*

Definition 1.5.5 For any state ω on the C^*-algebra B_∞, $tr_\omega(T) := \omega(\tau(T))$, $\forall T \in \mathcal{L}^{1+}$ defines a positive linear form on $\mathcal{L}^{1+}$, and satisfies $tr_\omega(ST) = tr_\omega(TS)$ for all $S \in \mathcal{B}(\mathcal{H})$. This functional is called **Dixmier trace**.

In general this functional depends on the choice of the state ω. It is easily seen that $tr_\omega(T)$ is independent of ω iff $\tau_\lambda(T)$ converges for $\lambda \to \infty$, and the limit is then equal to $tr_\omega(T)$. The following proposition relates Dixmier trace with heat kernel expansion in a generalized sense.

Proposition 1.5.6 *Let D be a self-adjoint operator such that $|D|^{-p} \in \mathcal{L}^{1+}$ for some $p > 0$. Then for any bounded operator T and any state ω on B_∞*

$$\omega(\lambda \mapsto \frac{1}{\lambda} tr(Te^{-\lambda^{-2/p}D^2})) = \Gamma(\frac{p}{2}+1) tr_\omega T|D|^{-p}.$$

In concrete situations it is often possible to show that the left hand side is even independent of ω implying the same for the right hand side. We have taken this statement from [24]. It's proof due to Connes is available in [41]. Another crucial property of Dixmier trace is stated in the following proposition. It is immediate from the definition. We will see several applications of this.

Proposition 1.5.7 *Let $T \in \mathcal{L}^{1+}$ and S be a compact operator, then for any state ω on B_∞ $tr_\omega(ST) = 0$.*

1.6 Metric Aspects Of Geometry

In the last chapter of his book Alain Connes has shown how to decipher metric information of a compact Riemannian spin manifold from its canonical spectral triple. Later on these ideas have further been extended by Frohlich, Grandjean and Recknagel in [36]. They have extended various notions of curvature in the noncommutative context. The following proposition shows how one recovers the Riemannian metric and volume measure:

Proposition 1.6.1 *Let $(\mathcal{A}, \mathcal{H}, D)$ be the canonical spectral triple of a compact Riemennian spin manifold M. Then*
(i) the geodesic distance between any two points on M is given by

$$d(p,q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)| : \||[D,f]\|| \leq 1\}, \forall p, q \in M. \quad (1.6.1)$$

(ii) The Riemannian measure on M is given by,

$$\int_M f = c(n) = tr_\omega(f|D|^{-n}), \forall f \in \mathcal{A}, \quad (1.6.2)$$

where $c(n) = 2^{(n-[n/2]-1)}\pi^{n/2} n\Gamma(n/2)$

This proposition gives us an idea about how to define distance and volume measure in the noncommutative context. The formula on the right hand side of (1.6.2) makes perfect sense for a general spectral triple and defines a hyper-trace, to be more precise let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple of **dimension** p, i.e., $|D|^{-p} \in \mathcal{L}^{1+}$ and $tr_\omega|D|^{-p} \neq 0$. Then the functional $a \in \mathcal{A} \mapsto tr_\omega a|D|^{-p}$ is a hypertrace ([17]), i.e., for $a \in \mathcal{A}, b \in \mathcal{B}(\mathcal{H})$

$$tr_\omega ab|D|^{-p} = tr_\omega ba|D|^{-p}.$$

Therefore the positive linear functional on $\mathcal{A}$ given by $a \mapsto tr_\omega a|D|^{-p}$ will be considered as the state coming from volume measure.

1.6.1 Noncommutative Differential Forms

We shall now describe how to construct a differential algebra of forms out of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It is useful to introduce a universal graded differential algebra associated with any algebra $\mathcal{A}$.

1.6.2 Universal Differential Forms

Let $\Omega^1(\mathcal{A})$ be the $\mathcal{A}$-$\mathcal{A}$ bimodule $\ker(m : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A})$, where $m(a \otimes_{\mathbb{C}} b) = ab$. The differential $\delta : \mathcal{A} \to \Omega^1(\mathcal{A})$ is given by $\delta(a) = 1 \otimes a - a \otimes 1$. As a bimodule $\Omega^1(\mathcal{A})$ is generated by $\delta(\mathcal{A})$. Indeed if $\sum a_i b_i = m(\sum a_i \otimes b_i) = 0$, then $\sum a_i \otimes b_i = \sum a_i(1 \otimes b_i - b_i \otimes 1) =$

$\sum a_i\delta(b_i)$. Let $\Omega^p(\mathcal{A}) = \underbrace{\Omega^1\mathcal{A}\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})}_{p\ times}$. Then $\Omega^\bullet(\mathcal{A}) = \oplus_{p\geq 0}\Omega^p(\mathcal{A})$ is an $\mathcal{A}$-algebra with multiplication and $\mathcal{A}$-$\mathcal{A}$-bimodule structures given by,

$$\begin{aligned}(\omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p).(\omega_{p+1}\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_{p+q}) &:= \omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_{p+q},\\ a.(\omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p) &:= a\omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p,\\ (\omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p).a &:= \omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p a, \forall\omega_j\in\Omega^1(\mathcal{A}), a\in\mathcal{A}\end{aligned}$$

For a one form $\omega = \sum a_i\otimes b_i$, define,

$$\begin{aligned}\delta(\omega) &:= \sum(1\otimes_{\mathbb{C}} a_i - a_i\otimes_{\mathbb{C}} 1)\otimes_{\mathcal{A}}(1\otimes_{\mathbb{C}} b_i - b_i\otimes_{\mathbb{C}} 1)\\ &= \sum 1\otimes_{\mathbb{C}} a_i\otimes_{\mathbb{C}} b_i - a_i\otimes_{\mathbb{C}} 1\otimes_{\mathbb{C}} b_i - a_i\otimes_{\mathbb{C}} b_i\otimes_{\mathbb{C}} 1.\end{aligned}$$

Then δ is extended using Leibnitz rule w.r.t $\otimes_{\mathcal{A}}$,

$$\delta(\omega_1\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\omega_p) = \sum_{i=1}^{p}(-1)^{i+1}\omega_1\otimes_{\mathcal{A}}\cdots\delta(\omega_i)\otimes_{\mathcal{A}}\omega_p, \tag{1.6.3}$$

The graded differential algebra $(\Omega^\bullet(\mathcal{A}) = \oplus_{p\geq 0}\Omega^p(\mathcal{A}), \delta)$ is characterized by the following universal property.

Proposition 1.6.2 *Let* (Γ, Δ) *be a graded differential algebra,* $\Gamma = \oplus\Gamma^p$*, and let* $\rho : \mathcal{A}\to\Gamma^0$ *be a unital homomorphism. Then there exists a unique extension of* ρ *to a morphism of graded differential algebras* $\widetilde{\rho} : \Omega^\bullet(\mathcal{A})\to\Gamma$*, such that* $\widetilde{\rho}\circ\delta = \Delta\circ\widetilde{\rho}$*.*

Because of this proposition the differential graded algebra $(\Omega^\bullet(\mathcal{A}), \delta)$ is called the universal graded differential algebra and δ is called the universal differential. Finally we mention that if $\mathcal{A}$ has an involution $*$, the algebra $\Omega^\bullet(\mathcal{A})$ is also made an involutive algebra by defining

$$\begin{aligned}(\delta(a))^* &:= -\delta(a^*)\\ (a_0\delta(a_1)\cdots\delta(a_p))^* &:= (\delta(a_p))^*\cdots(\delta(a_1))^* {a_0}^*\end{aligned}$$

1.6.3 Connes-deRham forms and associated calculi

The universal differential algebra is not very interesting from the cohomological point of view. Interesting cohomologies are obtained from the representations of the algebra. For a spectral triple the following representation constructs an exterior algebra of forms. The map $\pi : \Omega^\bullet(\mathcal{A})\to\mathcal{B}(\mathcal{H})$, given by $\pi(a_0\delta(a_1)\cdots\delta(a_p)) := a_0[D, a_1]\cdots[D, a_p]$, clearly extends to a $*$-homomorphism since both δ and $[D, a]$ are derivations. Unfortunately $\pi(\omega) = 0$ does not imply $\pi(\delta(\omega)) = 0$ and hence one can not define forms as $\pi(\Omega^\bullet(\mathcal{A}))$. To overcome this difficulty one goes through a quotienting procedure.

Proposition 1.6.3 *Let $J = \oplus_{p\geq 0} J_p$ be the two sided ideal of $\Omega^\bullet(\mathcal{A})$ given by $J_p = \{\omega \in \Omega^p(\mathcal{A}) : \pi(\omega) = 0\}$, then $J + \delta J$ is a graded differential two sided ideal of $\Omega^\bullet(\mathcal{A})$.*

Definition 1.6.4 The graded differential algebra of **Connes-deRham forms** over the algebra $\mathcal{A}$ is defined by

$$\Omega_D^\bullet(\mathcal{A}) := \Omega^\bullet(\mathcal{A})/(J + \delta J) \cong \pi(\Omega^\bullet(\mathcal{A}))/\pi(\delta J).$$

It is naturally graded by the degrees of $\Omega^\bullet(\mathcal{A})$. The space of p-forms is given by $\Omega_D^p(\mathcal{A}) := \Omega^p(\mathcal{A})/(J_p + \delta J_{p-1})$. Since J is a differential ideal, the exterior differential δ defines a differential on $\Omega_D^\bullet(\mathcal{A})$, $d : \Omega_D^p(\mathcal{A}) \to \Omega_D^{p+1}(\mathcal{A}), d[\omega] := [\delta\omega]$ with $\omega \in \Omega^p(\mathcal{A})$ and $[\omega]$ the corresponding class in $\Omega_D^p(\mathcal{A})$. This complex $(\Omega_D^\bullet(\mathcal{A}), d)$ will be called the **Connes-deRham complex** or **noncommutative exterior complex** associated with the spectral triple. For the canonical spectral triple of a compact Riemannian spin manifold this yields the exterior differential algebra of forms. Whence the nomenclature of Connes-deRham complex.

Remark 1.6.5 The key ingredient to the construction just described is the representation of $\Omega^\bullet(\mathcal{A})$ in $\mathcal{B}(\mathcal{H})$. For any representation of $\Omega^\bullet(\mathcal{A})$ we can similarly construct a differential graded algebra. In chapter 5 we will consider the complex obtained from the representation $\theta \circ \pi : \Omega^\bullet(\mathcal{A}) \to \mathcal{Q}(\mathcal{H})$ where $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the projection onto the Calkin algebra.

1.6.4 Square Integrable Forms and Associated calculus

To define the space of square integrable sections we note the expression for defining Riemannian measure extends to bounded operators on $\mathcal{H}$.

Definition 1.6.6 The integral over the noncommutative space defined by $(\mathcal{A}, \mathcal{H}, D)$ of dimension p is a state $⨍$ on $\pi(\Omega^\bullet(\mathcal{A}))$ defined by $⨍(\eta) = tr_\omega \eta |D|^{-p}$

Strictly speaking this is only a positive linear functional not necessarily of norm one. But we will call this a state without normalization. For this integral to be a useful tool, we need an additional property:

Assumption 1.6.7 The state $⨍$ on $\pi(\Omega^\bullet(\mathcal{A}))$ is cyclic, i.e.,

$$⨍ \eta_1\eta_2 = ⨍ \eta_2\eta_1$$

for all $\eta_1, \eta_2 \in \pi(\Omega^\bullet(\mathcal{A}))$.

The state determines a positive semidefinite sesquilinear form on $\Omega^\bullet(\mathcal{A})$ by setting

$$(\eta_1, \eta_2)_D = -\!\!\!\!\!\!\int \pi(\eta_1)\pi(\eta_2)^*, \text{ for all } \eta_1, \eta_2 \in \Omega^\bullet(\mathcal{A}).$$

In the formulas below we drop the representation symbol π. Later on while dealing with specific computations if there is no chance of confusion we will also drop the suffix D from $(\cdot,\cdot)_D$. By K_k we denote the kernel of this sesquilinear form restricted to $\Omega^k(\mathcal{A})$. More precisely we set

$$\mathsf{K} = \oplus_{k\geq 0}\mathsf{K}_k, \ \mathsf{K}_k = \{\omega \in \Omega^k(\mathcal{A}) : (\omega,\omega) = 0\}.$$

Obviously, K_k contains the ideal J_0^k; in the classical case they coincide. Assumption of cyclicity of the integral forces K to become a two sided graded *-ideal. We now define

$$\widetilde{\Omega}^\bullet(\mathcal{A}) = \oplus_{k=0}^{\infty}\widetilde{\Omega}^k(\mathcal{A}), \ \widetilde{\Omega}^k(\mathcal{A}) = \Omega^k(\mathcal{A})/\mathsf{K}_k.$$

The sesquilinear form $(\cdot,\cdot)$ induces a positive definite scalar product on $\widetilde{\Omega}^k(\mathcal{A})$, and we denote by $\widetilde{\mathcal{H}}^k$ the Hilbert space completion of this space with respect to the scalar product,

$$\widetilde{\mathcal{H}}^\bullet = \oplus_{k=0}^{\infty}\widetilde{\mathcal{H}}^k, \ \widetilde{\mathcal{H}}^k = \overline{\widetilde{\Omega}^k(\mathcal{A})}^{(\cdot,\cdot)}.$$

$\widetilde{\mathcal{H}}^k$ is to be interpreted as the space of square integrable k-forms.

Proposition 1.6.8 *The space $\widetilde{\Omega}^\bullet(\mathcal{A})$ is a unital graded *-algebra. For any $\omega \in \widetilde{\Omega}^k(\mathcal{A})$ the left and right actions of ω on $\widetilde{\Omega}^p(\mathcal{A})$ with values in $\widetilde{\Omega}^{(p+k)}(\mathcal{A})$*

$$m_L(\omega)(\eta) := \omega\eta, \quad m_R(\omega)(\eta) := \eta\omega$$

are continuous in the norm given by $(\cdot,\cdot)$. In particular the Hilbert space $\widetilde{\mathcal{H}}^\bullet$ is a bimodule over $\widetilde{\Omega}^\bullet(\mathcal{A})$ with continuous actions.

The algebra $\widetilde{\Omega}^\bullet(\mathcal{A})$ may fail to be differential. This problem is settled as in the earlier case. The unital graded differential algebra of **square integrable forms** $\widetilde{\Omega}^\bullet_D(\mathcal{A})$ is given by the graded quotient of $\widetilde{\Omega}^\bullet(\mathcal{A})$ by $\mathsf{K} + \delta\mathsf{K}$.

$$\widetilde{\Omega}^\bullet_D(\mathcal{A}) := \oplus_{k=0}^{\infty}\widetilde{\Omega}^k_D(\mathcal{A}), \ \widetilde{\Omega}^k_D(\mathcal{A}) := \Omega^k(\mathcal{A})/(\mathsf{K}_k + \delta\mathsf{K}_{k-1})$$

δ will induce a differential $\widetilde{d} : \widetilde{\Omega}^k_D(\mathcal{A}) \to \widetilde{\Omega}^{k+1}_D(\mathcal{A})$. $(\widetilde{\Omega}^\bullet_D(\mathcal{A}), \widetilde{d})$ is a differential graded algebra. Using the inner product $(\cdot,\cdot)$ on $\widetilde{\mathcal{H}}^k$ inner products are introduced on $\widetilde{\Omega}^k_D(\mathcal{A})$ as follows: let $P_{\delta\mathsf{K}_{k-1}} : \widetilde{\mathcal{H}}^k \to \widetilde{\mathcal{H}}^k$ be the orthogonal projection onto $\delta\mathsf{K}_{k-1}$ in $\widetilde{\mathcal{H}}^k$, and for each element

$[\omega] \in \widetilde{\Omega}^k_D(\mathcal{A})$ we set $\omega^\perp := (1 - P_{\delta K_{k-1}})\omega \in \widetilde{\mathcal{H}}^k$. A positive definite scalar product is defined on $\widetilde{\Omega}^k_D(\mathcal{A})$ via the representative $\omega^\perp$:

$$([\omega], [\eta]) := (\omega^\perp, \eta^\perp)$$

for all $[\omega], [\eta] \in \widetilde{\Omega}^k_D(\mathcal{A})$. In the classical case this is just the usual inner product on the space of square integrable k-forms.

1.6.5 Vector Bundles, Connections, Curvature etc.

Definition 1.6.9 A vector bundle $\mathcal{E}$ over the noncommutative space described by $(\mathcal{A}, \mathcal{H}, D)$ is a finitely generated projective left $\mathcal{A}$ module.

Definition 1.6.10 A Hermitian structure over a vector bundle $\mathcal{E}$ is a sesquilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A},$$

such that for all $a, b \in \mathcal{A}$, and $s, t \in \mathcal{E}$.
(i) $\langle as, bt \rangle = a\langle s, t\rangle b^*$,
(ii) $\langle s, s\rangle \geq 0$,
(iii) the $\mathcal{A}$ linear map $g : \mathcal{E} \to \mathcal{E}^*_R, s \mapsto \langle s, \cdot \rangle$ where $\mathcal{E}^*_R = \{\phi \in Hom(\mathcal{E}, \mathcal{A}) : \phi(as) = \phi(s)a^*\}$ is an isomorphism of left $\mathcal{A}$ modules, i.e., g can be regarded as a metric on $\mathcal{E}$.

The $\mathcal{A}$ bimodules $\widetilde{\Omega}^k_D(\mathcal{A})$ carry Hermitian structures in a slightly generalized sense. Let $\bar{\mathcal{A}}$ be the weak closure of the algebra $\mathcal{A}$ acting on $\widetilde{\mathcal{H}}^0$.

Theorem 1.6.11 (Frohlich et. al. [36]) *There is a canonically defined sesquilinear map*

$$\langle \cdot, \cdot \rangle_D : \widetilde{\Omega}^k(\mathcal{A}) \times \widetilde{\Omega}^k(\mathcal{A}) \to \bar{\mathcal{A}}$$

such that for all $a, a \in \mathcal{A}$ and all $\omega, \eta \in \widetilde{\Omega}^k(\mathcal{A})$
(i) $\langle a\omega, b\eta \rangle_D = a\langle \omega, \eta\rangle_D b^*$,
(ii) $\langle \omega, \omega\rangle_D \geq 0$,
(iii) $\langle \omega a, \eta\rangle_D = \langle \omega, \eta a^*\rangle_D$.

We call $\langle \cdot, \cdot \rangle_D$ a generalized Hermitian structure on $\widetilde{\Omega}^k(\mathcal{A})$. It is the non-commutative analogue of the Riemannian metric on the bundle of differential forms. Note that $\langle \cdot, \cdot \rangle_D$ takes values in $\bar{\mathcal{A}}$ and thus property (iii) of definition 1.6.10 is not directly applicable.

Definition 1.6.12 A **connection** ∇ on a vector bundle $\mathcal{E}$ over a noncommutative space is a $\mathbb{C}$ linear map

$$\nabla : \mathcal{E} \to \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

such that $\nabla(as) = \widetilde{d}(a) \otimes s + a\nabla(s)$ for all $a \in \mathcal{A}, s \in \mathcal{E}$.

Given a vector bundle $\mathcal{E}$, we define a space of $\mathcal{E}$ valued differential forms by $\widetilde{\Omega}^\bullet_D(\mathcal{E}) = \widetilde{\Omega}^\bullet_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$. If ∇ is a connection $\mathcal{E}$, then it extends uniquely to a $\mathbb{C}$ linear map, again denoted ∇, $\nabla : \widetilde{\Omega}^\bullet_D(\mathcal{E}) \to \widetilde{\Omega}^{\bullet+1}_D(\mathcal{E})$, such that $\nabla(\omega s) = \widetilde{d}(\omega)s + (-1)^k \omega\nabla(s)$ for all $\omega \in \widetilde{\Omega}^k_D(\mathcal{A})$ and all $s \in \widetilde{\Omega}^\bullet_D(\mathcal{E})$.

Definition 1.6.13 The **curvature** of a connection ∇ on a vector bundle $\mathcal{E}$ i given by

$$R(\nabla) = -\nabla^2 : \mathcal{E} \to \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}.$$

Note that the curvature extends to a left $\mathcal{A}$ linear map $R(\nabla) : \widetilde{\Omega}^\bullet_D(\mathcal{E}) \to \widetilde{\Omega}^{\bullet+2}_D(\mathcal{E})$.

Definition 1.6.14 A connection on a Hermitian vector bundle $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ is called unitary if $\widetilde{d}\langle s,t\rangle = \langle\nabla s, t\rangle - \langle s, \nabla t\rangle$ for all $s, t \in \mathcal{E}$, where the right hand side of this equation is defined by $\langle\omega \times s, t\rangle = \omega\langle s, t\rangle, \langle s, \eta \times t\rangle = \langle s, t\rangle\eta^*$ for all $\omega, \eta \in \widetilde{\Omega}^1_D(\mathcal{A})$ and all $s, t \in \mathcal{E}$.

As remarked earlier in general one does not have $\widetilde{\Omega}^k_D(\mathcal{A}) = \Omega^k_D(\mathcal{A})$. Without further hypothesis $\widetilde{\Omega}^1_D(\mathcal{A})$ need not be finitely generated projective. So to proceed further we require:

Assumption 1.6.15 The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is such that (i) $\mathsf{K}_0 = 0$, this implies $\widetilde{\Omega}^0_D(\mathcal{A}) = \mathcal{A}$ and $\widetilde{\Omega}^1_D(\mathcal{A}) = \widetilde{\Omega}^1(\mathcal{A})$. Thus $\widetilde{\Omega}^1_D(\mathcal{A})$ carries a generalized Hermitian structure.
(ii) $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle, i.e., it is finitely generated and projective. It is called the cotangent bundle.
(iii) The generalized metric $\langle\cdot,\cdot\rangle$ on $\widetilde{\Omega}^1_D(\mathcal{A})$ defines an isomorphism of left $\mathcal{A}$ modules between $\widetilde{\Omega}^1_D(\mathcal{A})$ and the space of $\mathcal{A}$ anti-linear maps from $\widetilde{\Omega}^1_D(\mathcal{A})$ to $\mathcal{A}$, i.e., to each $\mathcal{A}$ anti-linear map $\phi : \widetilde{\Omega}^1_D(\mathcal{A}) \to \mathcal{A}$ satisfying $\phi(a\omega) = \phi(\omega)a^*$ for all $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$ and all $a \in \mathcal{A}$, their is a unique $\eta_\phi \in \widetilde{\Omega}^1_D(\mathcal{A})$ with $\phi(\omega) = \langle\eta_\phi, \omega\rangle_D$.

Under these assumptions one can obtain noncommutative generalizations like torsion and Riemannian curvature. The last assumption in 1.6.15 will provide a substitute for the procedure of contracting indices leading to Ricci and scalar curvature. For the rest of the section we assume the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfies assumption 1.6.15.

Definition 1.6.16 Let ∇ be a connection on the cotangent bundle $\widetilde{\Omega}^1_D(\mathcal{A})$. The **torsion** of the connection ∇ is the $\mathcal{A}$-linear map.

$$T(\nabla) := \widetilde{d} - m \circ \nabla : \widetilde{\Omega}^1_D(\mathcal{A}) \to \widetilde{\Omega}^2_D(\mathcal{A})$$

where $m : \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A}) \to \widetilde{\Omega}^2_D(\mathcal{A})$ denotes the product of 1-forms in $\widetilde{\Omega}^\bullet_D(\mathcal{A})$

Since we assume that $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle, we can define Riemannian curvature of a connection. To proceed further, we make use of part (ii) of assumption 1.6.15, which implies that there exists a finite set of generators $\{E_i\}$ of $\widetilde{\Omega}^1_D(\mathcal{A})$ and an associated dual basis $\{\varepsilon_i\} \subseteq \widetilde{\Omega}^1_D(\mathcal{A})^*$,

$$\widetilde{\Omega}^1_D(\mathcal{A})^* := \{\phi : \widetilde{\Omega}^1_D(\mathcal{A}) \to \mathcal{A} : \phi(a\omega) = a\phi(\omega) \text{ for all } a \in \mathcal{A}, \omega \in \widetilde{\Omega}^1_D(\mathcal{A})\}$$

such that each $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$ can be written as $\omega = \sum_i \varepsilon_i(\omega)E_i$. By part (iii) of assumption 1.6.15, we get unique 1-form $e_i \in \widetilde{\Omega}^1_D(\mathcal{A})$ such that $\varepsilon_i(\omega) = \langle \omega, e_i \rangle_D$ for all $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$. By proposition 1.6.8 every such e_i determines a bounded operator $m_L(e_i) : \widetilde{\mathcal{H}}^1 \to \widetilde{\mathcal{H}}^2$. The adjoint of this operator with respect to the scalar product $(\cdot,\cdot)$ on $\widetilde{\mathcal{H}}^\bullet$ is denoted by $e_i^\dagger : \widetilde{\mathcal{H}}^2 \to \widetilde{\mathcal{H}}^1$. Similarly for any 1-form $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$, right multiplication on $\widetilde{\mathcal{H}}^0$ with ω defines a bounded operator $m_R(\omega) : \widetilde{\mathcal{H}}^0 \to \widetilde{\mathcal{H}}^1$, and we denote by $\omega_R^\dagger : \widetilde{\mathcal{H}}^1 \to \widetilde{\mathcal{H}}^0$, the adjoint of this operator.

Definition 1.6.17 Let ∇ be a connection on the cotangent bundle $\widetilde{\Omega}^1_D(\mathcal{A})$ over a noncommutative space defined by $(\mathcal{A}, \mathcal{H}, D)$ satisfying assumptions 1.6.15.
(i) The **Riemannian curvature** is the left $\mathcal{A}$-linear map

$$R(\nabla) = -\nabla^2 : \widetilde{\Omega}^1_D(\mathcal{A}) \to \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A}).$$

(ii) Choosing a set of generators E_i of $\widetilde{\Omega}^1_D(\mathcal{A})$ and dual generators ε_i of $\widetilde{\Omega}^1_D(\mathcal{A})^*$ we can write $R(\nabla) = \sum_{i,j} \varepsilon_i \otimes R_{ij} \otimes E_j$. The **Ricci tensor** $Ric(\nabla)$ is given by

$$Ric(\nabla) = \sum_j Ric_j \otimes E_j \in \widetilde{\mathcal{H}}^1 \otimes \widetilde{\Omega}^1_D(\mathcal{A})$$

where $Ric_j = \sum_i e_i^\dagger((1 - P_{\delta \mathsf{K}_{k-1}})R_{ij})$.
(iii) The **scalar curvature** $r(\nabla)$ of the connection ∇ is defined as

$$r(\nabla) = \sum_j (E_j^*)_R^\dagger(Ric_j) \in \widetilde{\mathcal{H}}^0.$$

Both $Ric(\nabla)$ and $r(\nabla)$ do not depend on the choice of the generators.

1.7 Compact Quantum Metric Spaces

In noncommutative geometry, the natural way to specify a metric is by means of a suitable "Lipschitz seminorm". This idea was first suggested by Connes ([22]), and developed further in [24]. Connes pointed out ([22],[24]) that from a Lipschitz seminorm one obtains in a simple way an ordinary metric on the state space of a C^*-algebra. A natural question in this context is when does this metric topology coincides with the weak* topology. In his search for an answer to this question Rieffel ([76],[77],[78]) has identified a larger class of spaces, namely order unit spaces on which one can answer these questions. He has introduced the concept of Compact Quantum Metric Spaces (CQMS) as a generalization of compact metric spaces, and used ([78]) this new concept for rigorous study of convergence questions of algebras much in the spirit of Gromov-Hausdorff convergence. This section is devoted to the basic definitions of CQMS.

In the last section we have seen how the canonical spectral triple captures information about the metric. It is natural to ask that for a compact metric space which class of functions and what special structures on them encodes metric information. Answer to this question is well known. For a compact metric space (X, ρ), let $Lip(X)$ be the class of real valued Lipschitz functions, i.e.,

$$Lip(X) = \{f \in C(X) : sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)} < \infty, f(x) \in \mathbb{R}, \ \forall x \in X\}$$

and L_ρ be the Lipschitz seminorm given by

$$L_\rho(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)}, \text{ for } f \in Lip(X).$$

Then one can recover ρ from L_ρ by

$$\rho(x,y) = sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\}.$$

Therefore, one can safely say that the metric information is hidden in the pair $(Lip(X), L_\rho)$. To single out special features of this pair we need:

Definition 1.7.1 An **order unit space** is a real partially ordered vector space $\mathcal{A}$ with a distinguished element e, the order unit satisfying

(i)(**Order Unit property**) For each $a \in \mathcal{A}$ there is an $r \in \mathbb{R}$ such that $a \leq re$.

(ii)(**The Archimidean property**) If $a \in \mathcal{A}$ and if $a \leq re$ for all $r \in \mathbb{R}$ with $r \geq 0$, then $a \leq 0$.

It is easily seen that real valued Lipschitz functions on a compact metric space with its order structure inherited from the algebra of continuous functions is an order unit space. Motivating example of the above concept is the real subspace of selfadjoint elements in a C*-algebra with the order structure inherited from the C*-algebra. In an order unit space $(\mathcal{A}, e)$ the order structure determines a norm given by

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\}.$$

By a state of an order unit space $(\mathcal{A}, e)$ we mean a $\mu \in \mathcal{A}'$, the dual of $(\mathcal{A}, \|\cdot\|)$ such that $\mu(e) = 1 = \|\mu\|'$. Here $\|\cdot\|'$ stands for the dual norm on $\mathcal{A}'$. States are automatically positive.

Definition 1.7.2 Let $(\mathcal{A}, e)$ be an order unit space. By a Lip norm on $\mathcal{A}$ we mean a seminorm L, on $\mathcal{A}$ such that
(i) For $a \in \mathcal{A}$, we have $L(a) = 0$ iff $a \in \mathbb{R}e$;
(ii) The topology on $S(\mathcal{A})$ coming from the metric $\rho_L(\mu, \nu) = sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}$ is the w^* topology.

For a compact metric space the essential feature of the pair $(Lip(X), L_\rho)$ is that $Lip(X)$ is an order unit space and L_ρ is a Lip norm. $Lip(X)$ has lot of extra properties for example the norm coming from the order structure is a C^*-norm, etc. Dropping all those extra properties a Compact Quantum Metric Spaces **(CQMS)**, is defined as:

Definition 1.7.3 (Rieffel) A compact quantum metric space is a pair $(\mathcal{A}, L)$ consisting of an order unit space $\mathcal{A}$ and a Lip norm L defined on it.

The following theorem of Rieffel gives a simple criterion for verifying the Lip norm condition.

Theorem 1.7.4 (Theorem 4.5 of [78]) *Let L be a seminorm on the order unit space $\mathcal{A}$ such that $L(a) = 0$ iff $a \in \mathbb{R}e$. Then ρ_L gives $S(\mathcal{A})$ the w^*-topology if and only if*
(i) $(\mathcal{A}, L)$ has finite radius, i.e, $\exists$ some constant C such that $\rho_L(\mu, \nu) \leq C$ for all $\mu, \nu \in S(\mathcal{A})$.
(ii) $\mathcal{B}_1 = \{a | L(a) \leq 1,\ \text{and}\ \|a\| \leq 1\}$ is totally bounded in $\mathcal{A}$ for $\|\cdot\|$.

Rieffel himself has constructed several examples of compact quantum metric spaces. He has developed general methodology for construction of examples whenever there is an ergodic action of a compact Lie group. Adapting some of his ideas in the last chapter we will see some examples coming from ergodic action of Heisenberg group. We will also construct examples from some C^*-algebra extensions.

Chapter 2

The Noncommutative Torus and the Quantum Plane

Geometry modelled on the irrational rotation algebras, commonly called noncommutative torus has received much attention in noncommutative geometry. Connes himself has studied this model in great depth ([19], [24]). An excellent survey on this by Rieffel is [74]. One very natural question in this regard is about the possibility of listing 'all' spectral triples under some natural constraints. Are they distinguishable by their associated volume forms, scalar curvature etc.? By a result of Bratteli, Eliott Jorgensen one can list down all spectral triples satisfying a mild condition. For this class of spectral triples we ([16]) show volume form remains invariant. Scalar curvature as introduced by Frohlich et. al ([36]) also does not change. For some specific cases we show that Connes-DeRham cohomology changes, thereby showing that these spectral triples are not unitarily equivalent to the one studied by Connes. Another approach to study geometry in the classical case is via heat semigroup. One may also like to use the notions of quantum stochastic processes to investigate these 'noncommutative spaces'. In particular using the near zero asymptotics of the trace of heat kernel one can introduce concepts like volume form, integrated scalar curvature etc. We show although the volume form for a perturbed family of Laplacians remains invariant the integrated scalar curvature may vary.

We introduce ([16]) quantum 2d-dimensional spaces as quantization of Euclidean 2d-dimensional space. These are examples of 'locally compact quantum spaces'. In this case also one can introduce the idea of volume form and show it remains invariant under quantization.

2.1 Noncommutative torus as C^*-algebra

There are various approaches to define irrational rotation algebras $\mathcal{A}_\theta$. We define this as the universal C^*-algebra generated by two unitaries U and V satisfying the relation:

$$UV = e^{2\pi i\theta} VU \tag{2.1.1}$$

This means (i) $\mathcal{A}_\theta$ is a C^*-algebra generated by two unitaries $\widetilde{U}, \widetilde{V}$ satisfying 2.1.1, and (ii) if $\mathcal{B}$ is another C^*-algebra generated by two unitaries U and V satisfying 2.1.1 then, there is a *-homomorphism from $\mathcal{A}_\theta$ onto $\mathcal{B}$ which carries $\widetilde{U}$ to U, and $\widetilde{V}$ to V. If one can show that such a C^*-algebra exists then it is unique. Existence is proved as follows. Let us denote by $\mathcal{A}_\theta^{fin}$ the *-algebra generated by two unitaries U, V satisfying (2.1.1). On $L^2(\mathbb{T})$ consider the unitary operators $U = M_z$, the multiplication by the unimodular function z on $\mathbb{T}$, and V the operator of rotation by θ, i.e.,

$$Uf(z) = zf(z),\ Vf(z) = f(ze^{2\pi i\theta}).$$

A simple calculation shows U, V satisfies (2.1.1). Let I be the set of irreducible representations of $\mathcal{A}_\theta^{fin}$ on some separable Hilbert space. Now consider the operators $\widetilde{U} = \oplus_{\pi\in I}\pi(U)$, $\widetilde{V} = \oplus_{\pi\in I}\pi(V)$. Let $\mathcal{A}_\theta = C^*(\widetilde{U}, \widetilde{V})$. In order to see that $\mathcal{A}_\theta$ is indeed the desired universal algebra, let $\mathcal{B} = C^*(U, V)$ be any other satisfying (2.1.1). To verify that the map $\phi : \mathcal{A}_\theta \to \mathcal{B}$ taking $\widetilde{U}$ to U, and $\widetilde{V}$ to V is well defined, it suffices to show that,

$$\|p(U, V, U^*, V^*)\| \leq \|p(\widetilde{U}, \widetilde{V}, \widetilde{U}^*, \widetilde{V}^*)\|$$

for every polynomial p. Let $a = p(U, V, U^*, V^*)$, by the GNS, there is an irreducible representation π of $\mathcal{B}$ such that $\|\pi(a)\| = \|a\|$. Consider the pair $U' = \pi(U), V' = \pi(V)$, then, (U', V') is an irreducible pair satisfying (2.1.1). Hence by construction, we see that

$$\begin{aligned} \|p(\widetilde{U}, \widetilde{V}, \widetilde{U}^*, \widetilde{V}^*)\| &\geq \|p(U', V', U'^*, V'^*)\| \\ &\geq \|p(U, V, U^*, V^*)\|. \end{aligned}$$

Therefore, ϕ is well defined and contractive on the *-algebra generated by $\widetilde{U}$ and $\widetilde{V}$ into $\mathcal{B}$. So, it extends by continuity to a homomorphism of $\mathcal{A}_\theta$ onto $\mathcal{B}$.

The algebra $\mathcal{A}_\theta$ for irrational values of θ is called the irrational rotation algebra. Henceforth we will consider this case only.

We apply this universal property to obtain certain special automorphism of $\mathcal{A}_\theta$. For any two complex numbers λ, μ of unit modulus the unitary pair $(\lambda\widetilde{U}, \mu\widetilde{V})$, satisfies (2.1.1). Thus there is an endomorphism $\alpha_{\lambda,\mu}$ of $\mathcal{A}_\theta$ such that,

$$\alpha_{\lambda,\mu}(\widetilde{U}) = \lambda\widetilde{U}, \alpha_{\lambda,\mu}(\widetilde{V}) = \mu\widetilde{V}.$$

Let $\sigma = \alpha_{\bar{\lambda},\bar{\mu}}\alpha_{\lambda,\mu}$. Since $\sigma(\widetilde{U}) = \widetilde{U}, \sigma(\widetilde{V}) = \widetilde{V}$, we have $\sigma = id$. Thus $\alpha_{\lambda,\mu}$ is an automorphism. Moreover for each fixed $a \in \mathcal{A}_\theta$, the map from $\mathbb{T}^2$ to $\mathcal{A}_\theta$ given by $f(\lambda,\mu) = \alpha_{\lambda,\mu}(a)$ is norm continuous. To verify this note that it is true for all noncommutative polynomials in $\widetilde{U}, \widetilde{V}, \widetilde{U}^*, \widetilde{V}^*$. These are dense and endomorphisms are contractive. Therefore, $(\mathcal{A}_\theta, \mathbb{T}^2, \alpha)$ is a C*-dynamical system.

Define two maps of $\mathcal{A}_\theta$ into itself by the formula

$$\Phi_1(a) = \int_0^1 \alpha_{1,e^{2\pi it}}(a)dt, \ \Phi_2(a) = \int_0^1 \alpha_{e^{2\pi it},1}(a)dt.$$

These integrals make sense as Riemann sums because the integrand is norm continuous. Some of the nice properties of these maps are captured in the following theorem.

Theorem 2.1.1 *Φ_1 is positive contractive and faithful, and maps $\mathcal{A}_\theta$ onto $C^*(\widetilde{U})$. Moreover,*

$$\Phi(f(\widetilde{U})ag(\widetilde{U})) = f(\widetilde{U})\Phi_1(a)g(\widetilde{U})$$

for all $f, g \in C(\mathbb{T})$. For any finite linear combination of $\widetilde{U}^k\widetilde{V}^l$ for $k, l \in \mathbb{Z}$,

$$\Phi_1(\sum_{k,l} a_{kl}\widetilde{U}^k\widetilde{V}^l) = \sum_k a_{k0}\widetilde{U}^k.$$

In addition, for every $a \in \mathcal{A}_\theta$,

$$\Phi_1(a) = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{-n}^{n} \widetilde{U}^j a \widetilde{U}^{-j}. \tag{2.1.2}$$

Proof: straightforward, see [30]. □

The corresponding results for Φ_2 also hold. Combining them we obtain,

Corollary 2.1.2 *The map $\tau = \Phi_1\Phi_2 = \Phi_2\Phi_1$ is a faithful unital scalar valued trace on $\mathcal{A}_\theta$. For a finite linear combination of $\widetilde{U}^k\widetilde{V}^l, k, l \in \mathbb{Z}$ explicitly given by $\tau(\sum a_{kl}\widetilde{U}^k\widetilde{V}^l) = a_{00}$.*

Proof: Faithfulness follows from that of Φ_i's and other properties are proved by verifying them on monomials $\widetilde{U}^k\widetilde{V}^l$. □

In fact for any other trace σ using (2.1.2) it is easy to see that $\sigma(a) = \sigma(\Phi_i(a))$ for $i = 1, 2$. Hence $\sigma(a) = \sigma(\tau(a)) = \tau(a)$, i.e., τ is the unique trace.

Theorem 2.1.3 *$\mathcal{A}_\theta$ is simple. Thus if U and V are unitary elements satisfying (2.1.1), then $C^*(U,V)$ is canonically isomorphic with $\mathcal{A}_\theta$.*

Proof: Suppose that $\mathcal{I}$ is a nonzero ideal of $\mathcal{A}_\theta$. Then there is a positive, nonzero element$X \in \mathcal{I}$. Since $\widetilde{U}^j X \widetilde{U}^{-j} \in \mathcal{I}$, the limit formula for Φ_1 shows that $\Phi_1(\mathcal{I}) \subseteq \mathcal{I}$. Similarly one can show that $\Phi_2(\mathcal{I}) \subseteq \mathcal{I}$. Hence $\tau(X) \in \mathcal{I}$. But since τ is faithful X must be a scalar multiple of identity. Therefore $\mathcal{I} = \mathcal{A}_\theta$.

If U, V are unitaries satisfying (2.1.1), then there is a *-homomorphism of $\mathcal{A}_\theta$ onto $C^*(U,V)$ taking $\widetilde{U}$ to U and $\widetilde{V}$ to V. Since $\mathcal{A}_\theta$ is simple, this homomorphism must be an isomorphism. □

From now on we will drop the tilde, and use the symbols U, V for the generators of $\mathcal{A}_\theta$. Because of simplicity this does not cause any ambiguity. For irrational values of θ, $\mathcal{A}_\theta$ can also be described as the crossed product of $C(\mathbb{T})$ by $\mathbb{Z}$, for the automorphism induced by rotation by angle θ. Now an application of Pimsner-Voiculescu exact sequence yield

$$K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2 \cong K_1(\mathcal{A}_\theta).$$

The isomorphism question was decided by Rieffel ([71]) as follows. The trace τ induces a homomorphism $\tau : K_0(\mathcal{A}_\theta) \to \mathbb{Z}$ simply by defining $\tau([p]) = (\tau \otimes tr)([p])$. Then we have,

Theorem 2.1.4 (Rieffel) $\tau(K_0(\mathcal{A}_\theta)) = \mathbb{Z} + \mathbb{Z}\theta$.

Corollary 2.1.5 *$\mathcal{A}_\theta$ is isomorphic with $\mathcal{A}_\eta$ iff $\eta = \pm\theta$ mod $\mathbb{Z}$.*

2.1.1 The Smooth Algebra

We have already seen for each $a \in \mathcal{A}_\theta$, $z \in \mathbb{T}^2 \mapsto \alpha_z(a)$ is a continuous map from $\mathbb{T}^2$ to $\mathcal{A}_\theta$. Let

$$\mathcal{A}_\theta^\infty = \{a \in \mathcal{A}_\theta : z \mapsto \alpha_z(a) \text{ is } C^\infty\}.$$

Then by general results on C*-dynamical system ([63]) it follows that $\mathcal{A}_\theta^\infty$ is a dense subalgebra of $\mathcal{A}_\theta$ closed under holomorphic function calculus. Hence by Karoubi density theorem, $K_*(\mathcal{A}_\theta^\infty) = K_*(\mathcal{A}_\theta)$. We wish to construct spectral triples on this algebra, and for that a more concrete description will be helpful. Let $\mathcal{H} = L^2(\mathcal{A}_\theta, \tau)$ (see [60] for a discussion on noncommutative L^p-spaces). Then it is easy to see that the family $\{U^m V^n\}_{m,n \in \mathbb{Z}}$ constitutes a complete orthonormal basis of $\mathcal{H}$. Hence we have an unitary isomorphism between $\mathcal{H}$ and $l^2(\mathbb{Z}^2)$. Using this identification we can expand every $a \in \mathcal{A}_\theta \subseteq L^2(\mathcal{A}_\theta, \tau)$ as $a = \sum_{m,n \in \mathbb{Z}} a_{mn} U^m V^n$.

Such an expansion will often be referred as the Fourier expansion of a. Let δ_1, δ_2 be the derivations coming from the $\mathbb{T}^2$ action, i.e.,

$$\begin{aligned} \delta_1(a) &= \lim_{t \to 0} \frac{\alpha_{e^{2\pi it},1}(a) - a}{t}, \\ \delta_2(a) &= \lim_{t \to 0} \frac{\alpha_{1,e^{2\pi it}}(a) - a}{t}. \end{aligned}$$

A simple calculation shows on monomials of the form $U^m V^n$, these derivations are given by $\delta_1(U^m V^n) = mU^m V^n$, and $\delta_2(U^m V^n) = nU^m V^n$. Let d_1, d_2 be the induced selfadjoint operators on $\mathcal{H}$, which act on the monomials as the corresponding δ_i's with respective domains

$$\begin{aligned} Dom(d_1) &= \{\sum a_{mn} U^m V^n : \sum (1 + m^2)|a_{mn}|^2 \infty\}, \\ Dom(d_2) &= \{\sum a_{mn} U^m V^n : \sum (1 + n^2)|a_{mn}|^2 \infty\}. \end{aligned}$$

Clearly $a \in \mathcal{A}_\theta^\infty$ if and only if $a \in Dom(\delta_1^k \delta_2^l), \forall k, l \geq 0$, and this yields the alternative description:

$$\mathcal{A}_\theta^\infty = \{\sum a_{mn} U^m V^n : \sup_{m,n} |m^k n^l a_{mn}| < \infty, \forall k, l \in \mathbb{N}\}.$$

The following result of Bratteli, Eliott and Jorgensen ([9]) describes the space of derivations of $\mathcal{A}_\theta^\infty$.

Theorem 2.1.6 *For almost all θ (Lebesgue) the derivations of $\mathcal{A}_\theta^\infty$ to itself are of the form $c_1\delta_1 + c_2\delta_2 + [r, \cdot]$ for some $c_1, c_2 \in \mathbb{C}$ and $r \in \mathcal{A}_\theta^\infty$.*

We will denote the derivation $a \mapsto [r, a]$ for some $r \in \mathcal{A}_\theta^\infty$ by δ_r. The bounded operator induced on $L^2(\mathcal{A}_\theta^\infty, \tau)$ will be denoted by d_r.

2.2 Noncommutative Laplacian and Weyl Asymptotics for $\mathcal{A}_\theta$

For classical compact Riemannian manifold (M, g) of dimension d with metric g, one has the natural heat semigroup $\mathcal{T}_t$ as the expectation semigroup of the Brownian motion on the manifold ([80]) so that the Laplace-Beltrami operator Δ is the generator of $\mathcal{T}_t$. It is known ([80]) that $\mathcal{T}_t$ is an integral operator on $L^2(M, dvol)$ with a smooth integral kernel $\mathcal{T}_t(x, y)$, which admits an asymptotic expansion as $t \to 0+$:

$$\mathcal{T}_t(x, y) = \sum_{j=0}^{\infty} \mathcal{T}^{(j)}(x, y) t^{-d/2+j}, \tag{2.2.1}$$

and that

$$\begin{aligned} vol(M) &= \int_M \mathcal{T}^0(x,x)dvol(x) \\ &= \lim_{t\to 0+} t^{d/2} \int_M \mathcal{T}_t(x,x)dvol(x) \\ &= \lim_{t\to 0+} t^{d/2}(tr\mathcal{T}_t), \end{aligned}$$

where we have taken the trace in $L^2(M, dvol)$. Similarly the scalar curvature s at $x \in M$ is given as $s(x) = \frac{1}{6}\mathcal{T}^{(1)}(x,x)$. This gives the integrated scalar curvature

$$\begin{aligned} s &= \int_M s(x)dvol(x) = \frac{1}{6}\int_M \mathcal{T}^{(1)}(x,x)dvol(x) \\ &= \frac{1}{6}\lim_{t\to 0+} t^{d/2-1}\int[\mathcal{T}_t(x,x) - t^{-d/2}\mathcal{T}^0(x,x)]dvol(x) \\ &= \frac{1}{6}\lim_{t\to 0+} t^{d/2-1}[tr\mathcal{T}_t - t^{-d/2}vol(M)] \end{aligned}$$

In the noncommutative case one possibility is to define volume V and integrated scalar curvature s by analogy from their classical counterparts as :

$$V(\mathcal{A}_\theta) \equiv V \equiv \lim_{t\to 0+} t^{d/2} tr\mathcal{T}_t, \tag{2.2.2}$$

$$s(\mathcal{A}_\theta) \equiv s \equiv \frac{1}{6}\lim_{t\to 0+} t^{d/2-1}[tr\mathcal{T}_t - t^{-d/2}V] \tag{2.2.3}$$

where the heat semigroup $\mathcal{T}_t$ in the classical case is replaced by some noncommutative generalization of heat semigroup. A good candidate would be completely positive semigroups. The question about which of these semigroups have 'local' generators $\mathcal{L}$ remains open, though Sauvageot studied these in [81]. Following these studies, we know that $\mathcal{L}$ is characterized by:
(i) $\mathcal{D} \subseteq Dom(\mathcal{L}) \subseteq \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, dense in $\mathcal{A}$ such that $\mathcal{D}$ itself is a $*$-algebra,
(ii) a $*$-representation π in some Hilbert space $\mathfrak{h}$ and an associated π derivation δ such that $\delta(x) \in \mathcal{B}(\mathcal{H}, \mathfrak{h})$ and $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$,
(iii) a second order cocycle relation : $\mathcal{L}(x^*y) - \mathcal{L}(x)^*y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y)$, for $x, y \in \mathcal{D}$.
In analogy with the heat semigroup in the case of classical diffusion, we shall call $\mathcal{L}$ the noncommutative **Laplacian** or **Lindbladian**. Hudson and Robinson ([43]) studied the above question for $\mathcal{A}_\theta$ in the case where the representation π is the identity representation in $\mathcal{H}$ itself. We claim that if we choose $\pi(x) = x \otimes I$ in $\mathfrak{h} = \mathcal{H} \otimes C^2 \cong \mathcal{H} \oplus \mathcal{H}$, and $\delta_0 = \delta_1 \oplus \delta_2$, then $\mathcal{L}_0 = -\frac{1}{2}({\delta_1}^2 + {\delta_2}^2), \mathcal{D} = {\mathcal{A}_\theta}^\infty$ satisfies all the properties (i) - (iii). In analogy, one can have the perturbed triple $(\pi, \delta', \mathcal{L})$ where $\delta' = \delta'_1 \oplus \delta'_2$ with $\delta'_1 = \delta_1 + [r_1, \cdot]$ and $\delta'_2 = \delta_2 + [r_2, \cdot]$ and $\mathcal{L} = -\frac{1}{2}({\delta_1'}^2 + {\delta_2'}^2), \mathcal{D} = {\mathcal{A}_\theta}^\infty$. Thus we have two triples $(\pi, \delta_0, \mathcal{L}_0)$ and $(\pi, \delta', \mathcal{L})$ both

satisfying (i)-(iii). Then the question arises: can we associate the same geometric features with these two Laplacians or are there geometrically discernible changes as we go from the Laplacian $\mathcal{L}_0$ to the perturbed one $\mathcal{L}$? This will be addressed in this section. Before we proceed further we need to study the operators $\mathcal{L}_0$ and $\mathcal{L}$ in $L^2(\tau)$ more carefully. The next theorem summarizes their properties for $d = 2$ and we have denoted by $\mathcal{B}_p$ the Schatten ideals in $\mathcal{B}(\mathcal{H})$ with the respective norms.

Theorem 2.2.1 *(i) $\mathcal{L}_0$ is a negative selfadjoint operator in $L^2(\tau)$ with compact resolvent. In fact $\mathcal{L}_0(U^mV^n) = -\frac{1}{2}(m^2+n^2)U^mV^n; m,n \in Z$ so that $(\mathcal{L}_0 - z)^{-1} \in \mathcal{B}_p(L^2(\tau))$ for $p > 1$ and $z \in \rho(\mathcal{L}_0)$.*

(ii) If $r_1, r_2 \in \mathcal{A}_\theta^\infty$ and are selfadjoint, then $\mathcal{L} = \mathcal{L}_0 + B + A$, where,

$$B = -\frac{1}{2}(d_{r_1}^2 + d_{r_2}^2 + d_{\delta_1(r_1)} + d_{\delta_2(r_2)}) \text{ and } A = -d_{r_1}d_1 - d_{r_2}d_2,$$

so that A is compact relative to $\mathcal{L}_0$ and $\mathcal{L}$ is selfadjoint on $\mathcal{D}(\mathcal{L}_0)$ with compact resolvent.

If $r_1, r_2 \in \mathcal{A}_\theta$, then $-\mathcal{L} = -\mathcal{L}_0 - B - A$ as quadratic form on $D((-\mathcal{L}_0)^{\frac{1}{2}})$ and

$$(-\mathcal{L} + n^2)^{-1} = (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(I + Z_n)^{-1}(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \tag{2.2.4}$$

where $Z_n = (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(B + A)(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}$, is compact for each n with

$$B = -\frac{1}{2}(d_{r_1}^2 + d_{r_2}^2),\ A = \frac{1}{2}(d_1 d_{r_1} + d_{r_1}d_1 + d_2 d_{r_2} + d_{r_2}d_2).$$

This defines $\mathcal{L}$ as a selfadjoint operator in $L^2(\tau)$ with compact resolvent. Furthermore, in both cases of (ii), the difference of resolvents $(\mathcal{L} - z)^{-1} - (\mathcal{L}_0 - z)^{-1}$ is a trace class operator for $z \in \rho(\mathcal{L}) \cap \rho(\mathcal{L}_0)$.

Proof: The proof of (i) is obvious and hence is omitted. For (ii) It is easy to verify that $\mathcal{L} = \mathcal{L}_0 + B + A$ on $\mathcal{A}_\theta^\infty$ and that $A(-\mathcal{L}_0 + n^2)^{-1}$ is compact for every $n = 1, 2, \ldots$. Therefore

$$(\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + n^2)^{-1} = (\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + 1)^{-1}(\mathcal{L}_0 + 1)(-\mathcal{L}_0 + n^2)^{-1} \to 0$$

in operator norm as $n \to \infty$. By the Kato-Rellich theorem ([68]), $\mathcal{L}$ is selfadjoint and since $(-\mathcal{L} + n^2)^{-1} = (-\mathcal{L}_0 + n^2)^{-1}[1 + (\mathcal{L}_0 - \mathcal{L})(-\mathcal{L}_0 + n^2)^{-1}]^{-1}$ for sufficiently large n, one also concludes that $\mathcal{L}$ has compact resolvent. Furthermore for $z \in \rho(\mathcal{L}) \cap \rho(\mathcal{L}_0)$, we have

$$(\mathcal{L} - z)^{-1} - (\mathcal{L}_0 - z)^{-1} = (\mathcal{L}_0 - z)^{-1}[1 + (\mathcal{L} - \mathcal{L}_0)(\mathcal{L}_0 - z)^{-1}]^{-1}(\mathcal{L}_0 - \mathcal{L})(\mathcal{L}_0 - z)^{-1}.$$

Since $(\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}$ is bounded, $(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3(L^2(\tau))$ and $(-\mathcal{L}_0 + z)^{-1} \in \mathcal{B}_{3/2}(L^2(\tau))$, it follows that $(\mathcal{L} - n^2)^{-1} - (\mathcal{L}_0 - n^2)^{-1}$ is trace class for $n = 1, 2, \ldots$ by the Holder inequality.

When $r_1, r_2 \in \mathcal{A}_\theta$, we cannot write the expression for $\mathcal{L}$ as above on $\mathcal{A}_\theta^\infty$, since r_1, r_2 may not be in the domain of the derivations d_1, d_2. For this reason, we need to define $-\mathcal{L}$ as the sum of quadratic forms and standard results as in [68] can be applied here. From the structure of B and A it is clear that Z_n is compact for each n and hence an identical reasoning as above would yield that $\|Z_n\| \to 0$ as $n \to \infty$ and therefore $(I + Z_n)^{-1} \in \mathcal{B}$ for sufficiently large n and the right hand side of (2.2.4) defines the operator $-\mathcal{L}$ associated with the quadratic form with $D((-\mathcal{L})^{\frac{1}{2}}) = D((-\mathcal{L}_0)^{\frac{1}{2}})$. Clearly

$$\begin{aligned}
&(-\mathcal{L} + n^2)^{-1} - (-\mathcal{L}_0 + n^2)^{-1} \\
&= -(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(I + Z_n)^{-1} Z_n (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \\
&= -(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(I + Z_n)^{-1}(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(B + A)(-\mathcal{L}_0 + n^2)^{-1}
\end{aligned}$$

for sufficiently large n and since

$$(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3, \quad (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} A (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3,$$

it is clear that $(\mathcal{L} - n^2)^{-1} - (\mathcal{L}_0 - n^2)^{-1}$ is trace class. □

The next theorem studies the effect of the perturbation from $\mathcal{L}_0$ to $\mathcal{L}$ on the volume and the integrated sectional curvature for $\mathcal{A}_\theta$.

Theorem 2.2.2 *(i) The volume V of $\mathcal{A}_\theta (d = 2)$ as defined in (2.2.2) is invariant under the perturbation from $\mathcal{L}_0$ to $\mathcal{L}$.*
(ii) The integrated scalar curvature for $r \in \mathcal{A}_\theta^\infty$, in general is not invariant under the above perturbation.

Proof: We need to compute $tr(e^{t\mathcal{L}} - e^{t\mathcal{L}_0})$. Note that if $r_1, r_2 \in \mathcal{A}_\theta^\infty$, then

$$e^{t\mathcal{L}} - e^{t\mathcal{L}_0} = -\int_0^t e^{(t-s)\mathcal{L}}(\mathcal{L} - \mathcal{L}_0) e^{s\mathcal{L}_0} ds$$

which on two iterations yields:

$$\begin{aligned}
&e^{t\mathcal{L}} - e^{t\mathcal{L}_0} \\
&= -\int_0^t e^{(t-s)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0) e^{s\mathcal{L}_0} ds + \int_0^t dt_1 e^{(t-t_1)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0) \int_0^{t_1} dt_2 e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0) e^{t_2\mathcal{L}_0} \\
&\quad - \int_0^t dt_1 e^{(t-t_1)\mathcal{L}}(\mathcal{L} - \mathcal{L}_0) \int_0^{t_1} dt_2 e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0) \int_0^{t_2} dt_3 e^{(t_2-t_3)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0) e^{t_3\mathcal{L}_0} \\
&\equiv I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{2.2.5}$$

For estimating the trace norms of these terms, we note that the $\mathcal{B}_p$-norm of $(\mathcal{L} - \mathcal{L}_0) e^{s\mathcal{L}_0}$ is estimated as

$$\begin{aligned}\|(\mathcal{L}-\mathcal{L}_0)e^{s\mathcal{L}_0}\|_p &= \|(B+A)e^{s\mathcal{L}_0}\|_p \leq \|B\|\|e^{s\mathcal{L}_0}\|_p + c_1(\|d_1e^{s\mathcal{L}_0}\|_p + \|d_2e^{s\mathcal{L}_0}\|_p)\\ &\leq c''(\|e^{s\mathcal{L}_0}\|_p + \|d_2e^{s\mathcal{L}_0}\|_p)\\ &\leq c'(s^{-p^{-1}} + s^{-p^{-1}-\frac{1}{2}}) \leq c\, s^{-p^{-1}-\frac{1}{2}}\end{aligned}$$

for constants c, c_1, c', c'' since we are interested only for the region $0 < s \leq t \leq 1$. Using Holder inequality for Schatten norms and the fact that

$$\|(\mathcal{L}-n^2)^{-1}\| \leq \|(\mathcal{L}_0-n^2)^{-1}[1+(\mathcal{L}-\mathcal{L}_0)(\mathcal{L}_0-n^2)^{-1}]^{-1}\| \leq \frac{2}{n^2}$$

for sufficiently large n. We get for the third term in (2.2.5)

$$\begin{aligned}&\|I_3(t)\|_1\\ &\leq 2\int_0^t dt_1 \int_0^{t_1} dt_2\|(\mathcal{L}-\mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}\|_{p_1}\int_0^{t_2} dt_3\|(\mathcal{L}-\mathcal{L}_0)e^{(t_2-t_3)\mathcal{L}_0}\|_{p_2}\|(\mathcal{L}-\mathcal{L}_0)e^{t_3\mathcal{L}_0}\|_{p_3}\\ &\leq c(p_1,p_2,p_3)\int_0^t t_1^{-\frac{1}{2}}dt_1 \to 0\end{aligned}$$

as $t \to 0$ where $p_1^{-1}+p_2^{-1}+p_3^{-1}=1$. A very similar estimate shows that

$$\|I_1(t)\|_1 \leq \int_0^t ds\|e^{(t-s)\mathcal{L}_0}\|_{p_1}\|(\mathcal{L}-\mathcal{L}_0)e^{s\mathcal{L}_0}\|_{p_2} \leq ct^{-\frac{1}{2}}$$

(with $p_2 > 2$ *and* $p_1^{-1}+p_2^{-1}=1$) and

$$\|I_2(t)\|_1 \leq \int_0^t dt_1\|e^{(t-t_1)\mathcal{L}_0}\|_{p_1}\int_0^{t_1} dt_2\|(\mathcal{L}-\mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}\|_{p_2}\|(\mathcal{L}-\mathcal{L}_0)e^{t_2\mathcal{L}_0}\|_{p_3} \leq c',$$

(with $p_1^{-1}+p_2^{-1}+p_3^{-1}=1$, in particular the choice $p_1=p_2=p_3=3$ will do) a constant independent of t. From this it follows that $\lim_{t\to 0+} t\, tr(e^{t\mathcal{L}}-e^{t\mathcal{L}_0})=0$ and thus the invariance of volume under perturbation.

In the case when $r_1, r_2 \in \mathcal{A}_\theta$ only, then $\mathcal{L}-\mathcal{L}_0 = B + d_1B_1 + d_2B_2 + B_1'd_1 + B_2'd_2$ where B, B_1, B_1', B_2, B_2' are bounded. Therefore a term like $e^{(t-s)\mathcal{L}_0}d_1B_1e^{s\mathcal{L}_0} = [e^{s\mathcal{L}_0}B_1^*d_1e^{(t-s)\mathcal{L}_0}]^*$ admits similar estimates as above and the same result follows.

(ii) From the expression (2.2.3) for the integrated scalar curvature s, we see that for $d=2$

$$s(\mathcal{L})-s(\mathcal{L}_0) = \frac{1}{6}\lim_{t\to 0+} tr(e^{t\mathcal{L}}-e^{t\mathcal{L}_0}) \tag{2.2.6}$$

if it exists, and conclude that the contribution to (2.2.6) from the term $I_3(t)$ vanishes as we have seen in (i).We claim that though $\|I_2(t)\|_1 \leq$ constant, $trI_2(t) \to 0$ as $t \to 0+$. In fact since the integrals in $I_2(t)$ converges in trace norm

$$trI_2(t) = \int_0^t dt_1\int_0^{t_1} dt_2 tr((\mathcal{L}-\mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L}-\mathcal{L}_0)e^{(t-t_1+t_2)\mathcal{L}_0})$$

and by a change of variable we have that $|trI_2(t)| \leq t\int_0^t \|(\mathcal{L}-\mathcal{L}_0)e^{s\mathcal{L}_0}(\mathcal{L}-\mathcal{L}_0)e^{(t-s)\mathcal{L}_0}\|_1 ds$ for $r \in \mathcal{A}_\theta^\infty$, the perturbation $(\mathcal{L}-\mathcal{L}_0)$ is of the form $b_0+b_1d_1+b_2d_2$ with $b_i \in \mathcal{B}(\mathcal{H})$ for $i=0,1,2$ and the Hilbert-Schmidt norm estimates are as follows:

$$\|(\mathcal{L}-\mathcal{L}_0)e^{s\mathcal{L}_0}\|_2 \leq \|b_0\|\|e^{s\mathcal{L}_0}\|_2 + \sqrt{2}(\|b_1\|+\|b_2\|)\|(-\mathcal{L}_0)^{\frac{1}{2}}e^{s\mathcal{L}_0}\|_2 \leq c(s^{-\frac{1}{2}}+s^{-\frac{3}{4}}).$$

Therefore,

$$|trI_2(t)| \leq ct\int_0^t (s^{-\frac{1}{2}}+s^{-\frac{3}{4}})((t-s)^{-\frac{1}{2}}+(t-s)^{-\frac{3}{4}})$$

and this clearly converges to zero as $t \to 0+$. This leaves only $I_1(t)$ contribution so that

$$6(s(\mathcal{L})-s(\mathcal{L}_0)) = -\lim_{t\to 0+} t\ tr((\mathcal{L}-\mathcal{L}_0)e^{t\mathcal{L}_0}).$$

As before we note that $(\mathcal{L}-\mathcal{L}_0)$ contains two kinds of terms:

$$B = -\frac{1}{2}(d_{r_1}^2+d_{r_2}^2),\ \text{ and } A = -\frac{1}{2}(d_{r_1}d_1+d_1d_{r_1}+d_{r_2}d_2+d_2d_{r_2}).$$

We show that the term $tr(Ae^{t\mathcal{L}_0})=0$ for all $t>0$. It suffices to show that $tr(d_rd_1e^{t\mathcal{L}_0})=0$ for $r \in \mathcal{A}_\theta^\infty$ and for this we note that

$$\begin{aligned} tr(d_rd_1e^{t\mathcal{L}_0}) &= \sum_{m,n} < U^mV^n, d_rd_1e^{t\mathcal{L}_0}(U^mV^n) > \\ &= \sum_{m,n} me^{-t/2(m^2+n^2)}\tau(V^{-n}U^{-m}d_r(U^mV^n)) \\ &= \sum_{m,n} me^{-t/2(m^2+n^2)}\tau(V^{-n}U^{-m}rU^mV^n - r) = 0 \end{aligned}$$

identically. This leaves only the contribution due to B. Thus

$$s(\mathcal{L})-s(\mathcal{L}_0) = \frac{1}{12}\lim_{t\to 0+} t\ tr((d_{r_1}^2+d_{r_2}^2)e^{t\mathcal{L}_0}), \tag{2.2.7}$$

if it exists. However since $\{ttr((d_{r_1}{}^2+d_{r_2}{}^2)e^{t\mathcal{L}_0})\}$ is bounded as $t \to 0+$, we can and will interpret the above limit as a special kind of Banach limit as in Connes [24], p.563.

$$\begin{aligned} s(\mathcal{L})-s(\mathcal{L}_0) &= \frac{1}{12}Lim_{t^{-1}\to\omega} ttr((d_{r_1}{}^2+d_{r_2}{}^2)e^{t\mathcal{L}_0}) \\ &= \frac{1}{12}tr_\omega((d_{r_1}{}^2+d_{r_2}{}^2)\hat{\mathcal{L}_0}^{-1}. \end{aligned} \tag{2.2.8}$$

The notation $\hat{\mathcal{L}_0}$ stands for the operator that agrees with $\mathcal{L}_0$ on $\ker(\mathcal{L}_0)^\perp$ and identity on $\ker(\mathcal{L}_0)$. In the following we show that in general the right hand side of (2.2.8) is strictly positive. For example set $r_1 = (U+U^{-1})$ and $r_2=0$, then $r_1, r_2 \in \mathcal{A}_\theta^\infty$, and

$$
\begin{aligned}
&6(s(\mathcal{L})-s(\mathcal{L}_0))\\
&= \frac{1}{2}Lim_{t^{-1}\to\omega} t\sum_{m,n} e^{-t/2(m^2+n^2)} < U^mV^n, d^2_{r_1}(U^mV^n) >\\
&= 2^{-1}Lim_{t^{-1}\to\omega} t\sum_{m,n} e^{-t/2(m^2+n^2)}\tau((1-\lambda^{-n})^2\lambda^{2n}U^2+(1-\lambda^n)^2\lambda^{-2n}U^{-2}+(2-\lambda^n-\lambda^{-n}))\\
&= 2^{-1}Lim_{t^{-1}\to\omega} t\left(2\sum_{m=1}^{\infty} e^{-m^2t/2}+1\right)\left(8\sum_{n=1}^{\infty} sin^2(\pi\theta n)e^{-n^2t/2}\right)
\end{aligned}
$$

Next note that for $0<t<2$,

$$
\begin{aligned}
\sqrt{t}\sum_{n=1}^{\infty} sin^2(\pi\theta n)e^{-n^2t/2} &\geq \sqrt{t}\sum_{n=1}^{[\sqrt{2/t}]} sin^2(\pi\theta n)e^{-n^2t/2}\\
&\geq e^{-1}(\sqrt{2}-\sqrt{t})\sum_{n=1}^{[\sqrt{2/t}]}[\sqrt{2/t}]^{-1}sin^2\pi(n\theta-[n\theta])\\
&= e^{-1}(\sqrt{2}-\sqrt{t})E(sin^2\pi X_t),
\end{aligned}
$$

where for each $0<t\leq 2$, X_t is a $[0,1]$-valued random variable with Probability$(X_t = k\theta-[k\theta]) = [\sqrt{2/t}]^{-1}$ for $k=1,2,\ldots,[\sqrt{\frac{2}{t}}]$ and E is the associated expectation. Since θ is irrational, it is known ([42]) that as $t\to 0+$, the random variable X_t converges weakly to one with uniform distribution on $[0,1]$ and therefore,

$$
\begin{aligned}
\liminf_{t\to 0+}\sqrt{t}\sum_{n=1}^{\infty} sin^2(\pi\theta n)e^{-n^2t/2} &\geq \lim_{t\to 0+}\sqrt{t}\sum_{n=1}^{[\sqrt{2/t}]} sin^2(\pi\theta n)e^{-n^2t/2}\\
&\geq \sqrt{2}e^{-1}\int_0^1 sin^2\pi x dx\\
&= (\sqrt{2}e)^{-1}.
\end{aligned}
$$

We also have by Connes (page 563, [24]) $\lim_{t\to 0+}\sqrt{t}\sum_{m=1}^{\infty} e^{-m^2t/2} = \frac{\sqrt{\pi}}{\sqrt{2}}$. Now by the general properties of the limiting procedure as expounded in [24],

$$
s(\mathcal{L})-s(\mathcal{L}_0)\geq \frac{2\sqrt{\pi}}{3e}.
$$

□

Remark 2.2.3 From the expression for $s(\mathcal{L}_0)$, we see that for $d=2$, $s(\mathcal{L}_0)=\lim_{t\to 0+}(tre^{t\mathcal{L}_0}-\frac{V}{t})$. Since the expression for $tre^{t\mathcal{L}_0}$and the volume V are exactly the same as in the case of classical two-torus with its heat semigroup, the integrated scalar curvature for $\mathcal{L}_0$ is the same as in the classical case, which is clearly zero. Therefore $s(\mathcal{L})$ is strictly positive for the case considered here.

2.3 Spectral Triple on $\mathcal{A}_\theta^\infty$, its perturbation and Associated Calculus

Following Connes ([24]) we consider the even spectral triple $(\mathcal{A} = \mathcal{A}_\theta^\infty, \mathcal{H} = L^2(\tau)\oplus L^2(\tau), D_0, \Gamma)$ where D_0 , the unperturbed Dirac operator, $\begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix} \equiv i\gamma_1 d_1(a) + i\gamma_2 d_2(a)$ in $\mathcal{H}$. Here γ_1, γ_2 are the 2×2 clifford matrices. The selfadjoint grading operator is given by $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. One easily verifies that $a\Gamma = \Gamma a, \Gamma^* = \Gamma = \Gamma^{-1}, \Gamma D_0 = -D_0\Gamma$. Note also D_0 has compact resolvent since $D_0^2 = -2\begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & \mathcal{L}_0 \end{pmatrix}$ and $ker D_0 = ker\mathcal{L}_0 \otimes C^2$ is two dimensional. The perturbed spectral triple is taken to be $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ where $D = D_0 + \begin{pmatrix} 0 & d_r \\ d_{r^*} & 0 \end{pmatrix}$ for some $r \in \mathcal{A}_\theta^\infty$. It is not difficult to see that D_0 and D are both essentially selfadjoint on $\mathcal{A} \subseteq L^2(\tau)$ and that the perturbed triple is also an even one. Here, as in Connes ([24]), by the volume form $v(a)$ on $\mathcal{A}$ we mean the linear functional $v(a) = \frac{1}{2}tr_\omega(a|\hat{D}|^{-2}P)$ where tr_ω is the Dixmier trace and we have used the notation that for a selfadjoint operator T with compact resolvent, $\hat{T} = T|_{N(T)^\perp} \equiv TP$, where P is the projection on $N(T)^\perp$. Next we prove that the volume form is invariant under the above perturbation. For this we need a lemma.

Lemma 2.3.1 *Let T be a selfadjoint operator with compact resolvent such that $\hat{T}^{-1}$ is Dixmier traceable. Then for $a \in \mathcal{A}$ and every $z \in \rho(T)$, $tr_\omega(a\hat{T}^{-1}P) = tr_\omega(a(T-z)^{-1})$.*

Proof: Note that $(T-z)^{-1} = (\hat{T}-z)^{-1}P \oplus -z^{-1}P^\perp$ and $P^\perp$ is finite dimensional. Therefore $tr_\omega(a(T-z)^{-1}) = tr_\omega(PaP(\hat{T}-z)^{-1}P)$. On the other hand

$$tr_\omega(PaP\hat{T}^{-1}P - PaP(\hat{T}-z)^{-1}P) = -z tr_\omega(PaP\hat{T}^{-1}(\hat{T}-z)^{-1}P) = 0,$$

The last equality follows from proposition 1.5.7, since $\hat{T}^{-1}$ is Dixmier traceable and $(\hat{T}-z)^{-1}$ is compact. □

Theorem 2.3.2 *If we set $v_0(a) = \frac{1}{2}tr_\omega(a|\hat{D}_0|^{-2})$ and $v(a) = \frac{1}{2}tr_\omega(a|\hat{D}|^{-2})$ for $a \in \mathcal{A}$, then $v_0(a) = v(a)$.*

Proof: Note that $D^2 = -2\begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix}$, where

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{L}_0 + d_r d_{r^*} + (d_1 d_{r^*} + d_r d_1) + i(d_2 d_{r^*} - d_r d_2), \\ \mathcal{L}_2 &= \mathcal{L}_0 + d_{r^*} d_r + (d_1 d_r + d_{r^*} d_1) + i(d_2 d_{r^*} - d_r d_2), \end{aligned}$$

and by theorem 2.2.1, both $\mathcal{L}_1$ and $\mathcal{L}_2$ have compact resolvents with P_1, P_2 projections on $\mathcal{N}(\mathcal{L}_1)^\perp$ and $\mathcal{N}(\mathcal{L}_2)^\perp$ respectively. Therefore by the previous lemma for $Imz \neq 0$

$$\begin{aligned} v(a) &= tr_\omega(a(-\hat{\mathcal{L}}_1)^{-1}P_1) + tr_\omega(a(-\hat{\mathcal{L}}_2)^{-1}P_2) \\ &= tr_\omega(a(-\mathcal{L}_1 - z)^{-1} + a(-\mathcal{L}_2 - z)^{-1}) \\ &= tr_\omega(a(-\mathcal{L}_0 - z)^{-1} + a(-\mathcal{L}_0 - z)^{-1}) + tr_\omega(a(-\mathcal{L}_1 - z)^{-1} - a(-\mathcal{L}_0 - z)^{-1}) \\ &\quad + tr_\omega(a(-\mathcal{L}_2 - z)^{-1} - a(-\mathcal{L}_0 - z)^{-1}) \\ &= v_0(a) \end{aligned}$$

since $(-\mathcal{L}_i - z)^{-1} - (-\mathcal{L}_0 - z)^{-1}$ is trace class for $i = 1, 2$. □

We say that two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $D_2 = UD_1U^*$ and $\pi_2(.) = U\pi_1(.)U^*$, where π_j, $j = 1, 2$ are the representation of $\mathcal{A}_j$ in $\mathcal{H}_j$ respectively. Now, we want to prove that in general the perturbed spectral triple is not unitarily equivalent to the unperturbed one. Let $\Omega^1(\mathcal{A}_\theta^\infty)$ be the universal space of 1-forms and π be the representation of $\Omega^1 \equiv \Omega^1(\mathcal{A}_\theta^\infty)$ in $\mathcal{H}$ given by

$$\pi(a) = a, \quad \pi(\delta(a)) = [D, a],$$

where δ is the universal derivation.
Note that $[D, a] = i[\delta_1'(a)\gamma_1 + \delta_2'(a)\gamma_2]$, where $r_1 = \text{Re}\, r$, $r_2 = \text{Im}\, r$ $\delta_1' = \delta_1 + \delta_{r_1}, \delta_2' = \delta_2 + \delta_{r_2}$.

Theorem 2.3.3 *(i) Let $r = U^m$, then $\Omega_D^1(\mathcal{A}_\theta^\infty) := \pi(\Omega^1) = \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$.*
(ii) $\Omega^2(\mathcal{A}_\theta^\infty) = 0$ for $r = U^m$.

Proof: (i) Clearly $\pi(\Omega^1) \subseteq \mathcal{A}_\theta^\infty\gamma_1 + \mathcal{A}_\theta^\infty\gamma_2$. The other inclusion follows from the facts that $\delta_2'(U^k) = 0, \delta_1'(U^k)$ is invertible, and that $\delta_2(V^l)$ is invertible for sufficiently large l.
(ii) Let $J_1 = \ker \pi|_{\Omega^1}$, $J_2 = \ker \pi|_{\Omega^2}$. Then $J_2 + \delta J_1$ is an ideal, implying that $\pi(\delta J_1)$ is a nonzero submodule of $\pi(\Omega^2) \subseteq \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. Since $\mathcal{A}_\theta^\infty$ is simple there are two possibilities, namely either $\pi(\delta J_1) \cong \mathcal{A}_\theta^\infty$, or $\pi(\delta J_1) = \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. To rule out the first possibility we take a closer look at J_1 and $\pi(\delta J_1)$. $J_1 = \{\sum_i a_i\delta(b_i) | \sum_i a_i\delta_1'(b_i) = 0, \sum_i a_i\delta_2'(b_i) = 0\}$. Using the fact that δ_1', δ_2' are derivations we get

$$\sum_i \delta_1'(a_i)\delta_2'(b_i) = -\sum_i a_i\delta_1'(\delta_2'(b_i)) \tag{2.3.1}$$

$$\sum_i \delta_2'(a_i)\delta_1'(b_i) = -\sum_i a_i\delta_2'(\delta_1'(b_i)). \tag{2.3.2}$$

For $\sum_i a_i\delta(b_i) \in J_1$

$$\begin{aligned}\pi(\sum_i \delta(a_i)\delta(b_i)) &= \sum_i (\delta_1'(a_i)\gamma_1 + \delta_2'(a_i)\gamma_2)(\delta_1'(b_i)\gamma_1 + \delta_2'(b_i)\gamma_2)\\ &= \sum_i (\delta_1'(a_i)\delta_1'(b_i) + \delta_2'(a_i)\delta_2'(b_i)) + \sum (\delta_1'(a_i)\delta_2'(b_i) - \delta_2'(a_i)\delta_1'(b_i))\gamma_{12},\end{aligned}$$

where $\gamma_{12} = \gamma_1\gamma_2 = -\gamma_2\gamma_1$. Taking $x = U^{-1}\delta(U) + U\delta(U^{-1}) \in \Omega^1$ it is easy to verify that $x \in J_1$ and $\pi(\delta x) = -2$. This proves $\mathcal{A}_\theta^\infty \oplus 0 \subseteq \pi(\delta J_1)$. We show that the inclusion is proper by showing the nontriviality of the coefficient of γ_{12}. Using (2.3.1), (2.3.2) we get coefficient of γ_{12} to be $\sum a_i[\delta_1', \delta_2'](b_i) = \sum -ima_i[r_1, b_i]$. As before we can find n_0 such that for $l \geq n_0$, $\delta_2'(V^l)$ is invertible. If we now choose $a_1 = I, b_1 = V^{n_0}, a_2 = -\delta_2'(V^{n_0})\delta_2'(V^l)^{-1}, b_2 = V^l, a_3 = (-a_1\delta_1'(b_1) - a_2\delta_2'(b_2))U^{-1}, b_3 = U$, then the vanishing of the coefficient of γ_{12} will imply that $[r_1, V^{n_0}] = \delta_2'(V^{n_0})\delta_2'(V^l)^{-1}[r_1, V^l]$ for all $l \geq n_0$ and we note that while the left hand side is nonzero and independent of l, the right hand side converges to 0 as $l \to \infty$ leading to a contradiction. Therefore $\mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty = \pi(\delta J_1) \subseteq \pi(\Omega^2) \subseteq \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. Hence $\Omega_D^2(\mathcal{A}_\theta^\infty) = \frac{\pi(\Omega^2)}{\pi(\delta J_1)} = 0$. □

Thus we have the following:

Theorem 2.3.4 *The spectral triples $(\mathcal{A}_\theta^\infty, \mathcal{H}, D_0)$ and $(\mathcal{A}_\theta^\infty, \mathcal{H}, D)$ are not unitarily equivalent for $r = U^m$.*

The proof is clear since $\Omega_{D_0}^2(\mathcal{A}_\theta^\infty) = \mathcal{A}_\theta^\infty \neq 0 = \Omega_D^2(\mathcal{A}_\theta^\infty)$.

Classically there is a correspondence between connection form and covariant differentiation. This correspondence comes from the duality between the module of derivations and the module of sections in the cotangent bundle. Unfortunately there is no such duality in the non-commutative context. Here for defining the connection form we visualize it more as the connection form arising from covariant differentiation. We need to do so because if we take the existing definition ([36]) then the curvature form becomes trivial.

Let $\mathfrak{h}$ be the vector space of all derivations $d : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$. This space is same as $\{c_1\delta_1 + c_2\delta_2 + [r,.] : r \in \mathcal{A}_\theta^\infty\}$ for almost all θ (Lebesgue) ([9]). For the rest of this section we will be using those $\theta's$ only. Let δ_{mn} be the element of $\mathfrak{h}$ given by $\delta_{mn}(a) = [U^mV^n, a]$. We turn $\mathfrak{h}$ into an inner product space by requiring that $\{\delta_1, \delta_2, \delta_{mn}\}$ to be orthonormal, for example as in [46]. Let $\mathcal{E}$ be any normed $\mathcal{A}_\theta^\infty$-module. For $\delta \in \mathfrak{h}$, let $c_\delta : \mathcal{E} \otimes \mathfrak{h} \to \mathcal{E}$, be the contraction with respect to δ. Topologize $\mathcal{E} \otimes \mathfrak{h}$ with the weak topology inherited from $c_\delta, \delta \in \mathfrak{h}$. Then a *connection* is a complex-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathfrak{h}$ such that $c_\delta\nabla(\xi a) = c_\delta\nabla(\xi)a + \xi\delta(a), \ \forall \delta \in \mathfrak{h}$.

Theorem 2.3.5 *Suppose that ∇_1, ∇_2 are maps from $\mathcal{E}$ to $\mathcal{E}$ satisfying*

$$\nabla_i(\xi a) = \nabla_i(\xi)a + \xi\delta_i(a), \ \ i = 1, 2.$$

Then the map ∇ given by

$$\nabla(\xi) = \nabla_1 \otimes \delta_1 + \nabla_2 \otimes \delta_2 - \sum \xi U^m V^n \otimes \delta_{mn}$$

is well-defined and is a connection.

Proof: Let $\delta \in \mathfrak{h}$, such that $\delta = c_1\delta_1 + c_2\delta_2 + \sum c_{mn}\delta_{mn}$, where $\{c_{mn}\} \in \mathcal{S}(\mathbb{Z}^2) \subseteq l_1(\mathbb{Z}^2)$. Therefore the sum in the right hand side of the definition of ∇ converges in the topology referred above. The rest is straightforward. □

It is clear from the definition of ∇ in the above theorem that $\nabla_j = c_{\delta_j}\nabla$ for $(j = 1, 2)$. We also set $\nabla_r = c_{\delta_r}\nabla$ for $r \in \mathcal{A}_\theta^\infty$.

Definition 2.3.6 *Let $R : \mathfrak{h} \otimes \mathfrak{h} \to \mathcal{L}(\mathcal{E})$ be the map given by $R(\delta, \delta') = c_{[\delta,\delta']}\nabla - [c_\delta\nabla, c_{\delta'}\nabla]$. We call R the curvature 2-form associated with the connection ∇.*

Theorem 2.3.7 *We have*

$$R(\delta_1, \delta_2) = R(\delta_1 + \delta_{r_1}, \delta_2 + \delta_{r_2}).$$

Proof: $[\delta_1 + \delta_{r_1}, \delta_2 + \delta_{r_2}] = [\delta_1(r_2), \cdot] - [\delta_2(r_1), \cdot] + [[r_1, r_2], \cdot]$. So we have

$$\begin{aligned}
&R(\delta_1 + \delta_{r_1}, \delta_2 + \delta_{r_2})(\xi) \\
&= -\xi\delta_1(r_2) + \xi\delta_2(r_1) - \xi[r_1, r_2] - (\nabla_1 + \nabla_{r_1})(\nabla_2\xi - \xi r_2) + (\nabla_2 + \nabla_{r_2})(\nabla_1\xi - \xi r_1) \\
&= -[\nabla_1, \nabla_2]\xi + \nabla_1(\xi r_2) + (\nabla_2\xi)r_1 - \xi r_2 r_1 - \nabla_2(\xi r_1) \\
&\quad -(\nabla_1\xi)r_2 + \xi r_1 r_2 - \xi\delta_1(r_2) + \xi\delta_2(r_1) - \xi[r_1, r_2] \\
&= -[\nabla_1, \nabla_2]\xi = R(\delta_1, \delta_2)(\xi) \ \ (\text{since } \ [\delta_1, \delta_2] = 0).
\end{aligned}$$

□

Remark 2.3.8 In section 2, we have seen that the integrated scalar curvature under the perturbed Lindbladian is different from zero, whereas in section 3, the curvature 2-form has been shown to be invariant under the same perturbation.

2.4 Non-commutative $2d$-dimensional space

In this section we shall discuss the geometry of the simplest kind of noncompact manifolds, namely the Euclidean $2d$-dimensional space and its noncommutative counterpart. Let d be a positive integer and let $\mathcal{A}_c \equiv C_0(\mathbb{R}^{2d})$, the (nonunital) C^*-algebra of all complex-valued continuous functions on $\mathbb{R}^{2d}$ which vanish at infinity. Then $\partial_j (j = 1, 2, \ldots, 2d)$, the partial derivative in the j-th direction, can be viewed as a densely defined derivation on $\mathcal{A}_c$, with the domain $\mathcal{A}_c^\infty \equiv C_c^\infty(\mathbb{R}^{2d})$, the set of smooth complex valued functions on $\mathbb{R}^{2d}$ having compact support. We consider the Hilbert space $L^2(\mathbb{R}^{2d})$ and naturally imbed $\mathcal{A}_c^\infty$ in it as a dense subspace. Then $i\partial_j$ is a densely defined symmetric linear map on $L^2(\mathbb{R}^{2d})$ with domain $\mathcal{A}_c^\infty$, and we denote its self-adjoint extension by the same symbol. Also, let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^{2d})$ given by

$$\hat{f}(k) \equiv (\mathcal{F}f)(k) = (2\pi)^{-d} \int e^{-ik.x} f(x)dx,$$

and M_φ be the operator of multiplication by the function φ. We set $\widetilde{M_\varphi} = \mathcal{F}^{-1} M_\varphi \mathcal{F}$, thus $i\partial_j = \widetilde{M_{x_j}}$. $\Delta \equiv \widetilde{M}_{-\sum x_j^2}$ is the self-adjoint negative operator, called the $2d$-dimensional Laplacian. Clearly, the restriction of Δ on $\mathcal{A}_c^\infty$ is the differential operator $\sum_{j=1}^{2d} \partial_j^2$. Let $h = L^2(\mathbb{R}^d)$ and U_α, V_β be two strongly continuous groups of unitaries in h, given by the following:

$$(U_\alpha f)(t) = f(t + \alpha), \quad (V_\beta f)(t) = e^{it.\beta} f(t), \quad \alpha, \beta, t \in \mathbb{R}^d, \; f \in C_c^\infty(\mathbb{R}^d).$$

Here $t.\beta$ is the usual Euclidean inner product of $\mathbb{R}^d$. It is clear that

$$\begin{aligned} U_\alpha U_{\alpha'} &= U_{\alpha+\alpha'}, \\ V_\beta V_{\beta'} &= V_{\beta+\beta'}, \\ U_\alpha V_\beta &= e^{i\alpha.\beta} V_\beta U_\alpha. \end{aligned} \tag{2.4.1}$$

For convenience, we define a unitary operator W_x for $x = (\alpha, \beta) \in \mathbb{R}^{2d}$ by

$$W_x = U_\alpha V_\beta e^{-\frac{i}{2}\alpha.\beta},$$

so that the Weyl relation (2.4.1) is now replaced by $W_x W_y = W_{x+y} e^{\frac{i}{2}p(x,y)}$, where $p(x,y)$ is the symplectic form $p(x, y) = x_1.y_2 - x_2.y_1$, for $x = (x_1, x_2), y = (y_1, y_2)$. This is exactly the Segal form of the Weyl relation ([34]). For f such that $\hat{f} \in L^1(\mathbb{R}^{2d})$, we set

$$b(f) = \int_{\mathbb{R}^{2d}} \hat{f}(x) W_x dx \in \mathcal{B}(h).$$

Let $\mathcal{A}^\infty$ be the $*$-algebra generated by $\{b(f)|f \in C_c^\infty(\mathbb{R}^{2d})\}$ and let $\mathcal{A}$ be the C^*-algebra generated by $\mathcal{A}^\infty$ with the norm inherited from $\mathcal{B}(h)$. It is easy to verify using the commutation relation (2.4.1) that $b(f)b(g) = b(f \odot g)$ and $b(f)^* = b(f^\natural)$, where

$$\widehat{(f \odot g)}(x) = \int \hat{f}(x - x')\hat{g}(x')e^{\frac{i}{2}p(x,x')}dx'; \quad f^\natural(x) = \bar{f}(-x).$$

We define a linear functional τ on $\mathcal{A}^\infty$ by setting $\tau((b(f)) = \hat{f}(0)$ $(= (2\pi)^{-d}\int f(x)dx)$, and easily verify ([34], page 36) that it is a well-defined faithful trace on $\mathcal{A}^\infty$. It is natural to consider $\mathcal{H} = L^2(\mathcal{A}^\infty, \tau)$ and represent $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ by left multiplication. From the definition of τ, it is clear that the map $C_c^\infty(\mathbb{R}^{2d}) \ni f \mapsto b(f) \in \mathcal{A}^\infty \subseteq \mathcal{H}$ extends to a unitary isomorphism from $L^2(\mathbb{R}^{2d})$ onto $\mathcal{H}$ and in the sequel we shall often identify the two.

There is a canonical $2d$-paramater group of automorphism of $\mathcal{A}$ given by $\varphi_\alpha(b(f)) = b(f_\alpha)$, where $\hat{f}_\alpha(x) = e^{i\alpha.x}\hat{f}(x)$, $f \in C_c^\infty(\mathbb{R}^{2d})$, $\alpha \in \mathbb{R}^{2d}$. Clearly, for any fixed $b(f) \in \mathcal{A}^\infty$, $\alpha \mapsto \varphi_\alpha(b(f))$ is smooth, and on differentiating this map at $\alpha = 0$, we get the canonical derivations $\delta_j, j = 1, 2, \ldots, 2d$ as $\delta_j(b(f)) = b(\partial_j(f))$ for $f \in C_c^\infty(\mathbb{R}^{2d})$. We shall not notationally distinguish between the derivation δ_j on $\mathcal{A}^\infty$ and its extension to $\mathcal{H}$, and continue to denote by $i\delta_j$ both the derivation on $*$-algebra $\mathcal{A}^\infty$ and the associated self-adjoint operator on $\mathcal{H}$.

Let us now go back to the classical case. As a Riemannian manifold, $\mathbb{R}^{2d}$ does not posses too many interesting features; it is a flat manifold and thus there is no nontrivial curvature form. Instead, we shall be interested in obtaining the volume form from the operator-theoretic data associated with the $2d$-dimensional Laplacian Δ. Let $\mathcal{T}_t = e^{\frac{t}{2}\Delta}$ be the contractive C_0-semigroup generated by Δ, called the heat semigroup on $\mathbb{R}^{2d}$. Unlike compact manifolds, Δ has only absolutely continuous spectrum. But for any $f \in C_c^\infty(\mathbb{R}^{2d})$ and $\epsilon > 0$, $M_f(-\Delta+\epsilon)^{-d}$ has discrete spectrum. Furthermore, we have the following :

Theorem 2.4.1 *$M_f\mathcal{T}_t$ is trace-class and $tr(M_f\mathcal{T}_t) = t^{-d}\int f(x)dx$. Thus, in particular, $v(f) \equiv \int f(x)dx = t^d tr(M_f\mathcal{T}_t)$.*

Proof: We have $tr(M_f\mathcal{T}_t) = tr(\mathcal{F}M_f\mathcal{F}^{-1}M_{e^{-\frac{t}{2}\sum x_j^2}})$, and $\mathcal{F}M_f\mathcal{F}^{-1}M_{e^{-\frac{t}{2}\sum x_j^2}}$ is an integral operator with the kernel $k_t(x,y) = \hat{f}(x-y)e^{-\frac{t}{2}\sum y_j^2}$. It is continuous in both arguments and $\int |k_t(x,x)|dx < \infty$, we obtain by using a result in [40], (p. 114, ch.3) that $M_f\mathcal{T}_t$ is trace class and $tr(M_f\mathcal{T}_t) = \int k_t(x,x)dx = (2\pi)^d t^{-d}\hat{f}(0) = t^{-d}v(f)$. □

As in section 4, we get an alternative expression for the volume form v in terms of the Dixmier trace.

Theorem 2.4.2 *For $\epsilon > 0$, $M_f(-\Delta+\epsilon)^{-d}$ is of Dixmier trace class and its Dixmier trace is equal to $\pi^d v(f)$.*

For convenience, we shall give the proof only in the case $d = 1$. We need following two lemmas.

Lemma 2.4.3 *If $f, g \in L^p(\mathbb{R}^2)$ for some p with $2 \le p < \infty$, then $M_f\widetilde{M_g}$ is a compact operator in $L^2(\mathbb{R}^2)$.*

Proof: It is a consequence of the Holder and Hausdorff-Young inequalities. We refer to [69] for a proof. □

Lemma 2.4.4 *Let S be a square in $\mathbb{R}^2$ and f be a smooth function with* Supp$(f) \subseteq$ int(S). *Let Δ_S denote the Laplacian on S with the periodic boundary condition. Then $tr_\omega(M_f(-\Delta_S+\epsilon)^{-1}) = \pi\int f(x)dx$.*

Proof: This follows from [55] by identifying S with the two-dimensional torus in the natural manner. □

Proof of the theorem: Note that for $g \in \mathcal{D}(\Delta) \subseteq L^2(\mathbb{R}^2)$, we have $fg \in \mathcal{D}(\Delta_S)$ and $(\Delta_S M_f - M_f\Delta)(g) = (\Delta M_f - M_f\Delta)(g) = Bg$, where $B = -M_{\Delta f} + 2i\sum_{j=1}^{2} M_{\partial_j(f)} \circ \partial_j$. From this follows the identity

$$M_f(-\Delta+\epsilon)^{-1} - (-\Delta_S+\epsilon)^{-1}M_f \;=\; (-\Delta_S+\epsilon)^{-1}B(-\Delta+\epsilon)^{-1}. \qquad (2.4.2)$$

Now, from Lemma 2.4.3, it follows that $B(-\Delta+\epsilon)^{-1}$ is compact, and since $(-\Delta_S+\epsilon)^{-1}$ is of Dixmier trace class (by Lemma 2.4.4), we have that the right hand side of (2.4.2) is of Dixmier trace class with Dixmier trace zero. The theorem follows from the general fact that $tr_\omega(xy) = tr_\omega(yx)$, if y is of Dixmier trace class and x is bounded. □

Similar computation can be done for the non-commutative case. The Lindbladian $\mathcal{L}_0$ generated by the canonical derivation δ_j on $\mathcal{A}$ is given by

$$\mathcal{L}_0(a(f)) = \frac{1}{2}a(\Delta f),\ f \in C_c^\infty(R^{2d}). \qquad (2.4.3)$$

Since in $L^2(R^{2d})$, $\frac{1}{2}\Delta$ has a natural selfadjoint extension (which we continue to express by the same symbol), $\mathcal{L}_0$ also has an extension as a negative selfadjoint operator in $\mathcal{H} \cong L^2(R^{2d})$, and we define the heat semigroup for this case as $\mathcal{T}_t = e^{t\mathcal{L}_0}$. By analogy we can define the volume form on $\mathcal{A}^\infty$ by setting $v(a(f)) = \lim_{t\to 0+} t^d tr(a(f)\mathcal{T}_t)$. Then we have

Theorem 2.4.5 $v(a(f)) = \int f dx$.

Proof: The kernel $\tilde{K}_t$ of the integral operator $a(f)\mathcal{T}_t$ in $\mathcal{H}$ is given as $\tilde{K}_t(x,y) = \hat{f}(\underline{x} - \underline{y})e^{-t|y|^2/2}e^{ip(x,y)/2}$. As before we note that K_t is continuous in R^{2d} and $\tilde{K}_t(x,x) = k_t(x,x) = \hat{f}(0)e^{-t|x|^2/2}$. Using [40] we get the required result. □

Remark 2.4.6 (i) Note that in the theorem 2.4.2, $tr_\omega(M_f(-\Delta + \epsilon)^{-d}) = \pi^d v(f)$ which is independent of $\epsilon > 0$. This could also have been arrived at directly as in section 4 for the algebra $\mathcal{A}_\theta$ once we have observed in the proof of the theorem that $tr_\omega M_f(\Delta - \epsilon)^{-1} = tr_\omega M_f(\Delta_S - \epsilon)^{-1}$.

Chapter 3

The Quantum Heisenberg Manifold

Let $G = \{\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}\}$ be the Heisenberg group. For a positive integer c, let H_c be the subgroup of G obtained when x, y, cz are integers. The Heisenberg manifold M_c is the quotient G/H_c. Nonzero Poisson brackets on M_c invariant under left translation by G are parametrized by two real parameters μ, ν with $\mu^2 + \nu^2 \neq 0$ ([73]). For each positive integer c and real numbers μ, ν, Rieffel constructed a C^*-algebra $A_{\mu,\nu}^{c,\hbar}$ as an example of deformation quantization along a Poisson bracket ([73]). These algebras have further been studied in [1], [2], and [83]. It was also remarked in [73] that it should be possible to construct example of non-commutative geometry as expounded in [24] in these algebras also. It is known ([73]) that Heisenberg group acts ergodically on $A_{\mu,\nu}^{c,\hbar}$ and $A_{\mu,\nu}^{c,\hbar}$ accommodates a unique invariant tracial state τ. Using the group action we construct a family of spectral triples. It is shown that they induce same element in K-homology. We also show that the associated Kasparov module is non-trivial. This has been achieved by constructing explicitly the pairing with a unitary. We also compute the space of forms as described in [24],[36]. Then we characterize torsionless and unitary connections. As an immediate corollary it follows that a torsionless unitary connection can not exist. For a family of unitary connections we compute Ricci curvature and scalar curvature as introduced in [36]. This family has non-trivial curvature.

3.1 Generalities on Deformation Quantization

Rieffel introduced the notion of deformation quantization to give analytical meaning to formal deformation quantization. This section is devoted to a discussion of that and a description of quantum Heisenberg manifold and its basic properties.

Let M be a compact C^∞-manifold, and let $C^\infty(M)$ be the associative algebra of C^∞ complex valued functions on M, with pointwise multiplication and involution by complex conjugation. By a Poisson bracket on M is meant a Lie algebra structure $\{\cdot,\cdot\}$ on the linear space $C^\infty(M)$, such that for every $f \in C^\infty(M)$ the linear map $g \mapsto \{f,g\}$ from $C^\infty(M)$ to itself is a derivation. We also require $\{\cdot,\cdot\}$ to be real in the sense that $\{f^*,g^*\} = \{f,g\}^*$. Let TM be the tangent bundle of M. Then to give a Poisson structure is same as to give a skew 2-vector field Λ on M, i.e., a cross section of $\wedge^2 TM$, such that if we set

$$\{f,g\} = \langle \Lambda, df \wedge dg \rangle,$$

then $\{\cdot,\cdot\}$ satisfies the Jacobi identity.

Definition 3.1.1 (Rieffel) Let $\wedge$ and Λ be as above. By a strict deformation quantization of $\mathcal{A} = C^\infty(M)$, in the direction of Λ, we will mean an open interval J of real numbers containing zero, together with, for each $\hbar \in J$, an associative product $\star_\hbar$, an involution $^{*_\hbar}$, and a C^*-norm $\|\cdot\|_\hbar$ (for $\star_\hbar$ and $^{*_\hbar}$) on $\mathcal{A}$, which for $\hbar = 0$ are the original pointwise product, complex conjugation involution, and supremum norm, such that
(i) for every $f \in \mathcal{A}$, the function $\hbar \mapsto \|f\|_\hbar$ is continuous;
(ii) for every $f,g \in \mathcal{A}$, $\|(f \star_\hbar g - g \star_\hbar f)/\hbar - \{f,g\}\|_\hbar$ converges to zero as $\hbar$ goes to zero.

It is condition (ii) which formalizes the idea that the deformation is "in the direction of Λ". Since this condition is essentially an infinitesimal condition at zero, one does not expect strict deformation quantization for a given Λ to be unique, and indeed as it has been observed by Rieffel himself it is not so.

Definition 3.1.2 Let G be a Lie group, and let α be an action of G as a group of diffeomorphisms of M which preserve the Poisson structure. Consequently α induces an action on $\mathcal{A}$. We will say that a strict deformation quantization of $\mathcal{A}$, as defined above, is invariant under the action α if
(i) for every $\hbar \in J$, and $x \in G$, the operator α_x on $\mathcal{A}$ is an isometric $*$-automorphism for $\star_\hbar,^{*_\hbar}$ and $\|\cdot\|_\hbar$;
(ii) for every $f \in \mathcal{A}$ and $\hbar \in J$, the map $x \mapsto \alpha_x(f)$ is a C^∞ function on G, for the norm $\|\cdot\|_\hbar$;
(iii) there is an isometric action α, of the Lie algebra $\mathfrak{g}$ of G on $\mathcal{A}$, which for each $\hbar \in J$ is by $*$-derivations of $\mathcal{A}$ for $\star_\hbar$ and $^{*_\hbar}$, such that for $X \in \mathfrak{g}$ and $f \in \mathcal{A}$,

$$\alpha_X(f) = \frac{d}{dt}|_{t=0}\alpha_{exp(tX)}(f)$$

with respect to $\|\cdot\|_\hbar$.

3.1.1 Quantum Heisenberg Manifold, as Deformation of Heisenberg Manifolds

The Heisenberg group is the group of 3×3 upper triangular matrices with 1's on the diagonal. If we identify G with $\mathbb{R}^3$ by identifying $\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$ with (x,y,z), then G is $\mathbb{R}^3$ with product given by,

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+yx').$$

For any positive integer c, let D_c denote the discrete subgroup of G consisting of those (x,y,z) for which x, y and cz are integers. The corresponding Heisenberg manifold is $M_c = G/D_c$, on which G acts on the left. Clearly G invariant Poisson structures on M_c correspond to the Poisson structures Λ on G which are invariant under left translation by G and right translation by D_c. By the G invariance of Λ, it will be determined by its value at the identity of G, and so is given by an element, say Λ again, of $\wedge^2\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Then right translation invariance under D_c will be equivalent to invariance of Λ under the restriction to D_c of the adjoint representation of G on $\wedge^2\mathfrak{g}$, induced from the adjoint representation on $\mathfrak{g}$. Let X, Y, Z be the basis of $\mathfrak{g}$ given by,

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that $[Y,X] = Z$. Then any G-invariant $\Lambda \in \wedge^2\mathfrak{g}$ is of the form

$$\Lambda = \mu X \wedge Z + \nu Y \wedge Z + \rho X \wedge Y,$$

for $\mu, \nu, \rho \in \mathbb{R}$. Let x and y denote elements $(1,0,0)$ and $(0,1,0)$ of D_c, then simple calculations show that

$$Ad_x(X) = X, \qquad Ad_x(Y) = Y + Z,$$
$$Ad_y(X) = X - Z, \qquad Ad_y(Y) = Y.$$

From this and the fact that Z is central, it is easily seen that Λ is D_c invariant iff $\rho = 0$, so that

$$\Lambda = (\mu X + \nu Y) \wedge Z.$$

Quantum Heisenberg manifolds are strict deformation quantization of Heisenberg manifolds in the direction of Λ. We will throughout assume $\mu, \nu \neq 0$.

For $x \in \mathbb{R}$, we will denote $e^{2\pi i x}$ by $e(x)$.

Definition 3.1.3 For any positive integer c, let S^c denote the space of infinitely differentiable functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ that satisfy the following two conditions:
a) $\Phi(x+k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in \mathbb{Z}$, and
b) for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\widetilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$, the function $P(p)(\widetilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset K of $\mathbb{R} \times \mathbb{T}$.
For each $\hbar, \mu, \nu \in \mathbb{R}, \mu\nu \neq 0$, let $\mathcal{A}_\hbar^\infty$ denote the space S^c equipped with product and involution defined respectively by

$$(\Phi \star \Psi)(x, y, p) = \sum_q \Phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\Psi(x - \hbar q\mu, y - \hbar q\nu, p - q), \quad (3.1.1)$$

$$\Phi^*(x, y, p) = \bar{\Phi}(x, y, -p). \quad (3.1.2)$$

Let π be the representation of $\mathcal{A}_\hbar^\infty$ on $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ given by

$$(\pi(\Phi)\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)\xi(x, y, p-q). \quad (3.1.3)$$

Then π gives a faithful representation of the involutive algebra $\mathcal{A}_\hbar^\infty$, hence a norm on it. Via the partial Fourier transform in the last variable, $\{\mathcal{A}_\hbar^\infty\}$ is a strict deformation quantization of the Heisenberg manifold $C^\infty(M_c)$ in the direction of the Poisson structure given by,

$$\Lambda = -\pi^{-1}(\mu X + \nu Y) \wedge Z.$$

The norm closure of $\pi(\mathcal{A}_\hbar^\infty)$, to be denoted by $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ is called the Quantum Heisenberg Manifold. Let $N_\hbar$ denote the weak closure of $\pi(\mathcal{A}_\hbar^\infty)$.

We will identify $\mathcal{A}_\hbar^\infty$ with $\pi(\mathcal{A}_\hbar^\infty)$ without any mention. Since we are going to work with fixed parameters $c, \mu, \nu, \hbar$ we will drop them altogether and denote $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ simply by $\mathcal{A}_\hbar$, the subscript merely distinguishes the Heisenberg algebra from a general algebra.

Action of the Heisenberg group: Heisenberg group acts ergodically on these algebras. If we identify the Heisenberg group with $\mathbb{R}^3$, then for $\Phi \in S^c$, $(r, s, t) \in \mathbb{R}^3$, the action is given by,

$$(L_{(r,s,t)}\phi)(x, y, p) = e(p(t + cs(x-r)))\phi(x-r, y-s, p) \quad (3.1.4)$$

In fact this action is implemented by the unitary operator $U_{(r,s,t)}$ on $L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$, explicitly given by,

$$(U_{(r,s,t)}\xi)(x,y,p) = e(p(t+cs(x+\hbar p\mu - r))\xi(x-r,y-s,p).$$

The Trace: $\tau : \mathcal{A}_\hbar^\infty \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x,y,0)dxdy$ extends to a faithful normal tracial state on $N_\hbar$. τ is invariant under the Heisenberg group action. So, the group action can be lifted to $L^2(\mathcal{A}_\hbar^\infty)$. We will denote the action at the Hilbert space level by the same symbol.

Theorem 3.1.4 (Weaver) *Let $\mathcal{H} = L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$ and V_f, W_k, X_r be the operators defined by*

$$\begin{aligned}(V_f\xi)(x,y,p) &= f(x,y)\xi(x,y,p),\\ (W_k\xi)(x,y,p) &= e(-ck(p^2\hbar\nu + py))\xi(x+k,y,p),\\ (X_r\xi)(x,y,p) &= \xi(x-2\hbar r\mu, y-2\hbar r\nu, p+r).\end{aligned}$$

Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in N_\hbar$ iff T commutes with the operators V_f, W_k, X_r for all f in $L^\infty(\mathbb{R}\times\mathbb{T})$, and $k, r \in \mathbb{Z}$.

Lemma 3.1.5 *Let $S^c_{\infty,\infty,1}$ be the space of all functions $\psi : \mathbb{R}\times\mathbb{T}\times\mathbb{Z} \to \mathbb{C}$ satisfying the following three conditions (i) ψ is measurable, (ii) $\psi_n = \sup_{x\in\mathbb{R},y\in\mathbb{T}}|\psi(x,y,p)|$ is an l_1 sequence, and (iii) $\psi(x+k,y,p) = e(ckyp)\psi(x,y,p)$ for all $k \in \mathbb{Z}\}$. Then for $\phi \in S^c_{\infty,\infty,1}$, $\pi(\phi)$ defined by (3.1.3) gives a bounded operator on $L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$.*

Proof: Let $\tilde{\phi} : \mathbb{Z} \to \mathbb{R}_+$ be defined by $\tilde{\phi}(n) = \sup_{x\in\mathbb{R},y\in\mathbb{T}}|\phi(x,y,n)|$. Then

$$|(\pi(\phi)\xi)(x,y,p)| \le (\tilde{\phi} \star |\xi(x,y,.)|)(p),$$

where $\star$ denotes convolution on $\mathbb{Z}$ and $|\xi(x,y,.)|$ is the function $p \mapsto |\xi(x,y,p)|$. By Young's inequality $\|(\pi(\phi)\xi)(x,y,.)\|_{l_2} \le \|\tilde{\phi}\star|\xi(x,y,.)|\|_{l_2} \le \|\tilde{\phi}\|_{l_1}\|\xi(x,y,.)\|_{l_2}$. As an immediate consequence one has, $\|\pi(\phi)\| \le \|\phi\|_{\infty,\infty,1}$, where $\|\phi\|_{\infty,\infty,1} = \|\tilde{\phi}\|_{l_1}$ □

Remark 3.1.6 i) Product and involution defined by (3.1.1), and (3.1.2) turns $S^c_{\infty,\infty,1}$ into an involutive algebra.
ii) $\phi \mapsto \|\phi\|_{\infty,\infty,1}$ is a $*$-algebra norm. In the last chapter we will provide a proof.

Lemma 3.1.7 $\pi(S^c_{\infty,\infty,1}) \subseteq N_\hbar$.

Proof: Follows from Weaver's characterization of $N_\hbar$. □

Proposition 3.1.8 *$L^2(\mathcal{A}_\hbar^\infty, \tau)$ is unitarily equivalent with $L^2(\mathbb{T}\times\mathbb{T}\times\mathbb{Z}) \cong L^2([0,1]\times[0,1]\times\mathbb{Z})$.*

Proof: For $\phi \in S^c_{\infty,\infty,1}$, $\Gamma\phi : \mathbb{R}\times\mathbb{T}\times\mathbb{Z} \to \mathbb{C}$ given by

$$\Gamma\phi(x,y,p) = \begin{cases} e(-cxyp)\phi(x,y,p) & \text{for } y < 1, \\ \phi(x,y,p) & \text{for } y = 1. \end{cases}$$

satisfies $\Gamma\phi(x+k,y,p) = \Gamma\phi(x,y,p)$. Also note that

$$\tau(\phi^* \star \phi) = \int_0^1 \int_{\mathbb{T}} \sum_q |\phi(x-\hbar q\mu, y-\hbar q\nu, -q)|^2 dxdy = \int_0^1 \int_{\mathbb{T}} \sum_q |\phi(x,y,q)|^2 dxdy,$$

and therefore $\tau(\phi^* \star \phi) = \|\Gamma\phi\|^2$, i.e, $\Gamma : L^2(\mathcal{A}_\hbar^\infty, \tau) \to L^2(\mathbb{T}^2\times\mathbb{Z})$ is an isometry. To see that Γ is a unitary observe that

(i) $N_\hbar \subseteq L^2(\mathcal{A}_\hbar^\infty, \tau)$, since τ is normal;

(ii) $\phi_{m,n,k}$ defined by

$$\phi_{m,n,k} = \begin{cases} e(cxyp)e(mx+ny)\delta_{kp}, & \text{for } 0 \le y \le 1, \\ \delta_{kp}e(mx) & \text{for } y = 1, \end{cases}$$

is an element of $S^c_{\infty,\infty,1} \subseteq N_\hbar$;

(iii) $\{\Gamma\phi_{m,n,k}\}_{m,n,k\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T}^2\times\mathbb{Z})$. □

Remark 3.1.9 $\phi \mapsto \phi|_{[0,1]\times\mathbb{T}\times\mathbb{Z}}$ gives an unitary isomorphism.

Corollary 3.1.10 *Let M_{yp} be the multiplication operator on $\mathcal{H} = L^2(\mathbb{T}^2\times\mathbb{Z})$. If we consider $\mathcal{A}_\hbar^\infty$ as a subalgebra of $\mathcal{B}(\mathcal{H})$ by the left regular representation then $[M_{yp}, \mathcal{A}_\hbar^\infty] \subseteq \mathcal{B}(\mathcal{H})$.*

Proof: Note that for $\phi \in \mathcal{A}_\hbar^\infty$, $(M_{yp}\phi)(x,y,p) = yp\phi(x,y,p)$ gives an element in $S^c_{\infty,\infty,1}$, and hence a bounded operator. Now for $\psi \in \mathcal{A}_\hbar^\infty$,

$$\begin{aligned} &[M_{yp},\phi]\psi(x,y,p) \\ &= \sum_q (yp - (y-\hbar q\nu)(p-q))\phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q) \times \psi(x-\hbar q\mu, y-\hbar q\nu, p-q) \\ &= \sum_q q(y-\hbar(q-p)\nu)\phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q) \times \psi(x-\hbar q\mu, y-\hbar q\nu, p-q) \\ &= (M_{yp}(\phi)\star\psi)(x,y,p). \end{aligned}$$

This completes the proof. □

3.2 A class of spectral triples

Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system with G an n dimensional Lie group, and τ a G-invariant trace on $\mathcal{A}$. Let $\mathcal{A}^\infty$ be the space of smooth vectors, $\mathfrak{h} = L^2(\mathcal{A}, \tau) \otimes \mathbb{C}^N$ where $N = 2^{\lfloor n/2 \rfloor}$. Fix any basis $X_1, X_2, \ldots X_n$ of $L(G)$ the Lie algebra of G. Since G acts as a strongly continuous unitary group on $\mathcal{H} = L^2(\mathcal{A}, \tau)$ we can form selfadjoint operators d_{X_i} on $\mathcal{H}$. Let us define $D : \mathfrak{h} \to \mathfrak{h}$ by $D = \sum_i d_{X_i} \otimes \gamma_i$, where $\gamma_1, \ldots \gamma_n$ are selfadjoint matrices in $M_N(\mathbb{C})$ such that $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$. The operator D along with $\mathcal{A}^\infty$ and $\mathfrak{h}$ should be a candidate for a spectral triple. For such a D, clearly one has $[D, \mathcal{A}^\infty] \subseteq \mathcal{A}^\infty \otimes M_N(\mathbb{C})$.

Proposition 3.2.1 *For the quantum Heisenberg manifold, if we identify the Lie algebra of Heisenberg group with the Lie algebra of upper triangular matrices, then D as described above is a selfadjoint operator with compact resolvent with the following choice of $X_i^{\prime}s$:*

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ is greater than one.

Proof: Domain of the operator D consists of all those square integrable functions f defined on $[0,1]\times[0,1]\times\mathbb{Z}$ that satisfy $(i)\, f(x,0,p) = f(x,1,p), (ii) f(1,y,p) = e(cpy)f(0,y,p), (iii) pf, \frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are square integrable. On such a domain D is defined by $D(f \otimes u) = \sum_{j=1}^3 id_j(f) \otimes \sigma_j(u)$, where

$$\begin{aligned} id_1(f) &= -i\frac{\partial f}{\partial x}, \\ id_2(f) &= -2\pi cpxf(x,y,p) - i\frac{\partial f}{\partial y}, \\ id_3(f) &= -2\pi pc\alpha f(x,y,p), \end{aligned}$$

and σ_j's are the Pauli spin matrices.
Let $\eta : L^2([0,1] \times [0,1] \times \mathbb{Z}) \to L^2([0,1] \times [0,1] \times \mathbb{Z})$ be the unitary given by

$$\eta(f)(x,y,p) = \begin{cases} e(-cxyp)f(x,y,p), & \text{for } y < 1, \\ f(x,y,p), & \text{for } y = 1. \end{cases}$$

Then domain of the operator $D' = (\eta \times I_2)D(\eta \otimes I_2)^{-1}$ is given by all those square integrable functions f that satisfy the periodic boundary conditions, namely $(i)\, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, pf$ are square integrable, $(ii)\, f(0,y,p) = f(1,y,p)$, and $(iii)\, f(x,0,p) = f(x,1,p)$. On this domain D' is prescribed by $D'(f \otimes u) = \sum_{j=1}^{3} i d'_j(f) \otimes \sigma_j(u)$ where,

$$\begin{aligned} d'_1(f)(x,y,p) &= -2\pi i c y p f(x,y,p) - \frac{\partial f}{\partial x}(x,y,p), \\ d'_2(f)(x,y,p) &= -\frac{\partial f}{\partial y}(x,y,p), \\ d'_3(f)(x,y,p) &= 2\pi i p c \alpha f(x,y,p), \end{aligned}$$

Note that, on $Dom(D'), D' = T + S$ where $Dom(T) = Dom(D') \subseteq Dom(S)$ and T, S given respectively by

$$T = -i\frac{\partial}{\partial x} \otimes \sigma_1 - i\frac{\partial}{\partial y} \otimes \sigma_2 - 2\pi c\alpha M_p \otimes \sigma_3, S = 2\pi c M_{yp} \otimes \sigma_1.$$

These are selfadjoint operators on their respective domains. Also observe that T has compact resolvents. Our conclusion follows from the Rellich lemma since S is relatively bounded with respect to T with relative bound less than $\frac{1}{\alpha} < 1$. □

Theorem 3.2.2 *Let $\mathcal{H} = L^2(\mathcal{A}_\hbar^\infty, \tau) \otimes \mathbb{C}^2$,. $\mathcal{A}_\hbar^\infty$ with its diagonal action becomes a subalgebra of $\mathcal{B}(\mathcal{H})$. $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ is an odd spectral triple of dimension 3.*

Proof: The fact that $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ is a spectral triple follows from the previous proposition and the remark preceding that. We only have to show $|D|^{-3} \in \mathcal{L}^{1,+}$, the ideal of Dixmier traceable operators. For that observe:

(i) since T is the dirac operator on $\mathbb{T}^3$, $\mu_n(T^{-1}|_{\ker T^\perp}) = O(1/n^{1/3})$, μ_n stands for the n^{th} singular value.

(ii) S is relatively bounded with relative bound less than $\frac{1}{\alpha} < 1$, hence $\|S(T+i)^{-1}\| \leq \frac{1}{\alpha}$ and $\|(1 + S(T+i)^{-1})^{-1}\| \leq \frac{\alpha}{\alpha - 1}$.

(iii) $\mu_n(AB) \leq \mu_n(A)\|B\|$, for bounded operators A, B.

Applying (i), (ii), (iii) to $(D' + i)^{-1} = (T+i)^{-1}(1 + S(T+i)^{-1})^{-1}$ we get the desired conclusion for D' and hence for D. □

Corollary 3.2.3 *Let T, S, D, D' be as in the proof of proposition 3.2.1. Let us denote by D_0 the operator $(\eta \otimes I_2)^{-1} T (\eta \otimes I_2)$. Then $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)$ is an odd spectral triple of dimension 3.*

Proof: We only have to show $[D_0, \mathcal{A}_\hbar^\infty] \subseteq \mathcal{B}(\mathcal{H})$. Let $B = (\eta \otimes I_2)^{-1} S (\eta \otimes I_2)$. Then since $\eta \otimes I_2$ commutes with S, we have $B = S$. By corollary 3.1.10, $[B, \mathcal{A}_\hbar^\infty] \subseteq \mathcal{B}(\mathcal{H})$. Now the previous theorem along with $D = D_0 + B$ completes the proof. □

Remark 3.2.4 Similarly taking $D_t = D_0 + tB$ one can show that $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_t)$ forms an odd spectral triple of dimension 3, for $t \in [0, 1]$.

Remark 3.2.5 D, and D_0 constructed above depends on α.

Proposition 3.2.6 *If $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent, then the positive linear functional on $\mathcal{A}_\hbar \otimes M_2(\mathbb{C})$ given by $⨍ : a \mapsto tr_\omega a|D|^{-3}$ coincides with $\frac{1}{2}(tr_\omega|D|^{-3})\tau \otimes tr$, where tr_ω is a Dixmier trace.*

Proof : Observe that $D^2 = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$, where

$$X_1 = -(d_1^2 + d_2^2 + (d_3 + \frac{1}{2\alpha})^2 - \frac{1}{4\alpha^2}), \quad X_2 = -(d_1^2 + d_2^2 + (d_3 - \frac{1}{2\alpha})^2 - \frac{1}{4\alpha^2}).$$

It is easily seen that:

(i) compactness of resolvents of D^2 implies that for X_1, X_2,

(ii) eigenvalues of X_1, X_2 have similar asymptotic behavior.

Therefore $X_1^{-3/2}, X_2^{-3/2} \in \mathcal{L}^{(1,\infty)}$ and $tr_\omega a X_1^{-3/2} = tr_\omega a X_2^{-3/2}$ for any $a \in \mathcal{B}(L^2(\mathcal{A}_\hbar))$. Consider the unitary group on $\mathcal{H} \cong L^2([0,1] \times \mathbb{T} \times \mathbb{Z}) \otimes \mathbb{C}^2$ given by

$$U_t(x \otimes y \otimes e_p \otimes z) = e(pt)(x \otimes y \otimes e_p \otimes z).$$

Then $U_t D = D U_t$ and

$$⨍ A = tr_\omega U_t A U_t^* |D|^{-3} = tr_\omega((\int_0^1 U_t A U_t^* dt)|D|^{-3}) = ⨍ (A)_0,$$

where

$$A = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \mapsto (A)_0 = \begin{pmatrix} (\psi_{11})_0 & (\psi_{12})_0 \\ (\psi_{21})_0 & (\psi_{22})_0 \end{pmatrix}$$

is the completely positive map explicitly given for $\psi \in S^c$ by $(\psi)_0(x,y,p) = \delta_{p0}\psi(x,y,p)$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ commutes with $|D|^{-3}$, we get

$$\begin{aligned} ⨍ A &= tr_\omega (a_{11})_0 X_1{}^{-3/2} + tr_\omega (a_{22})_0 X_2{}^{-3/2} \\ &= tr_\omega ((a_{11})_0 + (a_{22})_0) X_1{}^{-3/2}. \end{aligned}$$

Consider the homomorphism $\Phi : C(\mathbb{T}^2) \to \mathcal{A}_\hbar$ given by $\Phi(f)(x,y,p) = \delta_{p0} f(x,y)$. Now by Riesz representation theorem for $⨍ \circ (\Phi \otimes I_2) : C(\mathbb{T}^2) \to \mathbb{C}$, we get a measure λ on $\mathbb{T}^2$ such that $tr_\omega 2(\psi)_0 X_1{}^{-3/2} = \int (\psi)_0(x,y,0) d\lambda$ implying

$$⨍ A = \frac{1}{2} \int ((a_{11})_0 + (a_{22})_0) d\lambda. \tag{3.2.1}$$

In the next lemma we show λ is proportional to the Lebesgue measure. That will prove that $⨍$ is proportional with $\tau \otimes tr$ and the proportionality constant is obtained by evaluating both sides on I. □

Lemma 3.2.7 *If $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent then λ as obtained in the previous proposition is proportional to the Lebesgue measure.*

Proof: It is known ([41], [17]) that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $|\mathcal{D}|^{-p} \in \mathcal{L}^{(1,\infty)}$ for some p, $a \mapsto tr_\omega a|\mathcal{D}|^{-p}$ is a trace on the algebra. This along with (3.2.1) gives

$$\int (\phi \star \psi)(x, y, 0) d\lambda(x, y) = \int (\psi \star \phi)(x, y, 0) d\lambda(x, y), \quad \forall \phi, \psi \in S^c. \tag{3.2.2}$$

Taking $\phi(x, y, p) = e(c[x]yp) f(x - [x]) g(y) \delta_{1p}$ where $g : \mathbb{T} \to \mathbb{C}, f : [0, 1] \to \mathbb{C}$ are smooth functions with $supp(f) \subseteq [\epsilon, 1 - \epsilon]$ for some $\epsilon > 0$ and $\psi = \phi^*$, we get from (3.2.2)

$$\int |\phi(x + \hbar\mu, y + \hbar\nu, 1)|^2 d\lambda(x, y) = \int |\phi \circ \gamma(x + \hbar\mu, y + \hbar\nu, 1)|^2 \lambda(x, y), \tag{3.2.3}$$

where $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $\gamma(x, y) = (x - 2\hbar\mu, y - 2\hbar\nu)$. The hypothesis of linear independence of $(1, \hbar\mu, \hbar\nu)$ over the rationals implies that γ-orbits are dense. This along with (3.2.3) proves the lemma. □

Remark 3.2.8 In the rest of the chapter $⨍$ will denote $\frac{1}{2}\tau \otimes tr$.

3.3 Space of Connes deRham Forms

In this section we will compute the space of Connes deRham forms associated with the spectral triple. Our computations are for the specific case of quantum Heisenberg manifolds, but similar calculations can be carried through whenever one has a spectral triple coming from Lie group action on a simple C^*-algebra. We begin with a rather simple but useful lemma which is essentially the sweep out algorithm of linear algebra for solving linear equations.

Lemma 3.3.1 *Let $\mathcal{A}$ be a dense subalgebra of a unital C^*algebra $\bar{\mathcal{A}}$ closed under holomorphic function calculus, then $\mathcal{A}$ is simple provided $\bar{\mathcal{A}}$ is so.*

Proof: Let $J \subseteq \mathcal{A}$ be an ideal. Then $\bar{J} = \bar{\mathcal{A}}$, since $\bar{\mathcal{A}}$ is simple. There exists $x \in J$ such that $\|x - I\| < 1$. Then $x^{-1} \in \bar{\mathcal{A}}$, hence in $\mathcal{A}$ because $\mathcal{A}$ is closed under holomorphic function calculus. Therefore $1 = xx^{-1} \in J$. □

Assumption 3.3.2 Henceforth we will assume $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent. In that case $\mathcal{A}_\hbar$ is simple ([73]), hence so is $\mathcal{A}_\hbar^\infty$.

Let us introduce some notations before we proceed further. Let $\phi \in S^c$, then $[D, \phi] = \sum \delta_i(\phi) \otimes \sigma_i$ where $\delta_j(\phi) = id_j(\phi)$ (see proof of proposition 3.2.1 for d_j) but looked upon as derivation on $\mathcal{A}_\hbar^\infty$. Also note that $[\delta_1, \delta_3] = [\delta_2, \delta_3] = 0, [\delta_1, \delta_2] = \delta_3$. In the sequel we will need a special class of elements of $\mathcal{A}_\hbar^\infty$ whose symbols are given by $\phi_{m,n}(x, y, p) = e(mx + ny)\delta_{p0}$.

Lemma 3.3.3 *Let $\mathcal{A}$ be a unital simple algebra, $M \subseteq \overbrace{\mathcal{A} \oplus \ldots \oplus \mathcal{A}}^{n\ times}$ a sub$\mathcal{A}$-$\mathcal{A}$ bimodule. Suppose there exists algebra elements $a_{ij}, 1 \leq i \leq n, 1 \leq j \leq i$ such that (i) $a_{ii} \neq 0$, (ii) $b_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0) \in M$. Then M is isomorphic to $\underbrace{\mathcal{A} \oplus \ldots \oplus \mathcal{A}}_{n\ times}$ as an $\mathcal{A}$-$\mathcal{A}$ bimodule.*

Proof: By induction on n,
For $n = 1$, $0 \neq M$ is an ideal in $\mathcal{A}$, hence $M = \mathcal{A}$. Let $\pi : M \to \mathcal{A}$ be $\pi(a_1, \ldots, a_n) = a_n$. Then by hypothesis, $\pi(M)$ is a nontrivial ideal in $\mathcal{A}$ and hence equals $\mathcal{A}$. So, we have a split short exact sequence

$$0 \to \ker(\pi) \to M \to \mathcal{A} \to 0.$$

Therefore $M = ker(\pi) \oplus Im\pi = ker(\pi) \oplus \mathcal{A} = \underbrace{\mathcal{A} \oplus \ldots \oplus \mathcal{A}}_{n\ times}$. In the last equality we have used induction hypothesis for $ker(\pi)$. □

Now we will compute the Connes-deRham complex introduced in definition 1.6.4 associated with the spectral triple $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ constructed in theorem 3.2.2. We follow the notations of definition 1.6.4, i.e., π stands for the representation of $\Omega^\bullet(\mathcal{A}_\hbar^\infty)$ in $\mathcal{B}(\mathcal{H})$ and $J_n = \ker \pi|_{\Omega^n(\mathcal{A}_\hbar^\infty)}$.

Proposition 3.3.4 *(i)*

$$\begin{aligned} \Omega_D^1(\mathcal{A}_\hbar^\infty) &= \{\sum a_i \otimes \sigma_i | a_i \in \mathcal{A}_\hbar^\infty, \sigma_i's \text{ are spin matrices } \} \\ &= \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty. \end{aligned}$$

(ii) $\pi(\Omega^k(\mathcal{A}_\hbar^\infty)) = \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) = \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$.

Proof: $\Omega_D^1(\mathcal{A}_\hbar^\infty) = \pi(\Omega^1(\mathcal{A}_\hbar^\infty)) \subseteq$ R.H.S.
Let $\phi_{mn}(x, y, p) = \delta_{p0}e(mx + ny)$ and $\phi \in S^c$ be such that $\phi(x, y, p) = \delta_{p1}\phi(x, y, p)$. Then applying the previous lemma to $[D, \phi_{01}], [D, \phi_{10}], [D, \phi] \in \pi(\Omega^1(\mathcal{A}))$ we get the result (i). For (ii) use (i) along with $\Omega^k(\mathcal{A}_\hbar^\infty) = \underbrace{\Omega^1(\mathcal{A}_\hbar^\infty) \otimes_{\mathcal{A}_\hbar^\infty} \ldots \otimes_{\mathcal{A}_\hbar^\infty} \Omega^1(\mathcal{A}_\hbar^\infty)}_{k\ times}$. □

Proposition 3.3.5 *(i)* $\pi(\delta J_1) = \mathcal{A}_\hbar^\infty$.
(ii) $\Omega_D^2(\mathcal{A}_\hbar^\infty) = \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$.

Proof: (i) Let $\omega = \sum a_i\delta(b_i) \in J_1$. Then $\pi(\omega) = \sum a_i\delta_j(b_i)\sigma_j = 0$ gives $\sum a_i\delta_j(b_i) = 0, \forall j$.

$$\begin{aligned}\pi(\delta\omega) &= \sum_i(\sum_j \delta_j(a_i)\sigma_j)(\sum_k \delta_k(b_i)\sigma_k)\\ &= \sum_i(\sum_j \delta_j(a_i)\delta_j(b_i)) \otimes I_2 + \sum_i(\sum_{j<k}(\delta_j(a_i)\delta_k(b_i) - \delta_k(a_i)\delta_j(b_i))\sigma_j\sigma_k),\end{aligned} \tag{3.3.1}$$

$$\begin{aligned}\sum_i[\delta_j,\delta_k](a_ib_i) &= \sum_i \delta_j(\delta_k(a_i)b_i) - \delta_k(\delta_j(a_i)b_i)[Since \sum a_i\delta_j(b_i) = 0, \forall j\\ &= \sum_i[\delta_j,\delta_k](a_i)b_i + \sum_i(\delta_k(a_i)\delta_j(b_i) - \delta_j(a_i)\delta_k(b_i)).\end{aligned} \tag{3.3.2}$$

Also note,

$$\begin{aligned}\sum_i[\delta_j,\delta_k](a_ib_i) &= \sum_i[\delta_j,\delta_k](a_i)b_i + \sum_i a_i[\delta_j,\delta_k](b_i)\\ &= \sum_i[\delta_j,\delta_k](a_i)b_i.\end{aligned} \tag{3.3.3}$$

Comparing right hand side of (3.3.2), (3.3.3) we see that the second term on the right hand side of (3.3.1) vanishes, thus proving $\pi(\delta J_1) \subseteq \mathcal{A}_\hbar^\infty$. For equality, in view of lemma 3.3.3 it is enough to note, $\omega = 2\phi_{02}\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02}) \in J_1, \pi(\delta\omega) = 2\phi_{03} \otimes I_2 \neq 0$.
(ii)Suppose $\phi \in S^c$ satisfies $\phi(x,y,p) = \delta_{1p}\phi(x,y,p)$. Let $\omega_1 = \delta(\phi_{10})\delta(\phi_{01})$, $\omega_2 = \delta(\phi_{10})\delta(\phi)$, $\omega_3 = \delta(\phi_{01})\delta(\phi)$. Now lemma 3.3.3 together with (i) implies the result. □

Lemma 3.3.6 $\pi(\delta J_2) = \{\sum a_j \otimes \sigma_j | a_j \in \mathcal{A}_\hbar^\infty\} = \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$.

Proof: Let $\omega = \sum a_i\delta(b_i)\delta(c_i) \in J_2$,

$$\begin{aligned}0 = \pi(\omega) &= \sum a_i(\sum_j \delta_j(b_i)\sigma_j)(\sum_j \delta_k(c_i)\sigma_k)\\ &= a_i\delta_j(b_i)\delta_j(c_i) + \sum_{j<k} a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i)\sigma_j\sigma_k.\end{aligned}$$

Comparing the coefficients of the various spin matrices we get,

$$\sum a_i\delta_j(b_i)\delta_j(c_i) = 0, \tag{3.3.4}$$

$$\sum a_i(\delta_j(b_i)\delta_k(c_i) - \delta_k(b_i)\delta_j(c_i)) = 0, \forall j \neq k. \tag{3.3.5}$$

From (3.3.5),

$$0 = \sum \delta_1(a_i(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i))$$

$$= \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) + \sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)).$$

Therefore,

$$\sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) = -\sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)). \quad (3.3.6)$$

Similarly we get two more equalities. Let A be the coefficient of I_2 in $\pi(\delta\omega)$. Then

$$\begin{aligned}
&\sqrt{-1}A \\
&= \sum \delta_1(a_i)(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) + \sum \delta_2(a_i)(\delta_3(b_i)\delta_1(c_i) - \delta_1(b_i)\delta_3(c_i)) \\
&\quad + \sum \delta_3(a_i)(\delta_1(b_i)\delta_2(c_i) - \delta_2(b_i)\delta_1(c_i)) \\
&= -(\sum a_i\delta_1(\delta_2(b_i)\delta_3(c_i) - \delta_3(b_i)\delta_2(c_i)) + \sum a_i\delta_2(\delta_3(b_i)\delta_1(c_i) - \delta_1(b_i)\delta_3(c_i)) \\
&\quad + \sum a_i\delta_1(\delta_1(b_i)\delta_2(c_i) - \delta_2(b_i)\delta_1(c_i))) \\
&= -(\sum a_i([\delta_1,\delta_2](b_i)\delta_3(c_i) + \delta_2(b_i)[\delta_1,\delta_3](c_i)) + \sum a_i([\delta_3,\delta_1](b_i)\delta_2(c_i) + \delta_3(b_i)[\delta_2,\delta_1](c_i)) \\
&\quad + \sum a_i([\delta_2,\delta_3](b_i)\delta_1(c_i) + \delta_1(b_i)[\delta_3,\delta_2](c_i))) \\
&= 0.
\end{aligned}$$

Here second equality follows from (3.3.6) and the last equality follows from (3.3.5) since δ_j's form a lie algebra. This shows,

$$\pi(\delta J_2) \subseteq \{\sum_{j=1}^{3} a_j\sigma_j | a_j \in \mathcal{A}_\hbar^\infty\} \cong \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty. \quad (3.3.7)$$

Let $\phi \in S^c$ be such that $\phi(x,y,p) = \delta_{1p}\phi(x,y,p)$. Then,

$$\begin{aligned}
\omega_1 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01}) \in J_2, \\
\omega_2 &= 2\phi_{20}\delta(\phi_{10})\delta(\phi_{10}) - \phi_{10}\delta(\phi_{20})\delta(\phi_{10}) \in J_2, \\
\omega_3 &= \phi_{02}\delta(\phi_{01})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi) \in J_2,
\end{aligned}$$

satisfy,

$$\begin{aligned}
\pi(\delta\omega_1) &= 2\phi_{04}\sigma_2, \\
\pi(\delta\omega_2) &= 2\phi_{40}\sigma_1, \\
\pi(\delta\omega_3) &= 2\phi_{03}\delta_1(\phi)\sigma_1 + 2\phi_{03}\delta_2(\phi)\sigma_2 + 2\phi_{03}\delta_3(\phi)\sigma_3.
\end{aligned}$$

Therefore by Lemma 3.3.3 we get equality in (3.3.7). □

Corollary 3.3.7 $\Omega_D^3(\mathcal{A}_\hbar^\infty) = \mathcal{A}_\hbar^\infty$.

Proof: Immediate from the previous lemma and proposition 3.5(ii). □

Lemma 3.3.8 *(i)* $\Omega_D^4(\mathcal{A}_\hbar^\infty) = 0$.
(ii) $\Omega_D^k(\mathcal{A}_\hbar^\infty) = 0$, *for all* $k > 4$.

Proof: (i) It suffices to show $\pi(\delta J_3) = \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$.
For that note,

$$\begin{aligned}
\omega_1 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi_{01}) \in J_3,\\
\omega_2 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi_{01}) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi_{01}) \in J_3,\\
\omega_3 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{01})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi_{01})\delta(\phi) \in J_3,\\
\omega_4 &= 2\phi_{02}\delta(\phi_{01})\delta(\phi_{10})\delta(\phi) - \phi_{01}\delta(\phi_{02})\delta(\phi_{10})\delta(\phi) \in J_3,
\end{aligned}$$

satisfy,

$$\begin{aligned}
\pi(\delta\omega_1) &= 2\phi_{05} \otimes I_2,\\
\pi(\delta\omega_2) &= 2\phi_{14}\sigma_2\sigma_1,\\
\pi(\delta\omega_3) &= 2\phi_{04}\delta_2(\phi) \otimes I_2 + 2\phi_{04}\delta_1(\phi)\sigma_2\sigma_1 + 2\phi_{04}\delta_3(\phi)\sigma_2\sigma_3,\\
\pi(\delta\omega_4) &= 2\phi_{13}\delta_1(\phi)I_2 + 2\phi_{13}\delta_2(\phi)\sigma_1\sigma_2 + 2\phi_{13}\delta_3(\phi)\sigma_1\sigma_3.
\end{aligned}$$

Now an application of lemma 3.3.3 completes the proof.
(ii) The same argument as in (i) does the job with the following choice,
$\omega_i' = \omega_i \underbrace{\delta(\phi_{01}) \ldots \delta(\phi_{01})}_{(k-4)\ times}, i = 1, \cdots 4.$ □

3.4 Torsionless and Unitary Connections

Recall that for computation of connection, curvature one requires space of square integrable forms, which in this case coincides with the space of Connes-deRham forms. We also characterize unitary and torsionless connections and prove that a connection cannot be simultaneously torsionless and unitary. We follow notations of sections 1.6.4, 1.6.5.

Proposition 3.4.1 *(i)* $\widetilde{\Omega}^k(\mathcal{A}_\hbar^\infty) = \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) \cong \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$.
(iii) $\widetilde{\mathcal{H}}^k = L^2(\mathcal{A}_\hbar^\infty, \tau) \otimes \mathbb{C}^4$.
(ii) $\widetilde{\Omega}_D^k(\mathcal{A}_\hbar^\infty) = \Omega_D^k(\mathcal{A}_\hbar^\infty)$.

Proof:(i) Faithfulness of $A \mapsto ⨍ A$, defined on $\pi(\Omega^\bullet(\mathcal{A}_\hbar^\infty)) = \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$ gives $J_k = \mathsf{K}_k$.
hence $\widetilde{\Omega}^k(\mathcal{A}_\hbar^\infty) = \Omega^k(\mathcal{A}_\hbar^\infty)/ker(\pi) \cong \pi(\Omega^k(\mathcal{A}_\hbar^\infty)) = \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$.

(ii) Follows from (i) and proposition 3.2.6.

(iii)This follows from (i) and the definitions. □

Remark 3.4.2 Since $\widetilde{\Omega}^1_D(\mathcal{A}^\infty_\hbar)$ is free with 3 generators, we can and will identify $\widetilde{\Omega}^1_D(\mathcal{A}^\infty_\hbar)\otimes_{\mathcal{A}^\infty_\hbar} \widetilde{\Omega}^1_D(\mathcal{A}^\infty_\hbar)$ with $\mathcal{A}^\infty_\hbar \otimes M_3(\mathbb{C})$ and a connection ∇ is specified by its value on the generators.

Proposition 3.4.3 *A connection is torsionless iff its values on the generators $\sigma_1, \sigma_2, \sigma_3$ are given by*

$$\nabla(\sigma_1) = \begin{pmatrix} \square & a & b \\ a & \square & c \\ b & c & \square \end{pmatrix}, \nabla(\sigma_2) = \begin{pmatrix} \square & d & e \\ d & \square & f \\ e & f & \square \end{pmatrix}, \nabla(\sigma_3) = \begin{pmatrix} \square & p-1 & q \\ p & \square & r \\ q & r & \square \end{pmatrix},$$

where all the matrix entries are from $\mathcal{A}^\infty_\hbar$ with restrictions on them as indicated above and $\square$ denotes an unrestricted entry.

Proof: Observe that

$$\begin{aligned}\delta(\sum_{i,j} a_i\delta_j(b_i)\sigma_j) &= -\sqrt{-1}(\sum_i(\delta_1(a_i)\delta_2(b_i) - \delta_2(a_i)\delta_1(b_i))\sigma_3 \\ &\quad + \sum_i(\delta_2(a_i)\delta_3(b_i) - \delta_3(a_i)\delta_2(b_i))\sigma_1 \\ &\quad + \sum_i(\delta_3(a_i)\delta_1(b_i) - \delta_1(a_i)\delta_3(b_i))\sigma_2),\end{aligned}$$

$$\begin{aligned}m\circ\nabla(\sum_{i,j} a_i\delta_j(b_i)\sigma_j) &= m(\sum_{i,j}\delta(a_i\delta_j(b_i))\otimes\sigma_j) + \sum_{i,j} a_i\delta_j(b_i)m\circ\nabla(\sigma_j) \\ &= m(\sum_{i,j,k}\delta_k(a_i\delta_j(b_i))\sigma_k\otimes\sigma_j) + \sum_{i,j} a_i\delta_j(b_i)m\circ\nabla(\sigma_j).\end{aligned}$$

Torsion of ∇ vanishes if and only if $(\delta - m\circ\nabla)(\sum a_i\delta_j(b_i)\sigma_j) \equiv 0$, or equivalently ,

$$\sum_i(\delta_j(a_i)\delta_k(b_i) - \delta_k(a_i)\delta_j(b_i)) = \sum_i(\delta_j(a_i\delta_k(b_i)) - \delta_k(a_i\delta_j(b_i))) + \sum_{i,l} a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_n$$

whenever $j \neq k$ and n satisfies $\sigma_j\sigma_k\sigma_n = \sqrt{-1}$. This happens if and only if

$$0 = \sum_i a_i[\delta_j,\delta_k](b_i) + \sum_{i,l} a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_n$$

whenever $j \neq k$ and n satisfies $\sigma_j\sigma_k\sigma_n = \sqrt{-1}$.

Using the Lie algebra relations between the δ_j's we get equivalence of the above system of equations with

$$\begin{aligned} 0 &= \sum_i a_i\delta_3(b_i) + \sum_{i,l} a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_3 \\ 0 &= \sum_{i,l} a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_2 \\ 0 &= \sum_{i,l} a_i\delta_l(b_i)(m\circ\nabla(\sigma_l))_1 \end{aligned}$$

Taking $b_i = \phi_{01}, a_i = 1$ we get $\delta_1(b_i) = \delta_3(b_i) = 0, \delta_2(b_i) = b_i$. Substituting these in the above relations we get $(m\circ\nabla(\sigma_2))_j = 0$ for $j = 1,2,3$. Similarly taking $b_i = \phi_{10}, a_i = 1$ we get $(m\circ\nabla(\sigma_1))_j = 0$ for $j = 1,2,3$. Substituting these values in the above equations we get,

$$\sum_i a_i\delta_3(b_i)(m\circ\nabla(\sigma_3))_1 = \sum_i a_i\delta_3(b_i)(m\circ\nabla(\sigma_3))_2 = \sum_i a_i\delta_3(b_i)(1 + (m\circ\nabla(\sigma_3))_3 = 0.$$

Note that $J = \{\sum a_i\delta_3(b_i) | n\in\mathbb{N}, a_1,\ldots,a_i,b_1,\ldots,b_i\in\mathcal{A}_\hbar^\infty\}$ is a nontrivial ideal in $\mathcal{A}_\hbar^\infty$ and hence it equals $\mathcal{A}_\hbar^\infty$. Therefore $(m\circ\nabla(\sigma_3))_3 = -1$ and $(m\circ\nabla(\sigma_3))_1 = (m\circ\nabla(\sigma_3))_2 = 0$. Now the result follows from the anticommutation relation between the spin matrices. □

Proposition 3.4.4 *A connection ∇ on $\widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty)$ is unitary iff its values on the generators $\sigma_1,\sigma_2,\sigma_3$ are given by*

$$\nabla(\sigma_1) = \begin{pmatrix} X & Y & Z \\ Y & U & P \\ Z & V & Q \end{pmatrix}, \nabla(\sigma_2) = \begin{pmatrix} Y & U & V \\ U & R & S \\ P & S & F \end{pmatrix}, \nabla(\sigma_3) = \begin{pmatrix} Z & P & Q \\ V & S & F \\ Q & F & G \end{pmatrix},$$

where all the matrix entries are selfadjoint elements of $\mathcal{A}_\hbar^\infty$.

Proof: Taking $s = a_i\sigma_i, t = b_j\sigma_j$ in the defining condition of a unitary connection we get

$$\delta(\delta_{ij}a_ib_j^*) \quad = \quad a_i(\langle\nabla(\sigma_i),\sigma_j\rangle - \langle,\sigma_i,\nabla(\sigma_j)\rangle)b_j^* + \delta_{ij}(\delta(a_i)b_j^* - a_i(\delta(b_j))^*), \qquad (3.4.1)$$

implying that$\langle\nabla(\sigma_i),\sigma_j\rangle = \langle\sigma_i,\nabla(\sigma_j)\rangle$, which means the j-th row of $\nabla(\sigma_i)$ is the star of the i-th column of $\nabla(\sigma_j)$. This completes the proof. □

Corollary 3.4.5 *A connection ∇ can not simultaneously be torsionless and unitary.*

Proof: If possible let ∇ be one such. Comparing the forms of $\nabla(\sigma_j), j = 1,2,3$ in propositions 3.4.3 and 3.4.4 we get that $V = c = P$ and also $V - P = -1$. This leads to a contradiction.

□

3.5 Connections with non trivial scalar curvature

In this section we exhibit connections for which the associated scalar curvature is nontrivial. In fact it turns out that the scalar curvature is an element in the smooth algebra $\mathcal{A}_\hbar^\infty$. Let $\langle\cdot,\cdot\rangle_D$ be the map introduced in theorem 1.6.11 associated with the spectral triple $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$. In the following proposition we identify $\widetilde{\Omega}^k(\mathcal{A}_\hbar^\infty)$ with $\mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$.

Proposition 3.5.1 $\langle\omega,\eta\rangle_D = \frac{1}{2}(I\otimes tr)(\omega\eta^*)$

Proof: Let

$$\omega = \omega_0 \otimes I_2 + \sum_{i=1}^{3} \omega_i \otimes \sigma_i,\ \eta = \eta_0 \otimes I_2 + \sum_{i=1}^{3} \eta_i \otimes \sigma_i.$$

Then $\frac{1}{2}(I\otimes tr)(\omega\eta^*) = \sum_{i=0}^3 \omega_i\eta_i^*$ and $(x, \sum_{i=0}^3 \omega_i\eta_i^*) = \sum \tau(x\eta_i\omega_i^*) = (x, \langle\omega,\eta\rangle_D)$ for all $x \in \mathcal{A}_\hbar$. This completes the proof since $\mathcal{A}_\hbar$ is dense in $\widetilde{\mathcal{H}}^0$ □

Now we will exhibit connections with nontrivial curvature. We will follow notations of section 1.6.5. Since $\widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty)$ is a free bimodule with three generators, the curvature $R(\nabla)$ of a connection ∇, $R(\nabla) = -\nabla^2 : \widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty) \to \widetilde{\Omega}_D^2(\mathcal{A}_\hbar^\infty) \otimes_{\mathcal{A}_\hbar^\infty} \widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty)$ is given by a 3×3 matrix $((R_{ij}))$ with entries in $\widetilde{\Omega}_D^2(\mathcal{A}_\hbar^\infty)$. Let $P_{\delta\mathsf{K}_1} : \widetilde{\mathcal{H}}^2 \to \widetilde{\mathcal{H}}^1$ be the projection onto closure of $\pi(\delta\mathsf{K}_1) \subseteq \widetilde{\Omega}_D^2(\mathcal{A}_\hbar^\infty)$, and $R_{ij}{}^\perp = (I - P_{\delta\mathsf{K}_1})(R_{ij})$. Let e_1, e_2, e_3 be the canonical basis of $\widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty)$. If we write $Ric_j = \sum_i m_L(e_i)^\dagger(R_{ij}{}^\perp) \in \widetilde{\mathcal{H}}^1$, then Ricci curvature of ∇ is given by

$$Ric(\nabla) = \sum_j Ric_j \otimes e_j \in \widetilde{\mathcal{H}}^1 \otimes_{\mathcal{A}_\hbar^\infty} \widetilde{\Omega}_D^1(\mathcal{A}_\hbar^\infty),$$

where † denotes Hilbert space adjoint. Finally the scalar curvature $r(\nabla)$ of ∇ is given by

$$r(\nabla) = \sum_i m_R(e_i^*)^\dagger(Ric_i) \in \widetilde{\mathcal{H}}^0 = L^2(\mathcal{A}_\hbar^\infty).$$

Proposition 3.5.2 *Let $f, g : \mathbb{T} \to \mathbb{R}$ be smooth maps. We visualize them as elements of S^c in the following way, $f(x,y,p) = \delta_{0p}f(x)$, $g(x,y,p) = \delta_{0p}g(y)$. Let ∇ be the connection given by $\nabla(\sigma_1) = f'\delta(g)\sigma_1 + g'\delta(f)\sigma_2, \nabla(\sigma_2) = g'\delta(f)\sigma_1, \nabla(\sigma_3) = 0$, then $r(\nabla)$ is $-2f'^2g'^2$.*

Proof: It is clear that the derivative functions f', g' also can be visualized as elements of S^c exactly in the same way as f and g. By direct computation one gets,

$$\nabla^2(\sigma_1) = -R_{11}\sigma_1 - R_{12}\sigma_2, \quad \nabla^2(\sigma_2) = -R_{21}\sigma_1, \nabla^2(\sigma_3) = 0,$$

where

$$R_{11} = f''g\sigma_3, \quad R_{12} = \sqrt{-1}(f'^2g'^2 - g''f')\sigma_3, \quad R_{21} = -\sqrt{-1}(g''f' + f'^2g'^2)\sigma_3,$$

and the other R_{ij}'s are zero.
Then

$$Ric_1 = -f''g\sigma_2 - (g''f' + f'^2g'^2)\sigma_1, \quad Ric_2 = (g''f' - f'^2g'^2)\sigma_2$$

implying the desired conclusion $r(\nabla) = -2f'^2g'^2$. □

Remark 3.5.3 1. All the above notions of Ricci curvature, scalar curvature was introduced in [36]. This is one infinite dimensional example where one can have connections with nontrivial scalar curvature. (see also [16])

2. Note that our choice of the spectral triple depends on a parameter α. However, for the connections we have considered the scalar curvature does not depend on the parameter α.

3. Note that the scalar curvature is a negative element of the smooth algebra.

3.6 Nontriviality of the chern character associated with the spectral triples

The spectral triple we constructed depends on a real parameter α. In this section we show that the Kasparov module associated with the spectral triple are homotopic ([24], [8]). We also argue that they give non-trivial elements in $K^1(\mathcal{A}_\hbar)$ by explicitly computing pairing with some unitary in the algebra representing elements of $K_1(\mathcal{A}_\hbar)$.

Lemma 3.6.1 *Let A be a selfadjoint operator with a bounded inverse and B a symmetric operator with $Dom(A) \subseteq Dom(B)$ on some Hilbert space $\mathcal{H}$. Also suppose that $\|Bu\| \leq a\|Au\|, \forall u \in Dom(A)$. Then $|A|^{-p}B|A|^{-(1-p)} \in \mathcal{B}(\mathcal{H})$ and $\||A|^{-p}B|A|^{-(1-p)}\| \leq a$.*

Proof: Clearly $\|Bu\| \leq a\||A|u\|, \forall u \in D(A)$ implying $\|B|A|^{-1}\| \leq a$. For $u, v \in D(A)$

$$\begin{aligned} \||A|^{-1}Bu\| &= \sup_{v\in D(A),\|v\|\leq 1} |\langle |A|^{-1}Bu, v\rangle| \\ &= \sup_{v\in D(A),\|v\|\leq 1} |\langle u, B|A|^{-1}v\rangle| \leq a\|u\|. \end{aligned}$$

Therefore $|A|^{-1}B \in \mathcal{B}(\mathcal{H}), \||A|^{-1}B\| \leq a$. Let $\mathcal{H}_p$ be the Hilbert space completion of $\cap D(A^n)$ with respect to $\|u\|_p = \||A|^pu\|$. Let $B_1 : \mathcal{H}_1 \to \mathcal{H}_0, B_0 : \mathcal{H}_0 \to \mathcal{H}_{-1}$ be the maps given by $B_i(u) = B(u)$ for $u \in \cap D(A^n)$. Then $\|B_1\|, \|B_1\| \leq a$. By Calderon-Zygmund interpolation theorem ([68]) we get maps $B_p : \mathcal{H}_p \to \mathcal{H}_{-(1-p)}$ for $0 \leq p \leq 1$ with $\|B_p\| \leq a$. On $\cap D(A^n), B_p$ agrees with $|A|^{-p}B|A|^{-(1-p)}$, proving the lemma. □

Lemma 3.6.2 *Let A, B be as above with $a < 1$. Let $A_t = A + tB, t \in [0,1]$. Then the assignment $t \mapsto \tan^{-1}(A_t)$ gives a norm continuous function.*

Proof: Let us denote $|A|^{-1/2}B|A|^{-1/2}$ by C. Then by the previous lemma $\|C\| \leq a$. We also have $\||A|(A-\lambda)^{-1}| \leq 1$ for $\lambda \in i\mathbb{R}$,

$$\begin{aligned} A_t - \lambda &= (A-\lambda) + t|A|^{1/2}C|A|^{1/2} \\ &= |A|^{1/2}((A-\lambda)|A|^{-1} + tC)|A|^{1/2} \\ &= |A|^{1/2}(1 + tC(A-\lambda)^{-1}|A|)(A-\lambda)|A|^{-1}|A|^{1/2}. \end{aligned}$$

Now note $\|tC(A-\lambda)^{-1}|A|\| \leq a < 1$ for $0 \leq t \leq 1$. Therefore

$$(A_t - \lambda)^{-1} = |A|^{-1/2}|A|(A-\lambda)^{-1}(1 + tC(A-\lambda)^{-1}|A|)^{-1}|A|^{-1/2}.$$

So, if we denote by $R_t(\lambda) = (A_t - \lambda)^{-1}$ and $F(\lambda) = |A|(A-\lambda)^{-1}$ then the above equality becomes,

$$\begin{aligned} R_t(\lambda) &= |A|^{-1/2}|A|R_0(\lambda)(1 + tC|A|F(\lambda))^{-1}|A|^{-1/2} \\ &= R_0(\lambda) + |A|^{-1/2}F(\lambda)\sum_{n=1}^{\infty}(-tCF(\lambda))^n|A|^{-1/2}. \end{aligned} \tag{3.6.1}$$

Let $\lambda \in \mathbb{R}, t, s \in [0,1], u, v \in D(A)$. Observe,

(i)

$$\begin{aligned} &\|\sum_{n=1}^{\infty}(-tCF(i\lambda))^n|A|^{-1/2}u - \sum_{n=1}^{\infty}(-sCF(i\lambda))^n|A|^{-1/2}u\| \\ \leq\ & \sum_{n=0}^{\infty}\|(t^{n+1} - s^{n+1})CF(i\lambda)\|^n\|C\|\|F(i\lambda)|A|^{-1/2}u\| \\ \leq\ & \sum_{n=0}^{\infty}|(t^{n+1} - s^{n+1})|a^n a\|F(i\lambda)|A|^{-1/2}u\| \\ \leq\ & |(t-s)|\sum_{n=0}^{\infty}n + 1a^{n+1}\|F(i\lambda)|A|^{-1/2}u\| \\ \leq\ & |(t-s)|\frac{a}{(1-a)^2}\|F(i\lambda)|A|^{-1/2}u\| \end{aligned}$$

(ii)

$$\begin{aligned} \int_0^{\infty}\|F(i\lambda)|A|^{-1/2}u\|^2 d\lambda &\leq \int_0^{\infty}\langle(A^2+\lambda^2)^{-1}u, |A|u\rangle d\lambda \\ &= \frac{1}{2}\int_0^{\infty}\langle(A^2+\xi)^{-1}u, |A|u\rangle\frac{d\xi}{\sqrt{\xi}} \end{aligned}$$

$$= \frac{1}{2}\pi\langle A^{2^{-1/2}}u, |A|u\rangle = \frac{\pi}{2}\|u\|^2.$$

(iii) Using (3.6.1), (i) ,(ii) we get

$$\begin{aligned}
&\int_0^\infty |\langle (R_t(i\lambda) - R_s(i\lambda))u, v\rangle| d\lambda \\
\leq\ & \int_0^\infty |(t-s)| \frac{a}{(1-a)^2} \|F(i\lambda)|A|^{-1/2}u\| \|F(-i\lambda)|A|^{-1/2}v\| d\lambda \\
\leq\ & |(t-s)| \frac{a}{(1-a)^2} \Big(\int_0^\infty \|F(i\lambda)|A|^{-1/2}u\|^2 d\lambda\Big)^{1/2} \Big(\int_0^\infty \|F(-i\lambda)|A|^{-1/2}v\|^2 d\lambda\Big)^{1/2} \\
\leq\ & |(t-s)| \frac{a}{(1-a)^2} \frac{\pi}{2} \|u\| \|v\|.
\end{aligned}$$

This shows that $\lim_{s\to t} \| \int_0^\infty (R_t(i\lambda) - R_s(i\lambda)) d\lambda\| = 0$. Similarly one can also show that $\lim_{s\to t} \| \int_0^\infty (R_t(-i\lambda) - R_s(-i\lambda)) d\lambda\| = 0$. Now the result follows once we observe that $\tan^{-1} A_t = \int_0^\infty (R_t(i\lambda) + R_t(-i\lambda)) d\lambda$. □

Lemma 3.6.3 *Let A, B be as above except now we do not require A to be invertible. Instead we assume A to have discrete spectrum. Then there exists $\kappa \geq 0$ such that $t \mapsto \tan^{-1}(A_t + \kappa)$ is norm continuous.*

Proof: Without loss of generality we can assume 0 is an eigenvalue of A. Otherwise we are done by the previous lemma. Choose $2 \leq n \in \mathbb{N}$ such that $b = a\frac{n}{n-1} < 1$. Choose $\kappa > 0$ such that

(i) smallest positive eigenvalue of A is greater than κ;

(ii) if β is the biggest negative eigenvalue then $\beta < n\kappa$.

Let $\widetilde{A} = A + \kappa, \widetilde{A}_t = \widetilde{A} + tB$. Then by choice of κ

(i) $\widetilde{A}$ is an invertible selfadjoint operator;

(ii) $\|B\widetilde{A}^{-1}\| \leq a\|A(A+\kappa)^{-1}\| \leq a\frac{n}{n-1} < 1$.

That is B is relatively bounded with respect to $\widetilde{A}$ with relative bound $b < 1$. Now an application of the previous result to the pair $\widetilde{A}, B$ does the job. □

Combining these two we get

Proposition 3.6.4 *Let A, B be operators on the Hilbert space $\mathcal{H}$ such that*

(i) A is selfadjoint with compact resolvent;

(ii) B is symmetric with $Dom(A) \subseteq Dom(B)$, and relatively bounded with respect to A with relative bound less than 1.

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{x\to\pm\infty} f(x) = \pm 1$ such that $t \mapsto f(A + tB)$ is norm continuous.

Proof: If A is invertible then by lemma 3.6.2, $f(x) = \frac{2}{\pi}\tan^{-1}(x)$ serves the purpose. In the other case by lemma 3.6.3, $f(x) = \frac{2}{\pi}\tan^{-1}(x+\kappa)$ does the job. □

Let the Hilbert space $\mathcal{H}$ and the operators D_0, B, D be as in corollary 3.2.3.

Corollary 3.6.5 *The Kasparov module associated with $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ is operatorially homotopic with $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)$*

Proof: Let $D_t = D_0 + tB$ for $t \in [0,1]$. Then $D = D_1$ and as in remark 3.2.4, $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_t)$ are spectral triples. Let f be the function obtained from the previous proposition for the pair D_0, B. Then $((\mathcal{A}_\hbar, \mathcal{H}, f(D_t)))_{t\in[0,1]}$ gives the desired homotopy. □

As remarked earlier, the operator D_0 depends on a real parameter $\alpha > 1$. Now we will make that explicit and denote D_0 by $D_0^{(\alpha)}$.

Proposition 3.6.6 *The Kasparov modules associated with $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0^{(\alpha)})$ are operatorially homotopic for $\alpha > 1$.*

Proof: By proposition 3.1.8, $\mathcal{H} = L^2(\mathbb{T}\times\mathbb{T}\times\mathbb{Z})\otimes\mathbb{C}^2$. Let B be the operator $-2\pi c M_p \otimes \sigma_3$. Here p denotes the $\mathbb{Z}$ variable in the L^2 space. Then B is selfadjoint with $D(D_0^{(\alpha)}) \subseteq D(B)$. Also B is relatively bounded with respect to $D_0^{(\alpha)}$ with relative bound less than $\frac{1}{\alpha} < 1$. Let $D_t^{(\alpha)} = D_0^{(\alpha)} + tB$ for $t \in [0,1]$. Then $D_t^{(\alpha)} = D_0^{(\alpha+t)}$. Let f be the function obtained from proposition 3.6.4 for the pair $D_0^{(\alpha)}, B$. Then from the norm continuity of $t \mapsto f(D_0^{(\alpha+t)})$ we see the Kasparov modules $((\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0^{(\alpha+t)}))_{t\in[0,1]}$ are homotopic. Since α is arbitrary this completes the proof. □

Remark 3.6.7 Proposition 3.6.6 and corollary 3.6.5 together imply the Kasparov module associated with the spectral triple $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ is independent of α.

In the next proposition we show $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ has non trivial chern character.

Proposition 3.6.8 *The Kasparov module associated with $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ gives a nontrivial element in $K^1(\mathcal{A}_\hbar)$.*

Proof: By corollary 3.6.5, $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ and $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)$ give rise to same element $[(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)]$ in $K^1(\mathcal{A}_\hbar)$. Let $\phi \in \mathcal{A}_\hbar^\infty$ be the unitary whose symbol in S^c is given by $\phi(x,y,p) = \delta_{0p}e^{2\pi i y}$. This gives an element $[\phi] \in K_1(\mathcal{A}_\hbar)$. It suffices to show $\langle[\phi], [(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)]\rangle \neq 0$ where the pairing $\langle\cdot,\cdot\rangle : K_1(\mathcal{A}_\hbar) \times K^1(\mathcal{A}_\hbar) \to \mathbb{Z}$ is the one coming from the Kasparov product. ϕ acts on $L^2(\mathcal{A}_\hbar)\otimes\mathbb{C}^2 \cong L^2([0,1]\times\mathbb{T}\times\mathbb{Z})\otimes\mathbb{C}^2$ as a composition of two commuting unitaries $U_1 = M_{e(y)} \otimes I_2$ and $U_2 = M_{e(p\nu\hbar)} \otimes I_2$. Then note U_2 commutes with D_0. Let E be the projection $E = I(D_0 \geq 0)$. U_2 also commutes with E. Now by theorem 1.3.6, EU_1U_2E is a Fredholm operator and $\langle[\phi], [(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)]\rangle = Index(EU_1U_2E) = Index(EU_1E)$, last equality holds because U_2 commutes with E. Now $Index(EU_1E) \neq 0$ because this is the index pairing of the Dirac operator on $\mathbb{T}^3$ with the unitary U_1. □

3.7 Invariance of Chern character in entire cyclic cohomology

Now we will show that Chern character associated with the spectral triples considered above is same. We begin with a general proposition of invariance of Chern character under relatively bounded perturbations, which is an adaptation of the arguments given in proposition 2.4 in [39].

Let $\mathcal{A}$ be a Banach algebra, and $(\mathcal{H}, \mathcal{D}_0)$ be an odd unbounded theta summable Fredholm module in the sense of section 1.4.1. Suppose we are given another self adjoint operator Δ such that $a \mapsto [\Delta, \pi(a)]$ defines a bounded derivation and Δ is relatively bounded with respect to $\mathcal{D}_0$ with relative bound β strictly less than one. Then we have:

Lemma 3.7.1 $(\mathcal{H}, \mathcal{D}_t = \mathcal{D}_0 + t\Delta)$ *for* $0 \leq t \leq 1$ *define odd theta summable fredholm modules.*

Proof: Clearly $\mathcal{D}_t$ defines a self adjoint operator and $a \mapsto [\mathcal{D}_t, \pi(a)]$ defines a bounded derivation. It only remains to show that $Tr \exp(-s\mathcal{D}_t^2)$ is finite for all $s > 0$. For that note for bounded operators B_1, B_2, with B_1 compact, we have

$$\mu_n(B_1B_2) \;\; \leq \;\; \mu_n(B_1)\|B_2\|, \tag{3.7.1}$$

where $\mu_n(\cdot)$ stands for the n^{th} largest singular value. Letting $\mu_{n,t} = n^{\text{th}}$ smallest singular value of $\mathcal{D}_t$, 3.7.1 along with the resolvent identity

$$(\mathcal{D}_t - i)^{-1} \;\; = \;\; (\mathcal{D}_0 - i)^{-1}\left(1 + t\Delta(\mathcal{D}_0 - i)^{-1}\right)^{-1} \tag{3.7.2}$$

gives

$$(\mu_{n,0}^2 + 1)\left(\frac{\beta - 1}{\beta}\right)^2 \;\; \leq \;\; \mu_{n,t}^2 + 1. \tag{3.7.3}$$

Now we are done by the finiteness of $\sum \exp(-s\mu_{n,0}^2) = Tr \exp(-s\mathcal{D}_0^2)$ □

Remark 3.7.2 From the proof of the previous lemma it also follows that $Tr \exp(-\mathcal{D}_t^2)$ is uniformly bounded.

Let $\widetilde{\mathcal{H}}$ be the $\mathbb{Z}/2$ graded Hilbert space given by $\widetilde{\mathcal{H}} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+ \cong \mathcal{H} \cong \mathcal{H}^-$. Let $\widetilde{\pi}$ be the representation given by $\widetilde{\pi} = \pi \oplus \pi$. Let $\widetilde{\mathcal{D}_0} = \begin{pmatrix} 0 & i\mathcal{D}_0 \\ -i\mathcal{D}_0 & 0 \end{pmatrix}$; similarly define $\widetilde{\Delta}$ and $\widetilde{\mathcal{D}_t}$, then $\widetilde{\mathcal{D}_t} = \widetilde{\mathcal{D}_0} + t\widetilde{\Delta}$. Let c_1 be the odd operator given by $c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then c_1 graded commutes with $\widetilde{\mathcal{D}_t}$'s and $\widetilde{\pi}(\mathcal{A})$. Consider the multilinear maps $\langle \cdot, \cdots, \cdot \rangle_{t,n} : \mathcal{B}(\widetilde{\mathcal{H}})^{\otimes(n+1)} \to \mathbb{C}$ given by

$$\langle A_0, \cdots, A_n \rangle_{t,n} = \int_{\Delta_n} Str(c_1 A_0 e^{-s_0 \widetilde{\mathcal{D}_t}^2} A_1 e^{-s_1 \widetilde{\mathcal{D}_t}^2} \cdots A_n e^{-s_n \widetilde{\mathcal{D}_t}^2}) d^n s,$$

where Δ_n denotes the n-simplex and the integration is with respect to the Lebesgue measure on that simplex. Str stands for super trace, explicitly given by $Str(A) = TrA|_{\mathcal{H}^+} - TrA|_{\mathcal{H}^-}$. The Chern character of the theta summable Fredholm modules $(\mathcal{H}, \mathcal{D}_t)$ is given by the entire cyclic cocycles on $\mathcal{A}$ given by the formula

$$Ch^n(\mathcal{D}_t)(a_0, \cdots, a_n) = \langle a_0, [\widetilde{\mathcal{D}_t}, a_1], \cdots, [\widetilde{\mathcal{D}_t}, a_n] \rangle_{t,n}.$$

Note that in the right hand side a_i actually stands for $\widetilde{\pi}(a_i)$. Our objective is to prove the following theorem.

Theorem 3.7.3 *The chern character $Ch^\bullet(\mathcal{D}_t)$ associated with the Fredholm modules $(\mathcal{H}, \mathcal{D}_t)$ are cohomologous for $0 \leq t \leq 1$.*

For ease of reference let us recall some results (lemma 2.1,2.2 from [39]).

Lemma 3.7.4 *(i) If the operators $A_j, G_j, j = 0, \cdots, n$ are bounded and at most $(k+1)$ of the A_j's are nonzero, then for $0 < \epsilon < (2e)^{-1}$,*

$$|\langle A_0 \widetilde{\mathcal{D}_t} + G_0, \cdots, A_n \widetilde{\mathcal{D}_t} + G_n \rangle_{t,n}| \leq (2e\epsilon)^{-(k+1)/2} \frac{\Gamma(1/2)^{k+1}}{\Gamma((2n-k+1)/2)} Tr e^{-(1-\epsilon)\widetilde{\mathcal{D}_t}^2} \Pi_0^n (\|A_j\| + \|G_j\|).$$

(ii) In each of the following cases we assume that the operators A_i are such that each term is well defined. For an operator A, $|A| = 0$ if A is even, $|A| = 1$ if A is odd.

(a) $\langle A_0, \cdots, A_n \rangle_{t,n} = (-1)^{(|A_0| + \cdots + |A_{j-1}|)(|A_j| + \cdots + |A_n|)} \times \langle A_j, \cdots, A_n, A_0, \cdots, A_{j-1} \rangle_{t,n}$;

(b) $\langle A_0, \cdots, A_n \rangle_{t,n} = \sum_0^n (-1)^{(|A_0| + \cdots + |A_{j-1}|)(|A_j| + \cdots + |A_n|)} \langle 1, A_j, \cdots, A_n, A_0, \cdots, A_{j-1} \rangle_{t,n+1}$;

(c) $\sum_0^n (-1)^{|A_0| + \cdots |A_{j-1}|} \langle A_0, \cdots, [\widetilde{\mathcal{D}_t}, A_j], \cdots A_n \rangle_{t,n} = 0$;

(d) $\langle A_0, \cdots, [\widetilde{\mathcal{D}_t}^2, A_j], \cdots A_n \rangle_{t,n} = \langle A_0, \cdots, A_{j-1} A_j, A_{j+1}, \cdots, A_n \rangle_{t,n-1}$
$\qquad - \langle A_0, \cdots, A_{j-1}, A_j A_{j+1}, \cdots, A_n \rangle_{t,n-1}$;

(e) $\frac{d}{dt} \langle A_0, \cdots, A_n \rangle_{t,n} + \sum_0^n \langle A_0, \cdots, A_j, [\widetilde{\mathcal{D}_t}, \dot{\widetilde{\mathcal{D}_t}}], A_{j+1}, \cdots, A_n \rangle_{t,n+1} = 0$.

Proof of the Theorem: Let $A_0, A_1, \cdots, A_n, G$ be bounded operators. Then,
(a) observe that

$$\begin{aligned} &|\langle A_0, \cdots, A_j, G\widetilde{\Delta}, A_{j+1}, \cdots, A_n\rangle_{t,n+1}| \\ &= |\langle A_0, \cdots, A_i, G\widetilde{\Delta}(\widetilde{\mathcal{D}_t}+i)^{-1}(\widetilde{\mathcal{D}_t}+i), A_{i+1}, \cdots, A_n\rangle_{t,n+1}| \\ &\leq 2(2e\epsilon)^{-1/2}\frac{\beta}{\beta-1}\|G\|\Pi_0^n\|A_i\|\frac{\Gamma(1/2)}{\Gamma((2n+1)/2)}Tre^{-(1-\epsilon)\widetilde{\mathcal{D}_t}^2} \end{aligned}$$

Therefore,

$$\widetilde{Ch}^*(\widetilde{\mathcal{D}_t}, \widetilde{\Delta})((a_0, \cdots, a_n)) = \sum_0^n (-1)^j \langle a_0, [\widetilde{\mathcal{D}_t}, a_1], \cdots, [\widetilde{\mathcal{D}_t}, a_j], \widetilde{\Delta}, [\widetilde{\mathcal{D}_t}, a_{j+1}], \cdots, [\widetilde{\mathcal{D}_t}, a_n]\rangle_{t,n+1}$$

defines an entire cochain.
(b) Note that

$$\langle A_0, \cdots, \widetilde{\Delta}A_j, \cdots, A_n\rangle_{t,n} = \langle A_0, \cdots, (\widetilde{\mathcal{D}_t}+i)(\widetilde{\mathcal{D}_t}+i)^{-1}\widetilde{\Delta}A_j, \cdots, A_n\rangle_{t,n}$$

Left hand side is well defined by (i) of lemma 3.7.4 implying that the right hand side is well defined too. Therefore,

$$\alpha^*(\widetilde{\mathcal{D}_t}, \widetilde{\Delta})((a_0, \cdots, a_n)) = \sum_0^n \langle a_0, [\widetilde{\mathcal{D}_t}, a_1], \cdots, [\widetilde{\Delta}, a_j], \cdots, [\widetilde{\mathcal{D}_t}, a_n]\rangle_{t,n}$$

defines an entire cochain.
(c) Again as in (b) it is easily seen that expressions like $\langle A_0, \cdots, A_j, \widetilde{\Delta}\widetilde{\mathcal{D}_t}, A_{j+1}, \cdots, A_n\rangle_{t,n+1}$, and $\langle A_0, \cdots, A_j, \widetilde{\mathcal{D}_t}\widetilde{\Delta}, A_{j+1}, \cdots, A_n\rangle_{t,n+1}$ make perfect sense. So, that we can talk about $\langle A_0, \cdots, A_j, [\widetilde{\mathcal{D}_t}, \widetilde{\Delta}], A_{j+1}, \cdots, A_n\rangle_{t,n+1}$, which is nothing but $\langle A_0, \cdots, A_j, [\widetilde{\mathcal{D}_t}, \dot{\widetilde{\mathcal{D}_t}}], A_{j+1}, \cdots, A_n\rangle_{t,n+1}$. Now we are in a position to apply (ii)(c) of lemma 3.7.4 to the following choice

$$A_j = \begin{cases} a_0 & \text{for } j = 0, \\ [\widetilde{\mathcal{D}_t}, a_j] & \text{for } j \leq k, \\ \widetilde{\Delta} & \text{for } j = k+1, \\ [\widetilde{\mathcal{D}_t}, a_{j-1}] & \text{for } j \geq k+2. \end{cases}$$

This gives,

$$X_1 + X_2 + X_3 = 0, \tag{3.7.4}$$

where

$$X_1 = (-1)^k \langle [\widetilde{\mathcal{D}_t}, a_0], [\widetilde{\mathcal{D}_t}, a_1], \cdots, [\widetilde{\mathcal{D}_t}, a_k], \widetilde{\Delta}, [\widetilde{\mathcal{D}_t}, a_{k+1}], \cdots, [\widetilde{\mathcal{D}_t}, a_n]\rangle_{t,n+1},$$

$$\begin{aligned}X_2 &= \sum_{j<k}(-1)^{j+k-1}\langle a_0,[\widetilde{\mathcal{D}_t},a_1],\cdots,[\widetilde{\mathcal{D}_t}^2,a_j],\cdots,[\widetilde{\mathcal{D}_t},a_k],\widetilde{\Delta},\cdots,[\widetilde{\mathcal{D}_t},a_n]\rangle_{t,n+1}\\ &\quad+\sum_{j>k}(1)^{k+j}\langle a_0,[\widetilde{\mathcal{D}_t},a_1],\cdots,[\widetilde{\mathcal{D}_t},a_k],\widetilde{\Delta},\cdots[\widetilde{\mathcal{D}_t}^2,a_j],\cdots,[\widetilde{\mathcal{D}_t},a_n]\rangle_{t,n+1},\\ X_3 &= \langle a_0,[\widetilde{\mathcal{D}_t},a_1],\cdots,[\widetilde{\mathcal{D}_t},a_k],[\widetilde{\mathcal{D}_t},\widetilde{\Delta}],[\widetilde{\mathcal{D}_t},a_{k+1}],\cdots,[\widetilde{\mathcal{D}_t},a_n]\rangle_{t,n+1}.\end{aligned}$$

We now sum (3.7.4) over $0 \le k \le n$. By lemma 3.7.4(ii)(b) we see after reordering terms that

$$\sum_k X_1 = -(B\widetilde{Ch}^*(\widetilde{\mathcal{D}_t},\widetilde{\Delta}))((a_0,\cdots,a_n)). \qquad (3.7.5)$$

Similarly, using lemma 3.7.4(ii)(d),

$$\sum_k X_2 = -(b\widetilde{Ch}^*(\widetilde{\mathcal{D}_t},\widetilde{\Delta}))((a_0,\cdots,a_n)) + \alpha^*(\widetilde{\mathcal{D}_t},\widetilde{\Delta})((a_0,\cdots,a_n)). \qquad (3.7.6)$$

Here b, B are the boundary operators in entire cyclic theory (see 1.4.2, 1.4.3). Combining (3.7.4), (3.7.5), and (3.7.6) along with the expression for X_3 we get,

$$\begin{aligned}&\frac{dCh^*(\mathcal{D}_t)}{dt}(a_0,\cdots,a_n)\\ &= \alpha^*(\widetilde{\mathcal{D}_t},\widetilde{\Delta})((a_0,\cdots,a_n)) + \sum_k\langle a_0,[\widetilde{\mathcal{D}_t},a_1],\cdots,[\widetilde{\mathcal{D}_t},a_k],[\widetilde{\mathcal{D}_t},\widetilde{\Delta}],[\widetilde{\mathcal{D}_t},a_{k+1}],\cdots,[\widetilde{\mathcal{D}_t},a_n]\rangle_{t,n+1}\\ &= (B+b)\widetilde{Ch}^*(\widetilde{\mathcal{D}_t},\widetilde{\Delta})((a_0,\cdots,a_n)).\end{aligned}$$

□

Let the Hilbert space $\mathcal{H}$ and the operators D_0, B, D be as in corollary 3.2.3. $\mathcal{A}_\hbar^1$ defined as $\{a \in \mathcal{A}_\hbar | [D_0,a],[B,a] \in \mathcal{B}(\mathcal{H})\}$ becomes a Banach algebra with the norm $\|a\|_n = \max\{\|a\| + \|[D_0,a]\|, \|a\| + \|[B,a]\|\}$. Let $\mathcal{D}_0 = D_0, \Delta = B$, then with these choice $\mathcal{A}_\hbar^1, \mathcal{H}, \mathcal{D}_0, \Delta$ satisfy all the hypothesis required for applying theorem 3.7.3 by which we get,

Corollary 3.7.5 *The Chern character associated with the spectral triples $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$, and $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D_0)$ are cohomologous.*

Remark 3.7.6 The spectral triple $(\mathcal{A}_\hbar^\infty, \mathcal{H}, D)$ depends on a real number $\alpha > 1$. An argument very similar to proposition 3.6.6 will show that Chern character associated with this whole family of spectral triples is independent of α.

Chapter 4

Quantum SU(2) and Sphere

Study of quantum groups originated in the early eighties in the work of Fadeev, Sklyanin & Takhtajan in the context of quantum inverse scattering theory. It picked up momentum during the mid eighties, and connections were established with various other areas in mathematics. They were first studied in the topological setting by Woronowicz, who treated the q-deformation of the $SU(2)$ group in [85] and then went on to characterize the family of compact quantum groups and studied their representation theory. Later Podles ([67]) constructed quantum spheres S^2_{qc} as homogeneous space for $SU_q(2)$. In the context of Lie groups and their homogeneous spaces they have their own geometry. Now, the natural question is can one do geometry in the context of $SU_q(2)$ and S^2_{qc}. Here to construct explicit spectral triple we begin with computation of K-groups, and then from explicit generators we construct spectral triples which induce generating elements in K-homology. We also compute a modified version of the space of Connes deRham forms and an associated calculi. The space of L^2 forms also have been described explicitly.

4.1 Quantum SU(2) as a C^*-algebra

The quantized version of the theory of topological groups was initiated by Woronowicz. His first example was quantum SU(2) group. In this section we briefly recall the definition of a compact quantum and describe the C^*-algebra associated with $SU_q(2)$.

We will throughout assume q is a real parameter lying between zero and one.

Definition 4.1.1 (Woronowicz) Let $\mathcal{A}$ be a separable unital C^*-algebra, and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a unital $*$-homomorphism. We call $G = (\mathcal{A}, \Delta)$ a compact quantum group if the following two conditions are satisfied:

(i) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$,
(ii) Linear spans of both $\{(a \otimes I)\Delta(b) : a, b \in \mathcal{A}\}$ and $\{(I \otimes a)\Delta(b) : a, b \in \mathcal{A}\}$ are dense in $\mathcal{A} \otimes \mathcal{A}$.

Δ is called the comultiplication map associated with G. The underlying C^*-algebra is often denoted $C(G)$.

A linear functional on the C^*-algebra $C(G)$ plays the role of a measure on G. Using the comultiplication Δ, one can define a convolution product between two linear functionals ρ_1 and ρ_2:

$$\rho_1 \star \rho_2(a) = (\rho_1 \otimes \rho_2)\Delta(a), \quad \forall a \in C(G).$$

It is easy to check that if G is a group, this notion reduce to the usual convolution product of two measures.

A bounded linear functional λ is said to be right invariant if for any continuous functional ρ on $C(G)$, we have $\lambda \star \rho = \rho(I)\lambda$. Similarly, λ is left invariant if $\rho \star \lambda = \rho(I)\lambda$ for all ρ. As before one can easily check that these coincides with the usual notions if G is a group.

Theorem 4.1.2 (Woronowicz) *Let $G = (\mathcal{A}, \Delta)$ be a compact quantum group. Then, there exists a unique state h on $\mathcal{A}$, called the Haar state such that*

$$h \star \rho = \rho \star h = \rho(I)h$$

for any continuous linear functional ρ on $\mathcal{A}$.

In our case the C^*-algebra of continuous functions on the quantum $SU(2)$, to be denoted by $C(SU_q(2))$, is the universal C^*-algebra generated by two elements α and β satisfying the following relations:

$$\begin{aligned} \alpha^*\alpha + \beta^*\beta = I, &\qquad \alpha\alpha^* + q^2\beta\beta^* = I, \\ \alpha\beta - q\beta\alpha = 0, &\qquad \alpha\beta^* - q\beta^*\alpha = 0, \\ \beta^*\beta &= \beta\beta^*. \end{aligned}$$

The comultiplication map Δ, is a unital $*$-homomorphism from $C(SU_q(2))$ to $C(SU_q(2)) \otimes C(SU_q(2))$, given by:

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha - q\beta^* \otimes \beta, \\ \Delta(\beta) &= \beta \otimes \alpha + \alpha^* \otimes \beta. \end{aligned}$$

The C^*-algebra $C(SU_q(2))$ can be described more concretely as follows. Let $\{e_i\}_{i\geq 0}$ and $\{e_i\}_{i\in\mathbb{Z}}$ be the canonical orthonormal bases for $L_2(\mathbb{N}_0)$ and $L_2(\mathbb{Z})$ respectively. We denote by the same symbol N the operator $e_k \mapsto ke_k$, $k \geq 0$, on $L_2(\mathbb{N}_0)$ and $e_k \mapsto ke_k$, $k \in \mathbb{Z}$, on $L_2(\mathbb{Z})$. Similarly, denote by the same symbol ℓ the operator $e_k \mapsto e_{k-1}$, $k \geq 1$, $e_0 \mapsto 0$ on $L_2(\mathbb{N}_0)$ and the operator $e_k \mapsto e_{k-1}$, $k \in \mathbb{Z}$ on $L_2(\mathbb{Z})$. Now take $\mathcal{H}$ to be the Hilbert space $L_2(\mathbb{N}_0) \otimes L_2(\mathbb{Z})$, and define π to be the following representation of $C(SU_q(2))$ on $\mathcal{H}$:

$$\pi(\alpha) = \ell\sqrt{I - q^{2N}} \otimes I, \quad \pi(\beta) = q^N \otimes \ell.$$

Then π is a faithful representation of $C(SU_q(2))$, so that one can identify $C(SU_q(2))$ with the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\pi(\alpha)$ and $\pi(\beta)$. Image of π contains $\mathcal{K} \otimes C(\mathbb{T})$ as an ideal with $C(\mathbb{T})$ as the quotient algebra, that is we have a useful short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0. \tag{4.1.1}$$

The homomorphism σ is explicitly given by $\sigma(\alpha) = \ell, \sigma(\beta) = 0$. The Haar state h on $C(SU_q(2))$ is given by,

$$h : a \mapsto (1 - q^2) \sum_{i=0}^{\infty} q^{2i} \langle e_i \otimes e_0, ae_i \otimes e_0 \rangle.$$

Remark 4.1.3 This representation admits a nice interpretation. Let M be a compact topological manifold and E, a Hermitian vector bundle on M. Let $\Gamma(M, E)$ be the space of continuous sections. Then $\Gamma(M, E)$ is a finitely generated projective $C(M)$ module. Define an inner product on $\Gamma(M, E)$ as

$$\langle s_1, s_2 \rangle := \int (s_1(m), s_2(m))_m d\nu(m),$$

where ν is a smooth measure on M and $(\cdot,\cdot)_m$ is the inner product on the fibre on m. Let $\mathcal{H}_E$ be the Hilbert space completion of $\Gamma(M, E)$. Then we have a natural representation of $C(M)$ in $\mathcal{B}(\mathcal{H}_E)$. The same program can be carried out in the noncommutative context also. Let $\mathcal{A}$ be a C^*-algebra and E a Hilbert C^*-$\mathcal{A}$- module with its $\mathcal{A}$ valued inner product $\langle \cdot,\cdot \rangle_{\mathcal{A}}$. Let τ be a state on $\mathcal{A}$. Consider the inner product on E given by $\langle e_1, e_2 \rangle = \tau(\langle e_1, e_2 \rangle_{\mathcal{A}})$. If we denote by $\mathcal{H}_E$ the Hilbert space completion of E, then we get a natural representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_E)$. Now in the context of $C(SU_q^2)$ let $p = |e_0\rangle\langle e_0| \otimes I \in C(SU_q^2)$. Then it is easy to verify that $\mathcal{H}_E = l^2(\mathbb{N}_0) \otimes l^2(\mathbb{Z})$, for $E = C(SU_q^2)p$ with its natural left Hilbert $C(SU_q^2)$-module structure. Moreover, the associated representation is nothing but the concrete representation of $C(SU_q^2)$ described above.

4.2 Generators of K-homology

One way to have some idea about spectral triple is to compute the generators of K-homology. We will write $\mathcal{A}$ for the C^*-algebra $C(SU_q(2))$ and $\mathcal{A}_f$ for the *-subalgebra of $C(SU_q(2))$ generated by the two elements α and β. Restriction of π to $\mathcal{A}_f$ gives a representation of $\mathcal{A}_f$ on $\mathcal{H}$, which we denote by the same symbol π. The short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0 \tag{4.2.1}$$

gives rise to the following six-term exact sequence

$$\begin{array}{ccccc} K^0(C(\mathbb{T})) & \xrightarrow{\sigma^0} & K^0(\mathcal{A}) & \xrightarrow{i^0} & K^0(\mathcal{K}\otimes C(\mathbb{T})) \\ \uparrow & & & & \downarrow \\ K^1(\mathcal{K}\otimes C(\mathbb{T})) & \xleftarrow{i^1} & K^1(\mathcal{A}) & \xleftarrow{\sigma^1} & K^1(C(\mathbb{T})). \end{array}$$

It is known that $K_0(\mathcal{A}) = \mathbb{Z} = K_1(\mathcal{A})$. Since these are free abelian groups, It follows from the results of Rosenberg-Schochet ([79]) that $K^0(\mathcal{A}) = \mathbb{Z} = K^1(\mathcal{A})$. Therefore the six term sequence above becomes

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\sigma^0} & \mathbb{Z} & \xrightarrow{i^0} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ K^1(\mathcal{K}\otimes C(\mathbb{T})) & \xleftarrow{i^1} & \mathbb{Z} & \xleftarrow{\sigma^1} & K^1(C(\mathbb{T})) \end{array}$$

Lemma 4.2.1 *i^1 and σ^0 are isomorphisms while i^0 and σ^1 are zero morphisms.*

Proof: We know that $K^1(\mathcal{K}\otimes C(\mathbb{T})) \cong K^1(C(\mathbb{T})) \cong \mathbb{Z}$. Therefore by the exactness of the diagram above it is enough to show that i^1 is onto. For that observe $\mathcal{H} = L_2(\mathbb{N}_0) \otimes L_2(\mathbb{Z})$ and $F = I \otimes S$, where, S denotes the operator

$$S : e_k \mapsto \begin{cases} e_k & \text{if } k \geq 0, \\ -e_k & \text{if } k < 0, \end{cases}$$

is an odd Fredholm module on $\mathcal{A}$ and hence on $i(\mathcal{K}\otimes C(\mathbb{T})) \subseteq \mathcal{A}$. Moreover this Fredholm module is a generator of $K^1(\mathcal{K}\otimes C(\mathbb{T}))$ implying surjectivity of i^1. □

Remark 4.2.2 Proof of the above lemma also shows that $(\mathcal{H}, F)$ is a generating Fredholm module for $K^1(\mathcal{A})$.

4.3 Spectral triples

In this section we construct spectral triples with nontrivial Chern character. For $p \in (0, \infty)$, let D_p be the operator $N^p \otimes S + I \otimes N$ on $\mathcal{H} = L_2(\mathbb{N}_0) \otimes L_2(\mathbb{Z})$ with S as defined above.

Proposition 4.3.1 *Let $\mu_n(T)$ denote the n^{th} largest singular value of an opeartor T. Then*

$$\mu_n(|D_p|^{-1-\frac{1}{p}}) \sim \frac{1}{n}.$$

Proof: Check that the action of $|D_p|$ on $\mathcal{H}$ is given by $e_i \otimes e_j \mapsto (i^p + |j|)e_i \otimes e_j$. If we denote by λ_r the number of elements in $\{(i,j) : i, j \in \mathbb{N}_0, i^p + j \leq r\}$, then a simple calculation tells us that $\frac{\lambda_r}{r^{1+\frac{1}{p}}} \to \frac{2p}{1+p}$ as $r \to \infty$. It follows from this that the $\mathrm{n^{th}}$ eigenvalue of $|D_p|$ is of the order of $n^{\frac{1}{1+1/p}}$, which gives us the required result. □

Notation: In all our discussion involving $\mathcal{A}_f$, α_i will stand for α^i for i nonnegative and $\alpha^{*|i|}$ for i negative.

Lemma 4.3.2 *Define a functional ϕ on $\mathcal{A}_f$ by*

$$\phi(a) := \lim_{t \to 0} t^{1+1/p}\, tr\,(a \exp(-tD_p^2)).$$

Then $\phi(\alpha_i \beta^j \beta^{*k}) = \delta_{i0}\delta_{j0}\delta_{k0}$. In particular, ϕ does not depend on p.

Proof: Observe that $\exp(-tD_p^2)e_r \otimes e_s = \exp(-t(r^p + |s|)^2)e_r \otimes e_s$, and

$$\alpha_i \beta^j \beta^{*k} e_r \otimes e_s \begin{cases} = q^{2rk} e_r \otimes e_s & \text{if } i = 0,\ j = k \\ \in \mathbb{C}e_{r-i} \otimes e_{s+j-k} & \text{otherwise.} \end{cases}$$

Hence we have

$$\mathrm{tr}\,(\alpha_i \beta^j \beta^{*k} \exp(-tD_p^2)) = \begin{cases} \sum_{r=0}^{\infty} \sum_{s \in \mathbb{Z}} q^{2rk} \exp(-t(r^p + |s|)^2) & \text{if } i = 0,\ j = k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore it follows that $\phi(\alpha_i \beta^j \beta^{*k}) = 0$ for $i \neq 0$. Now note that

$$\sum_{r=0}^{\infty} \sum_{s \in \mathbb{Z}} q^{2rk} \exp(-t(r^p + |s|)^2) = 2 \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} q^{2rk} \exp(-t(r^p + |s|)^2) + \sum_{r=0}^{\infty} q^{2rk} \exp(-tr^{2p}),$$

and

$$\begin{aligned} \sum_{s=1}^{\infty} q^{2rk} \exp(-t(r^p + |s|)^2) &< \int_0^{\infty} \exp(-t(r^p + x)^2)\, dx \\ &= \frac{1}{\sqrt{2t}} \int_{r^p\sqrt{2t}}^{\infty} \exp(-\frac{1}{2}y^2)\, dy \\ &< \frac{1}{\sqrt{2t}}. \end{aligned}$$

Hence for $i = 0$ and $j = k \neq 0$, $\phi(\alpha_i\beta^j\beta^{*k}) = \lim_{t\to 0} t^{1+1/p}\mathrm{tr}\,(\alpha_i\beta^j\beta^{*k}\exp(-tD_p^2)) = 0$. Finally, from the previous lemma, it follows that $\phi(I) = \mathrm{tr}_\omega(|D_p|^{-1-1/p}) = 1$. □

Proposition 4.3.3 *For each $p \in (0,1]$, $\mathcal{S}_p := (\mathcal{A}_f, \mathcal{H}, D_p)$ defines an odd spectral triple.*

Proof: Self-adjointness of D_p is trivial, and it follows from proposition 4.3.1 that D_p has compact resolvent. Let $\mathcal{H}_0 = \mathrm{span}\,\{e_i \otimes e_j : i \in \mathbb{N}_0, j \in \mathbb{Z}\}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and is invariant under the actions of D_p and the elements of $\mathcal{A}_f$. In view of this and the self-adjointness of D_p, it is enough to show that $[D_p, \alpha]$ and $[D_p, \beta]$ are bounded. Straightforward calculation now gives

$$\begin{aligned} [D_p, \alpha] &= \alpha(((N-I)^p - N^p) \otimes S), \\ [D_p, \beta] &= q^N N^p \otimes [S, \ell^*] + \beta. \end{aligned} \tag{4.3.2}$$

Therefore $\mathcal{S}_p$ is a spectral triple. □

Remark 4.3.4 The circle group $\mathbb{T}$ has an action on $\mathcal{A}$ given by $\phi_z : \alpha \mapsto z\alpha, \beta \mapsto \beta$, where $z \in \mathbb{T}$. D_p is equivariant with respect to this action. Equivariance follows from the fact that D_p commutes with the generator of the action $N \otimes I$. The consequence of equivariance under the full $SU_q(2)$ action will be treated in the next chapter.

Theorem 4.3.5 *The spectral triple $(\mathcal{A}_f, \mathcal{H}, D_p)$ has nontrivial Chern character.*

Proof: For this one only has to note that the operator F constructed in the proof of 4.2.1 is nothing but $sign(D_p)$. We give an explicit description of the pairing with $K_1(\mathcal{A})$. Let $E = \frac{1+F}{2} = I(N \geq 0)$ and $u = I_{\{1\}}(\beta^*\beta)(\beta - 1) + 1$. u gives an element $[u] \in K_1(\mathcal{A})$. By theorem 1.3.6 EuE is a Fredholm operator and $\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = Index(EuE)$. It is easily seen that the last quantity is -1. Since $K_1(\mathcal{A}) = \mathbb{Z}$, this shows $[u]$ generates $K_1(\mathcal{A})$ and describes the pairing with $K_1(\mathcal{A})$ completely. □

For ease of future reference we note down the following corollary which follows from the proof of this theorem.

Corollary 4.3.6 *Let $u = I_{\{1\}}(\beta^*\beta)(\beta - 1) + 1$. Then $[u]$ generates $K_1(\mathcal{A})$.*

4.4 Modified Connes-de Rham complex

In this section we will compute the complex $(\Omega_d^\bullet(\mathcal{A}_f), d)$ introduced in remark 1.6.5. Before we enter the computations, few words about why we compute these rather than the usual

$\Omega_D^\bullet(\mathcal{A}_f)$. First, since for a compact operator K one has $\mathrm{Tr}_\omega(K|D_p|^{-1-1/p}) = 0$, the results in proposition 5 page 550 [24] regarding the Yang-Mills functional holds in our present case. Second, in the context of the canonical spectral triple associated with a compact Riemannian spin manifold this prescription also gives back the exterior complex.

4.4.1 The case $p = 1$

We will write D for D_1 throughout this subsection.

First, we need the following lemma which will be very useful for the computations.

Lemma 4.4.1 *Assume $a, b \in \mathcal{A}_f$ and $c \in \mathcal{K}(\mathcal{H})$. If $a(I \otimes S) + b = c$, then $a = b = 0$.*

Proof: For a functional ρ on $\mathcal{L}(L_2(\mathbb{N}_0))$, and $T \in \mathcal{L}(\mathcal{H})$, denote by T_ρ the operator $(\rho \otimes \mathrm{id})T$. Now observe that for any $a \in \mathcal{A}_f$ and any functional ρ,

$$a_\rho \ell = \ell a_\rho. \tag{4.4.3}$$

Write $P = \frac{1}{2}(I + S)$. It is easy to see that the given condition implies that $(b_\rho - a_\rho) + 2a_\rho P = c_\rho$, which in turn implies that

$$(b_\rho - a_\rho)e_i = c_\rho e_i \quad \forall i < 0, \tag{4.4.4}$$

$$(b_\rho + a_\rho)e_i = c_\rho e_i \quad \forall i \geq 0. \tag{4.4.5}$$

Now from (4.4.3) and (4.4.4), it follows that for any $i, j \in \mathbb{Z}$ and $j < 0$,

$$\begin{aligned}
\|(b_\rho - a_\rho)e_i\| &= \|(b_\rho - a_\rho)\ell^{j-i}e_j\| \\
&= \|\ell^{j-i}(b_\rho - a_\rho)e_j\| \\
&= \|(b_\rho - a_\rho)e_j\| \\
&= \|c_\rho e_j\|.
\end{aligned}$$

Since c is compact, $\lim_{j\to-\infty} \|c_\rho e_j\| = 0$. Hence $(b_\rho - a_\rho)e_i = 0$ for all i. In other words, $(b_\rho - a_\rho) = 0$. Since this is true for any ρ, we get $a = b$. Using this equality, together with equations (4.4.3) and (4.4.5), a similar reasoning yields $a = 0$. □

Recall that the modified Connes-deRham complex introduced in remark 1.6.5 requires a homomorphism of the universal differential algebra constructed out of the spectral triple. There it was denoted as a composition $\theta \circ \pi$. In the rest of the chapter we will denote the corresponding homomorphism in the context of $\mathcal{A}_f$ by ψ. The kernel $\psi|_{\Omega^n(\mathcal{A}_f)}$ will be denoted by J_n.

Lemma 4.4.2 *Let $\mathcal{I}_\beta$ denote the ideal in $\mathcal{A}_f$ generated by β and β^*. Then for $n \geq 1$, we have*

$$\psi(\Omega^n(\mathcal{A}_f)) = (I \otimes S)^n \mathcal{A}_f + (I \otimes S)^{n+1}\mathcal{I}_\beta. \tag{4.4.6}$$

Proof: Let us first prove the equality for $n = 1$. Let $Z_k = q^{N+k}(N+k)$, $B_{jk} = \sum_{i=j-k+1}^{j} |e_{i-1}\rangle\langle e_i|$, and

$$C_j = \begin{cases} \sum_{i=0}^{j-1} |e_i\rangle\langle e_{i-1}| & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

It follows from (4.3.2) that

$$\begin{aligned} [D, \alpha_i\beta^j\beta^{*k}] &= -i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k} + 2(Z_i \otimes C_j)\alpha_i\beta^{j-1}\beta^{*k} \\ &\quad -2(Z_i \otimes B_{jk})\alpha_i\beta^j\beta^{*k-1}. \end{aligned} \tag{4.4.7}$$

Since the last two terms in equation (4.4.7) is compact, it follows from remark 1.6.5 that $d(\alpha_i\beta^j\beta^{*k}) = -i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k}$. Thus for any $a \in \mathcal{A}_f$,

$$da = (I \otimes S)b + c, \qquad \text{where } b \in \mathcal{A}_f, \quad c \in \mathcal{I}_\beta. \tag{4.4.8}$$

Note that for any $a' \in \mathcal{A}_f$, $\psi(a')(I \otimes S) = (I \otimes S)\psi(a')$ in $\mathcal{Q}(\mathcal{H})$. Hence $\psi(a'(\delta a))$ is again of the form $(I \otimes S)b + c$, where $b \in \mathcal{A}_f$, $c \in \mathcal{I}_\beta$, i.e. is a member of $(I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. Thus $\psi(\Omega^1(\mathcal{A}_f)) \subseteq (I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. For the reverse inclusion, observe that $(I \otimes S) = (1-q^2)^{-1}((d\alpha)\alpha^* + q^2(d\alpha^*)\alpha)$, $\beta = d\beta$ and $\beta^* = -d\beta^*$.

The inductive step follows easily from (4.4.8). □

Lemma 4.4.3 $J_0 = \{0\}$, *and for $n \geq 1$, we have*

$$\psi(\delta J_n) = (I \otimes S)^{n+1}\mathcal{A}_f + (I \otimes S)^{n+2}\mathcal{I}_\beta. \tag{4.4.9}$$

Proof: By lemma 4.4.1, $\psi : \mathcal{A}_f \to \mathcal{Q}(\mathcal{H})$ is faithful. Hence it follows that $J_0 = \{0\}$.

We will prove here (4.4.9) by induction. From lemma 4.4.2, we have $\psi(\delta J_1) \subseteq \psi(\Omega^2(\mathcal{A}_f)) = \mathcal{A}_f + (I \otimes S)\mathcal{I}_\beta$. Let us show that I, $(I \otimes S)\beta$ and $(I \otimes S)\beta^*$ are all members of $\psi(\delta J_1)$.

Choose $\omega \in \Omega^1(\mathcal{A}_f)$ such that $\psi(\omega) = (I \otimes S)$. Let $\omega_k = k\alpha_k\omega - \delta(\alpha_k), k = \pm 1$. Then it follows from (4.3.2) that $\psi(\omega_k) = k\alpha_k(I \otimes S) - k\alpha_k(I \otimes S) = 0$, so that $\omega_k \in J_1$. $\psi(\delta\omega_k) = \psi(k(\delta\alpha_k)\omega) = k^2\alpha_k = \alpha_k \in \psi(\delta J_1)$, i.e. both α and α^* are in $\psi(\delta J_1)$. It follows from this that $I \in \psi(\delta J_1)$.

Next we show that $(I \otimes S)\beta \in \psi(\delta J_1)$. Take $\omega = \frac{1}{2}(\alpha(\delta\beta) - \delta(\alpha\beta) + q\beta(\delta\alpha))$. Then $\psi(\omega) = 0$ and $\psi(\delta\omega) = (I \otimes S)\alpha\beta$. So $(I \otimes S)\alpha\beta \in \psi(\delta J_1)$. Similarly taking $\omega = \frac{1}{2}(\alpha^*(\delta\beta) - \delta(\alpha^*\beta) + q^{-1}\beta(\delta\alpha^*))$, it follows that $(I \otimes S)\alpha^*\beta \in \psi(\delta J_1)$. These two together imply $(I \otimes S)\beta \in \psi(\delta J_1)$.

A similar argument shows that $(I\otimes S)\beta^*$ is also in $\psi(\delta J_1)$. Thus $\mathcal{A}_f+(I\otimes S)\mathcal{I}_\beta = \psi(\delta J_1)$.

For the inductive step, notice that $\psi(\delta J_n)\subseteq \psi(\Omega^{n+1}(\mathcal{A}_f)) = (I\otimes S)^{n+1}\mathcal{A}_f+(I\otimes S)^{n+2}\mathcal{I}_\beta$. We will show that the following are all elements of $\psi(\delta J_n)$:

$$\begin{array}{lll}(I\otimes S)^{n+1}\alpha, & (I\otimes S)^{n+2}\alpha\beta, & (I\otimes S)^{n+2}\alpha\beta^*,\\ (I\otimes S)^{n+1}\alpha^*, & (I\otimes S)^{n+2}\alpha^*\beta, & (I\otimes S)^{n+2}\alpha^*\beta^*.\end{array}$$

From the right $\mathcal{A}_f$-module structure of $\psi(\delta J_n)$, it will then follow that $(I\otimes S)^{n+1}$, $(I\otimes S)^{n+2}\beta$ and $(I\otimes S)^{n+2}\beta^*$ are in $\psi(\delta J_n)$, giving us the other inclusion.

Choose $\omega\in J_{n-1}$ such that $\psi(\delta\omega) = (I\otimes S)^n$. Take $\omega_k = k\omega(\delta\alpha_k)$, $k=\pm 1$. Then $\omega_k\in J_n$ and $\psi(\delta\omega_k) = (I\otimes S)^{n+1}\alpha_k$. Similarly choosing ω such that $\psi(\delta\omega) = (I\otimes S)^{n+1}\beta$ and ω_k as before, we get $\omega_k\in J_n$ and $\psi(\delta\omega_k) = q^{-k}(I\otimes S)^{n+2}\alpha\beta$. Finally, take ω such that $\psi(\delta\omega) = (I\otimes S)^{n+1}\beta^*$ and ω_k as before to show that $(I\otimes S)^{n+2}\alpha_k\beta^*\in\psi(\delta J_n)$. □

Proposition 4.4.4

$$\Omega_d^n(\mathcal{A}_f) = \begin{cases}\mathcal{A}_f\oplus\mathcal{I}_\beta & \text{if } n=1,\\ \{0\} & \text{if } n\geq 2.\end{cases}$$

Proof: Proof follows from lemmas 4.4.2 and 4.4.3. □

4.4.2 The case $0<p<1$

Let us first introduce a few notations. Let X_{rs} denote the operator $(N+r)^p-(N+s)^p$, Z_r stand for $q^{N+r}(N+r)^p$ and let B_{rs} and C_r be as in the earlier subsection. We have, then,

$$\begin{aligned}[D_p,\alpha_i\beta^j\beta^{*k}] &= (I\otimes S)(X_{0i}\otimes I)\alpha_i\beta^j\beta^{*k}+(j-k)\alpha_i\beta^j\beta^{*k}+2(Z_i\otimes C_j)\alpha_i\beta^{j-1}\beta^{*k}\\ &\qquad -2(Z_i\otimes B_{jk})\alpha_i\beta^j\beta^{*k-1}.\\ &= (I\otimes S)(X_{0i}\otimes I)\alpha_i\beta^j\beta^{*k}+(j-k)\alpha_i\beta^j\beta^{*k}+\text{compact} \qquad (4.4.10)\end{aligned}$$

and hence,

$$\begin{aligned}[D_p,(X_{r_1s_1}\ldots X_{r_ks_k}\otimes I)\alpha_i\beta^j\beta^{*k}] &= (I\otimes S)(X_{r_1s_1}\ldots X_{r_ks_k}X_{0i}\otimes I)\alpha_i\beta^j\beta^{*k}\\ &\quad +(j-k)(X_{r_1s_1}\ldots X_{r_ks_k}\otimes I)\alpha_i\beta^j\beta^{*k}+\text{compact} \qquad (4.4.11)\end{aligned}$$

We will work with the algebra $\widetilde{\mathcal{A}}_f$ generated by $\{X_{rs}\otimes I) : r,s\in\mathbb{Z}\}$ and the elements of $\mathcal{A}_f$. Note that $\widetilde{\mathcal{A}}_f$ is nothing but the span of $\{(X_{0s_1}\ldots X_{0s_n}\otimes I)\alpha_i\beta^j\beta^{*k}\}$. Now first of all observe that in the proof of lemma 4.4.1, the only property of $\mathcal{A}_f$ that has been used is that

$$a(I\otimes\ell) = (I\otimes\ell)a \qquad (4.4.12)$$

for all $a \in \mathcal{A}_f$, so that one has equation (4.4.3). Since (4.4.12) is satisfied by elements of $\widetilde{\mathcal{A}}_f$ also, it follows that lemma 4.4.1 remains valid even when $\mathcal{A}_f$ is replaced by the bigger algebra $\widetilde{\mathcal{A}}_f$.

Lemma 4.4.5 *Let $\widetilde{\mathcal{I}}_\beta$ denote the ideal in $\widetilde{\mathcal{A}}_f$ generated by β and β^*. Then for $n \geq 1$, we have*

$$\psi(\Omega^n(\widetilde{\mathcal{A}}_f)) = (I \otimes S)^n \widetilde{\mathcal{A}}_f + (I \otimes S)^{n+1} \widetilde{\mathcal{I}}_\beta. \tag{4.4.13}$$

Lemma 4.4.6 *Let $\widetilde{J}_n$ be the kernel of ψ restricted to $\Omega^n(\widetilde{\mathcal{A}}_f)$. Then $\widetilde{J}_0 = \{0\}$, and for $n \geq 1$, we have*

$$\psi(\delta \widetilde{J}_n) = (I \otimes S)^{n+1} \widetilde{\mathcal{A}}_f + (I \otimes S)^{n+2} \widetilde{\mathcal{I}}_\beta. \tag{4.4.14}$$

Proof: Arguments used for proving lemma 4.4.3 goes through. □

Proposition 4.4.7

$$\Omega_d^n(\widetilde{\mathcal{A}}_f) = \begin{cases} \widetilde{\mathcal{A}}_f \oplus \widetilde{\mathcal{I}}_\beta & \text{if } n = 1, \\ \{0\} & \text{if } n \geq 2. \end{cases}$$

Proof: Lemma 4.4.5, and 4.4.6 yields this as in proposition 4.4.4. □

4.5 L^2-complex of Frohlich et. al.

In this section we will compute the complex of square integrable forms for spectral triple corresponding to $p = 1$. For that we begin with similar computations for the spectral triple $(\mathbb{C}[z, z^{-1}], \mathcal{H}_0 = L_2(\mathbb{Z}), \mathfrak{D} = N)$ associated with the algebra $\mathbb{C}[z, z^{-1}]$. Here we consider the embedding $\pi : \mathbb{C}[z, z^{-1}] \to \mathcal{B}(\mathcal{H})$ that maps z to ℓ.

Lemma 4.5.1 *(i) $\widetilde{\Omega}_{\mathfrak{D}}^n(\mathbb{C}[z, z^{-1}]) = 0$, for $n \geq 2$,*
(ii) $\widetilde{\Omega}_{\mathfrak{D}}^1(\mathbb{C}[z, z^{-1}]) = \mathbb{C}[z, z^{-1}]$.

Proof: (i) Let $\omega = \sum \alpha_{n_0, \cdots, n_k} z^{n_0} \delta z^{n_1} \cdots \delta z^{n_k} \in \Omega^k(\mathbb{C}[z, z^{-1}])$, where the sum is a finite one and δ is the universal differential. Then it is easily verified that

$$(\omega, \omega)_{\mathfrak{D}} = \int \Big(\sum n_1 \cdots n_k \alpha_{n_0, \cdots, n_k} z^{\sum_0^k n_j}\Big)^* \Big(\sum n_1 \cdots n_k \alpha_{n_0, \cdots, n_k} z^{\sum_0^k n_j}\Big) dz,$$

where dz is the Lebesgue measure on the circle. Therefore,

$$\mathsf{K}_k(\mathbb{C}[z, z^{-1}]) \;=\; \{\omega \in \Omega^k(\mathbb{C}[z, z^{-1}]) : (\omega, \omega)_{\mathfrak{D}} = 0\}$$

$$= \{\sum \alpha_{n_0,\cdots,n_k} z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} : \sum_{n_0+\cdots+n_k=r} n_1\cdots n_k \alpha_{n_0,\cdots,n_k} = 0, \forall r\}.$$

Consequently we have,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} - n_1\cdots n_k z^{\sum_0^k n_i - k}\delta z\cdots\delta z \in \mathsf{K}_k(\mathbb{C}[z,z^{-1}]), \tag{4.5.15}$$

$$\delta z^r \delta z\cdots\delta z - r z^r \delta z\cdots\delta z \in \mathsf{K}_k(\mathbb{C}[z,z^{-1}]), \tag{4.5.16}$$

$$z^r\delta z\cdots\delta z - \frac{1}{r+1}\delta z^{r+1}\delta z\cdots\delta z \in \mathsf{K}_{k-1}(\mathbb{C}[z,z^{-1}]). \tag{4.5.17}$$

From (4.5.17) we get $\delta z^r\delta z\cdots\delta z \in \delta\mathsf{K}_{k-1}(\mathbb{C}[z,z^{-1}])$. Combining this with (4.5.15) and (4.5.16) we get,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in \mathsf{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathsf{K}_{k-1}(\mathbb{C}[z,z^{-1}]) \text{ for large } n_0.$$

Since $\mathsf{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathsf{K}_{k-1}(\mathbb{C}[z,z^{-1}])$ is a bimodule we have

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in \mathsf{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathsf{K}_{k-1}(\mathbb{C}[z,z^{-1}]) \quad \forall n_0,\cdots,n_k.$$

This proves (i).

(ii) It suffices to note that

$$z^{n_0}\delta z^{n_1} - n_1 z^{n_0+n_1-1}\delta z \in \mathsf{K}_1(\mathbb{C}[z,z^{-1}]).$$

The induced $d : \widetilde{\Omega}^0_{\mathfrak{D}}(\mathbb{C}[z,z^{-1}]) \to \mathbb{C}[z,z^{-1}]$ is given by $d(z^n) = nz^n$. □

Now we are in a position to compute the complex of square integrable forms for $\mathcal{A}_f$ for the spectral triple associated with $p=1$.

Theorem 4.5.2 *(i)*$\widetilde{\Omega}^n_D(\mathcal{A}_f) = 0$ *for* $n \geq 2$.
(ii) $\widetilde{\Omega}^n_D(\mathcal{A}_f) = \mathbb{C}[z,z^{-1}]$ *for* $n = 0,1$ *here equality is as an* $\mathcal{A}_f$ *bimodule.*

Proof: Note that the homomorphism σ in (4.2.1) induces a surjective homomorphism denoted by the same symbol from $\mathcal{A}_f$ to $\mathbb{C}[z,z^{-1}]$. We have the following short exact sequence

$$0 \longrightarrow \mathcal{I}_\beta \longrightarrow \mathcal{A}_f \xrightarrow{\sigma} \mathbb{C}[z,z^{-1}] \longrightarrow 0.$$

Let $\sigma_k : \Omega^k(\mathcal{A}_f) \to \Omega(\mathbb{C}[z,z^{-1}])$ be the induced surjective map. One easily verifies that $(\omega,\omega)_D = (\sigma_k(\omega),\sigma_k(\omega))_{\mathfrak{D}}$. Therefore,

$$\mathsf{K}_k(\mathcal{A}_f) = \{\omega \in \Omega^k(\mathcal{A}_f) : (\omega,\omega)_D = 0\} = \sigma_k^{-1}(\mathsf{K}_k(\mathbb{C}[z,z^{-1}])).$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
\mathsf{K}_0 = I_\beta & \longrightarrow & \mathcal{A}_f & \xrightarrow{\sigma} & \mathbb{C}[z,z^{-1}] & \longrightarrow & \pi(\mathbb{C}[z,z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{K}_1(\mathcal{A}_f) & \longrightarrow & \Omega^1(\mathcal{A}_f) & \xrightarrow{\sigma_1} & \Omega^1(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^1_{\mathfrak{D}}(\mathbb{C}[z,z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{K}_2(\mathcal{A}_f) & \longrightarrow & \Omega^2(\mathcal{A}_f) & \xrightarrow{\sigma_2} & \Omega^2(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^2_{\mathfrak{D}}(\mathbb{C}[z,z^{-1}]) \\
 & \cdots & & \cdots & & \cdots & \cdots \\
\mathsf{K}_n(\mathcal{A}_f) & \longrightarrow & \Omega^n(\mathcal{A}_f) & \xrightarrow{\sigma_n} & \Omega^n(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^n_{\mathfrak{D}}(\mathbb{C}[z,z^{-1}]).
\end{array}$$

This along with the previous lemma proves the theorem. We will only illustrate (i).
Let $\omega_n \in \Omega^n(\mathcal{A}_f)$, then by the previous lemma $\sigma_n(\omega_n) = \omega_{1,n} + \delta\omega_{2,n-1}$ where $\omega_{1,n} \in \mathsf{K}_n(\mathbb{C}[z,z^{-1}]), \omega_{2,n-1} \in \mathsf{K}_{n-1}(\mathbb{C}[z,z^{-1}])$. Let $\omega'_{1,n} = \sigma_n^{-1}(\omega_{1,n}), \omega'_{2,n-1} = \sigma_{n-1}^{-1}(\omega_{2,n-1})$, then $\sigma_n(\omega_n - \omega'_{1,n} - \delta\omega'_{2,n-1}) = 0$ implying $\omega_n \in \mathsf{K}_n + \delta\mathsf{K}_{n-1}$. □

4.6 The Case of Quantum Sphere

In this section we will do similar computations for quantum spheres. At times we will be sketchy because some of the arguments are very similar to the earlier one. Quantum sphere was introduced by Podles in [67]. This is the universal C^*-algebra denoted by $C(S^2_{qc})$, generated by two elements A and B subject to the following relations:

$$A^* = A, \qquad B^*B = A - A^2 + cI,$$
$$BA = q^2AB, \qquad BB^* = q^2A - q^4 + cI.$$

Here the deformation parameters q, c satisfy $|q| < 1, c > 0$. For later purpose we also note down two irreducible representations such that the representation given by the direct sum of these two is faithful. Let $\mathcal{H}_+ = l^2(\mathbb{N}_0), \mathcal{H}_- = \mathcal{H}_+$. Define $\pi_\pm(A), \pi_\pm(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$ by

$$\pi_\pm(A)(e_n) = \lambda_\pm q^{2n} e_n \qquad \text{where} \qquad \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{1/2}$$
$$\pi_\pm(B)(e_n) = c_\pm(n)^{1/2} e_{n-1} \quad \text{where} \quad c_\pm(n) = \lambda_\pm q^{2n} - (\lambda_\pm q^{2n})^2 + c, \text{and} e_{-1} = 0.$$

Since $\pi = \pi_+ \oplus \pi_-$ is a faithful representation the next theorem is immediate.

Theorem 4.6.1 (Sheu) *(i)$C(S^2_{qc}) \cong C^*(\mathfrak{T}) \oplus_\sigma C^*(\mathfrak{T}) := \{(x,y) : \ x, y \in C^*(\mathfrak{T}), \sigma(x) = \sigma(y)\}$ where $C^*(\mathfrak{T})$ is the Toeplitz algebra and $\sigma : C^*(\mathfrak{T}) \to C(\mathbb{T})$ is the symbol homomorphism.*

(ii) We have a short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} C^*(\mathfrak{T}) \longrightarrow 0 \tag{4.6.18}$$

Proof: (i) An explicit isomorphism is given by $x \mapsto (\pi_+(x), \pi_-(x))$.
(ii) Define $\alpha((x,y)) = x$ then $\ker\alpha = \mathcal{K}$. □

Corollary 4.6.2 *(i)* $K_0(C(S^2_{qc})) = K^0(C(S^2_{qc})) = \mathbb{Z}\oplus\mathbb{Z}$.
(ii) $K_1(C(S^2_{qc})) = K^1(C(S^2_{qc})) = 0$.

Proof: The six term exact sequence associated with (4.6.18) along with the KK-equivalence of $\mathcal{K}, C^*(\mathfrak{T})$ with $\mathbb{C}$ proves the result □

Proposition 4.6.3 *Let $\mathcal{A}_{fin}$ be the *-subalgebra of $C(S^2_{qc})$ generated by A and B. Then* $(\mathcal{A}_{fin}, \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ *is an even spectral triple.*

Proof: We only have to show that $[D,a]$ is bounded for $a \in \mathcal{A}_{fin}$. For that it is enough to note that,
(i) $N\pi_\pm(A)$, $\pi_\pm(A)N$ are bounded.
(ii) $n(c_\pm(n)^{1/2} - \sqrt{c})$ is bounded as n becomes large.
(iii) $[N,\ell] = \ell$. □

Remark 4.6.4 This spectral triple has nontrivial Chern character. This can be seen as follows: let $P_0 = i(|e_0\rangle\langle e_0|) \in C(S^2_{qc})$, then applying proposition 4, (page 296) of [24] we get the index pairing $\langle [P_0], [(\mathcal{A}_{fin}, \mathcal{H}, D, \gamma)]\rangle = -1$, implying nontriviality of the spectral triple.

Now we will briefly indicate the computations of the complex $(\Omega^\bullet_d(\mathcal{A}_{fin}), d)$ introduced in 1.6.5.

Proposition 4.6.5 *(i)* $\Omega^n_d(\mathcal{A}_{fin}) = 0$ *for* $n \geq 2$.
(ii) $\Omega^1_d(\mathcal{A}_{fin}) = \mathbb{C}[z, z^{-1}]$, *here also equality is as an $\mathcal{A}_{fin}$ bimodule.*

Proof: Let π be the associated representation of $\Omega^\bullet(\mathcal{A}_{fin})$ in $\mathcal{B}(\mathcal{H})$. Then straightforward verification gives (i) $[D,A]$ is compact, (ii) $[D,B] = \ell\otimes\kappa +$ compact, and (iii) $[D,B^*] = -\ell^*\otimes\kappa +$ compact, where $\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, modulo compacts

$$\pi(\Omega^{2k+1}(\mathcal{A}_{fin})) \;=\; C^*_{fin}(\mathfrak{T})\otimes\kappa,$$

$$\pi(\Omega^{2k}(\mathcal{A}_{fin})) \quad = \quad C^*_{fin}(\mathfrak{T}) \otimes I_2,$$

where $C^*_{fin}(\mathfrak{T})$ is the *-algebra generated by $\mathfrak{T}$. Now for (i) note that

$$\omega_n = B\delta B^* \underbrace{\delta B \cdots \delta B}_{n-2\ times} + B^*\delta B \underbrace{\delta B \cdots \delta B}_{n-2\ times}$$

satisfies (a) $\pi(\omega_n)$ is compact and (b)$\pi(\delta\omega_n) = 2I$ is invertible, hence (i) follows.
For (ii) observe that if $a \in \mathcal{A}_{fin}$ and $\pi(a)$ is compact then Na and aN both compact. Hence, $\Omega^1_d(\mathcal{A}_{fin}) = \pi(\Omega^1(\mathcal{A}_{fin})) = \mathbb{C}[z, z^{-1}]$ because modulo compacts $\mathbb{C}[z, z^{-1}]$ is $C^*(\mathfrak{T})$. □

Chapter 5

Equivariant Spectral Triples on $C(SU_q(2))$

In the classical context of a compact Lie group G, a left invariant differential operator is one that commutes with the left regular representation of G. Now in the case of abelian G, the C^*-algebra generated by the left regular representation is nothing but $C(\widehat{G})$. Therefore we can rephrase the left invariance condition as a commutation condition with $C(\widehat{G})$. For $C(SU_q(2))$, Woronowicz has explicitly described the generators for $C(\widehat{G})$. Therefore, a proper analog of a left invariant Dirac operator would be a Dirac operator commuting with these generators. In this chapter we will prove that there exists Dirac operators with nontrivial Chern character commuting with the generators of the dual group. We also show that they are universal in the sense that given any odd spectral triple there is an equivariant one inducing the same element in odd K-homology.

5.1 The regular representation

We will denote by $\mathcal{H}$ the GNS space associated with the Haar state h defined in the beginning of the last chapter.

The representation theory of $SU_q(2)$ is strikingly similar to its classical counterpart. In particular, for each $n \in \{0, \frac{1}{2}, 1, \ldots\}$, there is a unique irreducible unitary representation $t^{(n)}$ of dimension $2n+1$. Denote by $t_{ij}^{(n)}$ the ijth entry of $t^{(n)}$. These are all elements of $\mathcal{A}_f$ and they form an orthogonal basis for $\mathcal{H}$. Denote by $e_{ij}^{(n)}$ the normalized $t_{ij}^{(n)}$'s, so that $\{e_{ij}^{(n)} : n = 0, \frac{1}{2}, 1, \ldots, i, j = -n, -n+1, \ldots, n\}$ is an orthonormal basis.

Remark 5.1.1 One has to be a little careful here, because, unlike in the classical case, the choice of matrix entries does affect the orthogonality relations. Therefore one has to specify the matrix entries he/she is working with. In our case, $t_{ij}^{(n)}$'s are the same as in Klimyk & Schmuedgen (page 74, [54]).

We will use the symbol ν *to denote the number* $1/2$ throughout this chapter, just to make some expressions occupy less space. Using formulas for Clebsch-Gordon coefficients, and the orthogonality relations (page 80–81 and equation (57), page 115 in [54]), one can write down the actions of α, β and β^* on $\mathcal{H}$ explicitly as follows:

$$\alpha : e_{ij}^{(n)} \mapsto a_+(n,i,j)e_{i-\nu,j-\nu}^{(n+\nu)} + a_-(n,i,j)e_{i-\nu,j-\nu}^{(n-\nu)}, \tag{5.1.1}$$

$$\beta : e_{ij}^{(n)} \mapsto b_+(n,i,j)e_{i+\nu,j-\nu}^{(n+\nu)} + b_-(n,i,j)e_{i+\nu,j-\nu}^{(n-\nu)}, \tag{5.1.2}$$

$$\beta^* : e_{ij}^{(n)} \mapsto b_+^+(n,i,j)e_{i-\nu,j+\nu}^{(n+\nu)} + b_-^+(n,i,j)e_{i-\nu,j+\nu}^{(n-\nu)}, \tag{5.1.3}$$

where

$$a_+(n,i,j) = \left(q^{2(n+i)+2(n+j)+2}\frac{(1-q^{2n-2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^\nu, \tag{5.1.4}$$

$$a_-(n,i,j) = \left(\frac{(1-q^{2n+2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^\nu, \tag{5.1.5}$$

$$b_+(n,i,j) = -\left(q^{2(n+j)}\frac{(1-q^{2n-2j+2})(1-q^{2n+2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^\nu, \tag{5.1.6}$$

$$b_-(n,i,j) = \left(q^{2(n+i)}\frac{(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^\nu, \tag{5.1.7}$$

$$b_+^+(n,i,j) = \left(q^{2(n+i)}\frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^\nu, \tag{5.1.8}$$

$$b_-^+(n,i,j) = -\left(q^{2(n+j)}\frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^\nu. \tag{5.1.9}$$

5.2 Equivariant spectral triples

In this section, we will formulate the notion of equivariance, and investigate the behavior of D, where D is the Dirac operator of an equivariant spectral triple.

Let A_0 and A_1 be the following operators on $\mathcal{H}$:

$$A_0 \quad : \quad e_{ij}^{(n)} \mapsto q^j e_{ij}^{(n)},$$

$$A_1 \quad : \quad e_{ij}^{(n)} \mapsto \begin{cases} 0 & \text{if } j = n, \\ (q^{-2n} + q^{2n+2} - q^{-2j} - q^{2j+2})^\nu e_{ij+1}^{(n)} & \text{if } j < n. \end{cases}$$

The operators A_0 and A_1 generate the C^*-algebra of continuous functions on the dual of $SU_q(2)$ and thus are the 'generators' of the regular representation of $SU_q(2)$ (For more details, see [66]; A_0 and A_1 are the operators $\mathbf{a}$ and $\mathbf{n}$ there). We say that an operator T on $\mathcal{H}$ is **equivariant** if it commutes with A_0, A_1 and A_1^*. It is clear that any equivariant self-adjoint operator with discrete spectrum must be of the form

$$D : e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}, \tag{5.2.1}$$

where $d(n,i)$'s are real. Assume then that D is such an operator. Let us first write down the commutators of D with α and β.

$$\begin{aligned} [D,\alpha]e_{ij}^{(n)} &= a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))e_{i-\nu,j-\nu}^{(n+\nu)} \\ &\quad + a_-(n,i,j)(d(n-\nu,i-\nu)-d(n,i))e_{i-\nu,j-\nu}^{(n-\nu)}, \qquad (5.2.2)\\ [D,\beta]e_{ij}^{(n)} &= b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))e_{i+\nu,j-\nu}^{(n+\nu)} \\ &\quad + b_-(n,i,j)(d(n-\nu,i+\nu)-d(n,i))e_{i+\nu,j-\nu}^{(n-\nu)}. \qquad (5.2.3)\end{aligned}$$

We are now in a position to prove the following.

Proposition 5.2.1 *Let D be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$. Then $[D,a]$ is bounded for all $a \in \mathcal{A}_f$ if and only if $d(n,i)$'s satisfy the following two conditions:*

$$d(n+\nu,i+\nu)-d(n,i) = O(1), \tag{5.2.4}$$

$$d(n+\nu,i-\nu)-d(n,i) = O(n+i+1). \tag{5.2.5}$$

Proof: Assume that $[D,a]$ is bounded for all $a \in \mathcal{A}_f$. Then, in particular, $[D,\alpha]$ and $[D,\beta]$ are bounded, so that there is a positive constant C such that

$$\|[D,\alpha]\| \le C, \quad \|[D,\beta]\| \le C.$$

It follows from equations (5.2.2) and (5.2.3) that

$$|a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))|^2 + |a_-(n,i,j)(d(n-\nu,i-\nu)-d(n,i))|^2 \le C^2, \tag{5.2.6}$$

$$|b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))|^2 + |b_-(n,i,j)(d(n-\nu,i+\nu)-d(n,i))|^2 \le C^2 \tag{5.2.7}$$

for all n, i and j. From the second inequality above, we get

$$|b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))| \le C \quad \forall n,i,j.$$

Now
$$|b_+(n,i,j)| = \left(\frac{q^{2n+2j}-q^{4n+2}}{1-q^{4n+2}}\right)^{\nu}\left(\frac{1-q^{2n+2i+2}}{1-q^{4n+4}}\right)^{\nu}.$$
Hence
$$1-q^2 \le \frac{1-q^2}{1-q^{4n+4}} \le \max_j |b_+(n,i,j)| = \frac{1-q^{2n+2i+2}}{1-q^{4n+4}} \le \frac{1}{1-q^4}.$$
Hence $|d(n+\nu,i+\nu)-d(n,i)| \le \frac{C}{1-q^2}$ for all n, i, i. e. we have (5.2.4). We also have from equation (5.2.6), $|a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))| \le C$. But
$$a_+(n,i,j) = q\left(\frac{q^{2n+2j}-q^{4n+2}}{1-q^{4n+2}}\right)^{\nu}\left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^{\nu}.$$
Hence
$$\max_j |a_+(n,i,j)| = q\left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^{\nu}.$$
Therefore
$$q\left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^{\nu}|d(n+\nu,i-\nu)-d(n,i)| \le C \quad \forall n,i.$$
Consequently, $q^{n+i}|d(n+\nu,i-\nu)-d(n,i)| \le q^{-1}C\frac{1}{(1-q^2)^{\nu}}$, i. e.
$$|d(n+\nu,i-\nu)-d(n,i)| = O(q^{-n-i}). \tag{5.2.8}$$

Let us next write the difference $d(n+\nu,i-\nu)-d(n,i)$ as follows:
$$\begin{aligned}&\sum_{r=0}^{n+i-1}\left(d(n+\nu-r\nu,i-\nu-r\nu)-d(n+\nu-(r+1)\nu,i-\nu-(r+1)\nu)\right)\\ &\quad-\sum_{r=0}^{n+i-1}\left(d(n-r\nu,i-r\nu)-d(n-(r+1)\nu,i-(r+1)\nu)\right)\\ &\quad+d(n+\nu-(n+i)\nu,i-\nu-(n+i)\nu)-d(n-(n+i)\nu,i-(n+i)\nu).\end{aligned}$$

Using this expression together with (5.2.8) for the case $n+i=0$ and (5.2.4), we get (5.2.5).

Next assume that the $d(n,i)$'s satisfy the conditions (5.2.4) and (5.2.5). We will show that $[D,\alpha]$ and $[D,\beta]$ are bounded, which in turn will ensure that $[D,a]$ is bounded for all $a\in\mathcal{A}_f$. It follows from (5.2.4) and (5.2.5) that there is a positive constant $C>0$ such that
$$|d(n+\nu,i+\nu)-d(n,i)| \le C, \quad q^{n+i}|d(n+\nu,i-\nu)-d(n,i)| \le C.$$

It follows from the above two inequalities that
$$\begin{aligned}|a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))| &\le C(1-q^4)^{-1/2},\\ |a_-(n,i,j)(d(n-\nu,i-\nu)-d(n,i))| &\le Cq^{-1}(1-q^2)^{-1/2}.\end{aligned}$$

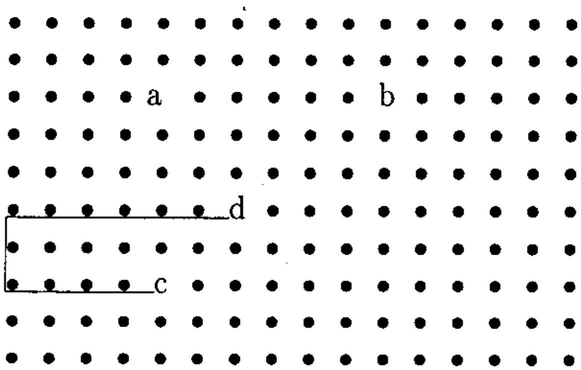
We now conclude from (5.2.2) that $[D, \alpha]$ is bounded. Proof of boundedness of $[D, \beta]$ is similar. □

Next, we exploit the condition that D must have compact resolvent. It is straightforward to see that a necessary and sufficient condition for an operator D of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ to have compact resolvent is that if we write the $d(n,i)$'s in a single sequence, it should not have any limit point other than ∞ or $-\infty$. As we shall see below, in presence of (5.2.4) and (5.2.5), we can say much more about the $d(n,i)$'s. In particular, we can extract information about the sign of D also.

Proposition 5.2.2 *Let D be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ such that $d(n,i)$'s satisfy conditions (5.2.4) and (5.2.5) and D has compact resolvent. Then*

1. *For each $k \in \mathbb{N}$, there exists an $r_k \in \mathbb{N}$, $r_k \geq k$ such that $d(n, n-k)$ are of the same sign for all $n \geq r_k$.*
2. *There exists an $r \in \mathbb{N}$ such that for all $k \geq r$ and for all n, $d(n, n-k)$ are of the same sign.* (5.2.9)

Proof: In the following diagram, each dot stands for a $d(n,i)$, the dot at the i^{th} row and j^{th} column representing $d(\frac{i+j}{2}, \frac{j-i}{2})$ (here i and j range from 0 onwards).



There are two restrictions imposed on these numbers, given by equations (5.2.4) and (5.2.5). Equation (5.2.4) says that: (i) the difference of two consecutive numbers along any row is bounded by a fixed constant, and (5.2.5) says that: (ii) the difference of two consecutive numbers along the j^{th} column is $O(j+1)$. Suppose C is a big enough constant which works for both (i) and (ii).

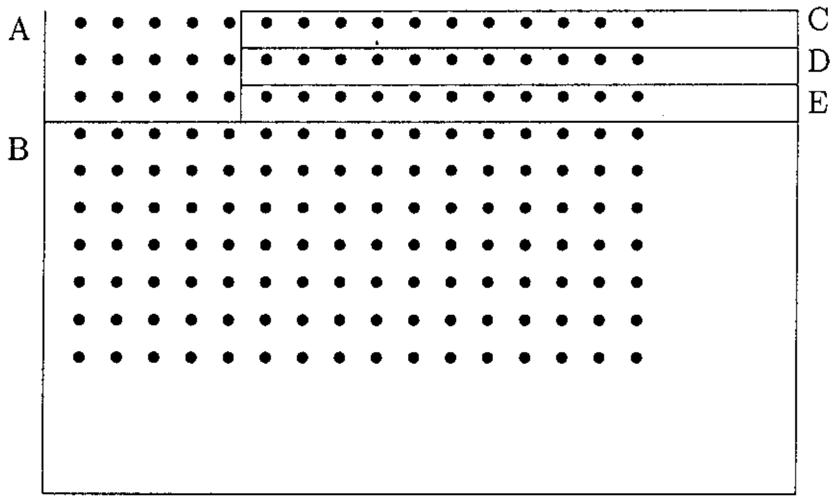
Now suppose a and b are two elements in the same row. Connect them with a path as in the diagram. If a and b are of opposite sign, then because of restriction (i) above, there has to be some dot between a and b for which the corresponding $d(n,i)$ lies in $[-C, C]$. Therefore, if the signs of the $d(n,i)$'s change infinitely often along a row, one can produce infinitely many $d(n,i)$'s in the interval $[-C, C]$. But this will prevent D from having a compact resolvent. This proves part 1.

For part 2, employ a similar argument, this time connecting two dots, say c and d, by a path as shown in the diagram, and observing that the difference between any two consecutive numbers along the path is bounded by C. □

Let m and n be two nonnegative integers. Let

$$\begin{aligned} F(m,n) &= \left\{d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : 0 \le i \le m, 0 \le j \le n\right\}, \\ S(m,n,r) &= \left\{d\left(\frac{j+r}{2}, \frac{j-r}{2}\right) : j > n\right\}, \quad 0 \le r \le m, \\ T(m) &= \left\{d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : i > m, j \ge 0\right\}. \end{aligned}$$

In the following diagram, for example, A is $F(2,4)$, B is $T(2)$, and C, D and E are $S(2,4,0)$, $S(2,4,1)$ and $S(2,4,2)$ respectively.



What the last proposition says is the following. There exist big enough integers m and n such that in each of the sets $T(m)$, $S(m,n,0), \ldots, S(m,n,m)$, all elements are of the same sign, i. e. each of the sets $T(m)$, $S(m,n,0), \ldots, S(m,n,m)$ is contained in either $\mathbb{R}_+$ or $-\mathbb{R}_+$.

Remark 5.2.3 One can extend the argument in the proof of the last proposition a little further and prove that if D is as in the previous proposition, then

$$\left.\begin{array}{l}\text{given any nonnegative real } \bar{N}\text{, there exist positive integers } m \text{ and } n \text{ such} \\ \text{that each of the sets } T(m), S(m,n,0),\ldots,S(m,n,m) \text{ is contained in ei-} \\ \text{ther } \{x\in\mathbb{R}: x>\bar{N}\} \text{ or } \{x\in\mathbb{R}: x<-\bar{N}\}.\end{array}\right\} \qquad (5.2.10)$$

Theorem 5.2.4 *An operator* D *on* $L_2(h)$ *gives rise to an equivariant spectral triple if and only if it is of the form* $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$, *where* $d(n,i)$*'s are real and satisfy conditions (5.2.4), (5.2.5) and (5.2.10).*

Proof: By virtue of (5.2.1), and propositions 5.2.1, 5.2.2, it only remains to prove that if the $d(n,i)$'s obey condition (5.2.10), then D has compact resolvent. But this is clear, because (5.2.10) implies that for any real number $\bar{N} > 0$, the interval $[-\bar{N}, \bar{N}]$ contains only a finite number of the $d(n,i)$'s. □

It is clear then that up to a compact perturbation, D will have nontrivial sign if and only if the following condition holds:

$$\left.\begin{array}{l}\text{1. there exist positive integers } m \text{ and } n \text{ such that in each of the sets} \\ \quad T(m), S(m,n,0),\ldots,S(m,n,m)\text{, all elements are of the same sign,} \\ \quad \text{and} \\ \text{2. there are two sets in this collection whose elements are of opposite} \\ \quad \text{sign.}\end{array}\right\} \qquad (5.2.11)$$

A natural question to ask now is whether there does indeed exist any D with nontrivial sign satisfying (5.2.4) and (5.2.5). It is easy to see that the operator D determined by the family $d(n,i)$, where

$$d(n,i) = \begin{cases} 2n+1 & \text{if } n \neq i, \\ -(2n+1) & \text{if } n = i, \end{cases} \qquad (5.2.12)$$

satisfy all the requirements in propositions 5.2.1 and 5.2.2. In fact, one can easily see that $D^{-3} \in \mathcal{L}^{1+}$, where $\mathcal{L}^{1+}$ stands for the ideal of Dixmier traceable operators. Thus we have the following.

Theorem 5.2.5 $SU_q(2)$ *admits an equivariant odd 3-summable spectral triple.*

The classical $SU(2)$ has (both topological as well as metric) dimension 3. For $SU_q(2)$, however, the topological dimension turns out to be 1, as can be seen from the following short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \longrightarrow \mathcal{A} \longrightarrow C(\mathbb{T}) \longrightarrow 0,$$

where $\mathcal{K}$ denotes the algebra of compact operators. The next theorem tell us that as far as metric dimension is concerned, it behaves more like its classical counterpart; in fact along with the previous theorem, it says that the metric dimension of $SU_q(2)$ is 3.

Theorem 5.2.6 *Let $(\mathcal{A}, \mathcal{H}, D)$ be an equivariant odd spectral triple. Then D can not be p-summable for $p < 3$.*

Proof: Conditions (5.2.4) and (5.2.5) impose the following growth restriction on the $d(n,i)$'s:

$$\max_i |d(n,i)| = O(n). \tag{5.2.13}$$

The conclusion of the theorem follows easily from this. □

The next proposition says that the derivative of any nonconstant function is nonzero.

Proposition 5.2.7 *Let D be given by (5.2.12). Then for $a \in \mathcal{A}_f$, $[D,a] = 0$ if and only if a is a scalar.*

Proof: Take $a = \sum_{(i,j,k)\in F} c_{ijk}\alpha_i\beta^j\beta^{*k}$, where F is a finite subset of $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ and all the c_{ijk}'s are nonzero. We will show that $[D,a] \neq 0$.

Let $m = \max\{|i|+j+k : (i,j,k) \in F\}$. Let (r,s,t) be a point of F such that $|r|+s+t = m$. Write $p = \frac{1}{2}(s-t-r)$, $p' = \frac{1}{2}(t-s-r)$. Then it is easy to see that

$$\begin{aligned}
&\langle e_{pp'}^{(n+m/2)}, [D,a]e_{00}^{(n)}\rangle \\
&= \langle e_{pp'}^{(n+m/2)}, [D, c_{rst}\alpha_r\beta^s\beta^{*t}]e_{00}^{(n)}\rangle \\
&= c_{rst}\prod_{i=1}^{t} b_+^+\left(n+\frac{i-1}{2}, -\frac{i-1}{2}, \frac{i-1}{2}\right)\prod_{i=t+1}^{t+s} b_+\left(n+\frac{i-1}{2}, -t+\frac{i-1}{2}, t-\frac{i-1}{2}\right) \\
&\quad\times\prod_{i=s+t+1}^{m} a_+^{\#}\left(n+\frac{i-1}{2}, p+\operatorname{sign}(r)\frac{m-i}{2}, p'+\operatorname{sign}(r)\frac{m-i}{2}\right) \\
&\qquad\qquad\times\Big(d(n+m/2,p)-d(n,0)\Big),
\end{aligned}$$

where $a_+^{\#}$ stands for a_+ or a_+^+ depending on the sign of r. The right hand side above is clearly nonzero because of our choice of D. □

The above proposition says, in particular, that the Dirac operator given by (5.2.12) is really a Dirac operator for the full tangent bundle rather than that of some lower dimensional subbundle.

5.3 Nontriviality of the Chern character

In this section we will examine the D given by the family (5.2.12) in more detail and see that the nontriviality in sign does indeed result in nontriviality at the Fredholm level. For this, we will compute the pairing between $\operatorname{sign} D$ and a generator of $K_1(\mathcal{A})$. Let u denote the element $I_{\{1\}}(\beta^*\beta)(\beta - I) + I$ of $\mathcal{A}$. In corollary 4.3.6 we proved that this is a generator of $K_1(\mathcal{A})$. We will choose an invertible element γ in $\mathcal{A}_f$ that is close enough to u so that γ and u are the same in $K_1(\mathcal{A})$. We then compute the pairing between $\operatorname{sign} D$ and this γ.

Theorem 5.3.1 *The Chern character of the spectral triple $(\mathcal{A}_f, \mathcal{H}, D)$ is nontrivial.*

Before we begin the proof, let us observe from equations (5.1.2) and (5.1.3) that the action of $\beta\beta^*$ on $\mathcal{H}$ is given by

$$(\beta\beta^*)(e_{ij}^{(n)}) = \sum_{\epsilon=-1}^{1} k_\epsilon(n,i,j) e_{ij}^{(n+\epsilon)}, \tag{5.3.1}$$

where

$$k_1(n,i,j) = -\Big(q^{4n+2i+2j+2}\frac{1-q^{2n+2j+2}}{1-q^{4n+2}}\frac{1-q^{2n-2i+2}}{1-q^{4n+4}}\frac{1-q^{2n-2j+2}}{1-q^{4n+4}}\frac{1-q^{2n+2i+2}}{1-q^{4n+6}}\Big)^{\nu}, \tag{5.3.2}$$

$$k_0(n,i,j) = q^{2(n+j)}\frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})} + q^{2(n+i)}\frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}, \tag{5.3.3}$$

$$k_{-1}(n,i,j) = -\Big(q^{4n+2i+2j-2}\frac{(1-q^{2n-2j})(1-q^{2n+2i})(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n-2})(1-q^{4n})(1-q^{4n})(1-q^{4n+2})}\Big)^{\nu}. \tag{5.3.4}$$

Proof of theorem 5.3.1 : Choose $r \in \mathbb{N}$ such that $q^{2r} < \frac{1}{2} < q^{2r-2}$. Define $\gamma_r = (\beta^*\beta)^r(\beta - I) + I$. By our choice of r, we have

$$\begin{aligned} \|\gamma_r - u\| &\le \|(\beta^*\beta)^r - I_{\{1\}}(\beta^*\beta)\| \cdot \|\beta - I\| \\ &\le 2q^{2r} < 1. \end{aligned}$$

Hence γ_r and u are the same in $K_1(\mathcal{A})$. Therefore it is enough for our purpose if we can show that the pairing between $\operatorname{sign} D$ and γ_r is nontrivial. Denote by P_k the projection onto the space spanned by $\{e_{n-k,j}^{(n)} : n, j\}$. Then $\operatorname{sign} D = I - 2P_0$. Therefore we now want to compute the index of the operator $P_0\gamma_r P_0$ thought of as an operator on $P_0\mathcal{H}$.

It follows from (5.3.1) that

$$(\beta\beta^*)^r(e_{ij}^{(n)}) = \sum_{\epsilon_t \in \{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1}\epsilon_s, i, j)\Big) e_{ij}^{(n+\sum_1^r \epsilon_s)}. \tag{5.3.5}$$

Since β is normal, we have

$$\begin{aligned}\gamma_r e_{ij}^{(n)} &= \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)\Big(b_+(n+\sum_{1}^{r}\epsilon_s, i, j)e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} \\ &\qquad + b_-(n+\sum_{1}^{r}\epsilon_s, i, j)e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \\ &\quad - \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)e_{ij}^{(n+\sum_1^r\epsilon_s)} + e_{ij}^{(n)}. \end{aligned} \tag{5.3.6}$$

Consequently,

$$\begin{aligned}\gamma_r e_{nj}^{(n)} &= \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)\Big(b_+(n+\sum_{1}^{r}\epsilon_s, n, j)e_{n+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} \\ &\qquad + b_-(n+\sum_{1}^{r}\epsilon_s, n, j)e_{n+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \\ &\quad - \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)e_{nj}^{(n+\sum_1^r\epsilon_s)} + e_{nj}^{(n)}. \end{aligned}$$

When we cut this off by P_0, we get

$$\begin{aligned}P_0\gamma_r e_{nj}^{(n)} &= \sum_{\sum\epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)b_+(n, n, j)e_{n+\nu,j-\nu}^{(n+\nu)} \\ &\quad + \sum_{\sum\epsilon_t=1} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)b_-(n+1, n, j)e_{n+\nu,j-\nu}^{(n+\nu)} \\ &\quad - \sum_{\sum\epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)e_{nj}^{(n)} + e_{nj}^{(n)}. \end{aligned}$$

A closer look at the quantities k_ϵ and $b_\pm$ tells us that if we do the calculations modulo compact operators, which we can because we want to compute the index, we find that there is no contribution from the second term, while in the case of the first and the third term, contribution comes from only the coefficient where the product $\prod_{t=1}^{r} k_{\epsilon_t}(n+\epsilon_1+\ldots+\epsilon_{t-1}, n, j)$ consists solely of k_0's, i. e. when each $\epsilon_t = 0$. A further examination of the terms k_0 and b_+ then yield the following:

$$\begin{aligned}P_0\gamma_r P_0 e_{nj}^{(n)} &= k_0(n, n, j)^r b_+(n, n, j)e_{n+\nu,j-\nu}^{(n+\nu)} + (1-k_0(n, n, j)^r)e_{nj}^{(n)} \\ &= -q^{2rn+2rj}(1-q^{2n-2j})^r q^{n+j}(1-q^{2n-2j+2})^{1/2}e_{n+\nu,j-\nu}^{(n+\nu)} \\ &\qquad + \left(1-q^{2rn+2rj}(1-q^{2n-2j})^r\right)e_{nj}^{(n)}, \end{aligned}$$

and

$$\begin{aligned} P_0\gamma_r^* P_0 e_{nj}^{(n)} &= -q^{2rn+2rj}(1-q^{2n-2j-2})^r q^{n+j}(1-q^{2n-2j})^{1/2} e_{n-\nu,j+\nu}^{(n-\nu)} \\ &\quad + \left(1-q^{2rn+2rj}(1-q^{2n-2j})^r\right) e_{nj}^{(n)} \end{aligned}$$

From these, one can easily show that the index of $P_0\gamma_r P_0$ is -1. Since P_0 is the eigenspace corresponding to the eigenvalue -1 of $\operatorname{sign} D$, the value of the K-homology–K-theory pairing $\langle [u], [(\mathcal{A},\mathcal{H},D)]\rangle$ coming from Kasparov product of K_1 and K_1 is $-\operatorname{index} P_0\gamma_r P_0$, which is nonzero. □

Remark 5.3.2 Strictly speaking, it is not essential to introduce the element u as a generator for $K_1(\mathcal{A})$. It is enough if one computes the pairing between $\operatorname{sign} D$ and a suitable γ_r and show that it is nontrivial. But the introduction of u makes the choice of γ_r's and hence the proof above more transparent.

It follows from proposition 5.2.2 that for the purposes of computing the index pairing, sign of any equivariant D must be of the form $I-2P$ where $P=\sum_{k\in F} P_k$, F being a finite subset of $\mathbb{N}$ (the actual P would be a compact perturbation of this). Conversely, given a P of this form, it is easy to produce a D satisfying the conditions in proposition 5.2.2 for which $\operatorname{sign} D = I-2P$. One could, for example, take the D given by $d(n,i)$'s, where

$$d(n,i) = \begin{cases} -(2n+1) & \text{if } n-i\in F, \\ 2n+1 & \text{otherwise.} \end{cases}$$

We are now in a position to prove the following.

Proposition 5.3.3 *Given any $m\in\mathbb{Z}$, there exists an equivariant spectral triple D acting on $\mathcal{H}$ such that $\langle \gamma_r, [(\mathcal{A},\mathcal{H},D)]\rangle = m$, where $\langle,\cdot,\cdot\rangle : K_1(\mathcal{A})\times K^1(\mathcal{A}) \to \mathbb{Z}$ denotes the map coming from the Kasparov product.*

Proof: It is enough to prove the statement for m positive. Let D be an equivariant Dirac operator whose sign is $I-2P$ where $P=\sum_{k\in F} P_k$, F being a subset of size m of $\mathbb{N}$. In order to compute the pairing $\langle \gamma_r, [(\mathcal{A},\mathcal{H},D)]\rangle$, we must first have a look at $P_{k+l}\gamma_r P_k$.

We get from equation (5.3.6)

$$\begin{aligned} \gamma_r e_{n-k,j}^{(n)} &= \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big) \\ &\quad \times \Big(b_+(n+\sum_1^r \epsilon_s, n-k, j) e_{n-k+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} + b_-(n+\sum_1^r \epsilon_s, n-k, j) e_{n-k+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \end{aligned}$$

$$- \sum_{\epsilon_t \in \{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) e_{n-k,j}^{(n+\sum_1^r \epsilon_s)} + e_{n-k,j}^{(n)}.$$

and consequently,

$$\begin{aligned} P_{k+l}\gamma_r e_{n-k,j}^{(n)} &= \sum_{\sum \epsilon_t = l} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) b_+(n+l, n-k, j) e_{n-k+\nu, j-\nu}^{(n+l+\nu)} \\ &+ \sum_{\sum \epsilon_t = l+1} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) b_-(n+l+1, n-k, j) e_{n-k+\nu, j-\nu}^{(n+l+\nu)} \\ &- \sum_{\sum \epsilon_t = l} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) e_{n-k,j}^{(n+l)} + \delta_{l0} e_{n-k,j}^{(n)}. \end{aligned}$$

Now because of the nature of the quantities k_ϵ and $b_\pm$, we see that for index calculations, none of the terms contribute anything for $l \neq 0$, while for $l = 0$, the first, third and the fourth term survive, with coefficient in the first term being $k_0(n, n-k, j)^r b_+(n, n-k, j)$ and that in the third being $(1 - k_0(n, n-k, j)^r)$. It follows from here that

$$\operatorname{index} P_k \gamma_r P_k = -1,$$

and $P_{k+l}\gamma_r P_k$ is compact for $l \neq 0$. Therefore the pairing between $\operatorname{sign} D$ and γ_r produces m. □

An immediate corollary of the above proposition and theorem 1.17 in [79] is the following universality property of equivariant spectral triples.

Corollary 5.3.4 *Given any odd spectral triple $(\mathcal{A}, \mathcal{K}, D)$, there is an equivariant triple $(\mathcal{A}, \mathcal{H}, D')$ inducing the same element in $K^1(\mathcal{A})$.*

Finally, we have the following characterization theorem for equivariant Dirac operators.

Theorem 5.3.5 *$(\mathcal{A}, \mathcal{H}, D)$ is an equivariant odd spectral triple with nontrivial Chern character if and only if D is given by (5.2.1) and the $d(n,i)$'s obey conditions (5.2.4), (5.2.5), (5.2.10) and (5.2.11).*

Proof: If D is of the form $e_{ij}^{(n)} \mapsto d(n,i) e_{ij}^{(n)}$, where $d(n,i)$'s are real and satisfy conditions (5.2.4), (5.2.5), (5.2.10) and (5.2.11), then proposition 5.2.1 says $[D, a]$ is bounded and nontriviality of Chern character follows from arguments of proposition 5.3.3. Conversely, if D is equivariant, then by propositions 5.2.1, 5.2.2 and remark 5.2.3, we have (5.2.4), (5.2.5) and (5.2.10). Since D has nontrivial Chern character, it has nontrivial sign so that we have (5.2.11). □

Chapter 6

Compact Quantum Metric Spaces

This chapter is devoted to construction of compact quantum metric spaces (CQMS). Rieffel has produced [76] examples coming from ergodic actions of compact Lie groups. Noncommutative torus falls in that class. Since quantum Heisenberg manifolds (QHM) in many aspects behave like the noncommutative torus it is natural to expect canonical CQMS structure on QHM. But the group acting ergodically on QHM is the noncompact Heisenberg group, hence Rieffel's theory does not apply. Here we will modify his arguments to obtain CQMS on QHM. Other algebras treated here are the $C(SU_q(2))$ and the Podles' sphere. To construct CQMS structure on them we adopt a different strategy. We show that from certain C^*-algebra extensions one can produce CQMS and then obtain CQMS structures on $C(SU_q(2))$ and Podles' sphere as immediate corollary.

6.1 Metrics on QHM

In this section we follow notations of chapter 3. Recall that for $\phi \in S^c$, $\|\phi\|_{\infty,\infty,1}$ was defined as $\sum_n \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\phi(x,y,n)|$.

Proposition 6.1.1 *$\|\cdot\|_{\infty,\infty,1}$ is a *-algebra norm on S^c.*

Proof: It is easy to see that the involution is an antilinear isometry in $\|\cdot\|_{\infty,\infty,1}$.

Let $\Phi, \Psi \in S^c$ and $\Phi'(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\Phi(x,y,p)|$, $\Psi'(p) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\Psi(x,y,p)|$ for $p \in \mathbb{Z}$

$$\begin{aligned}
&|(\Phi \star \Psi)(x,y,p)| \\
&\leq \sum_q |\Phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q)| \times |\Psi(x-\hbar q\mu, y-\hbar q\nu, p-q)| \\
&\leq \sum_q \Phi'(q)\Psi'(p-q).
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_p sup_{x\in\mathbb{R},y\in\mathbb{T}}|(\Phi\star\Psi)(x,y,p)| &\le \sum_p\sum_q \Phi'(q)\Psi'(p-q)\\ &= \|\Phi\|_{\infty,\infty,1}.\|\Psi\|_{\infty,\infty,1}.\end{aligned}$$

This proves that $\|\cdot\|_{\infty,\infty,1}$ is an algebra norm. □

Proposition 6.1.2 *The topology given by $\|\cdot\|_{\infty,\infty,1}$ is stronger than the topology given by the C^*-norm coming from $\mathcal{A}_\hbar$.*

Proof: It suffices to show for $\phi \in S^c$, $\|\phi\| \le \|\phi\|_{\infty,\infty,1}$ Let $\phi' : \mathbb{Z} \to \mathbb{R}_+$ be given by $\phi'(n) = \sup_{x\in\mathbb{R},y\in\mathbb{T}} |\phi(x,y,n)|$. Then for $\xi \in L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$ we have,

$$|(\phi\xi)(x,y,p)| \le (\phi' \star |\xi(x,y,.)|)(p),$$

where $\star$ denotes convolution on $\mathbb{Z}$ and $|\xi(x,y,.)|$ is the function $p \mapsto |\xi(x,y,p)|$. By Young's inequality

$$\|(\phi\xi)(x,y,.)\|_{l_2} \le \|\phi' \star |\xi(x,y,.)|\|_{l_2} \le \|\phi'\|_{l_1}\|\xi(x,y,.)\|_{l_2}.$$

Therefore, $\|\phi\| \le \|\phi\|_{\infty,\infty,1}$, since $\|\phi\|_{\infty,\infty,1} = \|\phi'\|_{l_1}$. □

General Scheme of Construction

Let (A, G, α) be a C^* dynamical system with G an n dimensional Lie group acting ergodically. Let $A^\infty = \{a \in A | g \mapsto \alpha_g(a)$ is smooth $\}$. Then for any $X \in Lie(G)$, the Lie algebra of G induces a derivation $\delta_X : A^\infty \to A^\infty$. Let $X_1, \ldots, X_n$ be a basis of $Lie(G)$. $L(a) = \vee_{i=1}^n \|\delta_{X_i}(a)\|_n$, should be a good candidate for a Lip norm defined in section 1.7. Here $\|\cdot\|_n$ stands for an algebra norm on A not necessarily the norm coming from the algebra. This is essentially Rieffel's construction; the only modification is he considers the case where $\|\cdot\|_n$ is the algebra norm. Here the problem of construction of Lip norms reduces to construction of the norm $\|\cdot\|_n$ such that L so defined becomes a Lip norm.

Illustration in the context of quantum Heisenberg manifolds

Let

$$X_1 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

be the canonical basis of the Lie algebra of the Heisenberg group. Then the associated derivations are given by

$$\delta_1(\phi)(x,y,p) = -\frac{\partial\phi}{\partial x}(x,y,p),$$

$$\begin{aligned}\delta_2(\phi)(x,y,p) &= 2\pi icpx\phi(x,y,p) - \frac{\partial\phi}{\partial y}(x,y,p),\\ \delta_3(\phi)(x,y,p) &= 2\pi ip\phi(x,y,p).\end{aligned}$$

Taking $\|\cdot\|_n$ as $\|\cdot\|_{\infty,\infty,1}$, we get a seminorm $L : {S^c}_{s.a} \to \mathbb{R}_+$ explicitly given by

$$L(\phi) = \vee_1^3 \|\delta_i(\phi)\|_{\infty,\infty,1}.$$

Notation: Henceforth A will stand for ${S^c}_{s.a}$

Proposition 6.1.3 *For all $\mu,\nu \in S(A)$*

$$\rho_L(\mu,\nu) = \sup\{|\mu(a)-\nu(a)| \ : \ L(a)\leq 1\} \leq 6.$$

Proof: Let $\phi \in S^c$ be such that $L(\phi)\leq 1$. Then $L(L_{(0,0,t)}(\phi))\leq 1$. Therefore, $L(\int_0^1 L_{(0,0,t)}(\phi)dt)\leq 1$. i.e, $L(\phi^{(0)})\leq 1$ where $\phi^{(0)}(x,y,p) = \delta_{p0}\phi(x,y,p)$. Recall,

$$\int_0^1\int_{\mathbb{T}} L_{(r,s,0)}(\phi^{(0)})drds) = \tau(\phi^{(0)})I.$$

Let $f_3(p) = |2\pi p\phi(x,y,p)|$, then from $L(\phi)\leq 1$ it follows that $\sum_p f_3(p)\leq 1$. Now,
(i) $\|\phi-\phi^{(0)}\| \leq \|\phi-\phi^{(0)}\|_{\infty,\infty,1} \leq \sum_{p\neq 0}\frac{f_3(p)}{2\pi|p|} \leq \sum_p f_3(p)\leq 1$, and
(ii) $\|\phi^{(0)} - L_{(r,s,0)}\phi^{(0)}\|_{\infty,\infty,1}\leq 2$.
Using these two we get,

$$\begin{aligned}|\mu(\phi)-\tau(\phi^{(0)})| &\leq |\mu(\phi)-\mu(\phi^{(0)})| + |\mu(\phi)-\mu(\tau(\phi^{(0)})I)|\\ &\leq \|\phi-\phi^{(0)}\| + \int_0^1\int_0^1 |\mu(\phi^{(0)}) - \mu(L_{(r,s,0)}(\phi^{(0)})|drds\\ &\leq 3.\end{aligned}$$

This completes the proof. □

Proposition 6.1.4 *L as defined above is a Lip norm.*

Proof: Since the action is ergodic and $\|\cdot\|_{\infty,\infty,1}$ is a norm it follows that $L(\phi)=0$ if and only if ϕ is a constant multiple of identity. The previous proposition gives finite radius of (A,L). Therefore, by theorem 1.7.4, it suffices to show that every sequence $\{\phi_n\}_{n\geq 1}$ in $\mathcal{B}_1 = \{\phi | L(\phi)\leq 1, \text{ and } \|\phi\|\leq 1\}$ admits a subsequence convergent in the norm coming from the C*-algebra.
Let

$$f_{1,n}(p) = sup_{x\in\mathbb{R},y\in\mathbb{T}}|\frac{\partial\phi_n}{\partial x}(x,y,p)|,$$

$$\begin{aligned} f_{2,n}(p) &= sup_{x\in\mathbb{R},y\in\mathbb{T}}|2\pi icpx\phi_n(x,y,p) - \frac{\partial\phi_n}{\partial y}(x,y,p)|, \\ f_{3,n}(p) &= sup_{x\in\mathbb{R},y\in\mathbb{T}}|2\pi ip\phi_n(x,y,p)|. \end{aligned}$$

$L(\phi_n) \leq 1$ is equivalent with $\sum_p f_{i,n}(p) \leq 1$ for $i = 1, 2, 3$.

$$sup_{|x|\leq 2,y\in\mathbb{T}}|\frac{\partial\phi_n}{\partial y}(x,y,p)| \leq 4\pi cf_{3,n}(p) + f_{2,n}(p) \leq 1 + 4\pi c$$

Now by Arzela-Ascoli theorem there exists $\phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that for each $p \in \mathbb{Z}$, $sup_{|x|\leq 2,y\in\mathbb{T}}|\phi_n(x,y,p) - \phi(x,y,p)| \to 0$. Clearly ϕ satisfies the periodicity condition.

Claim:

$$\sum_p sup_{x,y}|\phi(x,y,p)| < \infty.$$

Proof of Claim: Suppose not. Then for any $N \in \mathbb{N}$, there exists $p_1, \ldots, p_k > N$ such that $\sum_i sup_{x,y}|\phi(x,y,p_i)| > 2$. So, one can take n sufficiently large so that

$$\sum_{|p|\geq N} sup_{x,y}|\phi_n(x,y,p)| \geq \sum_i sup_{x,y}|\phi(x,y,p_i)| - 1/2 > 3/2$$

On the other hand note that,

$$\begin{aligned} \sum_{|p|\geq N} sup_{x,y}|\phi_n(x,y,p)| &= \sum_{|p|\geq N} \frac{f_{3,n}(p)}{p} \\ &\leq \frac{1}{N}\sum_p f_{3,n}(p) = \frac{1}{N}. \end{aligned} \tag{6.1.1}$$

This leads to a contradiction.

Note that ϕ defines a bounded operator by lemma 3.1.5. Therefore in view of proposition 6.1.2 it is enough to show that ϕ_n converges to ϕ in $\|\cdot\|_{\infty,\infty,1}$ norm. For $N \in \mathbb{N}$ let

$$\phi_{|p|\leq N}(x,y,p) = \begin{cases} \phi(x,y,p) & \text{for } |p| \leq N, \\ 0 & \text{for } |p| > N. \end{cases}$$

Let $\epsilon > 0$ be given. Choose N such that (i) $\|\phi - \phi_{|p|\leq N}\|_{\infty,\infty,1} \leq \epsilon$, and (ii) $\frac{1}{N} \leq \epsilon$. Then by (6.1.1) one has $\|\phi_n - \phi_{n,|p|\leq N}\|_{\infty,\infty,1} \leq \epsilon, \forall n$. Now choose an integer m such that for $m \leq n$, $\|\phi_{n,|p|\leq N} - \phi_{|p|\leq N}\|_{\infty,\infty,1} \leq \epsilon$. Therefore $\forall\, n \geq m$, $\|\phi_n - \phi\|_{\infty,\infty,1} \leq 3\epsilon$. □

Theorem 6.1.5 *$((A, I), L)$ is a compact quantum metric space.*

Proof: Follows from the previous two propositions. □

6.2 Extensions to CQMS

In this section we describe the general principle of construction of CQMS from certain C^*-algebra extensions. Let $\mathcal{A}$ be a unital C^*-algebra. Fix a faithful representation $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Suppose we have a dense order unit space $Lip(\mathcal{A}) \subseteq \mathcal{A}_{s.a}$, where $\mathcal{A}_{s.a}$ denotes the real partially ordered subset of selfadjoint elements in $\mathcal{A}$. Let L be a Lip norm on $Lip(\mathcal{A})$ such that $((Lip(\mathcal{A}), I), L)$ is a CQMS. Let ν be a state on $\mathcal{A}$, then define $\widetilde{\mathcal{A}_\nu}$ to be the collection of $((a_{ij})) \in \mathcal{K}(l^2(\mathbb{N})) \otimes \mathcal{A}$ such that (i) $a_{ij} \in Lip(\mathcal{A})$, (ii) $a_{ij} = a_{ji}$, (iii) $sup_{i\geq 1, j\geq 1}(i+j)^k(L(a_{ij}) + |\nu(a_{ij})|) < \infty \; \forall k$. Clearly $\mathcal{A}_\nu := \widetilde{\mathcal{A}_\nu} \oplus \mathbb{R}I$, where I is the identity on $\mathcal{B}(l^2(\mathbb{N}) \otimes \mathcal{H})$ is an order unit space. Define $L_k : \mathcal{A}_\nu \to \mathbb{R}_+$ by $L_k(I) = 0$,

$$L_k((a_{ij})) = sup_{i\geq 1, j\geq 1}(i+j)^k(L(a_{ij}) + |\nu(a_{ij})|).$$

Lemma 6.2.1 *Let d = diameter of $((Lip(\mathcal{A}), I), L)$. Then for a "Lipschitz function" $a \in Lip(\mathcal{A})$ one has $\|a\| \leq (L(a) + |\nu(a)|)(1+d)$.*

Proof: Let μ be an arbitrary state on $\mathcal{A}$. Then using $\sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\} \leq d$ we get,

$$\begin{aligned} |\mu(a)| &\leq |\mu(a) - \nu(a)| + |\nu(a)| \\ &\leq L(a)d + |\nu(a)| \\ &\leq (L(a) + |\nu(a)|)(1+d). \end{aligned}$$

□

Lemma 6.2.2 *There exists a constant $C > 0$ such that for $((a_{ij})) \in \widetilde{\mathcal{A}_\nu}$,*

$$\|((a_{ij}))\| \leq C L_2((a_{ij})).$$

Proof: Let $\{e_i\}_{i\geq 1}$ be the canonical orthonormal basis for $l^2(\mathbb{N})$. Let $\sum \lambda_i e_i \otimes u_i$ and $\sum \mu_i e_i \otimes v_i$ be two generic elements in $l^2(\mathbb{N}) \otimes \mathcal{H}$. Here $u_i, v_i \in \mathcal{H}$ are unit vectors. Then clearly $\|\sum \lambda_i e_i \otimes u_i\|^2 = \sum |\lambda_i|^2, \|\sum \mu_i e_i \otimes u_i\|^2 = \sum |\mu_i|^2$. Now observe that

$$\begin{aligned} |\langle \sum \lambda_i e_i \otimes u_i, ((a_{ij})) \sum \mu_j e_j \otimes v_j \rangle| &\leq \sum |\lambda_i||\mu_j||\langle u_i, a_{ij} v_j\rangle| \\ &\leq \sum |\lambda_i||\mu_j|(L(a_{ij}) + |\nu(a_{ij})|)(1+d) \\ &\leq (1+d)\sum |\lambda_i||\mu_j| \frac{L_2((a_{ij}))}{ij} \\ &\leq L_2((a_{ij}))(1+d)\sum_{n=1}^{\infty} \frac{1}{n^2}\sqrt{\sum |\lambda_i|^2}\sqrt{\sum |\mu_i|^2}. \end{aligned}$$

This proves the lemma with $C = (1+d)\sum_{n=1}^{\infty} \frac{1}{n^2}$. □

Lemma 6.2.3 *Let $\mathcal{B}_1 = \{a \in \mathcal{A}_\nu | L_k(a) \le 1, \|a\| \le 1\}$. Then $\mathcal{B}_1$ is totally bounded in norm for $k > 2$.*

Proof: Let $\epsilon > 0$ be given. Choose N such that $(\frac{1}{N})^{k-2} < \epsilon$. For $G = ((g_{ij})) \in \mathcal{A}_\nu$ let $P_N(G) \in \mathcal{K}(l^2(\mathbb{N})) \otimes \mathcal{A}$ be the element given by

$$P_N(G)_{ij} = \begin{cases} g_{ij} & \text{for } i,j \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned} L_k(G - P_N(G)) &= sup_{i \ge N or j \ge N}(i+j)^k (L(g_{ij}) + |\nu(g_{ij})|) \\ &\ge N^{k-2} sup_{i \ge N or j \ge N}(i+j)^2 (L(g_{ij}) + |\nu(g_{ij})|) \\ &= N^{k-2} L_2(G - P_N(G)). \end{aligned}$$

Note that for $G \in \mathcal{B}_1, L_k(G - P_N(G)) \le 1$, therefore

$$\begin{aligned} \|G - P_N(G)\| &\le CL_2(G - P_N(G)) \\ &\le CN^{-(k-2)} L_k(G - P_N(G)) < C\epsilon. \end{aligned}$$

Here the constant C is the one obtained in the previous lemma. Note C does not depend on N. By theorem 1.7.4 there exists $N \times N$ matrices $((a_{ij}^{(r)})) \in M_N(\mathcal{A})$, for $r = 1, \ldots, l$ such that for any $N \times N$ matrix $((a_{ij})) \in \mathcal{B}_1$, there exists r satisfying $\|((a_{ij})) - ((a_{ij}^{(r)}))\| < \epsilon$. Now for $G \in \mathcal{B}_1$, get $((a_{ij}^{(r)}))$ such that $\|P_N(G) - ((a_{ij}^{(r)}))\| < \epsilon$. Then,

$$\|G - ((a_{ij}^{(r)}))\| \le \|G - P_N(G)\| + \epsilon \le (1 + C)\epsilon.$$

This completes the proof. □

Theorem 6.2.4 *$((\mathcal{A}_\nu, I), L_k)$ is a compact quantum metric space for $k > 2$.*

Proof: In view of theorem 1.7.4 and the previous lemma we only have to show that $(\mathcal{A}_\nu, L_k)$ has finite radius. Let $\mu_1, \mu_2 \in S(\mathcal{A}_\nu), a \in \mathcal{A}_\nu$ with $L_k(a) \le 1$. By lemma 6.2.2 $\|a\| \le C$, because $L_2(a) \le L_k(a)$. Hence $|\mu_1(a) - \mu_2(a)| \le 2C$, that is $diam(\mathcal{A}_\nu, L_k) \le 2C$. □

Proposition 6.2.5 *Let*

$$0 \longrightarrow A_0 \xrightarrow{i} A_1 \xrightarrow{\pi} A_2 \longrightarrow 0$$

be a short exact sequence of C^*-algebras, with A_1, A_2 unital and a positive linear splitting $\sigma : A_2 \to A_1$. Let $\phi : A_1' \to A_0' \oplus A_2', \psi : A_0' \oplus A_2' \to A_1'$ be the bounded linear maps given by

$$\phi(\mu) = (\mu_1, \mu_2), \mu_1 = \mu|_{i(A_0)}, \mu_2 = \mu \circ \sigma$$

$$\psi(\mu_1, \mu_2) = \mu, \mu(a) = \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a))$$

Then μ_1, μ_2 are inverse to each other.

Proof: Let $\phi(\mu) = (\mu_1, \mu_2), \psi(\mu_1, \mu_2) = \mu'$. Then

$$\begin{aligned} \mu'(a) &= \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a)) \\ &= \mu(\sigma \circ \pi(a)) + \mu(a - \sigma \circ \pi(a)) \\ &= \mu(a). \end{aligned}$$

Therefore $\psi \circ \phi = Id_{A_1'}$. Similarly one can show that the other composition is also identity. □

Let $\mathcal{A}, Lip(\mathcal{A}), L$ be as above. Suppose we have a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{A} \xrightarrow{i} \widetilde{\mathcal{A}_1} \xrightarrow{\pi} \widetilde{\mathcal{A}_2} \longrightarrow 0$$

with $\widetilde{\mathcal{A}_1}, \widetilde{\mathcal{A}_2}$ unital and a positive unital linear splitting $\sigma : \widetilde{\mathcal{A}_2} \to \widetilde{\mathcal{A}_1}$. Let $(\mathcal{A}_2, L_2)$ be a compact quantum metric space with $\mathcal{A}_2$ a dense subspace of selfadjoint elements of $\widetilde{\mathcal{A}_2}$. Define $\mathcal{A}_1 = i(\widetilde{\mathcal{A}_\nu}) \oplus \sigma(\mathcal{A}_2)$. Then we have

Theorem 6.2.6 *In the above set up $L_1 : \mathcal{A}_1 \to \mathbb{R}_+$, given by*

$$L_1(a) = L_2(\pi((a)) + L_k(a - \sigma \circ \pi(a))$$

is a Lip norm for $k > 2$.

Proof: We break the proof in several steps.
Step (i) $L_1(a) = 0$ iff $a \in \mathbb{R}Id_{A_1}$: If part is obvious for the only if part note $L_1(a) = 0$ gives $\pi(a) = \lambda Id_{A_2}$ for some $\lambda \in \mathbb{R}$ and $L_0(a - \lambda Id_{A_1}) = 0$. Hence $a = \lambda Id_{A_1}$.
Step (ii) (A_1, L_1) has finite radius: Let $\mu, \lambda \in S(\mathcal{A}_1)$ and $(\mu_1, \mu_2) = \phi(\mu), (\lambda_1, \lambda_2) = \phi(\lambda)$, where ϕ is as in proposition 6.2.5. Then from the norm estimate of ϕ obtained in proposition 6.2.5 we get $\|\mu_i\|, \|\lambda_i\| \leq (1 + \|\sigma\|)$, for $i = 1, 2$ and positivity of σ implies $\|\mu_2\| = \|\lambda_2\| = 1$. Let $x \in \mathcal{A}_1$ with $L(x) \leq 1$, then

$$\begin{aligned} |\mu(x) - \lambda(x)| &= |\mu_2(\pi(x)) + \mu_1(x - \sigma \circ \pi(x)) - \lambda_2(\pi(x)) - \lambda_1(x - \sigma \circ \pi(x))| \\ &\leq |\mu_2(\pi(x)) - \lambda_2(\pi(x))| + |\mu_1(x - \sigma \circ \pi(x)) - \lambda_1(x - \sigma \circ \pi(x))| \\ &\leq diam(\mathcal{A}_2, L_2) + 2(1 + \|\sigma\|)C \end{aligned}$$

where C is the constant obtained in lemma 6.2.2. This proves $(\mathcal{A}_1, L_1)$ has finite radius.
Step (iii) In view of theorem 1.7.4 it suffices to show that $\mathcal{B}_1 = \{a \in \mathcal{A}_1 : \|a\| \leq 1, L_1(a) \leq$

1} is totally bounded. Since $(\mathcal{A}_\nu, L_k)$ and $(\mathcal{A}_2, L_2)$ are compact quantum metric spaces it follows that if we have a sequence $a_n \in \mathcal{B}_1$, then there exists a subsequence a_{n_k} such that both $\pi(a_{n_k})$ and $a_{n_k} - \sigma \circ \pi(a_{n_k})$ converges in norm. Hence a_{n_k} converges in norm implying the totally boundedness. □

6.3 Examples

Example 6.3.1 Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let $H^2(\partial\Omega)$ be the closure in $L^2(\partial\Omega)$ of boundary values of holomorphic functions that can be continuously extended to $\bar{\Omega}$. For $f \in C(\partial\Omega)$ let T_f be the associated Toeplitz operator, that is the compression of the multiplication operator M_f on $L^2(\partial\Omega)$ on $H^2(\partial\Omega)$. Let $\mathfrak{T}(\partial\Omega)$ be the associated Toeplitz extension, that is the C^*-algebra generated by the operators T_f along with the compacts. Then we have a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(H^2(\partial\Omega)) \xrightarrow{i} \mathfrak{T}(\partial\Omega) \xrightarrow{\pi} C(\partial\Omega) \longrightarrow 0$$

Since this sequence admits a positive unital splitting by the previous theorem we get CQMS structure on $\mathfrak{T}(\partial\Omega)$.

Example 6.3.2 In the context of quantum SU(2) it is easy to see that the associated short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \xrightarrow{i} C(SU_q(2)) \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0 \tag{6.3.1}$$

admits a positive splitting taking $z^n \in C(\mathbb{T})$ to $\ell^n \otimes I$, for all $n \geq 0$. Hence we get a compact quantum metric space structure on $C(SU_q(2))$.

Example 6.3.3 The short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} C^*(\mathfrak{T}) \longrightarrow 0 \tag{6.3.2}$$

is also split exact. Here a positive splitting is given by $\ell \in C^*(\mathfrak{T}) \mapsto (\ell, \ell)$. Now to apply the previous theorem note that by the earlier example on Toeplitz extensions we already have a Lip norm on a dense subspace of $C^*(\mathfrak{T})$.

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