Essays on Minimum Cost Spanning Tree Games

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Contents

Chapter 1

Introduction

There is a wide range of economic contexts in which *aggregate costs* have to be allocated amongst individual agents or components who derive the benefits from a common project. A firm has to allocate overhead costs amongst its different divisions. Regulatory authorities have to set taxes or fees on individual users for a variety of services. Partners in a joint venture must share costs (and benefits) of the joint venture. For example, when two doctors share an office they need to divide the cost of office space, medical equipment and secretarial help. If several municipalities use a common water supply system, they must reach an agreement on how to share the costs of operating it. When the members of NATO cooperate on common defense, they need to determine how to share the burden. In most of these examples, there is no external force such as the market, which determines the allocation of costs. Thus, the final allocation of costs is decided either by mutual agreement or by an *arbitrator* on the basis of some notion of ***distributive*** *justice.*

The main thrust of this area of research is the axiomatic analysis of allocation rules. Such an axiomatic analysis is supposed to enlighten an *arbitrator* on the possible interpretations of ***fairness*** while dividing the cost among the participants. Ideally, the axiomatic method can help our choice of allocation rules by, first, reducing the number of plausible solutions as much as possible and second, by providing us with a specific axiomatic characterization of each of these plausible solutions.

A central problem of cooperative game theory is how to divide the benefits of cooperation amongst individual players or agents. Since there is an obvious analogy between the division of costs and that of benefits, the tools of cooperative game theory have proved very useful in the analysis of cost allocation problems. Moulin [1999] and Young [1994] are excellent surveys of this literature. Much of this literature has focused on *general* cost allocation problems, so that the ensuing *cost game* is identical to that of a typical game in characteristic function form. This has facilitated the search for *appropriate* cost allocation rules considerably given the corresponding results in cooperative game theory.

In this monograph, we pursue this axiomatic analysis of cost allocation rules for a specific class of cost allocation problems known as *Minimum Cost Spanning Tree* games denoted as *m.c.s.t.* games. The common feature of these problems is that a group of users has to be connected to a single supplier of some service. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network. There is a positive cost of connecting each pair of users (towns) as well as a cost of connecting each user (town) to the common supplier (power plant). A cost game arises because cooperation reduces aggregate costs - it may be cheaper for town A to construct a link to town B which is *nearer* to the power plant, rather than build a separate link to the plant. An efficient network must be a *tree* which connects all users to the common supplier. That is why these games have been labeled minimum cost spanning tree games. In this monograph, we construct a few interesting cost allocation rules over the efficient network and provide axiomatic characterization of these rules.

1.1 Minimum Cost Spanning Tree Games

Let $\mathcal{N} = \{1, 2, \ldots\}$ be the set of all possible agents. We are interested in *graphs* or *networks* where the nodes are elements of a set $N \cup \{0\}$, where $N \subset \mathcal{N}$, and 0 is a distinguished node which we will refer to as the *source* or *root* .

Henceforth, for any set $N \subset \mathcal{N}$, we will use N^+ to denote the set $N \cup \{0\}$.

A typical graph over N^+ will be represented by $g_N = \{(ij) | i, j \in N^+\}$. Two nodes i and $j \in N^+$ are said to be *connected* in g_N if $\exists (i_1 i_2), (i_2 i_3), \ldots, (i_{n-1} i_n)$ such that $(i_k i_{k+1}) \in g_N$, $1 \le k \le n-1$, and $i_1 = i, i_n = j$. A graph g_N is called *connected* over N^+ if i, j are connected in g_N for all $i, j \in N^+$. The set of connected graphs over N^+ is denoted by Γ_N.

Consider any $N \subset \mathcal{N}$, where $\#N = n$. A *cost matrix* $C = (c_{ij})$ represents the cost of *direct* connection between any pair of nodes. That is, c_{ij} is the cost of directly connecting any pair $i, j \in N^+$. We assume that each $c_{ij} > 0$ whenever $i \neq j$. We also adopt the convention that for each $i \in N^+$, $c_{ii} = 0$. So, each cost matrix is nonnegative, symmetric and of order $n + 1$. The set of all cost matrices for N is denoted by $\mathcal{C}_N$. However, we will typically drop the subscript N whenever there is no cause for confusion about the set of nodes.

Consider any $C \in \mathcal{C}_N$. A *minimum cost spanning tree* (m.c.s.t.) over N^+ satisfies

$$g_N = \text{argmin}_{g \in \Gamma_N} \sum_{(ij) \in g} c_{ij}$$

Clearly a minimum cost spanning network must be a tree. Otherwise, we can delete an extra edge and still obtain a connected graph at a lower cost. An m.c.s.t. corresponding to $C \in \mathcal{C}_N$ will typically be denoted by $g_N(C)$. Note that an m.c.s.t. need not be unique.

Example 1.1.1: Consider individuals A, B, C, who want the service of cable network in their respective houses. There is only one cable operator, whom we will call the source, in the locality. It is not necessary for each individual to be connected directly to the source. For instance, A could be connected to B and B to the source, thereby providing an indirect connection of A to the source.

The costs of connections are represented by a cost matrix C.

$$C = \begin{array}{c} \\ 0 \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} 0 & A & B & C \end{array} \\ \begin{pmatrix} 0 & 2 & 4 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix} \end{array}$$

The objective of the cable operator is to find the minimum cost cable network in which A, B, C are connected to the source. The minimum cost will be 4 units. Our object of interest in this paper is to see how the total cost of 4 units is to be distributed amongst A, B and C. For the graphical representation of the problem and m.c.s.t. of C, see Fig. 1.1 and Fig. 1.2, respectively.

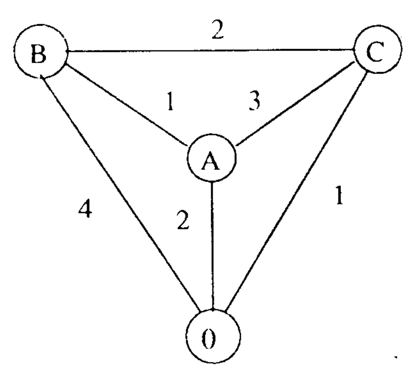

Fig. 1.1

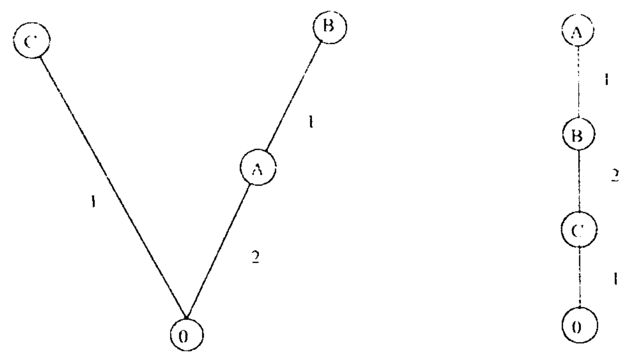

Fig. 1.2

Here we introduce a few more definitions regarding the minimum cost spanning tree. The (unique) *path* from i to j in tree g, is a set $U(i,j,g) = \{i_1, i_2, \ldots, i_K\}$, where each pair $(i_{k-1}i_k) \in g$, and $i_1, i_2, \ldots, i_K$ are all distinct agents with $i_1 = i, i_K = j$. The *predecessor set* of an agent i in g is defined as $P(i,g) = \{k|k \neq i,\ k \in U(0,i,g)\}$. The *immediate predecessor* of agent i, denoted by $\alpha(i)$, is the agent who comes immediately before i, that is, $\alpha(i) \in P(i,g)$ and $k \in P(i,g)$ implies either $k = \alpha(i)$ or $k \in P(\alpha(i),g)$.[1] The *followers* of agent i, are those agents who come immediately after i; $F(i) = \{j|\alpha(j) = i\}$.

Now we define a cost allocation rule in the context of m.c.s.t. games, which is the central theme of this monograph. A *cost allocation rule* is a family of functions $\{\mu^N\}_{N \subset \mathcal{N}}$,

$$\mu^N : \mathcal{C}_N \to \Re^N, \text{ satisfying } \sum_{i \in N} \mu_i^N(C) \geq \sum_{(ij) \in g_N(C)} c_{ij} \quad \forall C \in \mathcal{C}_N$$

We will drop the superscript N whenever there is no confusion about the set of agents. So, given any set of nodes N and any cost matrix C, a cost allocation rule specifies the costs attributed to agents in N. Note that the source 0 is not an *active* player, and hence does not bear any part of the cost. One condition

[1] Note that since g is a tree, the immediate predecessor must be unique.

which must be satisfied by a cost allocation rule is that the total payment by the agents must cover the total cost of an efficient network.

A cost allocation rule can be generated by any *single-valued* game-theoretic solution of a transferable utility game. Thus, consider the transferable utility game generated by considering the aggregate cost of a minimum cost spanning tree for each coalition $S \subseteq N$. Given C and $S \subseteq N$, let C_S be the cost matrix restricted to S^+. Then, consider a m.c.s.t. $g_S(C_S)$ over S^+, and the corresponding minimum cost of connecting S to the source. Let this cost be denoted c_S. For each $N \subset \mathcal{N}$, this defines the *minimum cost spanning tree games* (N, c) where for each $S \subseteq N$, $c(S) = c_S$. That is, c is the cost function, and is analogous to the characteristic function of a TU game. Then, if Φ is a single-valued solution, $\Phi(N, c)$ can be viewed as the cost allocation rule corresponding to the cost matrix which generates the cost function c.

An allocation rule based on the cost game would be the *Shapley value* [Shapley [1953], Myerson [1977]]. Let $C \in \mathcal{C}_N$ and $\#N = n$. The Shapley value is defined as,

$$\forall k \in N \quad \Phi_k(C) = \frac{1}{n!} \sum_{\pi \in \Pi} \left[c\left(S_{\pi(k)} \cup \{k\}\right) - c\left(S_{\pi(k)}\right)\right]$$

Here, Π denotes the set of all permutations over $\{1, 2, ..., n\}$ and $S_{\pi(k)} = \{i | \pi(i) < \pi(k)\}$.

Another allocation rule of this type is based on the *nucleolus* of the m.c.s.t. games. This is defined as follows.

Let $x \in \Re^N$ be such that $\sum_{i \in N} x_i = c(N)$. Let $e(S, x) = c(S) - \sum_{i \in S} x_i$ be the concession given to coalition S corresponding to allocation x. Consider the vector $e(x)$, each entry of $e(x)$ corresponding to some $e(S, x)$ where $S \subset N$. Let $\tilde{e}(x)$ be the ordered vector corresponding to $e(x)$, that is $\tilde{e}_k(x) \leq \tilde{e}_{k+1}(x)$

for all $1 \leq k < 2^{\#N} - 2$. Then the nucleolus ξ is defined as

$$\sum_{i \in N} \xi_i(C) = c(N) \text{ and } \tilde{e}(\xi(C)) \succ \tilde{e}(y), \text{ for all } y \text{ such that } \sum_{i \in N} y_i = c(N)$$

Here $z \succ \bar{z}$ if $z_j = \bar{z}_j$ for all $j < k$ and $z_k > \bar{z}_k$ for some $k \geq 1$. So, if an allocation x is the nucleolos then $\tilde{e}(x)$ lexicographically maximizes the concession given to different coalitions.

An allocation rule outside the cooperative game theory approach would be the*Proportional rule*, where the cost attributed to i is,

$$\lambda_i(C) = c(N)\left(\frac{c_{i0}}{\sum_{k \in N} c_{k0}}\right) \ \forall i \in N.$$

The method suggested by Bird (1976) is defined with respect to a *specific* tree. Let g_N be some m.c.s.t. corresponding to the cost matrix C. Then,

$$B_i(C, g_N) = c_{i\alpha(i)} \quad \forall i \in N$$

Notice that this does not define an allocation rule. If C gives rise to more than one m.c.s.t., then Bird's method will give two different allocations for the same cost matrix. However, one can still use Bird's method on each m.c.s.t. derived from C and then take some convex combination of the allocations corresponding to each m.c.s.t.. This serves as a valid cost allocation rule.

1.2 Related Literature

There is an extensive literature on m.c.s.t. games as well as its variations. However, much of this literature has focused on issues such as computational complexities and the design of efficient algorithms. This literature owes its genesis to the papers by Kruskal [1956], Prim [1957] and Dijkstra [1959], where they discuss algorithms to construct minimum cost spanning trees. Kruskal and Prim provide different algorithms for constructing minimum cost spanning trees, these algorithms being different versions of the *greedy algorithm*. A historic overview of this can be found in Graham and Hell [1985].

However, Claus and Kleitman [1973] treated this problem from a game theoretic perspective. Following Claus and Kleitman, Bird [1976] found an allocation rule which belongs to the core of the m.c.s.t. games. The proof of this result is due to Granot and Huberman [1981]. They also show that the core and the nucleolus of the cost game are the Cartesian products of the core and the nucleoli, respectively, of the induced games on the components of the efficient coalition structure. An alternative proof of existence of core in m.c.s.t. games was derived by Edmonds [1967] and Curiel [1988]. Granot and Huberman [1984] propose a couple of new allocation rules and provide a geometric characterization for the nucleous of the m.c.s.t. games. Feltkamp, Tijs, Muto [1994a] present an axiomatic characterization of the Bird rule. They also construct a non cooperative game in which Bird allocations correspond to the Nash equilibria of the game. Kuipers [1993,1994] computed all extreme elements of the core of the *information graph games*. These are games arising from m.c.s.t. situations in which the costs of links are either one or zero.

Other related network construction games are *Steiner tree games* analysed by Megiddo [1978b] where in addition to the nodes occupied by consumers and the source there is a set of junction nodes. These nodes are not occupied by any agents and are used to minimize the cost of the spanning network. Granot and Granot [1993] studied the *fixed-cost spanning forest games* where unlike m.c.s.t. games there can be more than one source.

The minimum cost spanning tree games need not be submodular. Granot and Huberman [1982] defined a class of games related to m.c.s.t. games based on the *permutational submodularity* of the cost function. Megiddo [1978a] studied the properties of the *monotone network structure*. This is similar to m.c.s.t. games, except that a subgroup, while forming its own network with the source, can use vertices which is not in the subset. Granot and Maschler [1998] extended this model to include *public vertices*, i.e. vertices which are not occupied by any agent and have positive and negative costs on edges and vertices. *Minimum cost spanning extension games* are generalizations of m.c.s.t. games in which an existing network has to be extended to connect new users to the source. These games are analysed by Feltkamp, Tijs, Muto [1994b,c]. Sharkey [1995] is an excellent overview of network models in economics.

More general cost sharing problems have been extensively discussed by Moulin [1999] and Young [1994]. Moulin introduces several methods of cost sharing and provides their axiomatic characterizations, whereas Young discusses a few important methods of cost sharing like the Shapley value, the weighted Shapley value, the Aumann-Shapley pricing and the Ramsey pricing. The axioms used in these surveys can be divided into various classes.

The first class consists of *Equity* or *fairness axioms*. These axioms provide a simple idea of equity among the set of agents concerned. A minimal requirement of this class is the *equal treatment of equals* criterion. It says that two agents with identical characteristics should bear an equal share of the total cost. A slightly more restrictive requirement of this type is *anonymity*. An allocation rule is anonymous if it is invariant under renaming of the agents. The axiom which played a central role in the microeconomics literature of fair division is the *no-envy* criterion. It says that at the chosen allocation every agent prefers her assigned bundle to that of any other agent.

Another set of axioms incorporate participation constraint. The property of being a *core selection* is the most well known member of the class. This requires that the costs allocated to any groups should not exceed the 'stand alone' cost for the group. In other words, no group should have an incentive to leave the grand coalition.

The third class consists of several *monotonicity* axioms. The type of monotonicity axioms used depends on the context. For instance, an allocation rule is monotonic with respect to aggregate cost if no agent's cost share goes down when aggregate cost increases, everything else remaining the same. Consider another context where a certain commodity is jointly produced in order to meet the sum of individual demands. Then *demand monotonicity* (Moulin [1999]) requires that if agent i's demand increases, while the demands of all other agents remain the same, then agent i's cost share should not decrease.

Another set of axioms are consists of *Structural Invariance axioms*. These properties express the commutativity of the allocation methods with respect to certain variations in the cost sharing problem under scrutiny. For instance *consistency* axioms require the sharing method to 'commute' with a variation in the set of agents. For a certain cost allocation problem, the *reduced games* are obtained by imagining the departure of some of the agents with their components of the allocation and reassessing the opportunities open to the remaining agents. An allocation rule is said to be consistent if for the reduced games it chooses the restriction of the original allocation. There are also *converse consistency* axioms which permit the opposite operation, to deduce the desirability of an outcome for some problem from the desirability of its restrictions to subgroups for the associated reduced problems these subgroups face. The *consistency* axioms are extensively discussed by Thomson [1998]. Thomson illustrates the importance of consistency and converse consistency axioms under different economic situations like cost allocations, matching problems, bargaining etc. Another axiom which belongs to this class is *scale invariance*. This axiom imposes commutativity with respect to the unit of measurement.

1.3 Some Properties of Cost Allocation Rules

In this section, we discuss some properties of allocation rules which will be used in the rest of the monograph.

The allocation rule proposed by Shapley has been widely discussed in various contexts. In the next chapter we characterize the Shapley value on minimum cost spanning tree games. In the context of network formation, the Shapley value has been already axiomatized by Myerson [1977] and Jackson-Wolinsky [1996]. The difference between our axioms and those used by Jackson and Wolinsky is also discussed in the next chapter.

We use the following four axioms to characterize the Shapley value.

Efficiency : An allocation rule μ satisfies efficiency if $\sum_{i \in N} \mu_i(C) = c(N)$.

This axiom ensures that the agents together pay exactly the cost of an efficient network. That is there should not be any excess payment made by the agents. This axiom is basically the *Pareto efficiency* condition, which is one of the fundamental concepts in economics. An allocation is Pareto efficient if there does not exist some other allocation in which everybody is at least as well off and one of the agents is strictly better off. Clearly if μ does not satisfy efficiency, then all individuals can be made strictly better off.

A $\star$-*graph* is a structure, where a node (called centre) is directly connected to every other node and there is no other connection in the graph. An example of an m.c.s.t. which is a $\star$-graph is one at which every agent is individually connected to the source.

Absence of cross subsidization : Suppose a $\star$-graph with 0 as the centre is an m.c.s.t. corresponding to some cost matrix C. An allocation rule μ satisfies absence of cross subsidization if $\mu_i(C) = c_{i0} \ \forall i \in N$.

When each agent is directly connected with the source then there is no 'interaction' amongst the agents. Then it is natural to assume that each agent is charged exactly his cost of connection with the source.

In order to introduce the next axiom, we need some further notation. The link between two agents $i, j \neq 0$ will be called an *irrelevant link* if $c_{ij} > \max(c_{i0}, c_{j0})$. An m.c.s.t. will never link i and j through (ij) if the link between them is irrelevant. A link between $i, j \in N$ will be called a *relevant link* if it is not an irrelevant link. Note that even if (ij) is a relevant link it may be the case that (ij) is not present in any of the m.c.s.t.s.

There is a *relevant route* from i to j in N if $\exists\, i_1, \ldots, i_K$ such that $i_k \in N$ for each $k \leq K$, $(i_k i_{k+1})$ is a relevant link for each $k < K$ and $i_1 = i, i_K = j$.

Suppose the set of agents has a partition $N = [N_1, N_2, \ldots, N_p]$ such that

(i) $i \in N_k, j \in N_k \Rightarrow$ there is a relevant route from i to j in N_k.

(ii) $i \in N_k, j \in N_l \Rightarrow$ there is no relevant route from i to j in N.

Then an element of the partition is called a *group*.

Group independence : Suppose that given the cost matrix C, agents are partitioned into p groups, $N = [N_1, N_2, ..., N_p]$. Let $i, j \in N_l$, and C' be another cost matrix such that $c'_{mn} = c_{mn} \forall (mn) \neq (ij)$. Then an allocation rule μ satisfies group independence if

$$\mu_k(C) = \mu_k(C') \;\; \forall k \in N_t, \;\; \forall t \neq l.^2$$

Suppose an efficient network structure is such that the set of agents can be partitioned into groups where each group is connected separately with the source and the cost of connection between any two agents of different groups is 'excessively' high. If cost of a connection between two agents from same group changes, everything else remaining the same, then this axiom says that it should not affect the cost allocations for members of other groups. This is

[2]Note that, in C', there is no change in groups except possibly N_l.

desirable as the change in cost within a group does not affect the agents outside it.

Equal treatment: Let $C, C' \in \mathcal{C}_N$ be such that $c'_{mn} = c_{mn} \forall (mn) \neq (ij); i, j \neq 0$. An allocation rule μ satisfies equal treatment if $\mu_i(C') - \mu_i(C) = \mu_j(C') - \mu_j(C)$.

Let us think of C as the 'original' cost situation. Then, $\mu_i(C)$ and $\mu_j(C)$ take into account all possible differences in the costs of links of i and j with other agents. Now, if only the cost of the edge (ij) changes, then equal treatment requires that the extra burden (or benefit) be divided equally among the agents i and j. This axiom belongs to the class of fairness axioms discussed in the previous section.

In the third chapter we introduce a new class of allocation rules which satisfy both *cost monotonicity* and *core selection*.

First we discuss cost monotonicity. Cost monotonicity is an extremely attractive property, and requires that the cost allocated to agent i does not increase if the cost of a link involving i goes down, nothing else changing. Notice that if a rule does not satisfy cost monotonicity, then it may not provide agents with the appropriate incentives to reduce the costs of constructing links.

Cost monotonicity: Let $C, C' \in \mathcal{C}_N$ be such that $c_{kl} = c'_{kl}$ for all $(kl) \neq (ij)$ and $c_{ij} > c'_{ij}$. An allocation rule μ satisfies cost monotonicity if for all $m \in N \cap \{i, j\}$, $\mu_m(C) \geq \mu_m(C')$.

This axiom belongs to the class of monotonicity axioms, discussed in the previous section. Note that the allocation rule corresponding to the Shapley value satisfies cost monotonicity.

One particularly important game-theoretic property is that of the *core*. If a rule does not always pick an element in the core of the game, then some subset of N will find it profitable to break up the grand coalition and construct its own minimum cost tree. This motivates the following definition.

Core selection : An allocation rule μ is a core selection if for all $N \subseteq \mathcal{N}$ and for all $C \in \mathcal{C}_N$, $\sum_{i \in S} \mu_i(C) \leq c(S)$, $\forall S \subseteq N$.

In the context of transferable utility games Young [1994] showed that core selection and a monotonicity property similar to cost monotonicity are not achievable simultaneously.

Let (N, v) be a transferable utility (TU) game, where N is the set of agents and $v : 2^N \longrightarrow \Re$ is a characteristic function. A solution for TU games specifies a feasible distribution for each game.

$$Core(v) = \{x | \sum_{i \in S} x_i(v) \geq v(S) \;\forall S \subset N \text{ and } \sum_{i \in N} x_i(v) = v(N)\}$$

A solution μ is a *core selection* in the context of TU games if μ is in the core of (N, v) whenever it is nonempty.

Let (N, v) and (N, w) be two TU games such that $v(T) = w(T)$ for all $T \neq S$ and $v(S) < w(S)$. A solution μ is *coalitionally monotonic* if $\mu_i(v) \leq \mu_i(w)$ for all $i \in S$.

Young showed that there does not exist any coalitionally monotonic solution which is also a core selection. In the third chapter, we construct a class of cost monotonic allocation rules which are also core selections. We are able to do this because the domain of m.c.s.t. games is a strict subset of the class of balanced games.

We will use two domain restrictions on the set of permissible matrices. Firstly these restrictions will simplify the description of the class of new allocation rules. More importantly, axiomatic characterization of two extreme points of our new class of allocations will be presented on these restricted domains in the fourth chapter.

Let $\mathcal{C}^1 = \{C \in \mathcal{C} | C$ induces a *unique* m.c.s.t.$\}$. That is $\mathcal{C}^1$ is the set of all cost matrices which have a unique minimum cost spanning tree.

However there can be cost matrices which have unique m.c.s.t. with edges which cost the same.

Let $\mathcal{C}^2 = \{C \in \mathcal{C}^1 |$ no two edges of the unique m.c.s.t. have the same cost $\}$.

Notice if C is not in $\mathcal{C}^2$, then even a "small" perturbation of C produces a matrix with the property that no two edges have the same cost. Of course such a matrix must be in $\mathcal{C}^2$. So, even the stronger domain restriction is relatively mild, and the permissible sets of cost matrices are large.

In addition to cost monotonicity and core selection, there are a few more desirable properties which are satisfied by the proposed class of allocation rules. These are defined bellow.

Let $C \in \mathcal{C}^1$ and $g_N(C)$ be the m.c.s.t. of C. Then, $i \in N$ is called an *extreme point* of $g_N(C)$ if i has no follower in $g_N(C)$.[3]

Suppose i is an extreme point of $g_N(C)$. Note that i is of no use to the rest of the network since no node is connected to the source through i. Extreme point monotonicity essentially states that no "existing" node k will agree to pay a higher cost in order to include i in the network.

Extreme point monotonicity : Let i be an extreme point of $C \in \mathcal{C}^1$. Let $\bar{C}$ be the restriction of C over the set $N^+ \setminus \{i\}$. An allocation rule μ satisfies extreme point monotonicity if $\mu_k(\bar{C}) \geq \mu_k(C)\ \forall k \in N \setminus \{i\}$.

Scale invariance is another appealing property that can be imposed on a cost allocation rule. This axiom says that the unit of measurement should not affect the cost allocation. That is if the cost of connections are measured in terms of dollar instead of Euros, it must not affect the payment made by the agents.

Scale invariance : Let C and C' be two cost matrices such that $C' = \delta C + \beta$, where $\delta, \beta \in \Re$ and $\delta \geq 0$. Then an allocation rule μ satisfies *scale invariance* if $\mu(C') = \delta\mu(C) + \beta$.

[3] We will often refer to i as an extreme point of C.

Tree invariance : Let $C, C' \in \mathcal{C}_N^1$ be such that $g_N(C) = g_N(C')$ and $(ij) \in g_N(C) \Rightarrow c_{ij} = c'_{ij}$. An allocation rule μ satisfies tree invariance if $\mu_k(C) = \mu_k(C')$ for all $k \in N$.

This axiom states that if two cost matrices have the same minimum cost spanning tree then the cost allocations corresponding to these matrices can not be different. This property adds to the computational simplicity of the rule. The allocation corresponding to the Shapley value of the game does not satisfy tree invariance.

In the third chapter we show that our new class of allocation rules satisfy core selection, scale invariance, extreme point monotonicity and tree invariance for all $\lambda \in [0, 1]$. Moreover these rules are cost monotonic iff $\lambda \in [0, 0.5]$.

In the last chapter we consider two extreme points of the new class of allocation rules, that is $\lambda = 0$ and $\lambda = 1$. First, we propose an alternative algorithm to calculate the allocation corresponding to $\lambda = 0$. Next, we assert that the allocation corresponding to $\lambda = 1$ is nothing but the allocation proposed by Bird.

Then we go on to axiomatize these two allocation rules over restricted domains $\mathcal{C}^2$ and $\mathcal{C}^1$ respectively. In addition to efficiency and extreme point monotonicity, the characterizations involve the use of two consistency properties, analogous to reduced game properties introduced by Davis and Maschler [1965] and Hart and Mas-Collel [1989].[4]

Consider any C with a unique m.c.s.t. $g_N(C)$, and suppose that $(i0) \in g_N(C)$. Let x_i be the cost allocation 'assigned' to i. Suppose i 'leaves' the scene (or stops bargaining for a different cost allocation), but other nodes are allowed to connect through it. Then, the effective *reduced matrix* changes for the remain-

[4] Thomson [1998] contains an excellent discussion of consistency properties in various contexts.

ing nodes. We can think of two alternative ways in which the others can use node i.

(i) The others can use node i only to connect to the source.

(ii) Node i can be used more widely. That is, node j can connect to node k through i.

In case (i), the connection costs on $N^+ \setminus \{i\}$ are described by the following equations:

$$\text{For all } j \neq i, \bar{c}_{j0} = \min(c_{j0}, c_{ji} + c_{i0} - x_i) \tag{1.1}$$

$$\text{If } \{j,k\} \cap \{i,0\} = \emptyset, \text{ then } \bar{c}_{jk} = c_{jk} \tag{1.2}$$

Equation 1.1 captures the notion that node j's cost of connecting to the source is the cheaper of two options - the first option being the original one of connecting directly to the source, while the second is the indirect one of connecting through node i. In the latter case, the cost borne by j is adjusted for the fact that i pays x_i. Equation 1.2 captures the notion that node i can only be used to connect to the source.

Let $C^{sr}_{x_i}$ represent the reduced matrix derived through equations 1.1, 1.2.

Consider now case (ii).

$$\text{For all } j,k \in N^+ \setminus \{i\}, \bar{c}_{jk} = \min(c_{jk}, c_{ji} + c_{ki} - x_i). \tag{1.3}$$

Equation 1.3 captures the notion that j can use i to connect to any other node k, where k is not necessarily the source.

Let $C^{tr}_{x_i}$ represent the reduced matrix derived through equation 1.3.

We can now define the two consistency conditions.

Source consistency : Let $C \in \mathcal{C}^1_N$, and $(0i) \in g_N(C)$. An allocation rule μ satisfies source consistency if $\mu_k(C^{sr}_{\mu_i(C)}) = \mu_k(C)$ for all $k \in N \setminus \{i\}$ whenever $C^{sr}_{\mu_i(C)} \in \mathcal{C}^1_{N\setminus\{i\}}$.

Tree consistency : Let $C \in \mathcal{C}^2_N$, and $(0i) \in g_N(C)$. An allocation rule μ satisfies tree consistency if $\mu_k(C^{tr}_{\mu_i(C)}) = \mu_k(C)$ for all $k \in N \setminus \{i\}$ whenever $C^{tr}_{\mu_i(C)} \in \mathcal{C}^2_{N\setminus\{i\}}$.

The two consistency conditions require that the cost allocated to any agent be the same on the original and reduced matrix. This ensures that once an agent connected to the source agrees to a particular cost allocation and then subsequently allows other agents to use its location for possible connections, the remaining agents do not have any incentive to reopen the debate about what is an appropriate allocation of costs.

The Bird allocation has been axiomatized using efficiency, extreme point monotonicity and *source consistency*. The other extreme point is axiomatized by efficiency, extreme point monotonicity and *tree consistency*. Thus these two allocations differ only in terms of their consistency conditions.

Chapter 2

Axiomatization of the Shapley Value

Myerson [1977] and Jackson and Wolinsky [1996] provide axiomatic characterizations of the Shapley value in the context of network models. However those axioms are not entirely appropriate for m.c.s.t. games. This provides the motivation for this chapter.[1] Here we present an axiomatic characterization of the Shapley value, using efficiency, absence of cross subsidization, group independence and equal treatment axioms, which are more suitable for the m.c.s.t. games. In particular, our axioms are imposed directly on the cost matrix, which is the primitive concept of this framework.

At first sight, the equal treatment axiom bears a close resemblance to the *equal bargaining power* axiom in Jackson and Wolinsky [2]. We briefly define the Jackson-Wolinsky framework of network games in order to emphasize the difference between it and m.c.s.t. games.

Let G be the set of graphs where nodes are the set of agents N. A value

[1] This chapter is based on 'Axiomatization of the Shapley value on minimum cost spanning tree games', Games and Economic Behavior, vol 38, 2002.

[2] This in turn is very close to the 'fair allocation rule' axiom in Myerson [1977].

function $v : G \longrightarrow \Re$ describes the value or worth of each possible network or graph in G. Let V be the set of all value functions. An allocation rule $\Gamma : V \times G \longrightarrow \Re^n$ specifies how the value of each graph g is to be distributed among the set of agents N.

Equal Bargaining Power : Let Γ be an allocation rule. Then,

$$\Gamma_i(v, g + ij) - \Gamma_i(v, g) = \Gamma_j(v, g + ij) - \Gamma_j(v, g) \ \forall g, \ \forall i, j \in N.$$

(Here, $g + ij = g \cup (ij)$.)

It is clear that equal bargaining power and equal treatment are similar in spirit. Both require that if the only difference between two situation is in the characteristics affecting only agents i and j, then the allocation rule must treat i and j similarly. However, these two axioms are formally quite different for at least two reasons.

First the value function in the Jackson-Wolinsky framework is obviously analogous to the cost function in our framework. Now, note that equal bargaining power considers a change in the structure of the graph while keeping the value function unchanged. However, if the cost matrix (and hence the cost function) remains unchanged, then the minimum cost spanning tree must also be unchanged, so that an axiom such as equal bargaining power cannot be imposed in our framework [3].

Second, the domains of the allocation rules in the m.c.s.t. games and the network games are completely different. The following network game will illustrate this fact. Let the set of agents be $N = \{1, 2\}$. Hence the set of graphs is $G = \{\emptyset, (12)\}$. A value function $v \in V$ can be, $v(\emptyset) = -1$ and $v(12) = -2.5$. Let u be the corresponding TU game, with $u(S) = \min_{g \in G} v(g|S)$, where $g|S = \{(ij) \in g | i, j \in S\}$. In this example $u(1) = -1, u(2) = -1, u(12) = -2.5$.

[3] On the other hand equal treatment imposes a restriction on how the cost allocation should change when the cost matrix itself changes.

Though v is a valid value function, it is not possible to obtain a cost matrix for which $(-u)$ is the corresponding cost function.[4]

2.1 The Characterization

Here we present the main result of this chapter.

Theorem 2.1.1 *The Shapley value is the only allocation rule which satisfies efficiency, absence of cross subsidization, group independence and equal treatment.*

Proof : We first prove that the Shapley value satisfies all the four axioms.

It is trivial to show that the Shapley value satisfies efficiency.

Suppose the m.c.s.t. corresponding to C is a $\star$-graph with 0 as the centre. Then, $g_{(S\setminus\{i\})}(C_{S\setminus\{i\}}) \cup (i0) = g_S(C_S), i \in S, \forall S \subseteq N, \forall i \in N$. Here C_S and $C_{S\setminus\{i\}}$ are restrictions of C over S^+ and $(S \setminus \{i\})^+$ respectively. This implies that $c(S_{\pi(i)} \cup \{i\}) - c(S_{\pi(i)}) = c_{i0}\ \forall \pi \in \Pi, \forall i \in N$. Hence, $\Phi_i(C) = c_{i0}\ \forall i \in N$. Therefore, the Shapley value satisfies absence of cross subsidization.

We now show that it satisfies group independence.

Let the set of agents be partitioned into p groups corresponding to cost matrix $C, N = [N_1, N_2, ..., N_p]$.

Let g_S be m.c.s.t. of C restricted to $S^+, S \subseteq N$.

Then, $g_S = \cup_{k=1}^{p}(g_{(S\cap N_k)})$ and $c(S) = \sum_{k=1}^{p} c(S \cap N_k)$.

Take C' such that $c'_{ij} = c_{ij}$, $\forall (ij) \neq (mn)$, $m, n \in N_l$.

Let g'_S be m.c.s.t. corresponding to C' restricted to S^+. Then,

$$g'_S = \cup_{k=1}^{p}(g'_{(S\cap N_k)}) = \cup_{k\neq l}(g_{(S\cap N_k)}) \cup g'_{S\cap N_l}$$

[4] $(-u)$ is the relevant function because as opposed to the value functions, we consider cost functions where a smaller value is better in cost games.

This follows because m.c.s.t. of groups other than N_l will not change in C'.

Hence, $c'(S) = \sum_{k \neq l} c(S \cap N_k) + c'(S \cap N_l)$.

Take $i \in N_t,\ t \neq l,\ \ \pi \in \Pi$. Then,

$$\begin{aligned} & c(\{i\} \cup S_{\pi(i)}) - c'(\{i\} \cup S_{\pi(i)}) \\ = \;& c(N_l \cap \left\{\{i\} \cup S_{\pi(i)}\right\}) - c'(N_l \cap \left\{\{i\} \cup S_{\pi(i)}\right\}) \\ = \;& c(N_l \cap S_{\pi(i)}) - c'(N_l \cap S_{\pi(i)}) \qquad [\text{since } i \notin N_l] \\ = \;& c(S_{\pi(i)}) - c'(S_{\pi(i)}) \end{aligned}$$

This implies,
$c(\{i\} \cup S_{\pi(i)}) - c(S_{\pi(i)}) = c'(\{i\} \cup S_{\pi(i)}) - c'(S_{\pi(i)})$,
and $\Phi_i(C) = \Phi_i(C')$.
Here the last implication follows from summing over $\pi \in \Pi$.

We now verify that the Shapley value satisfies equal treatment.

Let C and C' be two cost matrices, where $c'_{kl} = c_{kl}\ \ \forall (kl) \neq (ij); i, j \neq 0$. Note that the cost functions may differ only for those coalitions which have both i, j. Take $\pi \in \Pi$. Without loss of generality, assume $\pi(i) < \pi(j)$. Note that as $j \notin S_{\pi(j)}, c(S_{\pi(j)}) = c'(S_{\pi(j)})$.
Then,

$$\begin{aligned} & [c(S_{\pi(j)} \cup \{j\}) - c(S_{\pi(j)})] - [c'(S_{\pi(j)} \cup \{j\}) - c'(S_{\pi(j)})] \\ & \qquad = [c(S_{\pi(j)} \cup \{j\}) - c'(S_{\pi(j)} \cup \{j\})] \end{aligned} \tag{2.1}$$

and, since $j \notin (S_{\pi(i)} \cup \{i\})$,

$$[c(S_{\pi(i)} \cup \{i\}) - c(S_{\pi(i)})] - [c'(S_{\pi(i)} \cup \{i\}) - c'(S_{\pi(i)})] = 0 \tag{2.2}$$

Consider $\pi^\star \in \Pi$ such that $\pi^\star(k) = \pi(k)\ \forall k \neq i, j; \pi^\star(i) = \pi(j)$ and $\pi^\star(j) = \pi(i)$. Thus, $\pi^\star(i) > \pi^\star(j)$. Hence, by a similar argument,

$$[c(S_{\pi^\star(j)} \cup \{j\}) - c(S_{\pi^\star(j)})] - [c'(S_{\pi^\star(j)} \cup \{j\}) - c'(S_{\pi^\star(j)})] = 0 \tag{2.3}$$

As $(S_{\pi(j)} \cup \{j\}) = (S_{\pi^*(i)} \cup \{i\})$, we have

$$[c(S_{\pi^*(i)} \cup \{i\}) - c(S_{\pi^*(i)})] - [c'(S_{\pi^*(i)} \cup \{i\}) - c'(S_{\pi^*(i)})]$$

$$= [c(S_{\pi^*(i)} \cup \{i\}) - c'(S_{\pi^*(i)} \cup \{i\})] \tag{2.4}$$

$$= [c(S_{\pi(j)} \cup \{j\}) - c'(S_{\pi(j)} \cup \{j\})] \tag{2.5}$$

Now,

$\Phi_j(C) - \Phi_j(C')$

$= \frac{1}{n!} \sum_{\pi \in \Pi} \left[\left\{ c\left(S_{\pi(j)} \cup \{j\}\right) - c\left(S_{\pi(j)}\right) \right\} - \left\{ c'\left(S_{\pi(j)} \cup \{j\}\right) - c'\left(S_{\pi(j)}\right) \right\} \right]$

$= \frac{1}{n!} \left[\sum_{\pi | \pi(i) < \pi(j)} \left\{ c\left(S_{\pi(j)} \cup \{j\}\right) - c'\left(S_{\pi(j)} \cup \{j\}\right) \right\} + \sum_{\pi | \pi(j) < \pi(i)} 0 \right]$ [using (2.1), (2.3)]

$= \frac{1}{n!} \left[\sum_{\pi^* | \pi^*(i) > \pi^*(j)} \left\{ c\left(S_{\pi^*(i)} \cup \{i\}\right) - c'\left(S_{\pi^*(i)} \cup \{i\}\right) \right\} + \sum_{\pi^* | \pi^*(j) > \pi^*(i)} 0 \right]$ [using (2.5)]

$= \Phi_i(C) - \Phi_i(C')$ [using (2.2), (2.4)]

This completes the proof that the Shapley value satisfies all four axioms.

Next, we show that the Shapley value is the only allocation rule satisfying the axioms mentioned above. We prove this by induction on the number of relevant links.

Let Ψ be another allocation rule satisfying all the four axioms. If a cost matrix C has no relevant links, then the m.c.s.t. of C is a $\star$ -graph with 0 in the centre. By absence of cross subsidization, the allocation is unique and $\Psi_i(C) = c_{i0} = \Phi_i(C)\ \forall i$.

Let the proposition be true for all cost matrices with at most $(k-1)$ relevant links. We will prove that the allocation is unique for cost matrices with k relevant links.

Assume that our proposition is not true. Then, there exists a cost matrix C^k with k relevant links such that $\Phi(C^k) \neq \Psi(C^k)$.

Without loss of generality, let (mn) be a relevant link in C^k. Suppose that the set of agents can be partitioned in p groups as $N = [N_1, N_2, ..., N_p]$, where $m, n \in N_t$, for some t, $1 \leq t \leq p$.

Let C^{k-1} be such that $c_{ij}^{k-1} = c_{ij}^{k} \forall (ij) \neq (mn)$, and $c_{mn}^{k-1} = \max(c_{m0}^{k}, c_{n0}^{k}) + \varepsilon$; where $\varepsilon > 0$.Thus C^{k-1} is a cost matrix with $(k-1)$ relevant links.

From the induction hypothesis, we get $\Phi_i(C^{k-1}) = \Psi_i(C^{k-1})\ \forall i \in N$.

By the group independence axiom,

$$\Phi_i\left(C^k\right) = \Phi_i\left(C^{k-1}\right) = \Psi_i\left(C^{k-1}\right) = \Psi_i\left(C^k\right)\ \forall i \in N_l,\ \forall l \neq t \tag{2.6}$$

Now,

$$\begin{aligned}
\Phi_m\left(C^k\right) - \Phi_n\left(C^k\right) &= \Phi_m\left(C^{k-1}\right) - \Phi_n\left(C^{k-1}\right) && \text{[By equal treatment]}\\
&= \Psi_m\left(C^{k-1}\right) - \Psi_n\left(C^{k-1}\right) && \text{[By induction hypothesis]}\\
&= \Psi_m\left(C^k\right) - \Psi_n\left(C^k\right) && \text{[By equal treatment]}
\end{aligned}$$

Therefore

$$\Phi_m\left(C^k\right) - \Psi_m\left(C^k\right) = \Phi_n\left(C^k\right) - \Psi_n\left(C^k\right) \tag{2.7}$$

Note that (2.7) does not depend on C^{k-1}. Hence by specifying a similar $\bar{C}^{k-1}$ for some other relevant link $(pq) \neq (mn)$, we can get equation (2.7) for p, q. Now take any $i, j \in N_t$. Then, $\exists\, i_1, \ldots, i_K$ in N_t such that $(i_k i_{k+1})$ are relevant links, $\forall k < K$ and $i_1 = i, i_k = j$.

From (2.7), we get, $\Phi_{i_k}(C^k) - \Psi_{i_k}(C^k) = \Phi_{i_{k+1}}(C^k) - \Psi_{i_{k-1}}(C^k)$.

Hence,

$$\Phi_i\left(C^k\right) - \Psi_i\left(C^k\right) = \Phi_j\left(C^k\right) - \Psi_j\left(C^k\right) \text{ for any } i, j \in N_t \tag{2.8}$$

Thus,

$$\begin{aligned}
\textstyle\sum_{i\in N}\left[\Phi_i\left(C^k\right) - \Psi_i\left(C^k\right)\right] &= 0 && \text{[using efficiency]}\\
\Rightarrow \textstyle\sum_{i\in N_t}\left[\Phi_i\left(C^k\right) - \Psi_i\left(C^k\right)\right] &= 0 && \text{[using (2.6)]}\\
\Rightarrow \Phi_i\left(C^k\right) &= \Psi_i\left(C^k\right) \quad \forall i \in N_t && \text{[using (2.8)]}\\
\Rightarrow \Phi\left(C^k\right) &= \Psi\left(C^k\right) && \text{[using (2.6)]}
\end{aligned}$$

To complete the proof of the theorem, we now show that the four axioms are independent of each other.

(1) Let μ be an allocation rule such that $\mu_k(C) = c_{k0}$, $\forall k \in N$. It is easy to check that this allocation rule satisfies the absence of cross subsidization, group independence and equal treatment axioms, but may violate the efficiency axiom.

(2) Let μ be an allocation rule such that,
$\mu_k(C) = \Phi_k(C)\ \forall k \neq 1,2$; $\mu_1(C) = \Phi_1(C) + \delta$ and $\mu_2(C) = \Phi_2(C) - \delta$. where Φ is the Shapley allocation rule.

This allocation rule satisfies the efficiency, group independence, equal treatment axioms but violates the absence of cross subsidization axiom when $\delta \neq 0$.[5]

(3) Let μ be an allocation rule such that

$$\mu_k(C) = \Phi_k(C) \qquad \forall k \neq \{1,2,3,4\}$$

When (12) and (34) are relevant in C,

$$\mu_i(C) = \Phi_i(C) + \delta \qquad i = 1,2$$

$$\mu_i(C) = \Phi_i(C) - \delta \qquad i = 3,4$$

Otherwise,

$$\mu_i(C) = \Phi_i(C) \qquad \forall k = \{1,2,3,4\}$$

where Φ is the Shapley value. This allocation rule satisfies the efficiency, absence of cross subsidization, equal treatment axioms but may violate the group independence axiom when $\delta \neq 0$, as shown in example 2.1.1. [6]

Example 2.1.1:

[5] The proof follows from the fact that Φ satisfies all the four axioms.

[6] Efficiency and equal treatment follow from the fact that Φ satisfies these two axioms. If the m.c.s.t. of C is a $\star$-graph with source as center, then μ is same as Φ and hence absence of cross subsidization is satisfied.

$$C = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 3 \\ 2 & 0 & 3 & 3 & 3 & 1 \\ 2 & 3 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 4 \\ 2 & 3 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & 4 & 4 & 0 \end{pmatrix}$$

Corresponding to C, the set of agents are partitioned as $N = [125, 34]$. Here the edge (12) is not relevant. Therefore the allocation μ coincides with the Shapley value and $\mu_1(C) = \frac{8}{6}, \mu_2(C) = \frac{11}{6}, \mu_3(C) = 1.5, \mu_4(C) = 1.5, \mu_5(C) = \frac{11}{6}$. Now consider C', where the cost of (12) decreases to 1 and cost of other connections remain the same. In C', both (12) and (34) are relevant.

$\mu_1(C) = \frac{5}{6} + \delta, \mu_2(C) = \frac{8}{6} + \delta, \mu_3(C) = 1.5 - \delta, \mu_4(C) = 1.5 - \delta, \mu_5(C) = \frac{11}{6}$

Here the allocation of 3 changes, with a decrease in cost of another group.

(4) Let μ be an allocation rule such that

$$\mu_k(C) = \frac{1}{|G_N|} \sum_{g_n \in G_N} B_k(C, g_N) \quad \forall k \in N$$

This allocation rule satisfies the efficiency, absence of cross subsidization and group independence axioms. It is a convex combination of Bird allocations corresponding to minimum cost spanning trees of a cost matrix. The proof follows from the fact that the Bird allocation satisfies these three axioms. But this allocation may violate equal treatment axiom , as shown below in Example 2.1.2.

Example 2.1.2:

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1.5 \\ 1 & 1.5 & 0 \end{pmatrix}$$

According to μ, 1 pays 1.5, and 2 pays 1. Now in C' cost of (12) changes to 0.5, and cost of other connections remain the same. In C', 1 pays 0.5 ,but 2 still pays 1. Thus effects on 1 and 2 are not equal.

Remark 2.1.1: The minimum cost spanning tree games are different from the *monotone* version. The *monotone m.c.s.t. games* are defined as follows, $\forall S \subseteq N$, $\hat{c}(S) = \min_{S \subseteq T \subseteq N} c(T)$.[7] Here the assumption is that every set of players S is allowed to use nodes in $N \setminus S$ while constructing its separate network. It is relevant to point out the fact that our axioms do not characterize the Shapley value on the *monotone minimum cost spanning tree games*. The following example shows that the Shapley value does not satisfy equal treatment axiom on the monotone m.c.s.t. games.

Example 2.1.3:

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{pmatrix}$$

The monotone m.c.s.t. game corresponding to C will be $\hat{c}(1) = 1.5, \hat{c}(2) = 1, \hat{c}(12) = 1.5$. So, $\Phi_1(C) = 1, \Phi_2(C) = 0.5$ is the Shapley allocation. Now we consider C_1 such that only the cost of the edge (12) changes to 0.6. Monotone m.c.s.t. game corresponding to C_1 is $\hat{c}_1(1) = 1.6, \hat{c}_1(2) = 1, \hat{c}_1(12) = 1.6$. Here $\Phi_1(C_1) = 1.1$, and $\Phi_2(C_1) = 0.5$. Thus equal treatment is violated.

[7] For detailed discussion on monotone minimum cost spanning tree games see Granot and Huberman [1981] or Granot and Maschler [1998].

Chapter 3

Cost Monotonicity and Core Selection

The Shapley value and the Bird allocation have been studied extensively in the literature on m.c.s.t. games. While Bird's method always selects an allocation in the core of the game, the following example shows that the Bird rule does not satisfy cost monotonicity.

Example 3.0.1: Let $N = \{1, 2\}$. Consider two matrices specified below.

(i) $c_{01} = 4, c_{02} = 4.5, c_{12} = 3.$

(ii) $c'_{01} = 4, c'_{02} = 3.5, c'_{12} = 3.$

Then, $B_1(C) = 4, B_2(C) = 3$, while $B_1(C') = 3, B_2(C') = 3.5$. So, 2 is charged more when the matrix is C' although $c'_{02} < c_{02}$ and the costs of edges involving 1 remain the same.

On the other hand the cost allocation rule, which coincides with the Shapley value of the cost game, satisfies cost monotonicity. However, it may not lie in the core of the m.c.s.t. game. The following example shows that the Shapley value is not a core selection.

Example 3.0.2: Let $N = \{1, 2, 3\}$. Consider the following cost matrix.

$$C = \begin{pmatrix} 0 & 3 & 12 & 4 \\ 3 & 0 & 2 & 6 \\ 12 & 2 & 0 & 1 \\ 4 & 6 & 1 & 0 \end{pmatrix}$$

The allocation corresponding to the Shapley value is $\Phi_1(C) = \frac{4}{6}, \Phi_2(C) = \frac{25}{6}$ and $\Phi_3(C) = \frac{7}{6}$. However this allocation will be blocked by the coalition $\{2, 3\}$ as $\Phi_2(C) + \Phi_3(C) = \frac{32}{6} > 5 = c(\{23\})$.

An allocation rule corresponding to the nucleolus of m.c.s.t. games is a core selection. However, we do not know whether the neucleolus of m.c.s.t. games satisfies cost monotonicity. But it is easy to check that the nucleolus does not satisfy tree invariance. Here is an example to illustrate this fact.

Example 3.0.3: Consider the following cost matrix C.

$$C = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Let C' be another cost matrix where $c'_{20} = 4$. All other costs remain the same. These two cost matrices have same minimum cost spanning trees. But, $\xi_1(C) = 1, \xi_2(C) = 2$ which is different from $\xi_1(C') = 0.5, \xi_2(C') = 2.5$.

Young [1994] shows that in the context of transferable utility games, there is no solution concept which picks an allocation in the core of the game when the latter is nonempty and also satisfies a property which is analogous to cost monotonicity. In this chapter we show that cost monotonicity and core selection are not mutually exclusive in the context of m.c.s.t. games by constructing a new class of allocation rules which satisfies both the properties. We are able to do this despite the impossibility result due to Young because of the special structure of m.c.s.t. games - these are a *strict* subset of the class of *balanced* games.

The following lemma will be required to define our new allocation rules.

Lemma 3.0.1 *Let $C \in \mathcal{C}^1_N$, and $i \in N$. If $c_{ik} = \min_{l \in N^+ \setminus \{i\}} c_{il}$, then $(ik) \in g_N(C)$.*

Proof : Suppose $(ik) \notin g_N(C)$. As $g_N(C)$ is a connected graph over N^+, $\exists\, j \in N^+ \setminus \{i,k\}$ such that $(i\,j) \in g_N(C)$ and j is on the path between i and k. But, $\{g_N(C) \cup (ik)\} \setminus \{(ij)\}$ is still a connected graph which costs no more than $g_N(C)$, as $c_{ik} \leq c_{ij}$. This is not possible as $g_N(C)$ is the only m.c.s.t. of C.

■

3.1 A class of Allocation Rules

The new allocation rules are denoted by $\Psi^{N,\lambda}$ where N is the set of agents and λ is a parameter. From now on we will drop the index N whenever there is no confusion about the set over which the allocation is defined. Our rules are defined for all cost matrices in $\mathcal{C}$. However, in order to economise on notation, we describe the class of rules for a cost matrix in $\mathcal{C}^2$. We then indicate how to construct the rule for all cost matrices.

For $0 \leq \lambda \leq 1$, this class of allocation rules is defined recursively as follows. First, if $\#N = 1$ then $\Psi^\lambda_1(C) = c_{10}$. Suppose we have defined this class of allocation rules for all sets N with cardinality strictly less than m. Now we define it for set of agents N such that $\#N = m$. Assume that $c_{kl} = \min_{i \neq j} c_{ij}$, where k is the immediate predecessor of l.[1] This is unique as $C \in \mathcal{C}^2$. Now define the reduced cost matrix C^R over $N^+ \setminus \{l\}$ as follows,

$$c^R_{mn} = c_{mn} \text{ if } k \notin \{m,n\} \tag{3.1}$$

[1]Note that $(kl) \in g_N(C)$ from lemma 3.0.1.

$$c^R_{jk} = \min\{c_{jk}, c_{jl}\} \forall j \neq k, l \tag{3.2}$$

Since C^R is a cost matrix over a set which has cardinality $m-1$, $\Psi^\lambda(C^R)$ is already defined. Now define $\Psi^\lambda(C)$

$$\Psi^\lambda_l(C) = \begin{cases} (1-\lambda)\Psi^\lambda_k(C^R) + \lambda c_{kl} & \text{if } k \neq 0 \\ c_{kl} & \text{otherwise} \end{cases} \tag{3.3}$$

$$\Psi^\lambda_k(C) = \lambda \Psi^\lambda_k(C^R) + (1-\lambda) c_{kl} \text{ if } k \neq 0 \tag{3.4}$$

$$\Psi^\lambda_i(C) = \Psi^\lambda_i(C^R) \text{ for all } i \neq k, l \tag{3.5}$$

Lemma 3.1.1 *If $g_N(C)$ is the m.c.s.t. of $C \in \mathcal{C}^2$, then the m.c.s.t. of C^R will be $g(C^R)$, where*

$$g(C^R) = \{(pq) \in g_N(C) | \{p, q\} \cap \{k, l\} = \emptyset\} \cup \{(tk) \neq (kl) | (tk) \text{ or } (tl) \in g_N(C)\}$$

Proof : Let $C \in \mathcal{C}^2$. The m.c.s.t. of C can be divided into two parts. That is $g_N(C) = g^1(C) \cup g^2(C)$, where

$$\begin{aligned} g^1(C) &= \{(pq) \in g_N(C) | \{p, q\} \cap \{k, l\} = \emptyset\} \\ g^2(C) &= g_N(C) \setminus g^1(C) \end{aligned}$$

Thus, $g(C^R) = g^1(C) \cup \{(tk) \neq (kl) | (tk) \text{ or } (tl) \in g^2(C)\}$.

Clearly $g(C^R)$ is a connected graph over $N^+ \setminus \{l\}$.

$$\sum_{(ij)\in g(C^R)} c^R_{ij} = \sum_{(ij)\in g^1(C)} c^R_{ij} + \sum_{(tk)\in g(C^R)} c^R_{tk} = \sum_{(ij)\in g^1(C)} c_{ij} + \sum_{(tk)\in g(C^R)} \min(c_{tk}, c_{tl}) \tag{3.6}$$

Now, $(tk) \in g(C^R)$ implies either (tk) or $(tl) \in g^2(C)$. If $(tk) \in g^2(C)$ then $(tl) \notin g^2(C)$ as $(kl) \in g^2(C)$. Hence $c_{tk} < c_{tl}$ or $c^R_{tk} = c_{tk}$. Similarly if $(tl) \in g^2(C)$ then $c^R_{tk} = c_{tl}$. Therefore from (3.6)

$$\sum_{(ij)\in g(C^R)} c^R_{ij} = \sum_{(ij)\in g_N(C)} c_{ij} - c_{kl} \tag{3.7}$$

Suppose $g(C^R)$ is not the only m.c.s.t. of C^R. Let g be another m.c.s.t. corresponding to the cost matrix C^R. Hence g is a connected graph over $N^+ \setminus \{l\}$. We construct

$$\bar{g} = \{(ij) \in g | i,j \neq k\} \cup \{(it) | (ik) \in g, c_{ik}^R = c_{it}\} \cup \{(kl)\}$$

which is a connected graph over N^+.

Using (3.7) and the fact that g is an m.c.s.t. corresponding to C^R we get

$$\sum_{(ij)\in\bar{g}} c_{ij} = \sum_{(ij)\in g} c_{ij}^R + c_{kl} \leq \sum_{(ij)\in g(C^R)} c_{ij}^R + c_{kl} = \sum_{(ij)\in g_N(C)} c_{ij}$$

This contradicts the fact that $g_N(C)$ is the only m.c.s.t. of C. ■

This lemma shows that $C^R \in \mathcal{C}^2$ whenever $C \in \mathcal{C}^2$.

We now construct an example to illustrate our algorithm.

Example 3.1.1:

$$C = \begin{pmatrix} 0 & 3 & 4 & 6 \\ 3 & 0 & 5 & 1 \\ 4 & 5 & 0 & 2 \\ 6 & 1 & 2 & 0 \end{pmatrix}$$

Step 1 : Here, $\min_{p\neq q} c_{pq} = c_{13} = 1$. Let C^1 be the reduced cost matrix on $\{1,2\}$. We get, $c_{12}^1 = \min(c_{12}, c_{23}) = c_{23} = 2$ and $c_{10}^1 = \min(c_{10}, c_{30}) = c_{10} = 3$. Therefore

$$C^1 = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 0 & 2 \\ 4 & 2 & 0 \end{pmatrix}$$

Step 2 : Again $\min_{p\neq q} c_{pq}^1 = c_{12}^1 = 2$. The reduced cost matrix of C^1 is denoted by C^2.

$$C^2 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

Step 3 : We obtain the following allocation corresponding to $\lambda = 0.5$.

$$\begin{aligned}
\Psi_1^\lambda(C) &= \lambda[\lambda\Psi_1^\lambda(C^2) + (1-\lambda)c_{12}^1] + (1-\lambda)c_{13} = 1.75 \\
\Psi_2^\lambda(C) &= (1-\lambda)\Psi_1^\lambda(C^2) + \lambda c_{12}^1 = 2.5 \\
\Psi_3^\lambda(C) &= (1-\lambda)[\lambda\Psi_1^\lambda(C^2) + (1-\lambda)c_{12}^1] + \lambda c_{13} = 1.75
\end{aligned}$$

The allocation has been described for matrices in $\mathcal{C}^2$. Suppose that $C \notin \mathcal{C}^2$. Then there can be more than one m.c.s.t. corresponding to the cost matrix C. Moreover a m.c.s.t. may contain edges which cost the same. Then, our algorithm is not well-defined because at some step there may exist more than one edge which minimises the cost of connections. Even if the minimum cost edge is unique, it will not be possible to assert predecessor-follower relationship because C might have more than one m.c.s.t.. But, there is an easy way to extend the algorithm to deal with matrices not in $\mathcal{C}^2$.

Let σ be a strict ordering over the set of edges. Then, σ can be used as a tie-breaking rule. First, we use σ to fix a m.c.s.t. of C as follows.

Step 0: Let $A^0 = \{0\}$, $g^0 = \emptyset$.

Step k: Choose ordered pair (mn) such that $(mn) = \text{argmin}_{(i,j)\in A^{k-1}\times A_c^{k-1}} c_{ij}$, where for all $A \subseteq N$, $A_c = N \setminus A$. This ordered pair may not be unique. Let

$$S^k = \{(mn) | (mn) = \text{argmin}_{(i,j)\in A^{k-1}\times A_c^{k-1}} c_{ij}\}$$

Choose $(a^k b^k)$ where $(a^k b^k)$ is σ-maximized in S^k. Let $A^k = A^{k-1} \cup \{b^k\}$ and $g^k = g^{k-1} \cup \{(a^k b^k)\}$.

This algorithm stops at step $\#N$. The minimum cost spanning tree corresponding to σ is given by $g^{\#N}$.

Now, we use σ to fix the minimum cost edge. Let the set E be defined as $E = \{(ij) | c_{ij} = \min_{p\neq q} c_{pq}\}$. Then we choose (kl), the σ-maximized cost edge in E.

Thus any such tie-breaking rule makes the algorithm well-defined. Now, let Σ be the set of all strict orderings over the set of edges. Then, the eventual cost allocation is obtained by taking the simple average of the 'component' cost allocations. That is, for any $\sigma \in \Sigma$, let $\Psi_\sigma^\lambda(C)$ denote the cost allocation obtained from the algorithm when σ is used as the tie-breaking rule. Then,

$$\Psi^\lambda(C) = \frac{1}{\#\Sigma} \sum_{\sigma \in \Sigma} \Psi_\sigma^\lambda(C). \tag{3.8}$$

Here is an example to illustrate this procedure.

Example 3.1.2:

$$C = \begin{pmatrix} 0 & 4 & 4 & 5 \\ 4 & 0 & 2 & 2 \\ 4 & 2 & 0 & 5 \\ 5 & 2 & 5 & 0 \end{pmatrix}$$

The cost matrix C has two m.c.s.t.s - $g = \{(01), (12), (13)\}$ and $g^1 = \{(02), (12), (13)\}$. The edges (12) and (13) have the same cost.

We choose $\lambda = 0$. First, note that g will be the m.c.s.t. for all permutations σ which rank (10) over (20). Otherwise g^1 is the m.c.s.t. of C.

Consider the permutations for which g is the m.c.s.t.. Among those permutations if (12) is ranked over (13), then the minimum cost edge is (12). Otherwise (13) is the minimum cost edge. Taking each in turn we obtain the allocations $x^1 = (2, 2, 4)$ and $x^2 = (2, 4, 2)$. The weights on these allocations will be one-fourth each.

If g^1 is the m.c.s.t. of C then irrespective of the choice of minimum cost edge we get the allocation $x^3 = (2, 2, 4)$. Hence the weight attached to x^3 is half.

Taking the weighted average, we get $\Psi^0(C) = (2, 2.5, 3.5)$.

3.2 Properties of Ψ_N^λ

In this section, we show that for all $\lambda \in [0,1]$, each Ψ^λ is a core selection, satisfies tree invariance, scale invariance and extreme point monotonicity. Moreover Ψ^λ satisfies cost monotonicity iff $\lambda \in [0, 0.5]$. For notational simplicity, we assume that $C \in \mathcal{C}^2$ in all the subsequent proofs.

Lemma 3.2.1 *For* $0 \leq \lambda \leq 1,\ \forall i \in N,\ \Psi_i^\lambda(C) \geq \min_{t \in N^+ \setminus \{i\}} c_{it}$.

Proof: This is trivially true for $\#N = 1$. Let this be true for all cost matrices with number of agents strictly less than m. Take $C \in \mathcal{C}_N^2$ with $\#N = m$. Let $c_{kl} = \min_{i \neq j} c_{ij}$ and $k = \alpha(l)$.

$$\forall i \neq k, l;\ \Psi_i^\lambda(C) = \Psi_i^\lambda(C^R) \geq \min_{t \in N^+ \setminus \{i,l\}} c_{it}^R = \min_{t \in N^+ \setminus \{i\}} c_{it}$$

If $k \neq 0$ then

$$\Psi_k^\lambda(C) = \lambda \Psi_k^\lambda(C^R) + (1-\lambda)c_{kl} \geq \lambda \min_{t \in N^+ \setminus \{k,l\}} c_{kt}^R + (1-\lambda)c_{kl} \geq c_{kl} = \min_{t \in N^+ \setminus \{k\}} c_{kt}$$

Similarly, $\Psi_l^\lambda(C) \geq \min_{t \in N^+ \setminus \{l\}} c_{lt}$.
If $k = 0$ then $\Psi_l^\lambda(C) = c_{l0} = \min_{t \in N^+ \setminus \{l\}} c_{lt}$. Hence the result follows. ■

Theorem 3.2.1 *For all* $\lambda \in [0,1]$, *each* Ψ^λ *is a core selection, satisfies tree invariance, scale invariance and extreme point monotonicity. Moreover* Ψ^λ *satisfies cost monotonicity iff* $\lambda \in [0, 0.5]$.

Proof: We will prove all the results by induction over cardinality of N. Core selection, tree invariance, scale invariance, and cost monotonicity are trivially satisfied for $\#N = 1$. It can be easily checked that extreme point monotonicity is satisfied for $\#N = 2$. Assume that the result is true for all N with $\#N < m$. Now we will prove the result when $\#N = m$.

Take $C \in \mathcal{C}_N^2$. Let $c_{kl} = \min_{i \neq j} c_{ij}$ and k be the immediate predecessor of l.

[Core selection] : Suppose Ψ^λ does not belong to the core of C. Then $\exists S \subset N$ such that S can block Ψ^λ. Let C_S be the restriction of C over S^+. Let $g_S(C_S)$ be a m.c.s.t. corresponding to C_S.

There are two possible cases (1) $k \neq 0$ (2) $k = 0$.

Case 1 : Suppose $k \neq 0$. We will argue that $\{k, l\} \cap S \neq \emptyset$. To the contrary assume that S contains neither k nor l. Then using the fact that C and C^R coincide on S^+ we get

$$\sum_{t\in S} \Psi_t^\lambda(C^R) = \sum_{t\in S} \Psi_t^\lambda(C) > c(S) = c^R(S)$$

So, S is also a blocking coalition in C^R contradicting the induction hypothesis. Therefore $\{k, l\} \cap S \neq \emptyset$.

We will now show that $\widehat{S} = [S \cup \{k\} \setminus \{l\}]$ is a blocking coalition in C^R. Consider the following graph

$$g = \{(pq) \in g_S(C_S) | \{p, q\} \cap \{k, l\} = \emptyset\} \cup \{(tk) \neq (lk) | (tk) \text{ or } (tl) \in g_S(C_S)\}$$

Clearly, g is a connected graph over $\widehat{S}^+$ because $g_S(C_S)$ is a connected graph over S^+. Thus $\sum_{(ij)\in g} c_{ij}^R \geq c^R(\widehat{S})$.

Take any $(pq) \in g$. If $\{p, q\} \cap \{k, l\} = \emptyset$ then $c_{pq}^R = c_{pq}$. If $(tk) \in g$ then $c_{tk}^R = \min\{c_{tk}, c_{tl}\}$.

Suppose $k, l \in S$. Then $\sum_{(ij)\in g} c_{ij}^R = c(S) - c_{kl}$. Now,

$$\sum_{t\in \widehat{S}} \Psi_t^\lambda(C^R) = \sum_{t\in S} \Psi_t^\lambda(C) - c_{kl} > c(S) - c_{kl} \geq c^R(\widehat{S})$$

Therefore $\widehat{S}$ is a blocking coalition.

Otherwise $c^R(\widehat{S}) \leq \sum_{(ij)\in g} c_{ij}^R \leq c(S)$. Using lemma 3.2.1 we get

$$\sum_{t\in \widehat{S}} \Psi_t^\lambda(C^R) = \sum_{t\in S\setminus\{k,l\}} \Psi_t^\lambda(C^R) + \Psi_k^\lambda(C^R) \geq \sum_{t\in S\setminus\{k,l\}} \Psi_t^\lambda(C) + \Psi_j^\lambda(C) = \sum_{t\in S} \Psi_t^\lambda(C)$$

where j is either k or l. Thus,

$$\sum_{t\in \widehat{S}} \Psi_t^\lambda(C^R) \geq \sum_{t\in S} \Psi_t^\lambda(C) > c(S) \geq c^R(\widehat{S})$$

Again $\widehat{S}$ is a blocking coalition.

Therefore if $k \neq 0$ then it is always possible to obtain a blocking coalition in C^R which contradicts our induction hypothesis.

Case 2: $k = 0$. Suppose l does not belong to the blocking coalition S. Then $c(S) = \sum_{(ij)\in g_S(C_S)} c_{ij} \geq \sum_{(ij)\in g_S(C_S)} c^R_{ij} \geq c^R(S)$. Now,

$$\sum_{t\in S} \Psi^\lambda_t(C^R) = \sum_{t\in S} \Psi^\lambda_t(C) > c(S) \geq c^R(S)$$

Hence, S is a blocking coalition in C^R contradicting the induction hypothesis.

Therefore it must be the case that $l \in S$. Consider the following graph

$$g = \{(pq) \in g_S(C_S) | \{p, q\} \cap \{l\} = \emptyset\} \cup \{(t0) \neq (l0) | (t0) \text{ or } (tl) \in g_S(C_S)\}$$

Since $g_S(C_S)$ is a connected graph over S it follows that g is a connected graph over $[S \setminus \{l\}]$.

Take any $(pq) \in g$ such that $\{p, q\} \cap \{l\} = \emptyset$. Then $c^R_{pq} = c_{pq}$. Also

$$c^R_{t0} = \min\{c_{t0}, c_{tl}\} = \begin{cases} c_{t0} & \text{if } (t0) \in g_S(C_S) \\ c_{tl} & \text{if } (tl) \in g_S(C_S) \end{cases}$$

Since $(l0) \in g_S(C_S)$, we can not have both $(t0) \in g_S(C_S)$ and $(tl) \in g_S(C_S)$.

Thus

$$c^R(S \setminus \{l\}) \leq \sum_{(ij)\in g} c^R_{ij} = \sum_{(ij)\in g_S(C_S)} c_{ij} - c_{l0} = c(S) - c_{l0}$$

Now,

$$\sum_{t\in[S\setminus\{l\}]} \Psi^\lambda_t(C^R) = \sum_{t\in S} \Psi^\lambda_t(C) - c_{l0} > c(S) - c_{l0} \geq c^R(S \setminus \{l\})$$

Therefore $[S \setminus \{l\}]$ is a blocking coalition in C^R contradicting the induction hypothesis.

This completes the proof of core selection. ■

[Tree invariance] : Consider $\bar{C} \in \mathcal{C}^2_N$ such that $g_N(C) = g_N(\bar{C})$ and $(ij) \in g_N(C) \Rightarrow c_{ij} = \bar{c}_{ij}$. From lemma 3.0.1, $(kl) \in g_N(C)$. Hence $(kl) \in g_N(\bar{C})$

and $c_{kl} = \bar{c}_{kl}$. First we assert that $\bar{c}_{kl} = \min_{i \neq j} \bar{c}_{ij}$. Contrary to this, suppose $\bar{c}_{mn} = \min_{i \neq j} \bar{c}_{ij}$ where $(mn) \neq (kl)$. Then $(mn) \in g_N(\bar{C})$ from lemma 3.0.1. Therefore $(mn) \in g_N(C)$ and $c_{mn} = \bar{c}_{mn}$. Hence $c_{mn} = \bar{c}_{mn} < \bar{c}_{kl} = c_{kl}$, which contradicts our assumption.

Now, using lemma 3.1.1 we get that $g(C^R) = g(\bar{C}^R)$. It follows from the definition of $C^R, \bar{C}^R$ that $c_{ij}^R = \bar{c}_{ij}^R$ for all $(ij) \in g(C^R)$.

Therefore C^R and $\bar{C}^R$ are two cost matrices which have same minimum cost spanning trees. From the induction hypothesis $\Psi^\lambda(C^R) = \Psi^\lambda(\bar{C}^R)$. Using the fact that $c_{kl} = \bar{c}_{kl}$ we get $\Psi^\lambda(C) = \Psi^\lambda(\bar{C})$. Hence the result follows. ■

[Scale invariance] : Let C and D be two cost matrices such that $D = \delta C + \beta$. Note that (kl) is also the minimum cost edge in D and $D^R = \delta C^R + \beta$. By induction hypothesis $\Psi^\lambda(D^R) = \delta\Psi^\lambda(C^R) + \beta$. Thus for all $i \neq k, l$

$$\Psi_i^\lambda(D) = \Psi_i^\lambda(D^R) = \delta\Psi_i^\lambda(C^R) + \beta = \delta\Psi_i^\lambda(C) + \beta$$

If $k \neq 0$ then

$$\Psi_k^\lambda(D) = \lambda\Psi_k^\lambda(D^R) + (1-\lambda)d_{kl} = \lambda[\delta\Psi_k^\lambda(C^R) + \beta] + (1-\lambda)[\delta c_{kl} + \beta] = \delta\Psi_k^\lambda(C) + \beta$$

Similarly it can be shown that $\Psi_l^\lambda(D) = \delta\Psi_l^\lambda(C) + \beta$.
If $k = 0$ then $\Psi_l^\lambda(D) = d_{l0} = \delta c_{l0} + \beta = \delta\Psi_l^\lambda(C) + \beta$. ■

[Extreme point monotonicity] : Let t be an extreme point of C. Let D be the restriction of C over $N^+ \setminus \{t\}$. First, note that our allocation rule is well defined over D because $D \in \mathcal{C}^2$. There are two possible cases. Either $t \neq l$ or $t = l$.

If $t \neq l$ then from lemma 3.1.1 it follows that t is also an extreme point of C^R. Moreover D^R is the restriction of C^R over $[N^+ \setminus \{l, t\}]$. From the induction hypothesis, for all $i \in [N \setminus \{l, t\}]$, $\Psi_i^\lambda(D^R) \geq \Psi_i^\lambda(C^R)$. From the construction of Ψ^λ it follows that $\Psi_i^\lambda(D) \geq \Psi_i^\lambda(C)$ $\forall i \in N \setminus \{t\}$.

If $t = l$ then from lemma 3.1.1, $g_{N\setminus\{l\}}(C^R) = g_N(C) \setminus \{(kl)\}$. Since l is an extreme point of C we have $g_{N\setminus\{l\}}(D) = g_N(C) \setminus \{(kl)\}$. Thus $g_{N\setminus\{l\}}(C^R) = g_{N\setminus\{l\}}(D)$. Take any $(ij) \in g_{N\setminus\{l\}}(D)$. We have $d_{ij} = c_{ij}$. If $i, j \neq k$ then $c_{ij} = c^R_{ij}$. Since l is an extreme point, for all $t \neq k, l$; $c^R_{kt} = \min\{c_{kt}, c_{lt}\} = c_{kt} = d_{kt}$. Therfore $(ij) \in g_{N\setminus\{l\}}(D)$ implies $d_{ij} = c^R_{ij}$

Hence from tree invariance, $\Psi^\lambda_i(C^R) = \Psi^\lambda_i(D)$ for all $i \neq l$. Therefore $\Psi^\lambda_i(C) = \Psi^\lambda_i(D)$ for all $i \neq k, l$. Also $\Psi^\lambda_k(C) \leq \Psi^\lambda_k(C^R) = \Psi^\lambda_k(D)$ from lemma 3.2.1. ■

[Cost monotonicity] : Suppose $\lambda \in [0, 0.5]$. Take $\bar{C} \in \mathcal{C}^2_N$ where $c_{pq} = \bar{c}_{pq}$ $\forall (pq) \neq (ij)$ and $\bar{c}_{ij} > c_{ij}$. We have to prove that $\Psi^\lambda_t(\bar{C}) \geq \Psi^\lambda_t(C)$ for all $t \in \{i, j\} \cap N$.

If $(ij) \notin g_N(C)$ then $(ij) \notin g_N(\bar{C})$ and hence from tree invariance for all $t \in N$, $\Psi^\lambda_t(\bar{C}) = \Psi^\lambda_t(C)$. Therefore we assume that $(ij) \in g_N(C)$ and $i = \alpha(j)$. We prove the result for $k \neq 0$. For $k = 0$ the proof is similar and hence omitted. There are two possible cases.

Case (i) : $(ij) \neq (kl)$.

In this case (kl) is still the minimum cost edge of $\bar{C}$.

If $\{i, j\} \cap \{k, l\} = \emptyset$, then $\bar{c}^R_{ij} > c^R_{ij}$. For all other edges $c^R_{t_1t_2} = \bar{c}^R_{t_1t_2}$. Applying the induction hypothesis, $\Psi^\lambda_t(C^R) \leq \Psi^\lambda_t(\bar{C}^R)$ $\forall t \in \{i, j\} \cap N$ and hence $\Psi^\lambda_t(C) \leq \Psi^\lambda_t(\bar{C})$.

Otherwise let $q = \{i, j\} \setminus \{k, l\}$. Either $k = \alpha(l)$ or $l = \alpha(k)$ in $g_N(\bar{C})$.

If k is the immediate predecessor of l in $g_N(\bar{C})$ then $\bar{c}^R_{qk} = \min(\bar{c}_{qk}, \bar{c}_{ql}) \geq \min(c_{qk}, c_{ql}) = c^R_{qk}$. For all other edges (t_1t_2), $\bar{c}^R_{t_1t_2} = c^R_{t_1t_2}$. From the induction hypothesis

$$\Psi^\lambda_q(C) = \Psi^\lambda_q(C^R) \leq \Psi^\lambda_q(\bar{C}^R) = \Psi^\lambda_q(\bar{C})$$

$$\Psi^\lambda_k(C) = \lambda\Psi^\lambda_k(C^R) + (1-\lambda)c_{kl} \leq \lambda\Psi^\lambda_k(\bar{C}^R) + (1-\lambda)\bar{c}_{kl} = \Psi^\lambda_k(\bar{C})$$

Similarly it follows that $\Psi^\lambda_l(C) \leq \Psi^\lambda_l(\bar{C})$.

Otherwise l is the immediate predecessor of k in $g_N(\bar{C})$. This is only possible if $j = k$, that is cost of (ik) increases. Then $\bar{c}_{il}^R = \min(\bar{c}_{ik}, \bar{c}_{il}) \geq \min(c_{ik}, c_{il}) = c_{ik}^R$. Therefore $\Psi_i^\lambda(C) = \Psi_i^\lambda(C^R) \leq \Psi_i^\lambda(\bar{C}^R) = \Psi_i^\lambda(\bar{C})$ and $\Psi_l^\lambda(\bar{C}^R) \geq \Psi_k^\lambda(C^R)$. Now,

$$\begin{aligned}\Psi_k^\lambda(\bar{C}) - \Psi_k^\lambda(C) &= [(1-\lambda)\Psi_l^\lambda(\bar{C}^R) + \lambda\bar{c}_{kl}] - [\lambda\Psi_k^\lambda(C^R) + (1-\lambda)c_{kl}] \\ &= (1-\lambda)[\Psi_l^\lambda(\bar{C}^R) - c_{kl}] - \lambda[\Psi_k^\lambda(C^R) - \bar{c}_{kl}] \\ &\geq 0\end{aligned}$$

The last inequality follows from the fact that $\lambda \leq 0.5$, $\bar{c}_{kl} = c_{kl}$ and $\Psi_l^\lambda(\bar{C}^R) \geq \Psi_k^\lambda(C^R)$.

This completes the proof for $(ij) \neq (kl)$.

Case (ii) : $(ij) = (kl)$

Note that (kl) can still be the minimum cost edge of $\bar{C}$. Then it is immediate that $C^R = \bar{C}^R$. Hence $\Psi_k^\lambda(C) = \lambda\Psi_k^\lambda(C^R) + (1-\lambda)c_{kl} < \lambda\Psi_k^\lambda(\bar{C}^R) + (1-\lambda)\bar{c}_{kl} = \Psi_k^\lambda(\bar{C})$. Similarly $\Psi_l^\lambda(C) < \Psi_l^\lambda(\bar{C})$.

So, assume that $(mn) \neq (kl)$ is the minimum cost edge in $\bar{C}$ and $m = \alpha(n)$. It is sufficient to show that cost monotonicity is satisfied when $\bar{c}_{kl} = \min_{\{p \neq q | (pq) \neq (mn)\}} \bar{c}_{pq}$.[2] Note that $c_{mn} = \bar{c}_{mn} = \min_{\{p \neq q | (pq) \neq (kl)\}} c_{pq}$. Also $k = \alpha(l)$ in $\bar{C}$ and $m = \alpha(n)$ in C.

Let the reduced cost matrices C^R and $\bar{C}^R$ be represented by D and $\bar{D}$. From C to D^R we first remove (kl) and then (mn). On the other hand from $\bar{C}$ to $\bar{D}^R$ we first remove (mn) and then (kl). As (kl) is the only edge which has different cost in C and $\bar{C}$ we get $D^R = \bar{D}^R$. Now we compare the allocations of k, l between C and $\bar{C}$. We have four sub cases.

[2] If $\bar{c}_{kl} > \min_{\{p \neq q | (pq) \neq (mn)\}} \bar{c}_{pq}$ then from case (i) cost monotonicity is satisfied between $\bar{C}$ and an intermediate matrix C', where $c'_{kl} = \min_{\{p \neq q | (pq) \neq (mn)\}} c'_{pq}$. Repeated application of this will thus establish the desired conclusion.

Case (a) : If $\{k,l\} \cap \{m,n\} = \emptyset$, then
$\Psi_k^\lambda(C) = \lambda\Psi_k^\lambda(D) + (1-\lambda)c_{kl} = \lambda\Psi_k^\lambda(D^R) + (1-\lambda)c_{kl} < \lambda\Psi_k^\lambda(\bar{D}^R) + (1-\lambda)\bar{c}_{kl} = \Psi_k^\lambda(\bar{D}) = \Psi_k^\lambda(\bar{C})$.

Similarly $\Psi_l^\lambda(C) < \Psi_l^\lambda(\bar{C})$.

Case (b) : If $l = m$, then

$$\begin{aligned}
\Psi_k^\lambda(C) &= \lambda\Psi_k^\lambda(D) + (1-\lambda)c_{kl} \\
&= \lambda[\lambda\Psi_k^\lambda(D^R) + (1-\lambda)c_{ln}] + (1-\lambda)c_{kl} \\
&< \lambda\Psi_k^\lambda(\bar{D}^R) + (1-\lambda)\bar{c}_{kl} \\
&= \Psi_k^\lambda(\bar{D}) = \Psi_k^\lambda(\bar{C})
\end{aligned}$$

The inequality follows from the fact that $\bar{c}_{kl} > c_{kl}$ and from lemma 3.2.1. $\Psi_k^\lambda(\bar{D}^R) \geq \min_{i \in N^+ \setminus \{k,l,n\}} \bar{d}_{ik}^R \geq \bar{c}_{kl} > c_{ln}$.

Similarly, as $\bar{c}_{kl} > c_{ln} > c_{kl}$, we get

$$\begin{aligned}
\Psi_l^\lambda(C) &= (1-\lambda)\Psi_k^\lambda(D) + \lambda c_{kl} \\
&= (1-\lambda)[\lambda\Psi_k^\lambda(D^R) + (1-\lambda)c_{ln}] + \lambda c_{kl} \\
&< \lambda[(1-\lambda)\Psi_k^\lambda(\bar{D}^R) + \lambda\bar{c}_{kl}] + (1-\lambda)c_{ln} \\
&= \lambda\Psi_l^\lambda(\bar{D}) + (1-\lambda)c_{ln} \\
&= \Psi_l^\lambda(\bar{C})
\end{aligned}$$

Case (c) : If $k = m$ then

$$\begin{aligned}
&\Psi_k^\lambda(\bar{C}) - \Psi_k^\lambda(C) \\
&= [\lambda\Psi_k^\lambda(\bar{D}) + (1-\lambda)c_{kn}] - [\lambda\Psi_k^\lambda(D) + (1-\lambda)c_{kl}] \\
&= \Big[\lambda[\lambda\Psi_k^\lambda(\bar{D}^R) + (1-\lambda)\bar{c}_{kl}] + (1-\lambda)c_{kn}\Big] - \\
&\quad \Big[\lambda[\lambda\Psi_k^\lambda(D^R) + (1-\lambda)c_{kn}] + (1-\lambda)c_{kl}\Big] \\
&> (1-\lambda)^2[c_{kn} - c_{kl}] \\
&> 0
\end{aligned}$$

The inequalities are immediate from the fact that $\bar{c}_{kl} > c_{kn} > c_{kl}$.

As $\Psi_k^\lambda(\bar{D}^R) \geq \min_{i \in N^+ \setminus \{k,l,n\}} \bar{d}_{ik}^R \geq c_{kn}$ and $\bar{c}_{kl} > c_{kl}$

$$\begin{aligned}
\Psi_l^\lambda(C) &= (1-\lambda)\Psi_k^\lambda(D) + \lambda c_{kl} \\
&= (1-\lambda)[\lambda \Psi_k^\lambda(D^R) + (1-\lambda)c_{kn}] + \lambda c_{kl} \\
&< (1-\lambda)\Psi_k^\lambda(\bar{D}^R) + \lambda \bar{c}_{kl} \\
&= \Psi_l^\lambda(\bar{D}) = \Psi_l^\lambda(\bar{C})
\end{aligned}$$

Case (d) : The only remaining case is $k = n$, because $l = n$ is not possible. The proof is similar to case (c) except the situation where $m = 0$. Then, $\Psi_k^\lambda(C) = \lambda \Psi_k^\lambda(D) + (1-\lambda)c_{kl} = \lambda c_{k0} + (1-\lambda)c_{kl} < c_{k0} = \Psi_k^\lambda(\bar{C})$ and $\Psi_l^\lambda(C) = (1-\lambda)c_{k0} + \lambda c_{kl} < (1-\lambda)c_{k0} + \lambda \bar{c}_{kl} = \Psi_l^\lambda(\bar{C})$.

This completes the proof when $(ij) = (kl)$.

Therefore allocation rules Ψ^λ satisfy cost monotonicity for $0 \leq \lambda \leq 0.5$.

To complete the proof we now show that for any value of $\lambda > 0.5$, we can construct two cost matrices C and $\bar{C}$, which show that Ψ^λ violates cost monotonicity.

Let $N = \{1, 2\}$. We choose C and $\bar{C}$ such that

$$c_{20} > c_{10} = \bar{c}_{10} > \bar{c}_{20} > c_{12} = \bar{c}_{12} \tag{3.9}$$

Thus cost of the edge (02) decreases from C to $\bar{C}$, everything else remaining unchanged. Cost monotonicity will be violated if $\Psi_2^\lambda(C) < \Psi_2^\lambda(\bar{C})$. That is

$$\lambda c_{12} + (1-\lambda)c_{10} < \lambda \bar{c}_{20} + (1-\lambda)c_{12}$$

$$\Rightarrow \bar{c}_{20} > \frac{1}{\lambda}[(2\lambda - 1)c_{12} + (1-\lambda)c_{10}] \tag{3.10}$$

Equation 3.9 and equation 3.10 will be satisfied if we can choose $\bar{c}_{20}$ such that

$$c_{10} > \bar{c}_{20} > \frac{1}{\lambda}[(2\lambda - 1)c_{12} + (1-\lambda)c_{10}] > c_{12}$$

The last inequality follows from the fact that $c_{10} > c_{12}$. Since $\lambda > 0.5$ we have $c_{10} > \frac{1}{\lambda}[(2\lambda - 1)c_{12} + (1 - \lambda)c_{10}]$ and hence it is always possible to choose such $\bar{c}_{20}$. ■

Theorem 3.2.1 has so far been proved for $C \in \mathcal{C}_N^2$. Suppose instead that $C \notin \mathcal{C}^2$. Then, our proof shows that the outcome of the algorithm is in the core for each $\sigma \in \Sigma$. Since the core is a convex set, the average (that is, Ψ^λ) must be in the core if each Ψ_σ^λ is in the core. The outcome of the algorithm for each tie-breaking rule satisfies scale invariance and cost monotonicity for $\lambda \in [0, 1]$ and $\lambda \in [0, 0.5]$ respectively. Hence, the average must also satisfy these properties.

By similar argument tree invariance and extreme point monotonicity holds for all $C \in \mathcal{C}^1$.

Chapter 4

Axiomatizations of Ψ^0 and Ψ^1

In this chapter we provide axiomatic characterizations of extreme points of the new class of allocation rules, discussed in chapter 3. We show that these rules differ in terms of their consistency conditions.

4.1 Extreme points

First, we describe an algorithm whose outcome will be the cost allocation corresponding to Ψ^0.

Fix some $N \subset \mathcal{N}$, and choose some matrix $C \in \mathcal{C}_N^2$. Also, for any $A \subset N$, define A_c as the complement of A in N^+. That is $A_c = N^+ \setminus A$.

The algorithm proceeds as follows.

Let $A^0 = \{0\}$, $g^0 = \emptyset$, $t^0 = 0$.

Step 1: Choose the ordered pair (a^1b^1) such that $(a^1b^1) = \text{argmin}_{(i,j)\in A^0 \times A^0_c} c_{ij}$.

Define $t^1 = \max(t^0, c_{a^1b^1})$, $A^1 = A^0 \cup \{b^1\}$, $g^1 = g^0 \cup \{(a^1b^1)\}$.

Step k: Define the ordered pair $(a^kb^k) = \text{argmin}_{(i,j)\in A^{k-1} \times A^{k-1}_c} c_{ij}$, $A^k = A^{k-1} \cup \{b^k\}$, $g^k = g^{k-1} \cup \{(a^kb^k)\}$, $t^k = \max(t^{k-1}, c_{a^kb^k})$. Also,

$$\psi^*_{b^{k-1}}(C) = \min(t^{k-1}, c_{a^kb^k}). \tag{4.1}$$

The algorithm terminates at step $\#N = n$. Then,

$$\psi^*_{b^n}(C) = t^n \tag{4.2}$$

The new rule ψ^* is described by equations (4.1), (4.2).

At any step k, A^{k-1} is the set of nodes which have already been connected to the source 0. Then, a new edge is constructed at this step by choosing the *lowest-cost* edge between a node in A^{k-1} and nodes in A_c^{k-1}. The cost allocation of b^{k-1} is decided at step k. Equation (4.1) shows that b^{k-1} pays the minimum of t^{k-1}, which is the *maximum* cost amongst all edges which have been constructed in previous steps, and $c_{a^k b^k}$, the edge being constructed in step k. Finally, equation (4.2) shows that b^n, the last node to be connected, pays the maximum cost.[1]

We now construct a few examples to illustrate the algorithm.

Example 4.1.1: Suppose C^1 is such that the m.c.s.t. is unique and is a *line*. That is, each node has at most one follower. Then the nodes can be labelled $a_0, a_1, a_2, \ldots, a_n$, where $a_0 = 0$, $\#N = n$, with the predecessor set of a_k, $P(a_k, g) = \{0, a_1, \ldots, a_{k-1}\}$. Then,

$$\forall\, k < n,\ \psi^*_{a_k}(C^1) = \min(\max_{0 \le t < k} c_{a_t a_{t+1}}, c_{a_k a_{k+1}}) \tag{4.3}$$

and

$$\psi^*_{a_n}(C^1) = \max_{0 \le t < n} c_{a_t a_{t+1}} \tag{4.4}$$

Example 4.1.2: Let $N = \{1, 2, 3, 4\}$, and

$$C^2 = \begin{pmatrix} 0 & 4 & 5 & 5 & 5 \\ 4 & 0 & 2 & 1 & 5 \\ 5 & 2 & 0 & 5 & 5 \\ 5 & 1 & 5 & 0 & 3 \\ 5 & 5 & 5 & 3 & 0 \end{pmatrix}$$

[1] From Prim[12], it follows that g^n is also the m.c.s.t. corresponding to C.

There is only one m.c.s.t. of C^2.

Step 1: We have $(a^1b^1) = (01)$, $t^1 = c_{01} = 4$, $A^1 = \{0, 1\}$.

Step 2: Next, $(a^2b^2) = (13)$, $\psi_1^*(C^2) = \min(t^1, c_{13}) = 1$, $t^2 = \max(t^1, c_{13}) = 4$, $A^2 = \{0, 1, 3\}$.

Step 3: We now have $(a^3b^3) = (12)$, $\psi_3^*(C^2) = \min(t^2, c_{12}) = 2$, $t^3 = \max(t^2, c_{12}) = 4$, $A^3 = \{0, 1, 2, 3\}$.

Step 4: Next, $(a^4b^4) = (34)$, $\psi_2^*(C^2) = \min(t^3, c_{34}) = 3$, $t^4 = \max(t^3, c_{34}) = 4$, $A^4 = \{0, 1, 2, 3, 4\}$.

Since $A^4 = N^+$, $\psi_4^*(C^2) = t^4 = 4$, and the algorithm is terminated.

So, $\psi^*(C^2) = (1, 3, 2, 4)$. This example shows that it is not necessary for a node to be assigned the cost of its preceding or following edge. Here 2 pays the cost of the edge (34), while 3 pays the cost of the edge (12).

Now we show that ψ^* and the Bird allocation are two extreme points of Ψ^λ.

Remark 4.1.1: So far, we have defined ψ^* only on C_N^2. This can be extended outside C_N^2 using the same procedure discussed in chapter 3 for Ψ^λ.

Theorem 4.1.1 *The allocation rule Ψ^λ is equivalent to ψ^* if $\lambda = 0$ and to B if $\lambda = 1$.*

Proof : We will prove this result by induction on the number of agents. First if $\#N = 1$ then this result is trivially true. Suppose we have proved this result for all sets N such that $\#N < m$. Take $C \in C_N^2$ where $\#N = m$. Let $c_{kl} = \min_{p \neq q} c_{pq}$ and k to be the immediate predecessor of l. Let C^R be defined as (3.1),(3.2).

First we prove that $\Psi^0 = \psi^*$.

From (3.3)-(3.5) we get,

$$\Psi_i^0(C) = \Psi_i^0(C^R) \ \forall i \neq k, l \tag{4.5}$$

$$\Psi_k^0(C) = c_{kl} \text{ if } k \neq 0 \tag{4.6}$$

$$\Psi_l^0(C) = \begin{cases} \Psi_k^0(C^R) & \text{if } k \neq 0 \\ c_{l0} \text{ otherwise} & \end{cases} \tag{4.7}$$

In describing the algorithm which is used in constructing ψ^*, we fixed a specific matrix, and so did not have to specify the dependence of A^k, t^k, a^k, b^k etc. on the matrix. But, now we need to distinguish between these entities for the two matrices C and C^R. We adopt the following notation in the rest of the proof of the thorem. Let A^k, t^k, a^k, b^k, g_N etc. refer to the matrix C, while $\hat{A}^k, \hat{t}^k, \hat{a}^k, \hat{b}^k, \hat{g}_N$ etc. will denote the entities corresponding to C^R.

Without loss of generality assume that $b^p = k$ for some $p \geq 0$. In this proof, for notational convenience, assume that $b^0 = 0$. Since only edges involving k and l have different costs in C and C^R; and for $j = \alpha(k)$ in g_N,[2] $c_{kj}^R = \min\{c_{kj}, c_{lj}\} = c_{kj}$ we have $\hat{b}^p = k$. Therefore $\psi_i^*(C) = \psi_i^*(C^R)$ for all i such that $i = b^j = \hat{b}^j$ where $j < p$. Thus $t^p = \hat{t}^p$. Now $k \in A^p$ and $l \in A_c^p$. Since c_{kl} is the minimum cost edge $(a^{p+1}b^{p+1}) = (kl)$. If $k \neq 0$

$$\psi_k^*(C) = \min(t^p, c_{a^{p+1}b^{p+1}}) = c_{kl} \tag{4.8}$$

We also get $t^{p+1} = \max(t^p, c_{kl}) = t^p = \hat{t}^p$.

Now since both $k, l \in A^{p+1}$ from the construction of C^R it follows that $A^j = \hat{A}^{j-1} \cup \{l\}$ for all $j \geq p+1$. Also if $a^{j+1} \neq l$ then $(\hat{a}^j\hat{b}^j) = (a^{j+1}b^{j+1})$, otherwise $(\hat{a}^j\hat{b}^j) = (kb^{j+1})$. That is $c_{\hat{a}^j\hat{b}^j} = c_{a^{j+1}b^{j+1}}$ for all $j \geq p+1$. Therefore

$$\psi_l^*(C) = \min(t^{p+1}, c_{a^{p+2}b^{p+2}}) = \min(\hat{t}^p, c_{\hat{a}^{p+1}\hat{b}^{p+1}})$$

[2]If $k = 0$ then no such j exists.

$$= \begin{cases} \psi_k^*(C^R) & \text{if } k \neq 0 \\ \min(t^1, c_{a^2b^2}) = \min(c_{l0}, c_{a^2b^2}) = c_{l0} & \text{if } k = 0 \end{cases} \tag{4.9}$$

For all $j \geq p+1$ we get $t^{j+1} = \hat{t}^j$ and $\psi^*_{b^{j+1}}(C) = \psi^*_{\hat{b}^j}(C^R)$. Thus for all $i \neq k, l$,

$$\psi_i^*(C) = \psi_i^*(C^R) \tag{4.10}$$

Comparing (4.5) -(4.10) and using induction hypothesis on C^R, we get $\Psi^0(C) = \psi^*(C)$.

Now we prove that $\Psi^1 = B$.

Using (3.3), (3.4), induction hypothesis on C^R and lemma 3.1.1,

$$\Psi_i^1(C) = \Psi_i^1(C^R) = c^R_{i\hat{\alpha}(i)} = c_{i\alpha(i)} = B_i(C) \ \forall i \neq l$$

Where $\hat{\alpha}(i)$ and $\alpha(i)$ denotes the immediate predecessor of i in the m.c.s.t. of C^R and C respectively.

From (3.5), $\Psi_l^1(C) = c_{kl} = B_l(C)$.

Hence the result follows. ■

4.2 Characterizations

The following lemmas will be required for the characterization theorems.

Lemma 4.2.1 *Let $C \in \mathcal{C}_N^2$, and $(01) \in g_N(C)$. Let $\psi_1(C) = \min_{k \in N^+ \setminus \{1\}} c_{1k}$. Then, $C^{tr}_{\psi_1(C)} \in \mathcal{C}^2_{N \setminus \{1\}}$.*

Proof : We will denote $C^{tr}_{\psi_1(C)}$ by $\bar{C}$ for the rest of this proof.

Let $\psi_1(C) = \min_{k \in N^+ \setminus \{1\}} c_{1k} = c_{1k^*}$ (say).

Suppose there exists $(ij) \in g_N(C)$ such that $i, j \neq 1$. Without loss of generality assume i precedes j in $g_N(C)$. Then, $c_{1j} > c_{ij}$. As $\psi_1(C) \leq c_{i1}$, $c_{i1} + c_{1j} - \psi_1(C) \geq c_{1j} > c_{ij}$. Hence $\bar{c}_{ij} = c_{ij} \forall (ij) \in g_N(C)$, such that $i, j \neq 1$.

Now, suppose there is $j \in N^+$ such that $j \neq k^*$ and $(1j) \in g_N(C)$. Since $(1j), (1k^*) \in g_N(C)$, $(jk^*) \notin g_N(C)$. Hence, $c_{1j} < c_{k^*j}$. Thus,

$\bar{c}_{k^*j} = \min\{(c_{1j} + c_{1k^*} - \psi_1(C)), c_{k^*j}\} = \min(c_{1j}, c_{k^*j}) = c_{1j}$.

Next, let $\bar{g}_{N\setminus\{1\}}$, be a connected graph over $N^+ \setminus \{1\}$, defined as follows.

$$\bar{g}_{N\setminus\{1\}} = \{(ij) | \text{ either } (ij) \in g_N(C) \text{ s.t. } i,j \neq 1 \text{ or } (ij) = (k^*l) \text{ where } (1l) \in g_N(C)\}.$$

Note that no two edges have equal cost in $\bar{g}_{N\setminus\{1\}}$.

Also,

$$\sum_{(ij)\in\bar{g}_{N\setminus\{1\}}} \bar{c}_{ij} = \sum_{(ij)\in g_N(C)} c_{ij} - c_{1k^*}. \tag{4.11}$$

We prove that $\bar{C}$ belongs to $\mathcal{C}^2_{N\setminus\{1\}}$ by showing that $\bar{g}_{N\setminus\{1\}}$ is the only m.c.s.t. of $\bar{C}$.

Suppose this is not true, so that $g^*_{N\setminus\{1\}}$ is an m.c.s.t. corresponding to $\bar{C}$. Then, using 4.11,

$$\sum_{(ij)\in g^*_{N\setminus\{1\}}} \bar{c}_{ij} \leq \sum_{(ij)\in g_N(C)} c_{ij} - c_{1k^*} \tag{4.12}$$

Let $g^*_{N\setminus\{1\}} = g^1 \cup g^2$, where

$$\begin{aligned} g^1 &= \{(ij) | (ij) \in g^*_{N\setminus\{1\}}, c_{ij} = \bar{c}_{ij}\} \\ g^2 &= g^*_{N\setminus\{1\}} \setminus g^1 \end{aligned}$$

If $(ij) \in g^2$, then

$$\begin{aligned} \bar{c}_{ij} &= \min(c_{ij}, c_{1i} + c_{1j} - \psi_1(C)) \\ &= c_{1i} + c_{1j} - \psi_1(C) \\ &\geq \max(c_{1i}, c_{1j}) \end{aligned}$$

where the last inequality follows from the assumption that $\psi_1(C) = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

So,

$$\begin{aligned} \bar{c}_{ij} &= c_{ij} \text{ if } (ij) \in g^1 \\ &\geq \max(c_{1i}, c_{1j}) \text{ if } (ij) \in g^2. \end{aligned} \tag{4.13}$$

Now, extend $g^*_{N\setminus\{1\}}$ to a connected graph g'_N over N^+ as follows. Letting $g = \{(1i) | (ij) \in g^2, j \in U(i, k^*, g^*_{N\setminus\{1\}})\}$, define

$$g'_N = g^1 \cup (1k^*) \cup g$$

Claim: g'_N is a connected graph over N^+.

Proof of Claim: It is sufficient to show that every $i \in N^+ \setminus \{1\}$ is connected to 1 in g'_N. Clearly, this is true for $i = k^*$. Take any $i \in N^+ \setminus \{1, k^*\}$. Let $U(i, k^*, g^*_{N\setminus\{1\}}) = \{m_0, m_1, \ldots, m_{p+1}\}$[3] where $m_0 = i$ and $m_{p+1} = k^*$. If all these edges $(m_t m_{t+1})$ $\forall t \leq p$ are in g^1, then they are also in g'_N, and there is nothing to prove.

So, suppose there is $(m_t m_{t+1}) \in g^2$ while all edges in $\{(m_0 m_1), \ldots, (m_{t-1} m_t)\}$ belong to g^1. In this case, $(m_t 1)$ as well as all edges in $\{(m_0 m_1), \ldots, (m_{t-1} m_t)\}$ belong to g'_N. Hence, i is connected to 1. ■

To complete the proof of the lemma, note that

$$\sum_{(ij)\in g'_N} c_{ij} = \sum_{(ij)\in g^1} c_{ij} + c_{1k^*} + \sum_{(ij)\in g^2} c_{1i}.$$

Using (4.13),

$$\sum_{(ij)\in g'_N} c_{ij} \leq \sum_{(ij)\in g^1} \bar{c}_{ij} + c_{1k^*} + \sum_{(ij)\in g^2} \bar{c}_{ij} = \sum_{(ij)\in g^*_{N\setminus\{1\}}} \bar{c}_{ij} + c_{1k^*}$$

Finally, using (4.12),

$$\sum_{(ij)\in g'_N} c_{ij} \leq \sum_{(ij)\in g_N(C)} c_{ij}.$$

But, this contradicts the assumption that $g_N(C)$ is the unique m.c.s.t. for C.

■

Lemma 4.2.2 *Let* $C \in \mathcal{C}^1_N$, $(10) \in g_N(C)$. *Suppose* $\psi_1(C) = c_{01}$. *Then* $C^{sr}_{\psi_1(C)} \in \mathcal{C}^1_{N\setminus\{1\}}$.

[3] This path exists because $g^*_{N\setminus\{1\}}$ is a connected graph.

Proof : Throughout the proof of this lemma, we denote $C^{sr}_{\psi_1(C)}$ by $\bar{C}$.

We know $\psi_1(C) = c_{01}$. Suppose $(ij) \in g_N(C)$ such that $\{i, j\} \cap \{0, 1\} = \emptyset$. Then $\bar{c_{ij}} = c_{ij}$.

On the other hand if $(i0) \in g_N(C)$, and $i \neq 1$, then $\bar{c}_{0i} = \min\{(c_{i1} + c_{10} - \psi_1(C)), c_{0i}\} = \min(c_{i1}, c_{i0}) = c_{i0}$. Note that the last equality follows from the fact that $(i0) \in g_N(C)$ but $(i1) \notin g_N(C)$ implies that $c_{i1} > c_{i0}$.

If $(i1) \in g_N(C)$, then $\bar{c}_{i0} = \min\{(c_{i1}+c_{10}-\psi_1(C)), c_{i0}\} = \min(c_{i1}, c_{i0}) = c_{i1}$, as $(i1) \in g_N(C)$ but $(i0) \notin g_N(C)$.

Now we construct $\bar{g}_{N\setminus\{1\}}$, a connected graph over $N^+ \setminus \{1\}$ as follows.

$$\bar{g}_{N\setminus\{1\}} = \{(ij) | \text{ either } (ij) \in g_N(C) \text{ s.t. } i, j \neq 1 \text{ or } (ij) = (l0) \text{ where } (l1) \in g_N(C)\}$$

Then, $\bar{g}_{N\setminus\{1\}}$ must be the only m.c.s.t. of $\bar{C}$. For if there is another $g^*_{N\setminus\{1\}}$ which is also an m.c.s.t. of $\bar{C}$, then one can show that $g_N(C)$ cannot be the only m.c.s.t. coresponding to C.[4] ■

Lemma 4.2.3 *Suppose ψ satisfies tree consistency, extreme point monotonicity and efficiency. Let $C \in \mathcal{C}^2_N$. If $(10) \in g_N(C)$, then $\psi_1(C) \geq \min_{k\in N^+\setminus\{1\}} c_{1k}$.*

Proof: Consider any $C \in \mathcal{C}^2_N$, $(10) \in g_N(C)$, and ψ satisfying tree consistency, extreme point monotonicity and efficiency. Let $\psi(C) = x$, and $c_{1m} = \min_{k\in N^+\setminus\{1\}} c_{1k}$. We want to show that $x_1 \geq c_{1m}$.

Suppose 1 has no followers in $g_N(C)$. Then, from lemma 3.0.1 it follows that $m = 0$. Moreover 1 is an extreme point of $g_N(C)$. Let $\psi(C_{N\setminus\{1\}})$ be denoted by x^1. Here $C_{N\setminus\{1\}}$ denotes the restriction of C over $N^+ \setminus \{1\}$. Using extreme point monotonicity we get that $x^1_k \geq x_k$ for all $k \neq 1$. By efficiency

$$\sum_{k\neq 1} x_k \leq \sum_{k\neq 1} x^1_k = c(N) - c_{10} = \sum_{k\in N} x_k - c_{10}$$

[4]The proof of this assertion is analogous to that of the corresponding assertion in lemma 4.2.1, and is hence omitted.

Therefore $x_1 \geq c_{10}$.

Now consider the case where 1 has at least one follower in $g_N(C)$. Suppose 2 is a follower of 1. Assume that $x_1 < c_{1m}$. Let the reduced cost matrix of C, $C^{tr}_{x_1}$ be denoted by C'. The set $[N \setminus \{1\}]$ is denoted by N'. First, we note some properties of C'.

Suppose there is some $k \in N'$ such that $(1k) \notin g_N(C)$. Let l be the predecessor of k in $g_N(C)$. Thus $c_{kl} < c_{1k}$. Since $c_{1l} \geq c_{1m} > x_1$, we get

$$c'_{kl} = \min(c_{kl}, c_{k1} + c_{1l} - x_1) = c_{kl} \tag{4.14}$$

Consider the following two sets, $S_1 = \{i \in N | 1 \in P(i, g_N(C))\}$ and $S_2 = \{i \in N | 2 \in P(i, g_N(C))\} \cup \{2\}$. Since 2 is a follower of 1, we have $S_2 \subseteq S_1$.

Suppose $(kl) \notin g_N(C)$ and $(mn) \in U(k, l, g_N(C))$. Then $c_{kl} > c_{mn}$. Following inequalities are immediate using this and the fact that $x_1 < c_{1m}$.

Take any $i \in S_1$ and $j \in [N'^+ \setminus S_1]$. Since $(10) \in U(i, j, g_N(C))$

$$c'_{ij} = \min(c_{ij}, c_{i1} + c_{1j} - x_1) > c_{10} \tag{4.15}$$

If $S_1 \neq S_2$, then take any $i \in S_2$ and $j \in S_1 \setminus S_2$. Since $(12) \in U(i, j, g_N(C))$

$$c'_{ij} = \min(c_{ij}, c_{i1} + c_{1j} - x_1) > c_{12} \tag{4.16}$$

Suppose $i \in S_2$ and $j \in [N'^+ \setminus S_1]$. Since both $(12), (10) \in U(i, j, g_N(C))$

$$c'_{ij} = \min(c_{ij}, c_{i1} + c_{1j} - x_1) > \max(c_{10}, c_{12}) \tag{4.17}$$

Combining equation 4.15 and equation 4.16, we get that

$$\theta_1(C') \equiv \max\left(\min_{i \in S_1, j \in [N'^+ \setminus S_1]} c'_{ij}, \min_{i \in S_2, j \in [S_1 \setminus S_2]} c'_{ij}\right) > \max(c_{10}, c_{12})^5 \tag{4.18}$$

Therefore, using (4.18)

$$\theta_2(C') \equiv \min\left(\theta_1(C'), \min_{i \in S_2, j \in [N'^+ \setminus S_1]} c'_{ij}\right) > \max(c_{10}, c_{12}) \tag{4.19}$$

Choose $j \notin N^+$, and define $\bar{N} = N \cup \{j\}$. Let $\bar{C}$ be such that

[5] If $S_1 = S_2$ then assume that $\min_{i \in S_2, j \in [S_1 \setminus S_2]} c'_{ij} = 0$.

(i) $\bar{C}$ coincides with C on N^+.

(ii) $\bar{c}_{j2} = \min_{k\in N^+} \bar{c}_{jk} \neq c_{pq}$ for all $(pq) \in g_N(C)$.

(iii) $\theta_2(C') > \bar{c}_{j0} > \max(c_{10}, c_{12})$.[6]

(iv) For all $k \in N \setminus \{2\}, \bar{c}_{jk} > \max_{(pq)\in g_N(C)} c_{pq}$

Hence, $g_{\bar{N}}(\bar{C}) = g_N(C) \cup \{(2j)\}$ and $\bar{C} \in \mathcal{C}^2_{\bar{N}}$. Note that j is an extreme point of $\bar{C}$. Denoting $\psi(\bar{C}) = \bar{x}$, extreme point monotonicity implies that $x_k \geq \bar{x}_k \forall k \in N$.

Let $\bar{C}^{tr}_{\bar{x}_1} = \bar{C}'$, and $\bar{N}' = \bar{N} \setminus \{1\}, \psi(C') = x'$.

Case 1: Let $C', \bar{C}' \in \mathcal{C}^2$.

Since $\bar{x}_1 \leq x_1 < c_{1m} = \bar{c}_{1m}$, from equation 4.14, we have $\bar{c}'_{2j} = \bar{c}_{2j}$. We also have $\bar{c}'_{j0} = \min(\bar{c}_{j0}, \bar{c}_{j1} + \bar{c}_{10} - \bar{x}_1) = \bar{c}_{j0}$ because $\bar{c}_{j1} > \bar{c}_{j0}$ and $\bar{x}_1 < \bar{c}_{10}$. For all $k \in N' \setminus \{2\}, \bar{c}'_{jk} > \max_{(pq)\in g_N(C)} c_{pq}$ Hence $\bar{c}'_{2j} = \min_{k\in N'^+} \bar{c}'_{jk}$ and from lemma 3.0.1 $(2j) \in g_{\bar{N}'}(\bar{C}')$.

Since $C' \in \mathcal{C}^2$, using efficiency and tree consistency we get

$$c(N) = \sum_{i\in N} x_i = x_1 + \sum_{i\in N'} x'_i = x_1 + c'(N)$$

Thus $x_1 = c(N) - c'(N)$. Similarly $\bar{x}_1 = \bar{c}(N) - \bar{c}'(N)$. By extreme point monotonicity $x_1 \geq \bar{x}_1$. Hence

$$\begin{aligned} & c(N) - c'(N) \geq \bar{c}(N) - \bar{c}'(N) \\ \Rightarrow\ & [\bar{c}(N) - c(N)] + [c'(N) - \bar{c}'(N)] \leq 0 \\ \Rightarrow\ & [c'(N) + \bar{c}_{j2}] - \bar{c}'(N) \leq 0 \end{aligned}$$

But $g'_N(C') \cup \{(j2)\}$ is a connected graph over $\bar{N}' \cup \{0\}$. Hence $[c'(N) + \bar{c}_{j2}] - \bar{c}'(N) = [c'(N) + \bar{c}'_{j2}] - \bar{c}'(N) \leq 0$ is possible only if $g'_N(C') \cup \{(j2)\}$ is the m.c.s.t. of $\bar{C}'$. Therefore j is an extreme point of $g_{\bar{N}'}(\bar{C}')$.

[6]This is possible as equation 4.19 is true.

Since $(2j) \in g_{\bar{N}'}(\bar{C}')$ and j is an extreme point of $\bar{C}'$, one of the following must be true.

(i) There exists $m \in S_2$ and $n \in N'^+ \setminus S_1$ such that $(mn) \in g_{\bar{N}'}(\bar{C}')$.

(ii) There exist $m_1 \in S_1, m_2 \in S_2$ and $m_3 \in N'^+ \setminus S_1$ such that $(m_2 m_1), (m_1 m_3) \in g_{\bar{N}'}(\bar{C}')$.

If (i) is true then using (4.17), $\bar{c}'_{mn} = \min(\bar{c}_{mn}, \bar{c}_{m1} + \bar{c}_{1n} - \bar{x}_1) \geq \min(c_{mn}, c_{m1} + c_{n1} - x_1) = c'_{mn} \geq \theta_2(C')$. Hence $\bar{c}'_{j0} = \bar{c}_{j0} < \theta_2(C') \leq \bar{c}'_{mn}$. This contradicts the fact that $(j0) \notin g_{\bar{N}'}(\bar{C}')$.

If (ii) is true then by similar argument $\bar{c}'_{j0} < \theta_2(C') \leq \max(\bar{c}'_{m_2 m_1}, \bar{c}'_{m_1 m_3})$ which again contradicts the fact that $(j0) \notin g_{\bar{N}'}(\bar{C}')$.

Case 2: This includes all possible cases other than $C', \bar{C}' \in \mathcal{C}^2$

Let $T_1 = \{(q_1 q_2) \in g_N(C) | \{q_1, q_2\} \cap \{1\} = \emptyset\}$. From equation 4.14, we have

$$c'_{q_1 q_2} = c_{q_1 q_2} \tag{4.20}$$

Let $T_2 = \{(kl) | k, l \in N'^+, (kl) \notin T_1\}$. If $(kl) \in T_2$ then there exists $(s_1 s_2) \in g_N(C)$ with $s_1, s_2 \in U(k, l, g_N(C))$. Then either $s_1, s_2 \in U(k, 1, g_N(C))$ or $s_1, s_2 \in U(l, 1, g_N(C))$. Without loss of generality, assume $s_1, s_2 \in U(k, 1, g_N(C))$. Then, $c_{s_1 s_2} < c_{kl}$ and $c_{s_1 s_2} \leq c_{k1}$. As $x_1 < c_{1l}$

$$c'_{kl} = \min(c_{k1} + c_{1l} - x_1, c_{kl}) > c_{s_1 s_2} \tag{4.21}$$

For each $(kl) \in T_2$, choose $p(k,l) \notin N^+$ such that $p(k,l) \neq p(k',l')$ when $(k,l) \neq (k',l')$. Let $\widehat{N} = N \cup \{p(k,l) | (k,l) \in T_2\}$. Now define a new cost matrix $\widehat{C}$ over $\widehat{N}^+$ as follows,

(i) $\widehat{C}$ coincides with C on N^+.

(ii) $\widehat{c}_{p(k,l)k} = \min_{t \in N^+} \widehat{c}_{p(k,l)t}$ and $\widehat{c}_{p(k,l)k} \neq c_{q_1 q_2}$ for all $(q_1 q_2) \in T_1$

(iii) $c'_{kl} > \hat{c}_{p(k,l)l} > c_{s_j s_{j+1}} \forall s_j, s_{j+1} \in U(k,l,g_N(C))$ and $\hat{c}_{p(k,l)l} \neq c_{q_1 q_2}$ for all $(q_1 q_2) \in T_1$.[7]

(iv) Costs of all the remaining edges are chosen 'very' high, for example strictly greater than $\sum_{(t_1 t_2)} c_{t_1 t_2}$.

(v) Costs of all the new edges, that is those which were not present in C, are distinct.

Then, from specifications (ii) and (iv) we have $g_{\hat{N}}(\hat{C}) = g_N(C) \cup \{(p(k,l)k) | (kl) \in T_2\}$, so that all $p(k,l)$ are extreme points of $\hat{C}$. Let $\psi(\hat{C}) = \hat{x}$. From extreme point monotonicity,

$$x_1 \geq \hat{x}_1 \tag{4.22}$$

Now, consider the reduced matrix $\hat{C}' \equiv \hat{C}^{tr}_{\hat{x}_1}$. We want to show that the edges which will be connected in the m.c.s.t. of $\hat{C}'$ will be contained in $T_1 \cup \{(p(k,l)k), (p(k,l),l) | (k,l) \in T_2\}$. Since specifications (i), (ii), (iii) and (v) imply that all these costs are distinct, we will get $\hat{C}' \in \mathcal{C}^2$.

Consider any $(q_1 q_2) \in T_1 \cup T_2$. Since $x_1 \geq \hat{x}_1$, $c_{q_1 1} + c_{q_2 1} - \hat{x}_1 \geq c_{q_1 1} - c_{q_2 1} - x_1$. Thus, $\hat{c}'_{q_1 q_2} \geq c'_{q_1 q_2}$, with equality holding if $c_{q_1 q_2} \leq c_{q_1 1} + c_{q_2 1} - x_1$. Hence

$$\hat{c}'_{q_1 q_2} = c_{q_1 q_2} = c'_{q_1 q_2} \text{ if } (q_1 q_2) \in T_1, \hat{c}'_{q_1 q_2} > c'_{q_1 q_2} \text{ otherwise} \tag{4.23}$$

Since $\hat{x}_1 < \hat{c}_{1k}$, and $\hat{c}_{p(k,l)k} < \hat{c}_{p(k,l)1}$,

$$\hat{c}'_{p(k,l)k} = \min(\hat{c}_{p(k,l)k}, \hat{c}_{p(k,l)1} + \hat{c}_{1k} - \hat{x}_1) = \hat{c}_{p(k,l)k}.$$

Also, using specification (iv),

$$\hat{c}'_{p(k,l)l} = \min(\hat{c}_{p(k,l)l}, \hat{c}_{p(k,l)1} + \hat{c}_{1l} - \hat{x}_1) = \hat{c}_{p(k,l)l}$$

It is easy to check that all other edges in $\hat{C}'$ have costs strictly greater than $\sum_{(t_1 t_2)} c_{t_1 t_2}$.

[7] Note that this specification of costs is valid because (4.21) is true.

So, from specification (ii), $\tilde{c}'_{p(k,l)k} = \min_{t\in \hat{N}^+\setminus\{p(k,l),1\}} \tilde{c}'_{p(k,l)t}$. Hence, from lemma 3.0.1, $(p(k,l)k) \in g_{\hat{N}\setminus i}(\hat{C}')$.

Finally, specification (iii) implies $\tilde{c}'_{kl} \geq c'_{kl} > \hat{c}_{p(k,l)l} = \tilde{c}'_{p(k,l)l}$ for all $(kl) \in T_2$.

Thus for all $(kl) \in T_2$, $(p(k,l)k)$ is connected in a m.c.s.t of $\hat{C}'$ and $\tilde{c}'_{kl} > \tilde{c}'_{p(k,l)l}$. So, (kl) cannot be an edge in the m.c.s.t. corresponding to $\hat{C}'$. This establishes that the m.c.s.t. of $\hat{C}'$ can only contain edges from the set $[T_1 \cup \{(p(k,l)k), (p(k,l)l) | (kl) \in T_2\}]$.

Now $\hat{C}$ can be extended to $\tilde{C}$, using the procedure by which we obtained $\bar{C}$ from C in case 1. The reduced matrix of $\tilde{C}$ belongs to the class $\mathcal{C}^2$ because m.c.s.t. of the reduced matrix can only contain edges from the set $[T_1 \cup \{(p(k,l)k), (p(k,l)l) | (kl) \in T_2\} \cup \{(j0), (j2)\}]$. From case 1, we have $\hat{x}_1 \geq \hat{c}_{1m}$. Equation 4.22 now establishes that $x_1 \geq \hat{c}_{1m} = c_{1m}$. ∎

We state without proof the corresponding lemma when tree consistency is replaced with source consistency. The proof is almost identical to that of lemma 4.2.3.

Lemma 4.2.4 *Suppose ψ satisfies source consistency, extreme point monotonicityand efficiency. Let $C \in \mathcal{C}^1_N$. If $(i0) \in g_N(C)$, then $\psi_i(C) \geq \min_{k\in N^+\setminus\{i\}} c_{ik}$.*

We now present the characterization results.

Theorem 4.2.1 *Over the domain $\mathcal{C}^2$, a rule ψ satisfies tree consistency, efficiency and extreme point monotonicity if and only if $\psi = \psi^*$.*

Proof : In chapter 3, we have already proved that ψ^* satisfies extreme point monotonicity. Efficiency follows trivially from the algorithm which defines the allocation. Here, we prove that ψ^* satisfies tree consistency. Let $C \in \mathcal{C}^2$.

Let $(10) = \text{argmin}_{k\in N} c_{k0}$. Hence, the algorithm yields $b^1 = 1$, and $\psi^*_1(C) = \min(c_{10}, c_{a^2b^2})$. There are two possible choice of a^2.

Case 1: $a^2 = 1$. Then, we get $c_{1b^2} = \min_{k\in N\setminus\{1\}} c_{1k}$. Therefore $\psi_1^*(C) = \min(c_{10}, c_{1b^2}) = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

Case 2: $a^2 = 0$. Then, $c_{b^20} \leq c_{1k}\ \forall k \in N\setminus\{1\}$. Since $c_{10} \leq c_{b^20}$, we conclude $\psi_1^*(C) = \min(c_{10}, c_{b^20}) = c_{10} = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

So, in either case, 1 pays its minimum cost.

Let $\psi_1^*(C) = x_1 = \min_{k\in N^+\setminus\{1\}} c_{1k} = c_{1k^*}$. Denoting $C^{tr}_{x_1}$ by $\bar{C}$, we know from lemma 4.2.1, that $\bar{C} \in \mathcal{C}^2$. Hence, the algorithm is well defined on $\bar{C}$.

Let $\bar{a}^k, \bar{b}^k, \bar{t}^k$, etc denote the relevant variables of the algorithm corresponding to $\bar{C}$.

Claim: $\forall i \in N\setminus\{1\}$, $\psi_i^*(C) = \psi_i^*(\bar{C})$. That is, ψ^* satisfies tree consistency.

Proof of Claim: From the proof of lemma 4.2.1,

(i) $\bar{c}_{ij} = c_{ij}\ \forall (ij) \in g_N(C)$ s.t. $i, j \neq 1$.

(ii) $\bar{c}_{k^*j} = c_{1j}$ for $j \in N^+\setminus\{k^*\}$ s.t. $(1j) \in g_N(C)$.

Also,

$$g_{N\setminus\{1\}}(\bar{C}) = \{(ij)|(ij) \in g_N(C) \text{if } i,j \neq 1 \text{ and } (ij) = (k^*l) \text{ if } (1l) \in g_N(C)\}.$$

Let $b^2 = i$. Either $k^* = 0$ or $k^* = i$. In either case, $\bar{c}_{0i} < \bar{c}_{0j}$ for $j \notin \{0, 1, i\}$. Hence, $\bar{b}^1 = i$.

Now, $t^2 = \max(c_{a^1b^1}, c_{a^2b^2}) = \max(c_{10}, c_{a^2i}) = \bar{c}_{0i} = \bar{t}^1$.

Also, $a^3 \in \{0, 1, i\}$, while $b^3 \in \{0, 1, i\}_c$. If $a^3 \in \{0, i\}$, then $\bar{a}^2 = a^3$. If $a^3 = 1$, then $\bar{a}^2 = k^*$. In all cases, $b^3 = \bar{b}^2$, and $c_{a^3b^3} = \bar{c}_{\bar{a}^2\bar{b}^2}$. So,

$$\psi_i^*(C) = \min(t^2, c_{a^3b^3}) = \min(\bar{t}^1, \bar{c}_{\bar{a}^2\bar{b}^2}) = \psi_i^*(\bar{C}). \quad (4.24)$$

The claim is established for $\{b^3, \ldots, b^n\}$ by using the strucure of $g_{N\setminus\{1\}}(\bar{C})$, the definition of $\bar{C}$ given above, and the following induction hypothesis. The details are left to the reader.

For all $i = 2, \ldots, k-1$,

(i) $\bar{b}^{i-1} = b^i$.

(ii $\bar{t}^{i-1} = t^i$.

(iii) $\bar{a}^{i-1} = a^i$ if $a^i \neq 1$, and $\bar{a}^{i-1} = k^*$ if $a^i = 1$.

Next, we will prove that only *one* rule over $\mathcal{C}^2$ satisfies all three axioms. Let ψ be a rule satisfying all the three axioms. We will show by induction on the cardinality of the set of nodes that ψ is unique.

Let us start by showing that the result is true for $|N| = 2$. There are several cases.

Case 1: $c_{12} > c_{10}, c_{20}$. From lemma 4.2.3, $\psi_1(C) \geq c_{10}, \psi_2(C) \geq c_{20}$. By efficiency, $\psi_1(C) + \psi_2(C) = c_{10} + c_{20}$. Thus $\psi_1(C) = c_{10}$, and $\psi_2(C) = c_{20}$. So, the allocation is unique.

Case 2: $c_{20} > c_{12} > c_{10}$. Introduce a third agent 3 and costs $c_{20} < \bar{c}_{13} < \min(\bar{c}_{32}, \bar{c}_{30})$. Let the restriction of $\bar{C}$ on $\{1,2\}^+$ coincide with C. Hence, $g_{\{1,2,3\}} = \{(01),(12),(13)\}$. Let $\psi(\bar{C}) = \bar{x}$. From lemma 4.2.3, $\bar{x}_1 \geq \bar{c}_{10} = c_{10}$.

Denote the reduced matrix $\bar{C}^{tr}_{\bar{x}_1}$ as $\widehat{C}$. Now, $\widehat{c}_{02} = \min(\bar{c}_{01} + \bar{c}_{12} - \bar{x}_1, \bar{c}_{02}) = \bar{c}_{01} + \bar{c}_{12} - \bar{x}_1$. Similarly, $\widehat{c}_{23} = \min(\bar{c}_{13} + \bar{c}_{12} - \bar{x}_1, \bar{c}_{23})$. Noting that $\bar{x}_1 \geq \bar{c}_{10}, \bar{c}_{23} > \bar{c}_{12}$ and $\bar{c}_{13} > \bar{c}_{10}$, we conclude that

$$\widehat{c}_{02} < \widehat{c}_{23}.$$

Analogously, $\widehat{c}_{03} = \bar{c}_{01} + \bar{c}_{13} - \bar{x}_1 < \widehat{c}_{23}$.

Hence, $g_{\{2,3\}}(\widehat{C}) = \{(02),(03)\}$. So, $\widehat{C} \in \mathcal{C}^2$. Using tree consistency,

$$\psi_2(\widehat{C}) = \psi_2(\bar{C}), \psi_3(\widehat{C}) = \psi_3(\bar{C}) \tag{4.25}$$

From Case 1 above,

$$\psi_2(\widehat{C}) = \bar{c}_{01} + \bar{c}_{12} - \bar{x}_1, \psi_3(\widehat{C}) = \bar{c}_{01} + \bar{c}_{13} - \bar{x}_1 \tag{4.26}$$

From (4.25) and (4.26),

$$\begin{aligned} \psi_2(\bar{C}) + \psi_3(\bar{C}) &= \bar{c}_{01} + \bar{c}_{12} - \bar{x}_1 + \bar{c}_{01} + \bar{c}_{13} - \bar{x}_1 \\ \text{or } \bar{x}_1 + \psi_2(\bar{C}) + \psi_3(\bar{C}) &= \bar{c}_{01} + \bar{c}_{12} + \bar{c}_{13} + (\bar{c}_{01} - \bar{x}_1) \end{aligned}$$

But, from efficiency, $\bar{x}_1 + \psi_2(\bar{C}) + \psi_3(\bar{C}) = \bar{c}_{01} + \bar{c}_{12} + \bar{c}_{13}$. So, $\bar{x}_1 = \bar{c}_{01}$. So, $\psi_2(\hat{C}) = \psi_2(\bar{C}) = \bar{c}_{12} = c_{12}$.

By extreme point monotonicity, $\bar{x}_1 \leq \psi_1(C)$, and $\psi_2(\bar{C}) \leq \psi_2(C)$. Using efficiency, it follows that $\psi_1(C) = c_{01}$ and $\psi_2(C) = c_{12}$. Hence, ψ is unique.

The case $c_{10} > c_{12} > c_{20}$ is similar.

Case 3: $c_{20} > c_{10} > c_{12}$.

We again introduce a third agent (say 3). Consider the matrix $\bar{C}$, coinciding wih C on $\{1, 2\}^+$ such that

(i) $\bar{c}_{32} > \bar{c}_{13} > \bar{c}_{20}$.

(ii) $\bar{c}_{30} > \bar{c}_{10} + \bar{c}_{13}$.

Then, $\bar{C} \in \mathcal{C}^2$ since it has the unique m.c.s.t. $g_N(\bar{C}) = \{(01), (12), (13)\}$, where no two edges have the same cost.

Note that 3 is an extreme point of the m.c.s.t. corresponding to $\bar{C}$. Using extreme point monotonicity, we get

$$\psi_1(C) \geq \psi_1(\bar{C}), \psi_2(C) \geq \psi_2(\bar{C}). \tag{4.27}$$

Consider the reduced matrix $\bar{C}^{tr}_{\psi_1(\bar{C})}$ on $\{2, 3\}$. Denote $\bar{C}^{tr}_{\bar{x}_1} = \hat{C}$ for ease of notation. Since $\psi_1(\bar{C}) \geq \bar{c}_{12}$ from lemma 4.2.3, it follows that $\bar{c}_{12} + \bar{c}_{10} - \psi_1(\bar{C}) \leq \bar{c}_{10} < \bar{c}_{20}$, and $\bar{c}_{12} + \bar{c}_{13} - \psi_1(\bar{C}) \leq \bar{c}_{13} < \bar{c}_{23}$. Hence,

$$\hat{c}_{20} = \bar{c}_{12} + \bar{c}_{10} - \psi_1(\bar{C}), \hat{c}_{23} = \bar{c}_{12} + \bar{c}_{13} - \psi_1(\bar{C}), \hat{c}_{30} = \bar{c}_{13} + \bar{c}_{10} - \psi_1(\bar{C}) \tag{4.28}$$

Note that

$$\bar{c}_{21} + \bar{c}_{10} - \psi_1(\bar{C}) < \bar{c}_{21} + \bar{c}_{13} - \psi_1(\bar{C}) < \bar{c}_{10} + \bar{c}_{13} - \psi_1(\bar{C})$$

Hence, $g_{\{23\}}(\hat{C}) = \{(02), (23)\}$.

Applying case 2, $\psi_2(\hat{C}) = \hat{c}_{20} = \bar{c}_{12} + \bar{c}_{10} - \psi_1(\bar{C})$ and $\psi_3(\hat{C}) = \hat{c}_{23} = \bar{c}_{21} + \bar{c}_{13} - \psi_1(\bar{C})$. Using tree consistency, $\psi_2(\hat{C}) = \psi_2(\bar{C}), \psi_3(\hat{C}) = \psi_3(\bar{C})$. Also, efficiency on $\bar{C}$ gives,

$$\psi_1(\bar{C}) + \psi_2(\bar{C}) + \psi_3(\bar{C}) = \bar{c}_{10} + \bar{c}_{12} + \bar{c}_{13}$$

$$\text{or} \quad \psi_1(\bar{C}) + (\bar{c}_{12} + \bar{c}_{10} - \psi_1(\bar{C})) + (\bar{c}_{12} + \bar{c}_{13} - \psi_1(\bar{C})) = \bar{c}_{10} + \bar{c}_{12} + \bar{c}_{13}$$

$$\text{or} \quad \psi_1(\bar{C}) = \bar{c}_{12}$$

Hence $\psi_2(\bar{C}) = \bar{c}_{10}, \psi_3(\bar{C}) = \bar{c}_{13}$. From equation 4.27, $\psi_1(C) \geq \bar{c}_{12}, \psi_2(C) \geq \bar{c}_{10}$. Using efficiency on C we can conclude that, $\psi_1(C) = c_{12}$ and $\psi_2(C) = c_{10}$, i.e. the allocation is unique.

The case $c_{10} > c_{20} > c_{12}$ is similar.

This completes the proof of the case $|N| = 2$.[8]

Suppose the theorem is true for all $C \in \mathcal{C}^2_N$, where $|N| < m$. We will show that the result is true for all $C \in \mathcal{C}^2_N$ such that $|N| = m$.

Let $C \in \mathcal{C}^2_N$. Without loss of generality, assume $c_{10} = \min_{k \in N} c_{k0}$.[9] Thus $(10) \in g_N(C)$. There are two possible cases.

Case 1: $c_{10} = \min_{k \in N^+ \setminus \{1\}} c_{1k}$.

Then choose $j \in N$ such that $(j0) \in g_N(C)$ or $(j1) \in g_N(C)$.

Case 2: $c_{1j} = \min_{k \in N^+ \setminus \{1\}} c_{1k}$.

Then from lemma 3.0.1, $(1j) \in g_N(C)$.

In either case 1 or 2, let $\bar{C}$ denote the restriction of C on $\{1, j\}$. Then, from the case when $\#N = 2$, it follows that $\psi_1(\bar{C}) = \min_{k \in N^+ \setminus \{1\}} c_{1k}$.

Now, by iterative elimination of extreme points and repeated application of extreme point monotonicity, it follows that $\psi_1(C) \leq \psi_1(\bar{C}) = \min_{k \in N^+ \setminus \{1\}} c_{1k}$. But, $C \in \mathcal{C}^2_N$, and ψ satisfies efficiency, tree consistency and extreme point

[8] Note that these three cases cover all possibilities since equality between different costs will result in the matrix not being in $\mathcal{C}^2_N$.

[9] This is unique as $C \in \mathcal{C}^2_N$.

monotonicity. So, from lemma 4.2.3, it follows that $\psi_1(C) \geq \min_{k \in N^+ \setminus \{1\}} c_{1k}$. Hence, $\psi_1(C) = \min_{k \in N^+ \setminus \{1\}} c_{1k} = x_1$ (say).

We remove 1 to get reduced matrix $C^{tr}_{x_1}$. From lemma 4.2.1, $C^{tr}_{x_1} \in \mathcal{C}^2$. By tree consistency, $\psi_k(C^{tr}_{x_1}) = \psi_k(C)\ \forall k \neq 1$. From the induction hypothesis, the allocation is unique on $C^{tr}_{x_1}$ and hence on C.

This completes the proof of the theorem. ■

We now show that the three axioms used in the theorem are independent. The examples constructed below are all variants of ψ^*. So, a^k, b^k are derived from the algorithm used to construct ψ^*.

Example 4.2.1: We construct a rule ϕ which satisfies extreme point monotonicity and tree consistency but violates efficiency.

Let $\phi_k(C) = \psi^*_k(C) + \epsilon\ \forall k$, where $\epsilon > \sum_{(ij) \in g_N(C)} c_{ij}$.

Since ψ^* satisfies extreme point monotonicity, ϕ also satisfies extreme point monotonicity. Moreover, the restriction on the value of ϵ ensures that the reduced matrices always lie outside $\mathcal{C}$. So, tree consistency is vacuously satisfied by ϕ. Also, since $\sum_{k=1}^n \phi_k(C) = \sum_{k=1}^n \psi^*_k(C) + n\epsilon > c(N)$, ϕ violates efficiency.

To construct the next example we need to define the concept of an *m.c.s.t. partition*.

Given C, let $g_N(C)$ be the (unique) m.c.s.t. of C. Suppose $g_N(C) = g_{N_1} \cup g_{N_2} \ldots \cup g_{N_K}$, where each g_{N_k} is the m.c.s.t. on N_k for the matrix C restricted to N_k^+, with $\cup_{k=1}^K N_k = N$ and $N_i \cap N_j = \emptyset$. We will call such a partition the m.c.s.t. partition of N.

Example 4.2.2: Let $N = [N_1, \ldots, N_T]$ be the m.c.s.t. partition and $\# N_t = n_t$. Let C^t be the restriction of C over N_t^+. First, calculate ψ^* separately for each C^t. Consider any N_t. If $n_t = 1$, $\mu_k(C) = c_{k0}$ where $k \in N_t$. For $n_t \geq 2$,

(i) $\mu_k(C) = \psi^*_k(C^t)\ \forall k \neq b^{n_t - 1}, b^{n_t}$.

(ii) $\mu_{b^{n_t-1}}(C) = \psi^*_{b^{n_t-1}}(C^t) + M$ and $\mu_{b^{n_t}}(C) = \psi^*_{b^{n_t}}(C^t) - M$, where $M > \sum_{(ij)\in g_N(C)} c_{ij}$.

Efficiency is obviously satisfied. If $n_t > 2$, μ satisfies tree consistency because ψ^* satisfies tree consistency. If $n_t = 2$ then tree consistency is vacuously satisfied as the reduced matrix lies outside $\mathcal{C}$. But this allocation violates extreme point monotonicity. In order to see the latter, consider the following matrix C.

$$C = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

Then, $g_N(C) = \{(01),(12)\}$. Clearly, 2 is an extreme point of C. Let $\bar{C}$ be the restriction of C over $\{0,1\}$. Then, $\mu_1(C) = 1 + M > 1 = \mu_1(\bar{C})$ and hence extreme point monotonicity is violated.

We remark in the next theorem that the Bird rule B satisfies efficiency and extreme point monotonicity. Since $B \neq \psi^*$, it follows that B does not satisfy tree consistency. Here is an explicit example to show that B violates tree consistency.

Example 4.2.3:

$$C = \begin{pmatrix} 0 & 2 & 3.5 & 3 \\ 2 & 0 & 1.5 & 1 \\ 3.5 & 1.5 & 0 & 2.5 \\ 3 & 1 & 2.5 & 0 \end{pmatrix}$$

Then, $B_1(C) = 2$, $B_2(C) = 1.5$ and $B_3(C) = 1$. The reduced matrix is $C^{tr}_{B_1(C)}$ is shown below.

$$C^{tr}_{B_1(C)} = \begin{pmatrix} 0 & 1.5 & 1 \\ 1.5 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{pmatrix}.$$

Then, $B_2(C^{tr}_{B_1(C)}) = 0.5$ and $B_3(C^{tr}_{B_1(C)}) = 1$. Therefore tree consistency is violated.

However, B does satisfy source consistency on the domain $\mathcal{C}^1$. In fact, we now show that B is the only rule satisfying efficiency, extreme point monotonicity and source consistency.

Theorem 4.2.2 *Over the domain* $\mathcal{C}^1$, *a rule* ϕ *satisfies source consistency, efficiency and extreme point monotonicity iff* $\phi = B$.

Proof : We first show that B satisfies all the three axioms. Efficiency and extreme point monotonicity follows trivially from the definition. It is only necessary to show that B satisfies source consistency.

Let $(10) \in g_N$. Then, $B_1(C) = c_{01}$. Let us denote the reduced matrix $C^{sr}_{B_1}$ by $\bar{C}$. From lemma 4.2.2, $\bar{C} \in \mathcal{C}^1$. Also, the m.c.s.t. over $N \setminus \{1\}$ corresponding to $\bar{C}$ is

$$g_{N\setminus\{1\}} = \{(ij) | \text{ either } (ij) \in g_N(C) \text{ with } i, j \neq 1 \text{ or } (ij) = (l0) \text{ where } (1l) \in g_N(C)\}.$$

Also, for all $i, j \in N \setminus \{1\}$, $\bar{c}_{ij} = c_{ij}$ if $(ij) \in g_N$, and for $k \in N \setminus \{1\}, \bar{c}_{k0} = c_{1k}$ if $(1k) \in g_N(C)$. Hence, for all $k \in N \setminus \{1\}, \bar{c}_{k\bar{\alpha}(k)} = c_{k\alpha(k)}$, where $\bar{\alpha}(k)$ is the immediate predecessor of k in $g_{N\setminus\{1\}}$. So, $B_k(\bar{C}) = B_k(C)$ for all $k \in N \setminus \{1\}$ and B satisfies source consistency.

Next, we show that B is the only rule over $\mathcal{C}^1$, which satisfies all the three axioms. This proof is by induction on the cardinality of the set of agents.

We remark that the proof for the case $|N| = 2$ is virtually identical to that of theorem 4.2.1, with source consistency replacing tree consistency and lemma 4.2.4 replacing lemma 4.2.3.

Suppose B is the only rule satisfying the three axioms, for all $C \in \mathcal{C}^1$, where $|N| < m$. We will show that the result is true for all $C \in \mathcal{C}^1$ such that $|N| = m$.

Let $C \in \mathcal{C}^1$. Without loss of generality, assume $(10) \in g_N(C)$. There are two possible cases.

Case 1 : There are at least two extreme points of C, say m_1 and m_2.

First, remove m_1 and consider the matrix C^{m_1}, which is the restriction of C over $(N^+ \setminus \{m_1\})$. By extreme point monotonicity, $\psi_i(C) \leq \psi_i(C^{m_1})$ for all $i \neq m_1$. As C^{m_1} has $(m-1)$ agents, the induction hypothesis gives $\psi_i(C^{m_1}) = c_{i\alpha(i)}$. So, $\psi_i(C) \leq c_{i\alpha(i)}\ \forall i \neq m_1$. Similarly by eliminating m_2 and using extreme point monotonicity, we get $\psi_i(C) \leq c_{i\alpha(i)}\ \forall i \neq m_2$. Combining the two, we get $\psi_i(C) \leq c_{i\alpha(i)}\ \forall i \in N$.

But from efficiency, we know that $\sum_{i\in N} \psi_i(C) = c(N) = \sum_{i \in N} c_{i\alpha(i)}$. Therefore $\psi_i(C) = c_{i\alpha(i)}\ \forall i \in N$, and hence the allocation is unique.

Case 2: If there is only one extreme point of C, then $g_N(C)$ must be a line. i.e. each agent has atmost one follower. Without loss of generality, assume 1 is connected to 2 and 0. Let $\bar{C}$ be the restriction of C over the set $\{0, 1, 2\}$. By iterative elimination of the extreme points and use of extreme point monotonicity we get $\psi_i(C) \leq \psi_i(\bar{C})$. Using the induction hypothesis, we get $\psi_1(C) \leq c_{10}$ and $\psi_2(C) \leq c_{12}$.

Suppose $\psi_1(C) = x_1 = c_{10} - \epsilon$, where $\epsilon \geq 0$. Now consider the reduced matrix $C^{sr}_{x_1}$, which will be denoted by $\widehat{C}$. It can be easily checked that $g_{N\setminus\{1\}}$ is also a line where 2 is connected to 0. Thus $\psi_2(\widehat{C}) = \widehat{c}_{20} = \min(c_{20}, c_{12} + c_{10} - \psi_1(C)) = \min(c_{20}, c_{12} + \epsilon)$. So, $\psi_2(\widehat{C}) \geq c_{12}$ with equality holding only if $\epsilon = 0$. By SR, $\psi_2(C) = \psi_2(\widehat{C})$. But from extreme point monotonicity $\psi_2(C) \leq \psi_2(\bar{C}) = c_{12}$. This is possible only if $\epsilon = 0$. Therefore, $\psi_1(C) = c_{10}$. Using source consistency and the induction hypothesis, we can conclude that $\psi = B$. ■

We now show that the three axioms used in Theorem 3 are independent.

A rule which violates efficiency but satisfies source consistency and extreme point monotonicity can be constructed using example 4.2.1, ψ^* being replaced by B.

The rule obtained by replacing ψ^* with B in example 4.2.2, violates extreme point monotonicity but satisfies efficiency and source consistency.

Our new rule ψ^* satisfies all the axioms but source consistency. The fact that ψ^* satisfies efficiency and extreme point monotonicity is proved in the previous theorem. Here is an example to show that our rule may violates source consistency.

Example 4.2.4:

$$C = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 1.5 & 1 \\ 3 & 1.5 & 0 & 3.5 \\ 4 & 1 & 3.5 & 0 \end{pmatrix}$$

Then, $\psi_1^*(C) = 1$, $\psi_2^*(C) = 2$ and $\psi_3^*(C) = 1.5$. The reduced matrix is $\hat{C}$,

$$\hat{C} = \begin{pmatrix} 0 & 2.5 & 2 \\ 2.5 & 0 & 3.5 \\ 2 & 3.5 & 0 \end{pmatrix}$$

$\psi_2^*(\hat{C}) = 2.5$ and $\psi_3^*(\hat{C}) = 2$. Therefore source consistency is violated.

In theorem 4.2.1, we have restricted attention to matrices in C^2. This is because ψ^* does not satisfy tree consistency outside C^2. The next example illustrates.

Example 4.2.5: Consider

$$C = \begin{pmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 2 & 5 \\ 4 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{pmatrix}$$

Then, $g_N^1(C) = \{(10), (12), (23)\}$ and $g_N^2(C) = \{(30), (32), (21)\}$ are the two m.c.s.t. s corresponding to C. Taking the average of the two cost allocations derived from the algorithm, we get $\psi^*(C) = (2.5, 1.5, 2)$. If we remove 1, which is connected to 0 in g_N^1, the reduced matrix $\widehat{C}$ is:

$$\widehat{C} = \begin{pmatrix} 0 & 2.5 & 3 \\ 2.5 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Then, $\psi_2^*(\widehat{C}) = 1$ and $\psi_3^*(C) = 2.5$. So, tree consistency is violated.

Remark 4.2.1: Note that in the previous example C lies outside $\mathcal{C}^1$. If we take a matrix in $\mathcal{C}^1 \setminus \mathcal{C}^2$, then lemma 4.2.1 will no longer be valid - the reduced matrix may lie outside $\mathcal{C}^1$ even when a node connected to the source pays the minimum cost amongst all its links. Thus, ψ^* will satisfy tree consistency vacuously. But there may exist allocation rules other than ψ^* which satisfies efficiency, tree consistency and extreme point monotonicity over $\mathcal{C}^1$.

Similarly, B does not satisfy source consistency outside $\mathcal{C}^1$.

Example 4.2.6: Consider the same matrix as in example 4.2.5. Recall that $B(C) = (2.5, 1.5, 2)$.

If we remove 1, which is connected to 0 in g_N^1, the reduced matrix $\widehat{C}$ is:

$$\widehat{C} = \begin{pmatrix} 0 & 2.5 & 3 \\ 2.5 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Then, $B_2(\widehat{C}) = 2.5$ and $B_3(\widehat{C}) = 1$. Therefore source consistency is violated.

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