# Slippage and Change Point Problems with Directional Data

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# Contents

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# Chapter 1

## INTRODUCTION

In this thesis we develop statistical methods for dealing with two problems namely (1) THE SLIPPAGE PROBLEM and (2) THE CHANGE POINT PROBLEM in the set-up of directional data. In this chapter, we provide an introduction to these problems and discuss the importance of the present work.

The slippage problem is basically a problem to detect whether any unspecified observation in a given random sample comes from a distribution different from that for all the other remaining observations. This can also be viewed as a problem of "outlier" detection or that of "spuriosity". This problem assumes great importance in many practical situations where directional data are encountered, e.g. in applications to meteorological data, wind directions, movements of icebergs, propagation of cracks, biological and periodic phenomena, quality assurance and productivity measures, etc. However, little seems to be known regarding its theoretical foundations in the context of directional data under a parametric model, say e.g. a 'slip' in terms of the mean direction of the circular normal distribution (see however, Collet, 1980, Bagchi and Guttman, 1988, 1990 and Upton, 1993). The circular normal distribution with mean direction $\mu$ and concentration parameter $\kappa$, denoted by $\mathrm{CN}(\mu, \kappa)$, is one of the most popular distributions for modeling circular data. Some useful facts regarding this distribution can be found in Appendix-I. A survey of the work done on this problem can be found in Barnett and Lewis(1994).

In CHAPTER 2, we consider the problem of testing $H_0 : \Theta_j, j = 1, ..., k$ are identically distributed as $\text{CN}(\mu_0, \kappa)$ against $H_i : \Theta_1, \ldots, \Theta_{i-1}, \Theta_{i+1}, \ldots, \Theta_k$ are identically distributed as $\text{CN}(\mu_0, \kappa)$ and $\Theta_i$ is distributed as $\text{CN}(\mu_1, \kappa)$, $1 \leq i \leq k, \mu_1 > \mu_0$, $\mu_1, \mu_0$ and $\kappa$ are all known, using a decision theoretic route. We derive the Bayes test with respect to the prior distribution invariant with respect to permutations of $H_1, \ldots, H_k$. We also study the performance of the Bayes test when the null hypothesis is true and also when one of the alternative hypothesis is true.

In CHAPTER 3, we consider the problem of testing $H_0$ against $H_1^*$ : There exists $i$, $i$ unknown, such that $\Theta_i$ is distributed as $\text{CN}(\mu_1, \kappa)$ and $\Theta_1, \Theta_2, \ldots, \Theta_{i-1}, \Theta_{i+1}, \ldots, \Theta_n$ are distributed as $\text{CN}(\mu_0, \kappa), \Theta_1, \ldots, \Theta_n$ are all independent. We derive the LRT and study its performance using simulations. We illustrate the use of this test by analysing two well known data sets. We introduce the notion of a LOCALLY MOST POWERFUL TYPE TEST (LMPTT) and derive it for this problem. We also indicate a Multiple Testing approach for the outlier problem, which can be easily generalized to situations where the underlying distribution is not circular normal.

In CHAPTER 4, we provide a simulation based comparison of the various procedures for outlier detection, namely, the L-statistic(Collet, 1980), M-statistic(Mardia, 1975), the LRT, the LMP and the Bayes-rule with different values of $p$. It is found that the LMPTT performs best when outliers of small magnitude are sought to be detected, LRT performs best when outliers of moderate magnitude are sought to be detected and the Bayes-Test performs best when outliers of large magnitude are sought to be detected.

The onset of an abrupt change, which usually leads to poor quality products is a phenomenon which is common in the industrial context. Several methods like control charts, Cusum charts, EWMA charts etc. are all designed to detect such a sudden change. It is of interest to note that the original and which is also currently adopted, formulation given by Page (1955) does not consider any possible correlation between the successive time sequenced observations which makes it markedly different from the usual time series models. The problem of detecting whether at all there is a point of abrupt change in a given data set and thereafter the problem of detecting the change point has received a lot of attention. These problems have been extensively studied for the usual linear data e.g. in the normal and nonparametric setups see eg, Chernoff and Zacks (1964), Hinkley (1970), Sen and Srivastava (1973, 1975a, 1975b), Siegmund (1988) etc. A review of the estimation of change points can be found in Krishnaiah and Miao (1988). The change-point problem assumes great practical significance in the context of many real-life encounters with directional data also. However, no work on the parametric inference for this problem seems to have been done. Only Lombard (1986) has initiated investigations in a non-parametric setup. Here we present some results in the parametric framework.

In CHAPTER 5, we look at the change point problem for the mean direction of a circular normal distribution when the concentration parameter is known using likelihood based techniques. We are interested to test $H_0 : \Theta_1, \ldots, \Theta_n$ are i.i.d $\mathrm{CN}(\mu_0, \kappa)$ against the alternative $H_1$: There exist $r, 1 \leq r \leq n-1$, such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{CN}(\mu_0, \kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(\mu_1, \kappa), \mu_1 \neq \mu_0$. In the case when $\mu_0$ is known but $\mu_1$ is unknown we derive the Locally Most Powerful Type Test LMPTT for this problem. The asymptotic null distribution of the LMPTT statistic is shown to be same as that of the supremum of a time reversed Brownian Motion on [0,1]. When $\mu_0$ and $\mu_1$ are both unknown, then we derive the LRT for this problem. We find the cut-off values and the power of the LRT using simulations. We also indicate a generalization to the case of multiple change points. A multiple testing approach is provided which can be easily used even in situations where the underlying distribution is not circular normal.

In CHAPTER 6, we develop statistical methods for dealing with the above problem when the concentration parameter is unknown. The problem is complicated since $\kappa$ is neither a location nor a scale parameter. Hence the usual techniques of nuisance parameter elimination like similarity, sufficiency, invariance etc. do not work here. One can use conditional arguments but that would mean substantial loss of information contained in the data set (Laycock,1975). When the initial direction $\mu_0$ is known, we suggest a multiple test procedure based on the LMP Conditional Type Test LMPCTT derived under the assumption that the change point is known. When all the parameters are unknown we introduce the notion of Neyman-Rao Type Test(NRTT) and derive it for this problem. The Neyman-Rao Test is an extension of the well-known $C_\alpha$ test. We have not found any reference to the use of the $C_\alpha$ test or NR-test in the context of change point problems even in the linear case. Our approach for circular distributions here easily applies to linear distribution also. Since the NRTT is a large-sample test for small samples we obtain its cut-off values and also its power using simulations. We also illustrate the use of the LRT, LMPTT and the NRTT using two real-life data sets.

In CHAPTER 7, we look at the change point problem for the concentration parameter of a circular normal distribution when the mean direction is known using likelihood based techniques. We consider the problem of testing $H_0 : \Theta_1, \ldots, \Theta_n$ are i.i.d CN$(\mu, \kappa_0)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as CN$(\mu, \kappa_0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as CN$(\mu, \kappa_1), \kappa_1 \neq \kappa_0$ for some $r, 1 \leq r \leq n-1$. When $\kappa_0$ is known, we introduce the notion of UNIFORMLY MOST POWERFUL TYPE TEST (UMPTT) and derive it for this problem. When $\kappa_0$ is unknown we derive the LRT for this problem.

In CHAPTER 8, we look at the change point problem for the concentration parameter when the mean direction is unknown. We derive an NRTT for this problem.

In CHAPTER 9, we look at the change point and outlier problem for some skewed circular distributions. We consider three different skewed circular distributions, one due to Papakonstantinou(1979), another due to Rattihali and Sengupta(2001) and the third due to Batschelet(1981). We use the LMPTT as the main tool in tackling change point and outlier problems for these distributions.

In CHAPTER 10, we look at a few alternative approaches to the change point and outlier problems like (a) Semi-Bayesian and Hierarchial Bayes approach, (b) A new type of Integrated Likelihood approach, (c) Randomization tests and (d) Markov chain based approach. The last approach can be used for predicting change points.

In CHAPTER 11, we provide three new exploratory data analytic tools : (a) Changeogram, (b) Circular Difference Tables and (c) Circular CUSUM chart. These are useful for detection of change points in the context of directional data. Their uses are illustrated through some examples also.

In CHAPTER 12, we discuss some possible generalizations, and indicate scope for further research. The LMPTT is used as a tool for tackling change point and outlier problems for Cartwright-Mitsuyasu distribution(Cartwright, 1964, Mitsuyasu et. al. 1975), wrapped Cauchy distribution (Mardia, 1972) and the wrapped stable family of distributions (Mardia, 1972). Also a few interesting change point problems like Circular Uniform to

Circular Normal and Circular Uniform to Circular Uniform-Circular Normal mixture are shown to be easily tackled using the LMPTT approach. Finally, a general method of obtaining the NRTT statistic value for the case when exact computations are formidable is presented. This method yields an *Unified Approach* through which, we expect, important change point testing problems can be tackled, for all usually encountered circular distributions. Our pursuit of the solutions to our testing problems is thus brought to a halt at this point. Scope of further research in related areas is also indicated.

In CHAPTER 13, we provide some computer programs which are useful for implementing some of the methods developed in this thesis.

In CHAPTER 14, we provide tables of cut-off values and powers of someof the tests developed in this thesis.

In CHAPTER 15 we provide a collection of figures including graphs of some of the skewed circular distributions discussed in this thesis.

# Chapter 2

# SLIPPAGE PROBLEM - A DECISION THEORETIC APPROACH

## 2.1 Introduction

In this chapter we look at the slippage problem for the mean direction of the circular normal distribution. We assume that there is atmost one observation in a data set which comes from a different distribution than the rest. We follow a decision theoretic approach and derive the Bayes' test with respect to an invariant prior which is a natural choice in such contexts. The performance of the Bayes' test when the null hypothesis is true and also when any one of the alternative hypotheses is true is studied.

## 2.2 Decision Theoretic Rule

Suppose $\Theta_1, \ldots, \Theta_n$ are independent CN$(\mu_i, \kappa)$ random variables with density

$$f(\theta; \mu_i) = \frac{\exp\{\kappa \cos(\theta - \mu_i)\}}{2\pi I_0(\kappa)}, 0 \le \theta < 2\pi, \kappa > 0, 0 \le \mu_i < 2\pi, i = 0, 1 \quad (2.1)$$

where $I_0(\kappa)$ is the modified Bessel function of order 0.

We assume that $0 \leq \mu_0 < \mu_1 < 2\pi$ and $n \geq 3$. We are interested in finding the Bayes' rule for the multiple decision problem of accepting one of the $n+1$ hypotheses $H_0, H_1, \ldots, H_n$ with respect to the prior distributions invariant under permutations of $H_1, \ldots, H_n$ using the loss function which assigns loss $= 0$ if the correct hypothesis is accepted and loss $= 1$ otherwise. The prior distributions invariant under permutations of $H_1, \ldots, H_n$ give equal weight to $H_1, \ldots, H_n$ and hence they are of the form $\tau_p$, where

$$\begin{aligned} \tau_p(H_0) &= 1 - np, \\ \tau_p(H_i) &= p \end{aligned} \tag{2.2}$$

where $0 \leq p \leq (1/n)$, and $1 \leq i \leq n$.

Let $\Phi(\Theta) \equiv (\phi(1;\Theta), \ldots, \phi(n;\Theta))$, be a generalized critical function or a multiple decision rule (Ferguson,1967) with $\phi(i;\Theta)$, $i = 1, \ldots, n$ taking values 0 or 1, and $\sum \phi(i;\Theta) = 1$. Thus $\Phi$ chooses $H_i$ when $\Theta = \theta$ is observed if $\phi(i;\theta) = 1$. Let $R_j$ be the likelihood ratio at $\Theta_j$ under $\mu_1$ to $\mu_0$, i.e.,

$$R_j = f(\Theta_j;\mu_1)/f(\Theta_j;\mu_0) = \exp\left[\kappa\left\{\cos(\Theta_j - \mu_1) - \cos(\Theta_j - \mu_0)\right\}\right] \tag{2.3}$$

**Theorem 1 :** *The Bayes' test with respect to $\tau_p$ for $H_0$ against $H_i, 1 \leq i \leq n$ is given by*

$$\phi(0;\Theta) = 0 \textit{ whenever } (1 - np)/p < \max_j R_j \tag{2.4}$$

*and*

$$\begin{aligned} \phi(i;\Theta) &= 0 \textit{ whenever either } R_i < \max_j R_j \textit{ or} \\ &\quad (1 - np)/p > \max_j R_j, 1 \leq i \leq n \end{aligned} \tag{2.5}$$

**Proof :** The result follows from the general theory given in Ferguson(1967,pp. 299) after some simplifications.

The following theorem gives the performance of the Bayes' rule given in Theorem under the null hypothesis. Let,

$$
\begin{aligned}
K(\eta) = \quad & \exp\left\{\kappa\cos\left(\delta+\pi-\sin^{-1}\eta\right)\right\} \\
& +\exp\left\{\kappa\cos\left(\delta+\sin^{-1}\eta\right)\right\} \\
& +\exp\left\{\kappa\cos\left(\delta+3\pi-\sin^{-1}\eta\right)\right\}, \qquad (2.6)
\end{aligned}
$$

and $G(\eta) = \int_{-1}^{\eta} g(t)dt$, $\eta \in (-1,1)$ where,

$$
g(\eta) = \frac{K(\eta)}{2\pi I_0(\kappa)\sqrt{1-\eta^2}} \qquad (2.7)
$$

Further let $\delta = \frac{\mu_1-\mu_0}{2}$ and $u = \{(1-np)/p\}^{2\kappa\sin\delta}$.

**Theorem 2 :** *In the framework of Theorem 1,*

$$
\begin{aligned}
(a)\,\Pr(\phi(i;\Theta)=0 \mid H_0 \textit{ is true }) &= 1-\frac{1}{n}+\frac{[G(u)]^n}{n} \qquad (2.8)\\
(b)\,\Pr(\phi(0;\Theta)=0 \mid H_0 \textit{ is true }) &= 1-[G(u)]^n \qquad (2.9)
\end{aligned}
$$

**Proof:** Observe that,

$$
R_j = \exp\left(2\kappa\sin\delta\sin(\Theta_j-\mu-\delta)\right) \qquad (2.10)
$$

Thus,

$$
\max_j R_j = \exp\left\{(2\kappa\sin\delta)\max_j \sin(\Theta_j-\mu-\delta)\right\} \qquad (2.11)
$$

since $\exp(x)$ is an increasing function of $x$ and $2\kappa\sin\delta > 0$ as a consequence of our assumption $0 \le \mu_0 < \mu_1 < 2\pi$. Let $\eta_j = \sin(\Theta_j-\mu-\delta)$ for $j = 1,2,\ldots,n$. Since, $\Theta_1,\ldots,\Theta_n$ are i.i.d it follows that $\eta_1,\eta_2,\ldots,\eta_n$ are i.i.d. We first derive the distribution of $\eta$. Let us consider that branch of $\sin^{-1}\theta$ which is monotone on $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Then the inverse transformation is defined uniquely through the domain of $\theta$ as partitioned below, is

$$\begin{aligned}\psi(\eta) &= \mu+\delta+(\pi-\sin^{-1}\eta) \text{ if } \mu+\delta \leq \theta \leq \frac{\pi}{2}+\mu+\delta \\ &= \mu+\delta+\sin^{-1}\eta \text{ if } \frac{\pi}{2}+\mu+\delta \leq \theta \leq \frac{3\pi}{2}+\mu+\delta \\ &= \mu+\delta+3\pi-\sin^{-1}\eta \text{ if } \frac{3\pi}{2}+\mu+\delta \leq \theta \leq 2\pi+\mu+\delta \quad (2.12)\end{aligned}$$

After some calculations we get the density of $\eta$ to be $g(\eta)$. Note that the density does not exist for the points -1 and 1. But then the set $\{-1, 1\}$ has Lebesgue measure zero and hence any value can be put at these points without changing the distribution. We put 0 at these points. Define $W = \max_{\ell} \eta_\ell$. Then $W$ has distribution $H$ where $H(w) = [G(w)]^n$. Let $W_i^* = \max_{\ell \neq i} \eta_\ell$. Then the c.d.f of $W_i^*$ is $H_i^*(w) = [G(w)]^{n-1}$. Now

$$\begin{aligned}\Pr\left(\max_j R_j < \frac{1-np}{p}\right) &= \Pr\left(W < \frac{\ln\left(\frac{1-np}{p}\right)}{2\kappa\sin\delta}\right) \\ &= \int_{-1}^{u} dH(w) \qquad (2.13)\end{aligned}$$

where $u = \frac{\ln\left(\frac{1-np}{p}\right)}{2\kappa\sin\delta}$.

Note that the event, $R_i < \max_j R_j$ is equivalent to the event $\eta_i < W$. Further, $\Pr(\eta_i \geq W) = \Pr(\eta_i > \eta_s, s \neq i)$ since the distribution of $\eta_i$'s are continuous. Thus,

$$\begin{aligned}\Pr\left(\eta_i > \max_s \eta_s\right) &= \int_{-1}^{1}(1-G(\eta))dH_i^*(\eta) \\ &= \int_0^1 (n-1)(1-y)y^{n-2}dy = \frac{1}{n} \qquad (2.14)\end{aligned}$$

where we put $y = G(\eta)$ and $dH_i^*(\eta) = (n-1)[G(\eta)]^{n-2}dG(\eta)$. Now,

$$\Pr(\phi(i;\Theta) = 0) = \Pr(\eta_i < W) + \Pr(W < u) - \Pr(\eta_i < W < u)$$

Thus, the problem of finding $\Pr(\phi(i;\Theta) = 0 \mid H_0 \text{ is true})$ is solved if $\Pr(\eta_i < W < u)$ is obtained. Now,

$$\begin{aligned}
\Pr(\eta_i < W < u) &= \Pr(\eta_i < W_i^* < u) \\
&= \int_{-1}^{u} G(\eta)dH_i^*(\eta) \\
&= (n-1)\int_{-1}^{u} [G(\eta)]^{n-1}dG(\eta) \\
&= (n-1)\int_{0}^{G(u)} y^{n-1}dy \\
&= \frac{n-1}{n}[G(u)]^n
\end{aligned}$$

$$\begin{aligned}
\text{So, } \Pr(\phi(i;\theta) = 0 \mid H_0 \text{ is true}) &= 1 - \frac{1}{n} + \frac{[G(u)]^n}{n} \\
\Pr(\phi(i;\theta) = 1 \mid H_0 \text{ is true}) &= 1 - \Pr(\phi(i;\theta) = 0 \mid H_0 \text{ is true}) \\
&= \frac{1}{n} - \frac{[G(u)]^n}{n} \\
\Pr(\phi(0;\theta) = 1 \mid H_0 \text{ is true}) &= \Pr(W \le u) \\
&= [G(u)]^n \\
\Pr(\phi(0;\theta) = 0 \mid H_0 \text{ is true}) &= 1 - [G(u)]^n
\end{aligned}$$

The following theorem gives the performance of the Bayes' rule when $H_j$ is true. Let $\eta_i$ be as in the proof of the Theorem 2 and let G be its distribution function (d.f.) when $\Theta_i$ is $\mathrm{CN}(\mu_0, \kappa)$ and let $G^*$ denote its d.f. when $\Theta_i$ is $\mathrm{CN}(\mu_1, \kappa)$

**Theorem 3 :** *In the framework of Theorem 1,*
*Let,* $1 \le i, j \le n$,

$$(a)\ \Pr(\phi(i;\Theta) = 0 \mid H_j \textit{ is true}\ ) = 1 + \int_{-1}^{u} [G(w)]^{k-2}G^*(w)dG(w)$$

$$-(k-2)\int_{-1}^{1}[G(w)]^{k-3}[1-G(w)]G^*(w)dG(w)$$
$$-\int_{-1}^{1}[G(w)]^{k-2}[1-G(w)]dG^*(w)$$

*where* $i \neq j$

$$(b)\ \Pr(\phi(0;\Theta)=0 \mid H_j \textit{ is true }) = 1-(k-1)\int_{-1}^{u}[G(w)]^{k-2}G^*(w)dG(w) - \int_{-1}^{u}[G(w)]^{k-1}dG^*(w)$$

$$(c)\ \Pr(\phi(j;\Theta)=0 \mid H_j \textit{ is true }) = \int_{-1}^{u}[G(w)]^{k-1}dG^*(w) + (k-1)\int_{-1}^{1}[G(w)]^{k-2}G^*(w)dG(w)$$

**Proof :** (a). Using arguments similar to that in the proof of Theorem 1 we have when $H_j$ is true

$$\begin{aligned}\Pr(\eta_i < W) &= \Pr(\eta_i < W_i^*)\\ &= 1-\Pr(\eta_i \geq W_i^*)\\ &= 1-\{(k-2)\int_{-1}^{1}(1-G(\eta))[G(\eta)]^{k-3}G^*(\eta)dG(\eta)\\ &\quad +\int_{-1}^{1}(1-G(\eta))[G(\eta)]^{k-2}dG^*(\eta)\} \qquad (2.15)\end{aligned}$$

since $\eta_i$ and $W_i^*$ are independent.
Also,

$$\begin{aligned}\Pr(W<u) &= (k-1)\int_{-1}^{u}[G(w)]^{k-2}G^*(w)dG(w)\\ &\quad +\int_{-1}^{u}[G(w)]^{k-1}dG^*(w) \qquad (2.16)\end{aligned}$$

and

$$\begin{aligned}\Pr(\eta_i < W_i < u) &= (k-2)\int_{-1}^{u}[G(\eta)]^{k-2}G^*(\eta)dG(\eta)\\ &\quad +\int_{-1}^{u}[G(\eta)]^{k-1}dG^*(\eta) \qquad (2.17)\end{aligned}$$

Hence,

$$\begin{aligned}
\Pr(\phi(i;\boldsymbol{\Theta})=0 \mid H_j \text{ is true }) &= \Pr(\eta_i < W) + \Pr(W < u) \\
&\quad - \Pr(\eta_i < W < u) \\
&= 1 + \int_{-1}^{u} [G(w)]^{k-2} G^*(w) dG(w) \\
&\quad -(k-2)\int_{-1}^{1} [G(w)]^{k-3}[1-G(w)]G^*(w)dG(w) \\
&\quad -\int_{-1}^{1} [G(w)]^{k-2}[1-G(w)]dG^*(w) \qquad (2.18)
\end{aligned}$$

$$\begin{aligned}
(b)\ \Pr(\phi(0;\boldsymbol{\Theta})=0 \mid H_j \text{ is true}) &= \Pr(W > u \mid H_j \text{ is true}) \\
&= 1 - \Pr(W \leq u \mid H_j \text{ is true}) \\
&= 1 - \int_{-1}^{u} dH_1(w) \\
&= 1 - \int_{-1}^{u} (k-1)[G(w)]^{k-2} G^*(\omega) dG(w) \\
&\quad + \int_{-1}^{u} [G(w)]^{k-1} dG^*(w) \qquad (2.19)
\end{aligned}$$

where $H_1(w) = [G(w)]^{k-1}G^*(w)$ is the distribution function of $W$ under $H_j$.

$$\begin{aligned}
(c)\ \Pr(\phi(j;\boldsymbol{\Theta})=0 \mid H_j \text{ is true }) &= \Pr(\eta_j < W_j) + P(W < u) \\
&\quad - \Pr(\eta_j < W_j < u) \\
\Pr(\eta_j < W_j) &= (k-1)\int_{-1}^{1} G^*(w)[G(w)]^{k-2} dG(w) \\
\Pr(W < u) &= (k-1)\int_{-1}^{u} [G(w)]^{k-2} G^*(w) dG(w) \\
\Pr(\eta_j < W_j < u) &= (k-1)\int_{-1}^{u} G^*(w)[G(w)]^{k-2} dG(w) \\
\Pr(\phi(j;\Theta)=0 \mid H_j \text{ is true}) &= (k-1)\int_{-1}^{1} G^*(w)[G(w)]^{k-2} dG(w) \\
&\quad + \int_{-1}^{u} [G(w)]^{k-1} dG^*(w) \qquad (2.20)
\end{aligned}$$

$$\Longrightarrow \Pr(\phi(j;\Theta) = 0 \mid \ H_j \text{ is true}) \ = \ (k-1)\int_{-1}^{1} G^*(w)[G(w)]^{k-2} dG(w) + \int_{-1}^{u} [G(w)]^{k-1} dG^*(w) \qquad (2.21)$$

# Chapter 3

# SLIPPAGE PROBLEM - LIKELIHOOD BASED APPROACH

## 3.1 Introduction

In this chapter we provide a likelihood based approach to the slippage problem for the mean direction of the circular normal distribution. In section 3.2 we provide the LRT. We use simulations to obtain the cut-off points and the power of the LRT. The results are discussed in section 3.3. In section 3.4 we discuss the results of a simulation based sensitivity analysis for possibly mis-specified $\kappa$. In section 3.5 we provide two examples of the use of LRT in analysing real life data sets. In section 3.6 we introduce the notion of the "type tests". In section 3.7 we derive the LMPTT for the present problem. Finally in section 3.8 we provide a multiple testing approach to the slippage problem. The results of this section are general and widely applicable.

## 3.2 Likelihood Ratio Test

**Case I :** Suppose $\Theta_1, \ldots, \Theta_n$ are all independent. Let $H_0 : \Theta_1, \ldots, \Theta_n$ are i.i.d $\mathrm{CN}(\mu_0, \kappa)$ and $H_1^* :$ There exist $j, 1 \leq j \leq n$, such that $\Theta_1, \ldots, \Theta_{j-1}, \Theta_{j+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(\mu_0, \kappa)$ and $\Theta_j$ is distributed as $\mathrm{CN}(\mu_1, \kappa); \mu_1 > \mu_0$. We first consider the case of testing $H_0$ against $H_1^*$ when $0 \leq \mu_0 < \mu_1 < 2\pi, \mu_1, \mu_0$ and $\kappa$ are all known. As in the previous chapter, let $\mu_0 = \mu$ and $\mu_1 = \mu + 2\delta \pmod{2\pi}$, $0 < \delta < \pi$. In this case we prove the following theorem. Let $G$ be the distribution function of $\sin(\Theta_i - \mu - \delta)$ under $H_0$.

**Theorem 4 :** *In testing $H_0$ against $H_1^*$ the* LRT*-statistic is equivalent to $V = \max_j \sin(\Theta_j - \mu - \delta)$. The exact sampling distribution function of $V$ under $H_0$ is given by $M(\theta) = [G(\theta)]^n$.*

**Proof :** The likelihood under $H_0$ is,

$$\begin{aligned} f_0(\theta_1, \theta_2, \ldots, \theta_n) &= (2\pi I_0(\kappa))^{-n} \exp\{\kappa \sum_i \cos(\theta_i - \mu)\}, \\ & \quad 0 \le \theta_i < 2\pi, \kappa > 0, 0 \le \mu < 2\pi, \end{aligned} \tag{3.1}$$

and that under $H_1^*$ is,

$$\begin{aligned} f_1(\theta_1, \theta_2, \ldots, \theta_n) &= \max_j \; (2\pi I_0(\kappa))^{-n} \exp[\kappa\{\sum_{i \ne j} \cos(\theta_i - \mu) \\ & \quad + \cos(\theta_j - \mu - 2\delta)\}], \\ & \quad 0 \le \theta_i < 2\pi, \; \kappa > 0, 0 \le \mu < 2\pi, 0 < \delta < \pi \end{aligned} \tag{3.2}$$

Thus the LRT-statistic $\Lambda$ is given by,

$$\begin{aligned} -\ln\, \Lambda &= \max_j [\kappa\{\sum_{i \ne j} \cos(\theta_i - \mu) + \cos(\theta_j - \mu - 2\delta) - \sum_i \cos(\theta_i - \mu)\}] \\ &= \max_j [\kappa\{\cos(\theta_j - \mu - 2\delta) - \cos(\theta_j - \mu)\}] \\ &= \max_j [\kappa\{2 \sin(\theta_j - \mu - \delta) \sin\delta\}] \end{aligned} \tag{3.3}$$

Since $0 < \delta < \pi$ we have, $\sin\delta > 0$. Thus $-\ln\, \Lambda$ is equivalent to the statistic $\max_j \sin(\Theta_j - \mu - \delta) = V$ (say). Now note that under $H_0$, $\sin(\Theta_j - \mu - \delta)$ are i.i.d. Hence the distribution function of $V$ is $M(\theta) = [G(\theta)]^n$.

Observe that $G(\theta)$ can be evaluated numerically and hence the cut-off points-for the LRT are readily available.

**Case II :** In this case we are interested to test $H_0$ against $H_1^*$ when $\kappa$ is known but $\mu$ and $\delta$ are unknown. The form of the LRT is given by the following theorem. Let $\hat{\mu_0}$ and $\hat{\mu_1}^*$ denote the estimates of $\mu$ under $H_0$ and $H_1^*$ respectively. Further, let $f_j$ denote the likelihood when there is a slip at $j, (j = 1, 2, \ldots, n)$. Let $\hat{j}$ be that $j$ for which $f_j$ attains its maximum.

**Theorem 5 :** *In testing $H_0$ against $H_1^*$ the* LRT*-statistic $\Lambda$ is given by*

$$\begin{aligned} -\ln \Lambda &= \kappa\{(\sum_{i \neq \hat{j}} \cos \Theta_i)(\cos\ \hat{\mu_1}^* - \cos\ \hat{\mu_0}) \\ &\quad + (\sum_{i \neq \hat{j}} \sin \Theta_i)(\sin\ \hat{\mu}_1^* - \sin\ \hat{\mu}_0) + 1 - \cos(\Theta_{\hat{j}} - \hat{\mu}_0)\} \end{aligned} \qquad (3.4)$$

**Proof :** The log-likelihood under $H_0$ is,

$$\begin{aligned} \ln f_0(\theta_1, \ldots, \theta_n) &= -n \ln 2\pi I_0(\kappa) + \kappa \sum_i \cos(\theta_i - \mu) \\ &= -n \ln 2\pi I_0(\kappa) + \kappa\{\cos \mu \sum_i \cos \theta_i \\ &\quad + \sin \mu \sum_i \sin \theta_i\} \end{aligned}$$

Putting $\partial \ln f_0 / \partial \mu = 0$ and solving for $\mu$ gives

$$\hat{\mu_0} = \tan^{-1} \left( \sum \sin \theta_i / \sum \cos \theta_i \right).$$

Under $H_1^*$ the log-likelihood is

$$\begin{aligned} \ln f_1^*(\theta_1, \ldots, \theta_n) &= \max_j [-n \ln 2\pi I_0(\kappa) + \kappa\{\cos \mu \sum_{i \neq j} \cos \theta_i \\ &\quad + \sin \mu \sum_{i \neq j} \sin \theta_i + \cos(\theta_j - \mu - 2\delta)\}] \end{aligned}$$

Thus, $f_1^* = \max_j f_j$. Fix $1 \leq j \leq n$. We now compute $\hat{\mu_j}$ and $\hat{\delta_j}$ which are the maximum likelihood estimates (MLE's) of $\mu$ and of $\delta$ under $H_j$. Putting

$\frac{\partial \ln f_j}{\partial \mu} = 0$ and $\frac{\partial \ln f_j}{\partial \delta} = 0$ and solving for $\mu$ and $\delta$ gives

$$\hat{\mu_j} = \tan^{-1}\left(\frac{\sum_{i \neq j} \sin\theta_i}{\sum_{i \neq j} \cos\theta_i}\right)$$

and

$$\hat{\delta_j} = \frac{\theta_j - \hat{\mu_j}}{2}.$$

Let $\hat{j}$ be that $j$ for which $f_j$ attains its maximum value after substituting $\hat{\mu_j}$ and $\hat{\delta_j}$. Thus under $H_1^*$ the estimate of $\mu$ is $\hat{\mu_{\hat{j}}}(=\hat{\mu_1}^*)$ and that of $\delta$ is $\hat{\delta_{\hat{j}}}(=\hat{\delta})$. Therefore the LRT-statistic $\Lambda$ is given by

$$-\ln\Lambda = \kappa\{1 + \sum_{i \neq \hat{j}} \cos(\theta_i - \hat{\mu_1}^*) - \sum_i \cos(\theta_i - \hat{\mu_0})\}$$

which after some calculations gives

$$\begin{aligned} -\ln\ \Lambda \ &= \ \kappa\{(\sum_{i \neq \hat{j}} \cos\theta_i)(\cos\ \hat{\mu}_1^* - \cos\ \hat{\mu}_0) \\ &\quad +(\sum_{i \neq \hat{j}} \sin\theta_i)(\sin\ \hat{\mu}_1^* - \sin\ \hat{\mu}_0) + 1 - \cos(\theta_{\hat{j}} - \hat{\mu}_0)\} \end{aligned}$$

The exact sampling distribution of the LRT-statistic is formidable - no closed form or even any analytic representation for it in small samples seems to be possible.

**Remark 1 :** *We note that Collet(1980) considers a problem very similar to the above. He tests for no slippage versus a slippage alternative and derives a* LRT*-statistic for it, which he calls the L-statistic. However the difference from* *our approach is that, he first uses a data-based measure to detect the outlier candidate. Subsequently formal tests are conducted to statistically validate the hypothesis that it is in fact an outlier. The choice of the measure may not be reasonable for even symmetric data sets where clusters may appear far from the mean direction. In our procedure the detection and testing for the outlier is based on the* LRT *and is entirely probabilistic.*

## 3.3 Simulation

We use simulation to obtain the null distribution of $\Lambda$ as well as its power. The simulation results for the null distribution are based on 5000 repetitions with sample size $n, n=$ 10, 20, 30. The random sample from circular normal distribution is drawn using the IMSL library routine RNVMS. For simulating the power a particular observation is drawn from a circular normal population, CN($\Delta$, 1) with $\Delta = 20(20)180$ (in degrees) and repeating it 5000 times. Since the power function is symmetric about $180^o$ the above computation is sufficient. The cut-off points for the LRT at 5% level of significance is given in Table 1, Chapter 14. By looking at the null distribution of the test statistic it is seen(see Table 2, Chapter 14) that it gets increasingly concentrated with the increase in sample size. Further the null distribution is seen to be increasingly concentrated around 0 as the value of $\kappa$ increases.

Note that since the assumptions involved in the usual large sample approximation of the LRT $\Lambda$, i.e. $-2\ln\,\Lambda$ is distributed as $\chi^2$ is violated here, for example the parameter space $[0, 2\pi) \times (0, \pi) \times \{1, 2, \ldots, n\}$ is not an open set, it is not appropriate to use this approximation in this situation. We obtain the power of the LRT with $\kappa = 1$ through extensive simulations. The power of this LRT (see Table 3, Chapter 14) increases with increase in sample size.

The LRT exhibits (see Table 4, Chapter 14) encouraging power performance starting from $\kappa$ even as somewhat small as 4. Also, the convergence of the power to one increases rapidly with $\kappa$. This is expected since with the higher value of the concentration parameter the observations tend to be close together making it 'easier' for us to detect an outlier.

## 3.4 Sensitivity Analysis

The LRT statistic is derived under the assumption that the value of $\kappa$ is known. But in real life applications the value of $\kappa$ is usually unknown. Thus to apply the LRT in real life situations we need to specify a value of $\kappa$ which may be estimated from the data or arrived at from other considerations. Since the specified value of $\kappa$ may not be exactly equal to the actual value of $\kappa$, it is important to study the sensitivity of the level of significance and the power of the LRT to the possibly mis-specified value of $\kappa$.

Table I and Table II below give the results of a simulation-based sensitivity analysis for different values of true and specified $\kappa$. To study the sensitivity of the level of significance we generate 1000 random samples of size 10 each from $\mathrm{CN}(0, \kappa)$ where $\kappa$ is the true value of the concentration parameter. The LRT statistic is computed using the specified value of $\kappa$. The cut-off points used are such that the level of significance of LRT is 5% when the true value of $\kappa$ is known. In Table I, the actual level of significance of the LRT is given for different combinations of true and specified $\kappa$.

To study the sensitivity of the power of the LRT we generate 1000 random samples of size 10 each. In each sample of size 10, nine observations come from $\mathrm{CN}(0,2)$ and one comes from $\mathrm{CN}(\Delta, 2)$ where $\Delta = 45°, 90°, 135°, 180°$. The LRT statistic is computed using the specified value of $\kappa$. The cut-off point used is that of an LRT with level of significance 5% when the value of $\kappa$ is known to be 2. In Table II, the power of the LRT is given for different combinations of specified value of $\kappa$ and $\Delta$.

TABLE I Sensitivity Analysis of the Level of Significance of the LRT for outlier problem ($n = 10$)

| True $\kappa$ | Specified $\kappa$ 0.5 | 1.0 | 1.5 | 2.0 | 4.0 | 10.0 |
|---|---|---|---|---|---|---|
| 0.5 | 0.050 | 0.129 | 0.194 | 0.391 | 0.978 | 1.000 |
| 1.0 | 0.042 | 0.050 | 0.124 | 0.241 | 0.901 | 0.998 |
| 1.5 | 0.017 | 0.035 | 0.050 | 0.134 | 0.708 | 0.974 |
| 2.0 | 0.006 | 0.021 | 0.022 | 0.050 | 0.497 | 0.936 |
| 4.0 | 0.000 | 0.000 | 0.000 | 0.002 | 0.050 | 0.629 |
| 10.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.050 |

TABLE II Sensitivity Analysis of the Power of the LRT for outlier problem ($n = 10$, $\kappa = 2$ )

| Specified $\kappa$ | $\Delta$ 45° | 90° | 135° | 180° |
|---|---|---|---|---|
| 0.5 | 0.008 | 0.015 | 0.028 | 0.046 |
| 1.0 | 0.009 | 0.027 | 0.056 | 0.075 |
| 1.5 | 0.035 | 0.034 | 0.073 | 0.130 |
| 2.0 | 0.053 | 0.098 | 0.213 | 0.293 |
| 4.0 | 0.556 | 0.706 | 0.854 | 0.949 |
| 10.0 | 0.953 | 0.981 | 0.996 | 0.998 |

From Tables I and II we find that both the level of significance and the power of the LRT are sensitive to the variations in the values of $\kappa$. If the specified value of $\kappa$ is less than the true value then we find that the LRT is conservative but if the specified value of $\kappa$ is greater than the true value then the misspecification leads to anti-conservative nature of the LRT.

It is also important to know the sensitivity of the LRT with respect to the underlying distribution i.e. how the LRT will behave if data coming from a distribution other than circular normal is analysed using the LRT which is derived under the assumption that the underlying distribution is circular normal. It is particularly interesting to know the behaviour of the LRT if the underlying distribution is a skew circular distribution. A skew circular distribution of particular interest is the Rattihali-SenGupta's skewed circular distribution (Rattihali and SenGupta, 2000) discussed in Chapter 9. The $\mathrm{RS}(k_1, k_2, \mu)$ distribution has probability density function

$$f(\theta; k_1, k_2, \mu) = \frac{1}{C(k_1, k_2, \mu)} \exp\left[k_1 \cos(\theta - \mu) + k_2 \cos 2\theta\right]$$
$$0 \leq \theta < 2\pi, k_1, k_2 > 0, 0 \leq \mu < 2\pi. \tag{3.5}$$

It is clear from the above that if $k_2 = 0$ we get the $\mathrm{CN}(\mu, k_1)$ density.

A small simulation based sensitivity study was conducted regarding the level and the power of the LRT when the underlying distribution is a Rattihali-SenGupta's skew circular distribution. Random samples from $\mathrm{RS}(k_1, k_2, \mu)$ can be easily generated using the acceptance-rejection technique based on the $\mathrm{CN}(\mu, k_1)$ as the envelope function. Each time 10 random samples $\theta_1, \ldots, \theta_{10}$ are generated of which $\theta_1, \ldots, \theta_9$ are from $\mathrm{RS}(0.9, 1.1, 0.7854)$ and $\theta_{10} = (\theta^* + \Delta) \bmod 2\pi$ where $\theta^*$ is a random sample from $\mathrm{RS}(0.9, 1.1, 0.7854)$ distribution and $\Delta = 0°, 45°, 90°, 135°$ and $180°$. Since the value of the measure of concentration $\rho$, which is defined as $\rho = \sqrt{(E(\cos\Theta))^2 + (E(\sin\Theta))^2}$, for $\mathrm{RS}(0.9, 1.1, 0.7854)$ is quite close to that of a circular normal distribution with $\kappa = 1$, the LRT with $\kappa = 1$ is applied. The nominal level of significance of the test is fixed at 5%. For each $\Delta$ the above procedure is repeated 1000 times and the power of the test noted. Note that the power corresponding to $\Delta = 0$ is the actual level of significance of the test. The results are given in Table III below.

TABLE III Sensitivity Analysis of the Power of the LRT for the outlier problem w.r.t skewed circular distribution RS(0.9, 1.1, 0.7854) ($n = 10$)

| $\Delta$ | Power |
|---|---|
| 0° | 0.142 |
| 45° | 0.121 |
| 90° | 0.093 |
| 135° | 0.123 |
| 180° | 0.173 |

From the above table we find that the actual level of significance of the LRT is much larger than the nominal value when the underlying distribution is actually a Rattihali-SenGupta's skewed circular distribution, as intuitively expected.

## 3.5 Examples

We illustrate the above tests through two well-known examples on directional data. For both these examples, we assume that the 'known' needed value of $\kappa$ is $\hat{\kappa}$, the MLE, as obtained from the data. The relevant computations as needed below were done through DDSTAP (SenGupta, 1998), a statistical package for the analysis of directional data. We tested these data sets for circular uniformity using the Rayleigh test which resulted in rejection for both. These were however, found to be not incompatible with the assumption of the circular normal model.

*Example 1.* Fisher and Lewis (1983) give data from three samples of paleocurrent orientations from three bedded sandstones layers, measured on the Belford Anticline, New South Wales. We consider here the first sample. The data set is (all figures in degrees) : 284, 311, 334, 320, 294, 137, 123, 166, 143, 127, 244, 243, 152, 242, 143, 186, 263, 234, 209, 267, 315, 329, 235, 38, 241, 319, 308, 127, 217, 245, 169, 161, 263, 209, 228, 168, 98, 278, 154, 279. To use the LRT we need to specify a value of $\kappa$ which is to be treated as the true value of $\kappa$. We use the MLE of $\kappa$ under the null hypothesis as the 'true' value of $\kappa$. The MLE of $\kappa$ under the null hypothesis for this dataset is 0.885. With this value of $\kappa$ the LRT picked up the observation 38 as an outlier with an observed value of $\Lambda = 0.1675$ and with a $P$-value of 0.01. This outlier may be attributed to the segment of the sandstone layer corresponding to the outlier 38 being (inconsistently) disorientated by some 'external shocks'. Since the specified value of $\kappa$ is small the sensitivity analysis of the previous section indicates that the chance of arriving at a wrong conclusion due to the use of the estimated value of $\kappa$ as the true value of $\kappa$ is relatively small.

*Example 2.* We next consider the famous Roulette wheel data obtained from Mardia (1972). The data set is : 43°, 45°, 52°, 61°, 75°, 88°, 88°, 279°, 357°. This data set has previously been analyzed by Bagchi and Guttman(1990). They assumed circular normal distribution for this data set. We also assume the same for analyzing this data set. As in the previous example we use the MLE of $\kappa$ under the null hypothesis as the true value of $\kappa$. The MLE of $\kappa$ under the null hypothesis is found to be 2.076. The LRT when carried out with this value of $\kappa$ yielded $\Lambda = .0276$ which is not significant at 5% level of significance. The $P$-value of the test is 0.12. At this $P$-value the LRT identifies the observation 279° as an outlier.

**Remark 2 :** *Observe that in Example 2, at P-value* 0.12 *the observation* 279° *is identified as an outlier which indicates that unlike in the linear case, internal values can be outliers in the context of directional data. This incidentally also coincides with the analysis done by Bagchi and Guttman(1990).*

**Remark 3 :** *Once an outlier has been detected as above, one may discard it and proceed with further statistical analyses as needed using the rest of the data set. Alternatively, one may fit to the entire data set an extended model, say a contaminated or a mixture model with a circular normal distribution, which should give a better fit than the original one with only a circular normal distribution.*

## 3.6 Optimality based 'Type Tests'

At various points in this thesis, we encounter the problem of testing a null hypothesis $H_0$ against an alternative hypothesis $H_1$ which can be decomposed into several hypotheses $H_{1r}$. Usually it is possible to construct a test of $H_0$ against $H_{1r}$ using a well known test construction procedure. We then construct a test of $H_0$ against $H_1$ based on the above test of $H_0$ against $H_{1r}$. We name such a test of $H_0$ against $H_1$ according to the following principle: First, the name of the test on which it is based is written followed by the phrase 'type test'. For e.g., a test which is based on a LMP test of $H_0$ against $H_{1r}$ will be called LMP type test or LMPTT, in short. Similarly, for UMPTT, NRTT, and LMPCTT.

## 3.7 Locally Most Powerful Type Test

In this section we assume $\mu, \kappa$ to be known and $\delta$ to be unknown. Fix $1 \leq j \leq n$. For each $j$, let $H_j$ denote the alternative hypothesis that the $j^{th}$ observation is an outlier. We derive an LMP test of $H_0$ against $H_j$. Motivated by this we propose a test of $H_0$ against $H_1^*$.

**Theorem 6 :** *In testing $H_0$ against $H_j$ the* LMP *test is given by:*

$$\textit{Reject } H_0 \textit{ if } \sin(\Theta_j - \mu) > c$$

*for some constant $c$ depending on the size of the test.*

**Proof :** Observe the log-likelihood is given by

$$\ln f_j(\delta; \theta_1, \ldots, \theta_n) \quad = \quad K + \sum_{i \neq j} \cos(\theta_i - \mu) + \cos(\theta_j - \mu - 2\delta) \qquad (3.6)$$

where K is a constant.

Then, the score function is given by

$$S_j(\delta) = \frac{\partial \ln f_j}{\partial \delta} = 2\sin(\theta_j - \mu - 2\delta)$$

and hence

$$S_j(0) = 2\sin(\theta_j - \mu)$$

Thus the LMP-test statistic for testing $H_0$ against $H_j$ is $\sin(\Theta_j - \mu)$. We reject $H_0$ if $\sin(\Theta_j - \mu) > c$ for some constant $c$ depending on the size of the test.

Motivated by Theorem 6, for testing $H_0$ against $H_1^*$ we propose the LMPTT

$$\text{Reject } H_0 \text{ if } \max_{1\le j\le n} \sin(\Theta_j - \mu) > c$$

where $c$ is a constant to be determined from the size condition.

The exact sampling null distribution of the test-statistic can be easily obtained using standard techniques.

Recall that we designate $\theta_k$ as the outlier if $H_0$ is rejected and $\sin(\theta_k - \mu) = \max\limits_{1\le j\le n} \sin(\theta_j - \mu)$.

**Remark 4 :** *The general technique of constructing an* LMPTT *in outlier problems is as follows. We first derive the* LMP *test statistic for testing* $H_0$ *against* $H_j$*. Then we take the maximum (or minimum) of these* LMP *test statistics to get a new statistic. A test based on this statistic is called the* LMP *type test or* LMPTT *in short.*

## 3.8 Multiple Testing Approach

The LMPTT test statistic is constructed usually by taking the maximum (or minimum) of the $n$ LMP test statistics for testing $H_0$ against $H_j$. An alternative way is to follow a multiple testing approach. Let $\alpha$ be the desired level of significance. We can individually test $H_0$ against $H_j$ using the LMP test statistic at $\eta$ level of significance. We reject $H_0$ if any one of the tests turn out to be significant. This $\eta$ has to be chosen in such a way that the overall level of significance of the test procedure is $\alpha$. Since it is usually difficult to obtain the exact value of $\eta$ often $\frac{\alpha}{n}$ is used as an approximation. By Bonferroni's inequality it is easily seen that the overall level of significance of this procedure is at most $\alpha$. It is known that tests of this nature are quite conservative and the true level of significance obtained from this procedure is usually much less than $\alpha$, particularly when the test statistics are not independent. Now we note that in outlier problems usually the $n$ LMP tests of $H_0$ against $H_j$ are all independent. The true level of significance of the multiple test procedure (with $\eta = \frac{\alpha}{n}$) can then be easily obtained as follows :

**Lemma :** An approximate level $\alpha$ multiple testing procedure for testing $H_0$ against $H_1^*$ is obtained by testing $H_0$ against $H_j$ for each $j = 1, \ldots, n$ at level $\frac{-\ln(1-\alpha)}{n}$ if $n$ is large. $H_0$ is rejected if and only if atleast one of the tests of $H_0$ against $H_j$ turn out to be significant.

**Proof :** Pr(Reject $H_0$ | $H_0$ is true)
= Pr(at least one of the tests of $H_0$ against $H_j$ rejects $H_0$ | $H_0$ is true)
= 1 - Pr ( none of the tests of $H_0$ against $H_j$ rejects $H_0$ | $H_0$ is true)
$= 1 - \prod_{j=1}^{n}$ Pr($H_0$ is not rejected in favour $H_j$ | $H_0$ is true)
(since the tests are independent)
$= 1 - \left(1 - \frac{\alpha}{n}\right)^n$ (since level of significance of each test of $H_0$ against $H_j$ is $\frac{\alpha}{n}$)
$$= \alpha - \binom{n}{2}\frac{\alpha^2}{n^2} + \binom{n}{3}\frac{\alpha^3}{n^3} - \ldots + (-1)^{n+1}\left(\frac{\alpha}{n}\right)^n$$
If $n$ is large the above expression approximately equals $1 - e^{-\alpha}$.

Hence when $n$ is large, to obtain an approximate level $\alpha$ test procedure we have to carry out each of the $n$ individual tests of $H_0$ against $H_j$ at level of significance $\frac{-\ln(1-\alpha)}{n}$.

# Chapter 4

# Comparison of the Various Procedures for Identifying an Outlier

## 4.1 Introduction

In this chapter we provide a simulation based comparison of the various procedures for identifying an outlier. The statistics which are included in this comparison are the $L$-statistic (Collet 1980), $M$-statistic (Mardia, 1975), Bayes'-statistic, LRT-statistic and the LMPTT-statistic. The results of this chapter will help practitioners in choosing the most effective test-statistic for detecting outliers in a given situation.

## 4.2 The L-statistic and the M-statistic

THE L-STATISTIC

Suppose that $\theta_k$ is the observation with the greatest angular deviation from the sample mean direction. Then L-statistic is defined as

$$L = (R_k + 1)\,\hat{\kappa}_k - \hat{\kappa}R - n\ln\left\{\frac{I_0(\hat{\kappa}_k)}{I_0(\hat{\kappa})}\right\}$$

where $\hat{\kappa}$ is the usual maximum likelihood estimator of $\kappa$ given by $A(\hat{\kappa}) = d\ln I_0(\hat{\kappa})/d\hat{\kappa} = R/n$, $R_k^2 = C_k^2 + S_k^2$ where $C_k$ and $S_k$ are the values of $C$ and $S$ based on the $n-1$ observations exclusive of $\theta_k$, and $\hat{\kappa}_k$ is such that $A(\hat{\kappa}_k) = (R_k+1)/n$.

The sampling distribution of the L-statistic is non-standard and the cut-off points for this statistic can be obtained through simulation. Collett (1980) mentions that the null distribution of the $L$-statistic is effectively independent of $\kappa$ for $\kappa \geq 2$.

## The M-statistic

The M-statistic is defined as

$$M = \frac{R_k^* - R^* + 1}{n - R^*},$$

where, $R_k^* = \max_i \{R_i^*\}$ and $R_i^*$ is the length of the resultant omitting $\theta_i$.

For sufficiently large $\kappa$, the null distribution of $M$-statistic tends to the null distribution of $n(n-1)b^{*2}$, where $b^* = \max_i \left\{|x_i - \bar{x}|/\sum(x_i - \bar{x})^2\right\}$ is the well-known statistic used for tests of discordancy in univariate normal samples (Collett, 1980). It is also mentioned in Collett(1980), that the estimated percentage points of the $M$-statistic is effectively independent of $\kappa$ for values of $\kappa \geq 2$. Upton(1993) provides in his Table 2 the 5% and 1% cut-off points of the $M$-statistic for various sample sizes.

## 4.3 Performance of Different Test Statistics

For the purpose of comparison, 1000 samples of size 10 each containing one outlier are generated such that nine observations are drawn from CN$(0, \kappa)$ while the outlying observation is drawn from CN$(\mu_1, \kappa)$. $\mu_1$ is varied to measure the effectiveness of the procedures in detecting outliers of different severity. For $L$, $M$, LRT and LMPTT statistics we record the frequency of acceptance of the null hypothesis, the alternative hypothesis and also the number of times the correct observation is identified as the outlier. For the Bayes'-statistic we also need to specify the prior probability $p$ of any of the observations being an outlier. Since the performance of the Bayes'-statistic is seen to depend on the value of $p$ we examine the performance of the test for several values of $p$. The study is conducted for two values of $\kappa$, namely, 2 and 4. The results of these investigations are given in Tables I, II, III and IV below.

There are several well known criteria for comparing tests of outliers, (Barnett & Lewis, 1994). We compare the above test statistics based on $P_1$(the power of the test) and $P_3$(probability that presence of outlier in signaled and the outlier is correctly identified). A good test will have high $P_1$ and low $P_1 - P_3$(which is the probability that the test wrongly identifies a good observation as an outlier). Based on the these criteria, we see from the Tables I to IV that the LMPTT performs best for small values of $\mu_1$, which are most difficult to detect, as expected. The LRT performs best for moderate values of $\mu_1$ and the Bayes'-statistic performs best for large values of $\mu_1$ which could be important in a practical case. We also note that the Bayes'-statistic gives increasingly better result with larger value of $p$, which is expected since the data set actually contains an outlier.

The Bayes' rule with $p = p_0, p_0$ such that $1 - np_0 = 0.5$ may be used in cases when there are no prior information about an outlier being present in the data set. Since the efficacy of the Bayes' rule in detecting the presence of an outlier is seen to increase with $p$, a higher value of $p$ should be specified in cases where presence of outlier is suspected. The LMPTT appears to perform slightly better than the LRT when outliers of lesser severity is sought to be detected (i.e. $\mu_1$ is small) which is expected due to the nature of the LMPTT. However if we are interested in detecting outliers of moderate to large severity the LRT performs much better than the LMPTT. Moreover, we note that the Bayes'-statistic, the LRT and the LMPTT all perform better than the tests based on the $L$-statistic and $M$-statistic.

TABLE I. Performance of the Bayes' Test for Outlier
($\kappa = 2, n = 10, p = .05, .06$ & $.07$)

| $\mu_1$ (in degrees) | $H_0$ not rejected | | | $H_1$ Accepted : Correct Obsn. | | | $H_1$ Accepted : Wrong Obsn. | | |
|---|---|---|---|---|---|---|---|---|---|
| | p=.05 | .06 | .07 | p=.05 | .06 | .07 | p=.05 | .06 | .07 |
| 15 | 1000 | 1000 | 1000 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 1000 | 1000 | 1000 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 1000 | 1000 | 731 | 0 | 0 | 94 | 0 | 0 | 175 |
| 60 | 1000 | 735 | 373 | 0 | 121 | 294 | 0 | 144 | 333 |
| 75 | 754 | 453 | 271 | 146 | 295 | 395 | 100 | 252 | 334 |
| 90 | 527 | 343 | 204 | 315 | 420 | 506 | 158 | 237 | 290 |
| 120 | 343 | 213 | 136 | 533 | 625 | 662 | 124 | 162 | 202 |
| 150 | 240 | 157 | 97 | 666 | 722 | 765 | 94 | 121 | 138 |
| 180 | 214 | 125 | 84 | 712 | 763 | 785 | 84 | 112 | 131 |

TABLE II. Performance of the Bayes' Test for Outlier
($\kappa = 4, n = 10, p = .05, .06$ & $.07$)

| $\mu_1$ (in degrees) | $H_0$ not rejected | | | $H_1$ Accepted : Correct Obsn. | | | $H_1$ Accepted : Wrong Obsn. | | |
|---|---|---|---|---|---|---|---|---|---|
| | p=.05 | .06 | .07 | p=.05 | .06 | .07 | p=.05 | .06 | .07 |
| 15 | 1000 | 1000 | 1000 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 1000 | 898 | 650 | 0 | 45 | 153 | 0 | 57 | 197 |
| 45 | 692 | 540 | 372 | 207 | 271 | 353 | 101 | 189 | 275 |
| 60 | 457 | 372 | 232 | 423 | 477 | 561 | 120 | 151 | 207 |
| 75 | 319 | 231 | 181 | 595 | 646 | 695 | 86 | 123 | 124 |
| 90 | 200 | 165 | 128 | 741 | 779 | 801 | 59 | 56 | 71 |
| 120 | 89 | 63 | 54 | 888 | 913 | 928 | 23 | 24 | 18 |
| 150 | 53 | 23 | 26 | 940 | 970 | 963 | 7 | 7 | 11 |
| 180 | 23 | 19 | 16 | 974 | 974 | 980 | 3 | 7 | 4 |

TABLE III. Performance of the LRT, LMPTT, L & M test for Outlier ($\kappa = 2, n = 10$)

| $\mu_1$ (in degrees) | $H_0$ not rejected | | | | $H_1$ Accepted : Correct Obsn. | | | | $H_1$ Accepted : Wrong Obsn. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | LRT | LMPTT | L | M | LRT | LMPTT | L | M | LRT | LMPTT | L | M |
| 15 | 955 | 948 | 948 | 965 | 6 | 14 | 9 | 4 | 39 | 38 | 43 | 31 |
| 30 | 910 | 943 | 938 | 965 | 24 | 21 | 11 | 3 | 66 | 36 | 51 | 32 |
| 45 | 935 | 940 | 930 | 970 | 18 | 24 | 12 | 4 | 47 | 36 | 58 | 26 |
| 75 | 906 | 924 | 902 | 964 | 45 | 44 | 30 | 21 | 49 | 32 | 68 | 15 |
| 90 | 877 | 922 | 893 | 966 | 78 | 41 | 41 | 27 | 45 | 37 | 66 | 7 |
| 120 | 766 | 936 | 823 | 951 | 181 | 25 | 89 | 44 | 53 | 39 | 88 | 5 |
| 150 | 698 | 938 | 732 | 934 | 253 | 13 | 184 | 64 | 49 | 49 | 84 | 2 |
| 180 | 681 | 945 | 743 | 912 | 271 | 5 | 170 | 84 | 48 | 50 | 87 | 4 |

TABLE IV. Performance of the LRT, LMPTT, L & M test for Outlier ($\kappa = 4, n = 10$)

| $\mu_1$ (in degrees) | $H_0$ not rejected | | | | $H_1$ Accepted : Correct Obsn. | | | | $H_1$ Accepted : Wrong Obsn. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | LRT | LMPTT | L | M | LRT | LMPTT | L | M | LRT | LMPTT | L | M |
| 15 | 931 | 936 | 953 | 969 | 7 | 19 | 5 | 3 | 62 | 45 | 42 | 28 |
| 30 | 929 | 929 | 952 | 964 | 27 | 38 | 12 | 8 | 44 | 33 | 36 | 28 |
| 45 | 895 | 883 | 942 | 958 | 53 | 81 | 26 | 19 | 52 | 36 | 32 | 23 |
| 75 | 726 | 739 | 873 | 920 | 219 | 242 | 112 | 14 | 55 | 19 | 15 | 06 |
| 90 | 562 | 689 | 782 | 846 | 400 | 289 | 212 | 153 | 38 | 22 | 06 | 01 |
| 120 | 236 | 743 | 578 | 706 | 734 | 222 | 421 | 299 | 30 | 35 | 01 | 0 |
| 150 | 43 | 890 | 404 | 581 | 947 | 73 | 596 | 419 | 10 | 37 | 0 | 0 |
| 180 | 16 | 934 | 324 | 508 | 979 | 15 | 676 | 492 | 5 | 51 | 0 | 0 |

# Chapter 5

# Change Point Problem for Mean Direction in $CN(\mu, \kappa)$-$\kappa$ Known

## 5.1 Introduction

In this chapter, we look at the change point problem for the mean direction of the circular normal distribution using likelihood ratio techniques and give examples.

We discuss the problem of testing the null hypothesis that there is no change in the mean direction against the alternative hypothesis that there is one change in the mean direction. We assume that the concentration parameter $\kappa$ is known.

In section 5.2 we consider the case when both the initial mean direction and the possibly changed mean direction are known. We derive the LRT for this case and also derive its asymptotic null distribution. In section 5.3 we consider the case when the initial mean direction is known but the possibly changed mean direction is unknown. We derive the LMPTT for this case and also derive its asymptotic null distribution. In section 5.4 we consider the case when both the initial and the possibly changed mean directions are unknown. The form of the LRT is derived. The cut-off values and the power of the LRT is obtained by simulation. The results are discussed in section 5.5. The results of a simulation based sensitivity analysis for possibly mis-specified $\kappa$ is discussed in section 5.6. The generalization to the case of multiple change points is discussed in section 5.7. In section 5.8 we provide a multiple testing approach to the change point problem. The method discussed is quite general and can be applied in a variety of situations.

## 5.2 Change Point Problem for the Mean Direction - All Parameters Known

Let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be independent random variables. In this section we consider the case when the initial mean direction, the possibly changed mean direction as well as the concentration parameter are all known. Since the initial mean direction is known we can then w.n.l.g. assume that the initial mean direction is the zero direction. We want to test $H_0 : \Theta_1, \ldots, \Theta_n$ are distributed as CN$(0, \kappa)$, against $H_1 : \Theta_1, \ldots, \Theta_r$ are distributed as CN$(0, \kappa)$ and $\Theta_{r+1}, \ldots,$ $\Theta_n$ are distributed as CN$(\mu_1, \kappa)$ for some $r, 1 \leq r \leq n-1, \kappa > 0, 0 < \mu_1 < 2\pi$ are both known.

$$\begin{aligned} \text{Let } \nu &= \mathrm{E}\left[\sin(\Theta_i - \frac{\mu_1}{2})\right] \text{ (under } H_0) \\ &= -\sin\left(\frac{\mu_1}{2}\right) A(\kappa) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \tau^2 &= \mathrm{Var}\left[\sin\left(\Theta_i - \frac{\mu_1}{2}\right)\right] \text{ (under } H_0) \\ &= \cos^2\frac{\mu_1}{2} - \frac{I_0''(\kappa)}{I_0(\kappa)}\cos\mu_1 - \sin^2\frac{\mu_1}{2}A^2(\kappa) \end{aligned} \tag{5.2}$$

where $A(\kappa) = \frac{I_0'(\kappa)}{I_0(\kappa)}$ and $I_0'(\kappa)$ and $I_0''(\kappa)$ are the first and second derivatives of $I_0(\kappa)$ with respect to $\kappa$ respectively.

We will denote by $B_0^*(t)$ the time reversed Brownian Motion on [0,1] with drift 0 and diffusion coefficient 1. Further let,

$$S_l^k(\eta, \alpha) = \sum_{i=l}^{k} \sin(\eta_i - \alpha) \tag{5.3}$$

and

$$C_l^k(\eta, \alpha) \;=\; \sum_{i=l}^{k} \cos(\eta_i - \alpha) \tag{5.4}$$

Then we have the following theorem.

**Theorem 7 :** *In testing $H_0$ against $H_1$, the* LRT-*statistic is equivalent to*

$$\tilde{\Lambda} = \max_r \; S_{r+1}^n\left(\Theta, \frac{\mu_1}{2}\right). \tag{5.5}$$

*The asymptotic null distribution of*

$$\max_r \frac{S_{r+1}^n\left(\Theta, \frac{\mu_1}{2}\right) \; - (n-r)\nu}{n^{1/2}\;\tau}$$

*is the same as that of* $\sup\limits_{0 \le t \le 1} B_0^*(t)$.

**Proof:** Let $\theta_1, \ldots, \theta_n, 0 \le \theta_i < 2\pi$ be the given observations. We apply the likelihood ratio test. Fix $r$, $1 \le r \le n-1$. Under $H_0$ the likelihood is,

$$f_0(\theta_1, \ldots, \theta_n) = (2\pi I_0(\kappa))^{-n} \exp(\kappa C_1^n(\theta, 0))$$

and under the alternative we have

$$f_1(\theta_1, \ldots, \theta_n) = (2\pi I_0(\kappa))^{-n} \exp\left\{\kappa\left(C_1^r(\theta, 0) + C_{r+1}^n(\theta, \mu_1)\right)\right\}$$

Then the LRT test statistic for fixed $r$ is,

$$\Lambda_r = \exp\left[\kappa\left\{C_{r+1}^n(\Theta, 0) \;\; - \;\; C_{r+1}^n(\Theta, \mu_1)\right\}\right]$$

If $r$ is unknown we estimate it by $\hat{r}$ where $\hat{r}$ is that $r$ for which $\Lambda_r$ is minimum. Define $\Lambda = \Lambda_{\hat{r}}$. Then the rule is to reject $H_0$ if $\Lambda < c$ for some constant $c$

depending on the level of significance. Note that the test $\Lambda < c$ is equivalent to

$$S_{r+1}^{n}\left(\Theta, \frac{\mu_1}{2}\right) > k.$$

Let

$$\widetilde{\Lambda} = \max_{r} \; S_{r+1}^{n}\left(\Theta, \frac{\mu_1}{2}\right).$$

Also, note that under the null hypothesis the random variables

$$\sin\left(\Theta_i - \frac{\mu_1}{2}\right)$$

are i.i.d with finite expectation $\nu$ and finite variance $\tau^2$. Then by an application of the functional central limit theorem (Bhattacharya & Waymire, (1992)) we get that,

$$\frac{S_{r+1}^{n}\left(\Theta, \frac{\mu_1}{2}\right) - (n-r)\nu}{n^{1/2}\tau}$$

converges in distribution on $D[0,1]$ (the set of all functions on [0,1] that are right continuous and possess left hand limits at each point) to the time reversed Brownian motion in the interval [0,1] with drift 0 and diffusion coefficient 1 as $n \to \infty$. Since sup is a continuous function on $D[0,1]$ we see that under the null hypothesis

$$\max_{r} \frac{S_{r+1}^{n}(\Theta, \frac{\mu_1}{2}) - (n-r)\nu}{n^{1/2}\tau}$$

$$= \sup_{\frac{1}{n} \leq t \leq 1-\frac{1}{n}} \frac{S_{[nt]+1}^{n}(\Theta, \frac{\mu_1}{2}) - (n-[nt])\nu}{n^{1/2}\tau}$$

which converges in distribution to $\sup_{0 \leq t \leq 1} B_0^*(t)$ as $n \to \infty$.

**Remark 5 :** *The exact sampling distribution of $\Lambda$ is formidable. However the cut-off points and P-values based on the asymptotic distribution can be* *computed using the result given below. Hence, in practice, the test may be carried out easily for large samples.*

**Result :**

$$Pr\left(\sup_{0\leq t\leq 1} B_0^*(t) \geq \alpha\right) = 2Pr(Z \geq \alpha) \tag{5.6}$$

where $\alpha > 0$ and $Z \sim N(0,1)$.

**Proof** : $B_0^*(t) = B_0(1-t)$ where $B_0(.)$ is the standard Brownian Motion.

$$\begin{aligned} & Pr\left(\sup_{0\leq t\leq 1} B_0^*(t) \geq \alpha\right) \\ = \; & Pr\left(\sup_{0\leq t\leq 1} B_0(1-t) \geq \alpha\right) \\ = \; & Pr\left(\sup_{0\leq t\leq 1} B_0(t) \geq \alpha\right) \\ = \; & 2\ Pr\left(B_0(1) \geq \alpha\right) \quad \text{(see e.g. Billingsley, 1991, page 529)} \\ = \; & 2\ Pr(Z \geq \alpha). \end{aligned}$$

The next theorem gives an alternative statistic based on the LRT whose asymptotic distribution can be easily obtained. Let

$$\begin{aligned} u(n,t) &= 2\ln\ln(n-1)^{\frac{1}{2}} + \frac{\ln\ln\ln(n-1)}{2(2\ln\ln(n-1))^{\frac{1}{2}}} \\ &+ \frac{t}{(2\ln\ln(n-1))^{\frac{1}{2}}} \end{aligned} \tag{5.7}$$

**Theorem 8 :** *Let*

$$U = \max_{1\leq r\leq (n-1)} \frac{S_{r+1}^n(\Theta, \frac{\mu_1}{2}) - (n-r)\nu}{(n-r)^{1/2}\ \tau}$$

*Then as* $n \longrightarrow \infty$,

$$Pr(U < u(n,t)) \longrightarrow \exp\left(-\frac{1}{2\sqrt{\pi}}\exp(-t)\right)$$

**Proof :** The proof follows immediately on application of the results in Darling and Erdos(1956).

## 5.3 Change Point Problem for the Mean Direction - Changed Direction Unknown

In this section we look at the change point problem with $\mu_1$ unknown. We derive an LMPTT in this case. For some fixed $r, 1 \le r \le n-1$, let us denote by $H_{1r}$ the following hypothesis $H_{1r} : \Theta_1, \ldots, \Theta_r$ are distributed as $\mathrm{CN}(0,\kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are distributed as $\mathrm{CN}(\mu_1, \kappa)$, where $0 < \mu_1 < 2\pi$. Further let

$$\begin{aligned} \sigma^2 &= \mathrm{Var}(\sin\Theta_i) \text{ (under } H_0) \\ &= 1 - \frac{I_0''(\kappa)}{I_0(\kappa)} \end{aligned} \tag{5.8}$$

Then we have the following :

**Theorem 9 :** *(a) In testing* $H_0$ *against* $H_{1r}$ *the* LMP*-test is given by :*

$$\textit{Reject } H_0 \ \textit{ if } S_{r+1}^n(\Theta, 0) > c$$

*for some constant* $c$.
*(b) The* LMPTT *of* $H_0$ *against* $H_1$ *is based on the statistic* $\max_r S_{r+1}^n(\Theta,0)$. *Under the null hypothesis the asymptotic null distribution of*

$$\max_r \frac{S_{r+1}^n(\Theta,0)}{\sqrt{n}\,\sigma}$$

*is the same as that of* $\sup_{0\le t\le 1} B_0^*(t)$.

**Proof :** (a) Let us fix $r, 1 \le r \le n-1$. Let $\theta_1, \ldots, \theta_n,\ 0 \le \theta_i < 2\pi$ denote the observations. Observe the log-likelihood is

$$l\left(\mu_1; \theta_1, \ldots, \theta_n\right) = \text{ const } + \kappa \left\{C_1^r(\theta, 0) + C_{r+1}^n\left(\theta, \mu_1\right)\right\}$$

$$\text{Then } \frac{dl}{d\mu_1} = \kappa S_{r+1}^n\left(\theta, \mu_1\right) \text{ and hence } \frac{dl}{d\mu_1}(0) = \kappa S_{r+1}^n(\theta, 0)$$

Thus the LMP-test statistic for testing $H_0$ against $H_{1r}$ is $S_{r+1}^n(\Theta, 0)$ (since $\kappa$ is known). We reject $H_0$ if $S_{r+1}^n(\Theta, 0) > c$ for some constant $c$.

(b) Imitating the proof of Theorem 7 above we get that under the null hypothesis as $n \to \infty, \max\limits_r \dfrac{S_{r+1}^n(\Theta, 0)}{\sqrt{n}\ \sigma}$ converges in distribution (in Skorohod topology sense on $D[0,1]$) to $\sup\limits_{0 \le t \le 1} B_0^*(t)$.

**Remark 6 :** *The* LRT *is easily computable but doesn't have a tractable asymptotic null distribution. The form of the* LRT *for the more general case is shown in the next section – the asymptotic null distribution of which is also not tractable.*

## 5.4 Change Point Problem for the Mean Direction - Initial and Changed Direction Unknown

In this section we look at the change point problem with both $\mu_0$ and $\mu_1$ unknown. We propose the LRT for this problem. Let ${}_0\hat{\mu}_0$, ${}_1\hat{\mu}_0$ denote the estimate of $\mu_0$ under $H_0$ and $H_1$ respectively and ${}_1\hat{\mu}_1$, denote the estimate of $\mu_1$ under $H_1$.

**Theorem 10 :** *In testing $H_0$ against $H_1$ the* LRT*-statistic is $\Lambda = \min\limits_r \Lambda_r$ where*

$$\Lambda_r = \exp\left[\kappa\left\{C_1^n\left(\Theta,\ {}_0\hat{\mu}_0\right) - C_1^r\left(\Theta,\ {}_1\hat{\mu}_0\right) - C_{r+1}^n\left(\Theta,\ {}_1\hat{\mu}_1\right)\right\}\right]$$

**Proof :** Let us fix $r, 1 \leq r \leq n-1$. Then the log-likelihood under $H_0$ is

$$\ln\ f_0 = -n\ \ln(2\pi I_0(\kappa)) + \kappa C_1^n(\theta, \mu_0)$$

Putting $\frac{\partial \ln f_0}{\partial \mu_0} = 0$ we get $S_1^n(\theta, \mu_0) = 0$ which is equivalent to $\tan \mu_0 = \frac{S_1^n(\theta,0)}{C_1^n(\theta,0)}$. The estimate of $\mu_0$ under $H_0$ is,

$$_0\hat{\mu}_0 = \tan^{-1} \frac{S_1^n(\theta, 0)}{C_1^n(\theta, 0)}. \tag{5.9}$$

Under the alternative we have,

$$\ln\ f_1 = -n\ \ln\left(2\pi I_0(\kappa)\right) + \kappa \left\{C_1^r(\theta, \mu_0) + C_{r+1}^n\left(\theta, \mu_1\right)\right\}$$

Putting $\frac{\partial \ln f_1}{\partial \mu_0} = 0$ we get $S_1^r(\theta,\ \mu_0) = 0$ which yields the estimate of $\mu_0$ under $H_1$ as

$$_1\hat{\mu}_0 = \tan^{-1}\left(\frac{S_1^r(\theta, 0)}{C_1^r(\theta, 0)}\right) \tag{5.10}$$

and putting $\frac{\partial \ln f_1}{\partial \mu_1} = 0$ we get $S_{r+1}^n(\theta, \mu_1) = 0$ from which we get the estimate of $\mu_1 0$ under $H_1$ as

$$_1\hat{\mu}_1 = \tan^{-1}\left(\frac{S_{r+1}^n(\theta, 0)}{C_{r+1}^n(\theta, 0)}\right) \tag{5.11}$$

Thus the LRT statistic for testing $H_0$ against $H_{1r}$ is,

$$\Lambda_r = \exp\left[\kappa\left\{C_1^n(\Theta,\ {}_0\hat{\mu}_0) - C_1^r(\Theta,\ {}_1\hat{\mu}_0) - C_{r+1}^n(\Theta,\ {}_1\hat{\mu}_1)\right\}\right]$$

When $r$ is unknown we estimate it by $\hat{r}$, where $\hat{r}$ is that $r$ for which $\Lambda_r$ is minimum. Define $\Lambda \equiv \Lambda_{\hat{r}} = \min_r\ \Lambda_r$ which is the LRT-statistic.

**Remark 7** : *The exact sampling distribution of* $\Lambda$ *is formidable. The large sample distribution of* LRT *does not work here because of the violation in the assumptions (for example, the parameter space here is not an open set) of the relevant theorem.*

## 5.5 Simulation

We use simulation to compute the null distribution (see Tables 5 and 6, Chapter 14) of the LRT-statistic given in Theorem 10 as well as the power (see Table 7, Chapter 14) of the test. The null distribution results are based on 5000 repetitions with sample size $n, n = 10, 20, 30$ drawn from $\mathrm{CN}(0,1)$ distribution. The power is simulated by drawing the first $r$ observations from $\mathrm{CN}(0,1)$ and the rest $n - r$ from $\mathrm{CN}(\Delta, 1), \Delta = 10(10)180$ (in degrees) and repeating it 5000 times, where $r = 5$. Since the power function is symmetric about $\pi$ the above range for $\Delta$ is sufficient. By looking at the null distribution it is seen that it gets increasingly concentrated towards low values as sample size increases. The same feature is also observed as the value of $\kappa$ increases.

Observe that the estimation of $\mu_0$ and $\mu_1$ under $H_1$ depends on the exact location of the change point. We vary $r$ (the location of the change point) in a limited way ($r = 5, 9, 13, 17$) keeping $n$ ($= 20$) fixed to study it. It is seen (see Table 8, Chapter 14) that the power of the LRT is maximum if the change point is near the middle than at the extremes and this phenomenon becomes increasingly pronounced with the increase in the magnitude of change. In this context it is worthwhile to note that for the linear case Hinkley's (1970) derivation of the asymptotic distribution of the change point estimator assumes that the change point is not too close to 1 or $n$. As expected the power is seen to increase with increase in sample size.

We also use simulation to determine whether the LRT is consistent. We fix $\kappa = 1$, the level of the test to be at 5% and the change point at 20. We then generate the first 20 observations from $\mathrm{CN}(0,1)$ and the remaining $n - 20$ observations from $\mathrm{CN}(\Delta, 1), \Delta = 45°, 90°, 135°, 180°$ and repeat the procedure 1000 times. This is done for $n = 30, 40, 50$ and 75. The results are given in Table 9 of Chapter 14. We find from the table that the power of the LRT increases. While for small $\Delta$ this increase is slow with $n$, for large $\Delta$ it rapidly approaches 1 as $n$ increases. This indicates that the LRT may indeed be a consistent test.

## 5.6 Sensitivity Analysis

The LRT statistic is derived under the assumption that the value of $\kappa$ is known. But in real life applications the value of $\kappa$ is usually unknown. Thus to apply the LRT in real life situations we need to specify a value of $\kappa$ which may be estimated from the data or arrived at from other considerations. Since the specified value of $\kappa$ may not be exactly equal to the actual value of $\kappa$, it is important to study the sensitivity of the level of significance and the power of the LRT to the possibly mis-specified value of $\kappa$.

In Table I and Table II below we give the results of a simulation based sensitivity analysis for different values of true and also specified $\kappa$. To study the sensitivity of the level of significance we generate 1000 random samples of size 10 each from $\text{CN}(0, \kappa)$ where $\kappa$ is the true value of the concentration parameter. The LRT statistic is computed using the specified value of $\kappa$. The cut-off points used are such that the level of significance of LRT is 5% when the true value of $\kappa$ is known. In Table I, the actual level of significance of the LRT is given for different combinations of true and specified $\kappa$.

To study the sensitivity of the power of the LRT we generate 1000 random samples of size 10 each. In each sample of size 10, the first five observations come from $\text{CN}(0, 2)$ and the next five observations come from $\text{CN}(\Delta, 2)$ where $\Delta = 45°, 90°, 135°, 180°$. The LRT statistic is computed using the specified value of $\kappa$. The cut-off point used is that of an LRT with level of significance 5% when the value of $\kappa$ is known to be 2. In Table II, the power of the LRT is given for different combinations of specified value of $\kappa$ and $\Delta$.

TABLE I Sensitivity Analysis of the Level of Significance of the LRT in change point problem ($n = 10$)

| True $\kappa$ | Specified $\kappa$ 0.5 | 1.0 | 1.5 | 2.0 | 4.0 | 10.0 |
|---|---|---|---|---|---|---|
| 0.5 | 0.050 | 0.144 | 0.299 | 0.486 | 0.852 | 0.977 |
| 1.0 | 0.020 | 0.050 | 0.144 | 0.275 | 0.675 | 0.940 |
| 1.5 | 0.002 | 0.011 | 0.050 | 0.135 | 0.521 | 0.888 |
| 2.0 | 0.000 | 0.006 | 0.014 | 0.050 | 0.322 | 0.806 |
| 4.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.050 | 0.440 |
| 10.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.050 |

TABLE II Sensitivity Analysis of the Power of the LRT for change point problem ($n = 10$, $\kappa = 2$, change point at 5)

| Specified $\kappa$ | $\Delta$ 45° | 90° | 135° | 180° |
|---|---|---|---|---|
| 0.5 | 0.006 | 0.125 | 0.562 | 0.805 |
| 1.0 | 0.031 | 0.249 | 0.730 | 0.924 |
| 1.5 | 0.085 | 0.505 | 0.899 | 0.979 |
| 2.0 | 0.186 | 0.624 | 0.957 | 0.989 |
| 4.0 | 0.578 | 0.933 | 0.998 | 0.998 |
| 10.0 | 0.904 | 0.993 | 1.000 | 1.000 |

From Tables I and II we find that both the level of significance and the power of the LRT are sensitive to the variations in the values of $\kappa$. If the specified value of $\kappa$ is less than the true value then we find that the LRT is conservative but if the specified value of $\kappa$ is greater than the true value then the misspecification leads to anti-conservative nature of LRT.

We now study the sensitivity of the level and the power of the LRT when the underlying distribution is a skewed circular distribution like, Rattihali-SenGupta's skewed circular distribution RS($k_1, k_2, \mu$), using simulation. 20

random observations $\theta_1, \ldots, \theta_{20}$ are generated of which $\theta_1, \ldots, \theta_{10}$ are from RS(0.9, 1.1, 0.7854), and for $i = 11, \ldots, 20$, $\theta_i = \theta_i^* + \Delta \mod 2\pi$, where $\theta_i^*$ is a random observation from RS(0.9, 1.1, 0.7854) and $\Delta$ is varied over $0°, 45°, 90°, 135°$ and $180°$ as explained below. Since the value of the measure of concentration $\rho$ for RS(0.9, 1.1, 0.7854) is quite close to that of a circular normal distribution with $\kappa = 1$, the LRT with $\kappa = 1$ is used. The nominal level of significance of the test is fixed at 5%. For each $\Delta$ the above is repeated 1000 times and the power of the test noted. Note that the power when $\Delta = 0$ is the actual level of significance of the test. The results are given in Table III below.

TABLE III Power of the LRT for change point problem in RS(0.9, 1.1, 0.7854) ($n = 20$, change point at 10)

| $\Delta$ | Power |
|---|---|
| 0° | 0.059 |
| 45° | 0.069 |
| 90° | 0.068 |
| 135° | 0.054 |
| 180° | 0.050 |

We find that the actual level of significance of the test is close to the nominal value but the power of the LRT is substantially reduced when the underlying distribution is the Rattihali-SenGupta's skewed circular distribution.

## 5.7 Generalization

In case the data are suspected of having more than one change point, the above procedure can be easily modified as follows. Consider the case when we suspect that the data may have upto two change points. The data set of 60 successive flare launches analyzed by Lombard (1986) is of this kind. In this case we take $\Lambda$ to be the minimum of

$$\Lambda_r, 1 \leq r \leq n-1 \text{ and } \Lambda_{r_1,r_2}, 1 \leq r_1 \leq r_2 \leq n-1$$

where $\Lambda_{r_1,r_2}$ is the likelihood ratio test statistic with suspected change points at $r_1$ and $r_2$. In case we do not know the number of change points in advance but have a prespecified upper bound 'k', then also we can suitably modify the above scheme. We will take

$$\Lambda = \min\{\Lambda_r, \Lambda_{r_{21},r_{22}}\ \Lambda_{r_{31},r_{32},r_{33}}, \ldots, \Lambda_{r_{k1},r_{k2},\ldots,r_{kk}}\}$$

where

$$1 \leq r \leq n-1, 1 \leq r_{21} < r_{22} \leq n-1, 1 \leq r_{31} < r_{32} < r_{33} \leq n-1 \text{ etc.}$$

## 5.8 Multiple Testing Approach for Change Point Problem

Here we suggest a multiple testing approach for the change point problem. As will be seen the approach is completely general and can be used in a variety of situations. We note that we can reject $H_0$ if even one of the (n - 1) tests of $H_0$ against $H_{1r}, 1 \leq r \leq n-1$ turn out to be significant. The problem is to choose an appropriate level of these tests so that the overall procedure does not exceed desired level of significance say $\alpha$. One simple solution is to use $\frac{\alpha}{n-1}$ as the level of significance of the individual tests of $H_0$ against $H_{1r}$. By Bonferroni's inequality it is easy to see that such a procedure will have level atmost $\alpha$. However, unlike in the outlier problem the test statistics for testing $H_0$ against $H_{1r}$ are usually correlated. Due to this reason simple solutions such as using $\frac{\alpha}{n-1}$ as the level of significance of tests of $H_0$ against $H_{1r}$ usually yield very conservative procedures. Alternative multiple test procedures may be derived by using the joint distribution of the test statistics and obtaining the cut-off values of the individual tests from it.

# Chapter 6

# Change Point Problem for the Mean Direction in CN$(\mu, \kappa)$- $\kappa$ Unknown

## 6.1 Introduction

In the previous chapter we have looked at the change point problem in CN$(\mu, \kappa)$ for the mean direction $\mu_1$ assuming that the concentration parameter $\kappa$ is known. However, in most practical situations the concentration parameter is not known in advance. In this chapter we look at the change point problem for the mean direction when the concentration parameter $\kappa$ is unknown. We note that the usual methods of removal of nuisance parameters like use of sufficiency, invariance, similarity etc. do not work here. Moreover use of conditional arguments is not always desirable as Laycock(1975) observes"... for circular data this is arguably equivalent to discarding half the available information in the data". In section 6.2 we assume that the initial mean direction as well as the possibly changed mean direction are known and derive the LRT. In section 6.3 we consider the same problem but with the changed mean direction unknown and propose a conditional test. In section 6.4 we consider the practically most important case when both the initial mean direction as well as the possibly changed mean direction are unknown. We introduce the notion of NRTT and derive it. The behaviour of NRTT is studied through simulations and the results are given in section 6.5.

In section 6.6 we provide examples of the use of the techniques discussed in this and the previous chapter by analysing two real life data sets.

## 6.2 Change Point Problem for the Mean Direction - Initial and Changed Direction Known

Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. In this section we consider the case when the initial mean direction and the possibly changed mean direction are both known. Thus then concentration parameter is the only parameter unknown. Since the initial mean direction is known we can then w.n.l.g assume that the initial mean direction is the zero direction. We want to test $H_0 : \Theta_1, \ldots, \Theta_n$ are distributed as $\mathrm{CN}(0, \kappa)$, against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are distributed as $\mathrm{CN}(0, \kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are distributed as $\mathrm{CN}(\mu_1, \kappa)$ for some $r, 1 \leq r \leq n-1, \mu_1 > 0$ is known. For each fixed $r, 1 \leq r \leq n-1$ we will denote by $H_{1r}$ the hypothesis $H_{1r} : \Theta_1, \ldots, \Theta_r$ are distributed as $\mathrm{CN}(0, \kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are distributed as $\mathrm{CN}(\mu_1, \kappa), \mu_1 > 0$ known. Let $\hat{\kappa}_0$ and ${}_r\hat{\kappa}_1$ denote the MLE of $\kappa$ under $H_0$ and $H_{1r}$ respectively.

**Theorem 11** : *In testing $H_0$ against $H_1$ the* LRT-*statistic is*

$$\Lambda = \min_r \Lambda_r$$

*where*

$$\Lambda_r = \frac{I_0({}_r\hat{\kappa}_1)}{I_0(\hat{\kappa}_0)} \exp\left\{(\hat{\kappa}_0 - {}_r\hat{\kappa}_1)C_1^r(\theta, 0) + \hat{\kappa}_0 C_{r+1}^n(\theta, 0) - {}_r\hat{\kappa}_1 C_{r+1}^n(\theta, \mu_1)\right\} \tag{6.1}$$

*is the* LRT-*statistic for testing $H_0$ against $H_{1r}$.*

**Proof** : Let $\theta_1, \ldots, \theta_n$, $0 \leq \theta_i < 2\pi$ be the given observations. We apply the likelihood ratio test. Fix $r$, $1 \leq r \leq n-1$. Under $H_0$ the log-likelihood is,

$$\ell_0(\kappa; \theta_1, \ldots, \theta_n) = -n \ln 2\pi - n \ln I_0(\kappa) + \kappa C_1^n(\theta, 0) \tag{6.2}$$

Putting $\frac{\partial \ell_0}{\partial \kappa} = 0$ we get the MLE of $\kappa$ under $H_0$ as $\hat{\kappa}_0 = A^{-1}(\bar{C})$ where $\bar{C} = \frac{1}{n}C_1^n(\theta, 0)$ and $A(.) = \frac{I_0'(.)}{I_0(.)}$. Under the alternative $H_{1r}$ the likelihood is

$$\ell_r(\kappa; \theta_1, \ldots, \theta_n) = -n \ln 2\pi - n \ln I_0(\kappa) + \kappa C_1^r(\theta, 0) + \kappa C_{r+1}^n(\theta, \mu_1) \quad (6.3)$$

Putting $\frac{\partial \ell_r}{\partial \kappa} = 0$ we get the MLE of $\kappa$ under $H_0$ as ${}_r\hat{\kappa}_1 = A^{-1}\left\{\frac{r}{n}\bar{C}_1 + \left(1 - \frac{r}{n}\right)\bar{C}_2\right\}$ where $\bar{C}_1 = \frac{1}{r}C_1^r(\theta, 0)$, $\bar{C}_2 = \frac{1}{n-r}C_{r+1}^n(\theta, \mu_1)$, and $A(.) = \frac{I_0'(.)}{I_0(.)}$.

Then the LRT-statistic for testing $H_0$ against $H_{1r}$ is,

$$\Lambda_r = \frac{I_0({}_r\hat{\kappa}_1)}{I_0(\hat{\kappa}_0)} \exp\left\{(\hat{\kappa}_0 - {}_r\hat{\kappa}_1)C_1^r(\theta, 0) + \hat{\kappa}_0 C_{r+1}^n(\theta, 0) - {}_r\hat{\kappa}_1 C_{r+1}^n(\theta, \mu_1)\right\}$$

When $r$ is unknown we estimate it by $\hat{r}$ where $\hat{r}$ is that $r$ for which $\Lambda_r$ is minimum. Define $\Lambda = \Lambda_{\hat{r}}$. Then the LRT for testing $H_0$ against $H_1$ is given by :

$$\text{Reject } H_0 \text{ if } \Lambda < c$$

for some constant $c$ depending on the level of significance.

**Remark 8 :** *The exact sampling distribution of $\Lambda$ is formidable. The cut-off points required for carrying out the test can be easily obtained through simulation.*

**Remark 9 :** *Let $Z_n = \max_{1 \le r \le n-1}(-2 ln \Lambda_r)$. The asymptotic null distribution of $Z_n^{\frac{1}{2}}$, can be found by an application of Theorem 1.3.2 of Csorgo & Horvath (1997). This can be used for obtaining the cut-off points if we have a large sample. In this context note that all the assumptions of Theorem 1.3.2 hold good since we have assumed that both the initial and the changed mean directions are known. If any one of these is unknown then the assumptions of Theorem 1.3.2 do not hold good. (In fact, assumption C.4 does not hold.)*

## 6.3 Change Point Problem for the Mean Direction - Changed Direction Unknown

In this section we consider the case where the initial mean direction $\mu_0$ is known but the possibly changed direction $\mu_1$ is unknown. With no loss of generality $\mu_0$ is assumed to be 0. We propose a conditional test based on the LMP-test for $H_0$ against $H_{1r}$ for the case of known $\kappa$.

**Theorem 12** : *A test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } S^n_{r+1}(\Theta, 0) > c$$

*for some constant $c$ where the constant $c$ is chosen based on the conditional distribution of $S^n_{r+1}(\Theta, 0)$ given $C^n_1(\Theta, 0)$.*

**Proof** : Note that under $H_0$, $C^n_1(\theta, 0)$ is sufficient for $\kappa$. Hence the conditional distribution of $S^n_{r+1}(\Theta, 0)$ given $C^n_1(\Theta, 0) = t$ is free from $\kappa$. Hence a test for $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } S^n_{r+1}(\Theta, 0) > c$$

for some constant $c$ where the constant $c$ is chosen based on the conditional distribution of $S^n_{r+1}(\Theta, 0)$ given $C^n_1(\Theta, 0)$.

A test for $H_0$ against $H_1$ based on the above can be carried out using the multiple testing approach as follows :
Reject $H_0$ if any one of the $(n-1)$ tests of $H_0$ against $H_{1r}$, $1 \leq r \leq n-1$ is significant at $\frac{\alpha}{n-1}$ level.
This test has level of significance at most $\alpha$.

**Remark 10** *A test based on the LMP-test statistic (initially derived under the assumption that $\kappa$ is known) which is free from the nuisance parameter $\kappa$ upon conditioning will be called a LMP Conditional Type Test (LMPCTT).*

**Remark 11** : *The* LRT*-statistic can be easily computed. However in this case even for fixed $r$, the parameter space $(0, \infty) \times [0, 2\pi]$ is not an open set.*

*Therefore, the large sample approximation for the distribution of the* LRT-*statistic is not valid. Thus we have to use simulation or numerical techniques to get the cut-off points for carrying out the test. However this is complicated by the fact that $\kappa$ is unknown. One may start with the conditional likelihood given $C_1^n(\Theta, 0)$ which is free from $\kappa$. The LRT can then be derived using this conditional likelihood.*

## 6.4 Change Point Problem for the Mean Direction - Initial and Changed Direction both Unknown

In this section we deal with the case when the initial direction $\mu_0$, the changed direction $\mu_1$ and the concentration parameter $\kappa$ are all unknown. This is a very important case since in most practical situations $\mu_0, \mu_1$, and $\kappa$ are not known. We assume $\mu_1 > \mu_0$. In advance we rewrite $\mu_1$ as $\mu_0 + \delta$. Then the null hypothesis of no change can be rewritten as $H_0 : \delta = 0$ and the alternative hypothesis can be written as $H_1 : \delta > 0$. The construction of a test for $H_0$ against $H_1$ is complicated because of the presence of two nuisance parameters $\mu_0$ and $\kappa$. We use the Neyman-Rao(NR)-test (Hall and Mathiason, 1990) which is an asymptotically optimal test in the presence of nuisance parameters for this purpose. The NR-test is an extension of the well known $C_\alpha$ test (Neyman, 1959).

Let us first consider the problem of testing $H_0$ against $H_{1r}$ for a fixed $r$, $1 \leq r \leq n-1$. Since there is no cause for confusion we will denote $\mu_0$ simply by $\mu$. The log-likelihood of the observations will be denoted by $\ell(\delta, \mu, \kappa)$. We note that for the validity of the NR-test it is sufficient to check Assumption 4 of Hall and Mathiason(1990) which consist of three sub-parts 4(i), 4(ii) and 4(iii). It is simple to check that the log likelihood $\ell(\delta, \mu, \kappa)$ has a matrix of continuous second-order partial derivatives and this verifies Assumption 4(i). Now observe that in our case,

$$S_n(\delta) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \delta} = \frac{\kappa}{\sqrt{n}} S_{r+1}^n(\boldsymbol{\theta}, \mu + \delta).$$

$$S_n(\mu) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \mu} = \frac{\kappa}{\sqrt{n}}\{S_1^r(\boldsymbol{\theta}, \mu) + S_{r+1}^n(\boldsymbol{\theta}, \mu + \delta)\},$$

and

$$S_n(\kappa) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \kappa} = \frac{1}{\sqrt{n}}\{-nA(\kappa) + C_1^r(\boldsymbol{\theta}, \mu) + C_{r+1}^n(\boldsymbol{\theta}, \mu + \delta)\},$$

Under $H_0$, putting $\delta = 0$ in the above expressions we get,

$$S_n(\delta) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \delta} = \frac{\kappa}{\sqrt{n}} S_{r+1}^n(\Theta, \mu).$$

$$S_n(\mu) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \mu} = \frac{\kappa}{\sqrt{n}} S_n^1(\Theta, \mu),$$

and

$$S_n(\kappa) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \kappa} = \frac{1}{\sqrt{n}}\{-nA(\kappa) + C_1^n(\Theta, \mu)\}$$

Let us suppose that $n \longrightarrow \infty, r \longrightarrow \infty$ in such a way that $\frac{r}{n} \longrightarrow t$ where $t \neq 0, 1$. We shall use the Cramer-Wold technique to show the asymptotic joint normality of $(S_n(\delta), S_n(\mu), S_n(\kappa))$.

For any three real numbers $t_1, t_2, t_3$

$$\begin{aligned}
& t_1 S_n(\delta) + t_2 S_n(\mu) + t_3 S_n(\kappa) \\
= \; & \frac{t_1}{\sqrt{n}}\kappa \sum_{i=r+1}^{n} \sin(\Theta_i - \mu) \; + \frac{t_2\kappa}{\sqrt{n}} \sum_{i=1}^{n} \sin(\Theta_i - \mu) \\
& + \frac{t_3}{\sqrt{n}} \left\{ \sum_{i=1}^{n} \cos(\Theta_i - \mu) - nA(\kappa) \right\} \\
= \; & \sum_{i=1}^{r} \left[ \frac{t_2\kappa}{\sqrt{n}} \sin(\Theta_i - \mu) + \frac{t_3}{\sqrt{n}} \cos(\Theta_i - \mu) - \frac{t_3}{\sqrt{n}} A(\kappa) \right] \\
& + \sum_{i=r+1}^{n} \left[ \left( \frac{t_1\kappa}{\sqrt{n}} + \frac{t_2\kappa}{\sqrt{n}} \right) \sin(\Theta_i - \mu) + \frac{t_3}{\sqrt{n}} \cos(\Theta_i - \mu) - \frac{t_3}{\sqrt{n}} A(\kappa) \right]
\end{aligned}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{r} [t_2 \kappa \sin(\Theta_i - \mu) + t_3 \cos(\Theta_i - \mu) - t_3 A(\kappa)]$$
$$+ \frac{1}{\sqrt{n}} \sum_{i=r+1}^{n} [(t_1 + t_2)\, \kappa \sin(\Theta_i - \mu) + t_3 \cos(\Theta_i - \mu) - t_3 A(\kappa)]$$

$$\begin{aligned} \text{Let } Z_i &= t_2\ \kappa\ \sin(\Theta_i - \mu) + t_3\ \cos(\Theta_i - \mu) - t_3\ A(\kappa),\ 1 \le i \le r \\ &= (t_1 + t_2)\kappa\ \sin(\Theta_i - \mu) + t_3 \cos(\Theta_i - \mu) - t_3 A(\kappa), r+1 \le i \le n \end{aligned}$$

Now we note that $Z_i$'s are bounded independent random variables with $\mathrm{E}(Z_i) = 0, 1 \le i \le n$ and finite variances. Hence by the Lindeberg-Feller Central Limit Theorem we can conclude that $t_1 S_n(\delta) + t_2 S_n(\mu) + t_3 S_n(\kappa)$ is asymptotically normally distributed.

Thus we find that the distribution of any linear combination of $S_n(\mu), S_n(\kappa)$ and $S_n(\delta)$ is asymptotically normal and hence by the Cramer-Wold theorem we can conclude that the joint distribution of $(S_n(\delta), S_n(\mu), S_n(\kappa))$ is asymptotically trivariate normal.

Now since

$$\begin{aligned} E(S_n(\delta)) &= E(S_n(\mu)) = E(S_n(\kappa)) = 0, && (6.4) \\ V(S_n(\mu)) &= \kappa A(\kappa), && (6.5) \\ V(S_n(\kappa)) &= A'(\kappa), && (6.6) \\ V(S_n(\delta)) &= \kappa(1 - \frac{r}{n})A(\kappa), && (6.7) \\ \text{Cov } (S_n(\mu), S_n(\kappa)) &= 0, && (6.8) \\ \text{Cov } (S_n(\mu), S_n(\delta)) &= \kappa(1 - \frac{r}{n})A(\kappa), && (6.9) \\ \text{Cov } (S_n(\kappa), S_n(\delta)) &= 0 && (6.10) \end{aligned}$$

we have $(S_n(\delta), S_n(\kappa), S_n(\mu)) \longrightarrow N_3(\mathbf{0}, \mathbf{B})$ where the nonsingular symmetric matrix $\mathbf{B}$ has elements ,

$$\begin{aligned} b_{11} &= \kappa(1-t)A(\kappa), && (6.11) \\ b_{12} &= \kappa(1-t)A(\kappa), && (6.12) \end{aligned}$$

$$b_{13} = 0, \quad (6.13)$$
$$b_{22} = \kappa A(\kappa), \quad (6.14)$$
$$b_{23} = 0, \quad (6.15)$$
$$b_{33} = A(\kappa). \quad (6.16)$$

We will denote by $\mathbf{B_n}(\delta, \mu, \kappa)$ the *average sample information*, which is the negative of the matrix of the second order derivatives of $\ell(\delta, \mu, \kappa)$ divided by $n$. To verify assumption 4(iii) we have to show that $\mathbf{B}_n(\frac{h_1}{\sqrt{n}}, \mu + \frac{h_2}{\sqrt{n}}, \kappa + \frac{h_3}{\sqrt{n}})$ converges in probability to $\mathbf{B}$ uniformly in bounded $\mathbf{h} = (h_1, h_2, h_3)$. Let us suppose that $\| \mathbf{h} \| \leq K$ for some real number $K$. Then $| h_1 |, | h_2 |$ and $| h_3 |$ are all less than or equal to $K$. We will denote the $(i,j)$-th element of $\mathbf{B}_n\left(\frac{h_1}{\sqrt{n}}, \mu + \frac{h_2}{\sqrt{n}}, \kappa + \frac{h_3}{\sqrt{n}}\right)$ by $b_{ij}^{(n)}$.
Now,

$$\begin{aligned} b_{11}^{(n)} - b_{11} &= \frac{\kappa + \frac{h_3}{\sqrt{n}}}{n} C_{r+1}^n \left(\Theta, \frac{h_1}{\sqrt{n}} + \mu + \frac{h_2}{\sqrt{n}}\right) \\ &\quad -\kappa(1-t)A(\kappa) \end{aligned} \quad (6.17)$$

Using Taylor series expansion we have

$$\begin{aligned} b_{11}^{(n)} - b_{11} &= \frac{\kappa}{n} C_{r+1}^n(\Theta, \mu) + \frac{\kappa(h_1 + h_2)}{n\sqrt{n}} S_{r+1}^n(\Theta, \mu) \\ &\quad + \frac{h_3}{n\sqrt{n}} C_{r+1}^n(\Theta, \mu) + \frac{h_3(h_1+h_2)}{n^2} S_{r+1}^n(\Theta, \mu) \\ &\quad -\kappa(1-t)A(\kappa) + \text{terms involving higher powers of } n. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{\|\mathbf{h}\| \leq K} &\ | b_{11}^{(n)} - b_{11} | \\ &\leq \left| \frac{\kappa}{n} C_{r+1}^n(\Theta, \mu) - \kappa(1-t)A(\kappa) \right| \\ &\quad + \frac{2K^2}{\sqrt{n}} \frac{| S_{r+1}^n(\Theta, \mu) |}{n} \end{aligned}$$

$$+\frac{K}{\sqrt{n}}\frac{|C^n_{r+1}(\Theta,\mu)|}{n}$$
$$+\frac{2K^2}{n}\frac{|S^n_{r+1}(\Theta,\mu)|}{n}$$
$$+|\text{ terms involving higher powers of } n\ |. \qquad (6.18)$$

Since by the weak law of large numbers $\frac{1}{n-r}C^n_{r+1}(\Theta,\mu) \longrightarrow_P A(\kappa)$ and $\frac{1}{n-r}S^n_{r+1}(\Theta,\mu) \longrightarrow_P 0$ we note that each term in the right hand side of the above expression converge to 0 in probability. Hence we conclude that $b^{(n)}_{11} \longrightarrow_P b_{11}$. Using similar arguments, it can be shown that $b^{(n)}_{ij} \longrightarrow_P b_{ij}$ for all $1 \le i,j \le 3$.

Henceforth we will denote $S_n(\delta)$, $S_n(\mu)$, and $S_n(\kappa)$ by $S_\delta$, $S_\mu$ and $S_\kappa$ respectively, suppressing the subscript $n$. The first, second and third rows of the matrix $\mathbf{B}$ are denoted as $(B_{\delta\delta}, B_{\delta\mu}, B_{\delta\kappa})$, $(B_{\mu\delta}, B_{\mu\mu}, B_{\mu\kappa})$, $(B_{\kappa\delta}, B_{\kappa\mu}, B_{\kappa\kappa})$ respectively. Since $\mathbf{B}$ is a symmetric matrix we have $B_{\delta\mu} = B_{\mu\delta}$, $B_{\delta\kappa} = B_{\kappa\delta}$ and $B_{\mu\kappa} = B_{\kappa\mu}$. The vector of nuisance parameters $(\mu,\kappa)$ is denoted by $\eta$. The vector $\mathbf{S}$ is partitioned accordingly into $(S_\delta, \mathbf{S}_\eta)$. The matrix $\mathbf{B}$ is also partitioned into blocks $B_{\delta\delta}$(upper left), $\mathbf{B}_{\delta\eta}$(upper right) $\mathbf{B}_{\eta\delta}$(lower left), $\mathbf{B}_{\eta\eta}$(lower right). Following Hall and Mathiason (1990), we define the effective score for $\delta$

$$S^*_\delta = S_\delta - \mathbf{B}_{\delta\eta}\mathbf{B}^{-1}_{\eta\eta}\mathbf{S}_\eta$$

and effective information

$$B^*_\delta = B_{\delta\delta} - \mathbf{B}_{\delta\eta}\mathbf{B}^{-1}_{\eta\eta}\mathbf{B}_{\eta\delta}.$$

Let $\hat{\mu_2}$ and $R_2$ be the estimated mean direction and the sample resultants based on the observations $\theta_{r+1},\ldots,\theta_n$ and let R be the resultant computed on the basis of the entire sample $\theta_1,\ldots,\theta_n$.

**Theorem 13** : *The NR-test for testing $H_0$ against $H_{1r}$ is : Reject $H_0$ if*

$$\frac{\hat{\kappa}R_2\sin^2(\hat{\mu_2}-\hat{\mu})}{(\hat{\kappa}-\frac{n}{R})\cos(\hat{\mu}-\hat{\mu_2}) - \frac{\sin(\hat{\mu}-\hat{\mu_2})}{A'(\hat{\kappa})}} > \chi^2_{1,\alpha}$$

*where $\alpha$ is the level of significance.*

**Proof :** The estimated scores of the parameters $\mu$, $\kappa$ and $\delta$ under $H_0$ obtained by substituting the MLE's of $\mu$ and $\kappa$ under $H_0$ are

$$\hat{S}_\mu = 0, \tag{6.19}$$

$$\hat{S}_\kappa = 0 \tag{6.20}$$

and

$$\hat{S}_\delta = \frac{\hat{\kappa} R_2 \sin(\hat{\mu_2} - \hat{\mu})}{\sqrt{n}}. \tag{6.21}$$

The estimated Fisher information matrix has the following elements :

$$\hat{B_{\mu\mu}} = \frac{\hat{\kappa} R}{n}, \tag{6.22}$$

$$\hat{B_{\kappa\kappa}} = A'(\hat{\kappa}), \tag{6.23}$$

$$\hat{B_{\delta\delta}} = \frac{\hat{\kappa} R_2 \cos(\hat{\mu} - \hat{\mu_2})}{n}, \tag{6.24}$$

$$\hat{B_{\mu\kappa}} = 0, \tag{6.25}$$

$$\hat{B_{\mu\delta}} = \frac{\hat{\kappa} R_2 \cos(\hat{\mu} - \hat{\mu_2})}{n}, \tag{6.26}$$

$$\hat{B_{\delta\kappa}} = \frac{R_2 \sin(\hat{\mu} - \hat{\mu_2})}{n}. \tag{6.27}$$

Hence, the estimated effective score is

$$\hat{S}^*_\delta = \hat{S}_\delta \tag{6.28}$$

and the estimated effective information is

$$\hat{B}^*_\delta = \hat{B_{\delta\delta}} - \left\{ \frac{\hat{B^2_{\delta\mu}}}{\hat{B_{\mu\mu}}} + \frac{\hat{B^2_{\delta\kappa}}}{\hat{B_{\kappa\kappa}}} \right\} \tag{6.29}$$

Thus we have

$$\frac{{\hat{S}^*_\delta}^2}{\hat{B}^*_\delta} = \frac{\hat{\kappa} R_2 \sin^2(\hat{\mu_2} - \hat{\mu})}{(\hat{\kappa} - \frac{n}{R}) \cos(\hat{\mu} - \hat{\mu_2}) - \frac{\sin(\hat{\mu} - \hat{\mu_2})}{A'(\hat{\kappa})}} \tag{6.30}$$

and we reject $H_0$ at level of significance $\alpha$ if

$$\frac{\hat{\kappa} R_2 \sin^2(\hat{\mu_2} - \hat{\mu})}{(\hat{\kappa} - \frac{n}{R}) \cos(\hat{\mu} - \hat{\mu_2}) - \frac{\sin(\hat{\mu} - \hat{\mu_2})}{A'(\hat{\kappa})}} > \chi^2_{1,\alpha} \tag{6.31}$$

**Remark 12 :** *It is seen that the estimated effective information can sometimes become negative. In such cases we propose an alternative way of estimating the elements of the information matrix. First compute the expectations of the elements of the information matrix under $H_0$ and substitute in them the* MLE*'s of the parameters under $H_0$ to get estimates of the elements of the information matrix. Using this principle we have*

$$\hat{B_{\mu\mu}} = \frac{\hat{\kappa} R}{n}, \tag{6.32}$$

$$\hat{B_{\kappa\kappa}} = A'(\hat{\kappa}), \tag{6.33}$$

$$\hat{B_{\delta\delta}} = (1 - \frac{r}{n})\hat{\kappa}\frac{R}{n}, \tag{6.34}$$

$$\hat{B_{\mu\kappa}} = 0, \tag{6.35}$$

$$\hat{B_{\mu\delta}} = (1 - \frac{r}{n})\hat{\kappa}\frac{R}{n}, \tag{6.36}$$

$$\hat{B_{\delta\kappa}} = 0. \tag{6.37}$$

*Using these estimates of the elements of the information matrix the NR-test turns out to be: Reject $H_0$ if*

$$\frac{n^2 \hat{\kappa} R_2^2 \sin^2(\hat{\mu_2} - \hat{\mu})}{r(n-r)R} > \chi^2_{\alpha,1}$$

*where $\alpha$ is the level of significance. We refer to this statistic as the NR-test statistic.*

We use the above statistic to propose the following procedure for testing $H_0$ vs. $H_1$. Let us call the NR-test statistic for testing $H_0$ against $H_{1r}$ as $T_r$. We use $T = \max_{1 \le r \le n-1} T_r$ as the test statistic for testing $H_0$ against $H_1$. We call $T$ the Generalized NR-test statistic. We provide the cut-off points and power of this test using extensive simulations.

## 6.5 Simulation

We use simulation to compute the null distribution of the NRTT statistic as well as its power. The null distribution results are based on 5000 repetitions with sample size $n, n = 10, 20, 30, 40, 50, 75, 100$ drawn from $\mathrm{CN}(0,1)$ distribution. The 5% cut-off points of the NRTT for different values of $\kappa$ and $n$ are given in Table 10 of Chapter 14. It is seen that as the sample size increases the cut-off point becomes less sensitive to changes in $\kappa$, which is expected. The power of this test statistic is simulated by drawing the first 5 observations from $\mathrm{CN}(0,1)$ and the rest $n-5$ from $\mathrm{CN}(\Delta,1), \Delta = 10(10)180$ (in degrees) and repeating it 5000 times. The results are given in Table 11 of Chapter 14. It is seen that as $\Delta$ increases power also increases upto a point and then onwards the power decreases. This is expected since the NRTT is based on the NR-test which is only asymptotically locally optimal. However, the power of the test is seen to increase with increase in $n$.

## 6.6 Examples

In this section we look at two examples. In example 1 we analyse a data set of wind directions given by Weijers et. al. (1995) and in example 2 we look at the data set of flare launches analysed by Lombard(1986). Exploratory data analysis conducted on both these data sets using the Changeogram and circular difference tables indicated the presence of more than one change point. These techniques are discussed in detail in Chapter 11.

We applied the LRT, LMPTT, the test for single change point developed by Lombard and the NRTT to these data sets. We first tentatively identified the points of change using the exploratory techniques mentioned above and then divided the data set into several parts. We tested for circular normality of each of these parts separately using the test developed by Lockhart and Stephens(1985). Since the LRT and the LMPTT- tests could be carried out only when $\kappa$ is known we decided to proceed in an adaptive fashion for carrying out these tests. We tested for equality of the $\kappa$ values using the test given in Mardia(1972). On acceptance of the hypothesis of equal $\kappa$ values we decided to use the pooled value of $\kappa$ as the known $\kappa$ for the purpose of computing the LRT and LMPTT-test statistics. The NRTT and the Lombard's test can be applied when $\kappa$ is unknown.

*Example 1*: As a first example we analyse the data set given by Weijers et. al. (1995). They investigated the horizontal perturbation wind field within thermal structures encountered in the atmospheric surface layer boundary. A field experiment with four sonic anemometers on the vertices and one in the centroid of a square was performed to obtain the necessary data set. Structures were selected on a typical ramplike appearance in the temperature time series. Altogether a set of 47 'ramps' was obtained. Ensemble averages of turbulent temperature and horizontal and vertical velocities were constructed using conditional sampling and block averaging followed by a compositing technique. We are interested in the behaviour of the direction of the horizontal wind field as recorded by the anemometer at the centroid for the 32 bins after the ensemble averaging procedure. The method of construction of the bins and the ensemble averaging procedure show that the bins have a temporal ordering. We retrieved the actual data from the graphical representation presented in the paper. From the Changeogram (Figure 1, Chapter 15) one notices that there are possibly two change points, one around 17 and the other around 23. We decided to consider only the observation numbers 1 to 22 for our analysis. We split this set into two parts, one containing observations number 1 to 17 and the other containing observation numbers 18 to 22. The two sets were first tested separately for circular normality, which were not refuted at 5% level of significance. The NRTT was then applied and it gave a very significant result indicating the presence of change and identified the change point to be at 17. To apply the LRT we require to know the value of $\kappa$. For this purpose we estimated the value of $\kappa$ for the two sets separately. Then we tested for equality of $\kappa$ values of the two sets which was accepted at 5% level of significance. We used the pooled value of $\kappa$ as the true $\kappa$ for the purpose of LRT. The LRT detected a significant change and the change point was once again seen to be at 17. The Lombard test for single change point detected the presence of change but quite surprisingly as seen from the difference table, Table C, Chapter 11, it indicated 13 as the change point.

The LMPTT-test could not detect any significant change but indicated 17 as the most likely candidate for change point. We suspected the failure of the LMPTT was due to the small number (only five) of observations in the post change set. We extended the data set to 34 observations by augmenting 12 observations generated from the circular normal distribution using the post change mean as the mean direction and the pooled $\kappa$ as the concentration parameter. After augmenting these observations we applied all the four tests. Now all the four tests -NRTT, LRT, LMPTT, as well as the Lombard test, detected the presence of change and detected the change point to be at 17.

*Example 2 :* As the second example we treat the flare data analysed by Lombard (1986). The raw data is reproduced in Fisher(1993). The data consisted of 60 observations. Lombard detected two change points at 12 and 42 respectively using his change point statistic for single change iteratively. Based on that, he used a test statistic for detecting the presence of two change points and detected 12 and 37 as change points. Since our test statistics are also designed to detect at most one change point we decided to apply the iterative technique. Based on the information provided by Lombard(1986) we break up the data set into two subsets, one subset consisting of the first 42 observations and the other subset consisting of the remaining 48 observations after deletion of the first 12. The Changeogram (Figure 2, Chapter 15) and the circular difference tables for the observations 1-42 (Table A, Chapter 11) and also for observations 13-60 (Table B, Chapter 11) are given in Chapter 11.

Based on the Lombard's test and also on the findings from the Changeogram and the circular difference tables we decided to break up the data set consisting of observations 1 to 42 into two parts, the first containing observations 1 to 12 and the next consisting observations 13 to 42. We first tested the two parts separately for circular normality using the method given in Lockhart and Stephens (1985). The test could not refute the claim that the data came from a circular normal distribution. The NRTT was then applied which gave a very significant result indicating the presence of change and indicated the change point to be at 17. The LRT and LMPTT-test requires the value of $\kappa$. We proceeded in an adaptive manner and first computed the $\kappa$ values for both these sets and applied the two sample test for equality of $\kappa$ values as given in Mardia (1972). The test indicated the equality of two $\kappa$'s at 5% level of significance. Figure 1, Chapter 15 gives the Rosogram (a scatter plot superimposed on a Rose diagram) of the part of the data set between observations number 13-42.

Then we proceeded to pool the two $\kappa$ values to obtain the estimate of $\kappa$. Subsequently we treated this value of $\kappa$ as known for the purpose of computing the LRT and LMPTT-tests. By an application of LRT we were able to detect the existence of a change point at 5% level of significance. The change point was detected to be at 12. Then we applied the LMPTT-test on this data set. This test could not detect any change at 5% level of significance. However the point 36 was shown as the most prominent candidate for a change point. This is parallel to what the Lombard's two change point test has shown. Now considering the observations number 13-60 of the flare data we again divided it into two parts, one comprising of observations number 13-42 and the remaining 43-60. Like the earlier case we first tested each of these parts for circular normality which was not refuted. The NRTT was then applied and it gave a very significant result indicating the presence of change and identified the change point to be at 42. Next we tested for equality of $\kappa$ which again was not refuted at 5% level of significance. Then we proceeded to obtain the pooled estimate of $\kappa$ which was used as the known value of $\kappa$ for the LRT and the LMPTT-tests. The LRT also detected a significant change in the mean direction and the change point was seen to be at 42. The LMPTT-test however failed to detect any change in mean direction. The failure of the LMPTT-test in detecting a change may be due to the possibly large changes in the magnitudes of the observations present in the data as seen from the circular difference tables, Tables A and B, Chapter 11, and also from the Changeogram given in Figure 2, Chapter 15.

# Chapter 7

# Change Point Problem for the Concentration Parameter in CN$(\mu, \kappa)$-$\mu$ Known

## 7.1 Introduction

In this and the next chapter we discuss the change point problem for the concentration parameter of the circular normal distribution. In this chapter we discuss the case when the mean direction is known and in the next chapter we deal with the case when the mean direction is unknown. In section 7.2 we assume that the initial concentration parameter is known and derive a Uniformly Most Powerful Type Test (UMPTT)). In section 7.3 we discuss the case when the initial concentration parameter is unknown and derive the LRT.

## 7.2 Initial Concentration Parameter Known

Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the null hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as CN$(0, \kappa_0)$ against the alternative hypothesis $H_1$ : There exist $r, 1 \leq r \leq n-1, \Theta_1, \ldots, \Theta_r$

are identically distributed as $\mathrm{CN}(0, \kappa_0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(0, \kappa)$, $\kappa > \kappa_0$. Let $H_{1r}$ denote the alternative hypothesis that there is a change point at $r$. Note that the problem of testing $H_0$ against $H_{1r}$ is equivalent to that of testing $H_0' : \kappa = \kappa_0$ against $H_1' : \kappa > \kappa_0$.

**Theorem 14 :** *The* UMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } C_{r+1}^n(\Theta, 0) > c$$

*where $c$ is a constant depending on the level of significance.*

**Proof :** Let us fix $\kappa_1 > \kappa_0$. Then by Neyman-Pearson Lemma the Most Powerful (MP) test of $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } C_{r+1}^n(\Theta, 0) > c$$

where $c$ is a constant depending on the level of significance.
Since the critical region does not depend on the particular value of $\kappa_1 > \kappa_0$ we claim that the above test is Uniformly Most Powerful (UMP) for testing $H_0$ against $H_{1r}$. Motivated by the above theorem we suggest an UMPTT for testing $H_0$ against $H_1$.

**Theorem 15 :** *The* UMPTT *for testing $H_0$ against $H_1$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1 \le r \le n-1} C_{r+1}^n(\Theta, 0) > c$$

*where $c$ is a constant depending on the level of significance. The asymptotic null distribution of*

$$\max_{1 \le r \le n-1} \frac{C_{r+1}^n(\Theta, 0) - (n-r)A(\kappa_0)}{\sqrt{n}\sqrt{\left(\frac{I_0''(\kappa_0)}{I_0(\kappa_0)} - A^2(\kappa_0)\right)}}$$

*is same as that of* $\sup_{0 \le t \le 1} B_0^*(t)$.

**Proof :**

$$\text{Under } H_0,\ \mathrm{E}(\cos\Theta_i) = A(\kappa_0) \tag{7.1}$$

and

$$\mathrm{Var}(\cos\Theta_i) = \frac{I_0''(\kappa_0)}{I_0(\kappa_0)} - A^2(\kappa_0) \tag{7.2}$$

Hence by the Functional Central Limit Theorem and using the fact that sup is continuous on $D[0,1]$ we get the asymptotic null distribution of

$$\max_{1\leq r\leq n-1} \frac{C_{r+1}^n(\Theta,0) - (n-r)A(\kappa_0)}{\sqrt{n}\sqrt{\left(\frac{I_0''(\kappa_0)}{I_0(\kappa_0)} - A^2(\kappa_0)\right)}}$$

to be same as that of $\sup\limits_{0\leq t\leq 1} B_0^*(t)$.

## 7.3 Initial Concentration Parameter Unknown

We now consider the case when the initial value of the concentration parameter and the possibly changed value of the concentration parameter are both unknown. As before, let $\Theta_1, \ldots, \Theta_n$ be independent observations. We are interested to test $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(0, \kappa_0)$ against the alternative $H_1$ : There exist $r, 1 \leq r \leq n-1$, such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{CN}(0, \kappa_0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(0, \kappa_1)$, $\kappa_1 \neq \kappa_0$. Let $H_{1r}$ denote the alternative hypothesis that the change point is at $r$. We first derive an LRT for testing $H_0$ against $H_{1r}$.

**Theorem 16 :** *The* LRT *of $H_0$ against $H_{1r}$ is given by :*
*Reject $H_0$ if*

$$\begin{aligned} r\ln I_0({}_r\hat{\kappa}_0) &+ (n-r)\ln I_0({}_r\hat{\kappa}_1) - n\ln I_0({}_0\hat{\kappa}_0) \\ &+ ({}_0\hat{\kappa}_0 - {}_r\hat{\kappa}_0)\, C_1^r(\Theta,0) \\ &+ ({}_0\hat{\kappa}_0 - {}_r\hat{\kappa}_1)\, C_{r+1}^n(\Theta,0) < c \end{aligned} \tag{7.3}$$

*where $c$ is a constant depending on the level of significance.*

**Proof:** Under $H_0$, the log-likelihood is

$$\ell_0(\kappa_0; \theta_1, \ldots, \theta_n) = -n \ln 2\pi - n \ln I_0(\kappa_0) + \kappa_0 C_1^n(\theta, 0)$$

Hence under $H_0$, the MLE of $\kappa_0$ is

$$_0\hat{\kappa}_0 = A^{-1}\left(\frac{1}{n}\, C_1^n(\Theta, 0)\right)$$

Let $\ell_r$ denote the log-likelihood under $H_{1r}$

$$\begin{aligned} \ell_r(\kappa_0, \kappa_1; \theta_1, \ldots, \theta_n) &= -n \ln 2\pi - r \ln I_0(\kappa_0) - (n-r) \ln I_0(\kappa_1) \\ &\quad + \kappa_0 C_1^r(\theta, 0) + \kappa_1 C_{r+1}^n(\theta, 0) \end{aligned} \tag{7.4}$$

Hence under $H_{1r}$, the MLE's of $\kappa_0$ and $\kappa_1$ are

$$_r\hat{\kappa}_0 = A^{-1}\left(\frac{1}{r}\, C_1^r(\Theta, 0)\right) \tag{7.5}$$

and

$$_r\hat{\kappa}_1 = A^{-1}\left(\frac{1}{n-r}\, C_{r+1}^n(\Theta, 0)\right) \tag{7.6}$$

where $A(.) = \frac{I_0'(.)}{I_0(.)}$.

The LRT-statistic $\Lambda_r$ for testing $H_0$ against $H_{1r}$ is given by :

$$\begin{aligned} \ln \Lambda_r &= r \ln I_0(_r\hat{\kappa}_0) - (n-r) \ln \left(I_0(_r\hat{\kappa}_1)\right) \\ &\quad - n \ln I_0(_0\hat{\kappa}_0) + (_0\hat{\kappa}_0 - {}_r\hat{\kappa}_0)\; C_1^r(\Theta, 0) \\ &\quad + (_0\hat{\kappa}_0 - {}_r\hat{\kappa}_1)\; C_{r+1}^n(\Theta, 0) \end{aligned} \tag{7.7}$$

The LRT for $H_0$ against $H_{1r}$ is

$$\text{Reject } H_0 \text{ if } \ln \Lambda_r < c$$

for some constant $c$ depending on the level of significance.

**Remark 13 :** *For large n, the cut off points may be obtained using the fact that the asymptotic null distribution of* $-2\ln\Lambda_r$ *is* $\chi_1^2$.

The next theorem gives the LRT for $H_0$ against $H_1$.

**Theorem 17 :** *The* LRT *for* $H_0$ *against* $H_1$ *is*

$$\text{Reject } H_0 \text{ if } \min_{1\leq r\leq n-1} \ln\Lambda_r < c$$

*where c is a constant depending on the level of significance.*

**Proof :** Note that the LRT $\Lambda$ is given by

$$\begin{aligned}
\ln\Lambda &= \ell_0({}_0\hat{\kappa}_0;\theta_1,\ldots,\theta_n) - \max_{1\leq r\leq n-1} \ell_r({}_r\hat{\kappa}_0,{}_r\hat{\kappa}_1;\theta_1,\ldots,\theta_n) \\
&= \min_{1\leq r\leq n-1} \ell_0({}_0\hat{\kappa}_0;\theta_1,\ldots,\theta_n) - \ell_r({}_r\hat{\kappa}_0,{}_r\hat{\kappa}_1;\theta_1,\ldots,\theta_n) \\
&= \min_{1\leq r\leq n-1} \ln\Lambda_r. \qquad (7.8)
\end{aligned}$$

Hence the LRT is

$$\text{Reject } H_0 \text{ if } \min_{1\leq r\leq n-1} \ln\Lambda_r < c$$

where $c$ is a constant depending on the level of significance.

# Chapter 8

# Change Point Problem for the Concentration Parameter in CN$(\mu, \kappa)$- $\mu$ Unknown

## 8.1 Introduction

In this chapter we deal with the case when the initial concentration parameter $\kappa$, the possibly changed concentration parameter $\kappa_1 = (\kappa + \delta)$ and the mean direction $\mu$ are all unknown. This is a very important case since in most practical situations $\kappa_0$, $\kappa_1$, and $\mu$ are unknown. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the null hypothesis $H_0$ : $\Theta_1, \ldots, \Theta_n$ are identically distributed as CN$(\mu, \kappa)$ against the alternative hypothesis $H_1$ : There exist $r, 1 \leq r \leq n-1$ such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as CN$(\mu, \kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as CN$(\mu, \kappa + \delta), \delta > 0$. Then the null hypothesis of no change can be rewritten as $H_0 : \delta = 0$ and the alternative hypothesis can be written as $H_1 : \delta > 0$. The construction of a test for $H_0$ against $H_1$ is complicated because of the presence of two nuisance parameters $\kappa_0$ and $\mu$. We derive a NRTT for this problem. Like in earlier cases we denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$.

## 8.2 Neyman-Rao Type Test

Let us first consider the problem of testing $H_0$ against $H_{1r}$ for a fixed $r$, $1 \leq r \leq n-1$. The log-likelihood of the observations will be denoted by $\ell(\delta, \kappa, \mu)$. We note that for the validity of the NR-test it is sufficient to check Assumption 4 of Hall & Mathiason(1990) which consist of three sub-parts 4(i), 4(ii) and 4(iii). It is simple to check that the log likelihood $\ell(\delta, \kappa, \mu)$ has a matrix of continuous second-order partial derivatives and this verifies Assumption 4(i). Now observe that in our case,

$$\begin{aligned} S_\delta = S_n(\delta) &= \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \delta} \\ &= \frac{-(n-r)A(\kappa+\delta) + C_{r+1}^n(\Theta, \mu)}{\sqrt{n}} \end{aligned} \tag{8.1}$$

$$\begin{aligned} S_\mu = S_n(\mu) &= \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \mu} \\ &= \frac{\kappa}{\sqrt{n}} S_1^n(\Theta, \mu) + \frac{\delta}{\sqrt{n}} S_{r+1}^n(\Theta, \mu) \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} S_\kappa = S_n(\kappa) &= \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \kappa} \\ &= \frac{1}{\sqrt{n}}[C_1^n(\Theta, \mu) - rA(\kappa) - (n-r)A(\kappa+\delta)] \end{aligned} \tag{8.3}$$

Under $H_0$, putting $\delta = 0$ we have

$$S_n(\delta) = \frac{1}{\sqrt{n}}\frac{\partial \ell}{\partial \delta}$$

$$= \frac{-(n-r)A(\kappa) + C_{r+1}^n(\Theta, \mu)}{\sqrt{n}} \tag{8.4}$$

$$S_n(\mu) = \frac{1}{\sqrt{n}} \frac{\partial \ell}{\partial \mu}$$

$$= \frac{\kappa}{\sqrt{n}} S_1^n(\Theta, \mu) \tag{8.5}$$

and

$$S_n(\kappa) = \frac{1}{\sqrt{n}} \frac{\partial \ell}{\partial \kappa}$$

$$= \frac{1}{\sqrt{n}} [C_1^n(\Theta, \mu) - nA(\kappa)] \tag{8.6}$$

For any three real numbers $t_1, t_2, t_3$

$$\begin{aligned}
& t_1 S_n(\delta) + t_2 S_n(\mu) + t_3 S_n(\kappa) \\
&= \frac{t_1}{\sqrt{n}} \left\{ -(n-r)A(\kappa) + \sum_{i=r+1}^{n} \cos(\Theta_i - \mu) \right\} + \frac{t_2}{\sqrt{n}} \left\{ \kappa \sum_{i=1}^{n} \sin(\Theta_i - \mu) \right\} \\
&+ \frac{t_3}{\sqrt{n}} \left\{ \sum_{i=1}^{n} \cos(\Theta_i - \mu) - nA(\kappa) \right\} \\
&= \sum_{i=1}^{r} \left[ \frac{t_2 \kappa}{\sqrt{n}} \sin(\Theta_i - \mu) + \frac{t_3}{\sqrt{n}} \cos(\Theta_i - \mu) - \frac{t_3}{\sqrt{n}} A(\kappa) \right] \\
&+ \sum_{i=r+1}^{n} \left[ \frac{t_2 \kappa}{\sqrt{n}} \sin(\Theta_i - \mu) + \frac{t_1 + t_3}{\sqrt{n}} \cos(\Theta_i - \mu) - \frac{t_1 + t_3}{\sqrt{n}} A(\kappa) \right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{r} [t_2 \kappa \sin(\Theta_i - \mu) + t_3 \cos(\Theta_i - \mu) - t_3 A(\kappa)] \\
&+ \frac{1}{\sqrt{n}} \sum_{i=r+1}^{n} [t_2 \kappa \sin(\Theta_i - \mu) + (t_1 + t_3) \cos(\Theta_i - \mu) - (t_1 + t_3) A(\kappa)]
\end{aligned}$$

Let,

$$\begin{aligned}
Z_i &= t_2 \ \kappa \ \sin(\Theta_i - \mu) + t_3 \ \cos(\Theta_i - \mu) - t_3 \ A(\kappa), 1 \le i \le r \\
&= t_2 \ \kappa \ \sin(\Theta_i - \mu) + (t_1 + t_3) \left\{ \cos(\Theta_i - \mu) - A(\kappa) \right\}, r + 1 \le i \le n
\end{aligned}$$

Now we note that $Z_i$'s are bounded independent random variables with $\mathrm{E}(Z_i) = 0, 1 \leq i \leq n$ and finite variances. Hence by the Lindeberg-Feller Central Limit Theorem we can conclude that $t_1 S_\delta + t_2 S_\mu + t_3 S_\kappa$ is normally distributed as $n \longrightarrow \infty, r \longrightarrow \infty, \frac{r}{n} \longrightarrow t, t \neq 0, 1$.

Thus the distribution of any linear combination of $S_n(\delta), S_n(\mu)$, and $S_n(\kappa)$ is asymptotically normal and hence by the Cramer-Wold theorem we can conclude that the joint distribution of $(S_n(\delta), S_n(\mu), S_n(\kappa))$ is asymptotically trivariate normal. Now since (under $H_0$)

$$
\begin{aligned}
E\left(S_n(\delta)\right) &= E\left(S_n(\mu)\right) = E\left(S_n(\kappa)\right) = 0, && (8.7)\\
V\left(S_n(\mu)\right) &= \kappa A(\kappa) && (8.8)\\
V\left(S_n(\kappa)\right) &= A'(\kappa) && (8.9)\\
V\left(S_n(\delta)\right) &= \left(1 - \frac{r}{n}\right) A'(\kappa), && (8.10)\\
\text{Cov}\ \left(S_n(\mu), S_n(\kappa)\right) &= 0, && (8.11)\\
\text{Cov}\ \left(S_n(\mu), S_n(\delta)\right) &= 0, && (8.12)\\
\text{Cov}\ \left(S_n(\kappa), S_n(\delta)\right) &= \left(1 - \frac{r}{n}\right) A'(\kappa), && (8.13)
\end{aligned}
$$

we have $(S_n(\delta), S_n(\kappa), S_n(\mu)) \longrightarrow N_3(\mathbf{0}, \mathbf{B})$ where the nonsingular symmetric matrix $\mathbf{B}$ has elements ,

$$
\begin{aligned}
b_{11} &= (1-t)A'(\kappa), && (8.14)\\
b_{12} &= (1-t)A'(\kappa), && (8.15)\\
b_{13} &= 0, && (8.16)\\
b_{22} &= A'(\kappa), && (8.17)\\
b_{23} &= 0, && (8.18)\\
b_{33} &= \kappa A(\kappa) && (8.19)
\end{aligned}
$$

To verify assumption 4(iii) we have to show that $\mathbf{B}_n\left(\frac{h_1}{\sqrt{n}}, \kappa + \frac{h_2}{\sqrt{n}}, \mu + \frac{h_3}{\sqrt{n}}\right)$ converges in probability to $\mathbf{B}$ uniformly in bounded $\mathbf{h} = (h_1, h_2, h_3)$. Let us suppose that $\| \mathbf{h} \| \leq K$ for some real number $K$. Then $| h_1 |, | h_2 |$ and

$| h_3 |$ are all less than or equal to $K$. We will denote the $(i,j)$-th element of $\mathbf{B}_n\left(\frac{h_1}{\sqrt{n}}, \kappa + \frac{h_2}{\sqrt{n}}, \mu + \frac{h_3}{\sqrt{n}}\right)$ by $b_{ij}^{(n)}$.

Now,

$$b_{11}^{(n)} - b_{11} = \left(1 - \frac{r}{n}\right) A'\left(\kappa + \frac{h_2}{\sqrt{n}} + \frac{h_1}{\sqrt{n}}\right) - (1-t)A'(\kappa) \tag{8.20}$$

Using the Taylor series expansion we have

$$\begin{aligned} b_{11}^{(n)} - b_{11} &= \left(1 - \frac{r}{n}\right)\left\{A'(\kappa) + A''(\kappa).\frac{(h_2 + h_1)}{\sqrt{n}}\right\} \\ &\quad -(1-t)A'(\kappa) + \text{terms involving higher powers of } n \\ &= \left(t - \frac{r}{n}\right) A'(\kappa) + \left(1 - \frac{r}{n}\right)\frac{(h_2 + h_1)}{\sqrt{n}} A''(\kappa) \\ &\quad + \mid \text{terms involving higher powers of } n \mid . \end{aligned} \tag{8.21}$$

Hence,

$$\begin{aligned} \sup_{\|\mathbf{h}\| \leq K} & \mid b_{11}^{(n)} - b_{11} \mid \\ & \leq \mid t - \frac{r}{n} \mid\mid A'(\kappa) \mid \\ & + \frac{2K}{\sqrt{n}} \mid \left(1 - \frac{r}{n}\right) \mid\mid A''(\kappa) \mid \\ & + \mid \text{terms involving higher powers of } n \mid . \end{aligned} \tag{8.22}$$

We see that the right hand side of the above expression converges to 0 as $n \longrightarrow \infty$

Hence we conclude that $b_{11}^{(n)} \longrightarrow_P b_{11}$ uniformly in bounded $\mathbf{h}$. Using similar arguments, it can be shown that $b_{ij}^{(n)} \longrightarrow_P b_{ij}$ uniformly in bounded $\mathbf{h}$ for all $1 \leq i, j \leq 3$.

Let $\hat{\mu}_2$ and $R_2$ denote the sample mean direction and resultant computed from $\theta_{r+1}, \ldots, \theta_n$ , and $R$ the resultant computed from the entire sample.

**Theorem 18** : *The NR-test for testing $H_0$ against $H_{1r}$ is :*
*Reject $H_0$ if*

$$\frac{R_2^2 \cos^2(\hat{\mu} - \hat{\mu_2}) + \left(1 - \frac{r}{n}\right)^2 R^2 - 2\left(1 - \frac{r}{n}\right) R R_2 \cos(\hat{\mu} - \hat{\mu_2})}{\frac{r}{n}\left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) - \frac{R_2^2}{n\hat{\kappa}^2} \sin^2(\hat{\mu} - \hat{\mu_2})} > \chi^2_{1,\alpha} \quad (8.23)$$

*where $\alpha$ is the level of significance.*

**Proof :** The estimated scores of the parameters $\mu$, $\kappa$ and $\delta$ under $H_0$ obtained by substituting the MLE's of $\mu$ and $\kappa$ under $H_0$ are

$$\hat{S}_\mu = 0, \quad (8.24)$$
$$\hat{S}_\kappa = 0 \quad (8.25)$$

and

$$\hat{S}_\delta = R_2 \cos(\hat{\mu_2} - \hat{\mu}) - \left(1 - \frac{r}{n}\right) R \quad (8.26)$$

The estimated Fisher information matrix has the following elements :

$$\hat{B_{\mu\mu}} = \frac{\hat{\kappa} R}{n}, \quad (8.27)$$
$$\hat{B_{\kappa\kappa}} = A'(\hat{\kappa}), \quad (8.28)$$
$$\hat{B_{\delta\delta}} = \left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) \quad (8.29)$$
$$\hat{B_{\mu\kappa}} = 0, \quad (8.30)$$
$$\hat{B_{\mu\delta}} = \frac{R_2}{n} \sin(\hat{\mu} - \hat{\mu_2}), \quad (8.31)$$
$$\hat{B_{\delta\kappa}} = \left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) \quad (8.32)$$

Hence, the estimated effective score is

$$\hat{S}_\delta^* = \hat{S}_\delta$$

and the estimated effective information is

$$\begin{aligned}
\hat{B}_\delta^* &= \hat{B}_{\delta\delta} - \left\{ \frac{\hat{B}_{\delta\kappa}^2}{\hat{B}_{\kappa\kappa}} + \frac{\hat{B}_{\delta\mu}^2}{\hat{B}_{\mu\mu}} \right\} \\
&= \frac{r}{n}\left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) - \frac{R_2^2}{n\hat{\kappa}^2} \sin^2(\hat{\mu} - \hat{\mu_2}) \qquad (8.33)
\end{aligned}$$

Thus we have

$$\frac{\hat{S_\delta^*}^2}{\hat{B}_\delta^*} = \frac{R_2^2 \cos^2(\hat{\mu} - \hat{\mu_2}) + \left(1 - \frac{r}{n}\right)^2 R^2 - 2\left(1 - \frac{r}{n}\right) R R_2 \cos(\hat{\mu} - \hat{\mu_2})}{\frac{r}{n}\left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) - \frac{R_2^2}{n\hat{\kappa}^2} \sin^2(\hat{\mu} - \hat{\mu_2})}$$

and we reject $H_0$ at the level of significance $\alpha$ if

$$\frac{R_2^2 \cos^2(\hat{\mu} - \hat{\mu_2}) + \left(1 - \frac{r}{n}\right)^2 R^2 - 2\left(1 - \frac{r}{n}\right) R R_2 \cos(\hat{\mu} - \hat{\mu_2})}{\frac{r}{n}\left(1 - \frac{r}{n}\right) A'(\hat{\kappa}) - \frac{R_2^2}{n\hat{\kappa}^2} \sin^2(\hat{\mu} - \hat{\mu_2})} > \chi_{1,\alpha}^2$$

where $\alpha$ is the level of significance. We refer to this statistic as the NR-test statistic.

We use the above statistic to propose the NRTT for testing $H_0$ against $H_1$. Let us call the NR-test statistic for testing $H_0$ against $H_{1r}$ as $T_r$. We use $T = \max_{1 \le r \le n-1} T_r$ as the test statistic for testing $H_0$ against $H_1$. We reject $H_0$ if the observed value of $T$ is "too large". Since the asymptotic null distribution of $T$ is free from the nuisance parameters, the approximate cut-off points can be obtained through simulation for large $n$.

# Chapter 9

# On Change Point and Outlier Problems for Skewed Circular Distributions

## 9.1 Introduction

In this chapter we look at the change point problem for some skewed circular distributions. Though in most statistical applications we deal with circular distributions which are symmetric with respect to the mode, occasionally there is a need for using skew circular distributions (Batschelet, 1981). We discuss outlier and change point problems for three different families of skewed circular distributions, one of which is due to Papakonstantinou (1979), another due to Rattihali and SenGupta(2000), and the third one, which is an extension of the circular normal distribution, is given in Batschelet (1981, page 286). In section 9.2 we discuss the outlier problem for Papakonstantinou's skewed circular distribution. In section 9.3 we discuss change point problems in the same distributions. The outlier problem for Rattihali-SenGupta's skewed circular distribution is discussed in section 9.4 and the change point problems in the same distribution are discussed in section 9.5. The outlier problem in Batschelet's skewed circular distributions is discussed in section 9.6 and the change point problems in the same distribution in section 9.7. For all the outlier and change point problems with skewed distributions it is found that LMPTT has wide applications and is quite simple to derive in such set-ups. Other tests in this set-up are likely to be much more complicated.

## 9.2 Outlier Problem with Papakonstantinou's Skewed Circular Distribution.

Papakonstantinou's skewed circular distribution has probability density function

$$f(\theta; k, \nu) = \frac{1}{2\pi} + \frac{k}{2\pi}\cos(\theta + \nu\cos\theta)$$
$$0 \leq \theta \leq 2\pi, \nu \geq 0, -1 < k < 1, \tag{9.1}$$

which is equivalent to

$$f(\theta; k, \nu) = \frac{1}{2\pi} + \frac{k}{2\pi}\sin(\theta + \nu\sin\theta)$$
$$0 \leq \theta \leq 2\pi, \nu \geq 0, -1 < k < 1, \tag{9.2}$$

as shown in Batschelet(1981). $\nu = 0$ yields the symmetric cardioid distribution while $\nu > 0$ results in a skew distribution. We will use the latter form in the subsequent discussions. We denote this distribution by $P(k, \nu)$. The Figures 5 and 6 of Chapter 15 give the graphs of the above density for different values of $k$ and $\nu$.

Suppose the random variables $\Theta_1, \ldots, \Theta_n$ are all independent, and $k$ is known. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{P}(k, 0)$ against the alternative $H_1^* :$ There exists $r, 1 \leq r \leq n$ such that $\Theta_1, \ldots, \Theta_{r-1}, \Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{P}(k, 0)$ and $\Theta_r$ is distributed as $\mathrm{P}(k, \nu), \nu > 0$. We will denote by $H_r$ the hypothesis that the $r^{th}$ observation is the outlier. We will derive the LMPTT for this problem.

**Theorem 19 :** *(a) The* LMP *test for testing $H_0$ against $H_r$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \frac{\sin 2\Theta_r}{1 + k\sin\Theta_r} > c$$

*for some constant c depending on the level of significance.*
*(b) The* LMPTT *for testing* $H_0$ *against* $H_1^*$ *is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1\le r\le n} \frac{\sin 2\Theta_r}{1+k\sin\Theta_r} > c$$

*for some constant c depending on the level of significance.*

**Proof :** (a) Let $\theta_1, \ldots, \theta_n$ denote the observations. Observe the log-likelihood is

$$\begin{aligned}
\ell(\nu;\theta_1,\ldots,\theta_n) &= \sum_{i\neq r} \ln\left(\frac{1}{2\pi}+\frac{k}{2\pi}\sin\theta_i\right) + \ln\left(\frac{1}{2\pi}+\frac{k}{2\pi}\sin(\theta_r+\nu\sin\theta_r)\right) \\
\frac{\partial\ell}{\partial\nu} &= \frac{\frac{k}{2\pi}\cos(\theta_r+\nu\sin\theta_r).\sin\theta_r}{\frac{1}{2\pi}+\frac{k}{2\pi}\sin(\theta_r+\nu\sin\theta_r)} \\
\frac{\partial\ell}{\partial\nu}(0) &= \frac{k\cos\theta_r\sin\theta_r}{1+k\sin\theta_r} \\
&= \frac{k\sin 2\theta_r}{2(1+k\sin\theta_r)}
\end{aligned}$$

Thus the LMP test of $H_0$ against $H_r$ is :

$$\textit{Reject } H_0 \text{ if } \frac{\sin 2\Theta_r}{1+k\sin\Theta_r} > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT for testing $H_0$ against $H_1^*$ is :

$$\textit{Reject } H_0 \text{ if } \max_{1\le r\le n} \frac{\sin 2\Theta_r}{1+k\sin\Theta_r} > c$$

where $c$ is a constant depending on the level of significance.
If $\alpha$ is the level of significance then

$$\begin{aligned}
\alpha &= \Pr\left(\max_{1\le r\le n}\frac{\sin 2\Theta_r}{1+k\sin\Theta_r} > c \mid H_0\right) \\
&= 1-\Pr\left(\max_{1\le r\le n}\frac{\sin 2\Theta_r}{1+k\sin\Theta_r} \le c \mid H_0\right) \\
&= 1-\prod_{i=1}^{n}\Pr\left(\frac{\sin 2\Theta_r}{1+k\sin\Theta_r} \le c \mid H_0\right)
\end{aligned}$$

$$= 1 - \left[\Pr\left(\frac{\sin 2\Theta_r}{1 + k\sin\Theta_r} \leq c \mid H_0\right)\right]^n \quad \text{(since } \Theta_1, \ldots, \Theta_n \text{ are i.i.d under } H_0.)$$

Hence we need to find $c$ such that

$$\Pr\left(\frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} \leq c\right) = (1 - \alpha)^{\frac{1}{n}}.$$

Now under $H_0$ such a $c$ can be easily found using numerical integration.

## 9.3 Change Point Problem in Papakonstantinou's Skewed Circular Distribution

Let us suppose that $k$ is known. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are i.i.d $P(k, 0)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as $P(k, 0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $P(k, \nu), \nu > 0$, for some $r, 1 \leq r \leq n - 1$. We propose a LMPTT test for testing $H_0$ against $H_{1r}$. We denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$.

**Theorem 20** : *(a) In testing $H_0$ against $H_{1r}$ the* LMP *test is given by :*

$$\textit{Reject } H_0 \textit{ if } \sum_{i=r+1}^{n} \frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} > c \tag{9.3}$$

*for some constant $c$ depending on the level of significance.*

*(b) The* LMPTT *of $H_0$ against $H_1$ is based on the statistic*

$$\Lambda = \max_{1 \leq r \leq n-1} \sum_{i=r+1}^{n} \frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} \tag{9.4}$$

*The asymptotic null distribution of* $\max_{1 \leq r \leq n-1} \dfrac{\sum_{i=r+1}^{n} \frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} - (n - r)\eta}{\sqrt{n}\tau}$ *where* $\eta = E\left(\frac{\sin 2\Theta_i}{1 + k\sin\Theta_i}\right)$ *and* $\tau^2 = Var\left(\frac{\sin 2\Theta_i}{1 + k\sin\Theta_i}\right)$*; is same as that of* $\sup_{0 \leq t \leq 1} B_0^*(t)$.

**Proof :** (a) Let $\theta_1, \ldots, \theta_n$ denote the observations, $0 \le \theta_i < 2\pi$. Observe the log-likelihood is

$$l(\nu; \theta_1, \theta_2, \ldots, \theta_n) = \sum_{i=1}^{r} \ln\left(\frac{1}{2\pi} + \frac{k}{2\pi}\sin\theta\right) + \sum_{i=r+1}^{n} \ln\left[\left(\frac{1}{2\pi} + \frac{k}{2\pi}\sin(\theta + \nu\sin\theta)\right)\right]$$

Then,

$$\frac{\partial \ell}{\partial \nu} = \sum_{i=r+1}^{n} \frac{\frac{k}{2\pi}\cos(\theta + \nu\sin\theta)\sin\theta}{\frac{1}{2\pi} + \frac{k}{2\pi}\sin(\theta + \nu\sin\theta)} \tag{9.5}$$

Therefore,

$$\frac{\partial \ell}{\partial \nu}(0) = \sum_{i=r+1}^{n} \frac{k\cos\theta_i \sin\theta_i}{1 + k\sin\theta_i} = \frac{1}{2}\sum_{i=r+1}^{n} \frac{\sin 2\theta_i}{1 + k\sin\theta_i} \tag{9.6}$$

Thus, the LMP test for testing $H_0$ against $H_{1r}$ is

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n} \frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} > c$$

where $c$ is a constant depending on the level of significance.

(b) Since $\frac{\sin 2\Theta_i}{1+k\sin\Theta_i}$ are i.i.d under $H_0$ and sup is a continuous function on $D[0,1]$ we have by Functional Central Limit Theorem (Bhattacharya and Waymire, 1992) that

$$\max_{1 \le r \le n-1} \frac{\sum_{i=r+1}^{n} \frac{\sin 2\Theta_i}{1 + k\sin\Theta_i} - (n-r)\eta}{\sqrt{n}\tau} \tag{9.7}$$

converges in distribution to $\sup_{0 \le t \le 1} B_0^*(t)$.

## 9.4 Outlier Problem in Rattihali - SenGupta's Skewed Circular Distribution

Rattihali-SenGupta's skewed circular distribution (Rattihali and SenGupta, 2000) has probability density function given by :

$$f(\theta; k_1, k_2, \mu) = \frac{1}{C(k_1, k_2, \mu)} \exp[k_1 \cos(\theta - \mu) + k_2 \cos 2\theta]$$
$$0 \leq \theta < 2\pi, k_1, k_2 > 0, 0 \leq \mu < 2\pi. \quad (9.8)$$

(Rattihali and SenGupta, 2000). We will denote this distribution as RS$(k_1, k_2, \mu)$. The Figures 7 and 8 of Chapter 15, give the graphs of the above density for different values of $k_1, k_2$ and $\mu$.

Suppose $\Theta_1, \ldots, \Theta_n$ are all independent. Let $k_1, k_2$ be known. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as RS$(k_1, k_2, 0)$ against the alternative $H_1^*$ : there exists $r, 1 \leq r \leq n$, such that $\Theta_1, \ldots, \Theta_{r-1}, \Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as RS$(k_1, k_2, 0)$ and $\Theta_r$ is distributed as RS$(k_1, k_2, \mu), \mu > 0$. Let $H_r$ denote the hypothesis that the $r^{th}$ observation is the outlier.

**Theorem 21 :** *(a) The* LMP *test for testing* $H_0$ *against* $H_r$ *is given by :*

$$Reject\ H_0\ if\ \sin\ \Theta_r > c$$

*for some constant c depending on the level of significance.*
*(b) The* LMPTT *for testing* $H_0$ *against* $H_1^*$ *is given by :*

$$\max_{1 \leq r \leq n} \sin\ \Theta_r > c$$

*for some constant c depending on the level of significance.*

**Proof :** (a) Let $\theta_1, \ldots, \theta_n$ be the observations. The log-likelihood is

$$\ell(\mu; \theta_1, \ldots, \theta_n) = n(n-1) \ln C(k_1, k_2, 0) - \ln C(k_1, k_2, \mu)$$
$$+ k_1 \sum_{i \neq r} \cos \theta_i + k_1 \cos(\theta_r - \mu) + k_2 \sum_{i=1}^{n} \cos 2\theta_i$$
$$\frac{\partial \ell}{\partial \mu} = -\frac{C'(k_1, k_2, \mu)}{C(k_1, k_2, \mu)} + k_1 \sin(\theta_r - \mu)$$
$$\frac{\partial \ell}{\partial \mu}(0) = -\frac{C'(k_1, k_2, 0)}{C(k_1, k_2, 0)} + k_1 \sin \theta_r$$

Thus the LMP test for testing $H_0$ against $H_r$ is

$$\textit{Reject } H_0 \text{ if } \sin\,\Theta_r > c$$

where $c$ is a constant depending on the level of significance.

(b) Thus the LMPTT for testing $H_0$ against $H_1^*$ is

$$\textit{Reject } H_0 \text{ if } \max_{1\leq r\leq n} \sin\,\Theta_r > c$$

where $c$ is a constant depending on the level of significance. Since under $H_0, \Theta_1, \ldots, \Theta_n$, are i.i.d the value of $c$ can be explicitly computed using numerical integration.

## 9.5 Change Point Problems in Rattihali - Sen-Gupta's Skewed Circular Distribution

Suppose $\Theta_1, \ldots, \Theta_n$ are mutually independent and $k_1, \mu$ are known.

**Case I :** Note that for $k_2 = 0$, this distribution reduces to CN$(\mu, \kappa_1)$ while for $\mu = 0$, it reduces to a possibly bimodal distribution. We treat these two cases separately below. Without loss of generality we assume that $\mu = 0$ and further assume $\kappa_1$ is known. We are interested to test the hypothesis, $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as RS$(k_1, 0, 0)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as RS$(k_1, 0, 0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as RS$(k_1, k_2, 0), k_2 > 0$, for some $r, 1 \leq r \leq n-1$. We propose a test motivated by the LMP test for known $r$. We denote by $H_{1r}$ the alternative hypothesis that the possible change point is at $r$ where $r$ is known.

**Theorem 22 :** *(a) In testing $H_0$ against $H_{1r}$ the* LMP *test is given by : Reject $H_0$ if*

$$\sum_{i=r+1}^{n} \cos 2\Theta_i > c \tag{9.9}$$

*for some constant c depending on the level of significance.*

*(b) The* LMPTT *of $H_0$ against $H_1$ is based on the statistic*

$$\Lambda = \frac{\sum_{i=r+1}^{n} \cos 2\Theta_i - (n-r)\nu}{\sqrt{n}\tau} \tag{9.10}$$

*where $\nu$ and $\tau$ are the mean and standard deviation of $\cos 2\Theta_i$ under $H_0$. Then the asymptotic null distribution of $\Lambda$ is same as that of $\sup_{0\le t\le 1} B_0^*(t)$.*

**Proof :** (a) Let $\theta_1, \ldots, \theta_n$ denote the observations $0 \le \theta_i < 2\pi$. Observe that the log-likelihood is

$$\begin{aligned} \ell(k_2; \theta_1, \ldots, \theta_n) &= -r \ln C(k_1, 0, 0) - (n-r) \ln C(k_1, k_2, 0) \\ &\quad + k_1 \sum_{i=1}^{n} \cos\theta_i + k_2 \sum_{i=r+1}^{n} \cos 2\theta_i \end{aligned} \tag{9.11}$$

Now,

$$\frac{\partial \ell}{\partial k_2}(0) = \frac{-(n-r)C'(k_1,0,0)}{C(k_1,0,0)} + \sum_{i=r+1}^{n} \cos 2\theta_i \tag{9.12}$$

Hence, the LMP test for $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n} \cos 2\Theta_i > c$$

where c is a constant depending on the level of significance.

(b)Under $H_0$,

$$\begin{aligned} E(\cos 2\Theta_i) &= 1 + 2\frac{I_0''(k_1)}{I_0(k_1)} \\ &= \nu(\text{say}) \end{aligned} \tag{9.13}$$

and,

$$\begin{aligned} \mathrm{V}(\cos 2\Theta_i) &= 4\left[\frac{I_0^{(4)}(k_1)}{I_0(k_1)} - \left\{\frac{I_0''(k_1)}{I_0(k_1)}\right\}^2\right] \\ &= \tau^2(\text{say}) \end{aligned} \tag{9.14}$$

Hence by the Functional Central Limit Theorem (Bhattacharya and Waymire, 1992) and using the fact that sup is a continuous function on $D[0,1]$ we have

$$\max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} \cos 2\Theta_i - (n-r)\nu}{\sqrt{n}\tau} = \sup_{\frac{1}{n}\le t\le 1-\frac{1}{n}} \frac{\sum_{i=[nt]+1}^{n} \cos 2\Theta_i - (n-r)\nu}{\sqrt{n}\tau}$$

converges in distribution to $\sup_{0\le t\le 1} B^*[0,1]$

**Case II :** We next assume that $k_1, k_2$ are known and $\mu$ is unknown. Suppose $\Theta_1, \ldots, \Theta_n$ are mutually independent. We are interested to test the hypothesis, $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{RS}(k_1, k_2, 0)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{RS}(k_1, k_2, 0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{RS}(k_1, k_2, \mu), \mu > 0$, for some $r, 1 \le r \le n-1$. We propose a LMPTT for testing $H_0$ against $H_1$. We denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$ where $r$ is known.

**Theorem 23 :** *(a) In testing $H_0$ against $H_{1r}$ the* LMP *test is given by :*

$$\textit{Reject } H_0 \textit{ if } S_{r+1}^n(\Theta, 0) > c \tag{9.15}$$

*where $c$ is a constant depending on the level of significance.*

*(b) The* LMPTT *of $H_0$ against $H_1$ is based on the statistic*

$$\Lambda = \max_{1\le r\le n-1} \frac{S_{r+1}^n(\Theta, 0)}{\sqrt{n}\eta} \tag{9.16}$$

*where $\eta$ is the standard deviation of $\sin\Theta_i$ under $H_0$. Then the asymptotic null distribution of $\Lambda$ is the same as that of $\sup_{0\le t\le 1} B_0^*(t)$.*

**Proof :** (a) Let $\theta_1, \ldots, \theta_n$ be the observations, $0 \leq \theta_i < 2\pi$. Then the log-likelihood is

$$\begin{aligned}\ell(\mu; \theta_1, \ldots, \theta_n) &= -r \ln C(k_1, k_2, 0) - (n-r) \ln C(k_1, k_2, \mu) \\ &\quad + k_1 \left[C_1^r(\theta, 0) + C_{r+1}^n(\theta, \mu)\right] + k_2 \sum_{i=1}^{n} \cos 2\theta_i. \quad (9.17)\end{aligned}$$

Therefore,

$$\frac{\partial \ell}{\partial \mu}(0) = \frac{(n-r)C'(k_1, k_2, 0)}{C(k_1, k_2, 0)} + k_1 S_{r+1}^n(\theta, 0) \quad (9.18)$$

Hence the LMP test of $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } S_{r+1}^n(\Theta, 0) > c$$

where c is a constant depending on the level of significance.

(b) Under $H_0$ ,

$$E(\sin \Theta_i) = 0 \quad (9.19)$$

and let,

$$\text{Var}(\sin \Theta_i) = \eta(\text{say}) \quad (9.20)$$

Then by an application of Functional Central Limit Theorem and using the fact that sup is a continuous function on $D[0,1]$ we conclude that the asymptotic null distribution of

$$\max_{1 \leq r \leq n-1} \frac{S_{r+1}^n(\Theta, 0)}{\sqrt{n}\,\eta} = \sup_{\frac{1}{n} \leq r \leq 1 - \frac{1}{n}} \frac{S_{r+1}^n(\Theta, 0)}{\sqrt{n}\,\eta}$$

is the same as that of $\sup\limits_{0 \leq t \leq 1} B_0^*(t)$

## 9.6 Outlier Problem in Batschelet's Skewed Circular Distribution

The Batschelet's skewed circular distribution is an extension of the circular normal distribution. The probability density function of this distribution is

$$\begin{aligned} f(\theta;k,\nu) &= \frac{1}{c(k,\nu)}\exp\left[k\cos(\theta+\nu\cos\theta)\right] \\ & 0\le\theta<2\pi, k>0, -\infty<\nu<\infty. \end{aligned} \tag{9.21}$$

We will denote this distribution as Ba$(k,\nu)$. Note that $\nu=0$ yields CN$(0,k)$. The Figures 9 and 10 of Chapter 15 give the graphs of the above density for different values of $k$ and $\nu$.

Suppose $\Theta_1,\ldots,\Theta_n$ are independent random variables and $k$ is known. We want to test the hypothesis $H_0 : \Theta_1,\ldots,\Theta_n$ are identically distributed as Ba$(k,0)$ against the alternative $H_1^* : \Theta_1,\ldots,\Theta_{r-1},\Theta_{r+1},\ldots,\Theta_n$ are identically distributed as Ba$(k,0)$ and $\Theta_r$ is distributed as Ba$(k,\nu),\nu>0$ for some $r, 1\le r\le n$. Let $H_r$ denote the hypothesis that the $r^{th}$ observation is the outlier. The next theorem gives the LMP test for testing $H_0$ against $H_r$ and the LMPTT for testing $H_0$ against $H_1^*$.

**Theorem 24 :** *(a) The* LMP *test for testing $H_0$ against $H_r$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \sin 2\,\Theta_r < c$$

*where $c$ is a constant depending on the level of significance.*
*(b) The* LMPTT *for testing $H_0$ against $H_1^*$ is given by :*

$$\min_{1\le r\le n}\sin 2\,\Theta_r < c$$

*where $c$ is a constant depending on the level of significance.*

**Proof :** (a) Let $\theta_1,\ldots,\theta_n$ be the observations. The log-likelihood is

$$\begin{aligned} \ell(\nu;\theta_1,\ldots,\theta_n) &= -(n-1)\ln\ c(k,0) - \ln\ c(k,\nu) \\ &\quad +k\sum_{i\ne r}\cos\theta_i + k\cos(\theta_r+\nu\cos\theta_r) \end{aligned}$$

$$\frac{\partial \ell}{\partial \nu} = -\frac{c'(k,\nu)}{c(k,\nu)} - k\sin(\theta_r + \nu\cos\theta_r)\cos\theta_r$$

$$\frac{\partial \ell}{\partial \nu}(0) = -\frac{c'(k,0)}{c(k,0)} - k\sin\theta_r\cos\theta_r$$

$$= -\frac{c'(k,0)}{c(k,0)} - \frac{k}{2}\sin 2\theta_r$$

Thus the LMP test for testing $H_0$ against $H_r$ is

$$Reject\ H_0 \text{ if } \sin 2\,\Theta_r < c$$

for some constant $c$ depending on the level of significance.

(b) Thus the LMPTT for testing $H_0$ against $H_1^*$ is

$$Reject\ H_0 \text{ if } \min_{1\leq r\leq n} \sin 2\,\Theta_r < c$$

for some constant $c$ depending on the level of significance. The value of $c$ can be computed from the distribution of $\sin 2\Theta_r$ under $H_0$ using numerical integration.

## 9.7 Change Point Problem in Batschelet's Skewed Circular Distribution

Let us assume that $k$ is known and $\Theta_1, \ldots, \Theta_n$ are mutually independent. We are interested to test the hypothesis, $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $\text{Ba}(k,0)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as $\text{Ba}(k,0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\text{Ba}(k,\nu), \nu > 0$, for some $r, 1 \leq r \leq n-1$. We derive an LMPTT for testing $H_0$ against $H_1$. We denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$.

**Theorem 25 :** *(a) In testing $H_0$ against $H_{1r}$ the* LMP *test is given by :*

$$Reject\ H_0\ \ if \sum_{i=r+1}^{n} \sin 2\Theta_i < c$$

*where $c$ is a constant depending on the level of significance.*

*(b) The* LMPTT *for $H_0$ against $H_1$ is based on the statistic*

$$\Lambda = \min_{1 \le r \le n-1} \frac{\sum_{i=r+1}^{n} \sin 2\Theta_i}{\sqrt{n}\zeta}$$

*where $\zeta$ is the standard deviation of* $\sin 2\Theta_i$ *under $H_0$. The asymptotic null distribution of $\Lambda$ is same as that of* $\inf_{0 \le t \le 1} B_0^*(t)$.

**Proof :**

$$\begin{aligned} \ell(\nu; \theta_1, \ldots, \theta_n) &= -r \ln I_0(k) - (n-r) \ln c(k, \nu) \\ &\quad + k \left[ C_1^r(\theta, 0) + \sum_{i=r+1}^{n} \cos\left(\theta_i + \nu \cos \theta_i\right) \right] \end{aligned} \tag{9.22}$$

Therefore,

$$\frac{\partial \ell}{\partial \nu}(0) = -(n-r)\frac{c'(k,0)}{c(k,0)} - \frac{k}{2} \sum_{i=r+1}^{n} \sin 2\theta_i \tag{9.23}$$

Hence the LMP test for $H_0$ against $H_{1r}$ is

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n} \sin 2\Theta_i < c$$

where $c$ is constant depending on the level of significance.

(b) Under $H_0$,

$$\mathrm{E}(\sin 2\Theta_i) = 0 \tag{9.24}$$

and

$$\begin{aligned}\mathrm{Var}(\sin 2\Theta_i) &= 4\frac{I_0''(k) - I_0^{(4)}(k)}{I_0(k)} \\ &= \zeta^2(\text{say}) \end{aligned} \tag{9.25}$$

where $I_0^{(4)}(k)$ is the fourth derivative of $I_0(k)$ with respect to $k$.

Hence by using the Functional Central Limit Theorem and the fact that inf is a continuous function on $D[0,1]$ we get that the asymptotic null distribution of

$$\min_{1\leq r\leq n-1} \frac{\sum_{i=r+1}^{n} \sin 2\Theta_i}{\sqrt{n}\zeta} = \inf_{\frac{1}{n}\leq t\leq 1-\frac{1}{n}} \frac{\sum_{i=[nt]+1}^{n} \sin 2\Theta_i}{\sqrt{n}\zeta}$$

is same as that of $\inf_{0\leq t\leq 1} B_0^*(t)$

**Remark 14 :** *The case of nuisance parameters for each of the above testing problems may be tackled by the general unified theory of* NRTT *derived in section 12.10 and hence are not discussed here separately.*

# Chapter 10

# Alternative Approaches to Change Point Problems for the Mean Direction

## 10.1 Introduction

In this chapter we provide some alternative approaches to change point problems for the mean direction. In section 10.2 we discuss the Semi-Bayesian and Hierarchical Bayes' approaches. In section 10.3 we introduce a new method of computing the integrated likelihood which we call "modified integrated likelihood". This method can be used to tackle change point problems in situations where nuisance parameters are present. In section 10.4 we discuss the use of randomization tests in the context of change point problem. In section 10.5 we provide a Markov chain based approach which can be used for change point prediction.

## 10.2 Semi Bayesian & Hierarchical Bayes' Approaches for the Change Point Problem

In the next two sections we indicate the possibility of applying the semi Bayesian and hierarchical Bayes' approaches in solving the change point problem. In section 10.2.1 we take the semi-Bayesian route and illustrate it with an example whereas in section 10.2.2 we discuss the hierarchical Bayes' approach for estimation of the change point using an example.

### 10.2.1 Semi-Bayesian Approach

Often we have some a-priori information regarding the possible location of the change point which can be quantified in the form of a prior distribution. Let $p_i$ denote the probability that 'i' is the 'point of change', $1 \leq i \leq n-1$. Let $p_n$ denote the probability of no change, $\sum_{i=1}^{n} p_i = 1$. Assume that all the parameters $(\mu_0, \mu_1, \kappa)$ are known. We are interested to test $H_0 : \Theta_1, \ldots, \Theta_n$ are CN$(\mu_0, \kappa)$ against the alternative $H_1$. There exist $r, 1 \leq r \leq n-1$ such that $\Theta_1, \ldots, \Theta_r$ are CN$(\mu_0, \kappa)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are CN$(\mu_1, \kappa)$. Then the likelihood under $H_0$ is,

$$f_0(\theta) = p_n (2\pi I_0(\kappa))^{-n} \exp\left[\kappa C_1^n(\theta, \mu_0)\right] \tag{10.1}$$

and that under $H_1$ is,

$$f_1(\theta) = \sum_{i=1}^{n-1} p_i (2\pi I_0(\kappa))^{-n} \exp\left[\kappa \left\{C_1^i(\theta, \mu_0) + C_{i+1}^n(\theta, \mu_1)\right\}\right]. \tag{10.2}$$

Then the Bayes' factor is,

$$\frac{f_1(\theta)}{f_0(\theta)} = \sum_{i=1}^{n-1} (p_i/p_n) \exp\left[\kappa \left\{C_{i+1}^n(\theta, \mu_1) - C_{i+1}^n(\theta, \mu_0)\right\}\right] \tag{10.3}$$

which after some simplifications becomes

$$\frac{f_1(\theta)}{f_0(\theta)} = k. \sum_{i=1}^{n-1} (p_i/p_n) \exp\left(S_{i+1}^n\left(\theta, (\mu_1 + \mu_0)/2\right)\right) \tag{10.4}$$

where $k$ is a constant. A test for $H_0$ can then be based on the Bayes' factor. $H_0$ is rejected if the Bayes' factor is large.

**Example:** Let

$$p_i = c.\ e^{-\lambda}\ \lambda^i/i!\ ,1 \le i \le n,\ (\lambda \text{ known}) \tag{10.5}$$

where $c$ is chosen so that $\sum_{i=1}^{n} p_i = 1$.

Then

$$p_i/p_n = n!\lambda^{i-n}/i!\ ,1 \le i \le n \tag{10.6}$$

and

$$\frac{f_1(\theta)}{f_0(\theta)} = k. \sum_{i=1}^{n-1} \left(\lambda^{i-n}/i!\right) \exp\ \left(S_{i+1}^{n}\left(\theta, (\mu_1 + \mu_0)/2\right)\right), \tag{10.7}$$

where 'k' is a constant. $H_0$ is rejected if $\frac{f_1(\theta)}{f_0(\theta)}$ is large.

### 10.2.2 Hierarchical Bayes' Approach

This popular approach (Berger, 1985) is used in cases where it is difficult to specify the parameters of a prior distribution. The escape route is to specify priors for these parameters as well. For the change point problem we assume a prior on the change point. The prior is specified upto its form but its parameters are left unspecified. We further specify a prior on these unspecified parameters. Then to get the estimate of change point we merely find the posterior distribution of the change point given the data and find the posterior mode.

To illustrate this approach we assume that the change point is a random variable denoted by $\boldsymbol{K}$. Let the prior probability of the event $\boldsymbol{K} = k$ be given by

$$P(\boldsymbol{K} = k) = c.\binom{n}{k}\ p^k(1-p)^{n-k}, k = 1, \ldots n, \tag{10.8}$$

where $c$ is a constant such that $\sum_k P(\boldsymbol{K} = k) = 1$. Further assume that $p$ is distributed uniformly over (0,1). Let

$$h(t) = \exp\left[\kappa\left\{C_1^t(\theta,\mu_0) + C_{t+1}^n \cos(\theta,\mu_1)\right\}\right], \tag{10.9}$$

$\mu_0, \mu_1$ and $\kappa$ are all known. The posterior distribution of $\mathbf{K}$ is then given by

$$P(\ K = k|\theta_1 \ldots, \theta_n) = \frac{h(k)\ \binom{n}{k}\ B(k+1, n-k+1)}{\sum\limits_{\ell=1}^{n} \binom{n}{\ell}\ B(\ell+1, n-\ell+1)} \tag{10.10}$$

The posterior mode can be used to estimate the change point.

## 10.3 A Modified Integrated Likelihood Approach

In the usual integrated likelihood approach, the likelihood of all the observations is first computed and then this likelihood is integrated with respect to the prior density on the nuisance parameter. The resultant likelihood is used for the purpose of drawing inference on the parameter of interest. *In this new approach* for eliminating nuisance parameters we first consider the likelihood for each observation separately. A proper prior density is chosen for the nuisance parameter assumed to be the same for all observations. The likelihood corresponding to each observation is multiplied by the prior density and then integrated over the range of all possible values of the nuisance parameter to get the integrated likelihood for that observation. The likelihood of all the observations is then obtained by multiplying all of these individual integrated likelihoods together to yield the modified integrated likelihood. This likelihood is then used for drawing inference on the parameter of interest. We illustrate this technique by applying it on the change point problem for the Papakonstantinou's skewed circular distribution. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $P(k, 0)$ against the alternate $H_1$ : There exist $r, 1 \leq r \leq n-1$ such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as $P(k, 0)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $P(k, \nu), \nu > 0$. For each fixed $r$, let $H_{1r}$ denote the alternative hypothesis that the change point is at $r$. Note that in this case $k$ is a nuisance parameter. We put a prior density $\gamma$ with $E_\gamma(k) \neq 0$, on $k$. Now for a fixed $r$, the likelihood for one observation under $H_{1r}$ is

$$\begin{aligned} L(\nu_i, k; \theta_i) &= \frac{1}{2\pi} + \frac{k}{2\pi} \sin(\theta_i + \nu_i \sin\theta) \\ &\quad 0 \le \theta_i < 2\pi, -1 < k < 1 \end{aligned} \tag{10.11}$$

where

$$\begin{aligned} \nu_i &= 0 \text{ if } i = 1, \ldots, r \\ &= \nu \text{ if } i = r+1, \ldots, n \end{aligned} \tag{10.12}$$

The integrated likelihood for one observation is

$$\begin{aligned} L^*(\nu_i; \theta_i) &= \int_{-1}^{1} L(\nu_i, k; \theta_i)\gamma(k)dk \\ &= \frac{1}{2\pi} + \frac{E_\gamma(k)}{2\pi} \sin(\theta_i + \nu_i \sin\theta_i) \end{aligned} \tag{10.13}$$

Hence, the modified integrated likelihood is

$$L^*_\gamma(\nu; \theta) = \prod_{i=1}^{n} L^*(\nu_i; \theta_i) \tag{10.14}$$

$$\Longrightarrow \ln L^*_\gamma(\nu; \theta) = \sum_{i=1}^{n} \ln L^*(\nu_i; \theta_i) \tag{10.15}$$

$$\begin{aligned} &= \sum_{i=1}^{r} \ln\left(\frac{1}{2\pi} + \frac{E_\gamma(k)}{2\pi} \sin\theta_i\right) \\ &\quad + \sum_{i=r+1}^{n} \ln\left(\frac{1}{2\pi} + \frac{E_\gamma(k)}{2\pi} \sin(\theta_i + \nu \sin\theta_i)\right) \end{aligned} \tag{10.16}$$

We compute the maximum value of $\ln L^*_\gamma(\nu; \theta)$ and compare it with the specified value under $H_0$ i.e $\ln L^*_\gamma(0; \theta)$. Let $\nu^*_r$ be the value of $\nu$ for which $\ln L^*_\gamma(\nu; \theta)$ is maximum. If the difference $\ln L^*_\gamma(\nu^*_r; \theta) - \ln L^*_\gamma(0; \theta)$ is substantially large then we reject $H_0$ in favour of $H_{1r}$. To test for $H_0$ against $H_1$

the appropriate quantity to look for is

$$\max_{1 \le r \le n-1} \ln L^*_\gamma(\nu^*_r;\theta) - \ln L^*_\gamma(0;\theta)$$

If this difference is large then we reject $H_0$ in favour of $H_1$. The estimate of change point is that value of $r$ for which $\ln L^*_\gamma(\nu^*_r;\theta)$ has the largest value.

## 10.4 Randomization Tests for the Change Point Problem for the Mean Direction

Suppose we have $n$ observations from a circular distribution. We are interested to know whether all observations come from the same population say $F_0$ having mean direction $\mu_0$ or there is a point after which all the observations come from another distribution say $F_1$ with mean direction $\mu_1$. Usually $F_0, F_1$ have the same form, e.g. both may be circular normal and there is only a location parameter shift.

We consider the problem of testing $H_0 : \Theta_1, \ldots, \Theta_n$ are i.i.d. $F_0$ against the alternative. $H_1$ : There exist $r, 1 \le r \le n-1$ such that $\Theta_1, \ldots, \Theta_r$ are i.i.d. $F_0$ and $\Theta_{r+1}, \ldots, \Theta_n$ are i.i.d. $F_1$.

Lombard (1986) provides a non-parametric test for testing this hypothesis based on the uniform scores test. Here we propose a randomization test (Rao, 1973) utilising the fact that under the null hypothesis the observations are all independent and hence exchangeable. This test is distribution free and completely data based.

Since we are looking for a change in mean direction it is natural for us to use a test statistic which is based on the angular difference of the two mean directions. The following statistic is based on these considerations :
For $1 \le k \le n-1$, let

$$T_k = \left| \tan^{-1}\frac{S_1^k(\theta,0)}{C_1^k(\theta,0)} - \tan^{-1}\frac{S_{k+1}^n(\theta,0)}{C_{k+1}^n(\theta,0)} \right| \tag{10.17}$$

where $\tan^{-1}(.)$ is so defined as to yield an unique value.

Define

$$D_k = \min(T_k, 2\pi - T_k)$$

Thus $D_k$ gives the difference in the mean direction of the first $k$ and the last $(n-k)$ observations. Define the test statistic

$$D = \max_{1\leq k\leq n-1} D_k.$$

The value of $D$ is computed for all possible permutations of the observations. This gives rise to $n!$ values of $D$ which are then sorted in increasing or decreasing order. If the observed value of $D$ is towards higher end of this ordered list then we reject $H_0$. Since $n!$ increases very rapidly with $n$, so even for moderate values of $n$ it is not possible to compute $D$ for all permutations even with the most powerful computers. Thus it becomes necessary to use a random sample of permutations to use this test. A sample of $k$ permutations are chosen randomly from the possible $n!$ permutations and the above method is applied to the values of $D$ computed on these permutations of the observations.

*Example :* We applied the Randomization Test on the flare data (Lombard, 1986). Since this test is designed to indicate the presence of atmost one change point, and earlier analyses had indicated the presence of two change points in this data set, one at 12 and the other at 42, we decided to consider only the observations number 1-42 for the purpose of application of this randomization test. We generated 2000 random permutations of these observations and computed the statistic $D$ for each one of these. The resultant values were sorted and the observed value of D i.e. 2.94 is compared against the list. It is seen that more than 90% of the values in the list are less than the observed value. Hence we can say that there is a change point in the data set at 10% level of significance.

Note : It has been observed by one of the referees that randomization type tests based on the test statistics considered in Chapters 5 to 8 can be developed along the same line as randomization test based on 10.17. In all setups considered in Chapters 5 to 8 the null hypothesis corresponds to the i.i.d. case.

## 10.5 Prediction of Change Points for the Mean direction - Markov Chain Based Approach

We give a method for predicting change points using a Markov Chain based approach. We assume that at each point of time 't' there is a pair of circular random variables $(\Theta_t, W_t)$. We only observe the $\Theta_t$'s and the $W_t$'s remain unobserved. If $W_t = 0$ then $\Theta_t$ comes from a circular distribution having density $f_0$ whereas if $W_t = 1$ then $\Theta_t$ comes from a circular distribution having density $f_1$. We assume that $\Theta_1, \ldots, \Theta_n$ are independent given $W_1, \ldots, W_n$ and $W_1 (= 0), W_2, \ldots, W_k, \ldots$ form a Markov Chain with transition probability matrix,

| | $W_{t+1} = 0$ | $W_{t+1} = 1$ |
|---|---|---|
| $W_t = 0$ | 1-p | p |
| $W_t = 1$ | 0 | 1 |

We call the parameter $p$ the 'propensity for change'. Note that as $p$ increases the chance of a change point being present in the given data set increases. When $p = 0$ we can conclude not only that all the given observations come from the distribution with density $f_0$ but also all future observations will come from the same distribution. Thus in this situation we can predict that there will be no change of distribution in future. Let $n^* = \min\{n : W_n = 1\}$. If $p > 0$ then it is easy to see that $n^*$ has the geometric distribution with parameter $p$. We can use this fact to predict the change point. Thus we see that it is of some interest to test $H_0 : p = 0$ against $H_1 : p > 0$. Let $\theta_1, \ldots, \theta_n$ be the given observations. The following theorem gives the LRT for this problem. Let,

$$L(p) = p \sum_{r=1}^{n-1} \prod_{i=1}^{r} f_0(\theta_i) \prod_{i=r+1}^{n} f_1(\theta_i)(1-p)^{r-1} + (1-p)^{n-1} \prod_{i=1}^{n} f_0(\theta_i)$$

where $0 \leq p \leq 1$ and we use the convention $0^0 = 1$.

**Theorem 26** *In testing $H_0$ against $H_1$ the LRT is :*
*Reject $H_0$ if*

$$\Lambda = \frac{L(0)}{\sup_{0 \leq p \leq 1} L(p)} < c$$

*where $c$ is a constant depending on the level of significance.*

**Proof:** The proof follows immediately from the definition of the LRT.

In the above theorem, to compute the denominator of the LRT-statistic i.e. $\sup_{0 \leq p \leq 1} L(p)$ we need to compute the MLE of $p$ which is a solution of $L'(p) = 0$. This usually calls for solving a complicated polynomial equation involving higher powers of $p$. Instead of attempting to solve the complicated equation analytically one may try obtain the approximate value of the denominator numerically. Genetic Algorithms(GA) are often used for this purpose. We use a variation of the GA for continuous parameters called Genetic Algorithm Without Coding of Parameters, (GAWCP), (see e.g. Raol and Jalisatgi (1996), Haupt and Haupt (1998) etc.) to find the approximate value of the denominator. We adopt a very simple version of the algorithm with the consequence that it takes much longer to reach the maximum value with this algorithm than with more complicated algorithms.

The description of our algorithm is as follows: Start with parameter values $p_a$ and $p_b$.

$$\text{Suppose } L(p_a) > L(p_b)$$

We compute two new quantities

$$p_c = \frac{1}{2}(p_a + p_b) \tag{10.18}$$

and

$$p_d = p_a + \xi, \tag{10.19}$$

where $\xi \sim U(-1,1)$, where $U(a,b)$ denotes the uniform distribution on the open interval $(a,b)$.

(If $p_d \leq 0$ or $p_d \geq 1$ a new $\xi$ is generated and $p_d$ is recomputed). The values $L(p_c)$ and $L(p_d)$ are computed. Among the four values $L(p_a), L(p_b), L(p_c)$ and $L(p_d)$ parameter values corresponding to the largest and the next are retained and are called $p_a$ and $p_b$ respectively. The entire procedure is then repeated with the new values of $p_a$ and $p_b$.

It is easily seen that the value of $L(p)$ non-decreasing function of the number of iterations. When the value of $L(p)$ does not increase for a large number of iterations we conclude that the maximum value has been reached (approximately). This value is then used in the denominator of the LRT.

**Remark 15** : *If the densities $f_0$ and $f_1$ contain unknown parameters it becomes necessary to take supremum of the likelihood function over the parameter space of these parameters for computing the* LRT. *The above algorithm can be easily generalized to accommodate more than one parameters.*

**Remark 16** : *The distribution of $\xi$ need not be necessarily $U(-1,1)$. Other distributions including circular distributions may be used.*

# Chapter 11

# DATA ANALYTIC TOOLS

## 11.1 Introduction

In real-life situations statisticians often require some tools which are simple to use and can quickly indicate whether a feature is present in the data set or not. In this chapter we discuss three data analytic tools for exploratory change point detection. These tools can quickly indicate whether a change point is present in the data set or not. If the presence of a change point is indicated then one can proceed with formal testing as discussed in other chapters in this thesis. The data analytic tools introduced here are
(a) Changeogram
(b) Circular Difference Tables and
(c) Circular CUSUM.

## 11.2 Tools for Indicating The Presence of a Change Point

### 11.2.1 Changeogram

A Changeogram displays pictorially in terms of directed arrows, each of unit length, the direction in terms of the angle as given by the corresponding observation. A change in the mean direction can be visually observed by inspecting the sequence of arrows, carefully. In Figures 2 and 3, Chapter 15, we give examples of changeogram based on the flare data and the wind data.

### 11.2.2 Circular Difference Table

The circular difference table is another data analytic tool particularly suitable for detecting changes of moderate to large magnitudes when the dispersion is not too large. The circular difference table is constructed by considering the change of direction between two successive observations. For eg. if $\theta_t$ and $\theta_{t+1}$ are the two successive observations then we consider the circular difference to be $\min\left(\left|\theta_{t+1}-\theta_t\right|, 2\pi-\left|\theta_{t+1}-\theta_t\right|\right)$. The circular difference table shows a large value when a change in mean direction occurs. If the dispersion is not too large such a large value immediately indicates the presence of a change point. We illustrate this tool by applying it on the flare data and also on the wind data. It is seen that the tool is not that effective in indicating a change point in case of flare data because of the high amount of dispersion in the data set (see Table A and B). However for the wind data the change point is (see Table C) well indicated by the circular difference table. Note that the circular difference between observations 17 and 18 is 73° which is much larger than the circular difference between any two successive observations in the data set. Thus a change point is indicated at 17.

TABLE A

Circular Difference Table for flare data : Observations 1 - 42

| $t$ | Diff | $t$ | Diff | $t$ | Diff |
|---|---|---|---|---|---|
| 1 | 88.5 | 15 | 100 | 29 | 77 |
| 2 | 17.2 | 16 | 122.2 | 30 | 30.1 |
| 3 | 16.3 | 17 | 12.1 | 31 | 28.6 |
| 4 | 52.7 | 18 | 20.4 | 32 | 71.7 |
| 5 | 19 | 19 | 78.2 | 33 | 53.4 |
| 6 | 111.8 | 20 | 46.3 | 34 | 75.1 |
| 7 | 52.6 | 21 | 106.1 | 35 | 120.4 |
| 8 | 46.4 | 22 | 50.4 | 36 | 150.2 |
| 9 | 78 | 23 | 17.7 | 37 | 6.3 |
| 10 | 114.6 | 24 | 13.2 | 38 | 74.5 |
| 11 | 122.7 | 25 | 62.4 | 39 | 96.1 |
| 12 | 166.9 | 26 | 15.3 | 40 | 42 |
| 13 | 115.3 | 27 | 54 | 41 | 92.5 |
| 14 | 36.9 | 28 | 147.2 | | |

TABLE B

Circular Difference Table for flare data : Observations 13 - 48

| $t$ | Diff | $t$ | Diff | $t$ | Diff |
|---|---|---|---|---|---|
| 13 | 115.3 | 29 | 77 | 45 | 13.6 |
| 14 | 36.9 | 30 | 30.1 | 46 | 28.6 |
| 15 | 100 | 31 | 28.6 | 47 | 13.8 |
| 16 | 122.2 | 32 | 71.7 | 48 | 120 |
| 17 | 12.1 | 33 | 53.4 | 49 | 25.3 |
| 18 | 20.4 | 34 | 75.1 | 50 | 11.8 |
| 19 | 78.2 | 35 | 120.4 | 51 | 66.1 |
| 20 | 46.3 | 36 | 150.2 | 52 | 59.5 |
| 21 | 106.1 | 37 | 6.3 | 53 | 164.6 |
| 22 | 50.4 | 38 | 74.5 | 54 | 3 |
| 23 | 17.7 | 39 | 96.1 | 55 | 159.1 |
| 24 | 13.2 | 40 | 42 | 56 | 51.3 |
| 25 | 62.4 | 41 | 92.5 | 57 | 152.4 |
| 26 | 15.3 | 42 | 133 | 58 | 30 |
| 27 | 54 | 43 | 113.1 | 59 | 62.3 |
| 28 | 147.2 | 44 | 121.8 | | |

TABLE C

Circular Difference Table for wind data : Observations 1 - 22

| $t$ | Diff | $t$ | Diff | $t$ | Diff |
|---|---|---|---|---|---|
| 1 | 2 | 9 | 13 | 17 | 73 |
| 2 | 7 | 10 | 8 | 18 | 8 |
| 3 | 7 | 11 | 10 | 19 | 29 |
| 4 | 3 | 12 | 10 | 20 | 32 |
| 5 | 1 | 13 | 9 | 21 | 22 |
| 6 | 8 | 14 | 19 | | |
| 7 | 4 | 15 | 31 | | |
| 8 | 6 | 16 | 8 | | |

### 11.2.3 Circular CUSUM

The circular CUSUM is another useful data analytic tool for indicating presence of change point in case the initial mean direction is known. Let $\mu$ be the known initial mean direction. Note that in a circle there are two directions in which the angular difference can be calculated, one being clockwise and the other being anticlockwise. We first obtain $\delta$ by solving the equation $e^{i\delta}.e^{i\mu} = e^{i\theta}$. If $0 \leq \delta < \pi$ we say that the angular difference is anticlockwise of magnitude $\delta$. If $\pi \leq \delta < 2\pi$ then we say that the angular difference is clockwise of magnitude $2\pi - \delta$. Analogous to the linear case we compute cumulative sums in the anticlockwise direction $C_i^A$ as follows

$$\begin{aligned} C_i^A &= C_{i-1}^A + \delta_A \\ \text{where } \delta_A &= \delta \text{ if the } i^{\text{th}} \text{ angular difference is in the anticlockwise direction} \\ &= 0 \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} C_i^C &= C_{i-1}^C + \delta_C \\ \text{where } \delta_C &= \delta \text{ if the } i^{\text{th}} \text{ angular difference is in the clockwise direction.} \\ &= 0 \text{ otherwise} \end{aligned}$$

In case of no change in mean direction both $C_i^A$ and $C_i^C$ will grow at the same rate whereas if there is a change in mean direction then one of $C_i^A$ or $C_i^C$ will grow at a much faster rate than the other. Hence if we plot $\frac{C_i^C}{C_i^A}$ or $\frac{C_i^A}{C_i^C}$ with time then we will find a distinct change of inclination in the curve if a change point is present.

*Example :* We illustrate the above method by applying it on the flare data. Since from previous analysis we know that the data consists of two change points one at 12 and the other at 42. We consider only the portion of the data consisting of observation nos. 1 to 42. Since the mean direction is unknown we use the mean direction of the first 12 observations as the true mean and construct the circular CUSUM chart. We plot the ratio $\frac{C_i^C}{C_i^A}$ over time and as expected the graph (see Figure 4, Chapter 15) shows a change of inclination after the point of change.

# Chapter 12

# Some Generalizations and Scope of Further Work

## 12.1 Introduction

In this chapter we discuss the outlier and change point problems in symmetric circular distributions other than circular normal. The distributions covered in this chapter almost exhaustively covers all distributions that have been used in real-life applications. In section 12.2 we discuss the circular uniform to circular normal change point problem. This problem may arise in real-life situations for e.g., in study of wind directions where at the beginning there may be no preferred direction but after a certain period of time may exhibit a preferred direction. In section 12.3 we study the circular uniform to circular uniform - circular normal mixture change point problem. The motivation behind the study of this problem is it's potential for use in real-life situations. The circular uniform - circular normal mixture has been used in various applications e.g., see Ducharme and Milosevic(1990), SenGupta and Pal(2001) etc. In section 12.4 we discuss the outlier problem in the Cartwright - Mitsuyasu distribution. This distribution has found much applications in oceanography. In section 12.5 we discuss the change point problem in the same distribution. In section 12.6 we discuss the outlier problem in Wrapped Cauchy distribution. This distribution has also found several applications (see e.g., Kent and Tyler (1988)). In section 12.7 we discuss the change point problem in the same distribution. In section

12.8 we discuss the outlier problem for the symmetric wrapped stable family of distributions. This is an omnibus family of distributions including some well known distributions like wrapped Cauchy and wrapped normal as its members. In section 12.9 we discuss the change point problem for the same distribution. In section 12.10 we discuss the robustness with respect to the level of significance of the LMPTT for change point problem for the mean direction in the circular normal distribution. We use the symmetric wrapped stable family of distributions as the class of alternative distributions for the purpose of robustness study. In 12.11 we provide an unified approach of constructing NRTT. This is particularly useful for drawing inference (about the existence of a change point) in situations where nuisance parameters are present. In section 12.12 we discuss the scope of further research.

## 12.2 Change Point Problem in Circular Uniform to Circular Normal

Let $\Theta_1, \ldots, \Theta_n$ be independent variables. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as circular uniform against the alternative $H_1$ : There exists $r, 1 \leq r \leq n-1$, such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as circular uniform and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{CN}(0, \kappa), \kappa > 0$. We propose a LMPTT for the above problem. For each fixed $r, H_{1r}$ denotes the alternative hypothesis that there is a change point at $r$.

**Theorem 27 :** *(a) The* LMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\text{Reject } H_0 \text{ if } C_{r+1}^n(\Theta, 0) > c$$

*where $c$ is a constant depending on the level of significance.*

*(b) The* LMPTT *for $H_0$ against $H_1$ is given by :*

$$\max_{1 \leq r \leq n-1} \frac{C_{r+1}^n(\Theta, 0)}{\sqrt{n\pi}} > c$$

*where $c$ is a constant depending on the level of significance. The asymptotic null distribution of* $\max_{1\le r\le n-1} \frac{C_{r+1}^n(\Theta,0)}{\sqrt{n\pi}}$ *is same as that of* $\sup_{0\le t\le 1} B_0^*(t)$.

**Proof :** (a) The log-likelihood of $\theta_1,\ldots,\theta_n$ is :

$$\begin{aligned}
\ell(\kappa;\theta_1,\ldots,\theta_n) &= -n\ln 2\pi - (n-r)\ln I_0(\kappa) + \kappa\, C_{r+1}^n(\theta,0) \\
\frac{\partial \ell}{\partial \kappa} &= -(n-r)\frac{I_0'(\kappa)}{I_0(\kappa)} + C_{r+1}^n(\theta,0) \\
&= -(n-r)A(\kappa) + C_{r+1}^n(\theta,0) \qquad (12.1) \\
\left.\frac{\partial \ell}{\partial \kappa}\right|_{\kappa=0} &= C_{r+1}^n(\theta,0) \qquad (12.2)
\end{aligned}$$

Hence the LMP test for $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } C_{r+1}^n(\Theta,0) > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT for $H_0$ against $H_1$ is given by :

$$\max_{1\le r\le n-1} \frac{C_{r+1}^n(\Theta,0)}{\sqrt{n\pi}} > c$$

where $c$ is a constant depending on the level of significance. Now, since under $H_0$, $\mathrm{E}(\cos\Theta_i) = 0$ and $\mathrm{Var}(\cos\Theta_i) = \pi$ we have the asymptotic null distribution of $\max_{1\le r\le n-1} \frac{C_{r+1}^n(\Theta,0)}{\sqrt{n\pi}}$ as $n \longrightarrow \infty$ is same as that of $\sup_{0\le t\le 1} B_0^*(t)$.

**Remark 17** *The outlier problem is not quite meaningful in this set-up and hence is not considered here*

## 12.3 Change Point Problem in Circular Uniform to Circular Uniform-Circular Normal Mixture

Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We want to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as circular uniform against the alternative $H_1$: There exist $r, 1 \leq r \leq n-1$, such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as circular uniform and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed with pdf

$$f(\theta) = \frac{p}{2\pi} + (1-p)\frac{1}{2\pi I_0(\kappa)} \exp(\kappa \cos\theta), \kappa > 0,\ 0 < p < 1$$

$p$ is assumed to be known. Note that when $\kappa = 0$ the above pdf reduces to $\frac{1}{2\pi}$ which is the pdf of the circular uniform distribution. As before let for each $r$, $H_{1r}$ denote the alternative hypothesis that the change point is at $r$. From the above we note that a test of $H_0$ against $H_{1r}$ is equivalent to a test of $H_0' : \kappa = 0$ against $H_{1r}' : \kappa > 0$.

**Theorem 28 :** *(a) The* LMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } C_{r+1}^n(\Theta, 0) > c$$

*where $c$ is a constant depending on the level of significance.*
*(b) The* LMPTT *for $H_0$ against $H_1$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1 \leq r \leq n-1} \frac{C_{r+1}^n(\Theta, 0)}{\sqrt{n\pi}} > c$$

*where $c$ is a constant depending on the level of significance.*
*The asymptotic null distribution of $\frac{C_{r+1}^n(\Theta,0)}{\sqrt{n\pi}}$ is the same as that of $\sup_{0 \leq t \leq 1} B_0^*(t)$.*

**Proof:** (a) The log likelihood of $\theta_1, \ldots, \theta_n$ is

$$\begin{aligned}
\ell(\kappa;\theta_1,\ldots,\theta_n) &= -r\ln 2\pi + \sum_{i=r+1}^{n} \ln\left[\frac{p}{2\pi} + \frac{(1-p)}{2\pi I_0(\kappa)}\exp(\kappa\cos\theta_i)\right] \\
\frac{\partial \ell}{\partial \kappa} &= \sum_{i=r+1}^{n} \frac{\frac{1-p}{2\pi}\,\frac{I_0(\kappa)\exp(\kappa\cos\theta_i).\cos\theta_i - \exp(\kappa\cos\theta_i)I_0'(\kappa)}{[I_0(\kappa)]^2}}{\frac{p}{2\pi} + \frac{(1-p)}{2\pi I_0(\kappa)}\exp(\kappa\cos\theta_i)} \\
\left.\frac{\partial \ell}{\partial \kappa}\right|_{\kappa=0} &= \sum_{i=r+1}^{n} \frac{(1-p)}{2\pi}\frac{\cos\theta_i}{\frac{1}{2\pi}} \\
&= (1-p)\, C_{r+1}^{n}(\theta,0) \qquad (12.3)
\end{aligned}$$

Hence the LMP test for $H_0$ against $H_{1r}$ is

$$\text{Reject } H_0 \ \text{ if } C_{r+1}^{n}(\Theta,0) > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT for $H_0$ against $H_1$ is

$$\text{Reject } H_0 \ \text{ if } \max_{1\le r\le n-1} \frac{C_{r+1}^{n}(\Theta,0)}{\sqrt{n\pi}} > c$$

for some constant $c$ depending on the level of significance.
Now since under $H_0$, $\mathrm{E}(\cos\Theta_i) = 0$ and $\mathrm{Var}(\cos\Theta_i) = \pi$ we have that the asymptotic null distribution of $\max_{1\le r\le n-1} \frac{C_{r+1}^{n}(\Theta,0)}{\sqrt{n\pi}}$ is same that of $\sup_{0\le t\le 1} B_0^*(t)$.

**Remark 18** *The outlier problem is not quite meaningful in this set-up and hence is not considered here.*

## 12.4 Outlier Problem in Cartwright-Mitsuyasu Distribution

We will denote by $\mathrm{CM}(\mu, s)$ (Cartwright, 1964, Mitsuyasu, 1975) the probability distribution having density

$$f(\theta;\mu,s) = k.\cos^{2s}\frac{\theta-\mu}{2}, 0 \le \theta < 2\pi \qquad (12.4)$$

where $0 \leq \mu < 2\pi, s > 0$, and $k$ is so chosen that $\int_0^{2\pi} f(\theta; \mu, s) d\theta = 1$.

In this section we discuss the outlier problem for the Cartwright - Mitsuyasu distribution. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as CM$(0, s)$ against the alternative $H_1^* :$ There exist $r, 1 \leq r \leq n$ such that $\Theta_1, \ldots, \Theta_{r-1}, \Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as CM$(0, s)$ and $\Theta_r$ is distributed as CM$(\mu, s), \mu > 0$. We will assume s is known and we will denote by $H_r$ the hypothesis that the $r^{th}$ observation is the outlier.

**Theorem 29 :** *(a) The* LMP *test for $H_0$ against $H_r$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \tan \frac{\Theta_r}{2} > c$$

*for some constant c depending on the level of significance.*
*(b) The* LMPTT *for testing $H_0$ against $H_1^*$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1 \leq r \leq n} \tan \frac{\Theta_r}{2} > c$$

*where c is a constant depending on the level of significance.*

**Proof :** (a) The log-likelihood of the observations $\theta_1, \ldots, \theta_n$ is :

$$\begin{aligned} \ell(\mu; \theta_1, \ldots, \theta_n) &= n \ln k + 2s \left\{ \sum_{i \neq r} \ln \cos \frac{\theta_i}{2} + \ln \cos \frac{\theta_r - \mu}{2} \right\} \\ \frac{\partial \ell}{\partial \mu} &= s \tan \frac{\theta_r - \mu}{2} \\ \frac{\partial \ell}{\partial \mu}(0) &= s \tan \frac{\theta_r}{2} \end{aligned} \tag{12.5}$$

Hence the LMP test for testing $H_0$ against $H_r$ is :

$$\text{Reject } H_0 \text{ if } \tan \frac{\Theta_r}{2} > c$$

for some constant $c$ depending on the level of significance.

b) The LMPTT for testing $H_0$ against $H_1^*$ is :

$$\text{Reject } H_0 \text{ if } \max_{1\le r\le n} \tan\frac{\Theta_r}{2} > c$$

for some constant $c$. Since the random variables $\tan\frac{\Theta_i}{2}$ are i.i.d under $H_0$ therefore the value of $c$ can be easily obtained following standard techniques and using numerical integration.

## 12.5 Change Point Problem in Cartwright-Mitsuyasu Distribution

Let us assume $s$ is known. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as CM$(0, s)$ against the alternative $H_1$ : There exist $r, 1 \le r \le n-1$ such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as CM$(0, s)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as CM$(\mu, s), \mu > 0$. For each fixed $r$, let $H_{1r}$ denote the alternative hypothesis that the change point is at $r$.

**Theorem 30 :** *(a) The* LMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \sum_{i=r+1}^{n} \tan\frac{\Theta_i}{2} > c$$

*for some constant $c$ depending on the level of significance.*
*(b) The* LMPTT *for $H_0$ against $H_1$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} \tan\frac{\Theta_i}{2} - (n-r)\xi}{\sqrt{n}\tau} > c$$

*where $\xi = E\left(\tan\frac{\Theta_i}{2}\right)$, $\tau^2 = Var\left(\tan\frac{\Theta_i}{2}\right)$ under $H_0$, and $c$ is a constant depending on the level of significance.*
*The asymptotic null distribution of the* LMPTT*-statistic is same as that of* $\sup_{0\le t\le 1} B_0^*(t)$.

**Proof :** (a) The log-likelihood of $\theta_1, \ldots, \theta_n$ is

$$\begin{aligned}
\ell(\mu; \theta_1, \ldots, \theta_n) &= n \ln k + 2s \left\{ \sum_{i=1}^{r} \ln \cos \frac{\theta_i}{2} + \sum_{i=r+1}^{n} \ln \cos \frac{\theta_i - \mu}{2} \right\} \\
\frac{\partial \ell}{\partial \mu} &= s \sum_{i=r+1}^{n} \tan \left( \frac{\theta_i - \mu}{2} \right) \\
\frac{\partial \ell}{\partial \mu}(0) &= s \sum_{i=r+1}^{n} \tan \frac{\theta_i}{2} \qquad (12.6)
\end{aligned}$$

Hence, the LMP test for $H_0$ against $H_{1r}$ is :

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n} \tan \frac{\Theta_i}{2} > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT test of $H_0$ against $H_1$ is based on the statistic

$$\max_{1 \leq r \leq n-1} \sum_{i=r+1}^{n} \tan \frac{\Theta_i}{2}$$

Let $Z_i = \tan \frac{\Theta_i}{2}$. Now $Z_i'$s are independent random variables having identical distribution under $H_0$. Hence by the Functional Central Limit Theorem and using the fact that sup is a continuous function on D[0,1]

$$\max_{1 \leq r \leq n-1} \frac{\sum_{i=r+1}^{n} Z_i - (n-r)\xi}{\sqrt{n}\tau} = \sup_{\frac{1}{n} \leq t \leq 1 - \frac{1}{n}} \frac{\sum_{i=[nt]+1}^{n} Z_i - (n - [nt])\xi}{\sqrt{n}\tau}$$

where $\xi = \mathrm{E}(Z_i)$ and $\tau^2 = \mathrm{Var}(Z_i)$, has the same asymptotic null distribution as $\sup_{0 \leq t \leq 1} B_0^*(t)$.

## 12.6 Outlier Problem in Wrapped Cauchy Distribution

We will denote by $WC(\mu, \rho)$ the wrapped Cauchy distribution(Mardia, 1972) having p.d.f

$$\begin{aligned} f(\theta; \mu, \rho) &= \frac{1}{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}, \\ & 0 \leq \theta < 2\pi, 0 \leq \mu < 2\pi, -1 < \rho < 1. \end{aligned} \tag{12.7}$$

In this section we discuss the outlier problem for the wrapped Cauchy distribution. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $WC(0, \rho)$ against the alternative $H_1^*$ : There exist $r, 1 \leq r \leq n$ such that $\Theta_1, \ldots, \Theta_{r-1}, \Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $WC(0, \rho)$ and $\Theta_r$ is distributed as $WC(\mu, \rho), \mu > 0$. We assume $\rho$ is known and we will denote by $H_r$ the alternative hypothesis that the $r^{th}$ observation is an outlier.

**Theorem 31 :** *(a) The* LMP *test for testing $H_0$ against $H_r$ is given by :*

$$\textit{Reject } H_0 \textit{ if} \frac{\sin\Theta_r}{1+\rho^2-2\rho\cos\Theta_r} > c$$

*where c is a constant depending on the level of significance.*
*(b) The* LMPTT *for testing $H_0$ against $H_1^*$ is given by :*

$$\textit{Reject } H_0 \textit{ if} \max_{1\leq r\leq n} \frac{\sin\Theta_r}{1+\rho^2-2\rho\cos\Theta_r} > c$$

*where c is a constant depending on the level of significance.*

**Proof :** (a) The log-likelihood of the observations $\theta_1, \ldots, \theta_n$ is :

$$\begin{aligned} \ell(\mu; \theta_1, \ldots, \theta_n) &= -n\ln 2\pi + n\ln(1-\rho^2) - \sum_{i\neq r} \ln(1+\rho^2-2\rho\cos\theta_i) \\ & \quad - \ln\left(1+\rho^2-2\rho\cos(\theta_r-\mu)\right) \end{aligned}$$

$$\frac{\partial \ell}{\partial \mu} = \frac{2\rho \sin(\theta_r - \mu)}{1 + \rho^2 - 2\rho\cos(\theta_r - \mu)}$$

$$\text{Therefore, } \frac{\partial \ell}{\partial \mu}(0) = \frac{2\rho \sin\theta_r}{1 + \rho^2 - 2\rho\cos\theta_r} \qquad (12.8)$$

Hence the LMP test for testing $H_0$ against $H_1^*$ is given by :

$$\text{Reject } H_0 \text{ if } \frac{\sin\Theta_r}{1 + \rho^2 - 2\rho\cos\Theta_r} > c$$

for some constant $c$ depending on the level of significance.
(b)The LMPTT for testing $H_0$ against $H_r$ is given by :

$$\text{Reject } H_0 \text{ if } \max_{1 \le r \le n} \frac{\sin\Theta_r}{1 + \rho^2 - 2\rho\cos\Theta_r} > c$$

where $c$ is a constant depending on the level of significance. Since $\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i}$ are i.i.d under $H_0$, therefore the value of $c$ can be computed following standard techniques and numerical integration.

## 12.7 Change Point Problem in Wrapped Cauchy Distribution

In what follows we will assume $\rho$ is known. Suppose $\Theta_1, \ldots, \Theta_n$ are independent random variables. We are interested to test the hypothesis $H_0$ : $\Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{WC}(0, \rho)$ against the alternative $H_1$ : There exist $r, 1 \le r \le n-1$, such that $\Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{WC}(0, \rho)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{WC}(\mu, \rho), \mu > 0$. We will denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$.

**Theorem 32** : *(a) The* LMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \sum_{i=r+1}^{n} \frac{\sin\Theta_i}{1 + \rho^2 - 2\rho\cos\Theta_i} > c$$

*where $c$ is a constant depending on the level of significance.*
*(b) The* LMPTT *for $H_0$ against $H_1$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} \frac{\sin\theta_i}{1+\rho^2-2\rho\cos\Theta_i}}{\sqrt{n}\tau} > c$$

*where* $\tau^2 = Var\left(\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i}\right)$

*The asymptotic null distribution of* $\max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} \frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i}}{\sqrt{n}\tau}$ *is same as that of* $\sup_{0\le t\le 1} B_0^*(t)$.

**Proof :** (a) The log-likelihood of the observations $\theta_1,\ldots,\theta_n$ is :

$$\begin{aligned}
\ell(\mu;\theta_1,\ldots,\theta_n) &= -n\ln 2\pi + n\ln(1-\rho^2) - \sum_{i=1}^{r}\ln(1+\rho^2-2\rho\cos\theta_i) \\
&\quad - \sum_{i=r+1}^{n}\ln(1+\rho^2-2\rho\cos(\theta_i-\mu)) \\
\frac{\partial\ell}{\partial\mu} &= \sum_{i=r+1}^{n}\frac{2\rho\sin(\theta_i-\mu)}{1+\rho^2-2\rho\cos(\theta_i-\mu)} \\
\frac{\partial\ell}{\partial\mu}(0) &= \sum_{i=r+1}^{n}\frac{2\rho\sin\theta_i}{1+\rho^2-2\rho\cos\theta_i} \qquad (12.9)
\end{aligned}$$

Hence the LMP test of $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n}\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i} > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT for $H_0$ against $H_1$ is given by :

$$\text{Reject } H_0 \text{ if } \max_{1\le r\le n-1}\sum_{i=r+1}^{n}\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i} > c$$

where $c$ is a constant depending on the level of significance.
Now, $E\left(\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i}\right) = 0$ and the random variables $\frac{\sin\Theta_i}{1+\rho^2-2\rho\cos\Theta_i}$ are i.i.d under $H_0$. Hence by using the Functional Central Limit Theorem, and the

fact that sup is a continuous function on D[0,1], we find the asymptotic null distribution of $\max_{1 \le r \le n-1} \frac{\sum_{i=r+1}^{n} \frac{\sin \Theta_i}{1+\rho^2 - 2\rho\cos\Theta_i}}{\sqrt{n\tau}}$ is same as that of $\sup_{0 \le t \le 1} B_0^*(t)$

## 12.8 Outlier Problem in Wrapped Stable Family of Distributions

We will denote by WS$(\mu, \rho, a)$ the wrapped stable distribution (Mardia, 1972) having p.d.f.

$$
\begin{aligned}
f(\theta; \mu, \rho, a) &= \frac{1}{2\pi}\left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta - \mu)\right], \\
& \quad 0 \le \theta < 2\pi, 0 \le \mu < 2\pi, \rho \ge 0, 0 < a \le 2. \quad (12.10)
\end{aligned}
$$

In this section we discuss the outlier problem for the Wrapped Stable distribution. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We want to test the hypothesis $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as WS$(0, \rho, a)$ against the alternative $H_1^*$ : There exist $r, 1 \le r \le n$, such that $\Theta_1, \ldots, \Theta_{r-1}, \Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as WS$(0, \rho, a)$ and $\Theta_r$ is distributed as WS$(\mu, \rho, a), \mu > 0$. We assume both $\rho$ and $a$ to be known and we will denote by $H_r$ the alternative hypothesis that the $r^{th}$ observation is an outlier.

**Theorem 33 :** *(a) The* LMP *test for testing $H_0$ against $H_r$ is :*

$$
\textit{Reject } H_0 \textit{ if } \frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\,\Theta_r}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\,\Theta_r} > c
$$

*where $c$ is a constant depending on the level of significance.*
*(b) The* LMPTT *for testing $H_0$ against $H_r$ is :*

$$\textit{Reject } H_0 \textit{ if } \max_{1\le r\le n} \frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\ \Theta_r}{1+2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\ \Theta_r} > c$$

*where $c$ is a constant depending on the level of significance.*

**Proof:** (a) The log-likelihood of the observations $\theta_1, \ldots, \theta_n$ is

$$\begin{aligned}
\ell(\mu; \theta_1, \ldots, \theta_n) &= -n \ln 2\pi + \sum_{i\neq r} \ln \left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\theta_i\right] \\
&\quad + \ln\left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta_r - \mu)\right] \\
\frac{\partial \ell}{\partial \mu} &= \frac{2\sum_{k=1}^{\infty} k\ \rho^{k^a} \sin k(\theta_r - \mu)}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta_r - \mu)} \\
\left.\frac{\partial \ell}{\partial \mu}\right|_{\mu=0} &= \frac{2\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\ \theta_r}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\ \theta_r} \qquad (12.11)
\end{aligned}$$

Hence the LMP test for testing $H_0$ against $H_r$ is :

$$\text{Reject } H_0 \text{ if } \frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\ \Theta_r}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\Theta_r} > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT test for testing $H_0$ against $H_1^*$ is :

$$\text{Reject } H_0 \text{ if } \max_{1\le r\le n} \frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\ \Theta_r}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\ \Theta_r} > c$$

where $c$ is a constant depending on the level of significance. Since the random variables

$$\frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\ \Theta_r}{1+2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\ \Theta_r}$$

are i.i.d under $H_0$ the value of $c$ can be found using standard techniques and numerical integration.

## 12.9 Change Point Problem in Wrapped Stable Family of Distributions

In what follows we assume $\rho$ and $a$ to be known. Let $\Theta_1, \ldots, \Theta_n$ be independent random variables. We are interested to test the hypothesis $H_0$ : $\Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{WS}(0, \rho, a)$ against the alternative $H_1$ : There exist $r, 1 \leq r \leq n-1, \Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{WS}(0, \rho, a)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{WS}(\mu, \rho, a), \mu > 0$. Let $H_{1r}$ denote the alternative hypothesis that the change point is at $r$.

$$\begin{aligned} \text{Define } T_i &= \frac{\sum_{k=1}^{\infty} k\rho^{k^a} \sin k\Theta_i}{1+2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\Theta_i}, & (12.12) \\ \text{Let } \xi &= \mathrm{E}(T_i) \text{ and} \\ \tau^2 &= \mathrm{Var}(T_i) & (12.13) \end{aligned}$$

**Theorem 34 :** *(a) The* LMP *test for $H_0$ against $H_{1r}$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \sum_{i=r+1}^{n} T_i > c$$

*where $c$ is a constant depending on the level of significance.*
*(b) The* LMPTT *for $H_0$ against $H_1$ is given by :*

$$\textit{Reject } H_0 \textit{ if } \max_{1\le r\le n-1} \sum_{i=r+1}^{n} T_i > c$$

*where c is a constant depending on the level of significance. The asymptotic null distribution of*

$$\max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} T_i - (n-r)\xi}{\sqrt{n}\tau}$$

*is same as that of* $\sup_{0\le t\le 1} B_0^*(t)$.

**Proof :** (a) The log-likelihood of the observations $\theta_1, \ldots, \theta_n$ is

$$\begin{aligned} \ell(\mu; \theta_1, \ldots, \theta_n) &= -n \ln 2\pi + \sum_{i=1}^{r} \ln\left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\theta_i\right] \\ &\quad + \sum_{i=r+1}^{n} \ln\left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta_i - \mu)\right] \\ \frac{\partial \ell}{\partial \mu} &= \sum_{i=r+1}^{n} \frac{2\sum_{k=1}^{\infty} k\, \rho^{k^a} \sin k(\theta_i - \mu)}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta_i - \mu)} \\ \left.\frac{\partial \ell}{\partial \mu}\right|_{\mu=0} &= \sum_{i=r+1}^{n} \frac{2\sum_{k=1}^{\infty} \rho^{k^a} \sin k\theta_i}{1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k\theta_i} \qquad (12.14) \end{aligned}$$

Hence the LMP test for $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } \sum_{i=r+1}^{n} T_i > c$$

where $c$ is a constant depending on the level of significance.
(b) The LMPTT for $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } \max_{1\le r\le n-1} \frac{\sum_{i=r+1}^{n} T_i - (n-r)\xi}{\sqrt{n}\tau} > c$$

where $c$ is a constant depending on the level of significance. Since the random variables $T_i$'s are i.i.d under $H_0$ therefore, by Functional Central Limit Theorem, and the fact that sup is a continuous function on D, the asymptotic null distribution of

$$\max_{1 \leq r \leq n-1} \frac{\sum_{i=r+1}^{n} T_i - (n-r)\xi}{\sqrt{n}\tau}$$

is same as that of $\sup_{0 \leq t \leq 1} B_0^*(t)$.

## 12.10 An Unified approach for NRTT construction

In this section we outline a method of computing the NRTT statistic in cases where it is not possible to compute the effective score for the parameter of interest and the effective information in a closed form. We illustrate this technique by applying it in the context of change point problem for Papakonstantinou's Skewed Circular distribution but the method is equally applicable for other distributions as well. Let $\Theta_1, \ldots, \Theta_n$ be all independent. We are interested to test $H_0 : \Theta_1, \ldots, \Theta_n$ are identically distributed as $\mathrm{P}(k, \nu)$ against the alternative $H_1 : \Theta_1, \ldots, \Theta_r$ are identically distributed as $\mathrm{P}(k, \nu)$ and $\Theta_{r+1}, \ldots, \Theta_n$ are identically distributed as $\mathrm{P}(k, \nu + \delta)$ for some $r, 1 \leq r \leq n-1, \delta > 0$. We will denote by $H_{1r}$ the alternative hypothesis that the change point is at $r$. Note that a test of $H_0$ against $H_{1r}$ is essentially a test of $H_0' : \delta = 0$ against $H_{1r}' : \delta > 0$. Since $k$ and $\nu$ are unknown these are the nuisance parameters for this problem and $\delta$ is the parameter of interest. We follow the notations of Hall and Mathiason (1990). The general theory of NR-test tells us that the NR-test of $H_0$ against $H_{1r}$ is given by :

$$\text{Reject } H_0 \text{ if } \frac{\hat{S}_\delta^{*2}}{\hat{B}_\delta^*} > \chi^2_{1,\alpha}$$

where $\hat{S}_\delta^*$ is the estimated effective score for parameter $\delta$ and $\hat{B}_\delta^*$ is the estimated effective information. Unlike the case of Circular normal distribution it is not possible to compute $\hat{S}_\delta^*$ and $\hat{B}_\delta^*$ by substituting the MLE's of the nuisance parameters into the expressions of $S_\delta^*$ and $B_\delta^*$. This is because of the fact that it is not possible to compute $S_\delta^*$ and $B_\delta^*$ in a closed form.

Note for this problem, under $H_0$ (i.e. putting $\delta = 0$) we have

$$S_\delta = \frac{1}{\sqrt{n}} \sum_{i=r+1}^{n} \frac{k\cos(\Theta_i + \nu\sin\Theta_i).\sin\Theta_i}{1 + k\sin(\Theta_i + \nu\sin\Theta_i)}$$

$$S_\nu = \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{n} \frac{k\cos(\Theta_i + \nu\sin\Theta_i)\sin\Theta_i}{1 + k\sin(\Theta_i + \nu\sin\Theta_i)} \right\}$$

and

$$S_\kappa = \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{n} \frac{\sin(\Theta_i + \nu\sin\Theta_i)}{1 + k\sin(\Theta_i + \nu\sin\Theta_i)} \right\}$$

Now we note that we can write

$$S_\delta = \frac{n-r}{\sqrt{n}} . \frac{1}{n-r} \sum_{i=r+1}^{n} \frac{k\cos(\Theta_i + \nu\sin\Theta_i)\sin\Theta_i}{1 + k\sin(\Theta_i + \nu\sin\Theta_i)}$$

We can then get $\hat{S}_\delta$ by substituting consistent estimators of $k$ and $\nu$ in the expression of $S_\delta$. Similarly we can get $\hat{S}_\nu$ and $\hat{S}_\kappa$. Now to estimate the elements of the matrix B we need to compute the variances and covariances of $S_\delta, S_\nu$, and $S_\kappa$.

Let

$$Z_i = \frac{k\cos(\Theta_i + \nu\sin\Theta_i)\sin\Theta_i}{1 + k\sin(\Theta_i + \nu\sin\Theta_i)}$$

$$\text{Var}(S_\delta) = (1 - \frac{r}{n})\text{Var } (Z_n) \text{ (since } Z_i'\text{s are i.i.d under } H_0)$$

This can be consistently estimated by the corresponding sample variance. Similarly the covariances can be consistently estimated using the sample values, under $H_0$. Now since $S_\delta^*$ and $B_\delta^*$ are continuous functions of $S_\delta$, $S_\nu$, $S_\kappa$ and the elements of the matrix B we can use estimates of these quantities to yield consistent estimates of $S_\delta^*$ and $B_\delta^*$. An NR-test of $H_0$ against $H_{1r}$ can be carried out based on these. An NRTT statistics is constructed by taking the maximum of the $(n-1)$ NR-test statistics. An NRTT is carried out using this

NRTT statistic. Alternatively, we may use the average empirical information matrix (Hall & Mathiason (1990) as an estimate of **B**. However unlike their method our new approach has the advantage that it continues to hold even in cases where the second order partial derivatives are difficult to obtain or may even not exist.

## 12.11 Robustness

In this section we study the behaviour of the LMPTT-statistic derived in section 5.3 with the circular normal distribution as the underlying distribution. We use simulations to study the variation in the level of significance of the test when the observations come from the symmetric wrapped stable family $WS(\mu, \rho, a)$ of distributions. Note that this family of distributions is indexed by the parameter 'a'. We consider $a = 0.4, 0.8, 1.0, 1.2, 1.6$. Note that for $a = 1.0$ we get the wrapped Cauchy distribution and for $a = 2.0$ we get the wrapped normal distribution. We take $\mu = 0, \kappa = 1$ for the circular normal distribution and use the facts that for $WS(0, \rho, a)$, $E(\cos\Theta) = \rho$ for any $a, 0 < a \leq 2$ and for circular normal distribution $E(\cos\Theta) = A(\kappa)$ to find a matching value of $\rho$. We equate $A(\kappa)$ with $\rho$ and solve to get the matching value of $\rho$. This yields $\rho = 0.446$. The results given in the following Table 1 below are based on 5000 simulations.

Table 1 : Robustness of the level of significance

| | Actual level when nominal level is 5% | Actual level when nominal level is 1% |
|---|---|---|
| Circular Normal | .050 | .010 |
| Wrapped Normal | .075 | .018 |
| Wrapped Cauchy | .076 | .019 |
| WS (0, .446, .4) | .080 | .020 |
| WS (0, .446, .8) | .081 | .022 |
| WS (0, .446, 1.2) | .080 | .022 |
| WS (0, .446, 1.6) | .081 | .021 |

From the above table we see that though the actual levels exceed the nominal level the amount of exceedance is not very high. Thus this test may be used with reasonable safety even in situations when we are not sure about the underlying distribution but can restrict to the family of usual unimodal symmetric circular distributions.

**Remark 19 :** *Such studies may be carried out for other test statistics discussed in this thesis. This particular test statistic is chosen because of its simplicity which makes it particularly attractive for use in analysing real-life data sets.*

## 12.12 Scope of Further Research

In this section we indicate scope for further research.

(a) One possible generalization is the outlier and change point problems for the case of spherical data. Suppose the data comes from the Langevin distribution (Mardia, 1972) which is a popular parametric distribution used to model spherical data. It is possible to adapt some of the techniques developed for outlier and change point problems for the circular normal distribution to the outlier and change point problem for the mean vector of the Langevin distribution.

(b) Another possible generalization is the outlier and change point problems in multivariate circular distributions i.e. where the marginals of a p-variate distribution are all circular. Here also a LMP type approach may be used using a generalisation of it to the multiparameter case (see e.g. SenGupta & Vermier (1986))

(c) In this thesis we have considered outlier and change point problems in the context of directional data with independent observations. These problems may be studied when the observations are dependent.

(d) It is of interest to predict the change point based on the observations gathered upto a time point. This is of particular interest in studies of relating to movement of cyclones. It is of great practical importance to know when does the cyclone changes its path. This is an interesting and challenging practical problem which requires prediction of the change point.

# Chapter 13

# Computer Programs

## 13.1 Introduction

In this chapter we provide eight programs which would be useful in implementing some of the tests discussed in this thesis. Some of these programs are written in VAX-FORTRAN and some others are written in QBASIC.

## 13.2 Programs

1. QBASIC Program for finding the cut-off points for the NR-test statistic for the change point problem for small samples

```
DECLARE SUB vmsran (cdf())
DECLARE SUB vmsran2 (n, cdf())
DECLARE FUNCTION akappa (k AS DOUBLE)
DECLARE FUNCTION apkappa (k AS DOUBLE, a AS DOUBLE)
DECLARE FUNCTION atanf (s AS DOUBLE, c AS DOUBLE)
DECLARE FUNCTION kapest (r AS DOUBLE)

DIM theta(500)  AS DOUBLE
DIM cdf(5000)
pi# = 3.141592654#
INPUT "the sample size", n
INPUT "the random number seed", trn
```

```
RANDOMIZE trn
CALL vmsran(cdf())
OPEN "C:\RES\scalpha.out" FOR OUTPUT AS #2
OPEN "C:\RES\stemp.dat" FOR OUTPUT AS #3
OPEN "c:\res\simcalph.dat" FOR OUTPUT AS #4
calr = 0
calsup# = -1
FOR rep = 1 TO 5000
CALL vmsran2(n, cdf())
OPEN "C:\RES\scalpha.dat" FOR INPUT AS #1
PRINT #2, rep
FOR i = 1 TO n
    INPUT #1, theta(i)
    theta(i) = theta(i) * pi# / 180
NEXT i

si# = 0
co# = 0
FOR i = 1 TO n
    si# = si# + SIN(theta(i))
    co# = co# + COS(theta(i))
NEXT i

PRINT #3, "si", si#, "co", co#
muest# = atanf(si#, co#)
PRINT #3, "muest", muest#

R1# = SQR(si# ^ 2 + co# ^ 2)
PRINT #3, R1#
rbar# = R1# / n
PRINT #3, "R1", R1#

kest# = kapest(rbar#)
PRINT #3, "Kapest", kest#

bmm# = kest# * rbar#
PRINT #3, "bmm", bmm#
```

```
ak# = akappa(kest#)
ak1# = apkappa(kest#, ak#)
bkk# = ak1#
PRINT #3, "bkk", bkk#

bmk# = 0
PRINT #3, "bmk", bmk#

smu# = 0
PRINT #3, "smu", smu#

skap# = 0
PRINT #3, "skap", skap#


FOR r = 1 TO n - 1
PRINT #3, r

sir# = 0
cor# = 0
FOR i = r + 1 TO n
    sir# = sir# + SIN(theta(i))
    cor# = cor# + COS(theta(i))
NEXT i

PRINT #3, "si_r", sir#, "co_r", cor#
muestr# = atanf(sir#, cor#)
PRINT #3, "muest_r", muestr#

R2# = SQR(sir# ^ 2 + cor# ^ 2)
PRINT #3, "R2", R2#

bdd# = kest# * (R1# / n) * ((n - r) / n)
PRINT #3, "bdd", bdd#

bmd# = bdd#
```

```
PRINT #3, "bmd", bmd#

bdk# = 0
PRINT #3, "bdk", bdk#

sdel# = (1 / SQR(n)) * (kest# * R2# * SIN(muestr# - muest#))
PRINT #3, "sdel", sdel#

sdeleff# = sdel#
bdeff# = bdd# - ((bmd# ^ 2) / bmm#)

PRINT #3, "sdeleff", sdeleff#, "bdeff", bdeff#

calstat# = (sdeleff# ^ 2) / bdeff#

PRINT #3, "calpha", calstat#
PRINT #2, r, calstat#
IF calstat# > calsup# THEN
   calsup# = calstat#
   calr = r
END IF
NEXT r
CLOSE (1)
PRINT #4, calr, calsup#
calr = 0
calsup# = -1
PRINT rep
NEXT rep
END

FUNCTION akappa (k#)
IF k# > 1.55 THEN
   t = 1- (1/(2*k#))- (1/(8*(k#^2)))- (1/(8*(k#^3)))
   akappa = 1- (1/(2*k#))- (1/(8*(k#^2)))- (1/(8*(k#^3)))
ELSE
   t = (k#/2) * (1-((k#^2)/8) + ((k#^4)/48))
   akappa = (k#/2)*(1-((k#^2)/8)+((k#^4)/48))
```

```
END IF
END FUNCTION

FUNCTION apkappa (k#, a#)
apkappa = 1 - (a# ^ 2) - (a# / k#)
END FUNCTION

FUNCTION atanf (s#, c#)
pi# = 3.141592654#

IF s# >= 0 AND c# > 0 THEN
   atanf = ATN(s# / c#)
ELSE
   IF s# >= 0 AND c# < 0 THEN
      atanf = pi# + ATN(s# / c#)
   ELSE
      IF s# < 0 AND c# > 0 THEN
         atanf = (2 * pi#) + ATN(s# / c#)
      ELSE
         IF s# < 0 AND c# < 0 THEN
            atanf = pi# + ATN(s# / c#)
         ELSE
            IF s# > 0 AND c = 0 THEN
               atanf = pi# / 2
            ELSE
               IF s# < 0 AND c# = 0 THEN
                  atanf = 3 * pi# / 2
               END IF
            END IF
         END IF
      END IF
   END IF
END IF
END FUNCTION

FUNCTION kapest (rbar#)
IF rbar# < .53 THEN
```

```
   kapest = (2 * rbar#) + (rbar# ^ 3) + (5 * rbar# ^ 5 / 6)
ELSE
   IF rbar# < .85 THEN
      kapest = -.4 + (1.39 * rbar#) + (.43 / (1 - rbar#))
   ELSE
      kapest = 1 / ((rbar# ^ 3) - (4 * (rbar# ^ 2)) + (3 * rbar#))
   END IF
END IF

END FUNCTION

SUB vmsran (cdf())
pi# = 3.141592654#
inc = 2 * pi# / 1000
kappa = 4
x = 0
FOR i = 1 TO 1000
    x = x + inc
    y = (1 / (2 * pi#)) * EXP(kappa * COS(x))
    IF i = 1 THEN
       cdf(i) = y * inc
    END IF
    IF i > 2 THEN
       cdf(i) = cdf(i - 1) + y * inc
    END IF
NEXT i
FOR i = 1 TO 1000
    cdf(i) = cdf(i) / cdf(1000)
NEXT i
END SUB

SUB vmsran2 (n, cdf())

OPEN "c:\res\scalpha.dat" FOR OUTPUT AS #1
pi# = 3.141592654#
inc = 2 * pi# / 1000
FOR i = 1 TO n
```

```
    t = RND
    j = 1
    WHILE cdf(j) < t
          j = j + 1
    WEND
    vran= ((j-1)*inc)+((t-cdf(j-1))/(cdf(j)-cdf(j-1))*inc)
    PRINT #1, vran * 180 / pi#
NEXT i
CLOSE (1)
END SUB
```

2. QBASIC program for computing the NRTT statistic for the change point problem for a given sample

```
DECLARE FUNCTION akappa (k AS DOUBLE)
DECLARE FUNCTION apkappa (k AS DOUBLE, a AS DOUBLE)
DECLARE FUNCTION atanf (s AS DOUBLE, c AS DOUBLE)
DECLARE FUNCTION kapest (r AS DOUBLE)

DIM theta(500) AS DOUBLE
pi# = 3.141592654#
INPUT n
OPEN "calpha.dat" FOR INPUT AS #1
OPEN "calpha.out" FOR OUTPUT AS #2
OPEN "temp.dat" FOR OUTPUT AS #3
FOR i = 1 TO n
    INPUT #1, theta(i)
    theta(i) = theta(i) * pi# / 180
NEXT i

si# = 0
co# = 0
FOR i = 1 TO n
    si# = si# + SIN(theta(i))
    co# = co# + COS(theta(i))
NEXT i
```

```
PRINT #3, "si", si#, "co", co#
muest# = atanf(si#, co#)
PRINT #3, "muest", muest#

R1# = SQR(si# ^ 2 + co# ^ 2)
PRINT #3, R1#
rbar# = R1# / n
PRINT #3, "R1", R1#

kest# = kapest(rbar#)
PRINT #3, "Kapest", kest#

bmm# = kest# * rbar#
PRINT #3, "bmm", bmm#

ak# = akappa(kest#)
ak1# = apkappa(kest#, ak#)
bkk# = ak1#
PRINT #3, "bkk", bkk#

bmk# = 0
PRINT #3, "bmk", bmk#

smu# = 0
PRINT #3, "smu", smu#

skap# = 0
PRINT #3, "skap", skap#


FOR r = 1 TO n - 1
PRINT #3, r

sir# = 0
cor# = 0
FOR i = r + 1 TO n
    sir# = sir# + SIN(theta(i))
```

```
    cor# = cor# + COS(theta(i))
NEXT i

PRINT #3, "si_r", sir#, "co_r", cor#
muestr# = atanf(sir#, cor#)
PRINT #3, "muest_r", muestr#

R2# = SQR(sir# ^ 2 + cor# ^ 2)
PRINT #3, "R2", R2#

bdd# = (kest# * R2# * COS(muest# - muestr#)) / n
PRINT #3, "bdd", bdd#

bmd# = bdd#
PRINT #3, "bmd", bmd#

bdk# = (R2# * SIN(muest# - muestr#)) / n
PRINT #3, "bdk", bdk#

sdel#= (1/SQR(n))*(kest#*R2#*SIN(muestr#-muest#))
PRINT #3, "sdel", sdel#

sdeleff# = sdel#
bdeff#= bdd#-((bmd#^2)/bmm#)-((bdk#^2)/bkk#)

PRINT #3, "sdeleff", sdeleff#, "bdeff", bdeff#

calstat# = (sdeleff# ^ 2) / bdeff#

PRINT #3, "calpha", calstat#
PRINT #2, r, calstat#

NEXT r

END

FUNCTION akappa (k#)
```

```
PRINT k#
IF k# > 1.55 THEN
t = 1-(1/(2*k#))-(1/(8*(k#^2)))-(1/(8*(k#^3)))
PRINT t
akappa=1-(1/(2*k#))-(1/(8*(k#^2)))-(1/(8*(k#^3)))
ELSE
t = (k# / 2) * (1 - ((k# ^ 2) / 8) + ((k# ^ 4) / 48))
PRINT t
akappa = (k# / 2) * (1 - ((k# ^ 2) / 8) + ((k# ^ 4) / 48))
END IF
END FUNCTION

FUNCTION apkappa (k#, a#)
apkappa = 1 - (a# ^ 2) - (a# / k#)
END FUNCTION

FUNCTION atanf (s#, c#)
pi# = 3.141592654#

IF s# >= 0 AND c# > 0 THEN
   atanf = ATN(s# / c#)
ELSE
   IF s# >= 0 AND c# < 0 THEN
      atanf = pi# + ATN(s# / c#)
   ELSE
      IF s# < 0 AND c# > 0 THEN
         atanf = (2 * pi#) + ATN(s# / c#)
      ELSE
          IF s# < 0 AND c# < 0 THEN
             atanf = pi# + ATN(s# / c#)
          ELSE
             IF s# > 0 AND c = 0 THEN
                atanf = pi# / 2
             ELSE
                IF s# < 0 AND c# = 0 THEN
                   atanf = 3 * pi# / 2
                END IF
```

```
            END IF
         END IF
      END IF
   END IF
END IF
END FUNCTION

FUNCTION kapest (rbar#)
IF rbar# < .53 THEN
   kapest = (2 * rbar#) + (rbar# ^ 3) + (5 * rbar# ^ 5 / 6)
ELSE
   IF rbar# < .85 THEN
      kapest = -.4 + (1.39 * rbar#) + (.43 / (1 - rbar#))
   ELSE
      kapest = 1 / ((rbar# ^ 3) - (4 * (rbar# ^ 2)) + (3 * rbar#))
   END IF
END IF
END FUNCTION
```

3. VAX-FORTRAN program for computing the LRT-statistic for the change point problem for a given data set

```
      DIMENSION A(100)
      write(*,*)'Disp. Par. ?'
      read(*,*)c
      open(unit=1,file='data.dat',status='old')
      write(*,*)'No. of observations : '
      read(*,*)nr
      do i=1,nr
         read(1,*)a(i)
         a(i)=a(i)*3.1416/180
      enddo
      CALL ENULL(A,NR,XMU0N)
      XMIN=1
      DO J=1,NR-1
         CALL EALT(A,NR,J,XMU0A,XMU1A)
```

```
   CALL XLAM(A,NR,C,J,XMUON,XMUOA,XMU1A,XLR)
   IF(XLR.LT.XMIN) THEN
      XMIN=XLR
      CPT=J
      write(*,*)cpt
   ENDIF
ENDDO
XRT=XMIN
WRITE(*,*)'LRT =',XRT,' cpt =',cpt
STOP
END

SUBROUTINE ENULL(A,NR,XMUON)
DIMENSION A(100)
SI=0
CO=0
DO I=1,NR
     SI = SI + SIN(A(I))
     CO = CO + COS(A(I))
ENDDO
XMUON = ATAN2(SI,CO)
RETURN
END

SUBROUTINE EALT(A,NR,J,XMUOA,XMU1A)
DIMENSION A(100)
SIO=0
COO=0
DO I=1,J
     SIO = SIO + SIN(A(I))
     COO = COO + COS(A(I))
ENDDO
XMUOA = ATAN2(SIO,COO)
SI1=0
CO1=0
DO I=J+1,NR
     SI1 = SI1 + SIN(A(I))
```

```
      CO1 = CO1 + COS(A(I))
   ENDDO
   XMU1A = ATAN2(SI1,CO1)
   RETURN
   END

   SUBROUTINE XLAM(A,NR,C,J,XMUON,XMUOA,XMU1A,XLR)
   DIMENSION A(100)
   CON=0
   COA0=0
   COA1=0
   DO I=1,NR
      CON = CON + COS(A(I)-XMUON)
   ENDDO
   DO I=1,J
      COA0 = COA0 + COS(A(I)-XMUOA)
   ENDDO
   DO I=J+1,NR
      COA1 = COA1 + COS(A(I)-XMU1A)
   ENDDO
   C1=CON - (COA0+COA1)
   XLR = EXP(C*C1)
   RETURN
   END
```

4. VAX-FORTRAN program for computing the cut-off points of the LRT statistic for the change point problem for the mean direction

```
   DIMENSION A(100),XR(5000)
   WRITE(*,*) 'INPUT KAPPA'
   READ(*,*) C
   C1=C
   WRITE(*,*)'INPUT N '
   READ(*,*) NR
   DO I=1,5000
      WRITE(*,*)'I=', I
```

```
      C=C1
      CALL RNVMS(NR,C,A)
      CALL ENULL(A,NR,XMUON)
      CALL KAPEST(A,NR,SKAPPAA)
      C=SKAPPAA
      XMIN=1
      DO J=1,NR-1
         CALL EALT(A,NR,J,XMUOA,XMU1A)
         CALL XLAM(A,NR,C,J,XMUON,XMUOA,XMU1A,XLR)
         IF(XLR.LT.XMIN) THEN
           XMIN=XLR
           CPT=J
         ENDIF
      ENDDO
      XRT=XMIN
      XR(I)=XRT
      WRITE(*,*)'XRT ',XRT
   ENDDO
   CALL SORT(XR)
   STOP
   END

   SUBROUTINE ENULL(A,NR,XMUON)
   DIMENSION A(100)
   SI=0
   CO=0
   DO I=1,NR
       SI = SI + SIN(A(I))
       CO = CO + COS(A(I))
   ENDDO
   XMUON = ATAN2(SI,CO)
   RETURN
   END

   SUBROUTINE EALT(A,NR,J,XMUOA,XMU1A)
   DIMENSION A(100)
   SIO=0
```

```
CO0=0
DO I=1,J
     SI0 = SI0 + SIN(A(I))
     CO0 = CO0 + COS(A(I))
ENDDO
XMU0A = ATAN2(SI0,CO0)
SI1=0
CO1=0
DO I=J+1,NR
     SI1 = SI1 + SIN(A(I))
     CO1 = CO1 + COS(A(I))
ENDDO
XMU1A = ATAN2(SI1,CO1)
RETURN
END

SUBROUTINE XLAM(A,NR,C,J,XMU0N,XMU0A,XMU1A,XLR)
DIMENSION A(100)
CON=0
COA0=0
COA1=0
DO I=1,NR
   CON = CON + COS(A(I)-XMU0N)
ENDDO
DO I=1,J
   COA0 = COA0 + COS(A(I)-XMU0A)
ENDDO
DO I=J+1,NR
   COA1 = COA1 + COS(A(I)-XMU1A)
ENDDO
C1=CON - (COA0+COA1)
XLR = EXP(C*C1)
RETURN
END

SUBROUTINE SORT(XR)
DIMENSION XR(5000)
```

```
      OPEN(UNIT=2,STATUS='NEW',FILE='CPTQ.DAT')
      DO I=1,5000
         DO J=I+1,5000
            IF (XR(I).GT.XR(J)) THEN
               TEMP=XR(I)
               XR(I)=XR(J)
               XR(J)=TEMP
            ENDIF
         ENDDO
      ENDDO
      DO I=1,100
         WRITE(2,*)I,XR(I*50)
      ENDDO
      RETURN
      END


      SUBROUTINE KAPEST(A, NR, SKAPPAA)
      REAL MUN,MUA,MUAO,LAR,LAM,LRT,SA, CA
      REAL S18AEF
      REAL S18AFF
      double precision sn,sd,slrn,slrd,slrt,q2,q3,q4,q5,q6,q7,q8
      DOUBLE PRECISION CN,RN
      DIMENSION A(200),LAR(20),SA(200),CA(200),MUA(200)
      I=0
      J=0
      IFAIL=1
      IFI=2
      si=0
      co=0
      DO 10 J=1,NR
         Si = Si + SIN(A(J))
         Co = Co + COS(A(J))
10    CONTINUE
C   COMPUTING THE  ESTIMATES
         RSQ=Co*Co+Si*Si
         RN=(RSQ**0.5)/NR
```

```
         CALL SOLVE(RN,SKAPPAA,ifail)
      WRITE(*,*)'r =',RN
      WRITE(*,*)'KAPPA=',SKAPPAA
111   RETURN
      END

      SUBROUTINE SOLVE(SKAP,SKAPPA,ifail)
      DOUBLE PRECISION SN,SD,GS,SKAP,ERR
      REAL S18AEF
      REAL S18AFF
      IFAIL=1
      IP=0
      IN=0
      T=0
10    ifail=1
      SN = S18AFF(T,IFAIL)
      if (ifail.ne.0) then
         go to 50
      endif
      SD = S18AEF(T,IFAIL)
      if (ifail.ne.0) then
          go to 50
      endif
      GS=SN/SD
      ERR = GS-SKAP
      IF (ERR.LT.0) THEN
         IN=1
         TN=T
      ENDIF
      IF (ERR.GT.0) THEN
         IP=1
         TP=T
      ENDIF
      IF (ERR.EQ.0) THEN
         GO TO 30
      ENDIF
      IPN = IP*IN
```

```
      IF (IPN.EQ.1) THEN
         TMID = (TP + TN)/2
      ELSE
         T=T+0.1
         GO TO 10
      ENDIF
      ifail=1
20    SN = S18AFF(TMID,IFAIL)
      if (ifail.ne.0) then
          go to 50
      endif
      ifail=1
      SD = S18AEF(TMID,IFAIL)
      if (ifail.ne.0) then
         go to 50
      endif
      GS=SN/SD
      ERR = GS-SKAP
      IF (ERR.GT.0) THEN
         TP=TMID
      ENDIF
      IF (ERR.LT.0) THEN
         TN = TMID
      ENDIF
      IF (ERR.EQ.0) THEN
         GO TO 40
      ENDIF
      TMID = (TP + TN)/2
      EABS = ABS(ERR)
      IF (EABS.LT.0.0001) THEN
         GO TO 40
      ELSE
         GO TO 20
      ENDIF
30    SKAPPA = T
      GO TO 50
40    SKAPPA = TMID
```

```
50    RETURN
      END
```

5. VAX-FORTRAN program for obtaining the cut-off points of the LMPTT statistic for the change point problem for the mean direction

```
DIMENSION A(100),XR(5000)
      READ(*,*) C
      WRITE(*,*)'INPUT N '
      READ(*,*) NR
      DO I=1,5000
         CALL RNVMS(NR,C,A)
         XMAX = (-1)*NR - 1
         DO J = 1, (NR-1)
            CALL MEAN(A,J,XMUON)
            DO K = 1, NR
               A(K) = A(K) - XMUON
               IF (A(K).LT.0) THEN
                  A(K) = A(K) + 6.2832
               ENDIF
            ENDDO
            XMPS = 0
            T = J+1
            DO L = T,NR
               XMPS = XMPS + SIN(A(L))
            ENDDO
            IF (XMPS.GT.XMAX) THEN
               XMAX = XMPS
            ENDIF
         ENDDO
         XR(I) = XMAX/nr
         WRITE(*,*)'LMP ',I, XR(I)
      ENDDO
      CALL SORT(XR)
      STOP
      END
```

```
SUBROUTINE MEAN(A,J,XMUON)
DIMENSION A(100)
SI=0
CO=0
DO I=1,J
     SI = SI + SIN(A(I))
     CO = CO + COS(A(I))
ENDDO
XMUON = ATAN2(SI,CO)
RETURN
END


SUBROUTINE SORT(XR)
DIMENSION XR(5000)
OPEN(UNIT=2,STATUS='NEW',FILE='LMPQ34.DAT')
DO I=1,5000
   DO J=I+1,5000
      IF (XR(I).GT.XR(J)) THEN
          TEMP=XR(I)
          XR(I)=XR(J)
          XR(J)=TEMP
      ENDIF
    ENDDO
ENDDO
DO I=1,100
   WRITE(2,*)I,XR(I*50)
ENDDO
RETURN
END
```

6. VAX-FORTRAN program to compute the LMPTT statistic for change point problem

```
DIMENSION A(100),B(100)
```

```
OPEN(UNIT=1,STATUS='OLD',FILE='DATA.DAT')
WRITE(*,*)'INPUT N '
READ(*,*) NR
DO I = 1,NR
   READ(1,*) B(I)
   A(I) = B(I)*3.1416/180
ENDDO
XMAX = (-1)*NR - 1
DO J = 1, (NR-1)
   CALL MEAN(A,J,XMUON)
   DO K = 1, NR
      A(K) = A(K) - XMUON
      IF (A(K).LT.0) THEN
         A(K) = A(K) + 6.2832
      ENDIF
   ENDDO
   XMPS = 0
   T = J+1
   DO L = T,NR
      XMPS = XMPS + SIN(A(L))
   ENDDO
   IF (XMPS.GT.XMAX) THEN
      XMAX = XMPS
      CPT = J
   ENDIF
ENDDO
WRITE(*,*)'LMP ', XMAX
WRITE(*,*)'CHANGE POINT = ', CPT
STOP
END

SUBROUTINE MEAN(A,J,XMUON)
DIMENSION A(100)
SI=0
CO=0
DO I=1,J
     SI = SI + SIN(A(I))
```

```
        CO = CO + COS(A(I))
     ENDDO
     XMUON = ATAN2(SI,CO)
     RETURN
     END
```

7. VAX-FORTRAN program for computing the LRT statistic for outlier problem

```
DIMENSION A(100),XR(5000)
      OPEN (UNIT=1,FILE='data.dat',STATUS='OLD')
      WRITE(*,*)'INPUT C '
      READ(*,*) C
      WRITE(*,*)'INPUT N '
      READ(*,*) NR
      DO I=1,NR
         READ(1,*)A(I)
         a(i)=(a(i)*3.1416)/180
      ENDDO
      CALL ENULL(A,NR,XMUON)
      Write(*,*)'null estimate mu0',xmu0n
      XMIN=1
      DO J=1,NR-1
         CALL EALT(A,NR,J,XMU0A,XMU1A)
         Write(*,*)'alt est of mu0, mu1',xmu0a,xmu1a
         CALL XLAM(A,NR,C,J,XMUON,XMU0A,XMU1A,XLR)
         write(*,*)'j=',j,'xlr=',xlr
         IF(XLR.LT.XMIN) THEN
           XMIN=XLR
           SPT=J
           write(*,*)'spt=',spt
         ENDIF
      ENDDO
      XRT=XMIN
      WRITE(*,*)'SPT',SPT
```

```
WRITE(*,*)'XRT ',XRT
STOP
END

SUBROUTINE ENULL(A,NR,XMUON)
DIMENSION A(100)
SI=0
CO=0
DO I=1,NR
     SI = SI + SIN(A(I))
     CO = CO + COS(A(I))
ENDDO
XMUON = ATAN2(SI,CO)
RETURN
END

SUBROUTINE EALT(A,NR,J,XMU0A,XMU1A)
DIMENSION A(100)
SI0=0
CO0=0
DO I=1,NR
     SI0 = SI0 + SIN(A(I))
     CO0 = CO0 + COS(A(I))
ENDDO
SI0=SI0 - SIN(A(J))
CO0=CO0 - COS(A(J))
XMU0A = ATAN2(SI0,CO0)
XMU1A = A(J)
RETURN
END

SUBROUTINE XLAM(A,NR,C,J,XMUON,XMU0A,XMU1A,XLR)
DIMENSION A(100)
CON=0
COA0=0
COA1=0
DO I=1,NR
```

```
      CON = CON + COS(A(I)-XMUON)
   ENDDO
   DO I=1,NR
      COA0 = COA0 + COS(A(I)-XMU0A)
   ENDDO
   COA0=COA0 - COS(A(J)-XMU0A)
   COA1=COS(A(J)-XMU1A)
   C1=CON - (COA0+COA1)
   XLR = EXP(C*C1)
   RETURN
   END
```

8. VAX-FORTRAN program for computing the cut-off points of the LRT statistic for outlier problem

```
DIMENSION A(100),XR(5000)
      C=2.08
      WRITE(*,*)'INPUT N '
      READ(*,*) NR
      DO I=1,5000
         CALL RNVMS(NR,C,A)
         CALL ENULL(A,NR,XMUON)
         XMIN=1
         DO J=1,NR-1
            CALL EALT(A,NR,J,XMU0A,XMU1A)
            CALL XLAM(A,NR,C,J,XMUON,XMU0A,XMU1A,XLR)
            IF(XLR.LT.XMIN) THEN
              XMIN=XLR
              SPT=J
            ENDIF
         ENDDO
         XRT=XMIN
         XR(I)=XRT
         WRITE(*,*)'XRT ',XRT
      ENDDO
```

```
CALL SORT(XR)
STOP
END

SUBROUTINE ENULL(A,NR,XMUON)
DIMENSION A(100)
SI=0
CO=0
DO I=1,NR
     SI = SI + SIN(A(I))
     CO = CO + COS(A(I))
ENDDO
XMUON = ATAN2(SI,CO)
RETURN
END

SUBROUTINE EALT(A,NR,J,XMUOA,XMU1A)
DIMENSION A(100)
SIO=0
COO=0
DO I=1,NR
     SIO = SIO + SIN(A(I))
     COO = COO + COS(A(I))
ENDDO
SIO=SIO - SIN(A(J))
COO=COO - COS(A(J))
XMUOA = ATAN2(SIO,COO)
XMU1A = A(J)
RETURN
END

SUBROUTINE XLAM(A,NR,C,J,XMUON,XMUOA,XMU1A,XLR)
DIMENSION A(100)
CON=0
COAO=0
COA1=0
DO I=1,NR
```

```
   CON = CON + COS(A(I)-XMUON)
ENDDO
DO I=1,NR
   COAO = COAO + COS(A(I)-XMUOA)
ENDDO
COAO=COAO - COS(A(J)-XMUOA)
COA1=COS(A(J)-XMU1A)
C1=CON - (COAO+COA1)
XLR = EXP(C*C1)
RETURN
END

SUBROUTINE SORT(XR)
DIMENSION XR(5000)
OPEN(UNIT=2,STATUS='NEW',FILE='SPTQ.DAT')
DO I=1,5000
   DO J=I+1,5000
      IF (XR(I).GT.XR(J)) THEN
          TEMP=XR(I)
          XR(I)=XR(J)
          XR(J)=TEMP
      ENDIF
    ENDDO
ENDDO
DO I=1,100
   WRITE(2,*)I,XR(I*50)
ENDDO
RETURN
END
```

# Chapter 14

## TABLES

TABLE 1

CUT-OFF POINTS OF THE LRT FOR THE OUTLIER PROBLEM ($\alpha = .05$, various $\kappa$)

| $\kappa$ | $n$ =10 | $n$ = 20 | $n$ = 30 |
|---|---|---|---|
| 0.5 | 0.3680 | 0.3679 | 0.3679 |
| 1.0 | 0.1357 | 0.1354 | 0.1354 |
| 1.5 | 0.0504 | 0.0499 | 0.0498 |
| 2.0 | 0.0199 | 0.0187 | 0.0185 |
| 4.0 | 0.0139 | 0.0062 | 0.0034 |
| 10.0 | 0.0210 | 0.0102 | 0.0064 |

TABLE 2

PERCENTILES OF THE NULL DISTRIBUTION OF THE LRT FOR THE OUTLIER PROBLEM

($\kappa = 1$)

| PERCENTILES | n=10 | n=20 | n=30 |
|---|---|---|---|
| 1 | 0.1353 | 0.1353 | 0.1353 |
| 2 | 0.1354 | 0.1353 | 0.1353 |
| 5 | 0.1357 | 0.1354 | 0.1354 |
| 10 | 0.1367 | 0.1356 | 0.1354 |
| 25 | 0.1441 | 0.1371 | 0.1362 |
| 50 | 0.1786 | 0.1448 | 0.1398 |
| 75 | 0.2638 | 0.1707 | 0.1527 |
| 90 | 0.3884 | 0.2224 | 0.1774 |

TABLE 3

POWER OF THE LRT
FOR THE OUTLIER PROBLEM
$(\alpha = .05, \kappa = 1)$

| $\Delta$(in degrees) | n=10 | n=20 | n=30 | n=100 |
|---|---|---|---|---|
| 20 | 0.056 | 0.055 | 0.055 | 0.062 |
| 40 | 0.061 | 0.058 | 0.057 | 0.077 |
| 60 | 0.062 | 0.059 | 0.059 | 0.084 |
| 80 | 0.063 | 0.057 | 0.059 | 0.097 |
| 100 | 0.065 | 0.081 | 0.055 | 0.091 |
| 120 | 0.074 | 0.086 | 0.058 | 0.108 |
| 140 | 0.087 | 0.090 | 0.059 | 0.114 |
| 160 | 0.085 | 0.089 | 0.057 | 0.119 |
| 180 | 0.088 | 0.095 | 0.065 | 0.133 |

TABLE 4

VARIATION IN THE POWER OF THE LRT WITH $\kappa$
FOR THE OUTLIER PROBLEM
$(\alpha = .05, n = 10)$

| $\Delta$(in degrees) | $\kappa = 1$ | $\kappa = 2$ | $\kappa = 4$ | $\kappa = 6$ | $\kappa = 8$ | $\kappa = 10$ |
|---|---|---|---|---|---|---|
| 15 | 0.058 | 0.050 | 0.057 | 0.058 | 0.060 | 0.070 |
| 30 | 0.056 | 0.049 | 0.062 | 0.085 | 0.107 | 0.141 |
| 60 | 0.060 | 0.062 | 0.165 | 0.307 | 0.455 | 0.619 |
| 90 | 0.063 | 0.098 | 0.437 | 0.727 | 0.891 | 0.964 |
| 120 | 0.074 | 0.146 | 0.776 | 0.956 | 0.992 | 1.000 |
| 150 | 0.077 | 0.240 | 0.947 | 0.997 | 1.000 | 1.000 |
| 180 | 0.082 | 0.293 | 0.986 | 1.000 | 1.000 | 1.000 |

TABLE 5

5% CUT-OFF VALUES OF THE LRT FOR THE CHANGE POINT PROBLEM FOR DIFFERENT VALUES OF $\kappa$

| $\kappa$ | $n = 10$ | $n = 20$ | $n = 30$ |
|---|---|---|---|
| 0.5 | 0.1216 | 0.0559 | 0.0316 |
| 1.0 | 0.0335 | 0.0187 | 0.0152 |
| 1.5 | 0.0253 | 0.0149 | 0.0130 |
| 2.0 | 0.0214 | 0.0158 | 0.0158 |
| 4.0 | 0.0233 | 0.0167 | 0.0127 |
| 10.0 | 0.0272 | 0.0191 | 0.0150 |

TABLE 6

PERCENTILES OF THE NULL DISTRIBUTION OF LRT FOR THE CHANGE POINT PROBLEM ($\kappa = 1$)

| Percentiles | $n = 10$ | $n = 20$ | $n = 30$ |
|---|---|---|---|
| 1 | 0.0114 | 0.0034 | 0.0032 |
| 2 | 0.0181 | 0.0074 | 0.0064 |
| 5 | 0.0335 | 0.0187 | 0.0152 |
| 10 | 0.0600 | 0.0351 | 0.0270 |
| 25 | 0.1384 | 0.0962 | 0.0751 |
| 50 | 0.2498 | 0.1838 | 0.1628 |
| 75 | 0.4419 | 0.3348 | 0.2969 |
| 90 | 0.6193 | 0.4967 | 0.4503 |

TABLE 7

POWER OF THE LRT FOR THE CHANGE POINT PROBLEM

($\alpha = .05$, $\kappa = 1$, Change Point at 5)

| $\Delta$(in degrees) | $n = 10$ | $n = 20$ | $n = 30$ | $n = 50$ |
|---|---|---|---|---|
| 10 | 0.051 | 0.053 | 0.055 | 0.057 |
| 20 | 0.051 | 0.054 | 0.056 | 0.089 |
| 30 | 0.055 | 0.067 | 0.070 | 0.126 |
| 40 | 0.069 | 0.090 | 0.087 | 0.194 |
| 50 | 0.088 | 0.117 | 0.118 | 0.325 |
| 60 | 0.104 | 0.139 | 0.146 | 0.432 |
| 70 | 0.112 | 0.187 | 0.204 | 0.580 |
| 80 | 0.151 | 0.250 | 0.251 | 0.709 |
| 90 | 0.165 | 0.306 | 0.313 | 0.791 |
| 100 | 0.204 | 0.364 | 0.391 | 0.893 |
| 110 | 0.222 | 0.443 | 0.460 | 0.936 |
| 120 | 0.260 | 0.515 | 0.509 | 0.972 |
| 130 | 0.257 | 0.562 | 0.583 | 0.986 |
| 140 | 0.308 | 0.605 | 0.634 | 0.987 |
| 150 | 0.314 | 0.640 | 0.684 | 0.995 |
| 160 | 0.329 | 0.694 | 0.686 | 0.996 |
| 170 | 0.323 | 0.702 | 0.718 | 0.997 |
| 180 | 0.348 | 0.715 | 0.725 | 0.999 |

TABLE 8

VARIATION OF THE POWER OF THE LRT WITH THE LOCATION OF THE CHANGE POINT

$\alpha = .05,\ \kappa = 1,\ n = 20$

| $\Delta$(in degrees) | Change Point at | | | |
|---|---|---|---|---|
| | 5 | 9 | 13 | 17 |
| 10 | 0.053 | 0.050 | 0.051 | 0.050 |
| 20 | 0.054 | 0.062 | 0.059 | 0.049 |
| 30 | 0.067 | 0.084 | 0.081 | 0.057 |
| 40 | 0.090 | 0.109 | 0.103 | 0.071 |
| 50 | 0.117 | 0.149 | 0.142 | 0.078 |
| 60 | 0.139 | 0.202 | 0.183 | 0.107 |
| 70 | 0.187 | 0.267 | 0.241 | 0.118 |
| 80 | 0.250 | 0.347 | 0.310 | 0.136 |
| 90 | 0.306 | 0.429 | 0.393 | 0.175 |
| 100 | 0.364 | 0.512 | 0.463 | 0.220 |
| 110 | 0.443 | 0.612 | 0.546 | 0.249 |
| 120 | 0.515 | 0.673 | 0.615 | 0.293 |
| 130 | 0.562 | 0.737 | 0.687 | 0.342 |
| 140 | 0.605 | 0.793 | 0.741 | 0.350 |
| 150 | 0.640 | 0.833 | 0.787 | 0.400 |
| 160 | 0.694 | 0.855 | 0.812 | 0.412 |
| 170 | 0.702 | 0.870 | 0.828 | 0.439 |
| 180 | 0.715 | 0.872 | 0.836 | 0.446 |

TABLE 9

VARIATION OF THE POWER OF THE LRT WITH SAMPLE SIZE

$\alpha = .05$, $\kappa = 1$, Change Point at 20

| | $\Delta$(in degrees) | | | |
|---|---|---|---|---|
| $n$ | 45 | 90 | 135 | 180 |
| 30 | 0.156 | 0.553 | 0.866 | 0.953 |
| 40 | 0.193 | 0.829 | 0.979 | 0.998 |
| 50 | 0.265 | 0.832 | 0.989 | 0.998 |
| 75 | 0.286 | 0.879 | 0.995 | 0.998 |

TABLE 10

5% CUT-OFF VALUES OF THE NRTT FOR DIFFERENT VALUES OF $\kappa$ AND $n$

| $\kappa$ | $n = 5$ | 10 | 15 | 20 | 25 | 30 | 40 | 50 | 75 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 5.81 | 6.41 | 6.94 | 7.02 | 7.44 | 7.72 | 7.81 | 8.05 | 8.58 | 8.64 |
| 1.0 | 5.65 | 6.53 | 6.90 | 7.10 | 7.36 | 7.63 | 7.82 | 8.18 | 8.44 | 8.45 |
| 1.5 | 5.53 | 6.48 | 7.01 | 7.24 | 7.35 | 7.65 | 8.14 | 8.13 | 8.69 | 8.85 |
| 2.0 | 5.28 | 6.46 | 6.85 | 7.16 | 7.49 | 7.89 | 8.30 | 8.30 | 8.70 | 8.84 |
| 4.0 | 4.78 | 6.35 | 6.96 | 7.24 | 7.51 | 7.82 | 7.99 | 8.11 | 8.64 | 8.74 |
| 10.0 | 4.60 | 6.21 | 6.95 | 7.26 | 7.46 | 7.81 | 8.19 | 8.43 | 9.01 | 9.41 |

TABLE 11

POWER OF THE NRTT FOR
$\alpha = .05$, $\kappa = 1$
CHANGE POINT AT 5.

| $\Delta$(in degrees) | $n$=10 | $n$=20 | $n$=30 |
|---|---|---|---|
| 10 | 0.048 | 0.103 | 0.154 |
| 20 | 0.056 | 0.124 | 0.181 |
| 30 | 0.059 | 0.133 | 0.201 |
| 40 | 0.078 | 0.173 | 0.267 |
| 50 | 0.095 | 0.210 | 0.321 |
| 60 | 0.113 | 0.265 | 0.401 |
| 70 | 0.131 | 0.310 | 0.465 |
| 80 | 0.151 | 0.357 | 0.537 |
| 90 | 0.183 | 0.420 | 0.612 |
| 100 | 0.188 | 0.428 | 0.632 |
| 110 | 0.211 | 0.455 | 0.658 |
| 120 | 0.226 | 0.464 | 0.658 |
| 130 | 0.229 | 0.459 | 0.630 |
| 140 | 0.236 | 0.452 | 0.609 |
| 150 | 0.244 | 0.445 | 0.575 |
| 160 | 0.241 | 0.429 | 0.540 |
| 170 | 0.236 | 0.399 | 0.500 |
| 180 | 0.224 | 0.389 | 0.486 |

# Chapter 15

## FIGURES

In this chapter we provide figures of rosogram, changeogram, circular CUSUM and some of the skewed circular densities discussed in this thesis.

LIST OF FIGURES

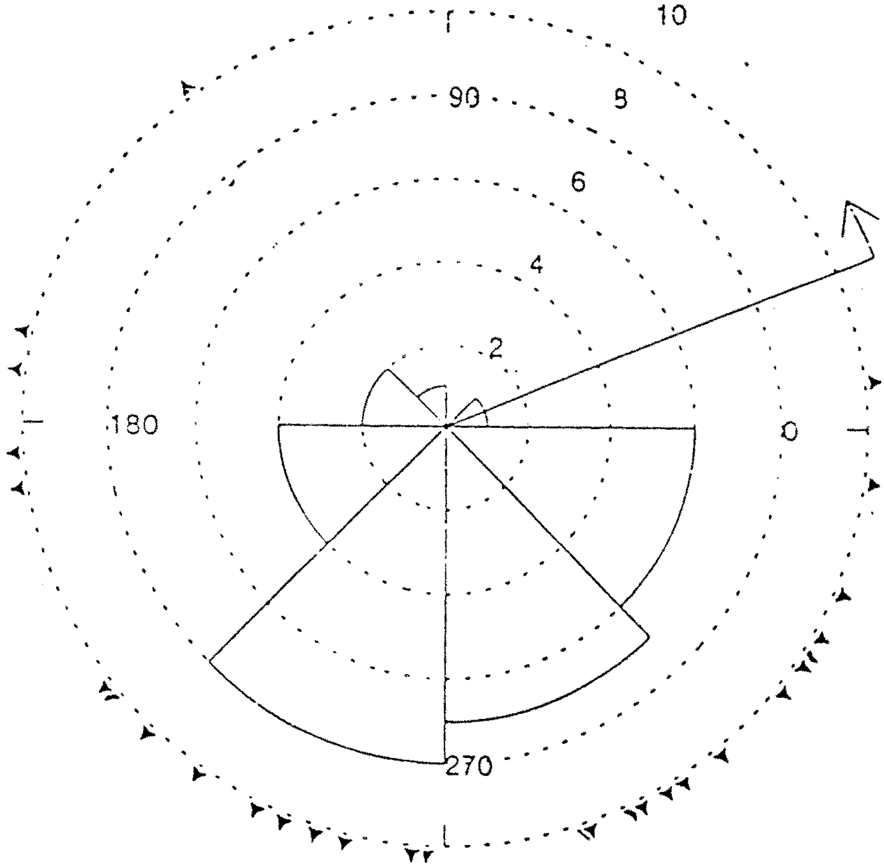


Fig.1 . Rosogram for Flare Data (Obs. 13-42)

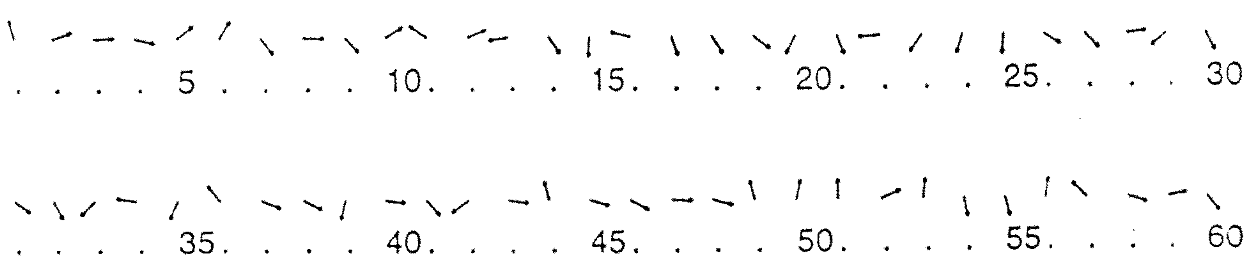


Fig.2 . Changeogram for Flare Data

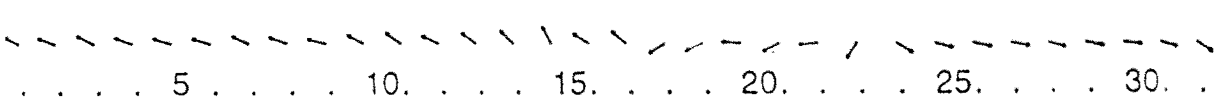


Fig.3. Changeogram for Wind Data


Fig 4: Circular CUSUM chart for Lombard's flare data
Observation Nos. 1-42

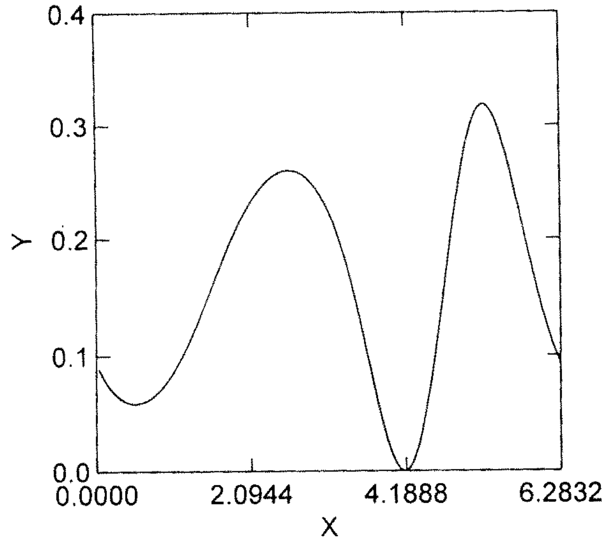


Fig 5: Papkonstantinou's skewed circular distribution P(1,2)

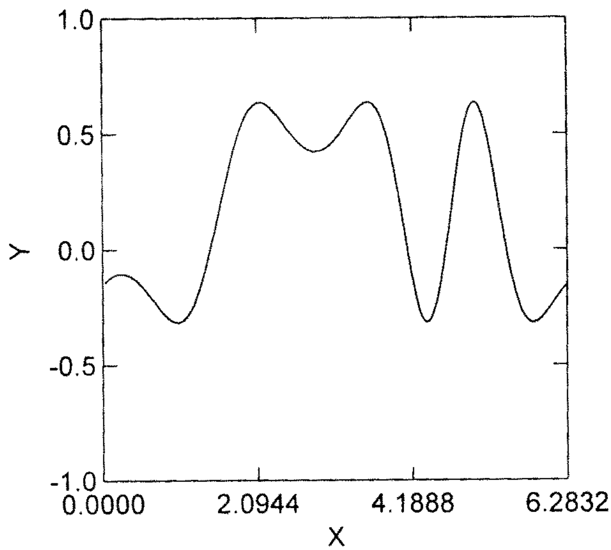


Fig 6: Papkonstantinou's skewed circular distribution P(3,4)

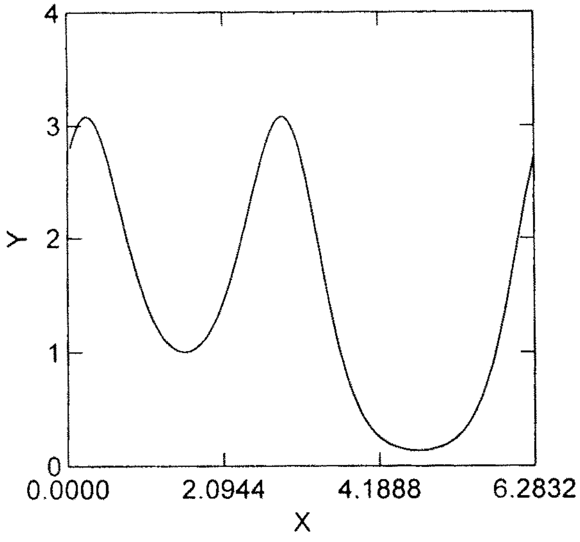


Fig 7 : Rattihali-SenGupta Skewed Circular Distributions RS(1,1,1.5708)

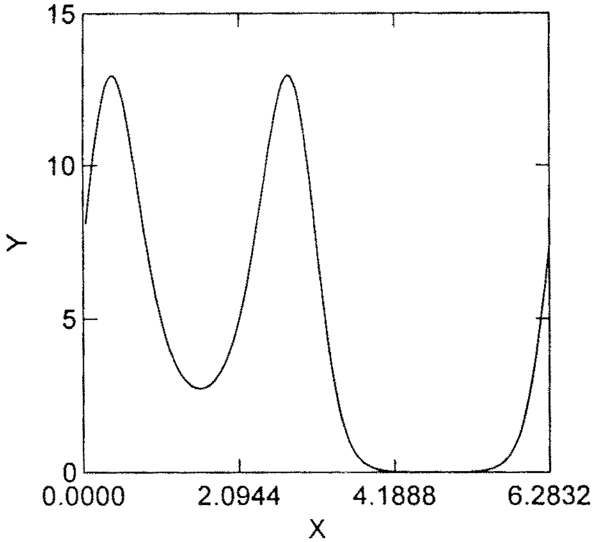


Fig 8 : Rattihali-SenGupta Skewed Circular Distributions RS(3,2,1.5708)

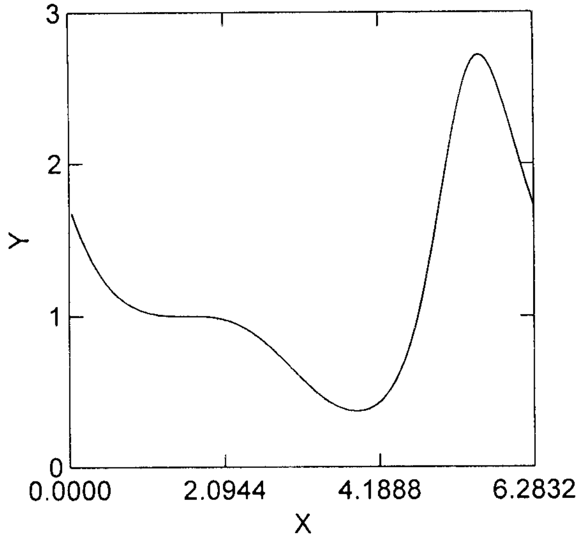


Fig 9 : Batschelet's Skewed Circular Distribution Ba(1,1)

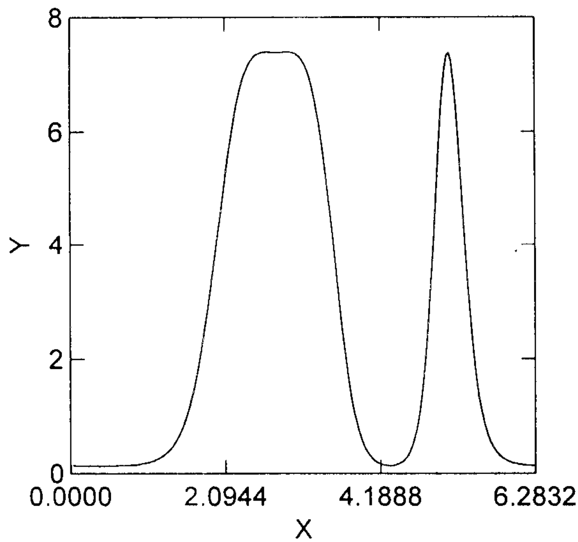


Fig 10 : Batschelet's Skewed Circular Distribution Ba(2,3)

APPENDIX I
SOME USEFUL CIRCULAR DISTRIBUTIONS

## 1. Circular normal distribution

The circular normal distribution, also known as the von Mises distribution, is one of the most popular distributions for modeling angular data. Introduced by von Mises in 1918, this distribution is most frequently used for analysis of directional data on the circle and plays a role similar to that of the normal distribution in linear statistical analysis. It is a symmetric, unimodal distribution with two parameters $\mu$ and $\kappa$, with probability density function(p.d.f.)

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left(\kappa \cos(\theta - \mu)\right), 0 \leq \theta < 2\pi, 0 \leq \mu < 2\pi, \kappa > 0$$

where $I_0(\kappa)$ is the modified Bessel function of order 0. The parameter $\mu$ is called the *mean direction* and the parameter $\kappa$ is called the *concentration parameter*. For $\kappa = 0$, the distribution reduces to the circular uniform distribution and as $\kappa$ increases, the distribution becomes increasingly concentrated near $\mu$. The circular normal distribution with parameters $\mu$ and $\kappa$ is denoted as $CN(\mu, \kappa)$.

Given a data set $\theta_1, \theta_2, \ldots, \theta_n$, the MLE of $\mu$ is the unique solution of the system of equations $R\cos\mu = \sum_{i=1}^{n} \cos\theta_i$ and $R\sin\mu = \sum_{i=1}^{n} \sin\theta_i$ where $R = \sqrt{\left(\sum_{i=1}^{n} \sin\theta_i\right)^2 + \left(\sum_{i=1}^{n} \cos\theta_i\right)^2}$ and that of $\kappa$ is $\hat{\kappa} = A^{-1}(\bar{R})$ where, $\bar{R} = R/n$, $A(.) = \frac{I_0'(.)}{I_0(.)}$ and $I_0'$ is the first derivative of $I_0$.

## 2. Wrapped normal distribution

The wrapped normal distribution is obtained by wrapping a normal distribution around the circle. In other words, if a random variable $X$ follows a normal distribution then the distribution of $\Theta = X \bmod 2\pi$ is the wrapped normal distribution. The p.d.f of the wrapped normal distribution with parameters $\mu$ and $\sigma$ (denoted by WN($\mu,\sigma$)) is

$$f(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \exp\left(\frac{-(\theta-\mu-2\pi m)^2}{2\sigma^2}\right),$$
$$0 \le \theta < 2\pi, 0 \le \mu < 2\pi, \sigma > 0$$

The WN($\mu,\sigma$) distribution is unimodal and symmetric about $\mu$.

## 3. Wrapped Cauchy distribution

The wrapped Cauchy distribution is obtained by wrapping the Cauchy distribution on the real line around the circle. The p.d.f of the wrapped Cauchy distribution with parameters $\mu$ and $\rho$ (denoted by WC($\mu,\rho$)) is

$$f(\theta) = \frac{1}{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}, \quad 0 \le \theta < 2\pi, 0 \le \mu < 2\pi, -1 < \rho < 1.$$

The WC($\mu,\rho$) distribution is unimodal and symmetric about $\mu$.

## 4. Wrapped Stable distributions

An important class of symmetric circular distributions is the Symmetric Wrapped Stable family of circular distributions. This family of distributions contain both the wrapped normal distribution and the wrapped Cauchy distribution as special cases. The p.d.f of symmetric wrapped stable distribution with parameters $\mu, \rho$ and $a$ (denoted by WS($\mu,\rho,a$)) is

$$f(\theta;\mu,\rho,a) = \frac{1}{2\pi}\left[1 + 2\sum_{k=1}^{\infty} \rho^{k^a} \cos k(\theta-\mu)\right],$$
$$0 \le \theta < 2\pi, 0 \le \mu < 2\pi, \rho \ge 0, 0 < a \le 2.$$

When $a = 2$ and $\rho = e^{-\frac{\sigma^2}{2}}$ we get the WN($\mu,\sigma$) distribution and when $a = 1$ we get the WC($\mu,\rho$) distribution.

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