DEFORMATION THEORY OF DIALGEBRAS

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Contents

0 Introduction **1**

1 Dialgebras and dialgebra cohomology **9**

1.1 Introduction 9

1.2 Dialgebras 10

1.3 Dialgebra cohomology 12

1.3.1 Planar binary trees 12

1.3.2 Cohomology with coefficients in a representation 14

2 Deformations of Dialgebras **16**

2.1 Introduction 16

2.2 Deformations 17

2.3 Infinitesimals 19

2.4 Integrability of 2-cocycles 23

2.5 Obstruction cochains 26

2.6 A deformation of the Laurent polynomials 29

3 Equivalent and trivial deformations **32**

3.1 Introduction 32

3.2 Equivalence of deformations 32

3.3 Rigidity 38

4 Automorphisms of the identity deformation **42**

4.1 Introduction 42

4.2 Infinitesimal 43

4.3 Obstructions to extending a derivation 45

5 Structure of a pre-Lie system on $CY^*(D, D)$ **50**

5.1 Introduction 50

5.2 Pre-Lie product on the cochain complex 51

5.3 Associative product on the cochain modules 73

6 Obstruction cocycles **88**

6.1 Introduction 88

6.2 Integrability of 2-cocycle 88

6.3 Integrability of 1-cocycle 93

7 G-algebras and dialgebra cohomology **95**

7.1 Introduction 95

7.2 Braces for dialgebra complex 96

7.3 Operad structure 99

7.4 Braces induced by the operad structure 111

7.5 G-algebra structure on cohomology 115

Bibliography **125**

Chapter 0

Introduction

The main objective of this thesis is to develop an algebraic deformation theory for associative dialgebras, which are binary quadratic algebras discovered by J.-L. Loday in [16], [17], and subsequently, to derive a G-algebra structure on the dialgebra cohomology with coefficients in itself.

Deformation theory dates back at least to Riemann's 1857 memoir on abelian functions in which he studied manifolds of complex dimension one and calculated the number of parameters (called moduli) upon which a deformation depends. The modern theory of deformations of structures on manifolds was developed extensively by Frolicher-Kodaira-Nijenhuis-Nirenberg-Spencer [13], [14], [15], [25], [26].

The study of deformations of algebraic structures was initiated by M. Gerstenhaber [5], [6], [7], [8], [9]. The basic theorems and features of a deformation theory are all due to him. The following is a brief description of the deformation theory of associative algebra, as introduced by M. Gerstenhaber.

Let A be an associative algebra over a field K, with underlying vector space V. Let $K[[t]]$ be the formal power series ring with coefficients in K and $Q = K((t))$ be the quotient power series field of $K[[t]]$. Let $V[[t]]$ be the power series module over

V and $V_Q = V[[t]] \otimes_{K[[t]]} Q$. Then V_Q is a vector space over Q.

Any bilinear function $f : V \times V \longrightarrow V$ extends to a bilinear function $V_Q \times V_Q \longrightarrow V_Q$ over Q. A bilinear function $f : V_Q \times V_Q \longrightarrow V_Q$ which is such an extension is said to be 'defined over K'. Suppose a multiplication $f_t : V_Q \otimes_Q V_Q \longrightarrow V_Q$ is given by a formal power series of the form

$$f_t(a,b) = F_0(a,b) + F_1(a,b)t + F_2(a,b)t^2 + \cdots$$

where each F_i is defined over K and $F_0(a,b) = ab$, the original multiplication of A.

Assume that f_t is associative, that is,

$$f_t(f_t(a,b),c) = f_t(a, f_t(b,c)), \quad \text{for all} \ \ a,b,c \in V_Q.$$

Then the associative algebra A_Q, with underlying vector space V_Q and multiplication f_t are called a one parameter family of deformations of A. The algebra A_Q is called a deformed algebra. The condition that f_t is associative is equivalent to having for all a, b, c in V and for all $\nu = 0, 1, 2, \ldots$

$$(0.1) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu\geq 0}} F_\lambda(F_\mu(a,b),c) - F_\lambda(a, F_\mu(b,c)) = 0.$$

For ν=0, this is just the associativity of the original multiplication. For ν=1, the above condition implies

$$aF_1(b,c) - F_1(ab,c) + F_1(a,bc) - F_1(a,b)c = 0$$

In terms of Hochschild theory this simply means that F_1 is a Hochschild 2-cocycle, that is, $F_1 \in Z^2(A,A)$. The 2-cocycle F_1 is called the infinitesimal of the deformation.

Thus we have the following result.

Proposition 0.0.1 *The infinitesimal of a one parameter family of deformation of an associative algebra is a Hochschild 2-cocycle.*

An arbitrary element $F_1 \in Z^2(A, A)$ need not be an infinitesimal of a deformation. If it be so, then we say that F_1 is integrable. The integrability of F_1 implies an infinite sequence of relations which may be interpreted as the vanishing of the 'obstructions' to the integration of F_1. For we can rewrite (0.1) as

$$\delta F_\nu(a, b, c) = \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda(F_\mu(a, b), c) - F_\lambda(a, F_\mu(b, c)). \tag{0.2}$$

For $\nu = 2$, (0.2) gives

$$\delta F_2(a, b, c) = F_1(F_1(a, b), c) - F_1(a, F_1(b, c)).$$

Let G be the function of three variables defined by the right hand side of the above equality, that is,

$$G(a, b, c) = F_1(F_1(a, b), c) - F_1(a, F_1(b, c)).$$

If $F_1 \in Z^2(A, A)$, then $G \in Z^3(A, A)$, and the cohomology class of G is the 1st obstruction to the integration of F_1; if F_1 is integrable, this class must be zero.

In general, suppose $F_1, F_2, \ldots, F_{n-1}$ satisfy (0.2) for $\nu = 1, 2, \ldots, n-1$ then the right hand side of (0.2), for $\nu = n$ defines a Hochschild 3-cochain, and is called the primary obstruction and it is an obstruction to finding F_n, which would extend the deformation. The most important result in deformation theory is the following theorem.

Theorem 0.0.2 *The primary obstruction is a 3-cocycle.*

We also have the following theorem.

Theorem 0.0.3 *If $H^3(A, A) = 0$ then any $F_1 \in Z^2(A, A)$ is integrable.*

A one parameter family of deformations of an associative algebra A defined by a multiplication f_t is trivial if there exists a linear automorphism Φ_t of V_Q

$$\Phi_t(a) = a + \Phi_1(a)t + \Phi_2(a)t^2 + \cdots,$$

where each $\Phi_i : V_Q \longrightarrow V_Q$ are linear maps defined over K, such that $\Phi_t f_t(a, b) = \Phi_t(a) \cdot \Phi_t(b)$. In that case, the deformed algebra is isomorphic to the algebra $A[[t]] \otimes_{K[[t]]} Q$

There is an obvious notion of equivalence of deformations and an associative algebra A is called rigid if every deformation is equivalent to the trivial one. An important result in this theory is the following theorem.

Theorem 0.0.4 *If $H^2(A, A) = 0$, then A is rigid.*

For example, the tensor algebra $T(V)$, the universal enveloping algebra $U(g)$ of a semi simple Lie algebra g are rigid.

M. Gerstenhaber remarked that his methods would extend to any equationally defined algebraic structure. His theory was extended to Lie algebras by A. Nijenhuis and R. Richardson [21], [22], [23]. Their work closely parallels those of the Frolicher-Kodaira-Nijenhuis-Nirenberg-Spencer theory. The deformation theory of bialgebra, which relates to quantum groups, was studied by Gerstenhaber and Schack in [11].

Any deformation theory should have the following features:

(i) A class of objects within which deformation takes place and a cohomology theory (deformation cohomology) associated to those objects which controls the deformation in the sense that infinitesimal deformation of a given object can be identified with the elements of a suitable cohomology group.

(ii) A theory of obstruction to the integration of an infinitesimal deformation.

(iii) Deformation of automorphisms of the deformed object and a notion of rigid objects.

(iv) Existence of a G-alegbra structure on the deformation cohomology.

For example, in the Gerstenhaber theory, objects are associative algebras and the natural candidate for the cohomology is the Hochschild cohomology and for Nijenhuis-Richardson theory, objects are Lie algebras and the associated cohomology is the Chevalley-Eilenberg cohomology.

The notion of Leibniz algebras, and dialgebras were introduced by J.-L. Loday in [16], [17], while studying periodicity phenomena in algebraic K-theory.

The dialgebras are a new kind of algebraic object, whose structure is determined by two associative operations intertwined by some relations, for which there exists an associated Koszul operad.

More precisely, a dialgebra over a field K is a vector space D over K equipped with two operations $\dashv$ (called left) and $\vdash$ (called right), satisfying the following five axioms:

$$\begin{aligned}
x \dashv (y \dashv z) &\overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z) \\
(x \vdash y) \dashv z &\overset{3}{=} x \vdash (y \dashv z) \\
(x \dashv y) \vdash z &\overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z
\end{aligned}$$

Observe that the relations 1 and 5 say that the operations $\dashv$ and $\vdash$ are associative respectively and 3 is called the inner associativity.

A cohomology theory associated to the dialgebras was developed by J.-L. Loday, called the dialgebra cohomology. The dialgebra cohomology with coefficients has been studied by A. Frabetti [4]. An interesting fact is the appearance of combinatorial objects called planar binary trees in the construction of this cohomology theory.

The major aim of the thesis (a part of which appears in [19]) is to develop deformation theory for associative dialgebras (which we simply call dialgebra), using dialgebra cohomology $HY^*(D, D)$, extending the methods of [5]-[9]. For deformation of a dialgebra, which consists of two associative products intertwined by some extra relations, one needs to consider deformations of each of the products simultaneously. Moreover, these deformations should be correlated in some sense and this correlation should be inbuilt in the definition of the infinitesimal.

The main results of this thesis consist of showing deformations of a dialgebra D are controlled by the dialgebra cohomology $HY^*(D, D)$ with coefficients in the dialgebra itself. More precisely, we show that:

(i) The infinitesimal of a deformation is a 2-cocycle, and the primary obstruction to the extension of a 2-cocycle to a deformation is a 3-cocycle.

(ii) We derive a sufficient condition for rigidity and show that the free dialgebra is rigid.

(iii) The infinitesimal of an automorphism of the identity deformation is a derivation (that is, a 1-cocycle), and the obstruction for extension of a derivation to an automorphism is a 2-cocycle.

To obtain the above results we introduce a pre-Lie system and a pre-Lie product on the dialgebra cochain complex $CY^*(D, D)$, which defines the dialgebra cohomology $HY^*(D, D)$. This structure on $CY^*(D, D)$ will play an important role to develop the deformation theory of dialgebras. Moreover, the pre-Lie product on the dialgebra complex is described constructively, using the combinatorial properties of the planar trees, typical of the dialgebra cohomology. We also introduce a new example of a dialgebra on the Laurent polynomials in two variables, and an explicit deformation of it. This is done in Chapters 2-6 of this thesis.

It may be mentioned that in [1], D. Balavoine studied formal deformations of algebras over a quadratic operad in general and showed that the cohomology theory which is involved is the one given by the Koszul dual operad. For a type of algebra, deformation theory may also be built up by using triple cohomology as explained in [3]. Moreover, an algebraic analogue of Haefliger's cohomology can be used to develop algebraic deformation theory too [24].

M. Gerstenhaber and A. A. Voronov have shown in [12], that the cochain complex $C^*(A, A)$, of an associative algebra A, admits a homotopy G-algebra structure, which in turn induces a G-algebra structure on the cohomology $H^*(A, A)$. In this thesis, we show that as in the case of Hochschild complex, the dialgebra complex $CY^*(D, D)$ with the differential altered by a suitable sign, admits a homotopy G-algebra structure which comes from a non-Σ operad structure on $CY^*(D, D)$. Moreover, we deduce that this structure induces a G-algebra structure on the cohomology $HY^*(D, D)$ of a dialgebra D. This is done in Chapter 7 of this thesis.

The thesis is organized as follows. In Chapter 1, we summarise the basic facts about dialgebras and their cohomology. In Chapter 2, we define formal deformation of dialgebras, obstruction cochains, prove few immediate results and state one of the two main theorems about obstruction cochains. In Chapter 3, we introduce the notion of equivalent and trivial deformations in this context and prove that the free dialgebra $Dias(V)$ is rigid. In Chapter 4, we introduce the notion of infinitesimal of an automorphism, define obstruction to integrability of 1-cocycles and state the other theorem about obstruction cochains. In Chapter 5 we introduce a $\circ_i$ product on $CY^*(D, D)$. It turns out that equipped with these $\circ_i$-products, $CY^*(D, D)$ admits the structure of a pre-Lie system. We then use $\circ_i$-products to define a pre-Lie product $\circ$ on $CY^*(D, D)$ which makes $CY^*(D, D)$ a pre-Lie ring. There is also defined an associative product $*$ on $CY^*(D, D)$ and we establish a relation connecting the pre-Lie product $\circ$, the associative product $*$ and the coboundary operators of $CY^*(D, D)$. In Chapter 6, we interpret the obstruction cochains in terms of $\circ$ and $*$ and prove the main theorems of the thesis. Finally in Chapter 7, we show that the dialgebra complex $CY^*(D, D)$ with the differential altered by a sign admits a homotopy G-algebra structure, which is induced by a non-Σ operad structure on $CY^*(D, D)$, and also deduce that the cohomology $HY^*(D, D)$ admits a G-algebra structure.

Chapter 1

Dialgebras and dialgebra cohomology

1.1 Introduction

The notion of Leibniz algebras and dialgebras (or more precisely, associative dialgebras) was discovered by J.-L. Loday while studying periodicity phenomena in algebraic K-theory [17]. The dialgebras are introduced as a new kind of algebraic object, whose structure is determined by two associative operations intertwined by some relations, for which there exists an associated Koszul operad.

A (co)homology theory associated to dialgebras has been developed by J.-L. Loday and, interestingly enough, combinatorial objects called planar binary trees play a crucial role in the construction of the (co)chain complex. Dialgebra (co)homology with coefficients was introduced by A. Frabetti in [4].

In this chapter we present, for completeness, the definition of dialgebras as introduced by J.-L. Loday in [16] and the description of dialgebra cohomology with coefficients as introduced by A. Frabetti in [4].

1.2 Dialgebras

After recalling the definition, we present a few examples of dialgebras, and describe the notion of free dialgebra over a vector space, all of which can be found in [16]. Throughout this thesis, K will denote the ground field, and the tensor product over K will be denoted by $\otimes$.

Definition 1.2.1 An associative dialgebra (or simply, dialgebra) D over K is a vector space over K along with two K-linear maps $\dashv: D \otimes D \longrightarrow D$ called left and $\vdash: D \otimes D \longrightarrow D$ called right satisfying the following axioms :

$$(1.1) \qquad \begin{cases} x \dashv (y \dashv z) & \overset{1}{=} \quad (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z) \\ (x \vdash y) \dashv z & \overset{3}{=} \quad x \vdash (y \dashv z) \\ (x \dashv y) \vdash z & \overset{4}{=} \quad x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z \end{cases}$$

for all x, y, and $z \in D$.

A morphism of dialgebras from D to D' is a K-linear map $f : D \longrightarrow D'$ such that $f(x \dashv y) = f(x) \dashv f(y)$ and $f(x \vdash y) = f(x) \vdash f(y)$ for all $x, y \in D$.

Example 1.2.2 Let (A,d) be a differential associative algebra. Define left and right products on (A,d) by

$$x \dashv y := x\ dy \qquad \text{and} \qquad x \vdash y := dx\ y.$$

Then A equipped with these two products is a dialgebra.

Example 1.2.3 Let A be an associative algebra and let M be an A-bimodule. Let $f : M \to A$ be an A-bimodule map. Then M can be made into a dialgebra by defining left and right products as

$$m \dashv m' := mf(m') \qquad \text{and} \qquad m \vdash m' := f(m)m'.$$

Example 1.2.4 The category of linear maps over a field K denoted by $\mathcal{LM}$ consists of the K-linear maps $f : V \to W$ as objects. Morphisms between $f : V \to W$ and $f' : V' \to W'$ are a pair of K-linear maps (φ, ψ), $\varphi : V \to V'$ and $\psi : W \to W'$ such that $f' \circ \varphi = \psi \circ f$. This category can be equipped with a tensor product as follows.

$$(V \xrightarrow{f} W) \otimes (V' \xrightarrow{f'} W') = V \otimes W' \oplus W \otimes V' \xrightarrow{f \otimes 1 + 1 \otimes f'} W \otimes W'.$$

Let $g : M \to R$ be an object in the category $\mathcal{LM}$. Then $g : M \to R$ is an associative algebra in this category is equivalent to saying that R is an associative algebra, M is an R-bimodule and g is a bimodule map, [18]. Hence, an associative algebra $g : M \to R$ in this category $\mathcal{LM}$ defines a dialgebra structure on the source M, by the previous example.

Example 1.2.5 Let A be an associative algebra and n be a positive integer. On the module of n-vectors $D = A^n$ one can define left and right products as follows :

$$(x \dashv y)_i = x_i(\sum_{j=1}^{n} y_j) \quad \text{for} \quad 1 \leq i \leq n,$$

$$(x \vdash y)_i = (\sum_{j=1}^{n} x_j)y_i \quad \text{for} \quad 1 \leq i \leq n.$$

Then D equipped with these products is a dialgebra. This is a special case of Example 1.2.3, where f is given by $f((y_i)) = \sum y_i$.

Let $Dias$ denote the category of dialgebras. The free object $Dias(V)$ in the category $Dias$ is described as follows. Let V be a vector space over K. The free dialgebra on V is the dialgebra $Dias(V)$ equipped with a K-linear map $i : V \longrightarrow Dias(V)$ such that for any K-linear map $f : V \longrightarrow D$, where D is a dialgebra over

K, there exists a unique dialgebra map $\varphi : Dias(V) \to D$, such that $f = \varphi \circ i$. An explicit description of $Dias(V)$ is as follows :

$$Dias(V) = T(V) \otimes V \otimes T(V)$$

where $T(V) = K \oplus V \oplus V^{\otimes 2} \oplus \cdots$, equipped with the two products induced by :

$$\begin{aligned} & (v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m) \dashv (w_{-p} \cdots w_{-1} \check{w}_0 w_1 \cdots w_q) \\ = \; & v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m w_{-p} \cdots w_{-1} w_0 w_1 \cdots w_q \end{aligned}$$

and

$$\begin{aligned} & (v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m) \vdash (w_{-p} \cdots w_{-1} \check{w}_0 w_1 \cdots w_q) \\ = \; & v_{-n} \cdots v_{-1} v_0 v_1 \cdots v_m w_{-p} \cdots w_{-1} \check{w}_0 w_1 \cdots w_q \end{aligned}$$

where

$$v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m = v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m$$

is a typical additive generator of $Dias(V)$.

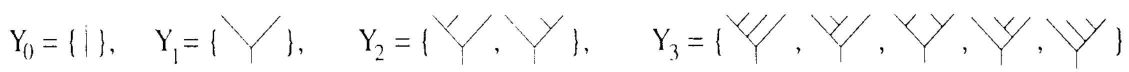
1.3 Dialgebra cohomology

In this section, we first recall planar binary trees and certain face maps which are defined on the set of planar binary trees making it into a pre-simplicial set. Next we recall the definition of cohomology of a dialgebra with coefficients.

1.3.1 Planar binary trees

A planar binary tree with n vertices (in short, n-tree) is a planar tree with $(n+1)$ leaves, one root and each vertex trivalent. Let Y_n denote the set of all n-trees. Let Y_0

be the singleton set consisting of a root only. The trees $Y_n, 0 \leq n \leq 3$ are as shown in the following diagram :



For any $y \in Y_n$, the $(n+1)$ leaves are labelled by $\{0, 1, \ldots, n\}$ from left to right and the vertices are labelled $\{1, 2, \ldots, n\}$ so that the ith vertex is between the leaves $(i-1)$ and i. The only element $|$ of Y_0 is denoted by $[0]$ and the only element of Y_1 is denoted by $[1]$.

There is a convenient way to denote every tree uniquely by an array of numbers, by using what is known as grafting of trees. Grafting of a p-tree y_1 and a q-tree y_2 is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$ which is obtained by joining the roots of y_1 and y_2 and creating a new root from that vertex. This is denoted by $[y_1 \; p+q+1 \; y_2]$ with the convention that all zeros are deleted except for the element in Y_0. For instance, $[0] \vee [0] = [1]$, $[1] \vee [0] = [12]$. Thus the trees pictured above from left to right are $[0], [1], [12], [21], [123], [213], [131], [312], [321]$. Throughout this thesis, we shall use these notations to represent elements of Y_n, $0 \leq n \leq 3$. Also, given a tree $y \in Y_n$, one can uniquely determine two trees $y_1 \in Y_p$ and $y_2 \in Y_q$ with $p, q < n$, such that $y = y_1 \vee y_2$.

It may be noted that the n-trees that we are considering here are just the finite binary trees obtained by considering free binary operation on Y_0, as described in [2].

Definition 1.3.1 For any i, $0 \leq i \leq n$, face maps $d_i : Y_n \longrightarrow Y_{n-1}$, are defined as follows :

$$\begin{aligned} d_i : Y_n &\longrightarrow Y_{n-1} \\ y &\mapsto d_i y, \end{aligned}$$

where $d_i y$ is obtained from y by deleting the ith leaf.

These face maps satisfy $d_i d_j = d_{j-1} d_i, \quad i < j$. Thus the set $\{Y_n; n \geq 0\}$ of planar binary trees equipped with the above face maps is a pre-simplicial set.

1.3.2 Cohomology with coefficients in a representation

The dialgebra cohomology was first constructed by J.-L. Loday [16]. Coefficients in this theory was introduced by A. Frabetti [4] to define dialgebra cohomology with coefficients. Coefficients are representations of a dialgebra as defined below.

Definition 1.3.2 Let D be a dialgebra over a field K. A representation of D is a K-module M, endowed with two left actions $\dashv, \vdash$: $D \otimes M \longrightarrow M$, and two right actions $\dashv, \vdash$: $M \otimes D \longrightarrow M$, satisfying the axioms (1.1), whenever one of the entries x, y or z is in M and the two others are in D.

Clearly, D itself is a representation of D where the left actions and the right actions are given by the left product and the right product of D respectively.

Let D be a dialgebra over K, and M a representation of D. For $n \geq 0$, let $K[Y_n]$ denote the K-vectorspace spanned by Y_n and $CY^n(D, M) := Hom_K(K[Y_n] \otimes D^{\otimes n}, M)$ be the module of n-cochains of D with coefficients in M. The coboundary operator $\delta : CY^n(D, M) \longrightarrow CY^{n+1}(D, M)$ is defined as the K-linear map $\delta = \sum_{i=0}^{n+1}(-1)^i \delta^i$, where

$$(\delta^i f)(y; a_1, a_2, \ldots, a_{n+1}) := \begin{cases} a_1 o_0^y f(d_0 y; a_2, \ldots, a_{n+1}), & \text{if } i = 0 \\ f(d_i y; a_1, \ldots, a_i o_i^y a_{i+1}, \ldots, a_{n+1}), & \text{if } 1 \leq i \leq n \\ f(d_{n+1} y; a_1, \ldots, a_n) o_{n+1}^y a_{n+1}, & \text{if } i = n+1 \end{cases}$$

for any $y \in Y_{n+1}$; $a_1, \ldots, a_{n+1} \in D$ and $f : K[Y_n] \otimes D^{\otimes n} \longrightarrow M$. Here, for any i, $0 \leq i \leq n+1$, $d_i : Y_{n+1} \longrightarrow Y_n$ are the face maps as in Definition 1.3.1 and the maps $o_i : Y_{n+1} \longrightarrow \{\dashv, \vdash\}$, are defined by

$$o_0(y) = o_0^y := \begin{cases} \dashv & \text{if } y \text{ is of the form } | \vee y_1, \text{ for some } n\text{-tree } y_1 \\ \vdash & \text{otherwise} \end{cases}$$

$$o_i(y) = o_i^y := \begin{cases} \dashv & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like `}\backslash\text{'} \\ \vdash & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like `/'} \end{cases}$$

for $1 \leq i \leq n$ and

$$o_{n+1}(y) = o_{n+1}^y := \begin{cases} \vdash & \text{if } y \text{ is of the form } y_1 \vee |, \text{ for some } n\text{-tree } y_1 \\ \dashv & \text{otherwise.} \end{cases}$$

The K-linear map δ satisfies $\delta \circ \delta = 0$. The cohomology of the dialgebra D with coefficients in M is defined by

$$HY^n(D, M) := H^n(CY^*(D, M)) = \frac{ZY^n(D, M)}{BY^n(D, M)}$$

where $ZY^n(D, M)$ and $BY^n(D, M)$ are submodules of $CY^n(D, M)$ consisting of cocycles and coboundaries respectively. Throughout this thesis, we shall be concerned with $HY^*(D, D)$, the cohomology of D with coefficients in D.

Chapter 2

Deformations of Dialgebras

2.1 Introduction

The study of deformations of algebraic structures was initiated by M. Gerstenhaber [5], [6], [7], [8], [9]. He introduced deformation theory for associative algebras and remarked that his methods would extend to any equationally defined algebraic structure. His theory was further extended to Lie algebras by A. Nijenhuis and R. Richardson [21], [22], [23]. The deformation theory of bialgebras, which relates to quantum groups, was studied by M. Gerstenhaber and S. D. Schack [11]. The main aim of this chapter is to construct a deformation theory for dialgebras, which parallels the existing theories for associative and Lie algebras.

In the majority of the available cases of deformation theory experience provides one with a natural candidate for the cohomology controlling the deformations. For example, one knows that Hochschild cohomology captures the deformation of associative algebras and the Chevalley-Eilenberg cohomology controls the deformation of Lie algebras. We shall see that in the case of dialgebras dialgebra cohomology as described in the previous chapter controls the deformations. It turns out that, as in most classical cases, deformations of a dialgebra are controlled by the lower dimensional cohomology groups. It is worth mentioning here that in classical algebraic deformation theories, the relevant cohomology controlling the deformations actually coincides with the Yoneda cohomology in a category of bimodules. In the present case, it would be interesting to find relationship between the Yoneda cohomology in the category of bimodules over a dialgebra and the dialgebra cohomology as introduced in Section 1.3.2.

In this chapter we introduce the definitions of deformations of a dialgebra, n-infinitesimal of a deformation, integrability and obstruction cochains. We also prove a few elementary results, involving the infinitesimals, which are analogous to the results obtained by M. Gerstenhaber in [6], in the theory of deformations of an associative algebra. Moreover, we state one of the main theorems of this thesis, involving obstruction cochains of a family of deformations. The proof of this theorem appears in chapter 6, and depends on the pre-Lie system structure of the dialgebra cochain complex $CY^*(D, D)$. At the end we introduce a new example of a dialgebra on the Laurent polynomials in two variables and an explicit deformation of it.

2.2 Deformations

The definitions of deformations of dialgebras and infinitesimals, as they appear in this section, closely parallel the same definitions, as they appear in the theory of deformations of an associative algebra, [6].

Let D be a dialgebra over K with left product $\dashv$ and right product $\vdash$. Let V be the underlying vector space of D, $K[[t]]$ denote the power series ring in one variable and $Q = K((t))$ denote the quotient power series field. Let $V[[t]]$ denote the power series module with coefficients in V over $K[[t]]$ and V_Q denote the Q-vector space $V[[t]] \otimes_{K[[t]]} Q$. Here one has to note that $V[[t]] \otimes_{K[[t]]} Q$ is made into a Q-vector space by setting $(\mathbf{a} \otimes q) \cdot q' = \mathbf{a} \otimes (q \cdot q')$ for all $\mathbf{a} \in V[[t]]$ and $q,\ q' \in Q$, that is, this Q-vector space is obtained from $V[[t]]$ by *extension of scalars* from $K[[t]]$ to Q. Note that any K-bilinear map $f : V \times V \longrightarrow V$, in particular, the two products $\dashv$ and $\vdash$, extends to a Q-bilinear map $\tilde{f} : V_Q \times V_Q \longrightarrow V_Q$ in a natural way. Any Q-bilinear map $V_Q \times V_Q \longrightarrow V_Q$ which is such an extension is said to be 'defined over K'.

With the above notations, we make the following definition.

Definition 2.2.1 Let there be given two bilinear maps $f_t^\ell, f_t^r : V_Q \times V_Q \longrightarrow V_Q$, which are expressible in the form

$$f_t^\ell(\mathbf{a},\mathbf{b}) = F_0^\ell(\mathbf{a},\mathbf{b}) + F_1^\ell(\mathbf{a},\mathbf{b})t + F_2^\ell(\mathbf{a},\mathbf{b})t^2 + \cdots \tag{2.1}$$

$$f_t^r(\mathbf{a},\mathbf{b}) = F_0^r(\mathbf{a},\mathbf{b}) + F_1^r(\mathbf{a},\mathbf{b})t + F_2^r(\mathbf{a},\mathbf{b})t^2 + \cdots \tag{2.2}$$

for all $\mathbf{a}, \mathbf{b} \in V_Q$, where F_i^ℓ and F_i^r are bilinear maps $V_Q \times V_Q \longrightarrow V_Q$ defined over K, and F_0^ℓ and F_0^r are induced by $\dashv$ and $\vdash$ respectively. Consequently, the maps f_t^ℓ and f_t^r are also defined over K. Moreover assume that V_Q equipped with the products f_t^ℓ and f_t^r is a dialgebra which we denote by D_t. Then D_t is called a one-parameter family of (formal) deformations of D.

We note that the identities (2.1) and (2.2) are equivalent to

$$f_t^\ell(a,b) = F_0^\ell(a,b) + F_1^\ell(a,b)t + F_2^\ell(a,b)t^2 + \cdots \tag{2.3}$$

$$f_t^r(a,b) = F_0^r(a,b) + F_1^r(a,b)t + F_2^r(a,b)t^2 + \cdots \tag{2.4}$$

for all $a,\ b \in V$, as all maps involved are defined over K.

Here is an obvious example of a deformation.

Example 2.2.2 *Identity deformation of a dialgebra D is the dialgebra $D_Q = D[[t]] \otimes_{K[[t]]} Q$ with the underlying vector space V_Q and with multiplications $g_t^\ell(\mathrm{a}, \mathrm{b}) = \mathrm{a} \dashv \mathrm{b}$ and $g_t^r(\mathrm{a}, \mathrm{b}) = \mathrm{a} \vdash \mathrm{b}$ for all $\mathrm{a}, \mathrm{b} \in V_Q$, induced by the products of D.*

2.3 Infinitesimals

Analogous to the definition of infinitesimal in the case of deformations of associative algebras, we introduce the following definitions.

Let D_t, with products f_t^ℓ and f_t^r as given by equations (2.1) and (2.2), denote a one parameter family of deformations of the dialgebra D. With these notations, we make the following definition.

Definition 2.3.1 The 'infinitesimal' or 'differential' of this family of formal deformations is the function $F_1 : K[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ defined by

$$F_1(y; a_1, a_2) = \begin{cases} F_1^\ell(a_1, a_2), & \text{if } y = [21] \\ F_1^r(a_1, a_2), & \text{if } y = [12] \end{cases}$$

where F_1^ℓ and F_1^r are considered as K-bilinear functions from $V \times V$ to V.

More generally, higher order infinitesimals are defined as follows.

Definition 2.3.2 Let $F_i^\ell = 0 = F_i^r$, $1 \leq i \leq n-1$, with either F_n^ℓ or F_n^r non-zero. Then the function $F_n : K[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ defined by

$$F_n(y; a_1, a_2) = \begin{cases} F_n^\ell(a_1, a_2), & \text{if } y = [21] \\ F_n^r(a_1, a_2), & \text{if } y = [12] \end{cases}$$

is called the n-infinitesimal of this family of deformations, where F_n^ℓ and F_n^r are considered as K-bilinear functions from $V \times V$ to V.

Thus infinitesimal of a family is simply 1-infinitesimal.

Let us now take a closer look at the conditions that are forced upon the bilinear maps F_ν^ℓ and F_ν^r, for all $\nu = 0, 1, \ldots$, by the fact that V_Q along with the products f_t^ℓ and f_t^r, is a dialgebra D_t. If D_t along with f_t^ℓ and f_t^r denotes a one parameter family of deformations of the dialgebra D, by the axioms (1.1), we must have,

$$(2.5) \qquad \begin{cases} f_t^\ell(a, f_t^\ell(b,c)) &= f_t^\ell(f_t^\ell(a,b),c), \\ f_t^\ell(f_t^\ell(a,b),c) &= f_t^\ell(a, f_t^r(b,c)), \\ f_t^\ell(f_t^r(a,b),c) &= f_t^r(a, f_t^\ell(b,c)), \\ f_t^r(f_t^\ell(a,b),c) &= f_t^r(a, f_t^r(b,c)), \\ f_t^r(a, f_t^r(b,c)) &= f_t^r(f_t^r(a,b),c) \end{cases}$$

for all $a, b, c \in V$, where V denotes the underlying vector space of D.

Now expanding both sides of each of the equations in (2.5), using the equations (2.3) and (2.4), we have the following equations respectively.

$$\begin{aligned} f_t^\ell(a, \textstyle\sum_{\mu\geq 0} F_\mu^\ell(b,c)t^\mu) &= f_t^\ell(\textstyle\sum_{\mu\geq 0} F_\mu^\ell(a,b)t^\mu, c), \\ f_t^\ell(\textstyle\sum_{\mu\geq 0} F_\mu^\ell(a,b)t^\mu, c) &= f_t^\ell(a, \textstyle\sum_{\mu\geq 0} F_\mu^r(b,c)t^\mu), \\ f_t^\ell(\textstyle\sum_{\mu\geq 0} F_\mu^r(a,b)t^\mu, c) &= f_t^r(a, \textstyle\sum_{\mu\geq 0} F_\mu^\ell(b,c)t^\mu), \\ f_t^r(\textstyle\sum_{\mu\geq 0} F_\mu^\ell(a,b)t^\mu, c) &= f_t^r(a, \textstyle\sum_{\mu\geq 0} F_\mu^r(b,c)t^\mu), \\ f_t^r(a, \textstyle\sum_{\mu\geq 0} F_\mu^r(b,c)t^\mu) &= f_t^r(\textstyle\sum_{\mu\geq 0} F_\mu^r(a,b)t^\mu, c). \end{aligned}$$

Now using the fact that the maps f_t^ℓ and f_t^r are $Q-$bilinear, we can rewrite the above equations as

$$\begin{aligned}
\sum_{\lambda,\mu\geq 0} F^\ell_\lambda(a, F^\ell_\mu(b,c))t^{\mu+\lambda} &= \sum_{\lambda,\mu\geq 0} F^\ell_\lambda(F^\ell_\mu(a,b),c)t^{\mu+\lambda},\\
\sum_{\lambda,\mu\geq 0} F^\ell_\lambda(F^\ell_\mu(a,b),c)t^{\mu+\lambda} &= \sum_{\lambda,\mu\geq 0} F^\ell_\lambda(a, F^r_\mu(b,c))t^{\mu+\lambda},\\
\sum_{\lambda,\mu\geq 0} F^\ell_\lambda(F^r_\mu(a,b),c)t^{\mu+\lambda} &= \sum_{\lambda,\mu\geq 0} F^r_\lambda(a, F^\ell_\mu(b,c))t^{\mu+\lambda},\\
\sum_{\lambda,\mu\geq 0} F^r_\lambda(F^\ell_\mu(a,b),c)t^{\mu+\lambda} &= \sum_{\lambda,\mu\geq 0} F^r_\lambda(a, F^r_\mu(b,c))t^{\mu+\lambda},\\
\sum_{\lambda,\mu\geq 0} F^r_\lambda(a, F^r_\mu(b,c))t^{\mu+\lambda} &= \sum_{\lambda,\mu\geq 0} F^r_\lambda(F^r_\mu(a,b),c)t^{\mu+\lambda}
\end{aligned}$$

respectively. Collecting coefficients of t^ν we see that the above equations are equivalent to the system of equations

$$(2.6_\nu)\qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a, F^\ell_\mu(b,c)) = 0$$

$$(2.7_\nu)\qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a, F^r_\mu(b,c)) = 0$$

$$(2.8_\nu)\qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F^\ell_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a, F^\ell_\mu(b,c)) = 0$$

$$(2.9_\nu)\qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F^r_\lambda(F^\ell_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)) = 0$$

$$(2.10_\nu)\qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}} F^r_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)) = 0$$

for all $a, b, c \in V$ and for $\nu = 0, 1, 2 \ldots$.

Since F^ℓ_0 and F^r_0 are extensions of the dialgebra products $\dashv$ and $\vdash$ respectively, it immediately follows that the above equations reduce to axioms (1.1) of the dialgebras for $\nu = 0$.

Infinitesimal of a family of formal deformation of a dialgebra measures the deviation from the existing products. We may note here that infinitesimals are always 2-cochains. The following result relates deformations of a dialgebra D to the dialgebra cohomology $HY^*(D, D)$.

Proposition 2.3.3 *The n-infinitesimal F_n of a formal deformation D_t of a dialgebra D is a 2-cocycle.*

Proof. By the definition of an n-infinitesimal, equations (2.6_ν) -(2.10_ν) yield the following set of equations for $\nu = n$.

$$\begin{aligned}
F_n^\ell(a \dashv b, c) - F_n^\ell(a, b \dashv c) + F_n^\ell(a, b) \dashv c - a \dashv F_n^\ell(b, c) &= 0 \\
F_n^\ell(a \dashv b, c) - F_n^\ell(a, b \vdash c) + F_n^\ell(a, b) \dashv c - a \dashv F_n^r(b, c) &= 0 \\
F_n^\ell(a \vdash b, c) - F_n^r(a, b \dashv c) + F_n^r(a, b) \dashv c - a \vdash F_n^\ell(b, c) &= 0 \\
F_n^r(a \dashv b, c) - F_n^r(a, b \vdash c) + F_n^\ell(a, b) \vdash c - a \vdash F_n^r(b, c) &= 0 \\
F_n^r(a \vdash b, c) - F_n^r(a, b \vdash c) + F_n^r(a, b) \vdash c - a \vdash F_n^r(b, c) &= 0.
\end{aligned}$$

In order to prove that $\delta F_n = 0$, we have to show that $(\delta F_n)(y; a, b, c) = 0$ for all $a, b, c \in D$ and for $y = [123], [213], [131], [312], [321]$.

For $y = [123]$,

$$\begin{aligned}
& (\delta F_n)([123]; a, b, c) \\
= \ & a \vdash F_n([12]; b, c) - F_n([12]; a \vdash b, c) + F_n([12]; a, b \vdash c) \\
& - F_n([12]; a, b) \vdash c \\
= \ & a \vdash F_n^r(b, c) - F_n^r(a \vdash b, c) + F_n^r(a, b \vdash c) - F_n^r(a, b) \vdash c \\
= \ & 0, \quad \text{by } (2.10_n).
\end{aligned}$$

For $y = [213]$,

$$\begin{aligned}
& (\delta F_n)([213]; a, b, c) \\
= \ & a \vdash F_n([12]; b, c) - F_n([12]; a \dashv b, c) + F_n([12]; a, b \vdash c) \\
& - F_n([21]; a, b) \vdash c \\
= \ & a \vdash F_n^r(b, c) - F_n^r(a \dashv b, c) + F_n^r(a, b \vdash c) - F_n^\ell(a, b) \vdash c \\
= \ & 0, \quad \text{by } (2.9_n).
\end{aligned}$$

For $y = [131]$,

$$\begin{aligned}
&(\delta F_n)([131]; a, b, c) \\
= \;& a \vdash F_n([21]; b, c) - F_n([21]; a \vdash b, c) + F_n([12]; a, b \dashv c) \\
& - F_n([12]; a, b) \dashv c \\
= \;& a \vdash F_n^{\ell}(b, c) - F_n^{\ell}(a \vdash b, c) + F_n^{r}(a, b \dashv c) - F_n^{r}(a, b) \dashv c \\
= \;& 0, \quad \text{by } (2.8_n).
\end{aligned}$$

For $y = [312]$,

$$\begin{aligned}
&(\delta F_n)([312]; a, b, c) \\
= \;& a \dashv F_n([12]; b, c) - F_n([21]; a \dashv b, c) + F_n([21]; a, b \vdash c) \\
& - F_n([21]; a, b) \dashv c \\
= \;& a \dashv F_n^{r}(b, c) - F_n^{\ell}(a \dashv b, c) + F_n^{\ell}(a, b \vdash c) - F_n^{\ell}(a, b) \dashv c \\
= \;& 0, \quad \text{by } (2.7_n).
\end{aligned}$$

For $y = [321]$,

$$\begin{aligned}
&(\delta F_n)([321]; a, b, c) \\
= \;& a \dashv F_n([21]; b, c) - F_n([21]; a \dashv b, c) + F_n([21]; a, b \dashv c) \\
& - F_n([21]; a, b) \dashv c \\
= \;& a \dashv F_n^{\ell}(b, c) - F_n^{\ell}(a \dashv b, c) + F_n^{\ell}(a, b \dashv c) - F_n^{\ell}(a, b) \dashv c \\
= \;& 0, \quad \text{by } (2.6_n).
\end{aligned}$$

Thus F_n is a 2-cocycle. This completes the proof of the proposition. ∎

2.4 Integrability of 2-cocycles

In view of the Proposition 2.3.3, one might be interested to know if every 2-cocycle is the infinitesimal of a one-parameter family of deformations of a dialgebra. This gives rise to the concept of integrability and to decide integrability of a 2-cocycle, one is led into an obstruction theory, which we will consider in the next section. Before we introduce the definition of integrability of a 2-cocycle, we need to make the following observations.

Let D be a dialgebra, $y \in Y_3$ and $a,\ b,\ c \in D$. Then for any 2-cochain $F_\nu \in CY^2(D, D)$, we have by definition of coboundary map,

$$\begin{aligned}(\delta F_\nu)(y; a, b, c) &= a o_0^y F_\nu(d_0 y; b, c) - F_\nu(d_1 y; a o_1^y b, c) \\ &\quad + F_\nu(d_2 y; a, b o_2^y c) - F_\nu(d_3 y; a, b) o_3^y c.\end{aligned}$$

More precisely, for $y = [321], [312], [131], [213], [123]$, the above equation takes the following forms respectively.

$$\begin{aligned}(\delta F_\nu)([321]; a, b, c) &= a \dashv F_\nu^\ell(b, c) - F_\nu^\ell(a \dashv b, c) + F_\nu^\ell(a, b \dashv c) - F_\nu^\ell(a, b) \dashv c \\ &= -\sum_{\substack{\lambda+\mu=\nu \\ \lambda=0, or \mu=0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^\ell(b, c)),\end{aligned}$$

$$\begin{aligned}(\delta F_\nu)([312]; a, b, c) &= a \dashv F_\nu^r(b, c) - F_\nu^\ell(a \dashv b, c) + F_\nu^\ell(a, b \vdash c) - F_\nu^\ell(a, b) \dashv c \\ &= -\sum_{\substack{\lambda+\mu=\nu \\ \lambda=0, or \mu=0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^r(b, c)),\end{aligned}$$

$$\begin{aligned}(\delta F_\nu)([131]; a, b, c) &= a \vdash F_\nu^\ell(b, c) - F_\nu^\ell(a \vdash b, c) + F_\nu^r(a, b \dashv c) - F_\nu^r(a, b) \dashv c \\ &= -\sum_{\substack{\lambda+\mu=\nu \\ \lambda=0, or \mu=0}} F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)),\end{aligned}$$

$$\begin{aligned}(\delta F_\nu)([213]; a, b, c) &= a \vdash F_\nu^r(b, c) - F_\nu^r(a \dashv b, c) + F_\nu^r(a, b \vdash c) - F_\nu^\ell(a, b) \vdash c \\ &= -\sum_{\substack{\lambda+\mu=\nu \\ \lambda=0, or \mu=0}} F_\lambda^r(F_\mu^\ell(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)),\end{aligned}$$

$$\begin{aligned}(\delta F_\nu)([123]; a, b, c) &= a \vdash F_\nu^r(b, c) - F_\nu^r(a \vdash b, c) + F_\nu^r(a, b \vdash c) - F_\nu^r(a, b) \vdash c \\ &= -\sum_{\substack{\lambda+\mu=\nu \\ \lambda=0, or \mu=0}} F_\lambda^r(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)).\end{aligned}$$

Thus, equations (2.6_ν) - (2.10_ν) can now be rewritten as

$$(2.11_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a,F^\ell_\mu(b,c)) = \delta F_\nu([321];a,b,c)$$

$$(2.12_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a,F^r_\mu(b,c)) = \delta F_\nu([312];a,b,c)$$

$$(2.13_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F^\ell_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a,F^\ell_\mu(b,c)) = \delta F_\nu([131];a,b,c)$$

$$(2.14_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F^r_\lambda(F^\ell_\mu(a,b),c) - F^r_\lambda(a,F^r_\mu(b,c)) = \delta F_\nu([213];a,b,c)$$

$$(2.15_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F^r_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a,F^r_\mu(b,c)) = \delta F_\nu([123];a,b,c)$$

for all $a, b, c \in D$.

Definition 2.4.1 Any 2-cocycle F need not be the infinitesimal of a deformation. If it be so, then we call F integrable.

Therefore, F is integrable if $F = F_1$ can be extended to a sequence F_2, F_3, $\ldots, F_\nu, \ldots$, where $F_\nu : K[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ is defined by

$$(2.16) \qquad F_\nu(y;a,b) = \begin{cases} F^\ell_\nu(a,b) & \text{if } y = [21] \\ F^r_\nu(a,b) & \text{if } y = [12] \end{cases}$$

for some D_t, along with f^ℓ_t and f^r_t, denoting a one parameter family of deformations of D, as defined in equations (2.1) and (2.2) and satisfying equations (2.11_ν) - (2.15_ν).

By the end of the next section, we shall be able to state a sufficient condition (Corollary 2.5.3) for the integrability of every 2-cocycle of a dialgebra. Moreover, we show in the next chapter that the integrability of a 2-cocycle F depends only on its cohomology class.

2.5 Obstruction cochains

Theory of obstructions has always played a very important role in deformation theory to decide whether a given 2-cocycle is integrable or not. Given a 2-cocycle we shall see that there exists a sequence of cohomology classes, vanishing of which is a necessary and sufficient condition for the given 2-cocycle to be integrable. In analogy to all the existing deformation theories, we proceed to define obstruction cochains, in the deformation theory of dialgebras, as follows.

Suppose we are given 2-cochains F_ν, $1 \leq \nu \leq n-1$. Define $F_\nu^\ell, F_\nu^r : D^{\otimes 2} \longrightarrow D$ by

$$(2.17) \qquad \begin{cases} F_\nu^\ell(a,b) &= F_\nu([21];a,b) \\ F_\nu^r(a,b) &= F_\nu([12];a,b) \end{cases}$$

for all $a, b \in D$. Moreover suppose that F_ν^ℓ, F_ν^r and F_ν satisfy equations (2.11_ν)-(2.15_ν), $1 \leq \nu \leq n-1$, then define a 3-cochain $G : k[Y_3] \otimes D^{\otimes 3} \longrightarrow D$ as follows:

$$\begin{aligned}
G([321];a,b,c) &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a,b),c) - F_\lambda^\ell(a,F_\mu^\ell(b,c)) \\
G([312];a,b,c) &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a,b),c) - F_\lambda^\ell(a,F_\mu^r(b,c)) \\
G([131];a,b,c) &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^r(a,b),c) - F_\lambda^r(a,F_\mu^\ell(b,c)) \\
G([213];a,b,c) &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda^r(F_\mu^\ell(a,b),c) - F_\lambda^r(a,F_\mu^r(b,c)) \\
G([123];a,b,c) &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda^r(F_\mu^r(a,b),c) - F_\lambda^r(a,F_\mu^r(b,c))
\end{aligned}$$

for all $a, b, c \in D$.

Let F and F' be two 2-cochains. We define a 3-cochain $F \circ F'$ as follows :

$$F \circ F'(y;a,b,c) = F(d_1 y; F'(d_3 y;a,b),c) - F(d_2 y; a, F'(d_0 y;b,c)),$$

for all $a,\ b,\ c\ \in D$ and $y \in Y_3$. For example, if $y = [321]$, then

$$F \circ F'(y; a, b, c) = F^{\ell}(F'^{\ell}(a, b), c) - F^{\ell}(a, F'^{\ell}(b, c)).$$

This is just one instance of the general pre-Lie product to be defined in Chapter 5.

Suppose $F_1, \ldots F_{n-1}$ are 2-cochains such that F_ν^ℓ, F_ν^r and F_ν satisfy equations (2.11_ν)-(2.15_ν), $1 \leq \nu \leq n-1$, where F_ν^ℓ and F_ν^r are defined as in (2.17). Then

$$\begin{aligned}
\delta F_\nu([321]; a, b, c) = \ & \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^\ell(b, c)) \\
& \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu([321]; a, b, c) \\
\delta F_\nu([312]; a, b, c) = \ & \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^r(b, c)) \\
& \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu([312]; a, b, c) \\
\delta F_\nu([131]; a, b, c) = \ & \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)) \\
& \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu([131]; a, b, c) \\
\delta F_\nu([213]; a, b, c) = \ & \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^r(F_\mu^\ell(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) \\
& \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu([213]; a, b, c) \\
\delta F_\nu([123]; a, b, c) = \ & \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^r(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) \\
& \textstyle\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu([123]; a, b, c)
\end{aligned}$$

Thus,

$$\delta F_\nu = \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu,$$

for all ν, $1 \leq \nu \leq n-1$. It turns out that the 3-cochain G as defined above can be expressed as $G = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda \circ F_\mu$. This 3-cochain appears as an obstruction to extending the given sequence. We now state below the first main theorem, the proof of which appears in chapter 6.

Theorem 2.5.1 *Let D be a dialgebra and $F_1, F_2, \ldots, F_{n-1}$ be elements of $CY^2(D, D)$, such that*

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu = \delta F_\nu, \tag{2.18}$$

for all $\nu = 1, 2, \ldots, n-1$. If $G \in CY^3(D, D)$ is given by

$$G = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda \circ F_\mu,$$

then $\delta G = 0$, that is, G is a 3-cocycle. The cohomology class of G vanishes if and only if the given sequence extends to a sequence $F_1, F_2, \ldots, F_n$ satisfying equation (2.18) for all $\nu = 1, 2, \ldots, n$. Thus with F_0 being the original dialgebra structure, $\sum F_i t^i$ is a deformation modulo t^n if and only if the equation (2.18) holds for $\nu < n$ and this can be extended to a deformation modulo t^{n+1} if and only if the cohomology class of G vanishes.

It is worth noting that the above theorem recapitulates Proposition 2.3.3 as the cases $n = 1$ and $F_{\nu<n} = 0$.

Definition 2.5.2 The cohomology class of G is called the primary obstruction to extend the sequence $F_1, F_2, \ldots, F_{n-1}$ satisfying equations (2.11_ν) to (2.15_ν), $1 \leq \nu \leq n-1$ to a sequence $F_1, F_2, \ldots, F_n$ satisfying equations (2.11_ν) to (2.15_ν), $1 \leq \nu \leq n$, with F_i^ℓ's and F_i^r's obtained from F_i as described above.

As a consequence of the above theorem, we can derive a sufficient condition for integrability of every 2-cochain.

Corollary 2.5.3 *If $HY^3(D, D) = 0$ for a dialgebra D, then all the obstructions vanish and hence any 2-cocyle is integrable.*

2.6 A deformation of the Laurent polynomials

Let $K[x, y, x^{-1}, y^{-1}]$ denote the vector space of all Laurent polynomials in two variables x and y over a field K with basis $x^p y^q$, p, q integers. Define two operations $\dashv$ and $\vdash$ on the basis elements by

$$x^m y^n \dashv x^r y^s = x^m y^{n+r+s} \text{ and } x^m y^n \vdash x^r y^s = x^{m+n+r} y^s.$$

Extending these two operations on $K[x, y, x^{-1}, y^{-1}]$ by bilinearity we get linear maps

$$\dashv, \vdash \colon K[x, y, x^{-1}, y^{-1}] \otimes K[x, y, x^{-1}, y^{-1}] \longrightarrow K[x, y, x^{-1}, y^{-1}].$$

We now show that $K[x, y, x^{-1}, y^{-1}]$ equipped with the operations $\dashv$ and $\vdash$ is a dialgebra. It is enough to check that the dialgebra axioms (1.1) hold for all basis elements $x^m y^n,\ x^p y^q,\ x^r y^s \in K[x, y, x^{-1}, y^{-1}]$. By the definition of $\dashv$ and $\vdash$ products, we have

$$x^m y^n \dashv (x^p y^q \dashv x^r y^s) = x^m y^n \dashv x^p y^{q+r+s} = x^m y^{n+p+q+r+s}, \tag{2.19}$$

$$(x^m y^n \dashv x^p y^q) \dashv x^r y^s = x^m y^{n+p+q} \dashv x^r y^s = x^m y^{n+p+q+r+s}, \tag{2.20}$$

$$x^m y^n \dashv (x^p y^q \vdash x^r y^s) = x^m y^n \dashv x^{p+q+r} y^s = x^m y^{n+p+q+r+s}, \tag{2.21}$$

$$(x^m y^n \vdash x^p y^q) \dashv x^r y^s = x^{m+n+p} y^q \dashv x^r y^s = x^{m+n+p} y^{q+r+s}, \tag{2.22}$$

$$x^m y^n \vdash (x^p y^q \dashv x^r y^s) = x^m y^n \vdash x^p y^{q+r+s} = x^{m+n+p} y^{q+r+s}, \tag{2.23}$$

$$(x^m y^n \dashv x^p y^q) \vdash x^r y^s = x^m y^{n+p+q} \vdash x^r y^s = x^{m+n+p+q+r} y^s, \tag{2.24}$$

$$x^m y^n \vdash (x^p y^q \vdash x^r y^s) = x^m y^n \vdash x^{p+q+r} y^s = x^{m+n+p+q+r} y^s, \tag{2.25}$$

$$(x^m y^n \vdash x^p y^q) \vdash x^r y^s = x^{m+n+p} y^q \vdash x^r y^s = x^{m+n+p+q+r} y^s. \tag{2.26}$$

Equations (2.19) and (2.20) imply that axiom (1) of (1.1) holds, (2.20) and (2.21) imply that axiom (2) holds. Again (2.22) and (2.23) imply that axiom (3) holds. Axiom (4) is implied by (2.24) and (2.25) and axiom (5) is implied by (2.25) and (2.26). Hence, the products defined above make $K[x, y, x^{-1}, y^{-1}]$ into a dialgebra over K.

Define linear maps

$$F_\nu^\ell, F_\nu^r : K[x, y, x^{-1}, y^{-1}] \otimes K[x, y, x^{-1}, y^{-1}] \longrightarrow K[x, y, x^{-1}, y^{-1}]$$

by

$$\begin{aligned} F^{\ell}_{\nu}(x^m y^n, x^r y^s) &= x^m y^{n-\nu} \dashv x^{r-\nu} y^s \\ &= x^m y^{n+r+s-2\nu} \\ F^{r}_{\nu}(x^m y^n, x^r y^s) &= x^m y^{n-\nu} \vdash x^{r-\nu} y^s \\ &= x^{m+n+r-2\nu} y^s. \end{aligned}$$

Then one checks that the linear maps $F^{\ell}_{\nu}, F^{r}_{\nu}$ satisfy equations (2.6_ν) to (2.10_ν). For this, it is enough to verify that for $a = x^m y^n$, $b = x^r y^s$ and $c = x^u y^v$ each term of the left hand side of the equations (2.6_ν) to (2.10_ν) except (2.8_ν) is zero. For (2.8_ν), the term corresponding to (λ, μ), $\lambda + \mu = \nu$, $\lambda \neq \mu$ cancels with the term corresponding to (μ, λ), $\lambda + \mu = \nu$, $\lambda \neq \mu$ and any term with $\lambda = \mu$ is again zero. Hence

$$D_t = K[x, y, x^{-1}, y^{-1}][[t]] \otimes_{K[[t]]} K((t))$$

with f^{ℓ}_t and f^{r}_t is a deformation of the dialgebra $D = K[x, y, x^{-1}, y^{-1}]$ where

$$f^{\ell}_t = \sum_{\nu \geq 0} F^{\ell}_{\nu} t^{\nu} \quad \text{and} \quad f^{r}_t = \sum_{\nu \geq 0} F^{r}_{\nu} t^{\nu}.$$

Chapter 3

Equivalent and trivial deformations

3.1 Introduction

In this chapter we introduce the notion of isomorphisms between deformations of a dialgebra D, define equivalence of deformations of a dialgebra and trivial deformations. With these notions, a dialgebra is then called rigid if every deformation of it is trivial. We deduce that vanishing of the second cohomology $HY^2(D, D)$ of a dialgebra D is a sufficient condition for rigidity of D. Using this condition, we show that the free dialgebra $Dias(V)$, over a vector space V, is rigid. It is worth mentioning here that the free associative algebra $T(V)$ over a vector space V is rigid, in the context of deformation theory of associative algebras.

3.2 Equivalence of deformations

Let D be a dialgebra over a field K. Let V denote the underlying vector space of D. Also, as in the previous chapter, $Q = K((t))$ denotes the quotient power series field in one variable over K, $V[[t]]$ denotes the power series module over $K[[t]]$ with coefficients in V and V_Q denotes the Q-vector space $V[[t]]\otimes_{K[[t]]}Q$. We note that any K-linear map $\phi : V \longrightarrow V$ extends to a Q-linear map $\psi : V_Q \longrightarrow V_Q$ in a natural way. Any map which is such an extension is said to be 'defined over K'. Let $D_t(f)$ be a deformation of a dialgebra D given by

$$f_t^\ell = \sum_{\nu\geq 0} F_\nu^\ell t^\nu, \quad f_t^r = \sum_{\nu\geq 0} F_\nu^r t^\nu \tag{3.1}$$

and $D_t(g)$ be a deformation of D given by

$$g_t^\ell = \sum_{\nu\geq 0} G_\nu^\ell t^\nu, \quad g_t^r = \sum_{\nu\geq 0} G_\nu^r t^\nu. \tag{3.2}$$

Note that any $K[[t]]$-linear homomorphism $V[[t]] \longrightarrow V[[t]]$ is of the form $\sum \psi_i t^i$, where $\psi_i \in \mathrm{Hom}_K(V,V)$.

With the above notations, we make the following definition.

Definition 3.2.1 A (formal) isomorphism $\Psi_t : D_t(f) \longrightarrow D_t(g)$ from deformation $D_t(f)$ to deformation $D_t(g)$ is a Q-linear automorphism $\Psi_t : V_Q \longrightarrow V_Q$

$$\Psi_t(\mathbf{a}) = \mathbf{a} + \psi_1(\mathbf{a})t + \psi_2(\mathbf{a})t^2 + \cdots,$$

where each $\psi_i : V_Q \longrightarrow V_Q$ is a linear map defined over K such that

$$\begin{cases} \Psi_t f_t^\ell(\mathbf{a},\mathbf{b}) & = & g_t^\ell(\Psi_t(\mathbf{a}),\Psi_t(\mathbf{b})) \\ \Psi_t f_t^r(\mathbf{a},\mathbf{b}) & = & g_t^r(\Psi_t(\mathbf{a}),\Psi_t(\mathbf{b})) \end{cases} \tag{3.3}$$

for all $\mathbf{a},\mathbf{b} \in V_Q$. In other words a formal isomorphism is an isomorphism which is equal to identity modulo t.

As in the case of deformations, equation (3.3) is equivalent to

$$\begin{cases} \Psi_t f_t^\ell(a,b) & = & g_t^\ell(\Psi_t(a),\Psi_t(b)) \\ \Psi_t f_t^r(a,b) & = & g_t^r(\Psi_t(a),\Psi_t(b)) \end{cases} \tag{3.4}$$

for all $a, b \in V$. The above definition enables us to define the notion of equivalence between deformations of a dialgebra.

Definition 3.2.2 Let $D_t(f)$ and $D_t(g)$, given by the expressions (3.1) and (3.2) respectively, denote two deformations of the dialgebra D. Then $D_t(f)$ and $D_t(g)$ are said to be equivalent if there exists a (formal) isomorphism $\Psi : D_t(f) \longrightarrow D_t(g)$.

Definition 3.2.3 A (formal) automorphism of a deformation $D_t(f)$ of a dialgebra D is simply a (formal) isomorphism from $D_t(f)$ to $D_t(f)$.

Recall from Example 2.2.2 that the identity deformation of a dialgebra D is the dialgebra $D_Q = D[[t]] \otimes_{K[[t]]} Q$ with the underlying vector space V_Q and with multiplications $g_t^\ell(\mathbf{a}, \mathbf{b}) = \mathbf{a} \dashv \mathbf{b}$ and $g_t^r(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vdash \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in V_Q$, induced by the products of D.

Definition 3.2.4 A deformation $D_t(f)$ is said to be trivial if it is equivalent to the identity deformation.

The relation between the infinitesimals of two equivalent deformations is given by the following proposition.

Proposition 3.2.5 *If $D_t(f)$ and $D_t(g)$ are two equivalent deformations of D, then the infinitesimals of $D_t(f)$ and $D_t(g)$ determine the same cohomology class.*

Proof. Let the equivalence of $D_t(f)$ and $D_t(g)$ be given by the isomorphism $\Psi_t : D_t(f) \longrightarrow D_t(g)$. Expanding both sides of the first equation in (3.4), by using (3.1) and (3.2), and using the fact that all the maps involved are defined over K, and Ψ_t

and g_t are Q-linear, we get

$$\begin{aligned} \Psi_t(\textstyle\sum_{j\geq 0} F_j^\ell(a,b)t^j) &= g_t^\ell(\textstyle\sum_{j\geq 0}\psi_j(a)t^j, \sum_{k\geq 0}\psi_k(b)t^k), \\ \text{or} \qquad \textstyle\sum_{i,j\geq 0}\psi_i(F_j^\ell(a,b))t^{i+j} &= \textstyle\sum_{i,j,k\geq 0} G_i^\ell(\psi_j(a),\psi_k(b))t^{i+j+k}. \end{aligned}$$

Now equating coefficients of t^n, we get

$$(3.5) \qquad \sum_{i+j+k=n} G_i^\ell(\psi_j(a),\psi_k(b)) = \sum_{i+j=n}\psi_i(F_j^\ell(a,b)).$$

Similarly, expanding both sides of the second equation in (3.4), by using (3.1) and (3.2), and using the fact that all the maps involved are defined over K, we get

$$\begin{aligned} \Psi_t(\textstyle\sum_{j\geq 0} F_j^r(a,b)t^j) &= g_t^r(\textstyle\sum_{j\geq 0}\psi_j(a)t^j, \sum_{k\geq 0}\psi_k(b)t^k), \\ \text{or} \qquad \textstyle\sum_{i,j\geq 0}\psi_i(F_j^r(a,b))t^{i+j} &= \textstyle\sum_{i,j,k\geq 0} G_i^r(\psi_j(a),\psi_k(b))t^{i+j+k}. \end{aligned}$$

Now equating coefficients of t^n, we get

$$(3.6) \qquad \sum_{i+j+k=n} G_i^r(\psi_j(a),\psi_k(b)) = \sum_{i+j=n}\psi_i(F_j^r(a,b)).$$

For $n=1$, equations (3.5) and (3.6) take the forms

$$\begin{aligned} F_1^\ell(a,b) &= G_1^\ell(a,b) + a \dashv \psi_1(b) + \psi_1(a) \dashv b - \psi_1(a \dashv b) \\ F_1^r(a,b) &= G_1^r(a,b) + a \vdash \psi_1(b) + \psi_1(a) \vdash b - \psi_1(a \vdash b) \end{aligned}$$

for all $a, b \in V$. Now since $\mathrm{Hom}_K(K[Y_1]\otimes D, D) \cong \mathrm{Hom}_K(D,D)$, ψ_1 can be identified with a unique 1-cochain again denoted by ψ_1 where $\psi_1([1];a) = \psi_1(a)$ for all $a \in D$. Observe that

$$\begin{aligned} \delta\psi_1([21];a,b) &= a \dashv \psi_1(b) - \psi_1(a \dashv b) + \psi_1(a) \dashv b \\ &= F_1^\ell(a,b) - G_1^\ell(a,b), \end{aligned}$$

$$\begin{aligned} \delta\psi_1([12];a,b) &= a \vdash \psi_1(b) - \psi_1(a \vdash b) + \psi_1(a) \vdash b \\ &= F_1^r(a,b) - G_1^r(a,b). \end{aligned}$$

Hence $\delta\psi_1 = F_1 - G_1$, that is F_1 and G_1 are cohomologous. This completes the proof of the proposition. ∎

On the other hand if two 2-cocycles are cohomologous and one is the infinitesimal of a deformation, then so is the other of an equivalent deformation.

Proposition 3.2.6 *Let D_t along with f_t^ℓ and f_t^r, given by the expressions* (3.1)*, denote a deformation of the dialgebra D. Let F_1 denote the infinitesimal of D_t. If a 2-cocycle G_1 is cohomologous to F_1, then there exists an equivalent deformation D'_t such that G_1 is the infinitesimal of D'_t.*

Proof. Let $G_1 = F_1 + \delta\psi$, for some 1-cochain ψ. Let $\Psi : V_Q \longrightarrow V_Q$ be a Q-linear map given by

$$\Psi_t(a) = a + \psi(a)t.$$

More precisely, Ψ_t is an isomorphism between Q-vector spaces, where the inverse map is given by

$$\Psi_t^{-1}(a) = a + \sum_{i\geq 1}(-1)^i\psi^i(a)t^i.$$

Then G_1 is the infinitesimal of the deformation D'_t given by g_t^ℓ and g_t^r where

$$g_t^\ell(a,b) = \Psi_t^{-1}f_t^\ell(\Psi_t(a),\Psi_t(b)) \tag{3.7}$$
$$g_t^r(a,b) = \Psi_t^{-1}f_t^r(\Psi_t(a),\Psi_t(b)) \tag{3.8}$$

This is because (3.7) and (3.8) can be rewritten as

$$\Psi_t^{-1}(f_t^\ell(\Psi_t(a),\Psi_t(b))) = \Psi_t^{-1}\textstyle\sum_{i\geq 0}F_i^\ell(a+\psi(a)t, b+\psi(b)t)t^i, \tag{3.9}$$
$$\Psi_t^{-1}(f_t^r(\Psi_t(a),\Psi_t(b))) = \Psi_t^{-1}\textstyle\sum_{i\geq 0}F_i^r(a+\psi(a)t, b+\psi(b)t)t^i. \tag{3.10}$$

Using the fact that F_i^ℓ, F_i^r and Ψ_t are Q-linear, and also using the fact that F_0^ℓ and F_0^r are the original $\dashv$ and $\vdash$ products respectively, coefficients of t in the equation

(3.9) is

$$\begin{aligned} & F_1^\ell(a,b) + a \dashv \psi(b) - \psi(a \dashv b) + \psi(a) \dashv b \\ = \;& F_1^\ell(a,b) + \delta\psi([21];a,b). \end{aligned}$$

Similarly, coefficients of t in the equation (3.10) is

$$\begin{aligned} & F_1^r(a,b) + a \vdash \psi(b) - \psi(a \vdash b) + \psi(a) \vdash b \\ = \;& F_1^r(a,b) + \delta\psi([12];a,b). \end{aligned}$$

By the definition of an infinitesimal, the infinitesimal of the deformation D'_t is $F_1 + \delta\psi$, which is G_1. This completes the proof of the proposition. ■

Remark 3.2.7 It follows that the integrability of an element of $ZY^2(D,D)$ depends only on its cohomology class. If any element in a cohomology class is integrable, then every other element in the same cohomology class is also integrable.

The above discussions reveal that the infinitesimal of a trivial deformation is a coboundary, though the converse may not be true. In other words, a non trivial deformation may have an infinitesimal which is a coboundary. What we can assure is the following.

Theorem 3.2.8 *A nontrivial deformation of a dialgebra is equivalent to a deformation whose n-infinitesimal is not a coboundary for some $n \geq 1$.*

Proof. Let D_t be a deformation of D with multiplications f_t^ℓ and f_t^r given by equation (3.1). Let F_n, the unique 2-cochain defined by F_n^ℓ and F_n^r as in (2.16), be the n-infinitesimal of the deformation, for $n \geq 1$. Then by Proposition (2.3.4), $\delta F_n = 0$. Now suppose that F_n is a coboundary, say $F_n = -\delta\psi_n$ for some $\psi_n \in$

$CY^1(D, D) \cong \mathrm{Hom}_K(D, D)$. Let Ψ_t be the formal automorphism of V_Q defined by $\Psi_t(a) = a + \psi_n(a)t^n$. Then setting

$$g_t^\ell(a, b) = \Psi_t^{-1} f_t^\ell(\Psi_t(a), \Psi_t(b)) = \sum_{\nu \geq 0} G_\nu^\ell(a, b)t^\nu,$$

$$g_t^r(a, b) = \Psi_t^{-1} f_t^r(\Psi_t(a), \Psi_t(b)) = \sum_{\nu \geq 0} G_\nu^r(a, b)t^\nu$$

we get a deformation D_t' isomorphic to D_t. Explicitly, g_t^ℓ and g_t^r are given by

$$\begin{aligned} g_t^\ell(a, b) &= a \dashv b - \{\psi_n(a \dashv b) - \psi_n(a) \dashv b - a \dashv \psi_n(b) - F_n^\ell(a, b)\}t^n \\ &\quad + F_{n+1}^\ell t^{n+1} + \cdots \\ g_t^r(a, b) &= a \vdash b - \{\psi_n(a \vdash b) - \psi_n(a) \vdash b - a \vdash \psi_n(b) - F_n^r(a, b)\}t^n \\ &\quad + F_{n+1}^r t^{n+1} + \cdots. \end{aligned}$$

Suppose $F_n^\ell \neq 0$. Then as $F_n = -\delta\psi_n$, we see that

$$\begin{aligned} F_n^\ell(a, b) &= F_n([21]; a, b) = -\delta\psi_n([21]; a, b) \\ &= -\{a \dashv \psi_n([1]; b) - \psi_n([1]; a \dashv b) + \psi_n([1]; a) \dashv b\} \\ &= -\{a \dashv \psi_n(b) - \psi_n(a \dashv b) + \psi_n(a) \dashv b\}. \end{aligned}$$

Thus the coefficient of t^n in $g_t^\ell(a, b)$ is zero. In case $F_n^\ell = 0$, then $\delta\psi_n([21]; a, b) = 0$ and hence coefficient of t^n is again zero. By a similar argument the coefficient of t^n in the expression of g_t^r is also zero. Thus $G_i^* = 0$ for $1 \leq i \leq n$ where $* = \{\ell, r\}$. Hence we can repeat our argument to kill off an infinitesimal that is a coboundary and the process must stop if the deformation D_t is nontrivial. This completes the proof of the theorem. ■

3.3 Rigidity

The notion of rigidity may be defined as follows.

Definition 3.3.1 *A dialgebra D is said to be rigid if every deformation of D is a trivial deformation.*

As a consequence of Theorem 3.2.8 we have the following sufficient condition for rigidity of a dialgebra D.

Corollary 3.3.2 *Let D be a dialgebra over a field K. If the second cohomology $HY^2(D, D)$ vanishes, then D is rigid.*

Proof. Let, if possible, D_t be a non-trivial deformation of D. Then by Theorem 3.2.8 D_t is equivalent to a deformation D'_t whose n-infinitesimal (which is a 2-cocycle by Proposition 2.3.3) is not a coboundary, for some $n \geq 1$. This contradicts the fact that $HY^2(D, D) = 0$. ■

It is well known that the tensor algebra $T(V)$ which is the free object in the category of associative algebras is rigid in the sense of deformation theory of associative algebras. Here we show that the free object in the category $Dias$, that is, the free dialgebra $Dias(V)$ over a vector space V, is rigid in the sense of deformation theory of dialgebras.

Proposition 3.3.3 *The free dialgebra $Dias(V)$ over the vector space V is rigid.*

Proof. Let us denote $Dias(V)$ by D. By Corollary 3.3.2, it is enough to show that $HY^2(D, D) = 0$. Let $f \in ZY^2(D, D)$. Let $\bar{D}$ denote the underlying vector space of D. Consider the short exact sequence of dialgebras

$$0 \longrightarrow \bar{D} \xrightarrow{j} \bar{D} \oplus \bar{D} \xrightarrow{\pi} D \longrightarrow 0$$

where $\bar{D}$ on the left is considered as a dialgebra with abelian products, that is, $a \dashv b = a \vdash b = 0$ for all $a, b \in \bar{D}$, and the dialgebra structure on $D \oplus \bar{D}$ is defined by

$$\begin{aligned}(a_1, b_1) \dashv (a_2, b_2) &= (a_1 \dashv b_2 + b_1 \dashv a_2 - f([21]; b_1, b_2), b_1 \dashv b_2)\\ (a_1, b_1) \vdash (a_2, b_2) &= (a_1 \vdash b_2 + b_1 \vdash a_2 - f([12]; b_1, b_2), b_1 \vdash b_2),\end{aligned}$$

j being the inclusion into the first factor and π the projection onto the second factor. This sequence splits as a sequence of vector spaces. So there exists a K-linear map $\sigma : D \longrightarrow \bar{D} \oplus \bar{D}$ such that $\pi \circ \sigma = id_D$. Hence σ must be of the form (g, id), where $g : D \longrightarrow \bar{D}$ is K-linear. Let $\sigma' = \sigma/V : V \longrightarrow \bar{D} \oplus \bar{D}$. Universal property of $D = Dias(V)$ gives a dialgebra map $\tilde{\sigma} : D \longrightarrow \bar{D} \oplus \bar{D}$ with $\tilde{\sigma} \circ i = \sigma'$, i being the inclusion $V \hookrightarrow D$. This implies that on V, $\tilde{\sigma}$ and σ' agree. We note here that as described in chapter 1, the generators of D look like $v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m$, for $v_i \in V$. Since π and $\tilde{\sigma}$ are both dialgebra maps we have $\pi \circ \tilde{\sigma} = id$. Hence $\tilde{\sigma}$ is of the form (φ, id) for some K-linear map $\varphi : \bar{D} \longrightarrow \bar{D}$. Now as $\tilde{\sigma}$ is a dialgebra map, we have

$$\tilde{\sigma}(a \dashv b) = \tilde{\sigma}(a) \dashv \tilde{\sigma}(b)$$

and

$$\tilde{\sigma}(a \vdash b) = \tilde{\sigma}(a) \vdash \tilde{\sigma}(b).$$

Since $\tilde{\sigma} = (\varphi, id)$ we get

$$\begin{aligned}(3.11) \qquad (\varphi(a \dashv b), a \dashv b) &= (\varphi(a), a) \dashv (\varphi(b), b)\\ &= (\varphi(a) \dashv b + a \dashv \varphi(b) - f([21]; a, b), a \dashv b)\end{aligned}$$

and

$$\begin{aligned}(3.12) \qquad (\varphi(a \vdash b), a \vdash b) &= (\varphi(a), a) \vdash (\varphi(b), b)\\ &= (\varphi(a) \vdash b + a \vdash \varphi(b) - f([12]; a, b), a \vdash b).\end{aligned}$$

Now, by the definition of the coboundary map,

$$\begin{aligned}\delta\varphi([21];a,b) &= a \dashv \varphi([1];b) - \varphi([1];a \dashv b) + \varphi([1];a) \dashv b \\ \delta\varphi([12];a,b) &= a \vdash \varphi([1];b) - \varphi([1];a \vdash b) + \varphi([1];a) \vdash b.\end{aligned}$$

Equating the first co-ordinates on both sides of (3.11) and (3.12), we deduce that $f(y;a,b) = \delta\varphi(y;a,b)$ for $y = [21]$, $[12]$, where φ has been interpreted as a 1-cochain. Thus $f = \delta\varphi$. This completes the proof of the proposition. ■

Chapter 4

Automorphisms of the identity deformation

4.1 Introduction

Recall that the identity deformation, as introduced in Example 2.2.2, is the dialgebra $D_Q = D[[t]] \otimes_{K[[t]]} Q$ with the underlying vector space V_Q and with multiplications $g_t^\ell(\mathbf{a}, \mathbf{b}) = \mathbf{a} \dashv \mathbf{b}$ and $g_t^r(\mathbf{a}, \mathbf{b}) = \mathbf{a} \vdash \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in V_Q$, induced by the products of D. In this chapter, we study automorphisms of the identity deformation D_Q. Analogous to the notion of infinitesimal of a deformation as introduced in chapter 2, we define infinitesimal of an automorphism, prove a few immediate results, and introduce obstructions to "integrability" of derivations (that is, 1-cocycles) of D. Further, we state another main theorem of this thesis, Theorem 4.3.1, involving obstruction cochains. This is the counterpart of Theorem 2.5.1 in the case of integrability of derivations. The proof of this theorem appears in chapter 6 and depends on the associative cup product $*$ on $CY^*(D, D)$ which will be introduced in the next chapter. We also subsequently derive a sufficient condition for extending a truncated automorphism to an automorphism of D_Q.

4.2 Infinitesimal

According to Definition 3.2.3, a (formal) automorphism of the identity deformation D_Q is a Q-linear map $\Psi_t : D_Q \longrightarrow D_Q$

$$(4.1) \qquad \Psi_t(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \cdots$$

for all $a \in D$, where each $\psi_i : D_Q \longrightarrow D_Q$ is a linear map defined over K and ψ_0 is the identity map satisfying

$$(4.2) \qquad \begin{cases} \Psi_t(a \dashv b) &= \Psi_t(a) \dashv \Psi_t(b), \\ \Psi_t(a \vdash b) &= \Psi_t(a) \vdash \Psi_t(b) \end{cases}$$

for all $a, b \in D$.

Definition 4.2.1 Let Ψ_t given by

$$\Psi_t(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \cdots$$

denote a (formal) automorphism of D_Q. The first non-zero coefficient ψ_n in the above expression is called the infinitesimal of the automorphism Ψ_t.

Thus infinitesimal of an automorphism measures its deviation from the identity map. Also since $CY^1(D, D) \cong \mathrm{Hom}_K(D, D)$, every infinitesimal is a 1-cochain. In fact, even more is true.

Lemma 4.2.2 *The infinitesimal of an automorphism of D_Q is a derivation, that is, a 1-cocycle.*

Proof. Let Ψ_t, given by equation (4.1) denote an automorphism of D_Q. Substituting Ψ_t in equation (4.2) we get

$$\begin{aligned}\sum_{\nu\geq 0}\psi_\nu(a\dashv b)t^\nu &= \sum_{\lambda\geq 0}\psi_\lambda(a)t^\lambda \dashv \sum_{\mu\geq 0}\psi_\mu(b)t^\mu\\ \sum_{\nu\geq 0}\psi_\nu(a\vdash b)t^\nu &= \sum_{\lambda\geq 0}\psi_\lambda(a)t^\lambda \vdash \sum_{\mu\geq 0}\psi_\mu(b)t^\mu,\end{aligned}$$

or equivalently

$$\begin{aligned}\sum_{\nu\geq 0}\psi_\nu(a\dashv b)t^\nu &= \sum_{\lambda,\mu\geq 0}(\psi_\lambda(a)\dashv \psi_\mu(b))t^{\lambda+\mu}\\ \sum_{\nu\geq 0}\psi_\nu(a\vdash b)t^\nu &= \sum_{\lambda,\mu\geq 0}(\psi_\lambda(a)\vdash \psi_\mu(b))t^{\lambda+\mu}.\end{aligned}$$

Equating coefficients of t^ν on both sides, the above set of equations reduce to

$$(4.3_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}}\psi_\lambda(a)\dashv\psi_\mu(b) = \psi_\nu(a\dashv b)$$

$$(4.4_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geq 0}}\psi_\lambda(a)\vdash\psi_\mu(b) = \psi_\nu(a\vdash b)$$

for all $a, b \in D$, and for all $\nu = 0, 1, 2, \ldots$. Equation (4.3_ν) and (4.4_ν) can be rewritten as

$$(4.5_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}}\psi_\lambda(a)\dashv\psi_\mu(b) = -\psi_\nu(a)\dashv b+\psi_\nu(a\dashv b)-a\dashv\psi_\nu(b)$$

$$(4.6_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}}\psi_\lambda(a)\vdash\psi_\mu(b) = -\psi_\nu(a)\vdash b+\psi_\nu(a\vdash b)-a\vdash\psi_\nu(b)$$

for all $a, b \in D$, and for all $\nu = 1, 2, \ldots$. As in chapter 3, we identify the linear map ψ_i with the corresponding 1-cochain defined by

$$\psi_i([1]; a) = \psi_i(a) \quad \text{for all} \quad a \in D.$$

Then equations (4.5_ν) and (4.6_ν) reduce to

$$(4.7_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}}\psi_\lambda([1];a)\dashv\psi_\mu([1];b) = -\delta\psi_\nu([21];a,b)$$

$$(4.8_\nu) \qquad \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}}\psi_\lambda([1];a)\vdash\psi_\mu([1];b) = -\delta\psi_\nu([12];a,b)$$

for all $a, b \in D$ and for all $\nu = 1, \ldots$. It follows from (4.7_ν) and (4.8_ν) that if ψ_n is the infinitesimal of Ψ_t then $\delta\psi_n = 0$. Hence the infinitesimal of an automorphism is a derivation of D. This completes the proof of the lemma. ■

We note here that the equations (4.7_ν) and (4.8_ν) give a necessary and sufficient condition for a linear automorphism Ψ_t as in (4.1) to be a dialgebra automorphism of D_Q.

4.3 Obstructions to extending a derivation

Any derivation of D need not be the infinitesimal of an automorphism of D_Q. We may ask when a derivation of D extends to an automorphism of D_Q. Suppose that a derivation ψ_1 has been extended to a truncated automorphism $\Psi_t = \mathrm{id} + \sum_1^{n-1} \psi_i t^i$ so that ψ_i's satisfy (4.7_ν) and (4.8_ν) for all $\nu = 1, \ldots, n-1$. Define a 2-cochain F by

$$F(y; a, b) = \begin{cases} \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b) & \text{if } \ y = [21] \\ \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b) & \text{if } \ y = [12] \end{cases}$$

for all $a, b \in D$.

Let F, G be two 1-cochains. Define a 2-cochain $F * G$ as follows :

$$\begin{aligned} (F * G)([21]; a, b) = &\quad F([1]; a) \dashv G([1]; b) \\ (F * G)([12]; a, b) = &\quad F([1]; a) \vdash G([1]; b). \end{aligned}$$

This is a special case of a more general definition of the $*$ product, to be defined in Chapter 5.

Let $\Psi_t = \mathrm{id} + \sum_1^{n-1} \psi_i t^i$ be a truncated automorphism, with 1-cochains ψ_i satisfying (4.7_ν) and (4.8_ν), $\nu = 1, \ldots, n-1$. Using the $*$ product defined above for

1-cochains, this is equivalent to saying

$$\delta\psi_\nu = -\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu,$$

for all $\nu = 1, \ldots, n-1$. It turns out that the 2-cochain F as defined above can be expressed as

$$F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu.$$

This 2-cochain appears as an obstruction to extending the given derivation to an automorphism. We now state the second main theorem, the proof of which appears in Chapter 6.

Theorem 4.3.1 *Let D be a dialgebra and $\psi_1, \psi_2, \ldots, \psi_{n-1}$ be 1-cochains, such that*

$$-\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu = \delta\psi_\nu$$

for all $\nu = 1, \ldots, n-1$. If $F \in CY^2(D, D)$ be given by

$$F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu,$$

then $\delta F = 0$, in otherwords, F is a 2-cocycle. The cohomology class of F must vanish if the truncated automorphism is to be extended. In other words, $id + \sum \psi_i t^i$ is an automorphism modulo t^n if and only if the first equation holds for $\nu < n$ and this can be extended to an automorphism $id + \sum \psi_i t^i$ modulo t^{n+1} if and only if the cohomology class of F vanishes.

Definition 4.3.2 The cohomology class of F is called the primary obstruction to extend the sequence $\psi_1, \psi_2, \ldots, \psi_{n-1}$ satisfying (4.7_ν) and (4.8_ν), $1 \le \nu \le n-1$, to a sequence $\psi_1, \psi_2, \ldots, \psi_n$ satisfying (4.7_ν) and (4.8_ν) for $1 \le \nu \le n$.

We may call a derivation integrable if it can be realized as an infinitesimal of an automorphism of D_Q.

From the above theorem we deduce the following corollary.

Corollary 4.3.3 *If $HY^2(D, D) = 0$, then every derivation of D may be extended to an automorphism of D_Q.*

The following theorem relates the theory of deformations developed in chapter 2 and the theory of automorphisms of the identity deformation D_Q developed in this chapter. Infact, it involves all the concepts that have been introduced so far in the thesis.

Theorem 4.3.4 *If every derivation of D extends to an automorphism of D_Q, then every trivial deformation of D has a trivial n-infinitesimal.*

Proof. Suppose that

$$\begin{aligned} f_t^\ell(a, b) &= a \dashv b + F_1^\ell(a, b)t + F_2^\ell(a, b)t^2 + \cdots \\ f_t^r(a, b) &= a \vdash b + F_1^r(a, b)t + F_2^r(a, b)t^2 + \cdots \end{aligned}$$

define a trivial deformation of a dialgebra D. Let F_n defined by

$$F_n([21]; a, b) = F_n^\ell(a, b), \quad F_n([12]; a, b) = F_n^r(a, b)$$

be the n-infinitesimal of this deformation. Recall from Definition 2.3.2, $F_i^\ell = 0 = F_i^r$, $1 \leq i \leq n-1$ and by Proposition 2.3.3 F_n is a cocycle. Let

$$\Psi_t(a) = a + \psi_1(a)t + \psi_2(a)t^2 + \cdots$$

be the isomorphism from D_t to D_Q where D_t denotes the deformation defined by f_t^ℓ and f_t^r. Thus we have

$$\Psi_t(f_t^\ell(a, b)) = \Psi_t(a) \dashv \Psi_t(b) \quad \text{and} \quad \Psi_t(f_t^r(a, b)) = \Psi_t(a) \vdash \Psi_t(b)$$

for all $a, b \in D$. Substituting the expression for Ψ_t, we get

$$\begin{aligned} \Psi_t(\sum_{i\geq 0} F_i^\ell(a,b)t^i) &= \sum_{i\geq 0} \psi_i(a)t^i \dashv \sum_{j\geq 0} \psi_j(b)t^j, \\ \Psi_t(\sum_{i\geq 0} F_i^r(a,b)t^i) &= \sum_{i\geq 0} \psi_i(a)t^i \vdash \sum_{j\geq 0} \psi_j(b)t^j. \end{aligned}$$

Using the expression of Ψ_t, we can rewrite the above equations as

$$\text{(4.9)} \quad \begin{cases} \sum_{i,\ j\geq 0} \psi_j(F_i^\ell(a,b))t^{i+j} = \sum_{i,\ j\geq 0} \psi_i(a) \dashv \psi_j(b)t^{i+j}, \\ \sum_{i,\ j\geq 0} \psi_j(F_i^r(a,b))t^{i+j} = \sum_{i,\ j\geq 0} \psi_i(a) \vdash \psi_j(b)t^{i+j}. \end{cases}$$

Equating coefficients of t from the above equations, we get

$$\begin{aligned} \psi_1(a \dashv b) &= \psi_1(a) \dashv b + a \dashv \psi_1(b) \\ \psi_1(a \vdash b) &= \psi_1(a) \vdash b + a \vdash \psi_1(b), \end{aligned}$$

which is equivalent to the fact that $\delta\psi_1 = 0$. In other words, ψ_1 (and hence $-\psi_1$) is a derivation of D. By hypothesis, there exists an automorphism of D_Q of the form

$$\eta_t^1(a) = a - \psi_1(a)t + \cdots.$$

It is easy to see that the composition $\eta_t^1 \circ \Psi_t$ which is again an isomorphism from D_t to D_Q has coefficient of t equal to 0.

Let the composition be given by

$$\Psi_t^1(a) = \eta_t^1 \circ \Psi_t(a) = a + \psi_2^1(a)t^2 + \psi_3^1(a)t^3 + \cdots.$$

Again as Ψ_t^1 is an isomorphism from D_t to D_Q, we have equations similar to (4.9) and equating coefficients of t^2 thereof we see that ψ_2^1 (and hence, $-\psi_2^1$) is a derivation of D.

Now, again by hypothesis, there exists an automorphism of D_Q of the form

$$\eta_t^2(a) = a - \psi_2^1(a)t^2 + \cdots.$$

As above the composition $\Psi_t^2 = \eta_t^2 \circ (\eta_t^1 \circ \Psi_t)$, which is again an isomorphism from D_t to D_Q, has both the coefficients of t and t^2 equal to zero. Since, $F_i^\ell = 0 = F_i^r$, $\leq i \leq n-1$ we can repeat the process $(n-1)$ times to construct an isomorphism

$$\Psi_t' = \Psi_t^{n-1} = \eta_t^{n-1} \circ \eta_t^{n-2} \circ \cdots \circ \eta_t^1 \circ \Psi_t$$

from D_t to D_Q and is of the form

$$\Psi_t'(a) = a + \psi_n'(a)t^n + \psi_{n+1}'(a)t^{n+1} + \cdots.$$

Since Ψ_t' is an isomorphism we have

$$\begin{aligned}
\textstyle\sum_{i,\ j\geq 0} \psi_j'(F_i^\ell(a,b))t^{i+j} &= \textstyle\sum_{i,\ j\geq 0} \psi_i'(a) \dashv \psi_j'(b)t^{i+j}, \\
\textstyle\sum_{i,\ j\geq 0} \psi_j'(F_i^r(a,b))t^{i+j} &= \textstyle\sum_{i,\ j\geq 0} \psi_i'(a) \vdash \psi_j'(b)t^{i+j}.
\end{aligned}$$

Equating coefficients of t^n, we get,

$$\begin{aligned}
\psi_n'(a \dashv b) + F_n^\ell(a,b) &= \psi_n'(a) \dashv b + a \dashv \psi_n'(b) \\
\psi_n'(a \vdash b) + F_n^r(a,b) &= \psi_n'(a) \vdash b + a \vdash \psi_n'(b).
\end{aligned}$$

By definition of the coboundary, and regarding ψ_n' as a 1-cochain,

$$\begin{aligned}
\delta\psi_n'([21];a,b) &= a \dashv \psi_n'([1];b) - \psi_n'([1];a \dashv b) + \psi_n'([1];a) \dashv b, \\
\delta\psi_n'([12];a,b) &= a \vdash \psi_n'([1];b) - \psi_n'([1];a \vdash b) + \psi_n'([1];a) \vdash b.
\end{aligned}$$

Hence we can rewrite the above equations as

$$\begin{aligned}
\delta\psi_n'([21];a,b) &= F_n^\ell(a,b) \\
\delta\psi_n'([12];a,b) &= F_n^r(a,b).
\end{aligned}$$

But by definition of n-infinitisimal, this simply implies that $\delta\psi_n' = F_n$, the n-infinitisimal. This proves that the n-infinitisimal of the family of deformations given by f_t^ℓ and f_t^r is trivial. ■

Chapter 5

Structure of a pre-Lie system on $CY^*(D, D)$

5.1 Introduction

The notion of pre-Lie system was introduced by M. Gerstenhaber [5]. He showed that the Hochschild complex of an associative algebra admits a pre-Lie system structure. He also showed that a pre-Lie system induces a pre-Lie ring structure. He used the above structures to develop deformation theory of associative algebras.

The aim of this chapter is to show that the direct sum of the cochain modules, $CY^*(D, D)$ of a dialgebra D, admits a pre-Lie system structure. Since the construction of $CY^*(D, D)$ involves planar binary trees, our case is more complicated than the case of Hochschild complex. We need to introduce certain operations (cf. Definition 5.2.1) on planar binary trees which are essential in giving a pre-Lie system structure on $CY^*(D, D)$. As a consequence, $CY^*(D, D)$ becomes a pre-Lie ring with a product which we call 'pre-Lie product'. Moreover, there exists an associative product $*$ on $CY^*(D, D)$, of degree zero, induced by the pre-Lie system structure. These structures play a crucial role in proving Theorems 2.5.1 and 4.3.1 in the next chapter. Finally we establish an important relationship connecting the associative product $*$, the pre-Lie product and the coboundary maps of the cochain complex.

5.2 Pre-Lie product on the cochain complex

In this section we recall the definition of pre-Lie system, as was introduced in [5], and define certain operations on the set of planar binary trees. These operations enable us to define the structure of a pre-Lie system on the cochain complex $CY^*(D, D)$, and consequently, a pre-Lie ring structure on the same.

Definition 5.2.1 Given a pair of integers, $p, q \geq 1$ with $p + q = n + 1$, we define two maps $R_1^i(n; p, q) : Y_n \longrightarrow Y_p$ and $R_2^i(n; p, q) : Y_n \longrightarrow Y_q$ for each i, $0 \leq i \leq p - 1$ as follows. For $y \in Y_n$

$$R_1^i(n;p,q)(y) = \begin{cases} d_{i+1}d_{i+2}\cdots d_{i+q-1}(y) & \text{if } p,q \geq 2 \text{ and } 0 \leq i \leq p-1 \\ y & \text{if } p = n,\ q = 1 \text{ and } 0 \leq i \leq n-1 \\ d_1 d_2 \cdots d_{n-1}(y) & \text{if } p = 1,\ q = n \text{ and } i = 0 \end{cases}$$

and

$$R_2^i(n;p,q)(y) = \begin{cases} d_0 d_1 \cdots d_{i-1} d_{i+q+1} \cdots d_{p+q-1}(y) & \text{if } p,q \geq 2 \text{ and } 0 < i < p-1 \\ d_{q+1} \cdots d_{p+q-1}(y) & \text{if } p,q \geq 2 \text{ and } i = 0 \\ d_0 d_1 \cdots d_{p-2}(y) & \text{if } p,q \geq 2 \text{ and } i = p-1 \\ d_0 d_1 \cdots d_{i-1} d_{i+2} \cdots d_n(y) & \text{if } p = n,\ q = 1 \text{ and } 0 < i < p-1 \\ d_2 \cdots d_n(y) & \text{if } p = n,\ q = 1 \text{ and } i = 0 \\ d_0 \cdots d_{n-2}(y) & \text{if } p = n,\ q = 1 \text{ and } i = n-1 \\ y & \text{if } p = 1,\ q = n \text{ and } i = 0. \end{cases}$$

To simplify notation, we will often denote the maps $R_1^i(m; r, s)$ and $R_2^i(m; r, s)$, $0 \leq i \leq r - 1$, corresponding to any triple of integers m, r and s with $m + 1 = r + s$, as defined above, simply by R_1^i and R_2^i. However, whenever necessary, we will explicitly write down these maps to avoid confusion.

We recall the following definition from [5].

Definition 5.2.2 A right pre-Lie system (or simply a pre-Lie system) $\{V_m, \circ_i\}$ is a sequence $\ldots, V_{-1}, V_0, V_1, \ldots$ of K-modules, equipped with a linear map $\circ_i = \circ_i(m,n) : V_m \otimes V_n \longrightarrow V_{m+n}$ for every triple of integers $m, n, i \geq 0$ with $i \leq m$ satisfying the following properties

$$(f^m \circ_i g^n) \circ_j h^p = \begin{cases} (f^m \circ_j h^p) \circ_{i+p} g^n & \text{if } 0 \leq j \leq i-1 \\ f^m \circ_i (g^n \circ_{j-i} h^p) & \text{if } i \leq j \leq n+1,\ i \neq 0 \\ & \text{and } 0 \leq j < n+1, \text{ if } i = 0 \end{cases}$$

where $f \in V_m$ is written as f^m to indicate its degree and $f \circ_i g = \circ_i(f \otimes g)$.

Next we define products $\circ_i$ on $CY^*(D,D)$, of degree -1, which eventually makes $CY^*(D,D)$ into a pre-Lie system.

Definition 5.2.3 Let D be a dialgebra over a field K. For all i, $0 \leq i \leq p-1$ the maps

$$\circ_i : CY^p(D,D) \times CY^q(D,D) \longrightarrow CY^{p+q-1}(D,D)$$

are defined in the following way. Given $f \in CY^p(D,D)$ and $g \in CY^q(D,D)$,

$$\begin{aligned} & (f \circ_i g)(y; a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q-1}) \\ = \; & f(R_1^i(y); a_1, \ldots, a_i, g(R_2^i(y); a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1}) \end{aligned}$$

where $y \in Y_{p+q-1}$, $R_1^i : Y_{p+q-1} \longrightarrow Y_p$ and $R_2^i : Y_{p+q-1} \longrightarrow Y_q$ are maps as in 5.2.1.

The maps $\circ_i$ as defined above will endow $CY^*(D,D)$ with the structure of a pre-Lie system. The proof of this depends on the next two lemmas. The proofs of the following lemmas are based on the pre-simplicial identity $d_l d_k = d_{k-1} d_l,\ l < k$.

Lemma 5.2.4 *Let $n+2 = p+q+r$. For $0 \leq j < p+q-2$, $0 < i \leq p-1$ and $j \leq i-1$, the following maps*

$$
\begin{aligned}
R_1^j &= R_1^j(n; p+q-1, r) : Y_n \longrightarrow Y_{p+q-1} \\
R_2^j &= R_2^j(n; p+q-1, r) : Y_n \longrightarrow Y_r \\
R_1^i &= R_1^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_p \\
R_2^i &= R_2^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_q \\
R_1^{i+r-1} &= R_1^{i+r-1}(n; p+r-1, q) : Y_n \longrightarrow Y_{p+r-1} \\
R_2^{i+r-1} &= R_2^{i+r-1}(n; p+r-1, q) : Y_n \longrightarrow Y_q \\
R_1^j &= R_1^j(p+r-1; p, r) : Y_{p+r-1} \longrightarrow Y_p \\
R_2^j &= R_2^j(p+r-1; p, r) : Y_{p+r-1} \longrightarrow Y_r
\end{aligned}
$$

satisfy

(*i*) $R_1^i R_1^j = R_1^j R_1^{i+r-1}$,

(*ii*) $R_2^i R_1^j = R_2^{i+r-1}$,

(*iii*) $R_2^j = R_2^j R_1^{i+r-1}$

where the terms on the either side of the equalities (i), (ii) and (iii) are suitable composition of maps, for example, $R_1^i R_1^j$ at the left hand side of the equality (i) denotes the composition of the maps $R_1^j(n; p+q-1, r)$ and $R_1^i(p+q-1; p, q)$.

Proof. First note that $0 < i < p$ implies $p \geq 2$. We need to consider a few cases.

Case 1. Let $q, r \geq 2$, $0 \leq j < p+q-2$, $0 < i \leq p-1$ and $j \leq i-1$. By Definition 5.2.1, we have

$$R_1^i R_1^j = d_{i+1}d_{i+2}\cdots d_{i+q-1}d_{j+1}d_{j+2}\cdots d_{j+r-1}, \tag{5.1}$$

$$R_1^j R_1^{i+r-1} = d_{j+1}d_{j+2}\cdots d_{j+r-1}d_{i+r}d_{i+r+1}\cdots d_{i+r+q-2}. \tag{5.2}$$

Since $j \leq i-1$, $j+r-1 < i+r$. Hence the simplicial identities $d_l d_k = d_{k-1}d_l$ for $l < k$ imply that the adjacent terms $d_{j+r-1}d_{i+r}$ in the right hand side of the equation (5.2) can be replaced by $d_{i+r-1}d_{j+r-1}$. We apply this argument again to the term $d_{j+r-1}d_{i+r+1}$. Continuing the process, (5.2) reduces to

$$R_1^j R_1^{i+r-1} = d_{j+1}d_{j+2}\cdots d_{j+r-2}d_{i+r-1}d_{i+r}\cdots d_{i+r+q-3}d_{j+r-1}.$$

Next, we repeat the argument starting with $d_{j+r-2}d_{i+r-1}$. Proceeding this way the string $d_{j+1}\cdots d_{j+r-1}$ in (5.2) can be pushed off to the right to get (5.1). This proves (i).

To prove (ii), first assume that $i < p-1$. Then, by Definition 5.2.1 we have

$$R_2^{i+r-1} = d_0 d_1 \cdots d_{j+r-1}d_{j+r}\cdots d_{i+r-2}d_{i+r+q}\cdots d_{p+q+r-2}.$$

Note that $j+r-1 \leq i+r-2$ as $j \leq i-1$. If $j+r-1 < i+r-2$, then in the above expression of R_2^{i+r-1} we can replace $d_{j+r-1}d_{j+r}$ by $d_{j+r-1}d_{j+r-1}$ and then replace $d_{j+r-1}d_{j+r+1}$ by $d_{j+r}d_{j+r-1}$. Repeating this process we can make d_{j+r-1} and d_{i+r+q} adjacent and hence can replace $d_{j+r-1}d_{i+r+q}$ by $d_{i+r+q-1}d_{j+r-1}$. Then starting with $d_{j+r-1}d_{i+r+q+1}$ and successively applying the simplicial identities we get

$$R_2^{i+r-1} = d_0 d_1 \cdots d_{j+r-2}d_{j+r-1}d_{j+r}\cdots d_{i+r-3}d_{i+r+q-1}\cdots d_{p+q+r-3}d_{j+r-1}.$$

Next, we apply the above argument again starting with the terms $d_{j+r-2}d_{j+r-1}$ to get

$$R_2^{i+r-1} = d_0 d_1 \cdots d_{j+r-3}d_{j+r-2}\cdots d_{i+r-4}d_{i+r+q-2}\cdots d_{p+q+r-4}d_{j+r-2}d_{j+r-1}.$$

We repeat the process $(r-1)$ times to obtain

$$\begin{aligned} R_2^{i+r-1} &= d_0 d_1 \cdots d_{i-1} d_{i+q+1} \cdots d_{p+q-1} d_{j+1} d_{j+2} \cdots d_{j+r-1} \\ &= R_2^i R_1^j. \end{aligned}$$

If $i = p-1$, then by Definition 5.2.1, we have

$$R_2^{p+r-2} = d_0 d_1 \cdots d_{p+r-3}$$

and

$$\begin{aligned} R_2^{p-1} R_1^j &= R_2^{p-1} d_{j+1} \cdots d_{j+r-1} \\ &= d_0 d_1 \cdots d_{p-2} d_{j+1} \cdots d_{j+r-1}. \end{aligned}$$

Now the proof of the desired equality is similar to the proof of the corresponding case for $i < p-1$.

To prove (iii) first assume that $j > 0$. Note that by Definition 5.2.1

$$R_2^j = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+q+r-2}$$

and

$$R_2^j R_1^{i+r-1} = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+r-1} d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2}.$$

As $p > i$, $p+r > i+r$. Thus, by applying $d_{j-1} d_i = d_i d_j$ for $i < j$, in the above expression of $R_2^j R_1^{i+r-1}$, the terms $d_{p+r-1} d_{i+r}$ can be replaced by $d_{i+r} d_{p+r}$. Next we consider $d_{p+r} d_{i+r+1}$ and replace it by $d_{i+r+1} d_{p+r+1}$. Repeating this $(q-1)$ times we get

$$R_2^j R_1^{i+r-1} = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+r-2} d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2} d_{p+q+r-2}.$$

Next, apply the above argument to the adjacent terms $d_{p+r-2} d_{i+r}$ to get

$$\begin{aligned} R_2^j R_1^{i+r-1} &= d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+r-3} \\ &\quad d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2} d_{p+q+r-3} d_{p+q+r-2}. \end{aligned}$$

Continuing this process $(p-i-1)$ times we obtain

$$\begin{aligned}R_2^j R_1^{i+r-1} &= d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{i+r-1} d_{i+r} d_{i+r+1} \cdots \\ &\quad \cdots d_{i+r+q-2} d_{i+r+q-1} \cdots d_{p+q+r-3} d_{p+q+r-2} \\ &= R_2^j.\end{aligned}$$

It remains to prove (iii) for the case $j = 0$. In this case $i \geq 1$ and we have

$$\begin{aligned}R_2^0 &= d_{r+1} \cdots d_{p+q+r-2} \\ R_2^0 R_1^{i+r-1} &= R_2^0 d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2} \\ &= d_{r+1} \cdots d_{p+r-1} d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2}.\end{aligned}$$

The proof here is an imitation of the corresponding proof for the case $j > 0$.

Case 2. Let $r = 1$, $q \geq 2$, $0 \leq j < p+q-2$, $0 < i \leq p-1$ and $j \leq i-1$.

Here,

$$\begin{aligned}R_1^i R_1^j &= R_1^i \quad \text{as} \quad R_1^j = \text{id} \\ &= d_{i+1} d_{i+2} \cdots d_{i+q-1},\end{aligned}$$

and

$$\begin{aligned}R_1^j R_1^i &= R_1^j d_{i+1} \cdots d_{i+q-1} \\ &= d_{i+1} \cdots d_{i+q-1} \quad \text{as} \quad R_1^j = \text{id}.\end{aligned}$$

This proves the equality (i), in this case.

To prove (ii) we note that

$$\begin{aligned}R_2^i R_1^j &= R_2^i \quad \text{as} \quad R_1^j = \text{id} \\ &= d_0 d_1 \cdots d_{i-1} d_{i+q+1} \cdots d_{p+q-1}\end{aligned}$$

and

$$R_2^i = d_0 d_1 \cdots d_{i-1} d_{i+q+1} \cdots d_{p+q-1}$$

This proves (ii) for $r = 1$.

To prove (iii), note that the maps

$$R_2^j = R_2^j(n; p+q-1, 1) : Y_n \longrightarrow Y_1$$

and

$$R_2^j R_1^i = R_2^j(p;p,1)R_1^i(n;p,q) : Y_n \longrightarrow Y_1$$

are the same constant maps as Y_1 is a singleton set. Thus (iii) follows.

Case 3. Let $q = 1$, $r \geq 2$. Here, by Definition 5.2.1,

$$\begin{aligned} R_1^i R_1^j &= R_1^i d_{j+1} d_{j+2} \cdots d_{j+r-1} \\ &= d_{j+1} d_{j+2} \cdots d_{j+r-1} \qquad \text{as } R_1^i = \text{id} \end{aligned}$$

and

$$\begin{aligned} R_1^j R_1^{i+r-1} &= R_1^j \qquad \text{as } R_1^{i+r-1} = \text{id} \\ &= d_{j+1} d_{j+2} \cdots d_{j+r-1}. \end{aligned}$$

This proves $R_1^i R_1^j = R_1^j R_1^{i+r-1}$, in this case.

To prove (ii), note that

$$R_2^i R_1^j = R_2^i(p;p,1)R_1^j(n;p,r) : Y_n \longrightarrow Y_1.$$

Moreover, by definition 5.2.1,

$$R_2^{i+r-1} = R_2^{i+r-1}(n;p+r-1,1) : Y_n \longrightarrow Y_1.$$

Again, since Y_1 is a singleton set, the maps on either sides of the equality (ii) are the same constants, hence the result follows immediately.

To prove (iii), observe that

$$R_1^{i+r-1} = \text{id}$$

and

$$R_2^j = R_2^j(n;p,r) = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+r-1}.$$

Also

$$R_2^j = R_2^j(p+r-1;p,r) = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+r-1}$$

by Definition 5.2.1. Thus (*iii*) holds in this case.

Case 4. Let $q = r = 1$. Note that by Definition 5.2.1,

$$\begin{aligned} R_1^i R_1^j &= R_1^i(p;p,1)R_1^j(n;p,1) \\ &= \text{id} \\ &= R_1^j(p;p,1)R_1^i(n;p,1) \\ &= R_1^j R_1^i. \end{aligned}$$

This proves (*i*).

To prove (*ii*), observe that

$$R_2^i R_1^j = R_2^i(p;p,1)R_1^j(n;p,1) : Y_n \longrightarrow Y_1,$$

and

$$R_2^i(n;p,1) : Y_n \longrightarrow Y_1.$$

(*ii*) follows trivially as Y_1 consists of a single element.

To prove (*iii*), we note that

$$R_2^j R_1^i = R_2^j(p;p,1)R_1^i(n;p,1) : Y_n \longrightarrow Y_n \longrightarrow Y_1$$

and

$$R_2^j = R_2^j(p;p,1) : Y_p \longrightarrow Y_1.$$

Again, since Y_1 consists of a single element, the result follows trivially.

These exhaust all possible cases, and the proof of the lemma is complete. ■

Lemma 5.2.5 *Let $n+2 = p+q+r$. For $0 \le j \le p+q-2$, $0 \le i \le p-1$ and $i \le j \le q$ if $i > 0$ and $0 \le j < q$ if $i = 0$ the maps*

$$
\begin{aligned}
R_1^j &= R_1^j(n; p+q-1, r) : Y_n \longrightarrow Y_{p+q-1} \\
R_2^j &= R_2^j(n; p+q-1, r) : Y_n \longrightarrow Y_r \\
R_1^i &= R_1^i(n; p, q+r-1) : Y_n \longrightarrow Y_p \\
R_2^i &= R_2^i(n; p, q+r-1) : Y_n \longrightarrow Y_{q+r-1} \\
R_1^i &= R_1^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_p \\
R_2^i &= R_2^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_q \\
R_1^{j-i} &= R_1^{j-i}(q+r-1; q, r) : Y_{q+r-1} \longrightarrow Y_q \\
R_2^{j-i} &= R_2^{j-i}(q+r-1; q, r) : Y_{q+r-1} \longrightarrow Y_r
\end{aligned}
$$

satisfy

(i) $R_1^i R_1^j = R_1^i$,

(ii) $R_2^i R_1^j = R_1^{j-i} R_2^i$,

(iii) $R_2^j = R_2^{j-i} R_2^i$.

Proof. Case 1. Let $p, q, r \ge 2$. To prove (i), we have by Definition 5.2.1

$$
\begin{aligned}
&(5.3) \qquad & R_1^i &= d_{i+1} d_{i+2} \cdots d_{i+q+r-2}, \\
&(5.4) \qquad & R_1^i R_1^j &= d_{i+1} d_{i+2} \cdots d_{i+q-1} d_{j+1} d_{j+2} \cdots d_{j+r-1}.
\end{aligned}
$$

Let us consider the adjacent terms $d_{i+q-1} d_{j+1}$, in the equation (5.4). First note that $i+q \ge j+1$. For, if $i > 0$, then $j+1 \le j+i \le q+i$ as $j \le q$ and if $i = 0$ then $j < q$. So $j+1 \le q$. Also note that if $i+q = j+1$, then the right hand sides of the equalities (5.3) and (5.4) are exactly the same. Hence we may assume $i+q > j+1$. In this case, the term $d_{i+q-1} d_{j+1}$ can be rewritten as $d_{j+1} d_{i+q}$, by using $d_{k-1} d_l = d_l d_k$ for

$l < k$. Next considering the two adjacent terms $d_{i+q}d_{j+2}$, and repeating the process $(r-1)$ times, we can rewrite equation (5.4) as

$$R_1^i R_1^j = d_{i+1}d_{i+2}\cdots d_{i+q-2}d_{j+1}d_{j+2}\cdots d_{j+r-1}d_{i+q+r-2}.$$

Next we consider the adjacent terms $d_{i+q-2}d_{j+1}$. Again, as in the previous case, we may assume that $i+q-1 > j+1$. Proceeding as in the previous step, the above expression can be rewritten as

$$R_1^i R_1^j = d_{i+1}d_{i+2}\cdots d_{i+q-3}d_{j+1}d_{j+2}\cdots d_{j+r-1}d_{i+q+r-3}d_{i+q+r-2}.$$

This process can be continued till the term left of d_{j+1} is d_{j+1}, and must be stopped once the term left of d_{j+1} is d_j. So, at the end of the process, $R_1^i R_1^j$ will look like

$$R_1^i R_1^j = d_{i+1}d_{i+2}\cdots d_j d_{j+1}\cdots d_{j+r-1}d_{j+r}\cdots d_{i+q+r-2}.$$

This is nothing but

$$R_1^i = d_{i+1}d_{i+2}\cdots d_{i+q+r-2}.$$

Hence (i) is proved.

To prove (ii), let us first assume that $0 < i < p-1$. Then, by Definition 5.2.1

$$\begin{aligned} R_2^i R_1^j &= R_2^i d_{j+1}d_{j+2}\cdots d_{j+r-1} \\ &= d_0 d_1\cdots d_{i-1}d_{i+q+1}\cdots d_{p+q-1}d_{j+1}d_{j+2}\cdots d_{j+r-1} \end{aligned}$$

and

$$\begin{aligned} R_1^{j-i} R_2^i &= R_1^{j-i} d_0 d_1\cdots d_{i-1}d_{i+q+r}\cdots d_{p+q+r-2} \\ &= d_{j-i+1}d_{j-i+2}\cdots d_{j-i+r-1}d_0 d_1\cdots d_{i-1}d_{i+q+r}\cdots d_{p+q+r-2}. \end{aligned}$$

Starting with the expression of $R_1^{j-i}R_2^i$, and following the same techniques as done previously, we bring all the operators $d_0, d_1, \cdots, d_{i-1}$ to the front, one at a time, and rewrite the whole equation as

$$R_1^{j-i}R_2^i = d_0d_1\cdots d_{i-1}d_{j+1}d_{j+2}\cdots d_{j+r-1}d_{i+q+r}\cdots d_{p+q+r-2}.$$

Next we note that $j+r-1 \leq q+r-1 < q+r < q+r+i$. So, using the identity $d_ld_k = d_{k-1}d_l$ for $l<k$, we can rewrite the adjacent terms $d_{j+r-1}d_{i+q+r}$ of the above equation as $d_{i+q+r-1}d_{j+r-1}$. Next we consider the terms $d_{j+r-2}d_{i+q+r-1}$, which can be rewritten as $d_{i+q+r-2}d_{j+r-2}$. This technique can be repeated $(r-1)$ times to obtain

$$R_1^{j-i}R_2^i = d_0d_1\cdots d_{i-1}d_{i+q+1}d_{j+1}d_{j+2}\cdots d_{j+r-1}d_{i+q+r+1}\cdots d_{p+q+r-2}.$$

All the operators from $d_{i+q+r+1}$ to $d_{p+q+r-2}$ can be treated similarly, to get

$$R_1^{j-i}R_2^i = d_0d_1\cdots d_{i-1}d_{i+q+1}d_{i+q+2}\cdots d_{p+q-1}d_{j+1}d_{j+2}\cdots d_{j+r-1}.$$

This is the required form of the expression of $R_2^iR_1^j$.

Now if $i=p-1$, then the expressions of $R_2^iR_1^j$ and $R_1^{j-i}R_2^i$ are given by

$$\begin{aligned} R_2^iR_1^j &= R_2^id_{j+1}d_{j+2}\cdots d_{j+r-1} \\ &= d_0d_1\cdots d_{i-1}d_{j+1}d_{j+2}\cdots d_{j+r-1} \\ R_1^{j-i}R_2^i &= R_1^{j-i}d_0d_1\cdots d_{i-1} \\ &= d_{j-i+1}d_{j-i+2}\cdots d_{j-i+r-1}d_0d_1\cdots d_{i-1}. \end{aligned}$$

To prove the required equality, the string of operators $d_0d_1\cdots d_{i-1}$ appearing in $R_1^{j-i}R_2^i$ can be moved to the extreme left by using $d_{k-1}d_l = d_ld_k$ for $l<k$. The resulting expression is precisely the expression for $R_2^iR_1^j$.

Now if $i=0$, then Definition 5.2.1 yields

$$\begin{aligned} R_2^0R_1^j &= R_2^0d_{j+1}\cdots d_{j+r-1} \\ &= d_{q+1}d_{q+2}\cdots d_{p+q-1}d_{j+1}\cdots d_{j+r-1}, \\ R_1^jR_2^0 &= R_1^jd_{q+r}d_{q+r+1}\cdots d_{p+q+r-2} \\ &= d_{j+1}\cdots d_{j+r-1}d_{q+r}d_{q+r+1}\cdots d_{p+q+r-2}. \end{aligned}$$

Observe that as $i = 0$, hypothesis implies $j < q$, hence $j + r - 1 < j + r < q + r$. Now starting with $d_{j+r-1}d_{q+r}$ and applying the identity $d_l d_k = d_{k-1} d_l$ for $l < k$, $(r - 1)$ times, the string of operators $d_{q+r} \cdots d_{p+q+r-2}$ appearing in the expression $R_1^j R_2^0$ can be moved to the extreme left to yield $R_2^0 R_1^j$.

In order to prove (iii), we need to consider a few cases separately. First let us assume $0 < i < p - 1$, and $i < j$. We note that by definition we have

$$\begin{aligned} R_2^j &= d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+q+r-2} \\ R_2^{j-i} R_2^i &= d_0 d_1 \cdots d_{j-i-1} d_{j-i+r+1} \cdots d_{q+r-1} d_0 d_1 \cdots d_{i-1} d_{i+q+r} \cdots d_{p+q+r-2} \end{aligned}$$

Since $i < j$, by using the identity $d_{k-1} d_l = d_l d_k$, for $l < k$, the string of operators $d_{j-i+r+1} \cdots d_{q+r-1}$ appearing in $R_2^{j-i} R_2^i$ can be replaced by the string $d_0 \cdots d_{i-1}$ following it, yielding

$$R_2^{j-i} R_2^i = d_0 d_1 \cdots d_{j-i-1} d_0 d_1 \cdots d_{i-1} d_{j+r+1} \cdots d_{q+r+i-1} d_{q+r+i} \cdots d_{p+q+r-2}.$$

Again, as $i < j$, by using the identity $d_{k-1} d_l = d_l d_k$ for $l < k$. the string of operators $d_0 d_1 \cdots d_{i-1}$ appearing in the above expression of $R_2^{j-i} R_2^i$ can be brought to the extreme left to obtain

$$\begin{aligned} R_2^{j-i} R_2^i &= d_0 d_1 \cdots d_{i-1} d_i \cdots d_{j-1} d_{j+r+1} \cdots d_{q+r+i-1} d_{q+r+i} \cdots d_{p+q+r-2} \\ &= R_2^j. \end{aligned}$$

Now if $0 < i < p - 1$, and $i = j$, we have

$$R_2^j = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+q+r-2}$$

and

$$\begin{aligned} R_2^0 R_2^j &= R_2^0 d_0 d_1 \cdots d_{j-1} d_{j+q+r} \cdots d_{p+q+r} \\ &= d_{r+1} \cdots d_{q+r-1} d_0 d_1 \cdots d_{j-1} d_{j+q+r} \cdots d_{p+q+r-2}. \end{aligned}$$

The proof is again similar to the previous case, we simply have to move the string of operators $d_0 \cdots d_{j-1}$, appearing in the expression of $R_2^0 R_2^j$ to the extreme left by using $d_{k-1}d_l = d_l d_k$, for $l < k$.

Next if $i = p-1$, and $i < j$,

$$\begin{aligned} R_2^j &= d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+q+r-2} \\ R_2^{j-p+1} R_2^{p-1} &= R_2^{j-p+1} d_0 d_1 \cdots d_{p-2} \\ &= d_0 d_1 \cdots d_{j-p} d_{j-p+r+2} \cdots d_{q+r-1} d_0 d_1 \cdots d_{p-2}. \end{aligned}$$

The proof is similar to the proof of case $0 < i < p-1, \quad i < j$. One simply moves the string $d_0 \cdots d_{p-2}$ in the expression $R_2^{j-p+1} R_2^{p-1}$ to the extreme left using $d_{k-1}d_l = d_l d_k$, for $l < k$.

If $i = p-1$, and $i = j$, the expressions look like

$$\begin{aligned} R_2^{p-1} &= d_0 d_1 \cdots d_{p-2} d_{p+r} \cdots d_{p+q+r-2} \\ R_2^0 R_2^{p-1} &= R_2^0 d_0 d_1 \cdots d_{p-2} \\ &= d_{r+1} \cdots d_{q+r-1} d_0 d_1 \cdots d_{p-2}. \end{aligned}$$

The proof again involves same steps as in the case $0 < i < p-1, \quad i = j$.

Now if $i = 0$, and $0 < j < q-1$, the case is trivial, because

$$R_2^j = d_0 d_1 \cdots d_{j-1} d_{j+r+1} \cdots d_{p+q+r-2}$$

and

$$\begin{aligned} R_2^j R_2^0 &= R_2^j d_{q+r} \cdots d_{p+q+r-2} \\ &= d_0 \cdots d_{j-1} d_{j+r+1} \cdots d_{q+r-1} d_{q+r} \cdots d_{p+q+r-2} \\ &= R_2^j. \end{aligned}$$

Similarly, if $i = 0, \ j = q-1$,

$$R_2^{q-1} = d_0 d_1 \cdots d_{q-2} d_{q+r} \cdots d_{p+q+r-2},$$

and

$$
\begin{aligned}
R_2^{q-1} R_2^0 &= R_2^{q-1} d_{q+r} d_{q+r+1} \cdots d_{p+q+r-2} \\
&= d_0 d_1 \cdots d_{q-2} d_{q+r} \cdots d_{p+q+r-2} \\
&= R_2^{q-1}.
\end{aligned}
$$

Finally, if $i = j = 0$, then

$$R_2^0 = d_{r+1} \cdots d_{p+q+r-2}$$

and

$$
\begin{aligned}
R_2^0 R_2^0 &= R_2^0 d_{q+r} d_{q+r+1} \cdots d_{p+q+r-2} \\
&= d_{r+1} \cdots d_{q+r-1} d_{q+r} d_{q+r+1} \cdots d_{p+q+r-2} \\
&= R_2^0.
\end{aligned}
$$

This proves that $R_2^j = R_2^{j-i} R_2^i$ in this case.

Case 2. Let $p, q \geq 2$ and $r = 1$.

To prove (i), we note that,

$$
\begin{aligned}
R_1^i R_1^j &= R_1^i(p+q-1; p, q) R_1^j(n; p+q-1, 1) \\
&= R_1^i(p+q-1; p, q) \\
&= R_1^i(n; p, q) \\
&= R_1^i,
\end{aligned}
$$

as $R_1^j(n; p+q-1, 1)$ is identity, by Definition 5.2.1.

Next we note that

$$
\begin{aligned}
R_2^i R_1^j &= R_2^i(p+q-1; p, q) R_1^j(n; p+q-1, 1) \\
&= R_2^i(n; p, q)
\end{aligned}
$$

as $R_1^j(n; p+q-1, 1)$ is identity, and

$$
\begin{aligned}
R_1^{j-i} R_2^i &= R_1^{j-i}(q; q, 1) R_2^i(n; p, q) \\
&= R_2^i(n; p, q)
\end{aligned}
$$

as $R_1^{j-i}(q;q,1)$ is identity. Hence, equality (ii) follows.

To prove (iii), note that

$$R_2^{j-i}R_2^i = R_2^{j-i}(q;q,1)R_2^i(n;p,q) : Y_n \longrightarrow Y_1,$$

and

$$R_2^j = R_2^j(n;p+q-1,1) : Y_n \longrightarrow Y_1.$$

The required equality is trivially true as Y_1 consists of a single element.

Case 3. Let $p, r \geq 2$ and $q = 1$. This forces $i = j$. The value could be either 0 or 1.

To prove (i) we note that, by Definition 5.2.1,

$$\begin{aligned} R_1^i R_1^j &= R_1^i(p;p,1)R_1^j(n;p,r) \\ &= R_1^j(n;p,r) \qquad \text{as} \quad R_1^i(p;p,1) \quad \text{is identity} \\ &= d_{j+1}d_{j+2}\cdots d_{j+r-1}, \end{aligned}$$

and

$$\begin{aligned} R_1^i &= R_1^i(n;p,r) \\ &= d_{i+1}d_{i+2}\cdots d_{i+(q+r-1)-1} \\ &= d_{i+1}d_{i+2}\cdots d_{i+r-1}. \end{aligned}$$

Since $i = j$, the result follows.

For proving (ii), note that

$$R_2^1 R_1^1 = R_2^1(p;p,1)R_1^1(n;p,r) : Y_n \longrightarrow Y_1,$$

and

$$R_1^0 R_2^1 = R_1^0(r;1,r)R_2^1(n;p,r) : Y_n \longrightarrow Y_1.$$

Since, maps on either sides of the equality land in Y_1, which is a singleton set, the result follows.

To prove (iii) we first assume that $i > 0$. This forces the condition $i = j = 1$. We observe here that

$$\begin{aligned} R_2^1 &= d_0 d_{r+2} \cdots d_{p+r-1} \\ R_2^0 R_2^1 &= R_2^0 d_0 d_{r+2} \cdots d_{p+r-1} \\ &= d_0 d_{r+2} \cdots d_{p+r-1} \quad \text{as } R_2^0 = R_2^0(r; 1, r) = \text{ id.} \end{aligned}$$

This shows that $R_2^j = R_2^{j-i} R_2^i$, in this case.

Now we assume $i = 0$. This forces $j = 0$. We have

$$\begin{aligned} R_2^0 &= d_{r+1} \cdots d_{p+r-1} \\ R_2^0 R_2^0 &= R_2^0 d_{r+1} \cdots d_{p+r-1} \quad \text{as } R_2^0 = R_2^0(r; 1, r) = \text{ id.} \end{aligned}$$

This proves (iii) for the case $i = 0$.

Case 4. Let $q = r = 1$, and $p \geq 2$. This again forces $i = j$.

Here we have by Definition 5.2.1,

$$\begin{aligned} R_1^i R_1^i &= R_1^i(p; p, 1) R_1^i(n; p, 1) \\ &= \text{id,} \end{aligned}$$

and

$$R_1^i = R_1^i(n; p, 1) = \text{ id.}$$

This proves (i).

To prove (ii), we have

$$R_2^i R_1^i = R_2^i(p; p, 1) R_1^i(n; p, 1) : Y_n \longrightarrow Y_1,$$

and

$$R_1^0 R_2^i = R_1^0(1;1,1) R_2^i(n;p,1) : Y_n \longrightarrow Y_1.$$

Again, since Y_1 consists of a single element, (ii) follows trivially.

To prove (iii),

$$R_2^0 R_2^1 = R_2^0(1;1,1) R_2^1(n;p,1) : Y_n \longrightarrow Y_1,$$

and

$$R_2^1 = R_2^1(n;p,1) : Y_n \longrightarrow Y_1.$$

Both the sides being constant maps, the result follows.

Case 5. $p = 1, q, r \geq 2$. Then hypothesis $0 \leq i \leq p-1$ implies $i = 0$. Hence $0 \leq j < q$.

To prove (i), by Definition 5.2.1,

$$\begin{aligned} R_1^0 &= R_1^0(n;1,q+r-1) : Y_n \longrightarrow Y_1, \\ R_1^0 R_1^j &= R_1^0(q;1,q) R_1^j(n;q,r) : Y_n \longrightarrow Y_1. \end{aligned}$$

In this case, both sides are constant maps onto Y_1. Hence, (i) is proved.

To prove (ii), by Definition 5.2.1,

$$\begin{aligned} R_2^0 R_1^j &= R_2^0 d_{j+1} d_{j+2} \cdots d_{j+r-1} \\ &= d_{j+1} d_{j+2} \cdots d_{j+r-1} \end{aligned}$$

as $R_2^0 = R_2^0(q;1,q)$ is identity and

$$\begin{aligned} R_1^j R_2^0 &= R_1^j \\ &= d_{j+1} d_{j+2} \cdots d_{j+r-1} \\ &= R_2^0 R_1^j \end{aligned}$$

as $R_2^0 = R_2^0(n; 1, q+r-1)$ is identity. This proves the required identity.

To prove (iii), by Definition 5.2.1,

$$\begin{aligned} R_2^j &= R_2^j(n; q, r) \\ R_2^j R_2^0 &= R_2^j(q+r-1; q, r)\ R_2^0(n; 1, q+r-1) \\ &= R_2^j(q+r-1; q, r) \\ &= R_2^j(n; q, r) \end{aligned}$$

as $R_2^0 = R_2^0(n; 1, q+r-1)$ is identity and $n+1 = q+r$. Hence, (iii) is proved.

Case 6. Let $p = q = 1$, and $r \geq 2$. Then, by hypothesis $0 \leq i \leq p-1$, $i = 0$. Also, hypothesis $0 \leq j < q$ implies $j = 0$.

In this case, as $p = q = 1$, the maps in (i) and (ii) ends with Y_1. So, (i) and (ii) follows immediately. Need only to prove (iii). We note that

$$R_2^0 = R_2^0(n; 1, r)$$

is identity. Also,

$$R_2^0 R_2^0 = R_2^0(r; 1, r)\ R_2^0(n; 1, r)$$

is identity, as both the maps are identity maps. This proves identity (iii).

Case 7. Let $p = r = 1$ and $q \geq 2$.

In this case, (i) and (iii) follows trivially, as both sides end with the only element in Y_1. So, need only to verify (ii). We note that

$$R_2^0 R_1^j = R_2^0(q; 1, q)\ R_1^j(n; q, 1)$$

is identity, as both the maps are identity maps, and

$$R_1^j R_2^0 = R_1^j(q; q, 1)\ R_2^0(n; 1, q)$$

is identity, as both the maps are so. Hence, (ii) follows.

Case 8. Let $p = q = r = 1$. This case is obvious as all maps in the statement of the lemma reduce to identity maps.

This exhausts all possible cases and completes the proof of the lemma. ■

We are now in a position to prove the following.

Proposition 5.2.6 *The maps* $\circ_i : CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D)$, $0 \leq i \leq p-1$, *as defined in* 5.2.3, *induce a pre-Lie system structure on* $CY^*(D, D)$.

Proof. Let $f \in CY^p(D, D)$, $g \in CY^q(D, D)$ and $h \in CY^r(D, D)$ and assume that $0 \leq j \leq i-1$. Then for $y \in Y_{p+q+r-2}$, and $a_1, \ldots, a_{p+q+r-2} \in D$, we have by definition of the maps $\circ_i$,

$$\begin{aligned}
& (f \circ_i g) \circ_j h(y; a_1, \ldots, a_{p+q+r-2}) \\
= \; & (f \circ_i g)(R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1}, \ldots, a_{p+q+r-2}) \\
= \; & f(R_1^i R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1}, \ldots, a_{i+r-1}, \\
& g(R_2^i R_1^j(y); a_{i+r}, \ldots, a_{i+r+q-1}), a_{i+r+q}, \ldots, a_{p+q+r-2}),
\end{aligned}$$

where

$$\begin{aligned}
R_1^j &= R_1^j(p+q+r-2; p+q-1, r) : Y_{p+q+r-2} \longrightarrow Y_{p+q-1}, \\
R_2^j &= R_2^j(p+q+r-2; p+q-1, r) : Y_{p+q+r-2} \longrightarrow Y_r, \\
R_1^i &= R_1^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_p, \\
R_2^i &= R_2^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_q
\end{aligned}$$

are the maps as defined in Definition 5.2.1.

On the other hand

$$
\begin{aligned}
& (f \circ_j h) \circ_{i+r-1} g(y; a_1, \ldots, a_{p+q+r-2}) \\
= \; & (f \circ_j h)(R_1^{i+r-1}(y); a_1, \ldots, a_{i+r-1}, g(R_2^{i+r-1}(y); a_{i+r}, \ldots, a_{i+r+q-1}), \\
& a_{i+r+q}, \ldots, a_{p+q+r-2}) \\
= \; & f(R_1^j R_1^{i+r-1}(y); a_1, \ldots, a_j, h(R_2^j R_1^{i+r-1}(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1}, \\
& \ldots, a_{i+r-1}, g(R_2^{i+r-1}(y); a_{i+r}, \ldots, a_{i+r+q-1}), a_{i+r+q}, \ldots, a_{p+q+r-2})
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^{i+r-1} &= R_1^{i+r-1}(p+q+r-2; p+r-1, q) : Y_{p+q+r-2} \longrightarrow Y_{p+r-1}, \\
R_2^{i+r-1} &= R_2^{i+r-1}(p+q+r-2; p+r-1, q) : Y_{p+q+r-2} \longrightarrow Y_q, \\
R_1^j &= R_1^j(p+r-1; p, r) : Y_{p+r-1} \longrightarrow Y_p, \\
R_2^i &= R_2^i(p+r-1; p, r) : Y_{p+r-1} \longrightarrow Y_r
\end{aligned}
$$

are the maps as defined in Definition 5.2.1.

It now follows from Lemma 5.2.4 that $(f \circ_i g) \circ_j h = (f \circ_j h) \circ_{i+r-1} g$ for $0 \leq j \leq i-1$.

Suppose now that $i \leq j \leq q$ if $i > 0$ and $0 \leq j < q$ if $i = 0$. Then

$$
\begin{aligned}
& (f \circ_i g) \circ_j h(y; a_1, \ldots, a_{p+q+r-2}) \\
= \; & (f \circ_i g)(R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}), \\
& a_{j+r+1}, \ldots, a_{p+q+r-2}) \\
= \; & f(R_1^i R_1^j(y); a_1, \ldots, a_i, g(R_2^i R_1^j(y); a_{i+1}, \ldots, a_j, h(R_2^j(y); a_{j+1}, \\
& \ldots, a_{j+r}), a_{j+r+1}, \ldots, a_{q+r+i-1}), a_{i+r+q}, \ldots, a_{p+q+r-2}).
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^j &= R_1^j(p+q+r-2; p+q-1, r) : Y_{p+q+r-2} \longrightarrow Y_{p+q-1}, \\
R_2^j &= R_2^j(p+q+r-2; p+q-1, r) : Y_{p+q+r-2} \longrightarrow Y_r, \\
R_1^i &= R_1^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_p, \\
R_2^i &= R_2^i(p+q-1; p, q) : Y_{p+q-1} \longrightarrow Y_q
\end{aligned}
$$

are the maps as defined in Definition 5.2.1.

On the other hand

$$\begin{aligned} & f \circ_i (g \circ_{j-i} h)(y; a_1, \ldots, a_{p+q+r-2}) \\ = \; & f(R_1^i(y); a_1, \ldots, a_i, (g \circ_{j-i} h)(R_2^i(y); a_{i+1}, \ldots, a_{q+r+i-1}), \\ & a_{q+r+i}, \ldots, a_{p+q+r-2}) \\ = \; & f(R_1^i(y); a_1, \ldots, a_i, g(R_1^{j-i} R_2^i(y); a_{i+1}, \ldots, a_j, h(R_2^{j-i} R_2^i(y); a_{j+1}, \\ & \ldots, a_{j+r}), a_{j+r+1}, \ldots, a_{q+r+i-1}), a_{q+r+i} \ldots, a_{p+q+r-2}). \end{aligned}$$

where

$$\begin{aligned} R_1^i &= R_1^i(p+q+r-2; p, q+r-1) : Y_{p+q+r-2} \longrightarrow Y_p, \\ R_2^i &= R_2^i(p+q+r-2; p, q+r-1) : Y_{p+q+r-2} \longrightarrow Y_{q+r-1}, \\ R_1^{j-i} &= R_1^{j-i}(q+r-1; q, r) : Y_{q+r-1} \longrightarrow Y_q, \\ R_2^{j-i} &= R_2^{j-i}(q+r-1; q, r) : Y_{q+r-1} \longrightarrow Y_r \end{aligned}$$

are the maps as defined in 5.2.1.

Lemma 5.2.5 now implies that $(f \circ_i g) \circ_j h = f \circ_i (g \circ_{j-i} h)$ for $i \leq j \leq q$ if $i > 0$ and $0 \leq j < q$ if $i = 0$. Thus considering elements of $CY^p(D, D)$ to be of degree $(p-1)$ we see that the maps

$$\circ_i : CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D)$$

for $i \leq p-1$ as defined in 5.2.3 make $CY^*(D, D)$ into a pre-Lie system. This completes the proof of the proposition. ■

The products $\circ_i$ can now be combined to give a product on $CY^*(D, D)$, which we call the pre-Lie product.

Definition 5.2.7 The 'pre-Lie product'

$$\circ : CY^p(D,D) \times CY^q(D,D) \longrightarrow CY^{p+q-1}(D,D)$$

on $CY^*(D,D)$ is defined by

$$f \circ g = \sum_{i=0}^{p-1} (-1)^{i(q-1)} f \circ_i g$$

for $f \in CY^p(D,D)$ and $g \in CY^q(D,D)$.

We recall the following definition from [5].

Definition 5.2.8 A graded ring A will be called a graded right pre-Lie ring (or simply a pre-Lie ring) if for elements a, b, c of A of degrees λ, μ, ν respectively we have

$$(c \circ a) \circ b - (-1)^{\lambda\mu}(c \circ b) \circ a = c \circ (a \circ b - (-1)^{\lambda\mu} b \circ a),$$

where $a \circ b$ denotes the product of a and b in A.

Now, let us recall Theorem 2 from [5].

Theorem 5.2.9 *Let $\{V_m, \circ_i\}$ be a pre-Lie system and f^m, g^n, h^p be elements of V_m, V_n, V_p respectively. Then*
(i) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum (-1)^{ni+pj} (f^m \circ_i g^n) \circ_j h^p$, where the sum is extended over those i and j with either $0 \leq j \leq i-1$ or $n+i+1 \leq j \leq m+n$,
(ii) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np}[(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)]$.

As a consequence of Proposition 5.2.6 and Theorem 5.2.9 stated above, we deduce the following result.

Corollary 5.2.10 *The direct sum of cochain modules $CY^*(D,D)$ of a dialgebra D, equipped with the pre-Lie product, becomes a pre-Lie ring.*

5.3 Associative product on the cochain modules

Next we define an associative product $*$ on the graded modules $CY^*(D,D)$, which is actually induced by the $\circ_i$ products defined in the previous section.

Definition 5.3.1 For $f \in CY^p(D,D)$ and $g \in CY^q(D,D)$,

$$*: CY^p(D,D) \times CY^q(D,D) \longrightarrow CY^{p+q}(D,D)$$

is given by $f * g = (\pi \circ_0 f) \circ_p g$, where $\pi \in CY^2(D,D)$ is the 2-cochain defined by

$$\begin{aligned} \pi([21];a,b) &= a \dashv b \\ \pi([12];a,b) &= a \vdash b \end{aligned}$$

for all $a, b \in D$.

Explicitly, for $y \in Y_{p+q}$, $a_1, a_2, \ldots, a_{p+q} \in D$,

$$(f * g)(y; a_1, a_2, \ldots, a_{p+q}) = (\pi \circ_0 f) \circ_p g(y; a_1, a_2, \ldots, a_{p+q})$$

$$= \pi(R_1^0 R_1^p(y); f(R_2^0 R_1^p(y); a_1, \ldots, a_p), g(R_2^p(y); a_{p+1}, \ldots, a_{p+q}))$$

$$= f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \bowtie g(R_2^p(y); a_{p+1}, \ldots, a_{p+q})$$

where $\bowtie$ is either $\dashv$ or $\vdash$ according as $R_1^0 R_1^p(y)$ is $[21]$ or $[12]$ respectively, and R_1^i, R_2^i for $i = 0, p$ are the maps

$$\begin{aligned} R_1^p &= R_1^p(p+q; p+1, q) : Y_{p+q} \longrightarrow Y_{p+1} \\ R_2^p &= R_2^p(p+q; p+1, q) : Y_{p+q} \longrightarrow Y_q \\ R_1^0 &= R_1^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_2 \\ R_2^0 &= R_2^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_p, \end{aligned}$$

as defined in Definition 5.2.1.

Remark 5.3.2 It is interesting to see that π is a cocycle. For, by definition of the coboundary,

$$(5.5) \qquad \begin{aligned} \delta\pi(y; a, b, c) &= a o_0^y \pi(d_0 y;\ b, c) - \pi(d_1 y;\ a o_1^y b, c) \\ &+ \pi(d_2 y;\ a, b o_2^y c) - \pi(d_3 y;\ a, b) o_3^y c. \end{aligned}$$

For $y = [321], [312], [131], [213], [123]$, equation (5.5) yields the following equations respectively

$$\begin{aligned}
\delta\pi([321]; a, b, c) &= a \dashv \pi([21];\ b, c) - \pi([21];\ a \dashv b, c) + \pi([21];\ a, b \dashv c) \\
&\quad -\pi([21];\ a, b) \dashv c \\
&= 0, \qquad \text{by axiom } 1 \text{ of } (1.1), \\
\delta\pi([312]; a, b, c) &= a \dashv \pi([12];\ b, c) - \pi([21];\ a \dashv b, c) + \pi([21];\ a, b \vdash c) \\
&\quad -\pi([21];\ a, b) \dashv c \\
&= 0, \qquad \text{by axiom } 2 \text{ of } (1.1), \\
\delta\pi([131]; a, b, c) &= a \vdash \pi([21];\ b, c) - \pi([21];\ a \vdash b, c) + \pi([12];\ a, b \dashv c) \\
&\quad -\pi([12];\ a, b) \dashv c \\
&= 0, \qquad \text{by axiom } 3 \text{ of } (1.1), \\
\delta\pi([213]; a, b, c) &= a \vdash \pi([12];\ b, c) - \pi([12];\ a \dashv b, c) + \pi([12];\ a, b \vdash c) \\
&\quad -\pi([21];\ a, b) \vdash c \\
&= 0, \qquad \text{by axiom } 4 \text{ of } (1.1), \\
\delta\pi([123]; a, b, c) &= a \vdash \pi([12];\ b, c) - \pi([12];\ a \vdash b, c) + \pi([12];\ a, b \vdash c) \\
&\quad -\pi([12];\ a, b) \vdash c \\
&= 0, \qquad \text{by axiom } 5 \text{ of } (1.1).
\end{aligned}$$

This proves that $\delta\pi = 0$. It is infact a coboundary, $\pi = \delta\varphi$, where $\varphi([1]; a) = a$ for all $a \in D$.

The following lemma shows that the graded product $*$ is associative.

Lemma 5.3.3 *The graded product $*$ on $CY^\bullet(D,D)$ is associative.*

Proof. Suppose $f \in CY^p(D,D)$, $g \in CY^q(D,D)$ and $h \in CY^r(D,D)$. Then, for $y \in Y_{p+q+r}$ and $a_1, a_2, \ldots, a_{p+q+r} \in D$, we note that

$$\begin{aligned}
&(f*g)*h(y; a_1, \ldots, a_{p+q+r}) \\
= \; &\pi(R_1^0 R_1^{p+q}(y); \pi(R_1^0 R_1^p R_2^0 R_1^{p+q}(y); f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p), \\
&g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})), h(R_2^{p+q}(y); a_{p+q+1}, \ldots a_{p+q+r}))
\end{aligned}$$

where

$$\begin{aligned}
R_1^{p+q} &= R_1^{p+q}(p+q+r; p+q+1, r) : Y_{p+q+r} \longrightarrow Y_{p+q+1} \\
R_2^{p+q} &= R_2^{p+q}(p+q+r; p+q+1, r) : Y_{p+q+r} \longrightarrow Y_r \\
R_1^0 &= R_1^0(p+q+1; 2, p+q) : Y_{p+q+1} \longrightarrow Y_2 \\
R_2^0 &= R_2^0(p+q+1; 2, p+q) : Y_{p+q+1} \longrightarrow Y_{p+q} \\
R_1^p &= R_1^p(p+q; p+1, q) : Y_{p+q} \longrightarrow Y_{p+1} \\
R_2^p &= R_2^p(p+q; p+1, q) : Y_{p+q} \longrightarrow Y_q \\
R_1^0 &= R_1^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_2 \\
R_2^0 &= R_2^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_p
\end{aligned}$$

are the maps involved in the above equation. On the other hand

$$\begin{aligned}
&f*(g*h)(y; a_1, \ldots, a_{p+q+r}) \\
= \; &\pi(R_1^0 R_1^p(y); f(R_2^0 R_1^p(y); a_1, \ldots, a_p), \\
&\pi(R_1^0 R_1^q R_2^p(y); g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}), h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r})))
\end{aligned}$$

where

$$\begin{aligned}
R_1^p &= R_1^p(p+q+r;p+1,q+r) : Y_{p+q+r} \longrightarrow Y_{p+1} \\
R_2^p &= R_2^p(p+q+r;p+1,q+r) : Y_{p+q+r} \longrightarrow Y_{q+r} \\
R_1^0 &= R_1^0(p+1;2,p) : Y_{p+1} \longrightarrow Y_2 \\
R_2^0 &= R_2^0(p+1;2,p) : Y_{p+1} \longrightarrow Y_p \\
R_1^q &= R_1^q(q+r;q+1,r) : Y_{q+r} \longrightarrow Y_{q+1} \\
R_2^q &= R_2^q(q+r;q+1,r) : Y_{q+r} \longrightarrow Y_r \\
R_1^0 &= R_1^0(q+1;2,q) : Y_{q+1} \longrightarrow Y_2 \\
R_2^0 &= R_2^0(q+1;2,q) : Y_{q+1} \longrightarrow Y_q.
\end{aligned}$$

Note that according to the convention, following Definition 5.2.1, we are using the same symbol to denote different maps. For example, in the expression of $(f * g) * h$, R_1^0 denotes the map $Y_{p+q+1} \longrightarrow Y_2$ as well as the map $Y_{p+1} \longrightarrow Y_2$. Now to prove that the right hand sides of the equalities given above are the same, we proceed as follows.

Step (i). First note that the composition $R_2^0 R_1^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f * g) * h$ is same as $R_2^0 R_1^p$ appearing in that of $f * (g * h)$. Because by Definition 5.2.1,

$$\begin{aligned}
R_2^0 R_1^p R_2^0 R_1^{p+q} &= d_{p+1}(d_{p+1} \cdots d_{p+q-1}) d_{p+q+1}(d_{p+q+1} \cdots d_{p+q+r-1}) \\
&= (d_{p+1} \cdots d_{p+q-1}) d_{p+q} d_{p+q+1}(d_{p+q+1} \cdots d_{p+q+r-1}) \\
&= d_{p+1} d_{p+1} \cdots d_{p+q-1} d_{p+q} d_{p+q+1} \cdots d_{p+q+r-1} \\
&= R_2^0 R_1^p.
\end{aligned}$$

(first applying $d_{j-1} d_i = d_i d_j, i < j$, $q-1$ times starting with the operators $d_{p+1} d_{p+1}$ at the left, then shifting the $(q+1)^{th}$ operator d_{p+q+1} to the left by using $d_i d_j = d_{j-1} d_i$, $i < j$, q times.)

Step (ii). Next observe that $R_2^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f*g)*h$ is the same as $R_2^0 R_1^q R_2^p$ appearing in that of $f*(g*h)$. This can be seen easily using a similar idea as in Step (i) above.

Step (iii). Next note that the map R_2^{p+q} appearing in the expression of $(f*g)*h$ is the same as $R_2^q R_2^p$ of $f*(g*h)$. The proof is similar to the previous cases.

Step (iv). Let $S : Y_{p+q+r} \longrightarrow Y_3$ be the operator

$$S = d_1 d_2 \cdots d_{p-1} d_{p+1} \cdots d_{p+q-1} d_{p+q+1} \cdots d_{p+q+r-1}.$$

Note that the maps $R_1^0 R_1^{p+q}$ and $R_1^0 R_1^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f*g)*h$ can be written as $d_1 S$ and $d_3 S$ respectively. This is because,

$$\begin{aligned} R_1^0 R_1^{p+q} &= (d_1 \cdots d_{p+q-1})(d_{p+q+1} \cdots d_{p+q+r-1}) \\ &= d_1 d_1 \cdots d_{p-1} d_{p+1} \cdots d_{p+q-1} d_{p+q+1} \cdots d_{p+q+r-1} \\ &= d_1 S \end{aligned}$$

and

$$\begin{aligned} R_1^0 R_1^p R_2^0 R_1^{p+q} &= (d_1 \cdots d_{p-1})(d_{p+1} \cdots d_{p+q-1})(d_{p+q+1})(d_{p+q+1} \cdots d_{p+q+r-1}), \\ &= d_3 d_1 d_2 \cdots d_{p-1} d_{p+1} \cdots d_{p+q-1} d_{p+q+1} \cdots d_{p+q+r-1} \\ &= d_3 S \end{aligned}$$

by using pre-simplicial identity.

Similarly the maps $R_1^0 R_1^p$ and $R_1^0 R_1^q R_2^p$ appearing in $f*(g*h)$ are respectively $d_2 S$ and $d_0 S$. Since $y \in Y_{p+q+r}$, $S(y) \in Y_3$ and there could be five possible cases for $S(y)$. For each of these five cases the result will follow from the five axioms of dialgebras.

Let $S(y) = [321]$. In this case, $d_1S(y) = [21]$, $d_3S(y) = [21]$, $d_2S(y) = [21]$ and $d_0S(y) = [21]$. Hence by definition of π, we get

$$\begin{aligned}(f * g) * h(y; a_1, \ldots, a_{p+q+r}) &= (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \dashv \\ &\quad g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \dashv \\ &\quad h(R_2^{p+q}(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

and

$$\begin{aligned}f * (g * h)(y; a_1, \ldots, a_{p+q+r}) &= f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \dashv \\ &\quad (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \dashv \\ &\quad h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

where $y = [321]$. It now follows from the dialgebra axiom 1 of (1.1) and Steps (i)-(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r})$$

where $y = [321]$.

Let $S(y) = [312]$. In this case, $d_1S(y) = [21]$, $d_3S(y) = [21]$, $d_2S(y) = [21]$ and $d_0S(y) = [12]$. Hence by definition of π, we get

$$\begin{aligned}(f * g) * h(y; a_1, \ldots, a_{p+q+r}) &= (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \dashv \\ &\quad g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \dashv \\ &\quad h(R_2^{p+q}(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

and

$$\begin{aligned}f * (g * h)(y; a_1, \ldots, a_{p+q+r}) &= f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \dashv \\ &\quad (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \vdash \\ &\quad h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

where $y = [312]$. It now follows from the dialgebra axiom 2 of (1.1) and Steps (i)-(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r})$$

where $y = [312]$.

Let $S(y) = [131]$. In this case, $d_1S(y) = [21]$, $d_3S(y) = [12]$, $d_2S(y) = [12]$ and $d_0S(y) = [21]$. Hence by definition of π, we get

$$\begin{aligned}(f * g) * h(y; a_1, \ldots, a_{p+q+r}) &= (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \vdash \\ &\quad g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \dashv \\ &\quad h(R_2^{p+q}(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

and

$$\begin{aligned}f * (g * h)(y; a_1, \ldots, a_{p+q+r}) &= f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \vdash \\ &\quad (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \dashv \\ &\quad h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

where $y = [131]$. It now follows from the dialgebra axiom 3 of (1.1) and Steps (i)-(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r})$$

where $y = [131]$.

Let $S(y) = [213]$. In this case, $d_1S(y) = [12]$, $d_3S(y) = [21]$, $d_2S(y) = [12]$ and $d_0S(y) = [12]$. Hence by definition of π, we get

$$\begin{aligned}(f * g) * h(y; a_1, \ldots, a_{p+q+r}) &= (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \dashv \\ &\quad g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \vdash \\ &\quad h(R_2^{p+q}(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

and

$$\begin{aligned}f * (g * h)(y; a_1, \ldots, a_{p+q+r}) &= f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \vdash \\ &\quad (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \vdash \\ &\quad h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

where $y = [213]$. It now follows from the dialgebra axiom 4 of (1.1) and Steps (i)-(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r})$$

where $y = [213]$.

Let $S(y) = [123]$. In this case, $d_1 S(y) = [12]$, $d_3 S(y) = [12]$, $d_2 S(y) = [12]$ and $d_0 S(y) = [12]$. Hence by definition of π, we get

$$\begin{aligned}(f * g) * h(y; a_1, \ldots, a_{p+q+r}) \quad = \quad & (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \vdash \\ & g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \vdash \\ & h(R_2^{p+q}(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

and

$$\begin{aligned}f * (g * h)(y; a_1, \ldots, a_{p+q+r}) \quad = \quad & f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \vdash \\ & (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \vdash \\ & h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))\end{aligned}$$

where $y = [123]$. It now follows from the dialgebra axiom 5 of (1.1) and Steps (i)-(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r})$$

where $y = [123]$. This completes the proof of the lemma. ∎

We will need the following lemma in Chapter 6.

Lemma 5.3.4 *If $f, g \in CY^1(D, D)$, then $\delta(f * g) = \delta f * g - f * \delta g$.*

Proof. By definition, we have to prove that

$$(\delta(f * g) - \delta f * g + f * \delta g)(y; a, b, c) = 0$$

for all $y \in Y_3$, $a, b, c \in D$. We now show that this identity is equivalent to the dialgebra axioms.

$$
\begin{aligned}
&(\delta(f * g) - \delta f * g + f * \delta g)([321]; a, b, c) \\
&= a \dashv (f * g)([21]; b, c) - (f * g)([21]; a \dashv b, c) + (f * g)([21]; a, b \dashv c) \\
&-(f * g)([21]; a, b) \dashv c - \delta f([21]; a, b) \dashv g([1]; c) + f([1]; a) \dashv \delta g([21]; b, c) \\
&= a \dashv (f([1]; b) \dashv g([1]; c)) - f([1]; a \dashv b) \dashv g([1]; c) + f([1]; a) \dashv g([1]; b \dashv c) \\
&-(f([1]; a) \dashv g([1]; b)) \dashv c - (a \dashv f([1]; b)) \dashv g([1]; c) + f([1]; a \dashv b) \dashv g([1]; c) \\
&-(f([1]; a) \dashv b) \dashv g([1]; c) + f([1]; a) \dashv (b \dashv g([1]; c)) - f([1]; a) \dashv g([1]; b \dashv c) \\
&+f([1]; a) \dashv (g([1]; b) \dashv c) \\
&= 0,
\end{aligned}
$$

by axiom 1 of (1.1).

$$
\begin{aligned}
&(\delta(f * g) - \delta f * g + f * \delta g)([312]; a, b, c) \\
&= a \dashv (f * g)([12]; b, c) - (f * g)([21]; a \dashv b, c) + (f * g)([21]; a, b \vdash c) \\
&-(f * g)([21]; a, b) \dashv c - \delta f([21]; a, b) \dashv g([1]; c) + f([1]; a) \dashv \delta g([12]; b, c) \\
&= a \dashv (f([1]; b) \vdash g([1]; c)) - f([1]; a \dashv b) \dashv g([1]; c) + f([1]; a) \dashv g([1]; b \vdash c) \\
&-(f([1]; a) \dashv g([1]; b)) \dashv c - (a \dashv f([1]; b)) \dashv g([1]; c) + f([1]; a \dashv b) \dashv g([1]; c) \\
&-(f([1]; a) \dashv b) \dashv g([1]; c) + f([1]; a) \dashv (b \vdash g([1]; c)) - f([1]; a) \dashv g([1]; b \vdash c) \\
&+f([1]; a) \dashv (g([1]; b) \vdash c) \\
&= 0,
\end{aligned}
$$

by axiom 2 of (1.1).

$$
\begin{aligned}
&(\delta(f*g)-\delta f*g+f*\delta g)([131];a,b,c)\\
&=a\vdash(f*g)([21];b,c)-(f*g)([21];a\vdash b,c)+(f*g)([12];a,b\dashv c)\\
&-(f*g)([12];a,b)\dashv c-\delta f([12];a,b)\dashv g([1];c)+f([1];a)\vdash\delta g([21];b,c)\\
&=a\vdash(f([1];b)\dashv g([1];c))-f([1];a\vdash b)\dashv g([1];c)+f([1];a)\vdash g([1];b\dashv c)\\
&-(f([1];a)\vdash g([1];b))\dashv c-(a\vdash f([1];b))\dashv g([1];c)+f([1];a\vdash b)\dashv g([1];c)\\
&-(f([1];a)\vdash b)\dashv g([1];c)+f([1];a)\vdash(b\dashv g([1];c))-f([1];a)\vdash g([1];b\dashv c)\\
&+f([1];a)\vdash(g([1];b)\dashv c)\\
&=0,
\end{aligned}
$$

by axiom 3 of (1.1).

$$
\begin{aligned}
&(\delta(f*g)-\delta f*g+f*\delta g)([213];a,b,c)\\
&=a\vdash(f*g)([12];b,c)-(f*g)([12];a\dashv b,c)+(f*g)([12];a,b\vdash c)\\
&-(f*g)([21];a,b)\vdash c-\delta f([21];a,b)\vdash g([1];c)+f([1];a)\vdash\delta g([12];b,c)\\
&=a\vdash(f([1];b)\vdash g([1];c))-f([1];a\dashv b)\vdash g([1];c)+f([1];a)\vdash g([1];b\vdash c)\\
&-(f([1];a)\dashv g([1];b))\vdash c-(a\dashv f([1];b))\vdash g([1];c)+f([1];a\dashv b)\vdash g([1];c)\\
&-(f([1];a)\dashv b)\vdash g([1];c)+f([1];a)\vdash(b\vdash g([1];c))-f([1];a)\vdash g([1];b\vdash c)\\
&+f([1];a)\vdash(g([1];b)\vdash c)\\
&=0,
\end{aligned}
$$

by axiom 4 of (1.1).

$$
\begin{aligned}
&(\delta(f * g) - \delta f * g + f * \delta g)([123]; a, b, c) \\
&= a \vdash (f * g)([12]; b, c) - (f * g)([12]; a \vdash b, c) + (f * g)([12]; a, b \vdash c) \\
&-(f * g)([12]; a, b) \vdash c - \delta f([12]; a, b) \vdash g([1]; c) + f([1]; a) \vdash \delta g([12]; b, c) \\
&= a \vdash (f([1]; b) \vdash g([1]; c)) - f([1]; a \vdash b) \vdash g([1]; c) + f([1]; a) \vdash g([1]; b \vdash c) \\
&-(f([1]; a) \vdash g([1]; b)) \vdash c - (a \vdash f([1]; b)) \vdash g([1]; c) + f([1]; a \vdash b) \vdash g([1]; c) \\
&-(f([1]; a) \vdash b) \vdash g([1]; c) + f([1]; a) \vdash (b \vdash g([1]; c)) - f([1]; a) \vdash g([1]; b \vdash c) \\
&+f([1]; a) \vdash (g([1]; b) \vdash c) \\
&= 0,
\end{aligned}
$$

by axiom 5 of (1.1). This completes the proof of the lemma. ■

Remark 5.3.5 It should be mentioned here that in [10] M. Gerstenhaber and S. D. Schack introduced the notion of 'Comp algebra' and showed that the Hochschild complex $C^*(A, A)$ of an associative algebra A is a natural example of a comp algebra. A proof similar to the proof of Proposition 5.2.6, where we have shown that $CY^*(D, D)$ is a pre-Lie system, shows that $CY^*(D, D)$ along with π as defined in 5.3.1 form a comp algebra. Then the associativity of $*$ product, graded derivation property of $*$ product and Theorem 5.3.7 (proved below) are special cases of some more general comp algebra statements.

The following result relates the pre-Lie product and the coboundary map.

Lemma 5.3.6 *For any* $f \in CY^p(D, D)$,

$$\delta f = -f \circ \pi + (-1)^{p-1} \pi \circ f = (-1)^{p-1}(\pi \circ f - (-1)^{p-1} f \circ \pi)$$

where π *is the 2-cochain as defined in Definition 5.3.1.*

Proof. Let $y \in Y_{p+1}$ and $a_1, a_2, \ldots, a_{p+1} \in D$. Then

$$\delta f(y; a_1, a_2, \ldots, a_{p+1})$$
$$= a_1 o_0^y f(d_0 y; a_2, \ldots, a_{p+1}) + \sum_{i=1}^{p} (-1)^i f(d_i y; a_1, \ldots, a_i o_i^y a_{i+1}, \ldots, a_{p+1})$$
$$+(-1)^{p+1} f(d_{p+1} y; a_1, \ldots, a_p) o_{p+1}^y a_{p+1}$$

and

$$(-f \circ \pi + (-1)^{p-1} \pi \circ f)(y; a_1, \ldots, a_{p+1})$$
$$= -\sum_{i=0}^{p-1} f(R_1^i(y); a_1 \ldots a_i, \pi(R_2^i(y); a_{i+1}, a_{i+2}),$$
$$\ldots, a_{p+1}) + (-1)^{p-1}[\pi(R_1^0(y); f(R_2^0(y); a_1, \ldots, a_p), a_{p+1})$$
$$+(-1)^{p-1} \pi(R_1^1(y); a_1, f(R_2^1(y); a_2, \ldots, a_p, a_{p+1}))]$$
$$= \sum_{j=1}^{p} f(R_1^{j-1}(y); a_1 \ldots a_{j-1}, \pi(R_2^{j-1}(y); a_j, a_{j+1}),$$
$$\ldots, a_{p+1}) + (-1)^{p-1}[\pi(R_1^0(y); f(R_2^0(y); a_1, \ldots, a_p), a_{p+1})$$
$$+(-1)^{p-1} \pi(R_1^1(y); a_1, f(R_2^1(y); a_2, \ldots, a_p, a_{p+1}))],$$

where the maps R_k^l in the above equality, according to their order of appearance, are

$$\begin{aligned}
R_1^i &= R_1^i(p+1; p, 2) : Y_{p+1} \longrightarrow Y_p \\
R_2^i &= R_2^i(p+1; p, 2) : Y_{p+1} \longrightarrow Y_2 \\
R_1^0 &= R_1^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_2 \\
R_2^0 &= R_2^0(p+1; 2, p) : Y_{p+1} \longrightarrow Y_p \\
R_1^1 &= R_1^1(p+1; 2, p) : Y_{p+1} \longrightarrow Y_2 \\
R_2^1 &= R_2^1(p+1; 2, p) : Y_{p+1} \longrightarrow Y_p.
\end{aligned}$$

To complete the proof observe the following:

(a) $R_1^{j-1}(y) = d_j(y)$, which follows from the definition of R_1^{j-1}.

(b) $R_2^{j-1}(y)$ is the tree [21] or the tree [12] according as o_j^y is $\dashv$ or $\vdash$.

We prove (b) by induction on the degree of y, where degree of y is n if $y \in Y_n$. Let $\deg y = 2$. Then j can take value 1 only and $R_2^{j-1} = R_2^0$ is the identity map.

Moreover, if $o_1^y = \dashv$, then y must be [21] and if $o_1^y = \vdash$, then y must be [12]. Hence (b) is true for deg $y = 2$. Assume that (b) holds for all y with deg $y \leq m$ and for all j, $1 \leq j \leq m-1$. Let deg $y = m+1$ and $1 \leq j \leq m$. Let $y = x_1 \vee x_2$, where $\ell =$deg $x_1 < m+1$, $k =$deg $x_2 < m+1$ and $\ell + k = m$. Let $o_j^y = \dashv$. Two cases arise. The jth leaf of y is either a leaf of x_1 or a leaf of x_2. Suppose that it is a leaf of x_1. In this case, it must be an interior leaf of x_1, that is, not those numbered 0 and p as $1 \leq j$ and $o_j^y = \dashv$. Note that $R_2^{j-1} : Y_{m+1} \longrightarrow Y_2$ is given by $R_2^{j-1} = d_0 d_1 \cdots d_{j-2} d_{j+2} \cdots d_{m+1}$ and, in the present case, we also have the map $R_2^{j-1} : Y_\ell \longrightarrow Y_2$ given by $R_2^{j-1} = d_0 d_1 \cdots d_{j-2} d_{j+2} \cdots d_\ell$. Note that the effect of applying the operator $d_{\ell+1} \cdots d_{m+1}$ on y is to delete the leaves of x_2 one after another, the leaves of x_1 remaining untouched during the process. In other words, $d_{\ell+1} \cdots d_{m+1}(y) = x_1$. Hence

$$\begin{aligned} R_2^{j-1}(y) &= d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_\ell d_{\ell+1} \cdots d_{m+1}(y) = d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_\ell(x_1) \\ &= R_2^{j-1}(x_1). \end{aligned}$$

Moreover $o_j^y = o_j^{x_1}$ by definition, as $j \neq \ell$. Hence by induction the result follows. Now if the jth leaf is a leaf of x_2 and an interior one the case is settled as above. Suppose now that the jth leaf in the 0th leaf of x_2, so that $j = \ell + 1$. We know that $R_2^{j-1} = d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_{m+1}$. Observe that if we apply the operator $d_{j+2} \cdots d_{m+1}$ on $y = x_1 \vee x_2$, it does not alter the leaves of x_1 and there are two leaves of x_2 which survive in the resulting tree and more over these are not deleted by applying the operator $d_0 d_1 \cdots d_{j-2}$ on the result. Since $j = \ell + 1$, it is now clear that $R_2^{j-1}(y)$ must be of the form $[0] \vee [1] = [21]$. The case $o_j^y = \vdash$ is similar.

(c) $R_2^0(y) = d_{p+1}(y)$, is immediate from the definition.

(d) $R_1^0(y)$ is [21] or [12] according to as o_{p+1}^y is $\dashv$ or $\vdash$.

To see (d), let $o_{p+1}^y = \dashv$. Then y is not of the form $y_1 \vee [0]$. Thus, if $y = x_1 \vee x_2$, then $deg\ x_2 \geq 1$ and the last two leaves of x_2 cannot be deleted by applying $R_1^0 = d_1 d_2 \cdots d_{p-1}$ on y. It follows that R_1^0 must be of the form $[0] \vee [1]$ as $R_1^0(y) \in Y_2$. Thus $R_1^0(y) = [21]$. Now if $o_{p+1}^y = \vdash$, y is of the form $y_1 \vee [0]$, for some tree $y_1 \in Y_p$. This implies that $R_1^0(y) = d_1 d_2 \cdots d_{p-1}(y) = [12]$.

(e) $R_2^1(y) = d_0 y$, is again immediate from the definition.

(f) $R_1^1(y)$ is [21] or [12] according as $o_0^y = \dashv$ or $o_0^y = \vdash$.

To see (f), we note that $R_1^1 = d_2 \cdots d_p$. If $o_0^y = \vdash$, then y is not of the form $[0] \vee y_1$, for some tree $y_1 \in Y_p$. Thus if $y = x_1 \vee x_2$, then $deg\ x_1 \geq 1$ and the first two leaves of x_1 cannot be deleted by applying R_1^1 on y. It follows that $R_1^1(y)$ must be of the form $[1] \vee [0] = [12]$. Again, if $o_0^y = \dashv$, y must be of the form $[0] \vee y_1$, for some tree $y_1 \in Y_p$. So, $R_1^1(y) = d_2 \cdots d_p(y) = [21]$.

The lemma now follows from the above observations. ∎

We conclude this chapter by establishing a relationship between the pre-Lie product, the $*$ product and the coboundary maps of the cochain complex.

Theorem 5.3.7 *Let D be a dialgebra over a field K. If $f \in CY^p(D, D)$ and $g \in CY^q(D, D)$, then*

$$f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g = (-1)^{q(p-1)} f * g + (-1)^{q-1} g * f.$$

Proof. From Lemma 5.3.6, we have

$$\begin{aligned} f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g &= [(-1)^{q-1} f \circ (\pi \circ g) - f \circ (g \circ \pi)] \\ &\quad -[(-1)^{p+q} \pi \circ (f \circ g) - (f \circ g) \circ \pi] \\ &\quad +(-1)^{q-1}[(-1)^{p-1} (\pi \circ f) \circ g - (f \circ \pi) \circ g]. \end{aligned}$$

As $(CY^*(D,D),\circ)$ is a pre-Lie ring by Corollary 5.2.10, we have

$$\begin{aligned} f\circ\delta g-\delta(f\circ g)+(-1)^{q-1}\delta f\circ g &= (-1)^{p+q}[(\pi\circ f)\circ g-\pi\circ(f\circ g)] \\ &= (-1)^{p+q}[\textstyle\sum(-1)^{(p-1)i+(q-1)j}(\pi\circ_i f)\circ_j g] \end{aligned}$$

where the sum is over those i,j such that $0\le j\le i-1$ or $j=p$, corresponding to $i=0$. The last equality follows from Theorem 5.2.9. Note that the degrees of π, f, g are respectively $1, p-1$ and $q-1$. Hence

$$\begin{aligned} f\circ\delta g-\delta(f\circ g)+(-1)^{q-1}\delta f\circ g &= (-1)^{p+q}[(-1)^{(q-1)p}(\pi\circ_0 f)\circ_p g+ \\ &\quad (-1)^{p-1}(\pi\circ_1 f)\circ_0 g)] \\ &= (-1)^{q(p-1)}(\pi\circ_0 f)\circ_p g+(-1)^{q-1}(\pi\circ_0 g)\circ_q f \\ &= (-1)^{q(p-1)}f*g+(-1)^{q-1}g*f. \end{aligned}$$

This completes the proof of the theorem. ■

Chapter 6

Obstruction cocycles

6.1 Introduction

We devote this chapter solely to the proofs of Theorems 2.5.1 and 4.3.1 using the products introduced in chapter 5 on the cochain complex $CY^*(D, D)$ for a dialgebra D. We first express the obstuction cochains introduced in chapters 2 and 4 in terms of the pre-Lie product and the $*$ product respectively, and then using the results proved in chapter 5 show that these obstuction cochains are actually cocycles. It should be mentioned here that Theorems 2.5.1 and 4.3.1 are formal properties of pre-Lie systems and comp-algebras, proved in general set-up by M. Gerstenhaber.

6.2 Integrability of 2-cocycle

In this section we interpret the obstruction cochains G, as defined in chapter 2, section 5, in terms of the pre-Lie product introduced in the previous chapter.

Let D be a dialgebra, F_λ and F_μ be any two 2-cochains in $CY^2(D, D)$. We first observe that by definition of the pre-Lie product

$$
\begin{aligned}
F_\lambda \circ F_\mu([321];a,b,c) &= (F_\lambda \circ_0 F_\mu \\
&\quad - F_\lambda \circ_1 F_\mu)([321];a,b,c) \\
&= F_\lambda(R_1^0([321];F_\mu(R_2^0([321];a,b),c)- \\
&\quad F_\lambda(R_1^1([321];a,F_\mu(R_2^1([321];b,c)) \\
&= F_\lambda(d_1([321];F_\mu(d_3([321];a,b),c) \\
&\quad - F_\lambda(d_2([321];a,F_\mu(d_0([321];b,c)) \\
&= F_\lambda([21];F_\mu([21];a,b),c) - F_\lambda([21];a,F_\mu([21];b,c)) \\
&= F_\lambda^\ell(F_\mu^\ell(a,b),c) - F_\lambda^\ell(a,F_\mu^\ell(b,c)).
\end{aligned}
$$

$$
\begin{aligned}
F_\lambda \circ F_\mu([312];a,b,c) &= (F_\lambda \circ_0 F_\mu \\
&\quad - F_\lambda \circ_1 F_\mu)([312];a,b,c) \\
&= F_\lambda(R_1^0([312];F_\mu(R_2^0([312];a,b),c) \\
&\quad - F_\lambda(R_1^1([312];a,F_\mu(R_2^1([312];b,c)) \\
&= F_\lambda(d_1([312];F_\mu(d_3([312];a,b),c) \\
&\quad - F_\lambda(d_2([312];a,F_\mu(d_0([312];b,c)) \\
&= F_\lambda([21];F_\mu([21];a,b),c) - F_\lambda([21];a,F_\mu([12];b,c)) \\
&= F_\lambda^\ell(F_\mu^\ell(a,b),c) - F_\lambda^\ell(a,F_\mu^r(b,c)).
\end{aligned}
$$

$$
\begin{aligned}
F_\lambda \circ F_\mu([131]; a, b, c) &= (F_\lambda \circ_0 F_\mu \\
&\quad - F_\lambda \circ_1 F_\mu)([131]; a, b, c) \\
&= F_\lambda(R_1^0([131]; F_\mu(R_2^0([131]; a, b), c) \\
&\quad - F_\lambda(R_1^1([131]; a, F_\mu(R_2^1([131]; b, c)) \\
&= F_\lambda(d_1([131]; F_\mu(d_3([131]; a, b), c) \\
&\quad - F_\lambda(d_2([131]; a, F_\mu(d_0([131]; b, c)) \\
&= F_\lambda([21]; F_\mu([12]; a, b), c) - F_\lambda([12]; a, F_\mu([21]; b, c)) \\
&= F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)).
\end{aligned}
$$

$$
\begin{aligned}
F_\lambda \circ F_\mu([213]; a, b, c) &= (F_\lambda \circ_0 F_\mu \\
&\quad - F_\lambda \circ_1 F_\mu)([213]; a, b, c) \\
&= F_\lambda(R_1^0([213]; F_\mu(R_2^0([213]; a, b), c) \\
&\quad - F_\lambda(R_1^1([213]; a, F_\mu(R_2^1([213]; b, c)) \\
&= F_\lambda(d_1([213]; F_\mu(d_3([213]; a, b), c) \\
&\quad - F_\lambda(d_2([213]; a, F_\mu(d_0([213]; b, c)) \\
&= F_\lambda([12]; F_\mu([21]; a, b), c) - F_\lambda([12]; a, F_\mu([12]; b, c)) \\
&= F_\lambda^r(F_\mu^\ell(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)).
\end{aligned}
$$

$$
\begin{aligned}
F_\lambda \circ F_\mu([123]; a, b, c) &= (F_\lambda \circ_0 F_\mu \\
&\quad - F_\lambda \circ_1 F_\mu)([123]; a, b, c) \\
&= F_\lambda(R_1^0([123]; F_\mu(R_2^0([123]; a, b), c) \\
&\quad - F_\lambda(R_1^1([123]; a, F_\mu(R_2^1([123]; b, c)) \\
&= F_\lambda(d_1([123]; F_\mu(d_3([123]; a, b), c) \\
&\quad - F_\lambda(d_2([123]; a, F_\mu(d_0([123]; b, c)) \\
&= F_\lambda([12]; F_\mu([12]; a, b), c) - F_\lambda([12]; a, F_\mu([12]; b, c)) \\
&= F_\lambda^r(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)).
\end{aligned}
$$

Thus, combining the above expressions, one has

$$
(F_\lambda \circ F_\mu)(y; a, b, c) = \begin{cases}
F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^\ell(b, c)) & \text{if } y = [321] \\
F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^r(b, c)) & \text{if } y = [312] \\
F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)) & \text{if } y = [131] \\
F_\lambda^r(F_\mu^\ell(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) & \text{if } y = [213] \\
F_\lambda^r(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) & \text{if } y = [123]
\end{cases}
$$

for all $a, b, c \in D$. Thus equation (2.11_ν)-(2.15_ν) can be rewritten as

$$
\delta F_\nu = \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu
$$

and the obstruction cochain $G \in CY^3(D, D)$ as defined in chapter 2, section 5 is given by

$$
G = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda \circ F_\mu .
$$

To prove Theorem 2.5.1, we shall need the following results from [5].

Lemma 6.2.1 (**Lemma 1**[3]) *Let f, g be elements of a right graded pre-Lie ring B, with g homogenous of odd degree. Suppose either*

(i) B has no element of order 2, i.e., $x \in B$ and $2x = 0$ implies $x = 0$, or

(ii) B is the pre-Lie ring of a pre-Lie system $\{V_m, \circ_i\}$. Then $(f \circ g) \circ g = f \circ (g \circ g)$.

We are now in a position to prove the first main theorem about obstruction cochains related to integrability of 2-cocycle in the case of deformation.

Proof of Theorem 2.5.1 By Theorem 5.3.7

$$\delta(F_\lambda \circ F_\mu) = F_\lambda \circ \delta F_\mu - \delta F_\lambda \circ F_\mu + (\pi \circ_0 F_\mu) \circ_2 F_\lambda - (\pi \circ_0 F_\lambda) \circ_2 F_\mu.$$

Hence,

$$\begin{aligned}
\delta G &= \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \delta(F_\lambda \circ F_\mu) \\
&= \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} (F_\lambda \circ \delta F_\mu - \delta F_\lambda \circ F_\mu) \\
&= \sum_{\substack{\alpha+\beta+\lambda=n \\ \alpha,\lambda,\mu>0}} [F_\alpha \circ (F_\beta \circ F_\lambda) - (F_\alpha \circ F_\beta) \circ F_\lambda].
\end{aligned}$$

By Lemma 6.2.1, we may assume that $\beta \neq \lambda$ in the term $F_\alpha \circ (F_\beta \circ F_\lambda) - (F_\alpha \circ F_\beta) \circ F_\lambda$. Now as in Proposition 3 of [6], the above sum can be written as a sum of terms of the form

$$[F_\alpha \circ (F_\beta \circ F_\lambda + F_\lambda \circ F_\beta) - ((F_\alpha \circ F_\beta) \circ F_\lambda + (F_\alpha \circ F_\lambda) \circ F_\beta)]$$

where $\alpha + \beta + \lambda = n$, $\alpha, \beta, \lambda > 0$ and each of these term vanishes by (ii) of Theorem 5.2.9. Hence $\delta G = 0$. Note that the cohomology class of G is zero if and only if $G = \delta F_n$ for some $F_n \in CY^2(D, D)$. Hence the last statement follows. This completes the proof.

6.3 Integrability of 1-cocycle

In this section, we interpret the obstructions to derivations in terms of the associative $*$ product, and show that these obstruction cochains are cocycles. Recall from chapter 4, section 3 that if a derivation ψ_1 has been extended to a truncated automorphism $\Psi_t = \sum_{i=0}^{n-1} \psi_i t^i$ of D_Q, then the primary obstruction is the 2-cochain F defined by

$$F(y; a, b) = \begin{cases} \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b) & \text{if } y = [21] \\ \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b) & \text{if } y = [12]. \end{cases}$$

Now we observe that

$$\begin{aligned} (\psi_\lambda * \psi_\mu)([21]; a, b) &= (\pi \circ_0 \psi_\lambda) \circ_1 \psi_\mu([21]; a, b) \\ &= \pi(R_1^0 R_1^1[21], \psi_\lambda([1]; a), \psi_\mu([1]; a)) \\ &= \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b), \end{aligned}$$

where π is the 2-cochain as defined in 5.3.1 and

$$\begin{aligned} R_1^0 &= R_1^0(2; 2, 1) : Y_2 \longrightarrow Y_2 \\ R_1^1 &= R_1^1(2; 2, 1) : Y_2 \longrightarrow Y_2 \end{aligned}$$

are both identity maps.

Similarly,

$$\begin{aligned} (\psi_\lambda * \psi_\mu)([12]; a, b) &= (\pi \circ_0 \psi_\lambda) \circ_1 \psi_\mu([12]; a, b) \\ &= \pi(R_1^0 R_1^1[12], \psi_\lambda([1]; a), \psi_\mu([1]; a)) \\ &= \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b). \end{aligned}$$

where

$$\begin{aligned} R_1^0 &= R_1^0(2; 2, 1) : Y_2 \longrightarrow Y_2 \\ R_1^1 &= R_1^1(2; 2, 1) : Y_2 \longrightarrow Y_2 \end{aligned}$$

are both identity maps.

Finally, we prove the second main theorem about obstruction cochains related to integrability of 1-cocycle in the case of automorphism.

Proof of Theorem 4.3.1. From above, we observe that the obstruction cochain F, in terms of the $*$ product is given by

$$F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu$$

and equations (4.7_ν) and (4.8_ν) can be written as

$$\delta\psi_\nu = -\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu$$

for $\nu = 1, 2, \ldots, n-1$. Hence

$$\begin{aligned} \delta F &= \textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \delta(\psi_\lambda * \psi_\mu) \\ &= -\textstyle\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \left\{ \left(\sum_{\substack{\alpha+\beta=\lambda \\ \alpha,\beta>0}} \psi_\alpha * \psi_\beta\right) * \psi_\mu - \psi_\lambda * \left(\sum_{\substack{\alpha+\beta=\mu \\ \alpha,\beta>0}} \psi_\alpha * \psi_\beta\right) \right\} \\ &= -\textstyle\sum_{\substack{\alpha+\beta+\mu=n \\ \alpha,\beta,\mu>0}} \{(\psi_\alpha * \psi_\beta) * \psi_\mu - \psi_\alpha * (\psi_\beta * \psi_\mu)\} \\ &= 0 \end{aligned}$$

as $*$ is associative. Here we note that the cohomology class of F is zero if and only if $F = \delta\psi_n$ for some $\psi_n \in CY^1(D, D)$. Hence, the last statement follows. This completes the proof.

Chapter 7

G-algebras and dialgebra cohomology

7.1 Introduction

It is well known since the pioneering work of M. Gerstenhaber [5] that the Hochschild cochain complex $C^*(A, A)$ of an associative algebra A admits a brace algebra structure. Moreover, in [12], M. Gerstenhaber and A. A. Voronov have shown that $C^*(A, A)$ admits a homotopy G-algebra structure which induces the G-algebra structure on the Hochschild cohomology as introduced in [5]. These structures on $C^*(A, A)$ are in fact induced from a natural operad structure on $C^*(A, A)$, where only the non-Σ part of the operad is responsible for inducing the above structures.

The aim of this chapter is to show that as in the case of Hochschild complex, the dialgebra cochain complex $CY^*(D, D)$, with the differential altered by a sign admits a homotopy G-algebra structure which comes from a non-Σ operad structure on $CY^*(D, D)$, for a dialgebra D. This homotopy G-algebra structure on the cochain level in turn induces a G-algebra structure on the cohomology $HY^*(D, D)$.

7.2 Braces for dialgebra complex

In this section, we generalize $\circ_i$ products as introduced in chapter 5 to define braces or multilinear operations in $CY^*(D,D)$ of a dialgebra D. These generalized $\circ_i$ products endow $CY^*(D,D)$ with a brace algebra structure. We recall from [12] the definition of a brace algebra.

Definition 7.2.1 A brace algebra is a graded vector space with a collection of braces (or multilinear operations) $x\{x_1, x_2, \ldots, x_n\}$ of degree $-n$ satisfying the identity (brace identity)

$$\begin{aligned} x\{x_1, x_2, \ldots, x_m\}\{y_1, y_2, \ldots, y_n\} \;=\; & \textstyle\sum_{0\le i_1\le j_1\le i_2\ldots\le i_m\le j_m\le n}(-1)^{\epsilon} x\{y_1, \ldots, y_{i_1}, \\ & x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_2}, \\ & x_2\{y_{i_2+1}, \ldots, y_{j_2}\}, y_{j_2+1}, \ldots, y_{i_m}, \\ & x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\} \end{aligned}$$

where $x\{\}$ is understood as just x, $\deg x\{x_1, \ldots, x_n\} = \deg x + \sum_{i=1}^{n} \deg x_i - n$, $|x| = \deg x - 1$, and $\epsilon = \sum_{p=1}^{m} |x_p| \sum_{q=1}^{i_p} |y_q|$.

Generalizing the maps R_1^i and R_2^i, as introduced in Definition 5.2.1, we define the following operations on the set of planar binary trees.

Definition 7.2.2 Let $n, i_1, i_2, \ldots, i_r$, $m_1, m_2, \ldots, m_r$ be non-negative integers with $n, m_1, \ldots, m_r \geq 1$ such that

$$0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{r-1} + m_{r-1} \leq i_r, i_r + m_r \leq N = n + \sum_{1}^{r} m_i - r.$$

For each j, $0 \leq j \leq r$ we define maps

$$R_{j+1}^{i_1,\ldots,i_r}(N; n, m_1, \ldots, m_r) : Y_N \longrightarrow Y_{m_j},$$

with $m_0 = n$ in the following way. For $j = 0$,

$$R_1^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r) = \Pi_{\substack{m_\ell \geq 2 \\ 1 \leq \ell \leq r}} (d_{i_\ell+1} \cdots d_{i_\ell+m_\ell-1})$$

if $2 \leq m_0 < N$, where Π stands for composition of terms and $R_1^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)$ is the identity or the obvious constant map according as m_0 is N or 1.

For $1 \leq j \leq r$, if $2 \leq m_j < N$ then

$$R_{j+1}^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r) = \begin{cases} (d_0 \cdots d_{i_j-1})(d_{i_j+m_j+1} \cdots d_N) & i_j \geq 1, i_j + m_j + 1 \leq N \\ (d_{m_j+1} \cdots d_N) & i_j = 0 \\ (d_0 \cdots d_{i_j-1}) & i_j + m_j + 1 > N \end{cases}$$

and $R_{j+1}^{i_1,\dots,i_r}(N; n, m_1, \dots, m_r)$ is identity or the obvious constant map according as $m_j = N$ or $m_j = 1$.

We may note here that for $r = 1$, the above definition coincides with the Definition 5.2.1. The maps defined above induce multilinear operations on the direct sum of the cochain modules $CY^*(D, D)$ of a dialgebra D. We show that these multilinear maps endow $CY^*(D, D)$ with the structure of a brace algebra.

Definition 7.2.3 Let D be a dialgebra over a field K. For non-negative integers $n, i_1, \dots i_r, m_1, \dots, m_r$ with $0 \leq i_1$, $i_1 + m_1 \leq i_2, \dots, i_{r-1} + m_{r-1} \leq i_r$, $i_r + m_r \leq N = n + \sum_1^r m_i - r$, the multilinear maps

$$\circ_{i_1,\dots,i_r} : CY^n(D, D) \otimes \bigotimes_{j=1}^{r} CY^{m_j}(D, D) \longrightarrow CY^N(D, D)$$

are defined as follows. Let $f \in CY^n(D, D), g_j \in CY^{m_j}(D, D), 1 \leq j \leq r$. For $y \in Y_N$, and $x_1, \ldots, x_N \in D$,

$$\begin{aligned} & f \circ_{i_1,\ldots,i_r} (g_1, \ldots, g_r)(y; x_1, \ldots, x_N) \\ = \;& f(R_1^{i_1,\ldots,i_r}(N; n, m_1, \ldots, m_r)y; x_1, \ldots, x_{i_1}, \\ & g_1(R_2^{i_1,\ldots,i_r}(N; n, m_1, \ldots, m_r)y; x_{i_1+1}, \ldots, x_{i_1+m_1}), \ldots, \\ & g_r(R_{r+1}^{i_1,\ldots,i_r}(N; n, m_1, \ldots, m_r)y; x_{i_r+1}, \ldots, x_{i_r+m_r}), \ldots, x_N). \end{aligned}$$

In the above definition, if for some j, $m_j = 0$, then $g_j \in CY^0(D, D)$ $\cong \mathrm{Hom}_K(K, D) = D$ and the corresponding input in f is simply g_j.

Next we use these generalized $\circ_i$ products to define braces on $CY^*(D, D)$ as follows.

Definition 7.2.4 For $f \in CY^n(D, D)$, $g_\nu \in CY^{m_\nu}(D, D)$, $\nu = 1, \ldots, r$,

$$f\{g_1, \ldots, g_r\} = \sum_{i_1,\ldots,i_r} (-1)^\eta f \circ_{i_1,\ldots,i_r} (g_1, \ldots, g_r)$$

where the summation extends over all $i_1, \ldots, i_r$ satisfying the inequalities as mentioned in 7.2.3, so that $\circ_{i_1,\ldots i_r}$ is defined, $\eta = \sum_{\nu=1}^r |g_\nu| i_\nu$, and $|g_\nu| = \deg g_\nu - 1 = m_\nu - 1$.

Remark 7.2.5 It may be noted that by the definition of braces on $CY^*(D, D)$, $f\{g\}$ coincides with the pre-Lie product $f \circ g$ as introduced in Definition 5.2.7.

Henceforth, we shall use the symbol $f \circ g$ in order to denote $f\{g\}$. The following proposition will follow from the Lemma 7.4.1.

Proposition 7.2.6 *The braces as defined above make the dialgebra cochain complex $CY^*(D, D)$, of a dialgebra D, into a brace algebra.*

7.3 Operad structure

In this section we show that the dialgebra complex $CY^*(D, D)$ of a dialgebra D admits the structure of a non-Σ operad. Later in this chapter, we use this operad structure to prove that the braces as defined in Definition 7.2.4 make $CY^*(D, D)$ into a brace algebra. We recall from [20] the following definition.

Definition 7.3.1 A non-Σ operad $\mathcal{C}$ of K-vector spaces consists of vector spaces $\mathcal{C}(j)$, $j \geq 0$, together with a unit map $\eta : K \longrightarrow \mathcal{C}(1)$ and multilinear maps

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \longrightarrow \mathcal{C}(j)$$

for $k \geq 1; j_s \geq 0$ and $j = \sum_{s=1}^{k} j_s$. The maps γ are required to be associative and unital in the following sense.

(a) The following associative diagram commutes, where $\sum j_s = j$, $\sum i_t = i$, $p_s = j_1 + j_2 + \cdots + j_s$ and $q_s = i_{p_{s-1}+1} + \cdots + i_{p_s}$, $1 \leq s \leq k$:

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{C}(j_s)) \otimes (\bigotimes_{r=1}^{j} \mathcal{C}(i_r)) & \xrightarrow{\gamma \otimes \mathrm{Id}} & \mathcal{C}(j) \otimes (\bigotimes_{r=1}^{j} \mathcal{C}(i_r)) \\
\Big\downarrow \text{shuffle} & & \downarrow \gamma \\
 & & \mathcal{C}(i) \\
 & & \uparrow \gamma \\
\mathcal{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{C}(j_s) \otimes (\bigotimes_{\nu=1}^{j_s} \mathcal{C}(i_{p_{s-1}+\nu}))) & \xrightarrow{\mathrm{Id} \otimes (\otimes_s \gamma)} & \mathcal{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathcal{C}(q_s))
\end{array}$$

(b) The following two unit diagrams commute :

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes K^k & \xrightarrow{\cong} & \mathcal{C}(k) \\
\mathrm{Id} \otimes \eta^k \Big\downarrow & \nearrow \gamma & \\
\mathcal{C}(k) \otimes \mathcal{C}(1)^k & &
\end{array}$$

$$\begin{array}{ccc} K \otimes \mathcal{C}(j) & \xrightarrow{\cong} & \mathcal{C}(j) \\ {\scriptstyle \eta\otimes \mathrm{Id}}\downarrow & \nearrow {\scriptstyle \gamma} & \\ \mathcal{C}(1)\otimes\mathcal{C}(j) & & \end{array}$$

Next we proceed to show that $CY^*(D, D)$ admits a non-Σ operad structure. In order to define the required multilinear maps, we first introduce the following maps.

Definition 7.3.2 Given an integer j, with $j = \sum_{r=1}^{k} j_r$, $k \geq 1$ and $j_r \geq 1$, define maps

$$\begin{aligned} \Gamma^0(k; j_1, \ldots, j_k) &: Y_j \longrightarrow Y_k \\ \Gamma^r(k; j_1, \ldots, j_k) &: Y_j \longrightarrow Y_{j_r}, \quad 1 \leq r \leq k \end{aligned}$$

by

$$\begin{aligned} \Gamma^0(k; j_1, \ldots, j_k) &= d_1 \cdots d_{j_1-1} d_{j_1+1} \cdots d_{j_1+j_2-1} d_{j_1+j_2+1} \cdots d_{\sum_{s=1}^{r} j_s - 1} d_{\sum_{s=1}^{r} j_s + 1} \cdots \\ &\quad d_{\sum_{s=1}^{k-1} j_s - 1} d_{\sum_{s=1}^{k-1} j_s + 1} \cdots d_{j-1} \\ &= d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_r} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1} \quad \text{for all } 1 \leq r \leq k-1, \end{aligned}$$

and

$$\begin{aligned} \Gamma^r(k; j_1, \ldots, j_k) &= d_0 \cdots d_{\sum_{s=1}^{r-1} j_s - 1} d_{\sum_{s=1}^{r} j_s + 1} \cdots d_{\sum_{s=1}^{k} j_s} \\ &= d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_j \end{aligned}$$

where $p_r = j_1 + j_2 + \cdots + j_r$, $1 \leq r \leq k$, and the symbol $\check{d}_i$ appearing in any expression means that the map d_i has been omitted.

Remark 7.3.3 Given integers j, $k \geq 1$, $j_r \geq 1$ with $j = \sum_{r=1}^{k} j_r$, we shall often write the map $\Gamma^r(k; j_1, \ldots, j_k)$ simply as Γ^r, for all $r = 0, 1, \ldots, k$. However, to avoid confusion we shall write the maps Γ^r explicitly, along with the values of k, $j_1, \ldots, j_k$, whenever necessary.

Our main aim is to prove the following result.

Theorem 7.3.4 *For a dialgebra D over a field K, the dialgebra complex $CY^*(D,D)$ is a non-Σ operad of K-vector spaces.*

In order to prove the above theorem we make use of the following lemma.

Lemma 7.3.5 *Let $j_r \geq 1$, $1 \leq r \leq k$ be integers with $j = \sum_{r=1}^{k} j_r$. Let $i = \sum_{t=1}^{j} i_t$, with integers $i_t \geq 1$. Set $p_s = j_1 + j_2 + \cdots + j_s$ and $q_s = i_{p_{s-1}+1} + \cdots + i_{p_s}$. Then for $1 \leq s \leq j_r$, $1 \leq r \leq k$ the corresponding maps*

$$\Gamma^0(k; j_1, \dots, j_k) : Y_j \longrightarrow Y_k \qquad \Gamma^r(k; j_1, \dots, j_k) : Y_j \longrightarrow Y_{j_r}$$

$$\Gamma^0(j; i_1, \dots, i_j) : Y_i \longrightarrow Y_j \qquad \Gamma^{p_{r-1}+s}(j; i_1, \dots, i_j) : Y_i \longrightarrow Y_{i_{p_{r-1}+s}}$$

$$\Gamma^0(k; q_1, \dots, q_k) : Y_i \longrightarrow Y_k \qquad \Gamma^r(k; q_1, \dots, q_k) : Y_i \longrightarrow Y_{q_r}$$

$$\Gamma^0(j_r; i_{p_{r-1}+1}, \dots, i_{p_{r-1}+j_r}) : Y_{q_r} \longrightarrow Y_{j_r}$$

$$\Gamma^s(j_r; i_{p_{r-1}+1}, \dots, i_{p_{r-1}+j_r}) : Y_{q_r} \longrightarrow Y_{i_{p_{r-1}+s}}$$

satisfy

(a) $\Gamma^0(k; j_1, \dots, j_k)\Gamma^0(j; i_1, \dots, i_j) = \Gamma^0(k; q_1, \dots, q_k)$,

(b) $\Gamma^r(k; j_1, \dots, j_k)\Gamma^0(j; i_1, \dots, i_j) = \Gamma^0(j_r; i_{p_{r-1}+1}, \dots, i_{p_{r-1}+j_r})\Gamma^r(k; q_1, \dots, q_k)$,

(c) $\Gamma^{p_{r-1}+s}(j; i_1, \dots, i_j) = \Gamma^s(j_r; i_{p_{r-1}+1}, \dots, i_{p_{r-1}+j_r})\Gamma^r(k; q_1, \dots, q_k)$.

Proof. The proof of the above lemma is a repeated application of the simplicial identity $d_i d_j = d_{j-1} d_i$, $i < j$.

By Definition 7.3.2, the operator $\Gamma^0\Gamma^0$ on the left hand side of the equality (a) is given by two strings of operators as

$$\begin{aligned}\Gamma^0\Gamma^0 \quad = \quad & (d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1}) \\ & (d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1})\end{aligned}$$

Now that operator d_1 at the extreme left in

$$d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1}$$

can be brought to the extreme right by successive application of $d_i d_j = d_{j-1} d_i, i < j$, yielding

$$d_1 \cdots \check{d}_{p_1 - 1} \cdots \check{d}_{p_2 - 1} \cdots \check{d}_{p_{k-1} - 1} \cdots d_{p_k - 2} d_1$$

Now, by applying $d_{j-1} d_i = d_i d_j, i < j$, the operator d_1 at the right of the above string can be pushed into the string

$$d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1 + i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1},$$

to recover the operator d_{i_1}, and thus yielding

$$\Gamma^0 \Gamma^0 = (d_1 \cdots \check{d}_{p_1 - 1} \cdots \check{d}_{p_2 - 1} \cdots \check{d}_{p_{k-1} - 1} \cdots d_{p_k - 2})(d_1 \cdots d_{i_1} \cdots \check{d}_{i_1 + i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1})$$

We repeat the above method, each time starting with the operator d_1 at the left of the first string to recover an omitted operator in the second string. After $(p_1 - 1)$ number of steps, we get

$$\begin{aligned} \Gamma^0 \Gamma^0 &= (d_2 \cdots d_{p_2 - p_1} d_{p_2 - (p_1 - 2)} \cdots d_{p_r - p_1} d_{p_r - (p_1 - 2)} \cdots \\ &\quad d_{p_{k-1} - p_1} d_{p_{k-1} - (p_1 - 2)} \cdots d_{p_k - p_1})(d_1 \cdots d_{i_1} \cdots d_{i_1 + i_2} \cdots \\ &\quad \check{d}_{q_1} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1}), \end{aligned}$$

since $q_1 = i_1 + \cdots i_{p_1}$. Again we apply the above method starting with the operators $d_2, \ldots, d_{p_2 - p_1}$, at the left end of the first string to replace all the omitted operators between $d_{q_1 + 1}$ and $d_{q_1 + q_2 - 1}$, of the second string. Proceeding this way, all the operators of the first string can be exhausted to yield

$$\begin{aligned} \Gamma^0 \Gamma^0 &= d_1 \cdots d_{q_1 - 1} d_{q_1 + 1} \cdots d_{q_1 + q_2 - 1} d_{q_1 + q_2 + 1} \cdots \\ &\quad d_{\sum_{s=1}^{r} q_s - 1} d_{\sum_{s=1}^{r} q_s + 1} \cdots d_{\sum_{s=1}^{k-1} q_s - 1} d_{\sum_{s=1}^{k-1} q_s + 1} \cdots d_{i-1} \end{aligned}$$

Observe that $\sum_{s=1}^{k} q_s = i$.

But this is the operator Γ^0 of the right hand side of the equality (a). This proves part (a).

To prove (b) first note that, the operator Γ^0on the right hand side of the equality (b) can be expressed as

$$\Gamma^0 = d_1 \cdots \check{d}_{q(r,1)} \cdots \check{d}_{q(r,2)} \cdots \check{d}_{q(r,j_r-1)} \cdots d_{q_r-1},$$

where $q(r,s) = i_{p_{r-1}+1} + i_{p_{r-1}+2} + \cdots + i_{p_{r-1}+s}$. Hence, by Definition 7.3.2, we have

$$\begin{aligned}\Gamma^0\Gamma^r = \; & (d_1 \cdots \check{d}_{q(r,1)} \cdots \check{d}_{q(r,2)} \cdots \check{d}_{q(r,j_r-1)} \cdots d_{q_r-1})(d_0 \cdots d_{\sum_{s=1}^{r-1} q_s-1} d_{\sum_{s=1}^{r} q_s+1} \cdots \\ & \cdots d_{\sum_{s=1}^{k} q_s=i})\end{aligned}$$

and

$$\Gamma^r\Gamma^0 = (d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_{p_k})(d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{s} i_t} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1}).$$

Now, $\Gamma^r\Gamma^0$ comprises of two strings of operators, denoted by the parentheses. Using the pre-simplicial identity $d_{j-1}d_i = d_i d_j,\ i < j$, the operator d_{p_k} of the first string can be shifted to the right of the second string, in the form of d_i, to get

$$\Gamma^r\Gamma^0 = (d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_{p_k-1})(d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} d_i).$$

This is because the number of operators in the second string is $i-j$ and $p_k+i-j=i$.

Now let us look at the operator d_{p_k-1} in the first string of the above expression of $\Gamma^r\Gamma^0$ and we keep on commuting it with the operators in the second string by using $d_{j-1}d_i = d_i d_j,\ i < j$ until we reach the operator $d_{\sum_{t=1}^{j-1} i_t - 1}$. The operator immediately before it is then $d_{p_k-1+\epsilon}$, where $\epsilon = \sum_{t=1}^{j-1} i_t - j$ as there are ϵ number of operators before $d_{\sum_{t=1}^{j-1} i_t - 1}$. If we apply $d_{j-1}d_i = d_i d_j,\ i < j$, once more then we get back the omitted operator $d_{\sum_{t=1}^{j-1} i_t}$ and $\Gamma^r\Gamma^0$ becomes

$$\Gamma^r\Gamma^0 = (d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_{p_k-2})(d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{s} i_t} \cdots \check{d}_{\sum_{t=1}^{j-2} i_t} \cdots \\ \cdots d_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} d_i).$$

Similarly all the operators $d_{p_r+1}, \ldots, d_{p_k-2}$ can be embedded in the second string to replace the omitted terms $d_{\sum_{t=1}^{p_r+1} i_t}, \ldots, d_{\sum_{t=1}^{j-2} i_t}$, respectively, using the identity $d_{j-1}d_i = d_i d_j,\ \ i < j$.

So, $\Gamma^r\Gamma^0$ now takes the form

$$(d_0 \cdots d_{p_{r-1}-1})(d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots d_{\sum_{t=1}^{p_r} i_t - 1} \check{d}_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i).$$

Now observe that by the simplicial identity $d_i d_j = d_{j-1} d_i, i < j$, the string

$$d_0 d_1 \cdots d_{p_{r-1}-1}$$

is equivalent to $d_0 d_1 \cdots d_{p_{r-1}-2} d_1$. Thus,

$$\Gamma^r\Gamma^0 = (d_0 d_1 \cdots d_{p_{r-1}-2} d_1)(d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots d_{\sum_{t=1}^{p_r} i_t - 1} \check{d}_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i).$$

Now again by applying $d_{j-1}d_i = d_i d_j, i < j$, the operator d_1 at the extreme right of the string $d_0 d_1 \cdots d_{p_{r-1}-2} d_1$ can be pushed into the string

$$d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots d_{\sum_{t=1}^{p_r} i_t - 1} \check{d}_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i$$

to recover the operator d_{i_1}. Thus $\Gamma^r\Gamma^0$ becomes

$$\Gamma^r\Gamma^0 = (d_0 d_1 \cdots d_{p_{r-1}-2})(d_1 \cdots d_{i_1} \cdots \check{d}_{i_1+i_2} \cdots d_{\sum_{t=1}^{p_r} i_t - 1} \check{d}_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i).$$

By repeated application of the previous step, all the operators $d_1, \ldots, d_{p_{r-2}-1}$ of the first string in the above expression can be embedded in the second string using

$d_{j-1}d_i = d_i d_j, i < j$, to replace the omitted operators $d_{i_1}, \ldots, d_{\sum_{t=1}^{p_{r-1}-1} i_t}$. Having done this, $\Gamma^r\Gamma^0$ becomes

$$d_0 d_1 \cdots d_{i_1} \cdots d_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{p_{r-1}} i_t} \cdots \check{d}_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t+1} \cdots d_i$$
$$= (d_0 d_1 \cdots d_{\sum_{s=1}^{r-1} q_s - 1})(d_{\sum_{t=1}^{p_{r-1}} i_t+1} \cdots \check{d}_{\sum_{t=1}^{p_{r-1}+u} i_t} \cdots d_{\sum_{t=1}^{p_r} i_t - 1})(d_{\sum_{t=1}^{r} q_t+1} \cdots d_i),$$

as $\sum_{s=1}^{r-1} q_s = \sum_{t=1}^{p_{r-1}} i_t$, where $1 \leq u \leq j_r$.

Now observe that the entire string of operators

$$(d_{\sum_{t=1}^{p_{r-1}} i_t+1} \cdots \check{d}_{\sum_{t=1}^{p_{r-1}+u} i_t} \cdots d_{\sum_{t=1}^{p_r} i_t-1}),$$

$1 \leq u \leq j_r$, can be moved to the extreme left, using $d_i d_j = d_{j-1}d_i, \quad i < j$,. Since there are $\sum_{s=1}^{r-1} q_s$ many operators in the first string, the suffix of each of the operator appearing in the sequence $d_{\sum_{t=1}^{p_{r-1}} i_t+1} \cdots \check{d}_{\sum_{t=1}^{p_{r-1}+u} i_t} \cdots d_{\sum_{t=1}^{p_r} i_t-1}$ will be reduced by $\sum_{s=1}^{r-1} q_s = \sum_{t=1}^{p_{r-1}} i_t$. Having done this we get

$$\begin{aligned}
\Gamma^r\Gamma^0 &= (d_1 \cdots \check{d}_{i_{p_{r-1}+1}} \cdots \check{d}_{(i_{p_{r-1}+1}+i_{p_{r-1}+2})} \cdots \check{d}_{(i_{p_{r-1}+1}+\cdots+i_{p_{r-1}+j_r-1})} \cdots d_{q_r-1}) \\
&\quad (d_0 \cdots d_{\sum_{s=1}^{r-1} q_s-1} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{i=\sum_{s=1}^{k} q_s}) \\
&= (d_1 \cdots \check{d}_{q(r,1)} \cdots \check{d}_{q(r,2)} \cdots \check{d}_{q(r,j_r-1)} \cdots d_{q_r-1})(d_0 \cdots d_{\sum_{s=1}^{r-1} q_s-1} d_{\sum_{s=1}^{r} q_s+1} \cdots \\
&\quad \cdots d_{i=\sum_{s=1}^{k} q_s}),
\end{aligned}$$

as $q(r,s) = i_{p_{r-1}+1} + i_{p_{r-1}+2} + \cdots + i_{p_{r-1}+s}$, which is precisely the expression for $\Gamma^0\Gamma^r$.

Finally, to prove (c), we have by Definition 7.3.2,

$$\Gamma^{p_{r-1}+s} = (d_0 \cdots d_{\sum_{t=1}^{p_{r-1}+s-1} i_t - 1})(d_{\sum_{t=1}^{p_{r-1}+s} i_t+1} \cdots d_i)$$

and

$$\Gamma^s\Gamma^r = (d_0 \cdots d_{q(r,s-1)-1} d_{q(r,s)+1} \cdots d_{q_r})(d_0 \cdots d_{\sum_{s=1}^{r-1} q_s-1} d_{\sum_{s=1}^{r} q_s+1} \cdots d_{\sum_{s=1}^{k} q_s = i}).$$

We note that the expression of $\Gamma^s\Gamma^r$ can be rewritten in the form

$$(d_0 \cdots d_{q(r,s-1)-1} d_{q(r,s)+1} \cdots d_{q_r})(d_0 \cdots d_{\sum_{t=1}^{p_r-1} i_t - 1} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i).$$

Now using $d_{j-i}d_i = d_i d_j,\ i < j$, the operator d_{q_r} can be shifted to the right of the second string by interchanging successively $\sum_{t=1}^{p_r-1} i_t$ number of times, to get

$$\Gamma^s\Gamma^r = (d_0 \cdots d_{q(r,s-1)-1} d_{q(r,s)+1} \cdots d_{q_r-1})(d_0 \cdots d_{\sum_{t=1}^{p_r-1} i_t - 1} d_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i),$$

as $q_r + \sum_{t=1}^{p_r-1} i_t = \sum_{t=1}^{p_r} i_t$. Treating all the operators $d_{q(r,s)+1}, \cdots, d_{q_r-1}$, similarly, we have

$$\Gamma^s\Gamma^r = (d_0 \cdots d_{q(r,s-1)-1})(d_0 \cdots d_{\sum_{t=1}^{p_r-1} i_t - 1} d_{\sum_{t=1}^{p_r-1+s} i_t + 1} \cdots d_{\sum_{t=1}^{p_r} i_t} d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i),$$

as $q(r,s) + 1 + \sum_{t=1}^{p_r-1} i_t = \sum_{t=1}^{p_r-1+s} i_t + 1$. Again, all the operators $d_0, \cdots, d_{q(r,s-1)-1}$ can be moved to the right, by $\sum_{t=1}^{p_r-1} i_t$ places, using $d_{j-1}d_i = d_i d_j,\ i < j$, to yield

$$\begin{aligned}
\Gamma^s\Gamma^r &= (d_0 \cdots d_{\sum_{t=1}^{p_r-1} i_t - 1} d_{\sum_{t=1}^{p_r-1} i_t} \cdots d_{\sum_{t=1}^{p_r-1+s-1} i_t - 1})(d_{\sum_{t=1}^{p_r-1+s} i_t + 1} \cdots d_{\sum_{t=1}^{p_r} i_t} \\
&\quad d_{\sum_{t=1}^{p_r} i_t + 1} \cdots d_i) \\
&= \Gamma^{p_r-1+s}.
\end{aligned}$$

This proves (c) and completes the proof of the lemma. ■

Proof of Theorem 7.3.4. For each $j \geq 0$, set

$$\mathcal{C}(j) = CY^j(D, D) = \mathrm{Hom}_K(K[Y_j] \otimes D^{\otimes j}, D).$$

Note that

$$\begin{aligned}
\mathcal{C}(1) &= \mathrm{Hom}_K(K[Y_1] \otimes D, D) \\
&\cong \mathrm{Hom}_K(D, D).
\end{aligned}$$

Define the unit map $\eta : K \longrightarrow \mathcal{C}(1)$ by $\eta(1) = id_D$. Now, for $k \geq 1, j_r \geq 0$ and $j = \sum j_r$ define multilinear maps

$$(7.1) \qquad \gamma : CY^k(D,D) \otimes \bigotimes_{r=1}^{k} CY^{j_r}(D,D) \longrightarrow CY^j(D,D)$$

as follows : For $f \in CY^k(D,D), g_r \in CY^{j_r}(D,D)$

$$\begin{aligned}
& \gamma(f; g_1, \ldots, g_k)(y; x_1 \ldots, x_j) \\
= \; & f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{j_1}), g_2(\Gamma^2(y); x_{j_1+1}, \ldots, x_{j_1+j_2}), \ldots, \\
& g_k(\Gamma^k(y); x_{\sum_{s=1}^{k-1} j_s+1}, \ldots, x_{\sum_{s=1}^{k} j_s})) \\
= \; & f(\Gamma^0(y); g_1(\Gamma^1(y); x_1, \ldots, x_{p_1}), g_2(\Gamma^2(y); x_{p_1+1}, \ldots, x_{p_2}), \ldots, \\
& g_k(\Gamma^k(y); x_{p_{k-1}+1}, \ldots, x_{p_k}))
\end{aligned}$$

where $\Gamma^0 = \Gamma^0(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_k$, and $\Gamma^r = \Gamma^r(k; j_1, \ldots, j_k) : Y_j \longrightarrow Y_{j_r}$ are the maps as defined in 7.3.2, $x_1, \ldots, x_j \in D$ and $y \in Y_j$.

It may be noted here that if $j_r = 0$ for some r, then $g_r \in CY^0(D,D) \cong \mathrm{Hom}_K(K,D) = D$ and the corresponding input in f is simply g_r.

To check associativity, let $f \in CY^k(D,D), g_r \in CY^{j_r}(D,D), r = 1, \ldots, k$, and $h_t \in CY^{i_t}(D,D), t = 1, \ldots, j = \sum_{r=1}^{k} j_r$. As in the above lemma, let

$$i = \sum_{t=1}^{j} i_t, p_s = j_1 + j_2 + \cdots + j_s, q_s = i_{p_{s-1}+1} + \cdots + i_{p_s}.$$

Also set $q_{(r,s)} = i_{p_{r-1}+1} + i_{p_{r-1}+2} + \cdots + i_{p_{r-1}+s}$, $1 \leq s \leq j_r$. Then

$$(7.2) \qquad \gamma \circ (\gamma \otimes id)((f; g_1, \ldots, g_k), h_1, h_2, \ldots, h_j) = \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j).$$

On the other hand, shuffle yields

$$((f, g_1, \ldots, g_k), h_1, \ldots, h_j) \xrightarrow{\text{shuffle}} (f, (g_1, h_1, \ldots, h_{j_1}), (g_2, h_{j_1+1}, \ldots, h_{p_2}), \ldots, (g_k, h_{p_{k-1}+1}, \ldots, h_{p_k=j})).$$

Now, composing with $\gamma \circ (id \otimes (\otimes_r \gamma))$ we get,

$$
\begin{aligned}
& \gamma \circ (id \otimes (\otimes_r \gamma)) \circ (\text{shuffle})((f, g_1, \ldots, g_k), h_1, \ldots, h_j) \\
(7.3) \qquad = \; & \gamma(f; \gamma(g_1; h_1, \ldots, h_{p_1}), \gamma(g_2; h_{p_1+1}, \ldots, h_{p_2}), \ldots, \\
& \gamma(g_k; h_{p_{k-1}+1}, \ldots, h_{p_k=j})).
\end{aligned}
$$

To show that (7.2) and (7.3) are the same cochain in $CY^i(D, D)$, let $y \in Y_i$ and $x_1, x_2, \ldots, x_i \in D$. Then,

$$
\begin{aligned}
& \gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_j)(y; x_1, \ldots, x_i) \\
(7.4) \qquad = \; & \gamma(f; g_1, \ldots, g_k)(\Gamma^0 y; h_1(\Gamma^1 y; x_1, \ldots, x_{i_1}), h_2(\Gamma^2 y; x_{i_1+1}, \ldots, x_{i_1+i_2}), \ldots, \\
& h_j(\Gamma^j y; x_{\sum_{t=1}^{j-1} i_t + 1}, \ldots, x_i))
\end{aligned}
$$

where,

$$
\begin{aligned}
\Gamma^0 y &= \Gamma^0(j; i_1, \ldots, i_j) y = d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y, \\
\Gamma^u y &= \Gamma^u(j; i_1, \ldots, i_j) y = d_0 \cdots d_{\sum_{t=1}^{u-1} i_t} d_{\sum_{t=1}^{u} i_t + 1} \cdots d_i y, \quad 1 \le u \le j.
\end{aligned}
$$

Now by definition of γ, as given in (7.1), the equation (7.4) is

$$
\begin{aligned}
= \; & f(\Gamma^0 \Gamma^0 y; g_1(\Gamma^1 \Gamma^0 y; h_1(\Gamma^1 y; x_1, \ldots, x_{i_1}), \ldots, \\
& h_{j_1}(\Gamma^{j_1} y; x_{\sum_{t=1}^{j_1-1} i_t + 1}, \ldots, x_{\sum_{t=1}^{j_1} i_t = q_1})), \ldots, \\
(7.5) \qquad & g_k(\Gamma^k \Gamma^0 y; h_{p_{k-1}+1}(\Gamma^{p_{k-1}+1} y; x_{\sum_{t=1}^{p_{k-1}} i_t + 1}, \ldots, x_{\sum_{t=1}^{p_{k-1}+1} i_t}), \ldots, \\
& h_j(\Gamma^j y; x_{\sum_{t=1}^{j-1} i_t + 1}, \ldots, x_i)))
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma^0 \Gamma^0 y &= \Gamma^0(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j) y \\
&= d_1 \cdots \check{d}_{p_1} \cdots \check{d}_{p_2} \cdots \check{d}_{p_{k-1}} \cdots d_{p_k - 1} d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y
\end{aligned}
$$

and for $1 \le r \le k$

$$
\begin{aligned}
\Gamma^r \Gamma^0 y &= \Gamma^r(k; j_1, \ldots, j_k) \Gamma^0(j; i_1, \ldots, i_j) y \\
&= d_0 \cdots d_{p_{r-1}-1} d_{p_r+1} \cdots d_{p_k} d_1 \cdots \check{d}_{i_1} \cdots \check{d}_{i_1+i_2} \cdots \check{d}_{\sum_{t=1}^{j-1} i_t} \cdots d_{i-1} y.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}(7.6)\qquad & \gamma(f;\gamma(g_1;h_1,\ldots,h_{p_1}),\ldots,\gamma(g_k;h_{p_{k-1}+1},\ldots,h_{p_k=j}))(y;x_1,\ldots,x_i)\\ = \ & f(\Gamma^0 y;\gamma(g_1;h_1,\ldots,h_{p_1})(\Gamma^1 y;x_1,\ldots,x_{q_1}),\ldots,\\ & \gamma(g_k;h_{p_{k-1}+1},\ldots,h_{p_k=j})(\Gamma^k y;x_{\sum_{s=1}^{k-1}q_s+1},\ldots,x_{\sum_{s=1}^{k}q_s=i}))\end{aligned}$$

where

$$\begin{aligned}\Gamma^0 y &= \Gamma^0(k;q_1,\ldots,q_k)y\\ &= d_1\cdots\check{d}_{q_1}\cdots\check{d}_{q_1+q_2}\cdots\check{d}_{\sum_{s=1}^{k-1}q_s}\cdots d_{\sum_{s=1}^{k}q_s-1}y\end{aligned}$$

and for $1\le r\le k$,

$$\begin{aligned}\Gamma^r y &= \Gamma^r(k;q_1,\ldots,q_k)y\\ &= d_0\cdots d_{\sum_{s=1}^{r-1}q_s-1}d_{\sum_{s=1}^{r}q_s+1}\cdots d_{\sum_{s=1}^{k}q_s=i}y.\end{aligned}$$

By definition of γ, (7.6) can further be written as

$$\begin{aligned} & = f(\Gamma^0 y;g_1(\Gamma^0\Gamma^1 y;h_1(\Gamma^1\Gamma^1 y;x_1,\ldots,x_{i_1}),\ldots,\\ & \quad h_{j_1}(\Gamma^{j_1}\Gamma^1 y;x_{\sum_{t=1}^{j_1-1}i_t+1},\ldots,x_{q_1}))\ldots,\\ (7.7)\qquad & \quad g_k(\Gamma^0\Gamma^k y;h_{p_{k-1}+1}(\Gamma^1\Gamma^k y;x_{\sum_{s=1}^{k-1}q_s+1},\ldots,x_{\sum_{t=1}^{p_{k-1}+1}i_t}),\ldots,\\ & \quad h_j(\Gamma^{j_k}\Gamma^k y;x_{\sum_{t=1}^{j-1}i_t+1},\ldots,x_i)))\end{aligned}$$

where

$$\begin{aligned}\Gamma^0\Gamma^r y &= \Gamma^0(j_r;i_{p_{r-1}+1},\ldots,i_{p_{r-1}+j_r})\Gamma^r(k;q_1,\ldots,q_k)y\\ &= (d_1\cdots\check{d}_{q(r,1)}\cdots\check{d}_{q(r,2)}\cdots\check{d}_{q(r,j_r-1)}\cdots d_{q_r-1})\\ &\quad (d_0\cdots d_{\sum_{s=1}^{r-1}q_s-1}d_{\sum_{s=1}^{r}q_s+1}\cdots d_{\sum_{s=1}^{k}q_s=i})y,\end{aligned}$$

and

$$\begin{aligned}\Gamma^s\Gamma^r y &= \Gamma^s(j_r;i_{p_{r-1}+1},\ldots,i_{p_{r-1}+j_r})\Gamma^r(k;q_1,\ldots,q_k)y\\ &= (d_0\cdots d_{q(r,s-1)-1}d_{q(r,s)+1}\cdots d_{q_r})\\ &\quad (d_0\cdots d_{\sum_{s=1}^{r-1}q_s-1}d_{\sum_{s=1}^{r}q_s+1}\cdots d_{\sum_{s=1}^{k}q_s=i})y\end{aligned}$$

for $1 \leq s \leq j_r$ and $1 \leq r \leq k$.

Comparing (7.5) and (7.7), and using the Lemma 7.3.5, it follows that the cochains in (7.2) and (7.3) are the same.

To check commutativity of unit diagrams, let $f \in \mathcal{C}(k) = CY^k(D, D)$, $\alpha_1, \ldots, \alpha_k \in K$. Then,

$$\gamma \circ (\mathrm{id} \otimes \eta^k)(f \otimes (\alpha_1, \ldots, \alpha_k)) = \gamma(f; \alpha_1, \ldots, \alpha_k)$$

where we identify $\alpha_i \in K$ with the map

$$\begin{aligned} \alpha_i \quad : \quad & K[Y_1] \otimes D \longrightarrow D \\ & (y; a) \mapsto \alpha_i a \end{aligned}$$

for all $i = 1, 2, \ldots, k$. If ϕ denotes the isomorphism

$$\mathcal{C}(k) \otimes K^k \cong \mathcal{C}(k),$$

then

$$\phi(f \otimes (\alpha_1, \ldots, \alpha_k))(y; x_1, \ldots, x_k) = f(y; \alpha_1 x_1, \ldots, \alpha_k x_k).$$

Now,

$$\gamma(f; \alpha_1, \ldots, \alpha_k)(y; x_1, \ldots, x_k) = f(\Gamma^0 y; \alpha_1(\Gamma^1 y; x_1), \ldots, \alpha_k(\Gamma^k y; x_k))$$

where $\Gamma^0 y = y$, as $\Gamma^0 = \Gamma^0(k; 1, \ldots, 1)$ and $\Gamma^r y = d_0 \cdots d_{r-2} d_{r+1} \cdots d_k y, 1 \leq r \leq k$.

Therefore,

$$\gamma(f; \alpha_1, \ldots, \alpha_k)(y; x_1, \ldots, x_k) = f(y; \alpha_1 x_1, \ldots, \alpha_k x_k).$$

Hence,

$$\gamma \circ (\mathrm{id} \otimes \eta^k)(f \otimes (\alpha_1, \ldots, \alpha_k)) = \phi(f \otimes (\alpha_1, \ldots, \alpha_k)).$$

Also for $f \in \mathcal{C}(j)$ and $\alpha \in K$,

$$\gamma(\eta \otimes \mathrm{id})(\alpha \otimes f) = \gamma(\alpha; f)$$

where α is regarded as an element of $\mathcal{C}(1)$ as above.

Now,

$$\gamma(\alpha; f)(y; x_1, \ldots, x_j) = \alpha(\Gamma^0 y; f(\Gamma^1 y; x_1, \ldots, x_j))$$

where $\Gamma^0 y = \Gamma^0(1; j)y = d_1 \ldots d_{j-1} y$ and $\Gamma^1 y = \Gamma^1(1; j)y = y$. Thus

$$\begin{aligned} \gamma(\alpha; f)(y; x_1, \ldots, x_j) &= \alpha(y'; f(y; x_1, \ldots, x_j)) \\ &= \alpha f(y; x_1, \ldots, x_j). \end{aligned}$$

where y' is the only tree in Y_1.

Note that $\psi : K \otimes \mathcal{C}(j) \xrightarrow{\cong} \mathcal{C}(j)$ is given by

$$\psi(\alpha \otimes f)(y; x_1, \ldots, x_j) = \alpha f(y; x_1, \ldots, x_j).$$

This completes the proof of the theorem.

7.4 Braces induced by the operad structure

We recall from [12] that if $\mathcal{C}(j), j \geq 0$ is a (non-Σ) operad with multiplication map γ, then the graded vector space $\mathcal{C} = \bigoplus \mathcal{C}(j)$ admits a brace algebra structure given by

$$f\{g_1, \ldots, g_n\} = \sum (-1)^{\epsilon} \gamma(f; \mathrm{id}_D, \ldots, \mathrm{id}_D, g_1, \mathrm{id}_D, \ldots, \mathrm{id}_D, g_n, \mathrm{id}_D, \ldots, \mathrm{id}_D)$$

where the summation runs over all possible substitutions of $g_1, \ldots, g_n$ into f in the prescribed order, and $\epsilon = \sum_{p=1}^{n} |g_p| i_p$, i_p being the total number of variables one has to input in front of g_p. Here id_D represents $\eta(1)$. The brace identity is a consequence of the commutativity of associative and unit diagrams. Therefore, in view of Theorem 7.3.4, we see that $CY^*(D, D)$ admits a brace algebra structure. The following lemma now shows that the braces as introduced in Definition 7.2.4 makes the dialgebra cochain complex into a brace algebra.

Lemma 7.4.1 *The braces on $CY^*(D, D)$ induced by the operad structure coincide with the braces as introduced in Definition 7.2.4.*

Proof. Let $f \in \mathcal{C}(k) = CY^k(D, D)$ and $g_i \in \mathcal{C}(m_i) = CY^{m_i}(D, D), 1 \leq i \leq n$.

Then according to M. Gerstenhaber and A. A. Voronov [12], the brace induced by the multilinear maps γ are given by

$$f\{g_1, \ldots, g_n\} = \sum (-1)^{\epsilon} \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

where $\mathrm{id} = \mathrm{id}_D = \eta(1)$ and the summation is over all possible substitutions of $g_1, \ldots, g_n$ into f, in the given order and $\epsilon = \sum_{p=1}^{n} |g_p| i_p$, i_p being the total number of inputs in front of g_p.

Observe that in the term

$$(-1)^{\epsilon} \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

of the above summation, the total number of identity entries in γ is $k - n$, the total number of identity entries in front of g_1 is i_1 and the total number of identity entries in front of g_r is $i_r - \sum_{t=1}^{r-1} m_t, 2 \leq r \leq n$. Moreover, the following inequalities hold:

$$0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{r-1} + m_{r-1} \leq i_r, i_n + m_n \leq k + \sum_{t=1}^{n} m_t - n = N, \text{(say)}.$$

By definition of γ as given in (7.1), we have for $y \in Y_N$,

$$
\begin{aligned}
(7.8) \qquad & \gamma(f;\mathrm{id},\ldots,\mathrm{id},g_1,\mathrm{id},\ldots,\mathrm{id},g_n,\mathrm{id},\ldots,\mathrm{id})(y;x_1,\ldots,x_N) \\
= \; & f(\Gamma^0 y; x_1,\ldots,x_{i_1}, g_1(\Gamma^{i_1+1}y; x_{i_1+1},\ldots,x_{i_1+m_1}), x_{i_1+m_1+1},\ldots,x_{i_2}, \\
& g_2(\Gamma^{i_2-m_1+2}y; x_{i_2+1},\ldots,x_{i_2+m_2}), x_{i_2+m_2+1},\ldots,x_{i_n}, \\
& g_n(\Gamma^{i_n-\sum_{t=1}^{n-1} m_t+n}y; x_{i_n+1},\ldots,x_{i_n+m_n}),\ldots,x_N),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma^p \;=\; & \Gamma^p(k; \underbrace{1,\ldots,1}_{i_1}, m_1, \underbrace{1,\ldots,1}_{i_2-m_1-i_1}, m_2,\ldots,m_{r-1}, \underbrace{1,\ldots,1}_{i_r-m_{r-1}-i_{r-1}}, \\
& m_r, 1,\ldots,m_n, \underbrace{1,\ldots,1}_{N-m_n-i_n})
\end{aligned}
$$

for $0 \le p \le k$. Note that in the definition of γ as given in (7.1), the map Γ^s yields the only tree in Y_1 when operated on y if $j_r = 1$ by Definition 7.3.2. In other words, Γ^r is the obvious constant map. For instance, by Definition 7.3.2, the map Γ^{i_1+2} appearing in (7.8), is given by

$$
\begin{aligned}
\Gamma^{i_1+2} &= d_0 \cdots d_{(i_1+m_1+1)-1} d_{(i_1+m_1+2)+1} \cdots d_N \\
&= d_0 \cdots \check{d}_{i_1+m_1+1} \check{d}_{i_1+m_1+2} \cdots d_N
\end{aligned}
$$

and consists of $N-1$ face maps d_i, hence $\Gamma^{i_1+2} y = y'$, where y' is the only tree in Y_1. Hence the corresponding input $\mathrm{id}(y';x_i)$ in γ is simply x_i.

Now according to Definition 7.3.2, we have

$$
\begin{aligned}
\Gamma^0 &= \check{d}_1 \cdots \check{d}_{i_1} d_{i_1+1} \cdots d_{i_1+m_1-1} \check{d}_{i_1+m_1} \cdots \check{d}_{i_2} d_{i_2+1} \cdots d_{i_2+m_2-1} \check{d}_{i_2+m_2} \\
&\quad \cdots \check{d}_{i_3} \cdots \check{d}_{i_r+m_r} \cdots \check{d}_{i_{r+1}} \\
&\quad \cdots \check{d}_{i_n+m_n} \cdots \check{d}_N \\
&= d_{i_1+1} \cdots d_{i_1+m_1-1} d_{i_2+1} \cdots d_{i_2+m_2-1} \cdots d_{i_r+1} \\
&\quad \cdots d_{i_r+m_r-1} \cdots d_{i_n+1} \cdots d_{i_n+m_n-1} \\
&= R_1^{i_1,\ldots,i_n}, \quad \text{as introduced in Definition 7.2.2.}
\end{aligned}
$$

Also the operator $\Gamma^{i_r - \sum_{t=1}^{r-1} m_t + r}$, corresponding to g_r, is given by

$$\begin{aligned}\Gamma^{i_r - \sum_{t=1}^{r-1} m_t + r} &= d_0 \cdots d_{(i_r - \sum_{t=1}^{r-1} m_t) + \sum_{t=1}^{r-1} m_t - 1} d_{(i_r - \sum_{t=1}^{r-1} m_t) + \sum_{t=1}^{r-1} m_t + m_r + 1} \\ &\quad \cdots d_N\end{aligned}$$

Recall that the number of identity entries infront of g_r is $i_r - \sum_{t=1}^{r-1} m_t$ and their degrees sum up to $i_r - \sum_{t=1}^{r-1} m_t$ and the sum of the degrees of $g_1, \ldots, g_{r-1}$ is $\sum_{t=1}^{r-1} m_t$. Thus,

$$\begin{aligned}\Gamma^{i_r - \sum_{t=1}^{r-1} m_t + r} &= d_0 \cdots d_{i_r - 1} d_{i_r + m_r + 1} \cdots d_N \\ &= R_{r+1}^{i_1, \ldots, i_n}, \quad \text{as introduced in Definition 7.2.2.}\end{aligned}$$

It follows that the N-cochain

$$\gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

is same as $f \circ_{i_1, \ldots, i_n} (g_1, \ldots, g_n)$. This sets up a sign preserving bijective correspondence between the terms of the summation

$$\sum (-1)^{\epsilon} \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

where the summation is over all possible substitutions of $g_1, \ldots, g_n$ into f, in the given order, $\epsilon = \sum_{p=1}^{n} |g_p| i_p$, i_p being the total number of inputs infront of g_p, and the terms of the summation

$$\sum (-1)^{\eta} f \circ_{i_1, \ldots i_n} (g_1, \ldots, g_n)$$

where the summation is over all $i_1, \ldots, i_n$ such that $0 \leq i_1, i_1 + m_1 \leq i_2, \ldots, i_{n-1} + m_{n-1} \leq i_n, i_n + m_n \leq k + \sum_{i=1}^{n} m_i - n$ and $\eta = \sum_{p=1}^{n} |g_p| i_p$.

Thus the braces as defined in section 2 are precisely the braces induced by the (non-Σ) operad structure. ■

7.5 G-algebra structure on cohomology

In this final section we show that the dialgebra cohomology $HY^*(D, D)$ of a dialgebra D has a G-algebra structure which is induced from a homotopy G-algebra structure on the dialgebra cochain complex $CY^*(D, D)$ with the differential altered by a sign.

Let us first recall the following definitions from [12].

Definition 7.5.1 A homotopy G-algebra is a brace algebra $V = \oplus_n V^n$ provided with a differential d of degree one and a dot product $x \cdot y$ of degree zero making V into a differentially graded associative algebra. The dot product must satisfy the following compatibility identities:

$$(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{\epsilon} x_1\{y_1, \ldots, y_k\} \cdot x_2\{y_{k+1}, \ldots, y_n\}, \tag{7.9}$$

where $\epsilon = (|x_2| + 1) \sum_{p=1}^{k} |y_p|$, and

$$\begin{aligned} & d(x\{x_1, \ldots, x_{n+1}\}) - (dx)\{x_1, \ldots, x_{n+1}\} \\ & -(-1)^{|x|} \textstyle\sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x\{x_1, \ldots, dx_i, \ldots x_{n+1}\} \\ = \; & (-1)^{(|x|+1)|x_1|} x_1 \cdot x\{x_2, \ldots, x_{n+1}\} \\ & -(-1)^{|x|} \textstyle\sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_i|} x\{x_1, \ldots, x_i \cdot x_{i+1}, \ldots x_{n+1}\} \\ & +(-1)^{|x|+|x_1|+\cdots|x_n|} x\{x_1, \ldots, x_n\} \cdot x_{n+1}. \end{aligned} \tag{7.10}$$

Definition 7.5.2 A multiplication on an operad $\mathcal{C}$ of vector spaces is an element $m \in \mathcal{C}(2)$ such that $m \circ m = 0$, where $m \circ m := m\{m\}$ and $\{\ \}$ denote the associated braces.

The following lemma shows that the operad $CY^*(D, D)$ is equipped with a multiplication.

Lemma 7.5.3 *The 2-cochain* $\pi \in CY^2(D, D)$ *defined by*

$$(7.11) \qquad \begin{cases} \pi([21]; a, b) &= a \dashv b \\ \pi([12]; a, b) &= a \vdash b \end{cases}$$

for all $a, b \in D$ *is a multiplication on the operad* $CY^*(D, D)$.

Proof. By Remark 7.2.5, we only need to verify that $\pi \circ \pi = 0$. Now, by definition of pre-Lie product in section 2 of chapter 5, we have for $y \in Y_3$ and $a, b, c \in D$,

$$\pi \circ \pi(y; a, b, c) = (\pi \circ_0 \pi - \pi \circ_1 \pi)(y; a, b, c).$$

Thus for $y = [321]$,

$$\begin{aligned} \pi \circ \pi([321]; a, b, c) &= (\pi \circ_0 \pi - \pi \circ_1 \pi)([321]; a, b, c) \\ &= \pi([21]; \pi([21]; a, b), c) - \pi([21]; a, \pi([21]; b, c) \\ &= (a \dashv b) \dashv c - a \dashv (b \dashv c) \\ &= 0 \end{aligned}$$

by dialgebra axiom 1 of (1.1).

For $y = [312]$,

$$\begin{aligned} \pi \circ \pi([312]; a, b, c) &= (\pi \circ_0 \pi - \pi \circ_1 \pi)([312]; a, b, c) \\ &= \pi([21]; \pi([21]; a, b), c) - \pi([21]; a, \pi([12]; b, c) \\ &= (a \dashv b) \dashv c - a \dashv (b \vdash c) \\ &= 0 \end{aligned}$$

by dialgebra axiom 2 of (1.1).

For $y = [131]$,

$$\begin{aligned}
\pi \circ \pi([131]; a, b, c) &= (\pi \circ_0 \pi - \pi \circ_1 \pi)([131]; a, b, c) \\
&= \pi([21]; \pi([12]; a, b), c) - \pi([12]; a, \pi([21]; b, c) \\
&= (a \vdash b) \dashv c - a \vdash (b \dashv c) \\
&= 0
\end{aligned}$$

by dialgebra axiom 3 of (1.1).

For $y = [213]$,

$$\begin{aligned}
\pi \circ \pi([213]; a, b, c) &= (\pi \circ_0 \pi - \pi \circ_1 \pi)([213]; a, b, c) \\
&= \pi([12]; \pi([21]; a, b), c) - \pi([12]; a, \pi([12]; b, c) \\
&= (a \dashv b) \vdash c - a \vdash (b \vdash c) \\
&= 0
\end{aligned}$$

by dialgebra axiom 4 of (1.1).

For $y = [123]$,

$$\begin{aligned}
\pi \circ \pi([123]; a, b, c) &= (\pi \circ_0 \pi - \pi \circ_1 \pi)([123]; a, b, c) \\
&= \pi([12]; \pi([12]; a, b), c) - \pi([12]; a, \pi([12]; b, c) \\
&= (a \vdash b) \vdash c - a \vdash (b \vdash c) \\
&= 0
\end{aligned}$$

by dialgebra axiom 5 of (1.1). This completes the proof of the lemma. ■

In order to show that the dialgebra cochain complex $CY^*(D, D)$ admits a homotopy G-algebra structure, we shall make use of Proposition 2(3) from [12], which we describe below. Let $\mathcal{C}$ denote an operad, m a multiplication on $\mathcal{C}$ and $m \circ x$ denote $m\{x\}$.

Proposition 7.5.4 *The product*

$$x \cdot y := (-1)^{|x|+1} m\{x, y\}$$

of degree 0 *and the differential*

$$dx = m \circ x - (-1)^{|x|} x \circ m, \qquad d^2 = 0, \quad deg\ d = 1,$$

define the structure of a differential graded (DG) associative algebra on C.

First, we observe the following two facts.

Remark 7.5.5 Note that by Lemma 5.3.6, the coboundary operator

$$\delta : CY^n(D, D) \longrightarrow CY^{n+1}(D, D)$$

can be expressed in the form

$$\delta f = (-1)^{|f|}(\pi \circ f - (-1)^{|f|} f \circ \pi) = (-1)^{|f|} df. \tag{7.12}$$

Remark 7.5.6 The $*$ product, as introduced in Definition 5.3.1, can be expressed in terms of braces as

$$f * g = (-1)^{(|f|+1)(|g|)} \pi\{f, g\}. \tag{7.13}$$

This is because, by the definition of braces on $CY^*(D,D)$,

$$\begin{aligned}
\pi\{f,g\}(y;x_1,\ldots,x_{p+q}) &= (-1)^{p(q-1)}\pi \circ_{0,p} (f,g)(y;x_1,\ldots,x_{p+q}) \\
&= (-1)^{p(q-1)}\pi(R_1^{0,p}(p+q;2,p,q)y; \\
&\quad f(R_2^{0,p}(p+q;2,p,q)y;x_1,\ldots,x_p), \\
&\quad g(R_3^{0,p}(p+q;2,p,q)y;x_{p+1},\ldots,x_{p+q})) \\
&= (-1)^{p(q-1)}\pi(d_1\cdots d_{p-1}d_{p+1}\cdots d_{p+q-1}(y); \\
&\quad f(d_{p+1}\cdots d_{p+q}(y);x_1,\ldots x_p), \\
&\quad g(d_0\cdots d_{p-1}(y);x_{p+1},\ldots,x_{p+q})) \\
&= (-1)^{p(q-1)}\pi(d_1\cdots d_{p-1}d_{p+1}\cdots d_{p+q-1}(y); \\
&\quad f(d_{p+1}d_{p+1}\cdots d_{p+q-1}(y);x_1,\ldots x_p), \\
&\quad g(d_0\cdots d_{p-1}(y);x_{p+1},\ldots,x_{p+q})) \\
&= (-1)^{p(q-1)}\pi(R_1^0(p+1;2,p)R_1^p(p+q;p+1,q)(y); \\
&\quad f(R_2^0(p+1;2,p)R_1^p(p+q;p+1,q)(y);x_1,\ldots,x_p), \\
&\quad g(R_2^p(p+q;p+1,q)((y);x_{p+1},\ldots,x_{p+q})) \\
&= (-1)^{p(q-1)}f*g(y;x_1,\ldots,x_{p+q}).
\end{aligned}$$

Here we make use of the fact that the operator d_{p+q} in the string of operators $d_{p+1}\cdots d_{p+q}$ can be moved to the extreme left of the same string using $d_id_j = d_{j-1}d_i, i<j$, to yield $d_{p+1}d_{p+1}\cdots d_{p+q-1}$.

Therefore by equation (7.13), the dot product $f \cdot g$ determined by the multiplication m as in Proposition 7.5.4 is in this case, nothing but the $*$ product, upto the sign $(-1)^{(|f|+1)(|g|+1)}$. Moreover, the differential d determined by m as in Proposition 7.5.4 is merely the coboundary operator δ, upto the sign $(-1)^{(|f|)}$, that is, $df = (-1)^{(|f|)}\delta(f)$.

Consequently, by Proposition 7.5.4 and Theorem 7.3.4, we deduce that

Corollary 7.5.7 *The graded cochain module $CY^*(D,D)$ equipped with the $*$ product $f * g$, altered by the sign $(-1)^{(|f|+1)(|g|+1)}$ and the coboundary $df = (-1)^{|f|}\delta f$ is a differential graded associative algebra.*

Next we recall Theorem 3 of [12].

Theorem 7.5.8 *A multiplication on an operad $\mathcal{C}$ defines the structure of a homotopy G-algebra on $\oplus_k \mathcal{C}(k)$. A multiplication on a brace algebra is equivalent to the structure of a homotopy G-algebra on it.*

Thus in view of Theorem 7.3.4, Theorem 7.5.8 and Lemma 7.5.3 we have

Corollary 7.5.9 *The cochain complex $(CY^*(D,D),d)$, where $df = (-1)^{|f|}\delta f$ is a homotopy G-algebra with the dot product $f \cdot g = (-1)^{(|f|+1)(|g|+1)} f * g$.*

As a consequence, we have

Corollary 7.5.10 *The cochain complex $(CY^*(D,D),d)$ is a differential graded Lie algebra with respect to the commutator $[x,y] = x \circ y - (-1)^{|x||y|} y \circ x$.*

Proof. The brace identity, for $m = n = 1$ implies that

$$x\{x_1\}\{y_1\} = x\{x_1, y_1\} + x\{x_1\{y_1\}\} + (-1)^{|x_1||y_1|} x\{y_1, x_1\},$$

as $0 \le i_1 \le j_1 \le 1$.

Using Remark 7.2.5, we deduce from above that

$$(x \circ x_1) \circ y_1 - x \circ (x_1 \circ y_1) = x\{x_1, y_1\} + (-1)^{|x_1||y_1|} x\{y_1, x_1\}. \tag{7.14}$$

In order to verify that the commutator satisfies the graded jacobi identity, we proceed as follows.

$$(-1)^{|x||z|}[[x,y],z]+(-1)^{|y||x|}[[y,z],x]+(-1)^{|z||y|}[[z,x],y]$$

$$\begin{aligned}
= & \ (-1)^{|x||z|}[x\circ y-(-1)^{|x||y|}y\circ x,z]+(-1)^{|y||x|}[y\circ z-(-1)^{|y||z|}z\circ y,x] \\
& +(-1)^{|z||y|}[z\circ x-(-1)^{|z||x|}x\circ z,y]
\end{aligned}$$

$$\begin{aligned}
= & \ (-1)^{|x||z|}((x\circ y)\circ z-(-1)^{|x\circ y||z|}z\circ(x\circ y)) \\
& -(-1)^{|x|(|y|+|z|)}((y\circ x)\circ z-(-1)^{|y\circ x||z|}z\circ(y\circ x)) \\
& +(-1)^{|y||x|}((y\circ z)\circ x-(-1)^{|y\circ z||x|}x\circ(y\circ z)) \\
& -(-1)^{|y|(|x|+|z|)}((z\circ y)\circ x-(-1)^{|z\circ y||x|}x\circ(z\circ y)) \\
& +(-1)^{|z||y|}((z\circ x)\circ y-(-1)^{|z\circ x||y|}y\circ(z\circ x)) \\
& -(-1)^{|z|(|x|+|y|)}((x\circ z)\circ y-(-1)^{|x\circ z||y|}y\circ(x\circ z))
\end{aligned}$$

$$\begin{aligned}
= & \ (-1)^{|x||z|}((x\circ y)\circ z-x\circ(y\circ z))+(-1)^{|z||y|}((z\circ x)\circ y-z\circ(x\circ y)) \\
& -(-1)^{(|z|+|y|)|x|}((y\circ x)\circ z-y\circ(x\circ z))-(-1)^{(|x|+|z|)|y|}((z\circ y)\circ x-z\circ(y\circ x)) \\
& +(-1)^{|y||x|}((y\circ z)\circ x-y\circ(z\circ x))-(-1)^{(|x|+|y|)|z|}((x\circ z)\circ y-x\circ(z\circ y)) \\
& \qquad\qquad\qquad\qquad \text{(as } |y\circ z|=|y|+|z| \text{ etc.)}
\end{aligned}$$

$$\begin{aligned}
= & \ (-1)^{|x||z|}(x\{y,z\}+(-1)^{|y||z|}x\{z,y\})+(-1)^{|z||y|}(z\{x,y\}+(-1)^{|x||y|}z\{y,x\}) \\
& -(-1)^{(|y||z|)|x|}(y\{x,z\}+(-1)^{|x||z|}y\{z,x\})-(-1)^{|y|(|x|+|z|)}(z\{y,x\} \\
& +(-1)^{|y||x|}z\{x,y\})+(-1)^{|x||y|}(y\{z,x\}+(-1)^{|z||x|}y\{x,z\}) \\
& -(-1)^{(|x|+|y|)|z|}(x\{z,y\}+(-1)^{|z||y|}x\{y,z\}) \\
& \qquad\qquad\qquad\qquad \text{(by (7.14))}
\end{aligned}$$

$$= 0,$$

by cancellation of terms.

Moreover, the dot product is always *homotopy* graded commutative, that is

$$x \cdot y - (-1)^{(|x|+1)(|y|+1)} y \cdot x = (-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy). \tag{7.15}$$

This follows directly from equation (7.10) as

$$\begin{aligned}
&(-1)^{|x|}(d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy) \\
&= (-1)^{|x|}((-1)^{(|x|+1)|y|} y \cdot x + (-1)^{|x|} x \cdot y) \\
&= x \cdot y - (-1)^{(|x|+1)(|y|+1)} y \cdot x.
\end{aligned}$$

Also, the differential is a derivation of the bracket. This follows from the *homotopy* graded commutativity of the dot product.

$$\begin{aligned}
& d[x,y] - [dx,y] - (-1)^{|x|}[x,dy] \\
= \;& d(x \circ y - (-1)^{|x||y|} y \circ x) - (dx \circ y - (-1)^{|dx||y|} y \circ dx) \\
& -(-1)^{|x|}(x \circ dy - (-1)^{|x||dy|} dy \circ x) \\
= \;& d(x \circ y) - dx \circ y - (-1)^{|x|} x \circ dy \\
& -(-1)^{|x||y|}(d(y \circ x) - dy \circ x - (-1)^{|y|} y \circ dx) \\
= \;& (-1)^{|x|}(x \cdot y - (-1)^{(|x|+1)(|y|+1)} y \cdot x) \\
& -(-1)^{|x||y|+|y|}(y \cdot x - (-1)^{(|x|+1)(|y|+1)} x \cdot y) \\
& \qquad\qquad\qquad\qquad \text{(by (7.15))} \\
= \;& 0.
\end{aligned}$$

This shows that every homotopy G-algebra is a differential graded Lie algebra with respect to the commutator $[x,y] = x \circ y - (-1)^{|x||y|} y \circ x$. ■

Next we recall the following definition from [12].

Definition 7.5.11 *A G-algebra is a graded vector space H with a dot product $x \cdot y$ defining the structure of a graded commutative algebra with a bracket $[x,y]$ of degree*

-1 *defining the structure of a graded Lie algebra such that the bracket with an element is a derivation of the dot product :*

$$[x, yz] = [x, y]z + (-1)^{|x|(|y|+1)}y[x, z].$$

Corollary 7.5.12 *The $*$ product $x * y$, altered by the sign $(-1)^{(|x|+1)(|y|+1)}$ and the bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ define the structure of a G-algebra on the dialgebra cohomology $HY^*(D, D)$ of a dialgebra D.*

Proof. First observe that $HY^n(D, D) = H^n((CY^*(D, D), \delta)) = H^n((CY^*(D, D), d))$. The fact that the dot product $x \cdot y = (-1)^{|x|+1}\pi\{x, y\}$ lifts to the cohomology follows from Proposition 7.5.4. Equation (7.15) implies that this dot product is graded commutative. Moreover, by Corollory 7.5.10, the bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ of degree -1 defines the structure of a graded Lie algebra on $HY^*(D, D)$. It remains to show that the bracket with an element is a derivation of the dot product.

First we show that the commutator $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ for all $x, y \in CY^*(D, D)$ is a graded derivation of the dot product up to null homotopy, that is

$$\begin{aligned} & [x, y \cdot z] - [x, y] \cdot z - (-1)^{|x|(|y|+1)} y \cdot [x, z] \\ = \; & (-1)^{|x|+|y|+1}(d(x\{y, z\}) - (dx)\{y, z\} - (-1)^{|x|} x\{dy, z\} - (-1)^{|x|+|y|} x\{y, dz\}). \end{aligned}$$

By definition of the commutator, we have

$$\begin{aligned}
&[x, y \cdot z] - [x, y] \cdot z - (-1)^{|x|(|y|+1)} y \cdot [x, z] \\
&= x \circ (y \cdot z) - (-1)^{|x||y \cdot z|}(y \cdot z) \circ x - (x \circ y - (-1)^{|x||y|} y \circ x) \cdot z \\
&-(-1)^{|x|(|y|+1)} y \cdot (x \circ z - (-1)^{|x||z|} z \circ x) \\
&= (x \circ (y \cdot z) - (-1)^{|x|(|y|+1)} y \cdot (x \circ z) - (x \circ y) \cdot z) \\
&-(-1)^{|x||y \cdot z|}(y \cdot z) \circ x - (-1)^{|x||y|}(y \circ x) \cdot z + (-1)^{|x|(|y|+|z|+1)} y \cdot (z \circ x) \\
&= (x \circ y.z - (-1)^{|x|(|y|+1)} y \cdot (x \circ z) - (x \circ y) \cdot z) \\
&-(-1)^{|x|(|y|+|z|+1)}((y \cdot z) \circ x - y \cdot (z \circ x) + (-1)^{|x|(|z|+1)} y \circ x.z) \\
&= x \circ y.z - (-1)^{|x|(|y|+1)} y \cdot (x \circ z) - (x \circ y) \cdot z \\
&\qquad \text{as } ((y \cdot z) \circ x - y \cdot (z \circ x) + (-1)^{|x|(|z|+1)} y \circ x.z) = 0, \quad \text{by equation } (7.9) \\
&= (-1)^{|x|+|y|+1}(d(x\{y, z\}) - (dx)\{y, z\} - (-1)^{|x|} x\{dy, z\} - (-1)^{|x|+|y|} x\{y, dz\})
\end{aligned}$$

by equation (7.10). This implies that $[x, y \cdot z] = [x, y] \cdot z + (-1)^{|x|(|y|+1)} y \cdot [x, z]$ for all $x, y, z \in HY^*(D, D)$. Thus $HY^*(D, D)$ admits a G-algebra structure. ■

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