PARAMETRIC HOMOTOPY PRINCIPLE OF SOME PARTIAL DIFFERENTIAL RELATIONS

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Preface

A system of r-th order partial differential inequalities is a subspace in the space of r-jets of C^r maps between manifolds. The problem of homotopy classification of C^∞ solutions of such systems is studied in this thesis. The text roughly divides into two parts. In the first part, the problem is considered in an equivariant setting when the system is open and invariant under the action of a compact Lie group, but may not be invariant under the action of the pseudogroup of equivariant local diffeomorphisms. The second part concerns a non-equivariant set-up without the openness condition on the system. The material is informally discussed in the introduction.

I am deeply indebted to my superviser Prof. Amiya Mukherjee, who introduced me to Differential Topology and provided an exposure to the problems studied in this thesis. I owe enormously to his teaching, guidance and his patience with me in strengthening my understanding of the subject. His inspiration and affection were valuable impulses to this endeavour. I would like to express in warmest terms my heartfelt gratitude to him.

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Contents

0 Introduction

The open extension theorem of Gromov [8] provides a unifying principle for the work of Smale [17], Hirsch [10], Phillips [15], Feit [6], and others on immersion and submersion problems. In the present thesis we study this theorem in an equivariant context. To describe the problem and results in proper perspective, we outline briefly the relevant background.

Let $p : E \longrightarrow X$ be a smooth fibre bundle, and $p^{(r)} : E^{(r)} \longrightarrow X$ be the bundle of r-jets of local sections of p. Then a *partial differential relation* of order r, or simply a relation, is a subset $\mathcal{R}$ of $E^{(r)}$, and a smooth section $f : X \longrightarrow E$ of p is called a *solution* of $\mathcal{R}$, or *$\mathcal{R}$-holonomic*, if its r-jet $j^r f : X \longrightarrow E^{(r)}$ maps X into $\mathcal{R}$. Let $\Omega(\mathcal{R})$ be the space of C^∞ solutions of $\mathcal{R}$ with the C^∞ compact-open topology, and $\Gamma(\mathcal{R})$ be the space of continuous sections of $p^{(r)}$ whose images lie in $\mathcal{R}$. Then the r-jet operator j^r maps $\Omega(\mathcal{R})$ continuously into $\Gamma(\mathcal{R})$. The relation $\mathcal{R}$ is said to satisfy *parametric h-principle* (*h* for homotopy), or said to be *integrable*, if $j^r : \Omega(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$ is a weak homotopy equivalence.

Let $\pi : X \times \mathbb{R} \longrightarrow X$ be the canonical projection, and $\tilde{\mathcal{R}}$ be a relation in the r-jet bundle of the induced bundle $\pi^*(E)$. Then $\tilde{\mathcal{R}}$ is called an *extension* of $\mathcal{R}$ if the induced map $\pi^{(r)}$ between the r-jet bundles maps $\tilde{\mathcal{R}}$ onto $\mathcal{R}$. Now suppose that $\mathcal{R}$ is open, and $\tilde{\mathcal{R}}$ is an open extension of $\mathcal{R}$ which is invariant under the action of the pseudogroup of fibre-preserving local diffeomorphisms of $X \times \mathbb{R}$ (that is, diffeomorphisms commuting with π). Then the open extension theorem of Gromov states that $\mathcal{R}$ satisfies the parametric h-principle.

The theorem was also proved by du Plessis [5] a little earlier under the extra conditions that $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are open subbundles invariant under some actions of the pseudogroups of local diffeomorphisms of X and $X \times \mathbb{R}$ respectively. He used a technique of Haefliger [9] and Poénaru [16] (developed from the early results of Smale [17] and Hirsch [10]), which is based on a handle body decomposition of X. On the other hand, Gromov gave a sheaf theoretic treatment of his theorem. One of the crucial steps here is the Flexibility Theorem which concerns certain homotopy lifting property of germs of solutions of $\widetilde{\mathcal{R}}$ over compact subsets of X, and Gromov proved this by introducing the concept of 'sharply-moving diffeotopy'. The main idea was to find local diffeotopies of $X \times \mathbb{R}$ which would keep $\widetilde{\mathcal{R}}$ invariant and at the same time 'sharply move' the manifold X locally at hypersurfaces.

For a non-closed manifold X (that is, when every compact component of X has boundary), the extension $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ is not necessary, and the corresponding theorem, a forerunner of the open extension theorem, was proved by Gromov in his thesis [7]. This theorem was generalized to the equivariant case by Bierstone [2]. He took $p : E \longrightarrow X$ to be a smooth G-locally trivial G-fibre bundle, where G is a compact Lie group, and the relation $\mathcal{R}$ to be a G-invariant open subbundle of $E^{(r)}$, which is also invariant under some action of the pseudogroup of G-equivariant local diffeomorphisms of X. Then the theorem of Bierstone [2] says that if for a G-manifold X each orbit bundle $X_{(H)}$ is non-closed, where $X_{(H)}$ consists of points $x \in X$ for which the isotropy subgroup G_x is conjugate to H, then $\mathcal{R}$ abides by the equivariant parametric h-principle. The proof runs along the line of Haefliger [9] and Poénaru [16] via decomposition of X into 'good' handle bundles over orbits.

The equivariant version of the theorem of du Plessis [5] was similarly proved by Izumiya [11]. She considered the same relation $\mathcal{R}$ as in Bierstone

[2] under the assumption that $\mathcal{R}$ satisfies certain 'G-extensibility condition', and proved that $\mathcal{R}$ is integrable over any G-manifold X. The strategy here is to replace the good handle bundle decomposition of Bierstone by certain 'proper' handle bundle decomposition.

The main result of this thesis is obtained by using sheaf theoretic arguments and the technique of sharply-moving diffeotopy of Gromov. The theorem states that if an open G-invariant relation $\mathcal{R}$ in $E^{(r)}$ admits an open G-invariant extension $\tilde{\mathcal{R}}$ in $(E \times \mathbb{R})^{(r)}$, and if $\mathcal{D}$ is a subpseudogroup of the pseudogroup of equivariant local diffeomorphisms of $X \times \mathbb{R}$ which keeps $\tilde{\mathcal{R}}$ fixed under its action on $E^{(r)}$, and sharply moves X in $X \times \mathbb{R}$, then $\mathcal{R}$ satisfies the equivariant parametric h-principle. It may be noted that the definition of sharply-moving diffeotopy is very complicated in the general set up of Gromov. Here we have simplified the idea according to our requirement. This enables us to have a clear picture of the general definition, and to give a more explicit and direct proof of the flexibility theorem in our context. The final part of the proof of the main theorem does not need much modification of the proof of Gromov.

The main theorem leads to the equivariant version of the open extension theorem, which again generalizes theorems of Bierstone [2] and Izumiya [11] in the sense that we do not require the invariance of the basic partial differential relation $\mathcal{R}$ with respect to the pseudogroup of equivariant local diffeomorphisms.

Our next application is a generalization of the transversality theorem of Gromov ([8], p.87). Consider a G-locally trivial G-fibre bundle $p : E \longrightarrow X$, where G is as before a compact Lie group, and let ξ and η be G-subbundles of the tangent bundles TE and TX respectively. Then if for each isotropy subgroup H of the G-action on X the dimensions of the H-fixed point sets satisfy $\dim X^H + \dim \xi^H < \dim E^H$, the equivariant sections of p, which are transverse to ξ on η, satisfy the equivariant parametric h-principle. It may be noted that here ξ^H is a fibre bundle over E^H, and $\dim \xi^H$ means its fibre dimension. When $\eta = TX$ and $G = \{e\}$, the theorem reduces to the transversality theorem of Gromov.

Another consequence of our transversality theorem is obtained by working with a trivial G-fibre bundle. Let X and Y be G-manifolds, and η and ξ be G-subbundles of TX and TY respectively. Let $\mathcal{R}$ be the subspace of the jet space $J^1(X,Y)$ consisting of 1-jets $j^1_x f$ of germs of local G-maps $f: X \longrightarrow Y$ at x, such that $(j^1_x f)|\eta_x$ is injective and $(j^1_x f)(\eta_x) \cap \xi_{f(x)} = \{0\}$. Then $\mathcal{R}$ satisfies equivariant parametric h-principle, if for each isotropy subgroup H of G over X we have $\dim \xi^H + \dim \eta^H < \dim Y^H$. This theorem leads us to the integrability of the relation of equivariant local immersions $X \longrightarrow Y$. Explicitly, suppose that $\mathrm{Imm}_G(X,Y)$ is the space of equivariant immersions, and $\mathrm{R}_G(TX,TY)$ is the space of equivariant bundle monomorphisms $F: TX \longrightarrow TY$ such that if $f: X \longrightarrow Y$ is the map induced by F, then, for each $x \in X$, $F|T(Gx)_x$ is given by the differential of the map $gx \mapsto gf(x)$ of the orbit Gx onto the orbit $Gf(x)$. Then the equivariant immersion theorem states that the differential map $d: \mathrm{Imm}_G(X,Y) \longrightarrow \mathrm{R}_G(TX,TY)$ is a weak homotopy equivalence, provided $\dim X^H < \dim Y^H$ for each isotropy subgroup H of G over X.

It may be noted that there had been two previous attempts to prove the equivariant immersion theorem by Bierstone [2] and Izumiya [11]. Bierstone used much more complicated dimension restriction than ours, and this may be described as follows. Recall that an invariant component of a G-space

X is the inverse image under the orbit map $X \longrightarrow X/G$ of a component of X/G, and that the saturation of a fixed point set X^H is the G-subspace $X^{(H)} = G.X^H$ of X. Let $\{X_i^j\}$ be the set of invariant components of the saturations $X^{(H_j)}$ partially ordered by inclusion, where H_j runs over the isotropy subgroups of G over X. Then the equivariant immersion theorem of Bierstone demands that $\dim X^{H_j}$ for isotropy subgroup H_j over each minimal component X_i^j should be strictly less than the dimension of each component of Y^{H_j}. On the other hand, if $n = \max\{\dim X^H\}$ where H runs over isotropy subgroups of G over X, and if $m = \min\{\dim Y^K\}$ where K runs over isotropy subgroups of G over Y, then the equivariant immersion theorem of Izumiya assumes that $n < m$.

In the remaining part of the thesis we consider non-open relations $\mathcal{R}$ in a non-equivariant set up, and observe that the proof of the main theorem can be modified to obtain h-principle for $\mathcal{R}$. This indicates that the general theory has a wider range of applications than at first apparent. The case in point is the following problem suggested by Gromov [8]. Let (V, g) and (W, h) be two symplectic manifolds with symplectic structures g and h. A symplectic isometry is a map $f : V \longrightarrow W$ such that $f^*h = g$. Then, if $\mathcal{R}$ is the relation corresponding to the symplectic isometric immersions $V \longrightarrow W$, the problem is to establish the h-principle for $\mathcal{R}$. We tackle the situation by making an appeal to a suitable version of Moser's stability theorem which says that a small perturbation of the symplectic form g within its class in de Rham's cohomology is diffeomorphic to g. This theorem shows that although $\mathcal{R}$ is not open, it has all the 'good' properties of an open relation so much so that the tactics of the proof of the main theorem are still applicable to it.

It should be remarked that Gromov studied a more general theorem in which f and h are as above and g is an arbitrary 2-form on V, and Lees [12] studied a special case of Gromov when $\dim W = 2\dim V$ and $g = 0$ (that is, when f is Lagrange immersion). The general theorem arises as the h-principle of the solution sheaf of some partial differential operator which is 'infinitesimally invertible', and Gromov proved this by using sophisticated machinary like Nash-Moser implicit function theorem.

We also prove parametric h-principle for contact isometric immersions.

Section 1 contains the basic vocabulary which are repeatedly used in the text. Here we state the concept of sharply-moving diffeotopy, and the main theorem with its corollaries. Section 2 is in the nature of a review. Here we describe topological sheaf extended over non-open subsets by means of germs of sections on them, and with a suitable structure called quasi-topology so as to behave well with respect to direct limits. A topological sheaf $\mathcal{F}$ is flexible if for every pair of compact sets $C' \hookrightarrow C$ the restriction map $\mathcal{F}(C) \longrightarrow \mathcal{F}(C')$ is a Serre fibration. The proof of the main theorem is based on the following homomorphism theorem: A homomorphism between flexible sheaves over X, $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, is a weak homotopy equivalence if it is a local weak homotopy equivalence. In Section 3 we introduced the sheaf Φ of equivariant solutions of $\mathcal{R}$, and the sheaf Ψ of equivariant sections of $p^{(r)}$ coming from r-jets of equivariant local sections of p and whose images lie in $\mathcal{R}$. Section 4 is devoted to the proof of the fact that Ψ is flexible, and then in Section 5 we prove that the r-jet map $j^r : \Phi \longrightarrow \Psi$ is a local weak homotopy equivalence. Sections 6 and 7 are prerequisites to the proof of the final step of the main theorem, namely the fact that Φ is flexible, and this we prove in Section 8. Section 9 takes up equivariant open extension theorem, and Section 10 some further applications of the main theorem – the principal results here are the equivariant transversality theorem and the equivariant immersion theorem. In Section 11, we discuss Moser's stability theorem for symplectic forms, and relative Poincaré lemma for differential forms. Section 12 and 13 deal with the h-principle for symplectic and contact isometric immersions.

The thesis may be treated as an exercise in understanding some of the intricate methods of Gromov, and solving some problems which arise naturally in this endeavour.

1 Formulation of Main Theorem

(1.1) Unless it is stated otherwise, G will denote a compact Lie group, X a differentiable G-manifold with a G-invariant Riemannian metric, and $p : E \longrightarrow X$ a G-locally trivial differentiable G-fibre bundle. Recall that a G-fibre bundle $p : E \longrightarrow X$ is a locally trivial G-map, and that this is G-locally trivial if for every x in X there exists a G_x-invariant open nighbourhood U_x of x such that $p^{-1}(U_x)$ is differentiably G_x-equivalent to the trivial G_x-fibre bundle $U_x \times p^{-1}(x)$. As shown in Bierstone [1], a differentiable G-fibre bundle is G-locally trivial if and only if it has the equivariant covering homotopy property.

Let $p^{(r)} : E^{(r)} \longrightarrow X$ be the bundle of r-jets of local sections of p. Then $p^{(r)}$ inherits a natural differentiable G-fibre bundle structure, where the action of G on $E^{(r)}$ is given by $g \cdot j^r_x f = j^r_{gx}(gfg^{-1})$, for a local section f of p at $x \in X$ and $g \in G$. Then a *partial differential relation*, or simply a *relation*, is a G-invariant subspace $\mathcal{R}$ of $E^{(r)}$.

Let $E^{(r)}_G \subset E^{(r)}$ be the subspace of $E^{(r)}$ consisting of r-jets of equivariant local sections of p defined on G-invariant open sets of X. Then $E^{(r)}_G$ is a G-invariant subspace of $E^{(r)}$. We shall denote the subset $\mathcal{R} \cap E^{(r)}_G$ by $\mathcal{R}_G$.

Definition 1.2 A section $\sigma : X \longrightarrow E^{(r)}$ of $p^{(r)}$ is *holonomic* if it is the r-jet of some section of p. Moreover, if σ maps X into $\mathcal{R}$, then it is called *$\mathcal{R}$-holonomic*. A section $f : X \longrightarrow E$ of p is $\mathcal{R}$-regular if its r-jet $j^r f : X \longrightarrow E^{(r)}$ is $\mathcal{R}$ holonomic.

Sometimes we refer to an $\mathcal{R}$-regular map as $\mathcal{R}$-holonomic by an abuse of language. This may also be called a *solution* of $\mathcal{R}$.

Notation 1.3 We shall denote the restriction of the bundle $p : E \longrightarrow X$

over a subspace A of X by $p|A$ (this symbol should not be confused with the restriction map). The restriction of the map $p^{(r)} : E^{(r)} \longrightarrow X$ to E_G and $\mathcal{R}_G$ will be denoted by $p_G^{(r)}$ and $p_{G,\mathcal{R}}^{(r)}$ respectively.

The symbol $\Gamma_G^\infty(p)$ will denote the space of C^∞ equivariant sections of p equipped with C^∞ compact-open topology, and $\Gamma_{G,\mathcal{R}}^\infty(p)$ the subspace of $\Gamma_G^\infty(p)$ consisting of all $\mathcal{R}$-regular sections of p. The spaces $\Gamma_{G,\mathcal{R}}^\infty(p|A) \subset \Gamma_G^\infty(p|A)$ are defined as above with p replaced by $p|A$.

On the other hand, $\Gamma_G^0(p^{(r)})$ is the space of continuous equivariant sections of $p^{(r)}$ with the C^0 compact-open topology, and $\Gamma_G^0(p_{G,\mathcal{R}}^{(r)})$ is the subspace of $\Gamma_G^0(p^{(r)})$ consisting of sections whose images lie in $\mathcal{R}_G$. The spaces $\Gamma_G^0(p_{G,\mathcal{R}}^{(r)}|A) \subset \Gamma_G^0(p^{(r)}|A)$ are also defined in a similar manner.

Definition 1.4 A relation $\mathcal{R} \subset E^{(r)}$ is said to satisfy *equivariant parametric h-principle* if the r-jet map $j^r : \Gamma_{G,\mathcal{R}}^\infty(p) \longrightarrow \Gamma_G^0(p_{G,\mathcal{R}}^{(r)})$ is a weak homotopy equivalence.

Definition 1.5 If G acts on $\mathbb{R}$ trivially, then $X \times \mathbb{R}$ is a G-manifold with diagonal G-action. Let $\pi : X \times \mathbb{R} \longrightarrow X$ be the canonical projection on the first factor. If $E \times \mathbb{R}$ is given the diagonal G-action as in the case of $X \times \mathbb{R}$, then $p \times \mathrm{id} : E \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ is a G-locally trivial G-fibre bundle. Let us denote $E \times \mathbb{R}$ by E'. Let $i : X \longrightarrow X \times \mathbb{R}$ be the canonical inclusion map of X into $X \times \mathbb{R}$, and denote by $\bar{\pi}^{(r)}$ the natural map from $i^*(E'^{(r)})$ to $E^{(r)}$ covering the identity map of X. This map $\bar{\pi}^{(r)}$ splits as follows:

$$\begin{array}{ccccc} i^*(E'^{(r)}) & \xrightarrow{\ \bar{i}\ } & E'^{(r)} & \xrightarrow{\ \pi^{(r)}\ } & E^{(r)} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow[\ i\ ]{} & X \times \mathbb{R} & \xrightarrow[\ \pi\ ]{} & X \end{array}$$

where $\bar{i}$ is the canonical map of pull-back, and $\pi^{(r)} : (E \times \mathbb{R})^{(r)} \longrightarrow E^{(r)}$ maps the r-jet $j^r_{(x,t)} f$ onto the r-jet $j^r_x(\pi \circ f \circ i_t)$, $i_t : X \longrightarrow X \times \mathbb{R}$ being given by $i_t(y) = (y, t)$ for $y \in X$.

We call a relation $\widetilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$ an *extension* of $\mathcal{R}$, if $\bar{\pi}^{(r)}$ sends $i^\bullet(\widetilde{\mathcal{R}}_G)$ onto $\mathcal{R}_G$.

Definition 1.6 Suppose $\mathcal{D}$ is a subpseudogroup of the pseudogroup $\mathcal{D}_G(X)$ (see [4]) of equivariant local diffeomorphisms on X. Then $\mathcal{D}$ is said to act on E if there is a map $\rho : \mathcal{D} \longrightarrow \mathcal{D}_G(E)$ which satisfies the following properties:

- ρ is continuous with respect to C^∞ compact-open topologies,
- If $\lambda \in \mathcal{D}$ then $\rho(\lambda) \in \mathcal{D}_G(E)$ covers λ, that is, $p \circ \rho(\lambda) = \lambda \circ p$,
- $\rho(1_U) = 1_{p^{-1}(U)}$ for any open subset $U \subset X$,
- $\rho(\lambda \circ \mu) = \rho(\lambda) \circ \rho(\mu)$ whenever $\lambda \circ \mu$ is defined.

The map ρ induces an action of $\mathcal{D}$ on the space of local sections of p, and hence an action on the jet space $E^{(r)}$. The actions are given by $(\lambda, f) \mapsto \rho(\lambda)^{-1} \circ f \circ \lambda$ and $(\lambda, j^r_{\lambda(x)} f) \mapsto j^r_x(\rho(\lambda)^{-1} \circ f \circ \lambda)$, where $\lambda \in \mathcal{D}$ is a local G-diffeomorphism at x and f is a local G-section of p at $\lambda(x)$. It is easy to see that if f is a section of p, then $\rho(\lambda) \circ f \circ \lambda$ is also a section of p; in fact, since $\rho(\lambda)$ covers λ, we have $p \circ \rho(\lambda)^{-1} \circ f \circ \lambda = \lambda^{-1} \circ p \circ f \circ \lambda = \lambda^{-1} \circ \lambda =$ identity. We may denote $\rho(\lambda)^{-1} \circ f \circ \lambda$ by $\lambda^\bullet(f)$, and $j^r_x(\rho(\lambda)^{-1} \circ f \circ \lambda)$ by $\lambda^\bullet(j^r_{\lambda(x)} f)$. Then a relation $\mathcal{R}$ is said to be $\mathcal{D}$-*invariant* if $\lambda^\bullet(\mathcal{R}) \subset \mathcal{R}$ for every $\lambda \in \mathcal{D}$.

Example 1.7 We shall be interested in the following subpseudogroup $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ of $\mathcal{D}_G(X \times \mathbb{R})$: $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ is the subspace of $\mathcal{D}_G(X \times \mathbb{R})$ consisting of fibre-preserving diffeomorphisms, which are local diffeomorphisms λ such that $\pi \circ \lambda = \pi$. Then the map $\rho : \mathcal{D}_G(X \times \mathbb{R}, \pi) \longrightarrow \mathcal{D}_G(E \times \mathbb{R})$

can be obtained in the following way. If $\lambda : U \times J \longrightarrow U \times J'$ is in $\mathcal{D}_G(X \times \mathbb{R}, \pi)$, where J and J' are open intervals of the real line $\mathbb{R}$, then we define $\rho(\lambda) : p^{-1}(U) \times J \longrightarrow p^{-1}(U) \times J'$ by $\rho(\lambda)(e,t) = (e, \lambda'(p(e), t))$, where $\lambda' : U \times J \longrightarrow J'$ is a G-equivariant map satisfying $\lambda(x,t) = (x, \lambda'(x,t))$ so that $\pi_2 \circ \lambda = \lambda'$ (π_2 denotes the projection on the second factor). Clearly, $\rho(\lambda)$ is a G-equivariant fibre-preserving local diffeomorphism and satisfies the first three properties listed in Definition 1.6. To prove the fourth property, it is enough to observe that

$$\lambda'(x, \mu'(x,t)) = (\lambda \circ \mu)'(x,t)$$

which is true, because on one hand $(\lambda \circ \mu)(x,t)$ is equal to $(x, (\lambda \circ \mu)'(x,t))$ and on the other hand it equals $(x, \lambda'(x, \mu'(x,t)))$.

Let f be a G-equivariant section of $p \times \mathrm{id}$. We will describe $\lambda^{\bullet}(f)$ for an arbitrary $\lambda \in \mathcal{D}_G(X \times \mathbb{R}, \pi)$. Observe that f can be expressed as $f(x,t) = (f'(x,t), t)$ where f' is an equivariant local map of $X \times \mathbb{R}$ in E. Hence,

$$\begin{aligned}(\rho(\lambda)^{-1} \circ f \circ \lambda)(x,t) &= (\rho(\lambda)^{-1} \circ f)(x, \lambda'(x,t)) \\ &= \rho(\lambda)^{-1}(f'(x, \lambda'(x,t)), \lambda'(x,t)) \\ &= (f'(x, \lambda'(x,t)), t),\end{aligned}$$

where the last equality follows from the fact that $p \circ f'(x,t) = x$ for all $(x,t) \in U \times J$.

Notation 1.8 If A is a subset of X, then the symbol $\mathrm{Op}\, A$ will denote some small but unspecified open neighbourhood of A in X.

Definition 1.9 Let U be an open G-invariant subset in X and let B be a compact G-invariant submanifold in U of codimension zero. Then the boundary ∂B of B is a G-invariant hypersurface contained in U. Let us denote this hypersurface by S. Fix a $\delta > 0$, and set $U_\delta = U \times (-\delta, \delta)$, which is in $X \times \mathbb{R}$.

Let $\mathcal{E}_G(U, U_\delta)$ denote the space of smooth equivariant embeddings of $\mathrm{Op}\,U$ in U_δ with quasi-topology (see Section 2 for definition), where $\mathrm{Op}\,U$ denotes a neighbourhood of U in $X \times \mathbb{R}$, which may be different for different embeddings. Then a subpseudogroup $\mathcal{D} \subset \mathcal{D}_G(X \times \mathbb{R})$ is said to *sharply move U at the hypersurface S* if for every positive number $a < \delta$ there exists an isotopy $\sigma : I \longrightarrow \mathcal{E}_G(U, U_\delta)$ in $\mathcal{D}$ with the following properties.

- σ_0 is the inclusion map,
- $\sigma_t(x, s) = (x, s)$ when x lies outside some τ-neighbourhood of S in U,
- dist $(X, \sigma_1(x, s)) > a$ for all x lying in some neighbourhood of S in U,

where 'dist' denotes the distance with respect to the G-invariant Riemannian metric on $X \times \mathbb{R}$.

A subpseudogroup $\mathcal{D} \subset \mathcal{D}_G(X \times \mathbb{R})$ is said to *sharply move X in $X \times \mathbb{R}$* if, given any open set U and a hypersurface S as described above, $\mathcal{D}$ sharply moves U at S.

Example 1.10 The pseudogroup $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ sharply moves X in $X \times \mathbb{R}$ (see Section 8).

We are now in a position to state our main theorem.

Theorem 1.11 *Let $\mathcal{R} \subset E^{(r)}$ be an open relation which admits a G-invariant open extension $\tilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$. Let $\mathcal{D}$ be a subpseudogroup of $\mathcal{D}_G(X \times \mathbb{R})$ which acts on $E \times \mathbb{R}$, and sharply moves X in $X \times \mathbb{R}$. Then, if $\tilde{\mathcal{R}}$ is $\mathcal{D}$-invariant, $\mathcal{R}$ abides by the equivariant parametric h-principle.*

The theorem implies in view of Example 1.10

Corollary 1.12 *If $\mathcal{R} \subset E^{(r)}$ is an open relation which admits a G-, and $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-invariant open extension $\tilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$, then $\mathcal{R}$ abides by the equivariant parametric h-principle.*

We may recover the 'Open Extension Theorem' of Gromov ([8], p.86) from the above corollary by taking $G = \{e\}$.

Corollary 1.13 *Let $p : E \longrightarrow X$ be a fibre bundle and $\mathcal{R}$ be an open relation. Then, $\mathcal{R}$ abides by the parametric h-principle if it admits an open $\mathcal{D}(X \times \mathbb{R}, \pi)$-invariant extension.*

Remark 1.14 The theorem and the corollaries of this section remain true if we replace $X \times \mathbb{R}$ by $X \times \mathbb{R}^d$ and modify the definitions accordingly.

2 Quasi-continuous Sheaves

In this section we collect some facts of sheaf theory which will be used in the subsequent sections.

Definition 2.1 A *presheaf* $\mathcal{F}$ of sets over a topological space X is a contravariant functor which assigns to each open set $U \subset X$ a set $\mathcal{F}(U)$ such that $\mathcal{F}(\emptyset) = \emptyset$, and to each inclusion $V \hookrightarrow U$ a map $r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$. Therefore we have

(i) $r_U^U = 1_U$ (identity on U), and

(ii) $r_W^U = r_W^V \circ r_V^U$, for $U \supset V \supset W$.

The map r_V^U is called the restriction map, and we often denote $r_V^U(f)$ by $f|V$ for $f \in \mathcal{F}(U)$.

Definition 2.2 A presheaf $\mathcal{F}$ is called a *sheaf* if, for every collection $\{U_i\}$ of open subsets of X with $U = \bigcup U_i$, the following conditions hold:

(i) If $f, g \in \mathcal{F}(U)$ and $f|U_i = g|U_i$ for all i, then $f = g$.

(ii) If $f_i \in \mathcal{F}(U_i)$, and if for $U_i \cap U_j \neq \emptyset$ we have

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j,$$

then there exists an $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$ for all i.

Definition 2.3 If $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X, then a sheaf homomorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of maps

$$\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for each open set U in X such that the following diagram is commutative whenever $U \supset V$.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ \downarrow r_V^U & & \downarrow r_V^U \\ \mathcal{F}(V) & \xrightarrow[\alpha_V]{} & \mathcal{G}(V). \end{array}$$

(2.4) We may extend $\mathcal{F}$ to non-open subsets C of X by defining $\mathcal{F}(C)$ as the inductive (direct) limit of $\mathcal{F}(U)$ over all open sets U containing C. When C is a point x, then $\mathcal{F}(x)$ is just the stalk over x. As mentioned in Gromov ([8], p.35), if $\mathcal{F}$ is a sheaf of topological spaces, that is, when each $\mathcal{F}(U)$ is a topological space and each r^U_V is continuous, then $\mathcal{F}(C)$ does not have any useful natural topology. However, $\mathcal{F}(C)$ can be equipped with a weaker structure, called quasi-topological structure, which behaves nicely under direct limit.

Definition 2.5 (Whitehead) A *quasi-topological structure* on a set A is given by distinguishing, for each topological space P, a subset in the set of all point-set maps $P \longrightarrow A$, such that these distinguished maps, which are called *quasi-continuous*, enjoy the following properties of ordinary continuous maps.

(i) If $\mu : P \longrightarrow A$ is quasi-continuous and if $\varphi : Q \longrightarrow P$ is a continuous map between topological spaces then the composed map $\mu \circ \varphi : Q \longrightarrow A$ is quasi-continuous.

(ii) If a map $\mu : P \longrightarrow A$ is locally quasi-continuous, that is, if each point of P has a neighbourhood such that μ restricted to it is quasi-continuous, then μ is quasi-continuous.

(iii) If P is covered by two closed subsets P_1 and P_2 of P, and if a map $\mu : P \longrightarrow A$ is quasi-continuous on P_1 as well as on P_2, then it is quasi-continuous on all of P.

By an abuse of language we shall refer to quasi-continuous maps simply as continuous maps whenever the implication will be clear from the context.

Definition 2.6 A map $\alpha : A \longrightarrow B$ between quasi-topological spaces is called *continuous* if $\alpha \circ \mu : P \longrightarrow B$ is continuous for all continuous maps $\mu : P \longrightarrow A$ and for all topological spaces P.

Definition 2.7 Two continuous maps $f_0, f_1 : P \longrightarrow A$, where A is a quasi-topological space, are said to be *homotopic* if there is a continuous map $F : P \times [0,1] \longrightarrow A$ such that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$.

(2.8) Suppose now that, for each open set U, $\mathcal{F}(U)$ has a quasi-topology. Then the direct limit of quasi-topological structures on $\mathcal{F}(U)$ over all open subsets $U \supset C$ gives a quasi-topological structure on $\mathcal{F}(C)$. With this quasi-topology on $\mathcal{F}(C)$, the space $C^0(P, \mathcal{F}(C))$ of continuous maps of the topological space P in $\mathcal{F}(C)$ is the direct limit of the spaces $C^0(P, \mathcal{F}(U))$ over all open subsets $U \supset C$. In other words, a point-set map $f : P \longrightarrow \mathcal{F}(C)$ is continuous if there exists an open subset $U \supset C$ and a continuous map $\tilde{f} : P \longrightarrow \mathcal{F}(U)$ such that $f = r_C^U \circ \tilde{f} = \tilde{f}|C$, where $r_C^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(C)$ is the canonical restriction map. It is easy to see that r_C^U is continuous with respect to the above quasi-topological structure on $\mathcal{F}(C)$ (because, in this case $P = \mathcal{F}(U)$ and we can take $\tilde{r}_C^U = r_U^U = 1_{\mathcal{F}(U)}$).

Definition 2.9 A sheaf $\mathcal{F}$ is *continuous* if for each open set U, $\mathcal{F}(U)$ has a quasi-topology such that $r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is continuous for all open subsets $V \subset U$, and if for every subset $C \subset X$, $\mathcal{F}(C)$ is equipped with the direct limit quasi-topology.

Remark 2.10 Let U, V be two open subsets of X containing C, and let $f : P \longrightarrow \mathcal{F}(U)$, $g : P \longrightarrow \mathcal{F}(V)$ be two continuous maps. If $\mathcal{F}(C)$ has the direct limit quasi-topology then f and g will represent the same continuous map $P \longrightarrow \mathcal{F}(C)$ if and only if there exists an open neighbourhood W of C such that $W \subset U \cap V$ and $r_W^U \circ f = r_W^V \circ g$.

Remark 2.11 For a pair of subsets $C' \subset C$ of X, the canonical restriction map $r^C_{C'} : \mathcal{F}(C) \longrightarrow \mathcal{F}(C')$ is continuous. To see this, let us take a continuous map $\bar{\mu} : P \longrightarrow \mathcal{F}(C)$. If $\mu : P \longrightarrow \mathcal{F}(U)$ is a representative of $\bar{\mu}$ (where U is an open neighbourhood of C), then μ also represents $r^C_{C'} \circ \bar{\mu}$ as $U \supset C'$. Now, the continuity of μ implies that $r^C_{C'} \circ \bar{\mu}$ is continuous. This proves that $r^C_{C'}$ is continuous. (To make the things clear, here we have used a different symbol, namely μ, for a representative of $\bar{\mu}$. However, in later sections we shall not distinguish between them.)

Definition 2.12 A *sheaf homomorphism* $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *between continuous sheaves* over X is a homomorphism such that each $\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is continuous.

Remark 2.13 If $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a continuous sheaf homomorphism, then for each subset C of X, it induces a continuous map $\alpha_C : \mathcal{F}(C) \longrightarrow \mathcal{G}(C)$. To see this, take a continuous map $\bar{\mu} : P \longrightarrow \mathcal{F}(C)$, and let $\mu : P \longrightarrow \mathcal{F}(U)$ be a representative of $\bar{\mu}$, i.e. $r^U_C \circ \mu = \bar{\mu}$, where U is an open set containing C. Then $\alpha_U \circ \mu$ is a continuous map $P \longrightarrow \mathcal{G}(U)$ and it represents $\alpha_C \circ \bar{\mu}$ (because, $r^U_C \circ \alpha_U \circ \mu = \alpha_C \circ r^U_C \circ \mu = \alpha_C \circ \bar{\mu}$). Therefore $\alpha_C \circ \bar{\mu}$ is continuous, and hence α_C is continuous.

Definition 2.14 A (continuous) sheaf homomorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a *weak homotopy equivalence* if for each open set $U \subset X$, α_U is so.

Definition 2.15 A continuous map $\mu : A \longrightarrow A'$ between quasi-topological spaces is called a *weak homotpoy equivalence* if the following conditions hold:

(i) For any polyhedron P and for any continuous map $\alpha' : P \longrightarrow A'$, there exists a continuous map $\alpha : P \longrightarrow A$ such that $\mu \circ \alpha$ is homotopic to α'.

(ii) If $\alpha_0 : P \longrightarrow A$ and $\alpha_1 : P \longrightarrow A$ are two continuous maps such that $\mu \circ \alpha_0$ is homotopic to $\mu \circ \alpha_1$ as maps from P to A', then α_0 is homotopic to α_1.

Definition 2.16 A (continuous) sheaf homomorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a local weak homotopy equivalence if $\alpha(x) : \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$ is a weak homotopy equivalence for each $x \in X$, where $\mathcal{F}(x)$ and $\mathcal{G}(x)$ are the stalks over x.

Definition 2.17 Let $\alpha : A \longrightarrow A'$ be a continuous map between quasi-topological spaces, P an arbitrary compact polyhedron, and $\mathbb{I}$ the closed unit interval $[0,1]$ in the real line. Then α is called a *(Serre) fibration* if given any commutative square of continuous maps:

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{\psi_0} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha} \\ P \times \mathbb{I} & \xrightarrow[\psi']{} & A' \end{array}$$

where i is the inclusion of $P \times \{0\}$ in $P \times \mathbb{I}$, there exists a continuous map $\psi : P \times \mathbb{I} \longrightarrow A$ such that $\psi | P \times \{0\} = \psi_0$ and $\alpha \circ \psi = \psi'$.

Call α a ***microfibration*** if there exists a positive number $\varepsilon \leq 1$, and a map $\psi : P \times [0, \varepsilon] \longrightarrow A$ (where ε may depend on P, ψ_0, ψ') such that $\psi | P \times \{0\} = \psi_0$ and $\alpha \circ \psi = \psi' | P \times [0, \varepsilon]$.

Definition 2.18 A sheaf $\mathcal{F}$ is *flexible* (resp. *microflexible*) if the restriction map $r^C_{C'} : \mathcal{F}(C) \longrightarrow \mathcal{F}(C')$ is a fibration (resp. microfibration) for every pair of compact subsets (C, C'), $C' \subset C \subset X$.

We now state a theorem which will be needed to prove the main theorem.

Theorem 2.19 (Homomorphism Theorem, [8], p.77) *Let $\mathcal{F}$, $\mathcal{G}$ be two flexible (continuous) sheaves over a locally compact finite dimensional space X. Then every local weak homotopy equivalence $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak homotopy equivalence.*

3 Plan of Proof of Main Theorem

(3.1) For each G-invariant open subset U of X, let $\Phi(U)$ be the space of all C^∞ equivariant solutions of $\mathcal{R}$ over U with C^∞ compact-open topology, and let $\Psi(U)$ be the space of all equivariant continuous sections of $p^{(r)}$ whose images lie in $\mathcal{R}_G$, with C^0 compact-open topology. Note that $\Phi(X) = \Gamma^\infty_{G,\mathcal{R}}(p)$ and $\Psi(X) = \Gamma^0_G(p^{(r)}_{G,\mathcal{R}})$ according to our earlier notation in (1.3)

For an arbitrary open subset $U \subset X$, define $\Phi(U) = \Phi(GU)$ and $\Psi(U) = \Psi(GU)$. For a pair of open subsets (U, V) in X, the restriction morphisms $r_\Phi : \Phi(U) \longrightarrow \Phi(V)$ and $r_\Psi : \Psi(U) \longrightarrow \Psi(V)$ are given by

$$r_\Phi(f) = f|GV \quad \text{and} \quad r_\Psi(\sigma) = \sigma|GV$$

Then it is easy to verify that both Φ and Ψ are sheaves over X. We can extend Φ and Ψ for arbitrary subsets C of X as follows:

$\Phi(C)$ = the space of germs of equivariant C^∞ solutions of $\mathcal{R}$ near C

$\Psi(C)$ = the space of germs of equivariant C^0 sections of $p^{(r)}$ near C with images in $\mathcal{R}_G$.

In other words, $\Phi(C)$ (resp. $\Psi(C)$) is the direct limit of the sets $\Phi(U)$ (resp. $\Psi(U)$) over all open subsets $U \supset C$ in X. We endow $\Phi(C)$ (resp. $\Psi(C)$) with the quasi-topology which is the direct limit of quasi-topologies associated to C^∞ compact-open topology (resp. C^0 compact-open topology) on $\Phi(U)$ (resp. $\Psi(U)$) over all open neighbourhoods U of C in X.

Since the restriction morphisms are continuous, Φ and Ψ define continuous sheaves over X.

(3.2) With respect to the above definition of $\Phi(C)$, and the direct limit quasi-topology on it, we have the following observations.

- $f \in \Phi(C)$ means that f is a regular G-section on some G-invariant neighbourhood of C.
- $f_1 = f_2 \in \Phi(C)$ means that there is a G-invariant neighbourhood U of C such that both the sections are defined and equal on U.
- An extension of $f \in \Phi(C_1)$, from $\mathrm{Op}\, C_1$ to $\mathrm{Op}\, C_2$, where $C_1 \subset C_2 \subset X$, is a regular section f' on some G-invariant neighbourhood U' of C_2 in X which equals f on a sufficiently small G-invariant neighbourhood U'' of C_1 in X where f and f' are simultaneously defined.
- Two sections f_0, $f_1 \in \Phi(C)$ are homotopic if there is a G-invariant neighbourhood U of C on which both f_0 and f_1 are defined, and a homotopy $f'_t \in \Phi(U)$ such that $f'_0 = f_0$ and $f'_1 = f_1$ on U.
- A map $f : P \longrightarrow \Phi(C)$ is continuous if each f_p, for $p \in P$, is defined on some G-invariant open neighbourhood U of C, and the assignment $p \mapsto f_p$, as a map from P to $\Phi(U)$, is continuous.
- For any subset $C \subset X$, $\Phi(C) = \Phi(GC)$.

We have similar observations for Ψ.

(3.3) Let us note that the r-jet maps $j^r(U) : \Phi(U) \longrightarrow \Psi(U)$ (where U is open) induce a sheaf homomorphism $j^r : \Phi \longrightarrow \Psi$. It is a standard result that $j^r(U)$ is continuous. Hence j^r is a homomorphism of continuous sheaves.

In view of Theorem 2.19, our main theorem will be proved if we can establish the following propositions.

Proposition 3.4 *The sheaf* Φ *is flexible.*

Proposition 3.5 *The sheaf* Ψ *is flexible.*

Proposition 3.6 *The homomorphism* j^r *is a local weak homotopy equivalence.*

4 Flexibility of Sheaf Ψ

In this section we prove Proposition 3.5. Let (C, C') be a pair of G-invariant compact subsets of X. Consider the following commutative diagram of continuous maps:

$$\begin{array}{ccc} P\times\{0\} & \xrightarrow{\varphi_0} & \Psi(U) \\ \downarrow i & & \downarrow r \\ P\times \mathbb{I} & \xrightarrow[\varphi']{} & \Psi(V) \end{array}$$

where P is a compact polyhedron, and U, V are G-invariant open sets containing C, C' respectively. It is required to construct a map $\varphi : P\times \mathbb{I} \longrightarrow \Psi(U')$, where U' is a neighbourhood of C in U, such that $\varphi \circ i = \varphi_0$ on $U'$and $r \circ \varphi = \varphi'$ on some neighbourhood of C' in V.

Take a G-invariant continuous function $\delta : U \longrightarrow [0,1]$ with compact support in V and $\delta \equiv 1$ in a smaller G-invariant neighbourhood V' of C' in V (such a function exists by Lemma 1.1.7 of [13]). Then define φ by

$$\varphi(p,t)(v) = \begin{cases} \varphi'(p, t\delta(v))(v) & \text{if } v \in V \\ \varphi_0(p,0)(v) & \text{if } v \in U \setminus V \end{cases}$$

where $p \in P$ and $t \in \mathbb{I}$. Clearly $\varphi(p,t)$ is equivariant. Also $\varphi(p,0) = \varphi_0(p,0)$ on V, as well as on outside of V. On the other hand, $\varphi(p,t) \mid V' = \varphi'(p,t) \mid V'$ since $\delta \equiv 1$ on V'. Thus Proposition 3.5 is proved. □

5 Local Weak Homotopy Equivalence

In this section we prove Proposition 3.6. As in Bierstone [2], we use Poénaru's formalism [16] (and also du Plessis' [5]) of integrability of partial differential relations over a disc, but in a different context. We take resort to sheaf theoretic language for obvious reasons, and all our spaces of maps have quasi-topological structure. The central feature is that our partial differential relation is devoid of the action of equivariant local diffeomorphisms.

Since G is a compact Lie group, about each point x of X there is an open tube which is a G-invariant open neighbourhood of the orbit Gx, equivariantly diffeomorphic to $G \times_{G_x} W_x$, where W_x (which is called the slice at x) is the normal space $T_x(X)/T_x(Gx)$ of the orbit Gx at x. Since the compact group G_x acts on the real representation space W_x, by a result of Bröker and tom Dieck [4], there is an inner product $<,>$ in W_x such that $< gw, gw' > = < w, w' >$ for $g \in G_x$ and $w, w' \in W_x$. This means that G_x fixes the origin 0 in W_x. Also note that an arbitrary small G-invariant open neighbourhood of x looks like $G \times_{G_x} U$, where U is a G_x-invariant neighbourhood of 0 in W_x. It is not difficult to see that the restriction maps

(5.1) $$\Gamma^{\infty}_{G}(p|G \times_{G_x} U) \longrightarrow \Gamma^{\infty}_{G_x}(p|U)$$

(5.2) $$\Gamma^{0}_{G}(p^{(r)}_{G,\mathcal{R}}|G \times_{G_x} U) \longrightarrow \Gamma^{0}_{G_x}(p^{(r)}_{G,\mathcal{R}}|U)$$

are homeomorphisms (see (1.3) for notation).

Since $E|U$ is a G_x-fibre bundle, $(E|U)^{(r)}$ is a G_x-space. There is a G_x-equivariant open C^{∞} map $\phi : E^{(r)}|U \longrightarrow (E|U)^{(r)}$ given by $j^r_y s \mapsto j^r_y(s|U)$, where $y \in U$. This map is surjective on each fibre, and maps $E^{(r)}_G|U$ isomorphically onto $(E|U)^{(r)}_{G_x}$. The inverse map of $\phi| : E^{(r)}_G|U \longrightarrow (E|U)^{(r)}_{G_x}$ is given by $j^r_y f \mapsto j^r_y(\tilde{f}|U)$, where $\tilde{f}[g, z] = gf(z)$ for $[g, z] \in G \times_{G_x} \mathrm{Op}\, y$, $y \in U$. Let

$\mathcal{R}_x$ be the image of $\mathcal{R}|U$ under the map ϕ. Then $\mathcal{R}_x$ is open in $(E|U)^{(r)}$, and we have the relation:

$$(E_G^{(r)} \cap \mathcal{R}|U) \cong (E|U)^{(r)}_{G_x} \cap \mathcal{R}_x, \tag{5.3}$$

and therefore

$$\Gamma^0_{G_x}(p^{(r)}_{G,\mathcal{R}}|U) \cong \Gamma^0_{G_x}((p|U)^{(r)}_{G_x,\mathcal{R}_x}). \tag{5.4}$$

Combining (5.1), (5.2) and (5.4), we get the following commutative diagram:

$$\begin{array}{ccc} \Gamma^\infty_{G,\mathcal{R}}(p|G \times_{G_x} U) & \xrightarrow{j^r} & \Gamma^0_G(p^{(r)}_{G,\mathcal{R}}|G \times_{G_x} U) \\ \downarrow & & \downarrow \\ \Gamma^\infty_{G_x,\mathcal{R}_x}(p|U) & \xrightarrow[j^r]{} & \Gamma^0_{G_x}((p|U)^{(r)}_{G_x,\mathcal{R}_x}) \end{array} \tag{5.5}$$

where the vertical arrows are homeomorphisms. Passing to direct limit, the diagram (5.5) leads to a commutative diagram of stalks of sheaves. In the resulting diagram, the upper horizontal arrow between stalks at $x \in X$ will be a weak homotopy equivalence if the lower horizontal arrow between stalks at $0 \in W_x$ is so.

Therefore we may assume that the base manifold X is itself a G-representation space with origin $0 \in X^G$. Let $\Phi(0)$ be the stalk at 0 of the sheaf $\Gamma^\infty_{G,\mathcal{R}}(p|U)$, and $\Psi(0)$ be the stalk at 0 of the sheaf $\Gamma^0_G(p^{(r)}_{G,\mathcal{R}}|U)$, where U runs over open neighbourhoods of 0 in X. Then our proposition will be proved if we show that the r-jet map

$$j^r : \Phi(0) \longrightarrow \Psi(0)$$

is a weak homotopy equivalence. To this end, we show that the maps

$$\rho = e \circ j^r : \Phi(0) \longrightarrow (p^{(r)}_{G,\mathcal{R}})^{-1}(0) \quad \text{and} \quad e : \Psi(0) \longrightarrow (p^{(r)}_{G,\mathcal{R}})^{-1}(0)$$

are weak homotopy equivalences, where $(p^{(r)}_{G,\mathcal{R}})^{-1}(0)$ is the fibre over 0 of the map $p^{(r)}_{G,\mathcal{R}} = p^{(r)}|\mathcal{R}_G : \mathcal{R}_G \longrightarrow X$ and e is the direct limit of the evaluation maps $\Gamma^0_G(p^{(r)}_{G,\mathcal{R}}|\mathrm{Op}\,0) \longrightarrow (p^{(r)}_{G,\mathcal{R}})^{-1}(0)$ defined by $\varphi \mapsto \varphi(0)$.

(5.6) ρ is a weak homotopy equivalence.

It is required to show that ρ induces isomorphism ρ_* between homotopy groups in each dimension. We first show that ρ_* is a monomorphism.

Take any continuous map of the sphere

$$f_0 : S^i \longrightarrow \Phi(0),$$

where $\Phi(0)$ has the quasi-topology. Then, for each $s \in S^i$, $f_0(s)$ is defined over some G-invariant open neighbourhood U of 0, and the map $f_0 : S^i \longrightarrow \Gamma^\infty_{G,\mathcal{R}}(p|U)$ is continuous. We may suppose that U is a trivializing neighbourhood so that $p|U$ is the trivial G-bundle $U \times Y \longrightarrow U$ where $Y = p^{-1}(0)$. Then $\Gamma^\infty_G(p|U)$ may be identified with $C^\infty_G(U, Y)$, which is the space of equivariant C^∞ maps $U \longrightarrow Y$, and $\Gamma^\infty_{G,\mathcal{R}}(p|U)$ with a subspace $C^\infty_{G,\mathcal{R}}(U, Y)$ of $C^\infty_G(U, Y)$. Similarly, the fibre of $p^{(r)}|E^{(r)}_G$ over 0 becomes identified with $J^r_G(U, Y)_0$, which is the space of r-jets of germs of equivariant maps $U \longrightarrow Y$ at 0, and the fibre $(p^{(r)}_{G,\mathcal{R}})^{-1}(0)$ with a subspace $J^r_{G,\mathcal{R}}(U, Y)_0$ of $J^r_G(U, Y)_0$.

Fix some $s_0 \in S^i$ and let $f_1 : S^i \longrightarrow \Phi(0)$ be the constant map defined by $f_1(s) = f_0(s_0)$ for $s \in S^i$. Let $h : S^i \times \mathbb{I} \longrightarrow J^r_{G,\mathcal{R}}(U, Y)_0$ be a homotopy such that $h(s, 0) = j^r_0 f_0(s)$, and $h(s, 1) = j^r_0 f_1(s)$, that is, $\rho(f_0)$ is null-homotopic. We shall show that f_0 is homotopic to the constant map f_1. Let Y' be a compact invariant neighbourhood of the image of $h(S^i \times \mathbb{I})$ under the target map $J^r_G(U, Y)_0 \longrightarrow Y$, which maps each r-jet onto its target. Without loss of generality we may suppose that $f_0(S^i)(U) \subset Y'$. By the equivariant embedding theorem (see [13]), there is an equivariant embedding $\lambda : Y' \longrightarrow V$ where V is an orthogonal G-representation space. Thus we may regard h as a map $S^i \times \mathbb{I} \longrightarrow J^r_G(U, V)_0$. Let $\mu : N \longrightarrow Y'$ be a smooth equivariant retraction of an invariant tubular neighbourhood N of $\lambda(Y')$

onto Y'.

Now choosing the polynomial representation $H_{s,t}$ of degree r for each r-jet $h(s,t)$ by means of the inclusions $j_0^r f \mapsto j_0^r(\lambda \circ f) \mapsto P(j_0^r(\lambda \circ f))$:

$$J_G^r(U, Y')_0 \subset J_G^r(U, V)_0 \xrightarrow{P} C_G^\infty(U, V),$$

where P is the map which sends each jet to its polynomial map, we get a continuous map $H : S^i \times \mathbb{I} \times U \longrightarrow V$ defined by $H(s,t,x) = H_{s,t}(x)$, so that $H(s,t)$ is equivariant and $j_0^r H(s,t) = h(s,t)$. Now define $\bar{H} : S^i \times \mathbb{I} \times U \longrightarrow V$ as follows:

$$\bar{H}(s,t,x) = \begin{cases} (1-3t)f_0(s)(x) + 3tH(s,0,x) & \text{if } 0 \le t \le 1/3 \\ H(s, 3t-1, x) & \text{if } 1/3 \le t \le 2/3 \\ (3t-2)f_1(s)(x) + (3-3t)H(s,1,x) & \text{if } 2/3 \le t \le 1 \end{cases}$$

where $s \in S^i$, $t \in \mathbb{I}$, and $x \in U$. Then $\bar{H}(s,t,.)$ is equivariant, since the action of G is linear on V, and we have

$$\bar{H}(s,0) = f_0(s), \quad \bar{H}(s,1) = f_1(s), \quad j_0^r \bar{H}(s,t) \in \mathcal{R}.$$

Moreover, $\bar{H}(s,t)(0) \in Y'$ for all $(s,t) \in S^i \times \mathbb{I}$, because $H(s,t)(0)$ is the image of $h(s,t)$ under the target map. Therefore, there is an open neighbourhood W of 0 in U such that $\bar{H}(s,t)(W) \subset N$. Define $\bar{h}(s,t) = \mu \circ \bar{H}(s,t)|W$. Then $\bar{h}$ maps $S^i \times \mathbb{I}$ in $C_G^\infty(W, Y)$. Also, for $s \in \mathbb{I}$,

$\bar{h}(s,0) = f_0(s)$, $\bar{h}(s,1) = f_1(s)$, and $j_0^r \bar{h}(s,t) = j_0^r \bar{H}(s,t) = h(s,t) \in \mathcal{R}$.[1]

The last condition implies by the openness of $\mathcal{R}|U$ that there is an open neighbourhood W' of 0 in W such that $\bar{h}(s,t)|W' \in \Gamma_{G,\mathcal{R}}^\infty(p|W')$. This means that we have a continuous homotopy $\bar{h} : S^i \times \mathbb{I} \longrightarrow \Phi(0)$ between f_0 and f_1. This completes the proof of the fact that ρ_* is a monomorphism.

Next, we show that ρ_* is an epimorphism. Let $f : S^i \longrightarrow J_{G,\mathcal{R}}^r(U, Y)_0$ be

[1] We find it a right place to note that up to this point we have not used the fact that $\mathcal{R}$ is open. This observation will be recalled in Section 12.

a map for some open neighbourhood U of 0. As before we find a compact invariant submanifold Y' of Y such that f maps S^i into $\mathcal{J}^r_G(U,Y')_0$, and embed Y' in a Euclidean G-space V. Let N be an invariant tubular neighbourhood of Y' in V with an equivariant retraction $\mu : N \longrightarrow Y'$. Then, choosing the polynomial representative F_s of degree r of each $f(s)$, $s \in S^i$, we get a continuous map $F : S^i \times U \longrightarrow V$ defined by $F(s,x) = F_s(x)$ so that $F(S^i \times \{0\}) \subset Y'$. Therefore, by compactness of S^i, there is an open neighbourhood W of 0 in U such that $F(S^i \times W) \subset N$. Now, for each $s \in S^i$ the r-jet at 0 of $\mu \circ F_s|W$ is $f(s) \in \mathcal{R}$, and, therefore, by openness of $\mathcal{R}$, there exists an open neighbourhood W' of 0 in W such that $\mu \circ F_s|W' \in \Gamma^\infty_{G,\mathcal{R}}(p|W')$. This defines a continuous map $\bar{f} : S^i \longrightarrow \Phi(0)$ by $\bar{f}(s) = \mu \circ F_s|W'$ so that $\rho \circ \bar{f}(s) = f(s)$ for each $s \in S^i$. This proves that ρ_* is an epimorphism.

(5.7) e is a weak homotopy equivalence

This proof runs parallel to that given in (5.6) subject to some modifications. Let us take a continuouos map of the sphere, $f_0 : S^i \longrightarrow \Psi(0)$. Then there exists an G-invariant open neighbourhood U of 0 such that, for each $s \in S^i$, $f_0(s)$ is an equivariant section of the bundle $\mathcal{J}^r(U,Y) \longrightarrow U$ lying in $\mathcal{J}_{G,\mathcal{R}}(U,Y)$. Let $f_1 : S^i \longrightarrow \Psi(0)$ be the constant map defined by $f_1(s) = f_0(s_0)$, for some $s_0 \in S^i$. Suppose that there is a homotopy $h : S^i \times \mathbb{I} \longrightarrow \mathcal{J}^r_{G,\mathcal{R}}(U,Y)_0$ satisfying $h(s,0) = f_0(s)(0)$, and $h(s,1) = f_1(s)(0)$. Proceeding as in (5.6), replace each jet $h(s,t)$ by its polynomial representation $H_{s,t}$ of degree r. Then $j^r_0 H_{s,t} = h(s,t)$, for $(s,t) \in S^i \times \mathbb{I}$. Let $H'(s,t) = j^r H_{s,t}$ and define $\bar{H} : S^i \times \mathbb{I} \times U \longrightarrow \mathcal{J}^r_G(U,V)$ as follows:

$$\bar{H}(s,t,x) = \begin{cases} (1-3t)f_0(s)(x) + 3tH'(s,0)(x) & \text{if } 0 \le t \le 1/3 \\ H'(s,3t-1)(x) & \text{if } 1/3 \le t \le 2/3 \\ (3t-2)f_1(s)(x) + (3-3t)H'(s,1)(x) & \text{if } 2/3 \le t \le 1 \end{cases}$$

where $s \in S^i$, $t \in \mathbb{I}$, and $x \in U$. Then $\bar{H}(s,t)$ is equivariant for $(s,t) \in S^i \times \mathbb{I}$, and has the following properties:

$$\bar{H}(s,0) = f_0(s), \quad \bar{H}(s,1) = f_1(s), \quad \bar{H}(s,t)(0) \in J^r_{G,\mathcal{R}}(U,Y')_0.$$

Hence there exists a neighbourhood W of 0 in U such that $\bar{H}(s,t)|W$ is an equivariant section in $J^r_G(W,N)$, for $(s,t) \in S^i \times \mathbb{I}$. Let the embedding $\lambda : Y' \longrightarrow N$ and the retraction $\mu : N \longrightarrow Y'$ induce the maps λ_* and μ_* in the following sequence

$$J^r_G(W,Y') \xrightarrow{\lambda_*} J^r_G(W,N) \xrightarrow{\mu_*} J^r_G(W,Y')$$

so that the composition is identity. Set $\bar{h}(s,t) = \mu_* \circ \bar{H}(s,t)|W$. Then $\bar{h}(s,0) = f_0(s)$, $\bar{h}(s,1) = f_1(s)$, and $\bar{h}(s,t)(0) \in \mathcal{R}$, since $\bar{H}(s,t)(0) \in J^r_G(W,Y')$. The last condition implies by the openness of $\mathcal{R}|W$ that there is an open neighbourhood W' of 0 in W such that $\bar{h}(s,t)|W'$ is an equivariant section in $J^r_{G,\mathcal{R}}(W',Y)$. This means that we have a continuous homotopy $\bar{h} : S^i \times \mathbb{I} \longrightarrow \Psi(0)$ between f_0 and f_1, and therefore e_* is a monomorphism.

It remains to check that e_* is an epimorphism. Take a continuous map $f : S^i \longrightarrow J^r_{G,\mathcal{R}}(U,Y)_0$ for some open invariant neighbourhood U of 0. Then proceeding as in (5.6), we get a continuous map $\bar{f} : S^i \longrightarrow \Phi(0)$ such that $e \circ \jmath^r \circ \bar{f}(s) = f(s)$ for each $s \in S^i$. Now define $\widetilde{f}(s) = \jmath^r(\bar{f}(s))$, so that $\widetilde{f}$ maps S^i into $\Psi(0)$ and $e_*(\widetilde{f}) = f$. Thus e_* is an epimorphism.

This concludes the proof of Proposition 3.6.

Remark 5.8 The above proof becomes much shorter if we use a sheaf theoretic result from [8] (p.76). In fact, surjectiveness of $\jmath^r_*$ follows directly from this result (compare Lemma 12.16), whereas injectiveness of ρ_* implies the same for $\jmath^r_*$.

6 Basic Compressibility Results

Throughout this section $\mathcal{F}$ will denote a quasi-continuous sheaf over a differentiable manifold X.

Definition 6.1 Let A be a subset of X. Then a *deformation* over A is a map $\psi : P\times \mathbb{I} \longrightarrow \mathcal{F}(A)$, where P is a compact polyhedron. Therefore, for a deformation ψ over a compact set A, each $\psi(p,t)$ is actually defined on some open set $U \supset A$, that is, ψ is a map from $P\times \mathbb{I}$ to $\mathcal{F}(U)$. We say that ψ is *fixed at a point* $u \in U$ if $\psi(p,t)(u) = \psi(p,0)(u)$ for all $(p,t) \in P\times \mathbb{I}$. The closure of the set of non-fixed points of ψ in U will be called *support* of ψ, and will be denoted by supp ψ.

Definition 6.2 A deformation ψ over A is called *compressible* if for an arbitrary small neighbourhood $\bar{U}$ of A in U, there is a deformation $\bar{\psi} : P\times \mathbb{I} \longrightarrow \mathcal{F}(U)$ with the following properties:

(*i*) $\bar{\psi}(p,t) \mid \mathrm{Op}\, A = \psi(p,0) \mid \mathrm{Op}\, A$, for all $(p,t) \in P\times \mathbb{I}$,

(*ii*) $\bar{\psi} \mid P\times 0 = \psi \mid P\times 0$,

(*iii*) supp $\bar{\psi} \subset \bar{U}$.

Thus compressibility property of a deformation ψ depends only on the behaviour of ψ near A, that is, on $\mathrm{Op}\, A \subset U$.

The sheaf $\mathcal{F}$ is called *compressible* if every deformation over an arbitrary compact set is compressible.

Definition 6.3 A deformation $\psi : P\times \mathbb{I} \longrightarrow \mathcal{F}(A)$ is called *microcompressible* if there is a positive number $\varepsilon \in (0,1]$ such that the restricted deformation $\psi \mid P\times [0,\varepsilon]$ is compressible.

We observe that the microcompressibility provides a universal ε for all arbitrary small neighbourhood $\bar{U}$ of A.

The sheaf $\mathcal{F}$ is called *microcompressible* if every deformation over an arbitrary compact set is microcompressible.

Definition 6.4 A deformation ψ over A is called *S-microcompressible* if for an arbitrary neighbourhood $\tilde{U}$ of supp ψ in U, there exists a positive $\tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{U})$, $\tilde{\varepsilon} \leq \varepsilon$, such that the restricted deformation $\psi \mid P \times [0, \tilde{\varepsilon}]$ can be compressed to a deformation $\bar{\psi} : P \times [0, \tilde{\varepsilon}] \longrightarrow \mathcal{F}(A)$ over A, satisfying, in addition to properties (i) – (iii) of (6.2), the condition:

$$\text{supp } \bar{\psi} \subset \tilde{U}.$$

We now state and prove some relations among the above concepts.

Proposition 6.5 *If $\mathcal{F}$ is a compressible sheaf then it is flexible.*

Proof. Consider a lifting problem for a pair of compact sets (B, A) as described in the following diagram.

(6.6)

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{\bar{\psi}_0} & \mathcal{F}(B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ P \times \mathbb{I} & \xrightarrow[\psi]{} & \mathcal{F}(A) \end{array}$$

The continuity of $\bar{\psi}_0$ and ψ means that they are respectively defined over some open subsets $U \supset B$ and $V \supset A$, where $V \subset U$. The commutativity condition implies that we can get V such that $\bar{\psi}_0 = \psi | P \times 0$ on V. Since ψ is compressible, there exists, for an arbitrary open neighbourhood $\bar{U}$ of A in V, a map $\psi' : P \times \mathbb{I} \longrightarrow \mathcal{F}(V)$ satisfying the conditions of (6.2). Now define a covering homotopy $\bar{\psi} : P \times \mathbb{I} \longrightarrow \mathcal{F}(B)$ by

$$\bar{\psi}(p,t)(x) = \begin{cases} \psi'(p,t)(x) & \text{if } x \in \bar{U} \\ \bar{\psi}_0(p,0)(x) & \text{if } x \in U \setminus \bar{U}. \end{cases}$$

□

Proposition 6.7 *If $\mathcal{F}$ is microcompressible then it is flexible.*

Proof. Consider the lifting problem (6.6). If each $\psi(p,t)$ is defined over an open neighbourhood U of A, we define an auxiliary deformation $\eta : Q \times [0,1] \longrightarrow \mathcal{F}(U)$, where $Q = P \times \mathbb{I}$, by the following formula:

$$\eta(p,t,\theta) = \psi(p, \min(t+\theta, 1)), \; p \in P, \; t \in \mathbb{I}, \; \theta \in [0,1].$$

By hypothesis, there exists an $\varepsilon > 0$ such that $\eta | Q \times [0,\varepsilon]$ is compressible. Hence the restriction of ψ to an arbitrary interval $[x_1, y_1] \subset [0,1]$ of length less than or equal to ε is a compressible deformation. The proof goes as follows. Let $\bar{U}$ and U_1 be open neighbourhoods of A such that $\mathrm{cl}\, U_1 \subset \bar{U} \subset U$, where cl denotes closure. Let $\tilde{\eta}$ be a microcompression for η such that supp $\tilde{\eta} \subset U_1$. Define $\tilde{\psi} : P \times [x_1, y_1] \longrightarrow \mathcal{F}(A)$ by

$$\tilde{\psi}(p,t) = \tilde{\eta}(p, x_1, t - x_1), \quad p \in P, \quad t \in [x_1, y_1], \quad |y_1 - x_1| < \varepsilon.$$

Then, for $a \in \mathrm{Op}\, A$,

$$\begin{aligned} \tilde{\psi}(p,t)(a) &= \tilde{\eta}(p, x_1, t - x_1)(a) \\ &= \eta(p, x_1, t - x_1)(a) \\ &= \psi(p,t)(a), \end{aligned}$$

and, for $x \in U$,

$$\begin{aligned} \tilde{\psi}(p,x_1)(x) &= \tilde{\eta}(p, x_1, t - x_1)(x) \\ &= \eta(p, x_1, t - x_1)(x) \\ &= \psi(p,t)(x). \end{aligned}$$

To prove the third property take $x \in X - U_1$; then

$$\tilde{\psi}(p,t)(x) = \tilde{\eta}(p, x_1, t - x_1)(x) = \tilde{\eta}(p, x_1, 0)(x) = \tilde{\psi}(p, x_1)(x).$$

Hence $\tilde{\psi}$ is a microcompression for $\psi \mid P \times [x_1, y_1]$ with supp $\tilde{\psi} \subset \bar{U}$.

The converses of (6.2) and (6.7) are also true (see [5]). They are not considered here, because we do not need to use them.

Thus we obtain a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $\mathbb{I}$ such that for each k, $0 \le k \le n-1$, $\psi \mid P \times [t_k, t_{k+1}]$ is a compressible deformation. We will complete the proof by induction. Our induction hypothesis is the following:

P_k : the lifting problem

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{\bar{\psi}_0} & \mathcal{F}(B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ P \times I_k & \xrightarrow[\psi \mid P \times I_k]{} & \mathcal{F}(A) \end{array}$$

has a solution $\psi_k : P \times I_k \longrightarrow \mathcal{F}(B)$, where $I_k = [0, t_k]$.

Observe that P_0 is trivially true, and P_n proves the proposition. Suppose now that P_k is true for some $k < n$ and consider the following lifting problem:

$$\begin{array}{ccc} P \times \{t_k\} & \xrightarrow{\psi_k \mid} & \mathcal{F}(B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ P \times [t_k, t_{k+1}] & \xrightarrow[\psi \mid]{} & \mathcal{F}(A) \end{array}$$

Applying Proposition 6.5 we get a lift, say $\psi'_{k+1} : P \times [t_k, t_{k+1}] \longrightarrow \mathcal{F}(B)$, of $\psi | P \times [t_k, t_{k+1}]$ such that $\psi'_{k+1} | P \times \{t_k\} = \psi_k | P \times \{t_k\}$. Then construct $\psi_{k+1} : P \times I_{k+1} \longrightarrow \mathcal{F}(B)$ by:

$$\psi_{k+1}(p,t) = \begin{cases} \psi_k(p,t) & \text{if } (p,t) \in P \times [0, t_k] \\ \psi'_{k+1}(p,t) & \text{if } (p,t) \in P \times [t_k, t_{k+1}] \end{cases}.$$

Thus we have proved that P_{k+1} is true, and hence the proposition. □

Proposition 6.8 *If $\mathcal{F}$ is S-microcompressible then it is microcompressible.*

Proof: Let ψ be a deformation over A and let U be its actual domain of definition. Take U for a neighbourhood of supp ψ. Then by the definition of S-microcompressibility, there exists a positive number $\tilde{\varepsilon} \leq \varepsilon$ such that $\psi \mid P \times [0, \tilde{\varepsilon}]$ is a compressible deformation which proves that ψ is microcompressible. □

Proposition 6.9 *Let ψ be a deformation over a compact set A. Let A' be a subset of A containing $A \cap \tilde{U}$, where $\tilde{U}$ is an open neighbourhood of supp ψ in $U(\psi)$. If ψ is S-microcompressible over A', then it is S-microcompressible over A.*

Proof: Let $\bar{U}$ be an arbitrary open set containing A and U_0 an open neighbourhood of supp ψ. The S-microcompressibility property of ψ as a deformation over A' gives an $\tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{U} \cap U_0) > 0$ and a map $\tilde{\psi} : P \times [0, \tilde{\varepsilon}] \longrightarrow \mathcal{F}(U)$ satisfying:

(*i*) $\tilde{\psi} \mid U' = \psi \mid U'$, where U' is an open neighbourhood of A',

(*ii*) $\tilde{\psi} \mid P \times \{0\} = \psi \mid P \times \{0\}$,

(*iii*) $\text{supp}\,\tilde{\psi} \subset \bar{U} \cap \tilde{U} \cap U_0 \subset \bar{U} \cap U_0$.

To complete the proof it remains to show that $\tilde{\psi} = \psi$ on a neighbourhood of A. Since (supp $\tilde{\psi} \cup$supp $\psi) \cap A \subset \tilde{U}$, $A \backslash \tilde{U}$ is disjoint from supp $\tilde{\psi} \cup$supp ψ. In fact, we can get a neighbourhood V of $A \setminus \tilde{U}$ such that V is disjoint from supp $\tilde{\psi} \cup$ supp ψ (because both the subsets supp $\tilde{\psi} \cup$ supp ψ and $A \setminus \tilde{U}$ are closed). We take $W = U' \cup V$. It is clear then that $A \subset W$ and $\tilde{\psi} = \psi$ on W. □

We shall now introduce double deformations and discuss their compressibility properties. These will be used in Section 8.

Definition 6.10 A continuous map $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is called a *double deformation* over A. Observe that a double deformation can always be thought of as a deformation $P' \times [0,\varepsilon] \longrightarrow \Phi(A)$ where P' denotes the polyhedron $P\times [0,\varepsilon]$. To avoid confusion we shall adopt the following convention: When ψ will be treated as a deformation, the first copy of $[0,\varepsilon]$ will be denoted by I_ε. Thus $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is a double deformation, whereas $\psi : P\times I_\varepsilon \times [0,\varepsilon] \longrightarrow \Phi(A)$ will denote a deformation.[1] *Support* of a double deformation $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is defined to be the support of the deformation $P\times I_\varepsilon \times [0,\varepsilon]$. So we may continue to denote the support of a double deformation ψ by $\operatorname{supp}\psi$. The ~~microcompressibility and~~ compressibility for a double deformation can also be defined in terms of the associated deformation. Explicitly,

Definition 6.11 A double deformation $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is *compressible* if the deformation $\psi : P\times I_\varepsilon \times [0,\varepsilon] \longrightarrow \Phi(A)$ is compressible.

Definition 6.12 A double deformation $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is *microcompressible* if there exists a positive number $\delta \le \varepsilon$ such that $\psi|P\times [0,\delta] \times [0,\delta]$ is compressible.

Definition 6.13 A double deformation $\psi : P\times [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is *$\widetilde{S}$-microcompressible* if there exists a positive number $\delta \le \varepsilon$ such that $\psi|P\times I_\delta \times [0,\delta]$ is S-microcompressible.

Clearly an $\widetilde{S}$-microcompressible double deformation is microcompressible.

[1] This convention will be useful in Section 8, where we deal with both deformations and double deformations

7 A Flexibility Theorem

(7.1) Let $\tilde{\Phi}$ denote the sheaf of equivariant C^∞ sclutions of the extension $\tilde{\mathcal{R}}$. We will see, in this section, that both the sheaves Φ and $\tilde{\Phi}$ are microflexible.

An important step towards proving Proposition 3.4 is to show that the sheaf $\tilde{\Phi}|X$ is flexible. This section is essentially devoted to the proof of this fact. To establish the flexibility of $\tilde{\Phi}|X$, we shall only utilise the microflexibility property of $\tilde{\Phi}$, and the existence of sharply moving diffeotopies in the subpseudogroup $\mathcal{D}$.

We need the following lemma, the proof of which can be deduced from the 'Equivariant Covering Homotopy Theorem' of Bierstone [1].

Lemma 7.2 *If $p : E \longrightarrow X$ is a G-locally trivial G-fibre bundle and σ is an equivariant C^k section of p over a closed G-invariant subset A of X, then σ can be extended to an equivariant C^k section over an open G-invariant neighbourhood of A.*

Proposition 7.3 *The sheaves Φ and $\tilde{\Phi}$ are microflexible.*

Proof. Let (A, B) be a pair of G-invariant compact subsets of X. Consider a lifting problem

$$\begin{array}{ccc} Q \times \{0\} & \xrightarrow{F_0} & \Phi(A) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ Q \times \mathbb{I} & \xrightarrow[f]{} & \Phi(B) \end{array}$$

where Q is a compact polyhedron. The maps F_0 and f fit together to give a G-section of the G-fibre bundle $\mathrm{id} \times p : Q \times \mathbb{I} \times X \longrightarrow Q \times \mathbb{I} \times E$ over

$(Q \times \{0\} \times \operatorname{Op} A) \cup (Q \times \mathbb{I} \times \operatorname{Op} B)$. Since p is G-locally trivial, so is $\mathrm{id} \times p$. Now, applying Lemma 7.2, we get an $\varepsilon > 0$ and a map $F : Q \times [0, \varepsilon] \longrightarrow \Gamma^{\infty}_{G}(p|\operatorname{Op} A)$ such that $F|Q \times \{0\} = F_0$ and $r \circ F = f$ (Lemma 7.2 takes care of different orders of smoothness). Since $\mathcal{R}$ is open, we can choose ε sufficiently small so that $F(q,t)$ is $\mathcal{R}$-regular for all $(q,t) \in Q \times [0, \varepsilon]$. This proves that Φ is microflexible.

Since $\widetilde{\mathcal{R}}$ is also an open relation, the same proof works for $\widetilde{\Phi}$. Consequently $\widetilde{\Phi}$ is also microflexible. □

Theorem 7.4 *Let A and B be two compact G-invariant subsets of X, and $A \subset B$. Let $\bar{\psi}_0 : Q \times \{0\} \longrightarrow \widetilde{\Phi}(B)$ be a map and $\psi : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(A)$ a homotopy of $\bar{\psi}_0|\operatorname{Op} A$, so that we have a commutative square of continuous maps*

$$\begin{array}{ccc} Q \times \{0\} & \xrightarrow{\bar{\psi}_0} & \widetilde{\Phi}(B) \\ {\scriptstyle i}\downarrow & \overset{\bar{\psi}}{\nearrow} & \downarrow{\scriptstyle r} \\ Q \times \mathbb{I} & \xrightarrow[\psi]{} & \widetilde{\Phi}(A) \end{array}$$

where r is the restriction morphism. Then there exists a deformation $\bar{\psi} : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(B)$ such that $\bar{\psi}|Q \times \{0\} = \bar{\psi}_0$ and $r \circ \bar{\psi} = \psi$.

In other words, the sheaf $\widetilde{\Phi}|X$ is flexible.

Proof. In view of Proposition 6.5, it is enough to prove that $\widetilde{\Phi}|X$ is compressible.

To prove this, let us consider an arbitrary deformation $\psi : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(A)$, and a G-invariant open neighbourhood $\bar{U}$ of A in $U \cap X$ where $U \subset X \times \mathbb{R}$ is the domain of ψ. We shall show that there exists an $\varepsilon > 0$ and a map $\bar{\psi} : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(U \cap (X \times (-\varepsilon, \varepsilon)))$ which satisfies the following three properties.

- $\bar{\psi}(p,t)| U_0 = \psi(p,t)| U_0$ for all $p \in P$ and $t \in \mathbb{I}$, where U_0 is an open neighbourhood of A.
- $\bar{\psi} \mid Q \times 0 = \psi \mid Q \times 0$,
- $\operatorname{supp} \bar{\psi} \subset U \cap (\bar{U} \times (-\varepsilon, \varepsilon))$.

Since A is compact, we get a G-invariant open neighbourhood U_1 of A in X (with $\operatorname{cl} U_1$ compact) and an $a > 0$ such that

$$U_1 \subset \bar{U} \quad \text{and} \quad \operatorname{cl} U_1 \times [-2a, 2a] \subset U.$$

Choose G-invariant open sets V_0, V such that $\operatorname{cl} V_0$ and $\operatorname{cl} V$ are compact and

$$A \subset V_0 \subset \operatorname{cl} V_0 \subset V \subset \operatorname{cl} V \subset U_1.$$

Set
$$X_0 = \operatorname{cl} V_0 \times [-a/2, a/2]$$
and
$$Y_0 = \operatorname{cl} U_1 \times [-2a, 2a] \setminus V \times (-a, a)$$

The sets X_0 and Y_0 are compact, G-invariant and disjoint from each other. Let Δ denote the diagonal subset of $\mathbb{I} \times \mathbb{I}$. Define a map $\varphi_1 : Q \times \Delta \longrightarrow \widetilde{\Phi}(\operatorname{cl} U_1 \times [-2a, 2a])$ by

$$\varphi_1(q,t,t) = \psi(q,t) \quad \text{for} \quad (q,t) \in Q \times \mathbb{I}.$$

Define another map $\varphi_2 : Q \times \mathbb{I} \times \mathbb{I} \longrightarrow \widetilde{\Phi}(X_0 \cup Y_0)$ by

$$\varphi_2(q,t,s)(x) = \begin{cases} \psi(q,s)(x) & \text{if } x \in X_0 \\ \psi(q,t)(x) & \text{if } x \in Y_0 \end{cases}.$$

Observe that $r \circ \varphi_1 = \varphi_2 | Q \times \Delta$, where r is the restriction morphism $\widetilde{\Phi}(\operatorname{cl} U_1 \times [-2a, 2a]) \longrightarrow \widetilde{\Phi}(X_0 \cup Y_0)$. Consider now the following lifting problem

$$
\begin{array}{ccc}
Q \times \Delta & \xrightarrow{\ \varphi_1\ } & \widetilde{\Phi}(\operatorname{cl} U_1 \times [-2a, 2a]) \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle r} \\
Q \times \mathbb{I} \times \mathbb{I} & \xrightarrow[\ \varphi_2\ ]{} & \widetilde{\Phi}(X_0 \cup Y_0)
\end{array}
$$

The diagonal Δ divides the square $\mathbb{I} \times \mathbb{I}$ into two triangular regions. Let T denote any one of them. Then note that the pair $(Q \times T, \Delta)$ is homeomorphic to $(Q \times \mathbb{I} \times \mathbb{I}, Q \times \mathbb{I} \times \{0\})$. Therefore, since r is a microfibration, there exists a neighbourhood N of $Q \times \Delta$ in $Q \times \mathbb{I} \times \mathbb{I}$ and a map $\widetilde{\psi} : N \longrightarrow \widetilde{\Phi}(\operatorname{cl} U_1 \times [-2a, 2a])$ such that $\widetilde{\psi}|Q \times \Delta = \varphi_1$ and $r \circ \widetilde{\psi} = \varphi_2$.

Since N is an open neighbourhood of the compact set $Q \times \Delta$, we can find a positive number $\varepsilon \leq 1$ such that $(q, t, s) \in N$ whenever $|t - s| < \varepsilon$. We now partition the interval $[0, 1]$ as follows:

$$0 = t_0 < t_1 < \cdots < t_n = 1 \text{ such that } |t_k - t_{k+1}| < \varepsilon \text{ for all } k,$$

and define, for each k, a map

$$\lambda_k : Q \times [t_k, t_{k+1}] \longrightarrow \widetilde{\Phi}(\operatorname{cl} U_1 \times [-2a, 2a])$$

by the rule

$$\lambda_k(q, t)(x) = \widetilde{\psi}(q, t_k, t)(x).$$

Then λ_k has the following properties:

- $\lambda_k(q, t_k)(x) = \widetilde{\psi}(q, t_k, t_k)(x) = \varphi_1(q, t_k, t_k)(x) = \psi(q, t_k)$ for all x,
- For $x \in X_0$, $\lambda_k(q, t)(x) = \widetilde{\psi}(q, t_k, t)(x) = \varphi_2(q, t_k, t)(x) = \psi(q, t)(x)$,
- For $x \in Y_0$, $\lambda_k(q, t)(x) = \widetilde{\psi}(q, t_k, t)(x) = \varphi_2(q, t_k, t)(x) = \psi(q, t_k)(x)$, that is, non-fixed points of λ_k lies inside $V \times (-a, a)$.

We are now in a position to define the required deformation $\bar{\psi}$ using the above λ_k's and the sharply moving diffeotopies. Suppose that, for some

k, $1 \leq k \leq n-1$, we have an $\varepsilon_k > 0$ and a map $\psi_k : Q \times [0, t_k] \longrightarrow \widetilde{\Phi}(U_1 \times (-\varepsilon_k, \varepsilon_k))$ such that for all $q \in Q$ and $t \in [0, t_k]$,

(*i*) $\psi_k(q,t) = \psi(q,t)$ on $V_k \times (-\varepsilon_k, \varepsilon_k)$, where V_k is a G-invariant neighbourhood of A in V_0,

(*ii*) $\psi_k(q,0) = \psi(q,0)$,

(*iii*) $\widetilde{\text{supp}}\, \psi_k \subset V \times (-\varepsilon_k, \varepsilon_k)$, where $\widetilde{\text{supp}}\, \psi_k$ denotes the set of non-fixed points of ψ_k.

We shall construct $\psi_{k+1} : Q \times [0, t_{k+1}] \longrightarrow \widetilde{\Phi}(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}))$, for some positive number $\varepsilon_{k+1} < \varepsilon_k$.

Choose open G-invariant neighbourhoods V'_k and W_k (with compact closures) of A in V_k satisfying:

$$A \subset W_k \subset \text{cl}\, W_k \subset V'_k \subset \text{cl}\, V'_k \subset V_k.$$

Now, if τ is such that $0 < \tau < \min(d(A, \partial(\text{cl}\, W_k)), d(W_k, \partial(\text{cl}\, V'_k)))$, where d denotes the distance with respect to the G-invariant Riemannian metric on X, then the τ-neighbourhood of $\partial(\text{cl}\, W_k)$ in X is contained in $V'_k \setminus A$.

Let us consider the open subset $U' = U_1 \times (-2a, 2a)$ of $X \times \mathbb{R}$. By hypothesis of sharply moving isotopy (see (1.9) for definition), there exists a positive number $\varepsilon_{k+1} < \varepsilon_k$ and an isotopy

$$\sigma : \mathbb{I} \longrightarrow \mathcal{E}_G(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}), U')$$

which lies in $\mathcal{D}$ and sharply moves U_1 at $\partial(\text{cl}\, W_k)$. Then, $\sigma_t^{*}\lambda_k(q,s) \in \widetilde{\Phi}(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}))$ for each $t \in \mathbb{I}$, $q \in Q$ and $s \in [t_k, t_{k+1}]$, since $\widetilde{\mathcal{R}}$ is invariant under the action of $\mathcal{D}$.

Let $\bar{\sigma}_t : U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}) \longrightarrow U'$, $0 \leq t \leq t_{k+1}$, be the isotopy obtained by shrinking σ_t:

$$\bar{\sigma}_t = \begin{cases} \sigma_{t/t_k} & \text{if } 0 \le t \le t_k \\ \sigma_1 & \text{if } t_k \le t \le t_{k+1} \end{cases}$$

Then corresponding to properties of σ_t we have:

a) $\bar{\sigma}_0$ is the inclusion map,

b) $\bar{\sigma}_t(x,s) = (x,s)$ if x is outside the τ-neighbourhood of $\partial(\mathrm{cl}\, W_k)$,

c) if $t_k \le t \le t_{k+1}$, then $\bar{\sigma}_t$ sends $N \times (-\varepsilon_{k+1}, \varepsilon_{k+1})$ outside the support of λ_k, where N is a neighbourhood of $\partial(\mathrm{cl}\, W_k)$ in U_1.

Now define $\psi_{k+1} : Q \times [0, t_{k+1}] \longrightarrow \widetilde{\Phi}(U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1}))$ in the following way:

$$\begin{aligned} \psi_{k+1}(q,t)(x,s) &= \psi_k(q,t)(x,s) \quad , & x \notin V'_k,\ 0 \le t \le t_k \\ &= [\bar{\sigma}_t^* \psi(q,t)](x,s) \ , & x \in \mathrm{cl}\, V'_k,\ 0 \le t \le t_k \\ &= [\bar{\sigma}_t^* \lambda_k(q,t)](x,s), & x \in \mathrm{cl}\, W_k,\ t_k \le t \le t_{k+1} \\ &= [\bar{\sigma}_t^* \psi(q,t_k)](x,s), & x \in \mathrm{cl}\, V'_k \setminus W_k,\ t_k \le t \le t_{k+1} \\ &= \psi_k(q,t_k)(x,s) \quad , & x \notin V'_k,\ t_k \le t \le t_{k+1} \end{aligned}$$

where $q \in Q$ and $s \in (-\varepsilon_{k+1}, \varepsilon_{k+1})$. The map ψ_{k+1} has the following properties:

- By property (b) of $\bar{\sigma}_t$, and the definition of ψ_k, there exists a neighbourhood V_{k+1} of A in W_k such that $\psi_{k+1} = \psi$ on $V_{k+1} \times (-\varepsilon_{k+1}, \varepsilon_{k+1})$.

- $$\psi_{k+1}(q,0)(x,s) = \begin{cases} \psi_k(q,0)(x,s) = \psi(q,0)(x,s) & \text{if } x \notin V'_k \\ \bar{\sigma}_0^* \psi(q,0)(x,s) = \psi(q,0)(x,s) & \text{if } x \in \mathrm{cl}\, V'_k \end{cases}.$$

  Hence $\psi_{k+1}(q,0) = \psi(q,0)$ on $U_1 \times (-\varepsilon_{k+1}, \varepsilon_{k+1})$.

- Let $x \notin V$. Then for $0 \le t \le t_k$, $\psi_{k+1}(q,t)(x,s) = \psi_k(q,t)(x,s) = \psi_k(q,0)(x,s)$. Therefore $\psi_{k+1}(q,t)(x,s) = \psi_{k+1}(q,0)(x,s)$ for all $x \notin V$

and $t \in [0, t_k]$. For $t \geq t_k$ and $x \notin V$,

$$\begin{aligned} \psi_{k+1}(q,t)(x,s) &= \psi_k(q,t_k)(x,s) \\ &= \psi_k(q,0)(x,s), \qquad \text{since } \widetilde{\operatorname{supp}}\,\psi_k \subset V \times (-\varepsilon_k, \varepsilon_k) \\ &= \psi_{k+1}(q,0)(x,s), \qquad \text{since } x \notin V'_k. \end{aligned}$$

Thus ψ_{k+1} has all the desired properties. Observe that ψ_n gives our required $\bar{\psi}$ with $\varepsilon = \varepsilon_n$; because, $\widetilde{\operatorname{supp}}\,\psi_n \subset V \times (-\varepsilon_n, \varepsilon_n)$ implies that $\operatorname{supp} \psi_n \subset U_1 \times (-\varepsilon_n, \varepsilon_n)$, and hence we can extend ψ_n to $U \cap (X \times (-\varepsilon_n, \varepsilon_n))$ by defining it to be fixed on the complement of $U_1 \times (-\varepsilon_n, \varepsilon_n)$.

To start the induction we must now define $\psi_1 : Q \times [0, t_1] \longrightarrow \widetilde{\Phi}(\mathrm{Op}\, U_1)$. For this construction we simply repeat the arguments above for $k = 0$. Note that we must take $t_k = t_0 = 0$, $V_k = V_0$ and $\tilde{\sigma}_t = \sigma_1$ for $0 \leq t \leq t_1$. The definition of ψ_1 can be read out from the definition of ψ_{k+1} as:

$$\psi_1(q,t)(x,s) = \begin{cases} \psi(q,0)(x,s) & \text{if } x \notin V'_0 \\ [\sigma_1^* \psi(q,0)](x,s) & \text{if } x \in \operatorname{cl} V'_0 \setminus W_0 \\ [\sigma_1^* \lambda_0(q,t)](x,s) & \text{if } x \in \operatorname{cl} W_0 \end{cases}.$$

where σ_1 is defined on $U_1 \times (-\varepsilon_1, \varepsilon_1)$ for some $\varepsilon_1 < a/2$. □

Corollary 7.5 *Let $\psi : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(A)$ be a deformation over a compact subset $A \subset X$ with actual domain U. If $\bar{U}$ and $\widetilde{U}$ are G-invariant open subsets in U containing A and $\operatorname{supp}\psi$ respectively, then there exists a deformation $\bar{\psi} : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(U \cap X)$ such that*

(*i*) *$\bar{\psi} = \psi$ on a neighbourhood of A,*

(*ii*) *$\bar{\psi}|Q \times \{0\} = \psi|Q \times \{0\}$,*

(*iii*) *$\operatorname{supp}\bar{\psi} \cap X \subset \widetilde{U} \cap \bar{U} \cap X$.*

Proof. Without loss of generality we may assume that $\operatorname{cl}\bar{U}$ is compact. Then, both the subsets $\operatorname{cl}(\widetilde{U} \cap \bar{U} \cap X) = B$ and $\operatorname{cl}(\widetilde{U} \cap \bar{U} \cap X) \setminus (\widetilde{U} \cap \bar{U} \cap X) = C$ are compact. Let us denote $\operatorname{supp}\psi$ by S. Then by hypothesis $A \cap S \subset \bar{U} \cap \tilde{U}$, and therefore $C \cap A \cap S = \emptyset$. Hence, there exist open neighbourhoods V of A, and W of C such that $V \cap W \cap S = \emptyset$. Now, consider the lifting problem described in the following diagram:

$$\begin{array}{ccc} Q \times \{0\} & \xrightarrow{\psi|Q \times \{0\}} & \widetilde{\Phi}(A \cup B) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ Q \times \mathbb{I} & \xrightarrow[\psi']{} & \widetilde{\Phi}(A \cup C) \end{array}$$

where ψ' is defined as follows:

$$\psi'(q,t)(x) = \begin{cases} \psi(q,t)(x) & \text{if } x \in \operatorname{Op} A \\ \psi(q,0)(x) & \text{if } x \in \operatorname{Op} C \end{cases}.$$

The map ψ' is well defined by the observation that $V \cap W \cap S = \emptyset$. Since $\widetilde{\Phi}|X$ is flexible (Theorem 7.4), we get a lift $\bar{\psi} : Q \times \mathbb{I} \longrightarrow \widetilde{\Phi}(A \cup B)$ of ψ' such that $\bar{\psi}(q,0) = \psi(q,0)$ for all $q \in Q$. It is easy to see that $\bar{\psi}$ has all the required properties. $\square$

8 Flexibility of Sheaf Φ

All the necessary prerequisites having been obtained, we are now in a position to prove Proposition 3.4. Note that it is only required to establish the microcompressibility of the sheaf Φ, since microcompressibility implies flexibility. We describe some of the main steps in the proof of this fact before moving onto the detailed constructions.

We first consider microcompressibility in the case when the polyhedron P is just a point. Then, letting $\varphi : \mathbb{I} \longrightarrow \Phi(A)$ to be an arbitrary deformation over a compact set A, we find a finite covering of A by compact invariant neighbourhoods A_i in A such that φ restricted to each $\mathrm{Op}\, A_i$ can be lifted to $\widetilde{\Phi}$. Next, we decompose φ into 'product of double deformations' ψ_i with $\mathrm{supp}\, \psi_i \cap A \subset A_i$ such that microcompressibility of all the ψ_i's (over the compact A_i's) imply that of φ. We then lift each ψ_i, as a deformation over A_i, to a map ψ'_i in $\widetilde{\Phi}$ so that the support of ψ'_i is contained in a preassigned neighbourhood of $\mathrm{supp}\, \psi_i$. These ψ'_i's can be compressed in $\widetilde{\Phi}$ using the flexibility of $\widetilde{\Phi}|X$ to deformations $\widetilde{\psi}_i$ over A_i. Finally $\widetilde{\psi}_i$ when projected to Φ gives microcompression of ψ_i over A_i. This proves the microcompressibility of Φ when P is a point. We take up the general case for an arbitrary polyhedron at the end of this section.

(8.1) We note that, since $\widetilde{\mathcal{R}}$ is an extension of $\mathcal{R}$, there is a restriction map

$$\alpha(C) : \widetilde{\Phi}(C) \longrightarrow \Phi(C).$$

defined by $\alpha(\widetilde{f}) = \pi \circ \widetilde{f} \circ i$, for each subset C of X.

Lemma 8.2 *For each $x \in X$, the restriction morphism $\alpha(x) : \widetilde{\Phi}(x) \longrightarrow \Phi(x)$ is surjective.*

Proof. Let us recall that a small G-invariant open neighbourhoood of $x \in X$

looks like $G \times_{G_x} U$, where U is a G_x-invariant neighbourhood of 0 in the slice W_x at x (see Section 5). Observe that the slice at $(x, 0)$ in $X \times \mathbb{R}$ is $W_x \times \mathbb{R}$, and hence a small open neighbourhood of $(x, 0) \in X \times \mathbb{R}$ looks like $G \times_{G_x} (U \times J)$, where U is as described above and J is an open interval in $\mathbb{R}$ containing 0. As we have observed in Section 5, there is a G_x-equivariant open C^∞ map $\tilde{\phi} : E'^{(r)}|U \times J \longrightarrow (E'|U \times J)^{(r)}$ which maps $E'^{(r)}_G|U \times J$ isomorphically onto $(E'|U \times J)^{(r)}_{G_x}$. If $\tilde{\mathcal{R}}_x$ is the image of $\tilde{\mathcal{R}}|U \times J$ under the map $\tilde{\phi}$, then $\tilde{\mathcal{R}}_x$ is open, and $\tilde{\phi}$ induces a homeomorphism

$$\Gamma^\infty_{G,\tilde{\mathcal{R}}}(p'|G \times_{G_x} U \times J) \cong \Gamma^\infty_{G_x,\tilde{\mathcal{R}}_x}(p'|U \times J)$$

where $p' = p \times \mathrm{id} : E' = E \times \mathbb{R} \longrightarrow X \times \mathbb{R}$. We also have a commutative diagram:

$$\begin{array}{ccc} \Gamma^\infty_{G,\tilde{\mathcal{R}}}(p'|G \times_{G_x} U \times J) & \longrightarrow & \Gamma^\infty_{G_x,\tilde{\mathcal{R}}_x}(p'|U \times J) \\ \downarrow & & \downarrow \\ \Gamma^\infty_{G,\mathcal{R}}(p|G \times_{G_x} U) & \longrightarrow & \Gamma^\infty_{G_x,\mathcal{R}_x}(p|U) \end{array}$$

where the horizontal arrows are homeomorphisms. As we pass to the direct limit, the above diagram gives rise to a commutative diagram of stalks of sheaves. In the resulting square, the left vertical map between stalks will be surjective if the right vertical map between stalks at $0 \in W_x$ is so (we identify $0 \in W_x$ with $(0, 0) \in W_x \times \mathbb{R}$ via the canonical embedding of X in $X \times \mathbb{R}$ as the subspace $X \times \{0\}$). Therefore, we may assume that X is a linear G-representation space with $x = 0 \in X^G$. We denote the stalk of the sheaf $\Gamma^\infty_{G,\tilde{\mathcal{R}}}(p'|U \times J)$ by $\tilde{\Phi}(0)$, where U and J run over open G-invariant neighbourhoods of $x \in X$ and $0 \in \mathbb{R}$ respectively. Then it is required to prove that the restriction map $\alpha(0) : \tilde{\Phi}(0) \longrightarrow \Phi(0)$ is surjective.

Let f be a local regular G-section of p defined on a compact ball D around x. We will produce an $\tilde{f} \in \tilde{\Phi}(x)$ such that $\alpha(x)(\tilde{f}) = f$. For this purpose, we identify $(E|D)^{(r)}$ with the jet space $J^r(D, Y)$ and $(E'|D \times \mathbb{R})^{(r)}$ with the jet space $J^r(D \times \mathbb{R}, Y)$ where Y is the fibre of p (as well as the fibre of $p \times \mathrm{id}$). The induced map $\pi^{(r)} : J^r(D \times \mathbb{R}, Y) \longrightarrow J^r(D, Y)$ is defined as $\pi^{(r)}(j^r_{(x,t)}\tilde{f}) = j^r_x(\tilde{f} \circ i_t)$. Using extensibility property of $\mathcal{R}$, we get an r-jet $\tau \in i^*(\tilde{\mathcal{R}}_G)$ such that $\pi^{(r)}(\tau) = j^r_x f$ under the above identification. Define $\bar{f} : D \times \mathbb{R} \longrightarrow Y$ by $\bar{f}(y,t) = f(y)$ for $y \in D$ and $t \in \mathbb{R}$. The set $f(D)$ is compact and invariant; we denote it by Y'. Let $e : Y' \longrightarrow V$ be an equivariant embedding of Y' in an Euclidean G-manifold V such that $e(f(x)) = 0$. The embedding e induces maps $J^r(D, Y') \longrightarrow J^r(D, V)$ and $J^r(D \times \mathbb{R}, Y') \longrightarrow J^r(D \times \mathbb{R}, V)$. Both of them will be denoted by the same symbol e_*. Thus we have the commutative diagram

$$\begin{array}{ccc} J^r(D \times \mathbb{R}, Y') & \xrightarrow{\pi^{(r)}} & J^r(D, Y') \\ \Big\downarrow e_* & & \Big\downarrow e_* \\ J^r(D \times \mathbb{R}, V) & \xrightarrow[\pi^{(r)}]{} & J^r(D, V) \end{array}$$

where $\pi^{(r)} : J^r(D \times \mathbb{R}, V) \longrightarrow J^r(D, V)$ is the map induced by $\pi : X \times \mathbb{R} \longrightarrow X$ as described above.

We identify Y' with $e(Y')$. Let N be a G-invariant tubular neighbourhood of Y' in V and let $\mu : N \longrightarrow Y'$ be an equivariant retraction onto Y'. Consider the r-jet $\sigma = e_*\tau - j^r_{(x,0)}(e \circ \bar{f})$, and let P be the (equivariant) polynomial representative of σ so that $P \circ i = 0$ (since $P \circ i$ is a polynomial of degree r and its r-jet $j^r_x(P \circ i) = \pi^{(r)} j^r_{(x,0)} P = \pi^{(r)}(e_*\tau) - \pi^{(r)}(j^r_{(x,0)}(e \circ \bar{f})) = e_*(\pi^{(r)}\tau - j^r_x f) = 0$.) The map $P + e \circ \bar{f}$ is equivariant and has $e_*\tau$ as its r-jet at $(x,0)$. Moreover, $P+e\circ\bar{f}|D = e\circ f$ and $(P+e\circ\bar{f})(\mathrm{Op}\,(x,0)) \subset N$. Define $\tilde{f} = \mu\circ(P+e\circ\bar{f})$. The map $\tilde{f}$ is equivariant and has $\tau\in\tilde{\mathcal{R}}$ as its r-jet at $(x,0)$, because μ induces a retraction of $J^r(D\times\mathbb{R},N)$ onto $J^r(D\times\mathbb{R},Y')$ (if we think of $J^r(D\times\mathbb{R},Y')$ as an embedded subspace of $J^r(D\times\mathbb{R},N)$) and $j^r_{(x,0)}(P+e\circ\bar{f}) \in J^r_{(x,0)}(D\times\mathbb{R},Y') \subset J^r_{(x,0)}(D\times\mathbb{R},Y)$.[1] Since $\tilde{\mathcal{R}}$ is open, $\tilde{f}$ is $\tilde{\mathcal{R}}$-regular on a neighbourhood of $(x,0)$. Moreover $\tilde{f}\circ i = f$, and hence $\alpha(x)$ is onto. □

Definition 8.3 Let (A,B) be a pair of subsets in X. Then any two sections $\sigma\in\Phi(A)$ and $\tilde{\sigma}\in\tilde{\Phi}(B)$ are said to be *coherent* if the restriction $\sigma \mid \mathrm{Op}\,B \in \Phi(B)$ equals $\alpha(\tilde{\sigma})\in\Phi(B)$. We will denote the space of all coherent pairs $(\sigma,\tilde{\sigma})\in\Phi(A)\times\tilde{\Phi}(B)$ by $\Gamma(A,B)$. Then there is a natural map $\gamma(A,B)$: $\tilde{\Phi}(A)\longrightarrow\Gamma(A,B)$, denoted also by γ, such that $\gamma(\tilde{\sigma}) = (\alpha(\tilde{\sigma}),\tilde{\sigma}|\mathrm{Op}\,B)$.

We observe that if $B=\emptyset$, then $\Gamma(A,B)=\Phi(A)$ and $\gamma=\alpha$.

The restriction morphism $\alpha:\tilde{\Phi}\longrightarrow\Phi$ is called a *microextension* if $\alpha(x):\tilde{\Phi}(x)\longrightarrow\Phi(x)$ is surjective for each $x\in X$ and if $\gamma(A,B)$ is a microfibration for every pair of compact subsets (A,B) in X. It is clear from the definition that when α is a microextension, $\alpha(A):\tilde{\Phi}(A)\longrightarrow\Phi(A)$ is a microfibration for every compact subset A of X.

Lemma 8.4 *For every pair of compact subsets (A,B) in X, $\gamma(A,B)$ is a microfibration. Consequently, α is a microextension*

Proof. The proof of the fact that $\gamma(A,B)$ is a microfibration is entirely similar to that of Proposition 7.3; one has only to use Lemma 7.2 and openness of $\tilde{\mathcal{R}}$. With Lemma 8.2, this shows that α is a microextension. □

[1]Note that we have not used the openness property of $\tilde{\mathcal{R}}$ so far. This observation will be recalled in Section 12.

(8.5) We have the following information in hand:

- Φ is a microflexible sheaf.
- $\widetilde{\Phi}|X$ is a flexible sheaf.
- $\alpha : \widetilde{\Phi} \longrightarrow \Phi$ is a microextension.

We now turn to the proof of the fact that an arbitrary deformation $\varphi : \mathbb{I} \longrightarrow \Phi(A)$ over a compact subset A of X is microcompressible. The proof is described in Subsections 8.5.1-8.5.9.

(8.5.1) Set $\varphi_0 = \varphi(0)$. Choose any $a \in A$. Since α is a microextension, a has a compact G-invariant neighbourhood A' in A such that $\varphi_0 \mid \operatorname{Op} A'$ has a lift, say $\widetilde{\varphi}_0 \in \widetilde{\Phi}(A')$, so that $\alpha \circ \widetilde{\varphi}_0 = \varphi_0 | \operatorname{Op} A'$.

Choose finitely many such small G-invariant compact subsets A_i of A, $i = 0, \ldots, k$, such that $\bigcup_{i=0}^{k} \mathring{A}_i$ covers A, where $\mathring{A}_i$ denotes the interior of A_i in A.

We shall now introduce some notations and observe a few simple facts.

Notation 8.5.2 Consider a double deformation $\psi : [0, \varepsilon] \times [0, \varepsilon] \longrightarrow \Phi(A)$ (see (6.10)). We will denote the restriction of ψ to $[0, \varepsilon] \times \{0\}$ by ψ^0. The restriction of ψ to Δ, the diagonal of $[0, \varepsilon] \times [0, \varepsilon]$, gives another deformation $\psi^{\bullet} : [0, \varepsilon] \longrightarrow \Phi(A)$ so that $\psi^{\bullet}(t) = \psi(t, t)$ for $t \in [0, \varepsilon]$.

Observation 8.5.3 *If ψ^0 and ψ are microcompressible (resp. compressible) then $\psi^{\bullet}$ is also so.*

Proof. Let U' be the domain of ψ. Take any G-invariant open neighbourhood $\bar{U}$ of A in U' and compress ψ^0 to $\bar{\psi}^0$ with $\operatorname{supp} \bar{\psi}^0 \subset \bar{U}$. By hypothesis, there exists a positive $\bar{\varepsilon} \leq \varepsilon$ such that $\psi|[0, \bar{\varepsilon}] \times [0, \bar{\varepsilon}]$ can be compressed to $\bar{\psi} : [0, \bar{\varepsilon}] \times [0, \bar{\varepsilon}] \longrightarrow \Phi(A)$ with $\operatorname{supp} \bar{\psi}$ contained in U, where U is a G-invariant open neighbourhood of A in $\bar{U}$, for which $\bar{\psi}^0 \mid U = \psi^0 \mid U$. We define the required microcompression $\bar{\psi}^\bullet : [0, \bar{\varepsilon}] \longrightarrow \Phi(A)$ of $\psi^\bullet$ by

$$\bar{\psi}^\bullet(t)(u) = \begin{cases} \bar{\psi}(t,t)(u) & \text{if } u \in U \\ \bar{\psi}^0(t)(u) & \text{if } u \in U' \setminus U \end{cases}$$

where $t \in [0, \bar{\varepsilon}]$. □

Definition 8.5.4 A finite sequence of double deformations $\psi_0, \ldots, \psi_k : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is said to be *compatible* if $\psi_i^0 = \psi_{i-1}^\bullet$ for $i = 1, \ldots, k$. For such a sequence we define the product $\circ_{i=0}^k \psi_i$ to be $\psi_k^\bullet$.

Repeated application of Observation 8.5.3 yields the following

Observation 8.5.5 *If the double deformations $\psi_1, \ldots, \psi_k : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ are compatible, and if $\psi_1^0, \psi_1, \ldots, \psi_k$ are microcompressible (resp. compressible) then the product $\psi_k^\bullet$ is also a microcompressible (resp. compressible) deformation.*

In view of (8.5.5), our objective will be to express φ as the product of compatible double deformations (in the sense of 8.5.4.), each of which is microcompressible.

Lemma 8.5.6 *There exist compatible double deformations $\psi_i : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ such that*

$$\varphi \mid [0,\varepsilon] = \circ_{i=1}^k \psi_i, \quad \operatorname{supp} \psi_1^0 \cap A \subset \mathring{A}_0 = U_0$$

$$\operatorname{supp} \psi_i \cap A \subset \mathring{A}_i = U_i, \quad i = 1, \ldots, k.$$

Proof. We shall prove the lemma by induction. Let $S = \operatorname{supp} \varphi$ and $\widetilde{S} = \operatorname{supp} \varphi \cap A$. Our induction principle is

P_i : there are deformations $\psi_k, \psi_{k-1}, \ldots, \psi_{k-i+1}$ such that

$$\psi_k^\bullet = \varphi \quad \text{and} \quad \psi_{k-j}^\bullet = \psi_{k-j+1}^0 \quad j = 1, \ldots, i-1$$

with

$$\operatorname{supp} \psi_{k-j+1} \cap A \subset U_{k-j+1}, \quad j = 1, \ldots, i.$$

$$\operatorname{supp} \psi_{k-i+1}^0 \cap A \quad \subset \quad \bigcap_{j=1}^i \operatorname{Op}(A \setminus U_{k-j+1}).$$

Since $\bigcap_{j=1}^{k} \mathrm{Op}\,(A \setminus U_{k-j+1}) \subset U_0$, P_k is just the assertion we require to prove. The next sublemma whose proof is deferred for a while, completes the induction.

Sublemma 8.5.7 *If $\varphi : [0,1] \longrightarrow \Phi(A)$ is a deformation and if V_0, V_1 are open sets in A covering A, then there exists a double deformation $\psi : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ for some $\varepsilon > 0$ such that*

$$\psi^* = \varphi \mid [0,\varepsilon], \qquad \mathrm{supp}\,\psi^0 \cap A \subset \mathrm{Op}\,\widetilde{S} \cap V_0,$$
$$\mathrm{supp}\,\psi \cap A \subset \mathrm{Op}\,\widetilde{S} \cap V_1.$$

Applying the sublemma to φ, $\mathrm{Op}\,(A \setminus U_k)$ and U_k, we get P_1. Next, if P_i is true, then applying the sublemma once more to ψ^0_{k-i+1}, $\mathrm{Op}\,(A \setminus U_{k-i})$ and U_{k-i}, we get a deformation ψ_{k-i} such that

$$\psi^*_{k-i} = \psi^0_{k-i+1}, \qquad \mathrm{supp}\,\psi^0_{k-i} \cap A \subset \textstyle\bigcap_{j=1}^{i+1} \mathrm{Op}\,(A \setminus U_{k-j+1}),$$
$$\mathrm{supp}\,\psi_{k-i} \cap A \subset \textstyle\bigcap_{j=1}^{i} \mathrm{Op}\,(A \setminus U_{k-j+1}) \cap U_{k-i} \subset U_{k-i}.$$

This proves the lemma modulo Sublemma 8.5.7.

We now proceed to the proof of Sublemma 8.5.7. Suppose that each $\varphi(t)$ is defined on a G-invariant open neighbourhood U of A in X. Since our required ψ satisfies the condition $\psi^* = \varphi$, therefore, the restriction of ψ to the diagonal Δ of $[0,1] \times [0,1]$ is given by:

$$f : \Delta \longrightarrow \Phi(A)$$
$$(t,t) \mapsto \varphi(t).$$

Also, since $\mathrm{supp}\,\psi^0 \cap A \subset V_0 \cap \mathrm{Op}\,\widetilde{S}$, the restriction of $\psi(t,0)$, $t \in [0,1]$, to a neighbourhood of $C = A \setminus (V_0 \cap \mathrm{Op}\,\widetilde{S})$ in U is given by

$$g : [0,\varepsilon] \times \{0\} \longrightarrow \Phi(C),$$

where

$$g(t,0) = g(0,0) = \varphi(0).$$

On the other hand, since $\mathrm{supp}\,\psi \cap A \subset V_1 \cap \mathrm{Op}\,\widetilde{S}$, the restriction of $\psi(t,s)$, $(t,s) \in [0,1] \times [0,1]$, to a neighbourhood of $D = A \setminus (V_1 \cap \mathrm{Op}\,\widetilde{S})$ in U is given by the map

$$h : [0,1] \times [0,1] \longrightarrow \Phi(D),$$

where

$$h(t,s) = h(t,0) = \varphi(t).$$

Clearly, both the maps g and h match properly with f. Therefore, we have only to make sure that g and h does not contradict each other on the intersection of their domains. Now, observe that C and D are compact and $C \cap D \subset A \setminus \mathrm{Op}\, \tilde{S} \subset A \setminus \tilde{S} \subset U' \setminus S$. Since $U' \setminus S$ is open, C and D have neighbourhoods W_0 and W_1 respectively such that $W_0 \cap W_1 \subset U' \setminus S$. Since on the complement of S we have $\varphi(t) = \varphi(0)$, therefore $g(t,0)$ and $h(t,0)$ match properly on the intersection $W_0 \cap W_1$, and we get a commutative diagram as follows

$$\begin{array}{ccc} \Delta \cup [0,1] & \xrightarrow{f \cup g} & \Phi(C) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ \mathbb{I} \times [0,1] & \xrightarrow[h]{} & \Phi(C \cap D). \end{array}$$

Now, the restriction morphism $r : \Phi(C) \longrightarrow \Phi(D)$ is a microfibration (by Proposition 7.3), and $\Delta \cup [0,1] \times \{0\}$ is a strong deformation retract of the square $[0,1] \times [0,1]$; therefore, arguing as in Theorem 7.4, we obtain a deformation $\psi' : I_\delta \times [0,\delta] \longrightarrow \Phi(C)$, where $I_\delta = [0,\delta]$, and δ is a positive number ≤ 1, such that ψ' and $f \cup g$ coincide on their common domain, and $r \circ \psi' = h | I_\delta \times [0,\delta]$ on a neighbourhood of $C \cap D$. We now consider the following lifting problem

$$\begin{array}{ccc} \Delta_\delta & \xrightarrow{\ f\ } & \Phi(A) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ I_\delta \times [0,\delta] & \xrightarrow[\psi' \vee h]{} & \Phi(C \cup D) \end{array}$$

where Δ_δ is the diagonal of $[0,\delta] \times [0,\delta]$, and $\psi' \vee h$ is defined on $\operatorname{Op} C$ by ψ' and on $\operatorname{Op} D$ by h. Since r is a microfibration, we get a positive number $\varepsilon \leq \delta$, and a map $\psi : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ which equals f on Δ_ε and also satisfies $r \circ \psi = \psi' \vee h$. This map ψ has all the required properties, and our proof is complete. $\square$

It turns out that if $\psi_1^0, \psi_1, \ldots, \psi_k$ are microcompressible, then $\varphi|[0,\varepsilon]$ will be microcompressible, which will imply that φ is microcompressible.

Observation 8.5.8 *For all i, $\psi_i(0,0) = \varphi_0$.*

Proof. $\varphi = \psi_k^\bullet$ gives $\varphi(t) = \psi_k^\bullet(t) = \psi_k(t,t)$. Hence $\varphi(0) = \psi_k(0,0)$. Suppose now that $\varphi(0) = \psi_i(0,0)$ for some i. Then $\psi_{i-1}^\bullet = \psi_i^0$ gives $\psi_{i-1}(t,t) = \psi_{i-1}^\bullet(t) = \psi_i^0(t) = \psi_i(t,0)$. In particular, $\psi_{i-1}(0,0) = \psi_i(0,0) = \varphi(0)$. $\square$

In view of (8.5.1), and Observation 8.5.8, the problem of showing φ microcompressible is reduced to proving the following

Lemma 8.5.9 *Let A' be a compact G-invariant subset with non-empty interior U' in A such that $\varphi_0|\operatorname{Op} A'$ lifts to a map $\widetilde{\varphi}_0 \in \widetilde{\Phi}(A')$.*

(a) If $\psi : [0,\varepsilon] \times [0,\varepsilon] \longrightarrow \Phi(A)$ is a double deformation satisfying $\psi(0,0) = \varphi_0$ and $\operatorname{supp} \psi \cap A \subset \mathring{A}' = U' \subset A$, then ψ is microcompressible.

(b) If $\psi : [0,\varepsilon] \longrightarrow \Phi(A)$ is a deformation satisfying $\psi(0) = \varphi_0$ and $\operatorname{supp} \psi \cap A \subset U'$, then ψ is microcompressible.

Proof. (a) By hypothesis we have the following commutative diagram:

$$\begin{array}{ccc} 0 & \xrightarrow{\widetilde{\varphi}_0} & \widetilde{\Phi}(A') \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha} \\ [0,\varepsilon] & \xrightarrow[\psi^0]{} & \Phi(A'). \end{array}$$

Since α is a microfibration (see Lemma 8.4), we get a partial lift $\bar{\psi} : [0, \bar{\varepsilon}] \longrightarrow \widetilde{\Phi}(A')$ of ψ^0 for some $\bar{\varepsilon} > 0$, so that $\alpha \circ \bar{\psi} = \psi^0 \mid [0, \bar{\varepsilon}]$ on Op A' (in X) and $\bar{\psi}(0) = \widetilde{\varphi}_0$ on Op A' (in $X \times \mathbb{R}$).

We shall show that $\psi \mid I_{\bar{\varepsilon}} \times [0, \bar{\varepsilon}]$ is S-microcompressible, which will imply $\bar{S}$-microcompressibility of ψ, and hence microcompressibility of ψ (see Section 6). To this end, take an arbitrary G-invariant open neighbourhood $\widetilde{U}$ of supp $(\psi|I_{\bar{\varepsilon}} \times [0, \bar{\varepsilon}])$ in U, where U is the actual domain of ψ, and consider the following commutative diagram :

$$\begin{array}{ccc} I_{\bar{\varepsilon}} & \xrightarrow{\psi} & \widetilde{\Phi}(A') \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \gamma} \\ I_{\bar{\varepsilon}} \times [0,\bar{\varepsilon}] & \xrightarrow[\eta]{} & \Gamma(A', A' \setminus \widetilde{U}) \end{array}$$

where $\eta(t, s) = (\psi(t, s), \bar{\psi}(t))$. Since γ is a microfibration (see Lemma 8.4), there exists a positive number $\varepsilon'(\widetilde{U}) \leq \bar{\varepsilon}$ such that $\eta \mid I_{\bar{\varepsilon}} \times [0, \varepsilon']$ can be lifted to a map $\psi' : I_{\bar{\varepsilon}} \times [0, \varepsilon'] \longrightarrow \widetilde{\Phi}(A')$ satisfying $\psi'^0 = \bar{\psi}$. In particular ψ' has the folllowing properties.

- $\alpha \circ \psi'(t, s) = \psi(t, s)$ on an open neighbourhood of A' in $U \cap X$, for all $t \in I_{\bar{\varepsilon}}$ and $s \in [0, \varepsilon']$,
- $\psi'(t, 0) = \bar{\psi}(t)$ for all $t \in I_{\bar{\varepsilon}}$,

- $\operatorname{supp} \psi' \subset \operatorname{Op} \widetilde{U}$.

So we can extend each $\psi'(t,s)$ to $\operatorname{Op} U$ by defining it to be fixed outside $\operatorname{Op} \widetilde{U}$. Take a G-invariant open neighbourhood $\bar{U}$ of A' in U. Then $\operatorname{supp} \psi' \cap A' \subset \operatorname{Op}(\widetilde{U} \cap \bar{U})$. Applying Corollary 7.5, we get a deformation $\tilde{\psi} : I_\xi \times [0, \varepsilon'] \longrightarrow \tilde{\Phi}(A')$ such that

- $\tilde{\psi} \mid I_\xi \times \{0\} = \psi' \mid I_\xi \times \{0\}$,
- $\tilde{\psi}(t,s) = \psi'(t,s)$ on $\operatorname{Op} A'$,
- $\operatorname{supp} \tilde{\psi} \cap X \subset \widetilde{U} \cap \bar{U}$.

Now it is easy to see that $\alpha \circ \tilde{\psi} : I_\xi \times [0, \varepsilon'] \longrightarrow \Phi(A')$ gives an S-microcompression for $\psi \mid I_\xi \times [0, \bar{\varepsilon}]$ over A'.

Let us denote the map $\psi|(I_\xi \times [0, \bar{\varepsilon}])$ by ψ_ξ. Since $\operatorname{supp} \psi_\xi$ is contained in $\operatorname{supp} \psi$, it follows from the hypothesis that $\operatorname{supp} \psi_\xi \cap A \subset \mathring{A}'$. Since A and $\operatorname{supp} \psi_\xi$ are closed subsets of X, A' contains $\operatorname{Op}(\operatorname{supp} \psi_\xi) \cap A$ where $\operatorname{Op}(\operatorname{supp} \psi_\xi)$ denotes an open neighbourhood of $\operatorname{supp} \psi_\xi$ in U. Now, applying Proposition 6.9 we may conclude that ψ_ξ is S-microcompressible as a deformation over A. This completes the proof of (a).

(b) Take any neighbourhood $\widetilde{U}$ of $\operatorname{supp} \psi$ and consider the commutative square of continuous maps

$$\begin{array}{ccc} 0 & \xrightarrow{\tilde{\varphi}_0} & \tilde{\Phi}(A') \\ \Big\downarrow i & & \Big\downarrow \gamma \\ [0, \varepsilon] & \xrightarrow[\eta]{} & \Gamma(A', A' \setminus \widetilde{U}) \end{array}$$

where η is defined by $\eta(t) = (\psi(t), \tilde{\varphi}_0)$ for $t \in [0, \varepsilon]$. Now, proceeding as in the case of (a), we get the required microcompression for ψ. □

This establishes the microcompressibility of Φ in the special case when the polyhedron P is a point.

(8.6) Finally, we turn to the case when P is not a point. Let $\varphi : P \times \mathbb{I} \longrightarrow \Phi(A)$ be a deformation. Observe that φ gives rise to a map $\varphi^P : \mathbb{I} \longrightarrow \Phi(A)^P$ defined by

$$\varphi^P(t)(p)(x) = \varphi(p, t)(x) \qquad \text{for } t \in \mathbb{I} \text{ and } x \in \text{the domain of } \varphi.$$

Let us define a sheaf Φ^P over $X \times P$, associated to the sheaf Φ and the compact polyhedron P, as follows: For open subsets $U \subset X$ and $R \subset P$, we set $\Phi^P(R \times U)$ equal to $\Phi(U)^R$, which is the space of continuous maps $R \longrightarrow \Phi(U)$ with the following quasi-topology. A map $Q \longrightarrow \Phi^P(R \times U)$ is continuous if and only if the corresponding map $R \times Q \longrightarrow \Phi(U)$ is continuous. It is straightforward that if φ^P is microcompressible then so is φ.

Now, it can be shown following Gromov [8] (p.87-p.89) that:

(a) Φ^P is microflexible.

(b) $\tilde{\Phi}^P|X \times P$ is flexible.

(c) The restriction morphism $\alpha^P : \tilde{\Phi}^P \longrightarrow \Phi^P$ is a microextension, where, for subsets $A \subset X$ and $Q \subset P$, the map $\alpha^P(A \times Q) : \tilde{\Phi}^P(A \times Q) \longrightarrow \Phi^P(A \times Q)$ is defined by

$$\alpha^P(\tilde{\tau})(p) = \alpha(\tilde{\tau}(p)),$$

$\tilde{\tau} \in \tilde{\Phi}^P(A \times Q) = \tilde{\Phi}(A)^Q$ and $p \in Q$.

We observe that, while proving microcompressibility of an arbitrary deformation $\mathbb{I} \longrightarrow \Phi(A)$ we have only exploited the facts that Φ is microflexible (in 8.5.7), α is a microextension (in 8.5.1 and 8.5.9) and $\widetilde{\Phi}|X$ is a flexible sheaf (in 8.5.9). Hence (a), (b) and (c) together imply that φ^P is micro-pressible. This completes the proof of Proposition 3.4. □

9 Equivariant Open Extension Theorem

(9.1) The equivariant open extension theorem was stated in Corollary 1.12. To prove this, it is enough to show that the pseudogroup $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ sharply moves X in $X \times \mathbb{R}$. Define a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(u) = \begin{cases} \exp 1/(u^2 - 1) & \text{if } |u| < 1 \\ 0 & \text{if } |u| \geq 1. \end{cases}$$

Let U, S, δ, a be as in Definition 1.9 and let τ be a positive number such that the τ-neighbourhood of S in X is contained in U. Define

$$\sigma_t : \mathrm{Op}\, U \longrightarrow U_\delta, \quad 0 \leq t \leq 1$$

by

$$\sigma_t(x, s) = (x, tcf(d(x, S)/\tau) + s),$$

where d is the G-invariant metric on X, and c is a constant (the value of which will be determined later according to our requirements).

From the definition it is clear that σ_0 is the inclusion map, σ_t is fibre-preserving and $\sigma_t(x, s) = (x, s)$ if x lies outside the τ-neighbourhood of S in X. Also, each σ_t is an equivariant map because

$$\begin{aligned} \sigma_t(gx, s) &= (gx, tcf(d(gx, S)/\tau) + s) \\ &= (gx, tcf(d(x, S)/\tau) + s) \qquad \text{as } d(gx, S) = d(gx, gS) = d(x, S) \\ &= g\sigma_t(x, s). \end{aligned}$$

To prove that σ_t is an embedding it is enough to observe that σ_t is fibre-preserving and that, for a fixed x, the map $s \mapsto tcf(d(x, S)/\tau) + s$ is a one-one immersion.

Now observe that $\max_{t,x} tcf(d(x, S)/\tau) = cf(0)$. Choose c such that

$$a/f(0) < c < \delta/f(0).$$

Then for all t and for all x in the closed τ-neighbourhood U_τ of S in X, $d(x, \sigma_t(x)) < \delta$. Since U_τ is compact, there exists a positive number $\varepsilon' \leq \delta$ such that

$$d(x, \sigma_t(x,s)) < \delta \text{ for all } x \in U_\tau,\ s \in (-\varepsilon', \varepsilon') \text{ and for all } t.$$

Hence σ_t maps $U \times (-\varepsilon', \varepsilon')$ into U_δ (we have already observed that each σ_t is identity outside U_τ).

Now, for $t = 1$ and $x \in S$, $d(x, \sigma_1(x)) = cf(d(x,S)/\tau) = cf(0) > a$. Since S is compact, there exists an $\varepsilon \leq \varepsilon'$ and a neighbourhood N of S in U_τ such that

$$d(x, \sigma_1(x,s)) > a \text{ for all } x \in N \text{ and } s \in (-\varepsilon, \varepsilon).$$

Hence the isotopy $\sigma_t : U \times (-\varepsilon, \varepsilon) \longrightarrow U_\delta$ has all the required properties, and we have the equvariant open extension theorem.

Remark 9.2 The formulation of open extension theorem in Gromov [8] is slightly different from that of Corollary 1.13. The extensibility condition of Gromov [8] for open extension theorem is given by the relation $\pi^{(r)}(\widetilde{\mathcal{R}}) = \mathcal{R}$, whereas our extensibility condition is $\bar{\pi}^{(r)}(i^*\widetilde{\mathcal{R}}) = \mathcal{R}$ (see Definition 1.5). Clearly, Gromov's condition is stronger than ours. However, when $\widetilde{\mathcal{R}}$ is invariant under the action of fibre-preserving diffeomorphisms of $X \times \mathbb{R}$ (described as in Example 1.7), these two conditions become equivalent. To see this take an r-jet $j^r_x f$ from $\mathcal{R}$ and let $j^r_{(x,t)}\tilde{f}$ be an r-jet in $\widetilde{\mathcal{R}}$ such that

$$\pi^{(r)}(j^r_{(x,t)}\tilde{f}) = j^r_x(\pi \circ \tilde{f} \circ i_t) = j^r_x f.$$

Consider the equivariant fibre preserving diffeomorphism $\lambda_t : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ defined by $\lambda_t(x,s) = (x, s+t)$, and observe that $\lambda_t \circ i_0 = i_t$. Since $\widetilde{\mathcal{R}}$ is invariant under $\mathcal{D}_G(X \times \mathbb{R}, \pi)$, the map $g = \rho(\lambda_t)^{-1} \circ \tilde{f} \circ \lambda_t$ is a regular (local) G-section of $p' : E \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ at $(x,0)$. Also, recalling the fact that $\rho(\lambda_t)$ is fibre-preserving, we get

$$\pi \circ g \circ i_0 = (\pi \circ \rho(\lambda_t)^{-1}) \circ \tilde{f} \circ (\lambda_t \circ i_0) = \pi \circ \tilde{f} \circ i_t.$$

Hence $\pi^{(r)}(j^r_{(x,0)}g) = j^r_x f$. This proves the required equivalence.

10 Applications

(10.1) Let us recall some basic facts about G-manifolds. If X is a G-manifold then its tangent bundle TX is also a G-manifold under the differential action of G, and TX is a G-vector bundle over X. Since it has a Lie structure group, TX is actually G-locally trivial (see [1]).

Let H be a closed subgroup of G, and let X^H denote the H-fixed point subset. Then X^H is a submanifold of X and its tangent space at a point $x \in X^H$ is the subspace $(T_xX)^H$ of T_xX ([14], [18]). Hence $T(X^H) = (TX)^H$. This also implies that $(TX)^H$ is a vector bundle over X^H. In fact, we have a more general result.

Lemma 10.2 *If η is a G-locally trivial G-fibre bundle (resp. G-vector bundle) over X, then η^H is a fibre bundle (resp. vector bundle) over X^H.*

Proof. Let η be a G-locally trivial G-fibre bundle with fibre F and structure group K. Let U_x be a trivializing G_x-invariant open neighbourhood around $x \in X$ for the fibre bundle η. Then we have a commutative diagram

$$\begin{array}{ccc} p^{-1}(U_x) & \xrightarrow{\alpha_x} & U_x \times F \\ & {\scriptstyle p|}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & U_x & \end{array}$$

where α_x is a G_x-equivalence between $p^{-1}(U_x)$ and the trivial G_x-fibre bundle $U_x \times F$.

Since X^H is a submanifold of X, $U_x^H = U_x \cap X^H$ is an open subset in

X^H. Let us denote the map $p|\eta^H : \eta^H \longrightarrow X^H$ by p^H. Since $(p^H)^{-1}(U_x^H) = (p^{-1}(U_x))^H$, restricting α_x and p to $(p^{-1}(U_x))^H$ in the above diagram we get

$$\begin{array}{ccc} (p^H)^{-1}(U_x^H) & \xrightarrow{\alpha_x|} & U_x^H \times F^H \\ & {\scriptstyle p^H|}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U_x^H & \end{array}$$

Suppose that $U_x \cap U_y \neq \emptyset$, for some $x, y \in X^H$. Then $\alpha_y \circ \alpha_x^{-1} : (U_x \cap U_y) \times F \longrightarrow (U_x \cap U_y) \times F$ maps an element (z, f) to $(z, f.\theta(z))$ where $\theta(z) \in K$. Since $\alpha_y \circ \alpha_x^{-1}$ is G_x-equivariant, $\theta(z)$ maps F^H homeomorphically onto F^H. Therefore, the group $K' = \{\theta|F^H : \theta \in K\}$ acts on F^H. Hence η^H is a fibre bundle with fibre F^H and structure group K'.

We now deal with the case when η is a G-vector bundle. Since structure group of a G-vector bundle is the general linear group associated to its fibre (which is in this case a linear space), therefore, by a result of Bierstone [1], a G- vector bundle is G-locally trivial. Hence from the first part of the proof it follows that η^H is a fibre bundle. In fact, η^H will be a vector bundle. To see this it is enough to note that the fibre of η at x is a linear H-space, and hence F^H is a linear subspace of F. □

We shall now state the main theorem of this section. Let $p : E \longrightarrow X$ be, as before, a G-locally trivial G-fibre bundle. Let ξ and η be G-subbundles of TE and TX respectively. Consider the partial differential relation $\mathcal{R} \subset E^{(1)}$ consisting of 1-jets of germs of local sections $j_x^1\sigma$ for $x \in X$, such that $d\sigma_x(\eta_x) \cap \xi_{\sigma(x)} = \{0\}$.

Theorem 10.3 *If for each isotropy subgroup H of the action of G on X we have*

$$\dim X^H + \dim \xi^H < \dim E^H$$

then $\mathcal{R}$ satisfies equivariant parametric h-principle, where $\dim \xi^H$ means the fibre dimension of ξ^H.

We postpone the proof until (10.9).

Corollary 10.4 *Let $p : E \longrightarrow X$ be a non-equivariant fibre bundle, and ξ, η be subbundles of TE and TX respectively. Then $\mathcal{R}$ satisfies parametric h-principle if*

$$\dim X + \dim \xi < \dim E.$$

Proof. Take $G = \{e\}$. □

It may be noted that when $\eta = TX$, this corollary reduces to the transversality theorem of Gromov ([8], p.87).

Corollary 10.5 *Let X, Y be smooth G-manifolds, ξ a G-subbundle of TY, and η a G-subbundle of TX. Let $\mathcal{R}_0$ be the subspace of $J^1(X,Y)$ consisting of 1-jets of germs of local G-maps defined on G-invariant open sets in X, $j^1_x f$ for $x \in X$, such that*

$$j^1_x f|\eta_x \text{ is injective and } j^1_x f(\eta_x) \cap \xi_{f(x)} = \{0\}.$$

Then $\mathcal{R}_0$ satisfies equivariant parametric h-principle (in an obvious sense) if for each isotropy subgroup H of the action of G on X we have

$$\dim \eta^H + \dim \xi^H < \dim Y^H.$$

Proof. Consider the G-locally trivial G-fibre bundle $E = X \times Y \longrightarrow X$. The equivariant sections of E are in one-one correspondence with the equivariant maps of X in Y by the following rule:

$$\sigma \mapsto p_2 \circ \sigma \quad \text{where} \quad p_2 : X \times Y \longrightarrow Y \text{ is the projection.}$$

We may therefore write a section σ as $(1_X, \bar{\sigma})$ where $\bar{\sigma} : X \longrightarrow Y$ is a smooth G-map.

Consider the bundle $\bar{\xi}$ on $X \times Y$ defined by $\bar{\xi}_{x,y} = \eta_x \times \xi_y$. Then a section $\sigma : X \longrightarrow E$ satisfies

$$d\sigma_x(\eta_x) \cap \bar{\xi}_{x,\bar{\sigma}(x)} = \{0\},$$

if and only if $\bar{\sigma} : X \longrightarrow Y$ satisfies the following two conditions:

$$d\bar{\sigma}_x|\eta_x \text{ is injective and } d\bar{\sigma}_x(\eta_x) \cap \xi_{\bar{\sigma}(x)} = \{0\}.$$

We divide the proof of this fact into two parts.

(a) *Only if part.* Let $v \in \eta_x$ and $d\bar{\sigma}_x(v) = 0$. Then $(v, d\bar{\sigma}_x(v)) = d\sigma_x(v) \in d\sigma_x(\eta_x)$ and $(v, d\bar{\sigma}_x(v)) \in \bar{\xi}_{\sigma(x)}$. Then by hypothesis $v = 0$. Hence $d\bar{\sigma}_x|\eta_x$ is injective. Again let $v \in \eta_x$ and $d\bar{\sigma}_x(v) \in \xi_{\bar{\sigma}(x)}$. Then $(v, d\bar{\sigma}_x(v)) \in \bar{\xi}_{\sigma(x)}$, and $d\bar{\sigma}_x(v) = 0$ by hypothesis. Hence $d\bar{\sigma}_x(\eta_x) \cap \xi_{\bar{\sigma}(x)} = \{0\}$.

(b) *If part.* Let $v \in \eta_x$ and $d\sigma_x(v) \in \bar{\xi}_{\sigma(x)}$, i.e. $(v, d\bar{\sigma}_x(v)) \in \eta_x \times \xi_{\bar{\sigma}(x)}$. Hence $d\bar{\sigma}_x(v) \in \xi_{\bar{\sigma}(x)}$ for $v \in \eta_x$. By our assumption $d\bar{\sigma}_x(v) = 0$, and then $v = 0$. Hence $d\sigma_x(\eta_x) \cap \bar{\xi}_{\sigma(x)} = \{0\}$.

Also, the condition

$$\dim X^H + \dim \bar{\xi}^H < \dim E^H$$

is equivalent to

$$\dim \xi^H + \dim \eta^H < \dim Y^H.$$

Hence the corollary. □

Corollary 10.6 *Let X, Y be smooth non-equivariant manifolds, and ξ and η be subbundles of TY and TX respectively satisfying*

$$\dim \eta < \operatorname{codim} \xi.$$

Then $\mathcal{R}_0$ satisfies parametric h-principle.

Let us now consider the equivariant version of the Smale-Hirsh Immersion Theorem.

Corollary 10.7 *Let X, Y be smooth G-manifolds. Let* $\mathrm{Imm}_G(X,Y)$ *denote the space of equivariant smooth immersions of X in Y and let* $\mathrm{R}_G(TX,TY)$ *denote the space of equivariant continuous monomorphisms $F: TX \longrightarrow TY$ such that $F_x|T_x(Gx)$ is given by the differential of the map $gx \mapsto gf(x)$ of the orbit Gx onto the orbit $Gf(x)$, where $f: X \longrightarrow Y$ is the map covered by $F: TX \longrightarrow TY$. Then the differential map*

$$d: \mathrm{Imm}_G(X,Y) \longrightarrow \mathrm{R}_G(TX,TY)$$

is a weak homotopy equivalence, if $\dim X^H < \dim Y^H$ *for every isotropy subgroup H of the G-action on X.*

Proof. Let $\mathcal{I}$ be the subspace of $J^1(X,Y)$ consisting of 1-jets of equivariant local immersions of X in Y. Taking η equal to TX and ξ equal to zero in Corollary 10.5, we see that $\mathcal{I}$ satisfies equivariant parametric h-principle if $\dim(TX)^H < \dim Y^H$, that is, if $\dim X^H < \dim Y^H$ (see (10.1)). Now, to complete the proof, it is enough to identify $\mathrm{Imm}_G(X,Y)$ as the solution space of $\mathcal{I}$ and $\mathrm{R}_G(TX,TY)$ as the space of sections of the jet bundle $J^1(X,Y) \longrightarrow X$ with images in $\mathcal{I}$. □

With $G = \{e\}$ in the above corollary we deduce the famous Smale-Hirsch Immersion Theorem.

Corollary 10.8 *Let X, Y be non-equivariant smooth manifolds such that* $\dim X < \dim Y$. *Then the differential map*

$$d: \mathrm{Imm}\,(X,Y) \longrightarrow \mathrm{Mono}\,(TX,TY)$$

is a weak homotopy equivalence, where $\mathrm{Imm}\,(X,Y)$ *is the space of C^∞ immersions of X into Y and* $\mathrm{Mono}\,(TX,TY)$ *is the space of bundle* *monomorphisms of TX into TY.*

(10.9) *Proof of* **Theorem 10.3.**

To prove the theorem it is enough to observe the following facts in view of Corollary 1.12.

- $\mathcal{R}$ is a G-invariant subset of $E^{(1)}$.
- $\mathcal{R}$ is an open relation.
- $\mathcal{R}$ has a G-invariant extension $\tilde{\mathcal{R}}$.
- $\tilde{\mathcal{R}}$ is an open relation.
- $\tilde{\mathcal{R}}$ is $\mathcal{D}_G(X \times \mathbb{R}, \pi)$-invariant.

We shall verify the above results in subsections (10.10)-(10.14).

(10.10) $\mathcal{R}$ is a G-invariant subset of $E^{(1)}$.

Let σ be an equivariant (local) section of p defined on a G-invariant open neighbourhood of $x \in X$ and let $j^1_x\sigma \in \mathcal{R}$. Choose $g \in G$. We shall show that $g \cdot j^1_x\sigma = j^1_{gx}(g\sigma g^{-1}) \in \mathcal{R}$, that is

$$j^1_{gx}(g\sigma g^{-1})(\eta_{gx}) \cap \xi_{g\sigma(x)} = \{0\}$$

By our assumption, $j^1_x\sigma(\eta_x) \cap \xi_{\sigma(x)} = \{0\}$. Also, we know that each $g \in G$ induces isomorphisms on the tangent spaces of X, and of Y. Hence, $(j^1_{\sigma(x)}g \circ j^1_x\sigma)(\eta_x) \cap j^1_{\sigma(x)}g(\xi_{\sigma(x)}) = \{0\}$. Moreover, since ξ and η are G-subbundles this equation is equivalent to the required one.

(10.11) $\mathcal{R}$ is an open relation.

Let $\mathcal{R}'$ be the subset of $E^{(1)}$ consisting of 1-jets of germs of smooth local maps $f : X \longrightarrow E$ at some $x \in X$, such that $df_x|\eta_x$ is injective and $df_x(\eta_x) \cap \xi_{f(x)} = \{0\}$. Consider the inclusion $i : E^{(1)} \hookrightarrow J^1(X, E)$. Then $\mathcal{R} = i^{-1}(\mathcal{R}')$. So it is enough to prove that $\mathcal{R}'$ is open.

Let W be a subspace of $\mathbb{R}^m$ and V be a subspace of $\mathbb{R}^n$ satisfying $\dim V + \dim W < m$, where $m = \dim E$ and $n = \dim X$. Denote by q the quotient map of $\mathbb{R}^m$ onto $\mathbb{R}^m/W$. Set

$$\omega = \{L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \ : \ L \text{ is linear and } q \circ (L|V) \text{ is injective}\}$$

and consider the following sequence of continuous linear maps:

$$L(\mathbb{R}^n, \mathbb{R}^m) \xrightarrow{r} L(V, \mathbb{R}^m) \xrightarrow{q_*} L(V, \mathbb{R}^m/W)$$

given by $L \mapsto L|V \mapsto q \circ (L|V)$. Then $\omega = (q_* \circ r)^{-1}(O)$, where O is the open subset of $L(V, \mathbb{R}^m/W)$ consisting of injective linear maps. This completes the proof, because if U is an open coordinate neighbourhood in X and U' an open coordinate neighbourhood in E then $U \times U' \times \omega$ is homeomorphic to an open subset in the 1-jet space $J^1(X, Y)$ with image in $\mathcal{R}'$.

(10.12) $\mathcal{R}$ has a G-invariant extension.

Recall that any section $\tau : X \times \mathbb{R} \longrightarrow E \times \mathbb{R}$ is of the form $\tau(x, t) = (\tau'(x, t), t)$ so that $\pi \circ \tau = \tau'$, where $\tau' : X \times \mathbb{R} \longrightarrow E$ is a map such that, for each $t \in \mathbb{R}$, $\tau'(\cdot, t)$ is a section of p. Define a G-subbundle $\tilde{\eta}$ of $T(X \times \mathbb{R})$ by $\tilde{\eta}_{(x,t)} = \eta_x \times \mathbb{R}$.

Let $\tilde{\mathcal{R}} \subset (E \times \mathbb{R})^{(r)}$ consist of 1-jets of local sections, $j^1_{(x,t)}\tau$, satisfying the following two conditions:

- $j^1_{(x,t)}\tau' \mid \tilde{\eta}_{(x,t)}$ is injective, and
- $j^1_{(x,t)}\tau'(\tilde{\eta}_{(x,t)}) \cap \xi_{\tau'(x,t)} = \{0\}$.

We shall prove that $\tilde{\mathcal{R}}$ is a G-invariant extension of $\mathcal{R}$.

Let $j^1_{(x,t)}\tau \in \tilde{\mathcal{R}}$. Recall from Section 1 that G acts on $(E \times \mathbb{R})^{(1)}$ by the rule $g \cdot j^1_{(x,t)}\tau = j^1_{g(x,t)} g\tau g^{-1}$. Now, since each g induces isomorphisms on the tangent spaces of $X \times \mathbb{R}$, and of E, and since $\tilde{\eta}$ and ξ are G-subbundles, it follows that $j^1_{g(x,t)} g\tau' g^{-1} = j^1_{\tau'(x,t)} g \circ j^1_{(x,t)}\tau' \circ j^1_{g(x,t)} g^{-1}$ restricted to $\tilde{\eta}_{(gx,t)}$

is injective, and $j^1_{g(x,t)}g\tau' g^{-1}(\widetilde{\eta}_{(gx,t)}) \cap \xi_{\tau'(gx,t)} = \{0\}$. Since $g\tau g^{-1}(x,t) = (g\tau' g^{-1}(x,t), t)$, therefore $j^1_{g(x,t)}g\tau g^{-1} \in \widetilde{\mathcal{R}}$.

Next we shall show that $\pi^{(1)}$ maps $\widetilde{\mathcal{R}}_G$ into $\mathcal{R}_G$. Let us take an element $j^1_{(x,t)}\tau \in \widetilde{\mathcal{R}}_G$. This means that τ is a local G-section at (x,t), $j^1_{(x,t)}\tau' \mid \widetilde{\eta}_{(x,t)}$ is injective and $j^1_{(x,t)}\tau'(\widetilde{\eta}_{(x,t)}) \cap \xi_{\tau'(x,t)} = \{0\}$. $\pi^{(1)}$ maps $j^1_{(x,t)}\tau$ onto $j^1_x(\pi\circ\tau\circ i_t) = j^1_x(\tau'\circ i_t)$. It is required to show that $j^1_x(\tau'\circ i_t)(\eta_x)\cap\xi_{\tau'(x,t)} = \{0\}$. Suppose that for some $v \in \eta_x$, $j^1_x(\tau'\circ i_t)(v) \in \xi_{\tau'(x,t)}$. We shall show that $j^1_x(\tau'\circ i_t)(v) = 0$. Now, $j^1_x(\tau'\circ i_t)(v) = j^1_{(x,t)}\tau'(v,0)$ and $(v,0) \in \widetilde{\eta}_{(x,t)}$. Hence by hypothesis $j^1_x(\tau'\circ i_t)(v) = 0$.

Finally we shall show that $\pi^{(1)}$ is onto. Let $\sigma : U \longrightarrow E$ be a local G-section of p defined on a G-invariant neighbourhood U of $x \in X$ such that $j^1_x\sigma \in \mathcal{R}_G$. We will produce an equivariant local section τ at $(x,0)$ defined on some G-invariant open neighbourhood $\widetilde{U}$ of $(x,0)$ such that $j^1_{(x,0)}\tau \in \widetilde{\mathcal{R}}_G$ and $\pi^{(1)}j^1_{(x,0)}\tau = j^1_x\sigma$. If there exists such a τ, then

(*i*) $\tau(y,t)$ can be expressed as $(\tau'(y,t),t)$, for $(y,t) \in \widetilde{U}$, where τ' is an equivariant map from $\widetilde{U}$ to E, and it satisfies the relation $p\circ\tau'(y,t) = y$. Moreover $\tau'(x,0) = \sigma(x)$.

(*ii*) Since τ' is equivariant, it maps $\widetilde{U}^H$ into E^H, where H denotes the isotropy subgroup G_x at x. Let p^H denote the restriction of p to E^H. The relation $p^H\circ\tau'(x,t) = x$ gives $dp^H_{\sigma(x)} \circ d\tau'_{(x,0)}(0,w) = 0$ for $(0,w) \in T_xX^H \times T_0\mathbb{R}$. Then $d\tau'_{(x,0)}(0,w) \in \operatorname{Ker} dp^H_{\sigma(x)}$. Since $p^H : E^H \longrightarrow X^H$ is a fibre-bundle with fibre $(E_x)^H$, which is same as $(E^H)_x$ (where E_x denotes the fibre of p over x), we have $\operatorname{Ker} dp^H_{\sigma(x)} = T_{\sigma(x)}(E^H_x)\ (\subset T_{\sigma(x)}(E^H) \subset (TE)^H_{\sigma(x)}\ )$. Hence $d\tau'_{(x,0)}(0,w) \in T_{\sigma(x)}(E^H_x)$.

(*iii*) Also, by hypothesis, $d\tau'_{(x,0)}(0,1) \notin d\sigma_x(\eta_x) \oplus \xi_{\sigma(x)}$. Therefore, to obtain τ', it is necessary to find a vector $u \in T_{\sigma(x)}E^H_x$ which does not belong to the intersection $(d\sigma_x(\eta_x) \oplus \xi_{\sigma(x)}) \cap T_{\sigma(x)}E_x^H$. Since η and ξ are G-invariant subbundles, σ is equivariant and $d\sigma_x$ is injective, we can prove that

$$(d\sigma_x(\eta_x^H) \oplus \xi_{\sigma(x)}^H) \cap T_{\sigma(x)}E_x^H = (d\sigma_x(\eta_x) \oplus \xi_{\sigma(x)}) \cap T_{\sigma(x)}E_x^H.$$

So, it is enough to find $u \in T_{\sigma(x)}E_x^H$ which does not belong to $(d\sigma_x(\eta_x^H) \oplus \xi_{\sigma(x)}^H) \cap T_{\sigma(x)}E_x^H$.

Let us now investigate the condition $\dim X^H + \dim \xi^H < \dim E^H$. This inequality is equivalent to $\dim \xi^H < \dim E_x^H$. Moreover, since $d\sigma_x(\eta_x^H) \cap T_{\sigma(x)}(E_x^H) = \{0\}$, therefore $T_{\sigma(x)}(E_x^H)$ is not contained in $d\sigma_x(\eta_x^H) \oplus \xi_{\sigma(x)}^H$. This permits us to choose the required vector u.

We shall now construct τ' described in (i) above. We identify $E|U$ with the trivial G_x-bundle $U \times Y$, where Y is G_x-homeomorphic to the fibre E_x. Then σ can be expressed in the following way:

$$\sigma(y) = (y, \bar{\sigma}(y)) \in U \times Y,$$

where $y \in U$, and $\bar{\sigma} : U \longrightarrow Y$ is a G_x-equivariant map. Because of (ii), we may assume without loss of generality, that $u \in T_{\bar{\sigma}(x)}Y^H \subset T_xX \times T_{\bar{\sigma}(x)}Y$.

Note that we can always find a smooth function $\bar{f}$ (not necessarily equivariant) from a neighbourhood of $(x, 0) \in X \times \mathbb{R}$ to Y such that at the point $(x, 0)$ it satisfies the following relations:

$$\bar{f}(x, 0) = \bar{\sigma}(x), \quad \frac{\partial \bar{f}}{\partial x_i} = \frac{\partial \bar{\sigma}}{\partial x_i} \quad \text{and} \quad \frac{\partial \bar{f}}{\partial t} = u$$

where x_i's are coordinate functions on a neighbourhood of $x \in X$ and u is as chosen above. These conditions imply that the map $f' : \mathrm{Op}\,(x, 0) \longrightarrow E$ defined by the formula $f'(y, t) = (y, \bar{f}(y, t))$ has all the properties of τ' except (perhaps) equivariance. Thus $\tilde{\mathcal{R}}$ is a non-equivariant extension of $\mathcal{R}$. For the openness of $\tilde{\mathcal{R}}$ (which will be proved in (10.13)), we may also assume that f' agrees with σ on X (see Lemma 8.2).

We shall now modify f' to get our required equivariant map τ'. Define a map τ_1' on the domain of f' by the following rule:

$$\tau_1'(y,t) = \int_H h^{-1} f'(h.(y,t))dh,$$

where dh is the normalised Haar measure on $G_x = H$. Then τ_1' is a G_x-equivariant map agreeing with f' (and hence with σ) on $U \times \{0\}$, and is such that, for each $t \in \mathbb{R}$, $\tau_1'(\,.\,,t)$ is a local section of p, provided the composition $f \circ i_t$ is defined, (in fact for a fixed t, $f'(h.(y,t)) \in E_{hy}$ and therefore $h^{-1}.f'(h.(y,t)) \in E_y$; consequently, $\tau_1'(y,t) \in E_y$). Moreover, since H fixes both $(x,0)$ and u, we have

$$\frac{\partial \tau_1'}{\partial t}(x,0) = \int_H h^{-1} \frac{\partial f'}{\partial t}(x,0)\, dh = \int_H h^{-1}.u\, dh = u.$$

Let S_x be the slice at $x \in U$; now define $\tau' : G \times_H S_x \times \mathbb{R} \longrightarrow E$ by

$$\tau'([g,y],t) = g\tau_1'(y,t).$$

Then τ' is G-equivariant, and agrees with σ on U, because taking $t = 0$ we get $\tau'([g,y],0) = g\tau_1'(y,0) = g\sigma(y) = \sigma(gy) = \sigma([g,y])$ (we have identified $[g,y]$ with gy since the correspondence $[g,y] \mapsto gy$ defines a local G-homeomorphism between $G \times_{G_x} S_x$ and X at $[1,x]$). Moreover, since $[1,x] = x$ we have $\tau'(x,t) = \tau'([1,x],t) = \tau_1'(x,t)$. Therefore, $\frac{\partial}{\partial t}\tau'(x,0) = \frac{\partial}{\partial t}\tau_1'(x,0) = u$. Thus we have obtained our required τ'.

(10.13) <u>$\widetilde{\mathcal{R}}$ is open.</u>

Consider the subset $\mathcal{R}'' \subset J^1(X \times \mathbb{R}, E)$ consisting of 1-jets $j^1_{(x,t)}f$ such that (a) $j^1_{(x,t)}f \mid \widetilde{\eta}_{(x,t)}$ is injective and (b) $j^1_{(x,t)}f(\widetilde{\eta}_{(x,t)}) \cap \xi_{f(x,t)} = \{0\}$. We can prove as in (10.11) that $\mathcal{R}''$ is open. Consider the map $\pi_* : (E \times \mathbb{R})^{(1)} \longrightarrow J^1(X \times \mathbb{R}, E)$ which takes $j^1_{(x,t)}\widetilde{\sigma}$ to $j^1_{(x,t)}(\pi \circ \widetilde{\sigma})$. Then it is easy to see that $\widetilde{\mathcal{R}} = (\pi_*)^{-1}(\mathcal{R}'')$. Hence $\widetilde{\mathcal{R}}$ is open.

(10.14) $\widetilde{\mathcal{R}}$ is $\mathcal{D}_G(X \times \mathbb{R}, \pi)$ invariant.

Let $\lambda \in \mathcal{D}_G(X\times\mathbb{R}, \pi)$ and $j^1_{\lambda(x,t)}\sigma \in \widetilde{\mathcal{R}}$. We have to show that $\lambda^*(j^1_{\lambda(x,t)}\sigma) \in \widetilde{\mathcal{R}}$. If we write $\lambda(y,s) = (y, \lambda'(y,s))$ and $\sigma(y,s) = (\sigma'(y,s), t)$ for $(y,s) \in X \times \mathbb{R}$, then $\lambda^*(j^1_{\lambda(x,t)}\sigma) = j^1_{(x,t)}\widetilde{\sigma}$, where $\widetilde{\sigma}$ is defined by $\widetilde{\sigma}(y,s) = ((\sigma' \circ \lambda)(y,s), s)$ for $(y,s) \in X \times \mathbb{R}$ (see Example 1.7). Hence, it is required to prove that $j^1_{(x,t)}(\sigma' \circ \lambda)(\widetilde{\eta}_{(x,t)}) \cap \xi_{(\sigma'\circ\lambda)(x,t)} = \{0\}$ and $j^1_{(x,t)}(\sigma' \circ \lambda) \mid \widetilde{\eta}_{(x,t)}$ is injective.

Let $j^1_{(x,t)}(\sigma' \circ \lambda)(v,w) = 0$ for some $(v,w) \in \eta_x \times \mathbb{R}$. Since λ is a fibre-preserving local diffeomorphism, $d\lambda_{(x,t)}$ maps $\eta_x \times \mathbb{R}$ onto itself. Therefore, by the hypothesis (which is $j^1_{\lambda(x,t)}\sigma \in \widetilde{\mathcal{R}}$), we conclude that $v = 0$ and $d\lambda'_{(x,t)}(v,w) = 0$. This implies that $d\lambda'_{(x,t)}(0,w) = 0$. Moreover, since $s \mapsto \lambda'(y,s)$ is a local diffeomorphism of $\mathbb{R}$, we have $w = 0$. This proves that $j^1_{(x,t)}(\sigma' \circ \lambda) \mid \widetilde{\eta}_{(x,t)}$ is injective.

Next suppose that $j^1_{(x,t)}(\sigma'\circ\lambda)(v,w) \in \xi_{(\sigma'\circ\lambda)(x,t)}$ for some $(v,w) \in \eta_x \times \mathbb{R}$. Then arguing as in the previous paragraph we get that $j^1_{(x,t)}(\sigma' \circ \lambda)(v,w)) = 0$. Hence $j^1_{(x,t)}(\sigma' \circ \lambda)(\widetilde{\eta}_{(x,t)}) \cap \xi_{(\sigma'\circ\lambda)(x,t)} = \{0\}$.

11 Symplectic Manifolds and Moser's Theorem

(11.1) In this section we will review some basic facts from the theory of symplectic manifolds. We shall consider only manifolds without boundary. If M is such a manifold, then we shall denote the space of all exterior differential p-forms on M by $\Lambda^p(M)$, and the exterior differential operator by d. A closed 2-form g on M is called *symplectic* if it is nondegenerate, that is, if the induced map $I_g : \mathfrak{X}(M) \longrightarrow \Lambda^1(M)$ is an isomorphism, where $\mathfrak{X}(M)$ is the space of smooth vector fields on M. A manifold M together with a symplectic form is called a *symplectic manifold*. It is a standard result that a symplectic manifold is always even-dimensional. A diffeomorphism $f : M \longrightarrow M$ of a symplectic manifold (M, g) is called a *symplectic diffeomorphism* (or *symplectomorphism*) if the puli back f^*g equals g.

Our main objective in this section is to study the consequences of Moser's Theorem which will be required in the next section to obtain parametric h principle for symplectic isometric immersions.

We begin with a proof of Relative Poincaré Lemma ([20]).

Lemma 11.2 (Relative Poincaré Lemma) *Let ω be a closed p-form on a manifold M such that the pull-back of ω on a submanifold N of M vanishes. Then there exists a $(p-1)$-form α such that $\omega = d\alpha$ on a neighbourhood of N and $\alpha = 0$ on N.*

Proof. Let T be a tubular neighbourhood of N in M. Then T has a vector bundle structure over N. If $\mu_t : T \longrightarrow T$ denotes multiplication by t for $t \in [0, 1]$, then μ_1 is the identity map and μ_0 is the retraction of T onto N. Moreover, each μ_t restricted to N is the identity of N. Let X_t be the vector field $d\mu_t/dt$ along μ_t. Then it is a standard result that

$$\frac{d}{dt}(\mu_t^*\omega) = \mu_t^*(X_t.d\omega) + d(\mu_t^*(X_t.\omega)).$$

Since ω is a closed form, the above relation reduces to $\frac{d}{dt}(\mu_t^*\omega) = d(\mu_t^*(X_t.\omega))$. Integrating both sides of this equation and noting that μ_1 is identity, we obtain that

(11.3) $$\omega - \mu_0^*(\omega) = \int_0^1 d(\mu_t^*(X_t.\omega))dt = d(\int_0^1 \mu_t^*(X_t.\omega)dt).$$

We define $I\omega = \int_0^1 \mu_t^*(X_t.\omega)dt$; thus equation (11.3) transforms into

$$\omega - \mu_0^*\omega = d(I\omega)$$

Since μ_0 maps T onto N and $i^*\omega = 0$ (i being the inclusion of N in M), we have $\mu_0^*\omega = 0$. Consequently, $\omega = d(I\omega)$. On the other hand, since $\mu_t|N$ is identity, X_t vanishes on N, and hence $I\omega|N = 0$. Thus $I\omega$ is the required $(p-1)$-form α. □

Since I in the proof of Lemma 11.2 determines a continuous process, we get two immediate corollaries of Relative Poincaré Lemma.

Corollary 11.4 *Let M be a manifold, and let ω_q, $q \in Q$, be a family of closed p-forms on M such that pull-backs of the forms ω_q vanish on a submanifold N of M. Then there exists a continuous family of $(p-1)$-forms α_q, and a neighbourhood U of N in M, such that α_q vanishes on N and $\omega_q = d\alpha_q$ on U.*

Corollary 11.5 *Let $x \in M$, and U be an open subset of M containing x. Let $\{\omega_u : u \in U\}$ be a continuous family of closed p-forms on U such that $\omega_u(u) = 0$ for all $u \in U$. Then there exists an open neighbourhood $\widetilde{U}$ of x in U, and a continuous family of $(p-1)$-forms $\{\alpha_u\}$ such that $\omega_u = d\alpha_u$ on $\widetilde{U}$, and $\alpha_u(u) = 0$ for all $u \in \widetilde{U}$.*

Proof. Take $\widetilde{U}$ to be a convex coordinate neighbourhood of x. Consider the smooth map $\mu : \widetilde{U} \times \widetilde{U} \times \mathbb{I} \longrightarrow \widetilde{U}$ defined by $\mu(u, x, t) = (1-t)x + tu$, for $u, x \in \widetilde{U}$ and $t \in \mathbb{I}$. Then, for each $u \in \widetilde{U}$, the map $\mu_u : \widetilde{U} \times \mathbb{I} \longrightarrow \widetilde{U}$ defined by $\mu_u(x, t) = \mu(u, x, t)$ gives a strong deformation retraction of $\widetilde{U}$ onto u. Thus the correspondence $u \mapsto \mu_u$ gives a continuous family of retractions. Then proceeding as in Lemma 11.2 we get the desired result. □

Theorem 11.6 (Moser's Stability Theorem, see [21]) *Let (M, g) be a symplectic manifold and N a compact submanifold of M. Let g_q, $q \in Q$, be a continuous family of symplectic forms on M such that $g_q = g$ on the submanifold N. Then there exists a continuous family of diffeomorphisms $\{\delta_q, q \in Q\}$ defined on some neighbourhood of N such that $\delta_q^* g_q = g$, $\delta_q|N = 1_N$ and $d\delta_q|T_N M =$ identity. Moreover, if $g_q = g$ on a neighbourhood of N for some q, then the corresponding δ_q is identity on M.*

Proof. Let us write ω_q for $g_q - g$. Then by hypothesis each ω_q vanishes on N. Now, by Corollary 11.4, there exists a continuous family of 1-forms $\{\alpha_q, q \in Q\}$ such that, for all $q \in Q$, $\omega_q = d\alpha_q$ on a neighbourhood of N, and $\alpha_q|N = 0$.

Now, the homotopy $G : Q \times \mathbb{I} \longrightarrow \Lambda^2(M)$ defined by $G(q, s) = g + s.d\alpha_q$ is smooth with respect to the parameter s and continuous with respect to q. Since $\omega_q|N = 0$, $G(q, s) = g$ on N for all (q, s), and therefore $I_{G(q,s)}|N = I_g|N$ is an isomorphism. This means that $G(q, s)$ is a symplectic structure on a neighbourhod of N for all $q \in Q$, and $s \in \mathbb{I}$.

We construct a smooth family of diffeomorphisms $\delta_q(s)$ as follows: Let $\xi_q(s)$ be the vector field $-I^{-1}_{G(q,s)}(\alpha_q)$; then

$$\mathcal{L}_{\xi_q(s)}G(q,s) = d(\xi_q(s).G(q,s)) = -d\alpha_q.$$

Now, since $\alpha_q|N = 0$ for each $q \in Q$, the vector fields $\xi_q(s)$ vanish on N and hence they are integrable on N. Therefore $\xi_q(s)$ can be integrated on a neighbourhood of N (since N is compact, we can get an open neighbourhood of N in M on which each $\delta_q(s)$ is defined) to a 1-parameter family of diffeomorphisms $\{\delta_q(s)\}$ such that $\delta_q(0)$ is identity for all $q \in Q$ and each $\delta_q(s)$ is identity on N. By the definition of integrability we have $\frac{d}{ds}(\delta_q(s)) = \xi_q(s) \circ \delta_q(s)$ and therefore

$$\frac{d}{ds}[\delta_q(s)^* G(q,s)] = \delta_q(s)^*(\mathcal{L}_{\xi_q(s)}G(q,s) + \frac{d}{ds}G(q,s)) = 0.$$

So, $\delta_q(s)^* G(q,s) = \delta_q(0)^* G(q,0) = g$. In particular we have $\delta_q(1)^*(g_q) = \delta_q(1)^* G(q,1) = g$. Since all the steps involved here are continuous, the map $(q,x) \mapsto \delta_q(1)(x)$ is continuous in $q \in Q$ and smooth in $x \in M$. Hence, the correspondence $q \mapsto \delta_q(1)$ defines a continuous map with respect to C^∞ compact-open topology on the function space. Consequently, $\{\delta_q(1)\}$ is our required family of diffeomorphisms.

Since $\delta_q(s)|N$ is identity for all $(q,s) \in Q \times \mathbb{I}$, it follows that

$$\frac{d}{ds}(d\delta_q(s))_n = d(\frac{d}{ds}(\delta_q(s)(n)) = 0,$$

for each $q \in Q$ and $n \in N$. Hence $(d\delta_q(1))_n = (d\delta_q)(0)_n$ =identity for $n \in N$. □

Corollary 11.7 *Let (M,g) be a symplectic manifold and let g_u be a continuous family of symplectic forms on M, where u runs over an open subset $U \subset M$, such that $g_u = g$ at u. Then there is a continuous family $\{\delta_u : u \in \widetilde{U}\}$ of diffeomorphisms defined on some open subset $\widetilde{U} \subset U$ such that $\delta_u^* g_u = g$, $\delta_u(u) = u$ and $d\delta_u|T_uM$ is identity for all $u \in \widetilde{U}$.*

Proof. This follows from Theorem 11.6 along with the observation in Corollary 11.5. □

Corollary 11.8 *Let (M, g) be a symplectic manifold, and g_t, $t \in [0,1]$, a continuous homotopy of g in the space of symplectic forms on M such that the cohomology class of g_t is the same as that of g for all t. Let (C, C') be a pair of compact subsets in M and $g_t = g$ on an open neighbourhood of C'. Then there exists an $\varepsilon > 0$ and an isotopy $\{\delta_t : 0 \leq t \leq \varepsilon\}$ on an open neighbourhood of C such that $\delta_t^* g_t = g$ for all t and each δ_t is identity on a neighbourhood of C'.*

Proof. First suppose that there exists a continuous family of 1-forms α_t on M such that $g_t = g + d\alpha_t$ where each α_t vanishes on a neighbourhood of C'. Without loss of generality we may assume that $\alpha_0 = 0$ (if not, replace α_t by $\beta_t = \alpha_t - \alpha_0$). Consider an auxiliary 1-parameter family of 2-forms $g'_t = g + d(\rho\alpha_t)$, where ρ is a smooth function with compact support and $\rho \equiv 1$ on an open set $U_0 \supset C$. Then $g'_0 = g$, and therefore, g'_t is symplectic for small t, since $\alpha_0 = 0$. Now, define a homotopy $H : \mathbb{I} \times \mathbb{I} \longrightarrow \Lambda^2(M)$ by $H(t,s) = g + s.d(\rho\alpha_t)$. Since $H(0,s) = g$ for all $s \in \mathbb{I}$, there exists an $\varepsilon > 0$ such that $H(t,s)$ is symplectic for all $s \in \mathbb{I}$ and for all $t \leq \varepsilon$. Again, since ρ has a compact support, the vector fields $I^{-1}_{H(t,s)}(\rho\alpha_t)$ are integrable on M. Hence proceeding exactly as in Theorem 11.6, we obtain an isotopy δ_t, $0 \leq t \leq \varepsilon$, such that $\delta_t^* g'_t = g_0$ and δ_t is identity on a neighbourhood of C'. Since δ_0 is identity and C is compact, therefore we find a neighbourhood U of C such that $\delta_t(U) \subset U_0$ for sufficiently small t. It is easy to see that $\delta_t|U$ pulls back g_t onto g.

Now to prove the existence of the above-mentioned α_t's let us recall some preliminaries from Hodge theory. Let $\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator which is a linear operator on $\Lambda^p(M)$ for each p. If H^p is the space of harmonic p-forms (a form ω is harmonic if $\Delta\omega = 0$), then $\Lambda^p(M)$ has the *following decomposition:*

$$\Lambda^p(M) = d(\Lambda^{(p-1)}(M)) \oplus \delta(\Lambda^{(p+1)}(M)) \oplus H^p(M)$$

with respect to some inner product on $\Lambda^p(M)$ (see [19]). Therefore, it follows that the equation $\Delta\alpha = \omega$ has a solution if and only if ω is orthogonal to H^p. This permits us to define an operator $G : \Lambda^p(M) \longrightarrow (H^p)^\perp$, called Green's operator, by setting $G(\omega)$ equal to the unique solution of $\Delta\alpha = \omega - H(\omega)$ in $(H^p)^\perp$, where $H(\omega)$ denotes the orthogonal projection of ω in H^p. Then G is a bounded self-adjoint operator which commutes with all operators commuting with Δ.

Now, if ω is an exact p-form, then $H(\omega)$ is equal to zero (since H^p does not contain any exact form). So $\Delta\alpha = \omega$ has a unique solution in $(H^p)^\perp$. Also, $\Delta G(\omega) = d\delta G(\omega)$ since ω is closed. We define $G' : B^2(M) \longrightarrow \Lambda^1(M)$ by $G'(\omega) = \delta G(\omega)$. Then $dG'(\omega) = \omega$; in other words G' is a right inverse of the exterior differential operator d. If ω is zero on an open submanifold, then so is $G'(\omega)$.

Thus, for the continuous family $\omega_t = g_t - g_0$ of exact 2-forms, we get a continuous family of 1-forms α_t, given by $\alpha_t = G'\omega_t$, such that $d\alpha_t = \omega_t$. This completes the proof of the corollary. □

12 Symplectic Isometric Immersions

(12.1) Let (V, g) and (W, h) be two symplectic manifolds, where V is without boundary, and g and h are symplectic structures. Consider the space of C^∞ immersions $f : V \longrightarrow W$ satisfying the condition $f^* h = g$. These maps will be called symplectic isometric immersions, and their space will be denoted by $\mathrm{Symp}(V, W)$. Symplectic isometric immersions correspond to the relation $\mathcal{R} \subset J^{(1)}(V, W)$ consisting of all 1-jets $j^1_v f$, $v \in V$, of local immersions f such that $f^* h = g$ at v. Evidently $\mathcal{R}$ is not an open relation. The space of sections of the 1-jet bundle $J^1(V, W)$ over V, with images in $\mathcal{R}$, is isomorphic to the space of vector bundle monomorphisms $F : TV \longrightarrow TW$ such that $F^*(h) = g$. We shall denote the latter space by $\mathrm{RS}(TV, TW)$. The main result of this section is stated in the following theorem.

Theorem 12.2 *The differential map* $d : \mathrm{Symp}(V, W) \longrightarrow \mathrm{RS}(TV, TW)$ *is a weak homotopy equivalence if* $\dim V < \dim W$.

The remaining part of this section is devoted to the proof of this theorem.

(12.3) We embed V into $V \times \mathbb{R}^2$ by the canonical immersion i. Let us denote $V \times \mathbb{R}^2$ by $\tilde{V}$. Then $\tilde{V}$ has a symplectic form $\tilde{g} = g + dx \wedge dy$ on it where (x, y) are the global coordinate system for $\mathbb{R}^2$. Moreover, i pulls back $\tilde{g}$ onto g.

Let $\tilde{\mathcal{R}}$ be the relation corresponding to the symplectic immersions of $\tilde{V}$ in W. We shall denote the sheaf of solutions of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ by Φ and $\tilde{\Phi}$ respectively.

Lemma 12.4 *The relation* $\tilde{\mathcal{R}}$ *is an extension of* $\mathcal{R}$.

Proof. Let $j^1_v f$ be the 1-jet of a local immersion $V \supset U \xrightarrow{f} W$ satisfying $f^* h = g$ at $v \in U$; in other words, the 1-jet belongs to $\mathcal{R}$. We shall show that there exists a 1-jet $\jmath^1_{(v,0)}\tilde{f} \in \tilde{\mathcal{R}}$, such that $\pi^{(1)}(\jmath^1_{(v,0)}\tilde{f}) = \jmath^1_v(\tilde{f} \circ i) = \jmath^1_v f$. The isometric condition on $\tilde{f}$ at $(v,0)$ says that

$$h_{f(v)}(\tilde{f}_*\mathbf{x}, \tilde{f}_*\mathbf{y}) = \tilde{g}_{(v,0)}(\mathbf{x}, \mathbf{y}) \tag{12.5}$$

for any two tangent vectors $\mathbf{x}$ and $\mathbf{y}$ at $(v,0) \in \tilde{V}$. Furthermore, the extension condition gives the following relation

$$\tilde{f}_*(\mathbf{x}) = f_*(\mathbf{x}), \tag{12.6}$$

for all $\mathbf{x} \in T_vV$.

It is easy to verify that the global vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ satisfy the following relations

$$\begin{gathered}\tilde{g}(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}) = 1, \\ \tilde{g}(\tfrac{\partial}{\partial x}, X) = 0 \text{ and } \tilde{g}(\tfrac{\partial}{\partial y}, X) = 0,\end{gathered} \tag{12.7}$$

where X is an arbitrary vector field on V.

In view of (12.5), (12.6) and (12.7), we now make the following assumption. There are two tangent vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ at $f(v) \in W$ such that:

$$\begin{gathered}h(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 1, \\ h(f_*(\mathbf{x}), \bar{\mathbf{x}}) = 0 \text{ and } h(f_*(\mathbf{x}), \bar{\mathbf{y}}) = 0,\end{gathered} \tag{12.8}$$

for all $\mathbf{x}$ in T_vV.

Given $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ satisfying (12.8), we define a smooth map $\tilde{f}$ on a neighbourhood of $(v,0)$ in $\tilde{V}$ such that

$$d\tilde{f}_v|T_vV = df_v, \quad d\tilde{f}_v(\tfrac{\partial}{\partial x}|_v) = \bar{\mathbf{x}}, \quad d\tilde{f}_v(\tfrac{\partial}{\partial y}|_v) = \bar{\mathbf{y}}$$

Then it readily follows that $\pi^{(1)}(\jmath^1_{(v,0)}\tilde{f}) = \jmath^1_v f$ and $\tilde{f}^*h = \tilde{g}$ at $(v,0) \in \tilde{V}$. This local map $\tilde{f}$ will also be an immersion at $(v,0)$, because, if there exists some $\bar{\mathbf{x}}$ satisfying the condition (12.8) then it cannot belong to $f_*(T_vV)$. (If possible, let $\bar{\mathbf{x}} \in f_*(T_vV)$ so that $\bar{\mathbf{x}} = f_*(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in T_vV$. Then, using (12.8) we get

$$g(\mathbf{x}, \mathbf{x}_0) = h(f_*(\mathbf{x}), f_*(\mathbf{x}_0)) = h(f_*(\mathbf{x}), \bar{\mathbf{x}}) = 0$$

for all vectors $\mathbf{x} \in T_v V$. This implies, since g is non-degenerate, that $\mathbf{x}_0 = 0$, and hence $\bar{\mathbf{x}} = 0$ – a contradiction.) Therefore, $j^1_{(v,0)}\tilde{f} \in \tilde{\mathcal{R}}$, and consequently $\tilde{\mathcal{R}}$ is an extension of $\mathcal{R}$ modulo the assumption (12.8).

Now to complete the proof of the lemma it remains to show the existence of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ satisfying (12.8). To this end, consider the linear map

$$F : T_{f(v)}W \longrightarrow \mathbb{R}^n$$

defined by

$$\mathbf{w} \mapsto (\, h(f_*(e_1), \mathbf{w}), \ldots, h(f_*(e_n), \mathbf{w})\,)$$

where e_i, $i = 1, \ldots, n$, denote the vector fields corresponding to the coordinate functions on a neighbourhood U' of $v \in V$. Observe that $f_*(T_v V) \subset T_{f(v)}W$, and $F|f_*(T_v V)$ is an isomorphism onto $\mathbb{R}^n$, since g is non-degenerate. Hence F is an epimorphism. It also follows that $f_*(T_v V) \oplus \operatorname{Ker} F = T_{f(v)}W$, since $\operatorname{Ker} F \cap f_*(T_v V) = \{0\}$. Let us choose a tangent vector $\bar{\mathbf{x}}(\neq 0)$ in $\operatorname{Ker} F$, so that $h(f_*\mathbf{x}, \bar{\mathbf{x}}) = 0$ for all $\mathbf{x} \in T_v V$, and define a linear map

$$G : \operatorname{Ker} F \longrightarrow \mathbb{R}$$

by $\mathbf{w} \mapsto h(\bar{\mathbf{x}}, \mathbf{w})$. Since h is non-degenerate, G is onto. For, if G is not onto, then it is the zero map, which means that $h(\bar{\mathbf{x}}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \operatorname{Ker} F$. On the other hand, $h(f_*(\mathbf{x}), \bar{\mathbf{x}}) = 0$ for all $\mathbf{x} \in T_v V$. These two together imply that $h(\mathbf{w}, \bar{\mathbf{x}}) = 0$ for all $\mathbf{w} \in T_{f(v)}W$; but $\bar{\mathbf{x}} \neq 0$, which contradicts the fact that h is non-degenerate. So, we can choose a tangent vector $\bar{\mathbf{y}}$ at $f(v) \in W$ such that $h(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 1$. This completes the proof. □

(12.9) Observe that $\tilde{\mathcal{R}}$ is not invariant under the action of $\mathcal{D}(\tilde{V})$. It is only invariant under the action of symplectic diffeomorphisms of $\tilde{V}$. It is easy to see that a symplectic diffeotopy δ_t on $\tilde{V}$, with δ_0=identity, is obtained by integrating a vector field ∂ on $\tilde{V}$ satisfying $\mathcal{L}_\partial \tilde{g} = 0$. Since $\tilde{g}$ is a closed form, this is equivalent to saying that $\partial.\tilde{g}$ is closed. So ∂ is the inverse image under the isomorphism $I_{\widetilde{g}} : \mathfrak{X}(\widetilde{V}) \longrightarrow \Lambda^1(\widetilde{V})$ of a closed 1-form on $\widetilde{V}$.

A diffeotopy of $(\widetilde{V}, \widetilde{g})$ is called *exact* if it can be obtained by integrating a vector field ∂ which corresponds to an exact 1-form dH under $I_{\widetilde{g}}$ for some smooth function H on $\widetilde{V}$.

Lemma 12.10 *Exact diffeotopies of $(\widetilde{V}, \widetilde{g})$ sharply move V in $\widetilde{V}$.*

Proof. Start with U, δ, a and S as in Definition 1.9. Let us denote by $H : \widetilde{V} \longrightarrow \mathbb{R}$ the projection map of $\widetilde{V}$ onto its last factor; that is, $H(v, x, y) = y$. Then $dH = dy$. We define a vector field X by the formula $X = I_{\widetilde{g}}^{-1}(dH) = I_{\widetilde{g}}^{-1}(dy)$. Then $X.\widetilde{g} = dy$, and therefore $X = \frac{\partial}{\partial x}$. If $\{\delta_t\}$ is the one-parameter group of diffeomorphisms generated by X, then $\delta_t(v, x, y) = (v, x + t, y)$.

Take a constant c satisfying $a < c < \delta$, and define a function $b : \widetilde{V} \longrightarrow [0, c]$ as follows:

$$b(v, x, y) = f(d(v, S)/\tau) \quad \text{for} \quad (v, x, y) \in \widetilde{V},$$

where f is a smooth function $\mathbb{R} \longrightarrow [0, c]$ such that $f(0) \equiv c$ on a neighbourhood of 0 and f is identically zero outside the open unit interval. Then $b \equiv 0$ outside the τ-neighbourhood of S, and $b \equiv c$ on a neighbourhood of S.

Let $\{x_i\}$, $i = 1, \ldots, 2n$, denote the local coordinate system in V with respect to which the symplectic form g can be written as $g = \sum dx_i \wedge dx_{n+i}$. Since b is independent of the coordinates x and y in $\mathbb{R}^2$, db can be expressed locally as $db = \sum f_i.dx_i$ where f_i's are local functions on $\widetilde{V}$. Then it follows that $I_{\widetilde{g}}^{-1}(db)$ is a vector field with no component in x and y directions. Let us denote this vector field by Y.

Now, $d(bH) = H\,db + b\,dH = y\,db + b\,dy$ (in particular, we have $d(bH) = b\,dy$ on V). Hence $I_{\widetilde{g}}^{-1}(d(bH)) = bX + yY$. We shall denote $bX + yY$ by ∂. Let σ_t be the one-parameter group of diffeomorphisms generated by ∂. Clearly,

σ_t is an exact diffeotopy. We shall prove that σ_t sharply moves U at the hypersurface S. Observe that $(bX)(v,x,y) = b(v)\frac{\partial}{\partial x}(v,x,y)$. Hence, if $\bar{\delta}_t$ denotes the one-parameter group of diffeomorphisms generated by bX then $\bar{\delta}_t(v,x,y) = (v, x + tb(v), y)$. As a result we have

$$\begin{aligned} \text{dist}\,(\bar{\delta}_t(v), V) &= \text{dist}\,((v, tb(v), 0), V) \\ &= tb(v) \end{aligned} \qquad (12.11)$$

Now, noting that the vector field yY has no component in the direction perpendicular to V, and then using (12.11), we get the following relation:

$$\text{dist}\,(\sigma_t(v), V) \leq \max_{v,t} \text{dist}\,(\bar{\delta}_t(v), V) = c < \delta,$$

where $v \in U$. In fact, for sufficiently small real numbers x and y we have

$$\text{dist}\,(\sigma_t(v,x,y), V) < \delta,$$

where $t \in \mathbb{I}$, $v \in U$.

On the other hand $b(v,x,y) = c$ for v in some neighbourhood of S, and for all x, y. This implies that $\partial = bX$ on a neighbourhood of S, and hence $\text{dist}\,(\bar{\delta}_t(v), V) = \text{dist}\,(\sigma_t(v), V)$ when $v \in S$. Then using (12.11) we get, for $v \in S$,

$$\text{dist}\,(\sigma_1(v), V) = \text{dist}\,(\bar{\delta}_1(v), V) = b(v) = c > a,$$

Moreover, since $b \equiv 0$ outside the τ-neighbourhood of S, each σ_t is identity there. Thus we have proved that σ_t sharply moves U at S. □

Lemma 12.12 *The sheaves Φ and $\widetilde{\Phi}$ are microflexible.*

Let us consider the following lifting homotopy problem for the compact pair (C, C') in V:

$$\begin{array}{ccc} Q \times \{0\} & \xrightarrow{F_0} & \Phi(C) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r} \\ Q \times \mathbb{I} & \xrightarrow[f]{} & \Phi(C') \end{array}$$

where Q is a compact polyhedron.

Let us denote the sheaf of immersions of V into W by Φ'. Then Φ is a subsheaf of Φ'. Since the immersion relation is open, we can lift an initial part of f to $\Phi'(C)$. Let $F: Q \times [0, \varepsilon] \longrightarrow \Phi'(C)$ denote a lift of $f|Q \times [0, \varepsilon]$ for some positive number $\varepsilon \leq 1$ such that $F(q, 0) = F_0(q, 0)$. Since $F(q, 0)^* h$ is non-degenerate on a neighbourhood of C, for $q \in Q$, and since C is compact, therefore $F(q, t)^* h$ is non-degenerate on a neighbourhood of C for all $q \in Q$ and $t \in [0, \varepsilon_1]$, for some positive number $\varepsilon_1 \leq \varepsilon$.

Set $F(q, t)^* h = g_{q,t}$ and denote the cohomology class of a form ω by $[\omega]$. Since $F(q, t)$ is homotopic to $F(q, 0)$, therefore we have

$$[g_{q,t}] = [F(q, t)^* h] = F(q, t)^*[h] = F(q, 0)^*[h] = [F(q, 0)^* h] = [g].$$

Moreover, $g_{q,0} = g$ on $\operatorname{Op} C$ and $g_{q,t} = g$ on $\operatorname{Op} C'$ for $q \in Q$ and $t \in [0, \varepsilon_1]$. Since C and C' are compact subsets, we can apply Corollary 11.9 to obtain a continuous family $\{\delta_{q,t}\}$ of diffeomorphisms on $\operatorname{Op} C$ such that $\delta_{q,0}$ and $\delta_{q,t}|\operatorname{Op} C'$ are identity on their respective domains, and $\delta_{q,t}^* g_{q,t} = g$.

Set $\bar{f}(q, t) = F(q, t) \circ \delta_{q,t}$. Then $\bar{f}(q, t) \in \Phi(C)$ for all $(q, t) \in Q \times [0, \varepsilon_1]$. Also, $\bar{f}(q, 0) = F_0(q, 0)$ and $\bar{f}(q, t)|\operatorname{Op} C' = f(q, t)|\operatorname{Op} C'$. Hence $\bar{f}$ is the required lift.

With the same arguments as above we can conclude $\widetilde{\Phi}$ is also a microflexible sheaf. □

We have noted in Section 7 that for proving flexibility of $\widetilde{\Phi}|V$ we require the sheaf $\widetilde{\Phi}$ to be microflexible, and also we need the existence of diffeotopies which would sharply move V in $\widetilde{V}$ and act on the jet space so as to keep $\widetilde{\mathcal{R}}$ invariant. Therefore we may conclude, in view of (12.9), (12.10) and (12.12), that

Proposition 12.13 *The sheaf* $\widetilde{\Phi}|V$ *is flexible.*

We shall now prove the following

Lemma 12.14 $\widetilde{\mathcal{R}}$ *is a microextension of* $\mathcal{R}$

Proof. Consider the morphism $\alpha : \widetilde{\Phi} \longrightarrow \Phi$ of (8.1), and recall that the proof of surjectiveness of $\alpha(x)$ depended on the fact that $\widetilde{\mathcal{R}}$ is open (see last two lines of the proof of Lemma 8.2). Although $\widetilde{\mathcal{R}}$ is not open in the present situation, Theorem 11.6 allows us to conclude that $\alpha(x)$ is onto for all $x \in V$. In fact, $\widetilde{f}$ (as constructed in Lemma 8.2) has the following two properties:

- $\widetilde{f}$ equals f on $U \cap V$, where U is the domain of $\widetilde{f}$. Hence, pullbacks of both the forms $\widetilde{f}^*h$ and $\widetilde{g}$ to V are the same. Therefore, by Lemma 11.2, we obtain a 1-form φ on a neighbourhood, say $\widetilde{U}$, of $(x, 0)$ in U such that $d\varphi = \widetilde{f}^*h - \widetilde{g}$ and $\varphi|\widetilde{U} \cap V = 0$.

- $\widetilde{f}^*h = \widetilde{g}$ at $(x, 0)$.

Now, proceeding as in Theorem 11.6 we get a diffeomorphism δ on a neighbourhood, say U', of $(x, 0)$ in $\widetilde{U}$, such that $\delta^*(\widetilde{f}^*h) = \widetilde{g}$ and $\delta|U' \cap V$ is identity. Then $f' = \widetilde{f} \circ \delta$ is the required extension of f.

Next we shall prove that the map $\gamma : \widetilde{\Phi}(A) \longrightarrow \Gamma(A, B)$ is a microfibration for a pair of compact subsets (A, B) in V (recall from Section 8 that $\Gamma(A, B)$ is a subset of $\Phi(A) \times \widetilde{\Phi}(B)$ consisting of compatible pair of maps $(\widetilde{\sigma}, \sigma)$ so that $\alpha(\widetilde{\sigma}) = \sigma|B$). To prove this consider a lifting problem

$$\begin{array}{ccc} P\times\{0\} & \xrightarrow{\widetilde{f}_0} & \widetilde{\Phi}(A) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \gamma} \\ P\times \mathbb{I} & \xrightarrow[(f,\varphi)]{} & \Gamma(A,B) \end{array}$$

where $r\circ\widetilde{f}_0 = (f,\varphi)\circ i$. Let Φ' denote the sheaf of C^∞ immersions of $\widetilde{V}$ into W. Since the immersion relation is open, we get a lift $\bar{f} : P\times[0,\varepsilon] \longrightarrow \Phi'(A)$ such that for all $t\leq\varepsilon$ we have

- $\bar{f}(p,0) = \widetilde{f}_0(p)$
- $\bar{f}(p,t)|\mathrm{Op}_V A = f(p,t)$,
- $\bar{f}(p,t)|\mathrm{Op}_{\widetilde{V}} B = \varphi(p,t)$.

where $\mathrm{Op}_V A$ denotes an open neighbourhood of A in V and $\mathrm{Op}_{\widetilde{V}}B$ denotes an open neighbourhood of B in $\widetilde{V}$. Since $\bar{f}(p,0)^*h$ is non-degenerate on a neighbourhood of the compact set A, for all p, there exists a positive number ε_1 such that $\bar{f}(p,t)^*h$ is non-degenerate on a neighbourhood of A, for all $p\in P$ and $t\leq\varepsilon_1$. We also have,

$$\bar{f}(p,t)^*[h] = \bar{f}(p,0)^*[h] = [\widetilde{g}]$$

for all $p\in P$ and $t\in\mathbb{I}$.

Now, for brevity, let us write $g_{p,t}$ for $\bar{f}(p,t)^*h$. Then for $t\leq\varepsilon_1$ we can express $g_{p,t}$ as

$$g_{p,t} = \widetilde{g} + \omega_{p,t},$$

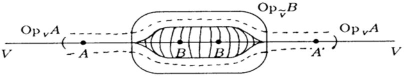
where $\omega_{p,t}$ is an exact 2-form on a neighbourhood of A in $\widetilde{V}$ satisfying

$$\omega_{p,t}|\mathrm{Op}_{\widetilde{V}}B = 0 \quad\text{and}\quad i_A^*\omega_{p,t} = 0,$$

where $i_A : \mathrm{Op}_V A \longrightarrow \widetilde{V}$ is the inclusion map. Proceeding as in Lemma 11.2 we get a continuous family of 1-forms $\alpha_{p,t}$ satisfying the following conditions.

(12.15) $$d\alpha_{p,t} = \omega_{p,t}, \quad \alpha_{p,t}|\mathrm{Op}_V A = 0, \quad \alpha_{p,t}|\mathrm{Op}_{\widetilde{V}} B = 0.$$

(To see the existence of $\alpha_{q,t}$'s, let us take a subset N in $\widetilde{V}$ which is the union of the shaded region and $\mathrm{Op}_V A$ as shown in the diagram below. Let T be a neighbourhood of N. This is illustrated by broken lines in the picture.



Consider a deformation $\pi_t : T \longrightarrow T$ such that π_0 is a retraction of T onto N, π_1 is the identity map of T and π_t fixes N for each t. Now proceeding as in the proof of the Relative Poincaré lemma we obtain the required $\alpha_{q,t}$'s.)

Thus $g_{p,t}$ is a family of symplectic forms on a neighbourhood of B in $\widetilde{V}$, such that $g_{p,t} = \widetilde{g} + d\alpha_{p,t}$ and $g_{p,0} = \widetilde{g}$, where $\alpha_{p,t}$'s are given by (12.15). Since B is compact, we can apply Corollary 11.8 to the family $g_{p,t}$ to obtain a family of diffeomorphism $\{\delta_{p,t}; p \in P, \ 0 \leq t \leq \widetilde{\varepsilon}\}$ for some $\widetilde{\varepsilon} \leq \varepsilon_1$ such that

- $\delta_{p,0}$ is the identity map on a neighbourhood of B, / A
- $\delta_{p,t}|\mathrm{Op}_V A =$ identity,
- $\delta_{p,t}|\mathrm{Op}_{\widetilde{V}} B =$ identity,
- $\delta^*_{p,t} g_{p,t} = \widetilde{g}$.

The required partial lift of the lifting homotopy problem can now be given by the map $\widetilde{f} : P \times [0, \widetilde{\varepsilon}] \longrightarrow \widetilde{\Phi}(B)$ which is defined by the formula $\widetilde{f}(p,t) = \bar{f}(p,t) \circ \delta_{p,t}$. □

Proposition 12.16 *The sheaf Φ is flexible*

Proof. All the requirements of the proof has been satisfied in view of (8.5) and (8.6). Hence the proposition. □

To complete the proof of Theorem 12.2, it remains only to verify

Proposition 12.17 *The 1–jet map j^1 is a local weak homotopy equivalence.*

Proof. Consider the map ρ of Section 5. Although $\mathcal{R}$ is not open in this case, we can still prove that ρ is a weak homotopy equivalence, with the help of following observation.

Let $\varphi_q : V \longrightarrow W$, $q \in Q$, be a continuous family of smooth maps such that $j^1_x\varphi \in \mathcal{R}$, in other words $\varphi_q^* h = g$ at x. Set $g_q = \varphi_q^* h$ and apply Theorem 11.6. Then we get a family of diffeomorphisms δ_q on a neighbourhood of x such that $\delta_q^* g_q = g$, $\delta_q(x) = x$, and $d\delta_q|T_xV$ =identity. Define $\widetilde{\varphi} = \varphi \circ \delta_q$ on $\mathrm{Op}\, x$. Then $\widetilde{\varphi}_q$'s are isometric immersions on $\mathrm{Op}\, x$ and $j^1_x\varphi_q = j^1_x\widetilde{\varphi}_q$. Moreover, if some φ_q is isometric on a neighbourhood of x, we may get $\widetilde{\varphi}_q = \varphi_q$ on $\mathrm{Op}\, x$.

Since $\rho = e \circ j^1$, injectiveness of ρ_* implies that of j^1_*. So we shall only have to prove that j^1_* is surjective. Let Γ denote the sheaf of smooth maps from V to W. This is a sheaf over V. Consider the parametric sheaf Γ^V (see (8.6)) over $V \times V$ and take its restriction to V. We shall denote this sheaf by Γ^*, and Γ^* will be called the *associated sheaf* of Γ. Observe that $\Gamma^*(v)$ is the direct limit of the spaces $\Gamma(U)^U$ where U runs over open neighbourhoods of v in V. Thus a local section in Γ^* can be conceived as a continuous family of germs $\varphi_v \in \Gamma(U)$, $v \in U$. It can be proved that the canonical inclusion of Γ in Γ^*, given by $\varphi \mapsto \{v \mapsto \varphi\}$, is a weak homotopy equivalence (see [8], p.76). (The above construction is equally true for an arbitrary sheaf). To this end, we split j^1 in the following way:

$$\begin{array}{ccccc} \Phi(v) & \xrightarrow{\iota} & \Phi^*(v) & \xrightarrow{j} & \Gamma_0^*(v) \\ & \searrow^{j^*} & \downarrow & \swarrow_{J} & \\ & & \Psi(v) & & \end{array}$$

where Φ^* is the associated sheaf of Φ and $\Gamma_0^*(v)$ is the subspace of $\Gamma^*(v)$ consisting of all those families of germs $\{\varphi_u : u \in \operatorname{Op} v\}$ for which φ_u is a local immersion and $\varphi_u^* h = g$ at u, in other words $j_u^1 \varphi_u \in \mathcal{R}$. Thus it is easy to see that any section in $\Psi(v)$ gives rise to a section in $\Gamma^*(v)$. Hence J is onto. (The same technique can be applied to show that J_* is onto at each homotopy level). Now we shall show that the map j is also surjective, which will complete our proof. We shall first reduce our problem to a situation where Corollary 11.7 can be applied. Take a family $\{\varphi_u : u \in U\}$ as above where U is an open neighbourhood of v. We may suppose without loss of generality that each φ_u is defined on the same open subset U. Now applying Corollary 11.7 to the family $g_u = \varphi_u^* h$, we get a continuous family of diffeomorphisms $\{\delta_u\}$ defined on some open subset $\widetilde{U} \subset U$ such that $\delta_u^* g_u = g$, $\delta_u(u) = u$, and $d\delta_u | T_u M =$ identity for all $u \in \widetilde{U}$. Set $\widetilde{\varphi}_u = \varphi_u \circ \delta_u$. Then for u in $\widetilde{U}$, $\widetilde{\varphi}_u \in \Phi(\widetilde{U})$. Moreover $j_u^1 \widetilde{\varphi}_u = j_u^1 \varphi_u$ for all $u \in \widetilde{U}$ so that j is onto. □

We have now achieved our goal.

13 Contact Isometric Immersions

In this last section, we shall show how the arguments of Section 12 can be adapted to get h-principle for contact isometric immersions. Therefore manifolds here are without boundary as in Section 12.

A *contact structure* on a manifold V of dimension $2n+1$ is a 1-form g on V such that $g \wedge (dg)^n$ is a nowhere zero form on V. The pair (V, g) is called a *contact manifold*. If (V, g) and (W, h) are two contact manifolds with $\dim V < \dim W$, then a smooth immersion $f : V \longrightarrow W$ is called a contact isometric immersion if $f^* h = g$. Let $\mathrm{Cont}\,(V, W)$ be the space of contact isometric immersions of V in W, and $\mathrm{RC}\,(TV, TW)$ be the space of continuous vector bundle monomorphisms $F : TV \longrightarrow TW$ satisfying $F^* h = g$. Then analogous to Theorem 12.2 we have

Theorem 13.1 *The differential map*

$$d : \mathrm{Cont}\,(V, W) \longrightarrow \mathrm{RC}\,(TV, TW)$$

is a weak homotopy equivalence.

The proof of this is just a matter of following through the arguments of Section 11 and 12. The essential steps are indicated below.

First note that a substitute of Stability Theorem (11.6), which is the main ingradient of the proof of (12.2), for contact manifolds may be stated as follows.

Theorem 13.2 *Let (M, g) be a compact contact manifold and g_t, $t \in [0,1]$, a smooth 1-parameter family of contact forms on M with $g_0 = g$. Then there exists an isotopy $\{\delta_t\}$ of M such such that $\delta_t^* g_t = g$ for all $t \in [0,1]$. Moreover, if $g_t = g$ on some submanifold N of M for all t, then $\delta_t|N$ and $d\delta_t|T_N M$ are the identity maps.*

The proof may be seen easily after noting the following point. Since g_t is a contact form on M there exists a unique vector field ξ_t in $\operatorname{Ker} g_t$ such that $\mathcal{L}_{\xi_t} g_t = \frac{dg_t}{dt}$ (see [22] Proposition 2.1).

Next, reverting to (13.1), embed V in $V \times \mathbb{R}^2$ as in the symplectic case. The manifold $\widetilde{V} = V \times \mathbb{R}^2$ has a natural contact structure given by the 1-form $\widetilde{g} = g + x\,dy - y\,dx$, where (x, y) denotes the canonical coordinate system on $\mathbb{R}^2$, and the inclusion map $i : V \hookrightarrow \widetilde{V}$ pulls back $\widetilde{g}$ onto g in V. If $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ denote the relations corresponding to contact immersions in W of V and $\widetilde{V}$ respectively, then it follows that $\widetilde{\mathcal{R}}$ is an extension of $\mathcal{R}$, and that it is invaraiant under the action of contact diffeomorphisms of $\widetilde{V}$. The arguments are as follows. Starting with a 1-jet $j^1_v f$ as in Lemma 12.4, one gets the relations

$$h_{f(v)}(\widetilde{f}_*\mathbf{x}) = \widetilde{g}_{(v,0)}(\mathbf{x})$$

and
$$\widetilde{g}(\tfrac{\partial}{\partial x}) = -y, \qquad \widetilde{g}(\tfrac{\partial}{\partial y}) = x,$$

as substitutes for (12.5) and (12.7) respectively. One then finds two linearly independent vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ in $T_{f(v)}W \setminus df_v(T_vV)$ such that

$$h(\bar{\mathbf{x}}) = 0, \qquad h(\bar{\mathbf{y}}) = 0.$$

The choice of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ is possible because, given that g is contact, $h|df_v(T_vV) : df_v(T_vV) \longrightarrow \mathbb{R}$ is surjective and hence $df_v(T_vV) \not\subset \operatorname{Ker} h_{f(v)}$.

Finally one proves following [8] (p.339) that contact diffeomorphisms of $\widetilde{V}$ (definition is obvious) sharply move V in $\widetilde{V}$. If $\{\delta_t\}$ is a contact diffeotopy with $\delta_0 =$ identity then $\delta_t^*\widetilde{g} = \widetilde{g}$ for all $t \in [0, 1]$. Differentiating this with respect to t we get $\mathcal{L}_\partial \widetilde{g} = 0$, in other words $d(\partial.\widetilde{g}) + \partial.d\widetilde{g} = 0$, where ∂ is the vector field $\frac{d}{dt}\delta_t$. Moreover, if the differential of each δ_t maps $\operatorname{Ker}\widetilde{g}$ in $\operatorname{Ker}\widetilde{g}$, then δ_t pulls back $\widetilde{g}|\operatorname{Ker}\widetilde{g}$ onto itself. Such a diffeotopy is called a contact diffeotopy for the pair $(\widetilde{V}, \operatorname{Ker}\widetilde{g})$ and in that case ∂ is called a contact vector field. Thus for a contact diffeotopy $\{\delta_t\}$ of $(\widetilde{V}, \operatorname{Ker}\widetilde{g})$ we have $(d(\partial.\widetilde{g}) + \partial.d\widetilde{g})|\operatorname{Ker}\widetilde{g} = 0$. If ∂_0 and $\partial^{\perp}$ denote the components of ∂ along $\operatorname{Ker}\widetilde{g}$ and $\operatorname{Ker} d\widetilde{g}$ (which are complementary subbundles in $T\widetilde{V}$, $\widetilde{g}$ being contact), then the above relation is equivalent to

$$(d(\partial^{\perp}.\widetilde{g}) + \partial_0.d\widetilde{g})|\operatorname{Ker}\widetilde{g} = 0.$$

Thus every smooth function H on M determines a contact vector field ∂ such that

$$\widetilde{g}(\partial^{\perp}) = H, \quad \text{and} \quad \partial_0 = -I^{-1}(dH),$$

where I denotes the isomorphism $I_{d\widetilde{g}} : L \longrightarrow L^*$. Now proceeding as in Lemma 12.10 we can prove the desired result. □

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