Recurrence and Transience

of

Reflecting Diffusions

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and the everlasting memory of

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Contents

Chapter 1

Introduction

An attempt to obtain conditions for certain stability properties of reflecting diffusions in unbounded domains with boundary has been made in this thesis. For diffusions in $\mathbb{R}^d$, such stability properties like recurrence, transience and positive recurrence have been studied extensively; see Bhattacharya (1978), Kliemann (1987), Pinsky (1987). One might see Pinsky (1995) for an up-to-date review of known methods and results in this case. (For corresponding recurrence classification results on Markov chains using martingale ideas based on stochastic analogues of Lyapunov functions, see Meyn and Tweedie (1993a), (1993b) and the references given therein). The main concern in this study is to establish related results in the case of unbounded domains like the half space, the orthant and the quadrant.

Natural definitions of recurrence, transience, and positive recurrence are used throughout and they will be stated precisely in Chapter 2. In general, the diffusion is said to be *recurrent* if it visits every neighborhood of the starting point infinitely often. It is said to be *transient* if starting from any point it wanders off to infinity. A diffusion is *positive recurrent* if the hitting time of any bounded open set has a finite expectation. In the case of recurrent (positive recurrent) diffusion, the existence of a unique σ-finite (finite) invariant measure can be shown.

In the case of a smooth bounded domain, the reflecting diffusion, being a Feller continuous strong Markov process on a compact space, has an invariant probability measure and hence is positive recurrent. Therefore, the problem of interest is in unbounded domains.

In Chapter 2, reflecting diffusions in the half space are considered, where the dispersion and drift are Lipschitz continuous functions and the reflection field is C^3-smooth (see, however, Remark 2.1.10). The existence and uniqueness of such diffusions is well known. A dichotomy between recurrence and transience of such reflecting diffusions is proved; (a priori such a dichotomy is not obvious). We give proofs only when it differs from the case of diffusions (in $\mathbb{R}^d$) without boundary conditions. (see case(ii) in the proof of Lemma 2.1.2(a) and the proof of (c) $\Rightarrow$ (*d*) in Proposition 2.1.3). The main difference is the following: It is not clear if an analogue of Lemma 2.3(b) of Bhattacharya (1978) holds in the case of reflecting diffusions. (Of course, maximum principles under stronger differentiability conditions are available as in Protter and Weinberger (1967)). Further, functional characterisations of recurrence and transience are obtained which in turn lead to verifiable sufficient criteria for recurrence/transience in terms of appropriate Lyapunov functions. Using these criteria a "real variable" proof of certain interesting results of Rogers (1991) concerning reflecting Brownian motion in the half plane are given. Also, it is shown that the hitting time of any open set has a finite expectation if there is one positive recurrent point; in the course of the proof an analogue of an estimate due to Dupuis-Williams (1994) is obtained. The problem of transience down a side in the case of reflecting diffusions in the half plane is also dealt with.

The case of the orthant is considered in Chapter 3. In the past few years there has been an increasing interest in the study of reflecting diffusions in the orthant. In particular, reflecting Brownian motion in the orthant is proposed as an approximate model of open queueing networks in heavy traffic; see Harrison and Nguyen (1993). Also the papers of Reiman (1984) and Petersen (1991) concerning certain limit theorems to justify diffusion approximations for some multiclass feedforward networks have served as great impetus for these studies.

Due to the nonsmoothness of the domain and the possible discontinuity of the reflection field, techniques from the theory of partial differential equations cannot be applied to get the existence and uniqueness of solutions.

A probabilistic method to resolve this problem involves solving the *Skorohod problem* ; see Harrison and Reiman (1981), Bernard and El Kharroubi (1991), and the introduction in Taylor and Williams (1993) for the nuances concerning Skorohod problem.

In Section 3.1, the deterministic Skorohod problem in the orthant is studied when the drift and reflection field are path dependent non-anticipating functionals satisfying linear growth and local Lipschitz conditions. (In general this would lead to non-Markovian solutions). Also, the reflection field is assumed to satisfy a "spectral radius bound" type condition. By solving this problem, the existence of a unique solution is proved. Using this approach a class of reflecting diffusions is obtained with constant dispersion, state dependent drift, and reflection field as strong solutions of corresponding stochastic differential equations; these processes are Feller continuous and strong Markov. In particular, this includes the Ornstein- Uhlenbeck process. In Section 3.3, using a combination of probabilistic and analytic methods some useful properties of such reflecting diffusions are established. In particular, the strong Feller property of such diffusions is proved. Further, an expression for measurable transition density is obtained. As in Chapter 2, analogous results concerning recurrence, transience, and positive recurrence are established.

Another probabilistic method for studying reflecting diffusions in domains with corner is the "*submartingale problem*" in the sense of Varadhan and Williams (1985). (It may be noted that the reflecting diffusions considered in Chapter 3 do solve the submartingale problem). In the case of the quadrant, because of the simpler geometry, we can get hold of another class of reflecting diffusions as solutions of the submartingale problem. This is done in Chapter 4. These are diffusions that behave near the origin like reflecting Brownian motion (without drift) with constant reflection field. By a canonical conditioning argument, these diffusions are obtained. In Chapters 2 and 3, the strong Feller property has been very crucial in establishing results concerning recurrence and transience, in particular, dichotomy. As it is not clear if this class of diffusions in the quadrant has strong Feller property, the difficulty is circumvented by using certain auxiliary diffusions in half planes to study the asymptotics.

Examples are given at the end of each chapter to illustrate the usefulness and limitations of the results.

Chapter 2

Recurrence and Transience of Diffusions in the Half Space

In this chapter we will be dealing with reflecting diffusions in the half space. We will derive certain criteria for recurrence and transience in terms of appropriate Lyapunov functions.

2.1 Criteria for recurrence and transience

Let $\mathcal{H}^d = \{x \in \mathbb{R}^d : x_1 > 0\}$, $d \geq 2$ and $\overline{\mathcal{H}^d} = \mathcal{H}^d \cup \partial\mathcal{H}^d$. We have the coefficients a, b defined on $\overline{\mathcal{H}^d}$ and γ defined on $\partial\mathcal{H}^d$ satisfying the following conditions.

(A2.1) For each $x \in \overline{\mathcal{H}^d}$, $a(x) = ((a_{ij}(x)))_{1\leq i,j\leq d}$ is a $d \times d$ real symmetric positive definite matrix; there exist $\lambda_1, \lambda_2 > 0$ such that for any $x \in \overline{\mathcal{H}^d}$, any eigenvalue of $a(x) \in [\lambda_1, \lambda_2]$; $a_{ij}(\cdot)$ are bounded and Lipschitz continuous.

(A2.2) For each $x \in \overline{\mathcal{H}^d}$, $b(x) = (b_1(x), b_2(x), \cdots, b_d(x))$ is a vector in $\mathbb{R}^d$; $b_i(\cdot)$ are bounded Lipschitz continuous.

(A2.3) For each $x \in \partial\mathcal{H}^d$, $\gamma(x) = (1, \gamma_2(x), \gamma_3(x), \cdots, \gamma_d(x))$ is a vector in $\mathbb{R}^d$, and each $\gamma_i \in C_b^3(\partial\mathcal{H}^d)$.

Let the generator L and the boundary operator J be given by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad x \in \mathcal{H}^d, \tag{2.1.1}$$

$$Jf(x) = \frac{\partial f(x)}{\partial x_1} + \sum_{i=2}^{d} \gamma_i(x)\frac{\partial f(x)}{\partial x_i}, \quad x \in \partial\mathcal{H}^d. \tag{2.1.2}$$

For notational convenience we will denote $\mathcal{H}^d$ by $\mathcal{H}$ in the sequel. Let $\Omega = C([0,\infty) : \mathbb{R}^d)$ be endowed with the topology of uniform convergence on compacts. Let $X(t)$ denote the t-th coordinate map on Ω, that is
$X(t)(w) = X(t,w) := w(t)$; let $\{\mathcal{B}_t\}$ be the natural filtration. We also denote $X(t)$ by X_t in the sequel.

Let $\{P_x : x \in \overline{\mathcal{H}}\}$ be the (L, J) diffusion on $\overline{\mathcal{H}}$, that is,

(i) $P_x\{w : X(0,w) = x,\ X(t,w) \in \overline{\mathcal{H}} \quad \forall t \geq 0\} = 1$

(ii) For each $f \in C_b^2(\mathbb{R}^d)$ with $Jf \geq 0$ on $\partial\mathcal{H}$,

$$\{f(X(t)) - \int_0^t 1_{\mathcal{H}}(X(s))Lf(X(s))ds\} \quad \text{is a } P_x - \text{submartingale.} \tag{2.1.3}$$

Moreover there exists a continuous, nondecreasing, progressively measurable process $\xi(t)$ on Ω such that

(i) $\xi(t) = \int_0^t 1_{\partial\mathcal{H}}(X(s))d\xi(s)$

(ii) For each $f \in C_b^2(\mathbb{R}^d)$,

$$\{f(X(t)) - \int_0^t 1_{\mathcal{H}}(X(s))Lf(X(s))ds - \int_0^t 1_{\partial\mathcal{H}}(X(s))Jf(X(s))d\xi(s)\}$$
$$\text{is a } P_x - \text{martingale.} \tag{2.1.4}$$

(A2.3) For each $x \in \partial\mathcal{H}^d$, $\gamma(x) = (1, \gamma_2(x), \gamma_3(x), \cdots, \gamma_d(x))$ is a vector in $\mathbb{R}^d$, and each $\gamma_i \in C_b^3(\partial\mathcal{H}^d)$.

Let the generator L and the boundary operator J be given by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad x \in \mathcal{H}^d, \tag{2.1.1}$$

$$Jf(x) = \frac{\partial f(x)}{\partial x_1} + \sum_{i=2}^{d} \gamma_i(x)\frac{\partial f(x)}{\partial x_i}, \quad x \in \partial\mathcal{H}^d. \tag{2.1.2}$$

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Moreover there exists a continuous, nondecreasing, progressively measurable process $\xi(t)$ on Ω such that

(i) $\xi(t) = \int_0^t 1_{\partial\mathcal{H}}(X(s))d\xi(s)$

(ii) For each $f \in C_b^2(\mathbb{R}^d)$,

$$\{f(X(t)) - \int_0^t 1_{\mathcal{H}}(X(s))Lf(X(s))ds - \int_0^t 1_{\partial\mathcal{H}}(X(s))Jf(X(s))d\xi(s)\}$$
$$\text{is a } P_x - \text{martingale.} \tag{2.1.4}$$

The (L, J) diffusion $\{P_x : x \in \overline{\mathcal{H}}\}$ defined above is strong Markov and Feller continuous (see Stroock and Varadhan (1971)); or equivalently under $\{P_x\}$ the process $\{X(t) : t \geq 0\}$ is strong Markov and Feller continuous. By the existence of a continuous transition density under the conditions (A2.1), (A2.2), (A2.3), strong Feller property follows; (see Ramasubramanian (1996)). We will denote by E_x^P, the expectation with respect to P_x.

For any open set V in $\overline{\mathcal{H}}$, define the stopping times

$$\tau_V = \inf\{t \geq 0 : X(t) \notin V\}, \text{ and } \sigma_V = \inf\{t \geq 0 : X(t) \in V\}$$

Note that we are not assuming V to be bounded. If V is bounded by Lemma 3 in Ramasubramanian(1986), we have $P_x(\tau_V < \infty) = 1$ for all $x \in \bar{V}$.

Lemma 2.1.1 : Let V be a bounded open set in $\overline{\mathcal{H}}$, g a bounded measurable function such that, for $x \in V$

$$g(x) = E_x^P[g(X(\tau_V))]. \tag{2.1.5}$$

Then g is a continuous function on V.

Proof : In view of the strong Markov and strong Feller properties of (L, J) diffusions, Theorem 13.1 of Dynkin (1965), (see page 30), and Lemma 2.2 of Bhattacharya (1978), it is enough to show that

$$\lim_{t \downarrow 0} \sup_{x \in K} P_x(|X(t) - x| > \epsilon) = 0, \tag{2.1.6}$$

for any $K \subseteq \overline{\mathcal{H}}, K$ compact and $\epsilon > 0$. But this follows from the uniform estimate given in page 181 of Stroock and Varadhan (1971). □

Lemma 2.1.2 : (a): Let U_1, U_2 be open sets in $\overline{\mathcal{H}}$ such that U_1 is nonempty and $\overline{U}_1 \cap \overline{U}_2 = \phi$. Let $\sigma_i = \inf\{t \geq 0 : X(t) \notin \overline{(\overline{U_i})^c} \cap \overline{\mathcal{H}}\}, i = 1, 2$. Then $x \mapsto P_x(\sigma_1 < \sigma_2)$ is a strictly positive continuous function on $\overline{(U_1)}^c \cap \overline{(U_2)}^c \cap \overline{\mathcal{H}}$.

(b) Let U be an open set in $\overline{\mathcal{H}}$. Then $x \mapsto P_x(\tau_U < \infty)$ is a strictly positive continuous function on U.

Proof : (a) Let $g(x) = P_x(\sigma_1 < \sigma_2)$, and $x \in \overline{U}_1^c \cap \overline{U}_2^c$ be arbitrary. Let V be a neighborhood of x such that $x \in V \subset \overline{V} \subset \overline{(U_1)}^c \cap \overline{(U_2)}^c \cap \overline{\mathcal{H}}$. Then we have

$$g(x) = E_x^P \left[E_{X(\tau_V)}(1_{\{\sigma_1 < \sigma_2\}} \right] = E_x^P[g(X(\tau_V))]. \tag{2.1.7}$$

Hence by Lemma 2.1.1, g is continuous on V. It remains to show that g is strictly positive.

Case (i) $x \in \mathcal{H} \cap \overline{U}_1^c \cap \overline{U}_2^c$. Let L-diffusion denote the diffusion in $\mathbb{R}^d$ with generator L. Since (L, J) diffusion behaves like L-diffusion till hitting $\partial\mathcal{H}$, by the support theorem of Stroock and Varadhan (1972), it follows that g is strictly positive.

Case (ii) $x \in \partial\mathcal{H} \cap \overline{U}_1^c \cap \overline{U}_2^c$. Let $\delta > 0$, be such that $\overline{B(x : \delta)} \cap \overline{U}_i = \phi, i = 1, 2$. Then by the strong Markov property, $g(x) = E_x^P[g(X(\tau_B))]$, where $\tau_B = \inf\{t \geq 0 : X(t) \notin B(x : \delta)\}$. Suppose $g(x) = 0$. Then $P_x(\sigma_1 < \sigma_2) = 0$, $P_x X(\tau_B)^{-1}$ a.s. Since (L, J) diffusion does not hit $\partial\mathcal{H} \cap \partial B(x : \delta)$ which is a $(d-2)$ dimensional manifold (see Theorem 3.7 of Ramasubramanian (1988)), it follows that $P_z(\sigma_1 < \sigma_2) = 0$ for some $z \in \mathcal{H} \cap \partial B(x : \delta)$. This contradicts Case (i). Hence g is strictly positive.

(b) Follows directly from (a) by taking $U_1 = \text{Int}(U^c) \cap \overline{\mathcal{H}}$ and $U_2 = \phi$.

Definition (a) A point $x \in \overline{\mathcal{H}}$ is said to be a *recurrent* point for (L, J) diffusion if for every $\epsilon > 0$,

$$P_x(X(t) \in B(x : \epsilon) \text{ for a sequence of } t's \uparrow \infty) = 1. \tag{2.1.8}$$

(b) A point $x \in \overline{\mathcal{H}}$ is a *transient* point for the (L, J) diffusion if

$$P_x \left(\lim_{t \to \infty} |X(t)| = \infty \right) = 1. \tag{2.1.9}$$

If all points $x \in \overline{\mathcal{H}}$ are recurrent (transient) then the diffusion is said to be *recurrent* (*transient*).

Proposition 2.1.3 : Assume (A2.1) - (A2.3). The following statements are equivalent.

(a) $x_0 \in \overline{\mathcal{H}}$ is a recurrent point.

(b) $P_{x_0}(X(t) \in U \text{ for some } t \geq 0) = 1$, for all nonempty open sets $U \subset \overline{\mathcal{H}}$.

(c) There exist $z_0 \in \overline{\mathcal{H}}, 0 < r_0 < r_1$, and $y \in \partial B(z_0 : r_1)$ such that $P_y(\sigma < \infty) = 1$, where $\sigma = \inf\{t \geq 0 : X(t) \in \overline{B(z_0 : r_0)}\}$.

(d) There exists a compact set $K \subset \overline{\mathcal{H}}$ such that $P_x(X(t) \in K \text{ for some } t \geq 0) = 1, \ x \in \overline{\mathcal{H}}$.

(e) $P_x(X(t) \in U$ for some $t \geq 0) = 1$, for all $x \in \overline{\mathcal{H}}$ and for all nonempty open sets $U \subset \overline{\mathcal{H}}$.

(f) $P_x(X(t) \in U$ for a sequence of t's $\uparrow \infty) = 1$, for all $x \in \overline{\mathcal{H}}$ and for all nonempty open sets $U \subset \overline{\mathcal{H}}$.

(g) (L, J) diffusion is recurrent.

Proof : We will prove only (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) ; proofs of other implications are either trivial or analogous to the corresponding implications in Bhattacharya (1978).

(a) $\Rightarrow$ (b) : Let $x_0 \in \overline{\mathcal{H}}$ be a recurrent point. Assume w.l.g, that $x_0 \notin U$. Let B be a ball such that $\overline{B} \subset U$. Choose $\epsilon > 0$ such that, $\overline{B(x_0, \epsilon)} \cap \overline{B} = \phi$. Let U_1 be a bounded open set such that $\overline{B(x_0, \epsilon)} \cup \overline{B} \subset U_1$. By Lemma 2.1.2, and as the diffusion exits out of bounded sets in finite time, we have

$$\inf_{y \in \partial U_1} P_y(\sigma_1 < \sigma_2) > 0, \tag{2.1.10}$$

where $\sigma_1 = \tau_{\overline{B}^c}$ and $\sigma_2 = \tau_{\overline{B(x_0,\epsilon)}^c}$.
The rest of the proof follows as the proof of $(a) \Rightarrow (b)$ in Proposition 3.1 of Bhattacharya (1978).

(c) $\Rightarrow$ (d) : Let $K = \overline{B(z_0 : r_0)}$; $y \in \partial B(z_0 : r_1)$. By (c) we have $P_y(\sigma < \infty) = 1$.

Case (i) $y \in \mathcal{H}$. Define

$$V(x) = 1 - P_x(\sigma < \infty). \tag{2.1.11}$$

By Lemma 2.1.2, V is continuous on $K^c \cap \overline{\mathcal{H}}$. By strong Markov property

$$0 = V(y) = E_y^P[V(X(\tau))], \tag{2.1.12}$$

where τ = exit time from $B(y:\delta)$ with $\overline{B(y:\delta)} \cap K = \phi, B(y:\delta) \subset \mathcal{H}$. By (2.1.12) we have $V(z) = 0$, $P_y X(\tau)^{-1}$ a.s. Now by the support theorem for L-diffusions (see Stroock and Varadhan(1972)) , and continuity of V, $V(z) = 0$, $\forall z \in \partial B(y:\delta)$. This holds for all sufficiently small $\delta < (r_1 - r_0) \wedge d(y, \partial\mathcal{H})$. If $z \in \mathcal{H} \cap K^c$, then one can find points $y_0, y_1, \cdots, y_{k+1} \in \mathcal{H} \cap K^c$ such that $y_0 = y, |y_{j+1} - y_j| < (|y_j - z_0| - r_0) \wedge d(y_j, \partial\mathcal{H})$ and $y_{k+1} = z$ By repeating the above argument we find

$$V(y_0) = V(y_1) = \cdots = V(z) = 0. \qquad (2.1.13)$$

Thus $V \equiv 0$ on $K^c \cap \mathcal{H}$. By continuity, $V \equiv 0$ on K^c and hence on $\overline{\mathcal{H}}$.

Case (ii) $y \in \partial\mathcal{H}$. As in equation (2.1.12), we have $0 = V(y) = E_y^P[V(X(\tau))]$ by strong Markov property. Hence $V(z) = 0$, $P_y X(\tau)^{-1}$ a.s. z. Since (L, J) diffusion does not hit $\partial\mathcal{H} \cap \partial B(y:\delta)$ (See Ramasubramanian (1988)) we have $V(z) = 0$ for some $z \in \mathcal{H} \cap K^c$. Thus, the problem is reduced to Case (i). Hence the Proposition. □

Theorem 2.1.4 : Assume (A2.1) - (A2.3).

(a) (Dichotomy) (L, J) diffusion is not recurrent $\Leftrightarrow$ (L, J) diffusion is transient.

(b) (L, J) diffusion is recurrent $\Leftrightarrow$ there exist a compact set $K \subseteq \overline{\mathcal{H}}$ with nonempty interior, a point $x \in K^c \cap \overline{\mathcal{H}}$ and a measurable real valued function u such that

(i) $u(z) \uparrow \infty$ as $|z| \uparrow \infty$; (ii) $E_x^P[u(X(\sigma_K))] \leq u(x)$

(c) (L, J) diffusion is transient $\Leftrightarrow$ there exist a compact set $F \subseteq \overline{\mathcal{H}}$ with nonempty interior, $y \in F^c \cap \overline{\mathcal{H}}$ and a measurable real valued function u such that

(i) $E_y^P[1_{\{\sigma_F < \infty\}} u(X(\sigma_F))] \leq u(y)$; (ii) $u(y) < \inf_{z \in F} u(z)$

Proof : (a) If (L, J) diffusion is transient then it trivially follows that (L, J) diffusion is not recurrent. Now let us suppose that (L, J) diffusion is not recurrent. Let $x \in \overline{\mathcal{H}}$ be arbitrary and choose r_0, r_1 such that $|x| < r_0 < r_1$. Put $\delta_1 = \sup_{\{|y|=r_1\} \cap \overline{\mathcal{H}}} P_y(\sigma_0 < \infty)$ where $\sigma_0 = \inf\{t \geq 0 : X(t) \notin \overline{B(0:r_0)}^c\}$. Since no point in $\overline{\mathcal{H}}$ is recurrent by the previous proposition, we have $P_y(\sigma_0 < \infty) < 1$ for all y such that $|y| = r_1$. Now as, $y \mapsto P_y(\sigma_0 < \infty)$

is a continuous function, we have $\delta_1 < 1$. Hence proceeding as in the proof of Theorem 3.2(b) of Bhattacharya (1978), we get the result. □

(b) Necessity : Let u be a function such that $u(z) = \hat{u}(|z|)$ where $\hat{u}$ is a strictly increasing function with $\lim_{r\to\infty} \hat{u}(r) = \infty$. Let $K = \overline{B(0:1)}$ and choose x such that $|x| > 1$. As the diffusion is recurrent, $|X(\sigma_K)| = 1$ a.s. P_x. Hence we have,

$$E_x^P[u(X(\sigma_K))] = \hat{u}(1) < u(x). \tag{2.1.14}$$

Sufficiency : Suppose the diffusion is not recurrent and hence by part (a), it is transient. Let $A = \{\sigma_K < \infty\}$. By transience, we see that $P_x(A^c) > 0$; and again by transience and (i) (of (b)) note that $u(X(\sigma_K)) = \infty$ on A^c. Hence, we have $E_x^P[u(X(\sigma_K))] = \infty$, which is a contradiction. □

(c) Necessity : Let $F = \overline{B(0:1)}$. Put $u(x) = P_x(\sigma_F < \infty), x \in \overline{\mathcal{H}}$. Then, we have $u(x) = E_x^P[1_{\{\sigma_F<\infty\}}1] = E_x^P[1_{\{\sigma_F<\infty\}}u(X(\sigma_F))]$, since $u(X(\sigma_F)) = 1$ on $\{\sigma_F < \infty\}$. By transience $u(x) < 1$ for $|x| > 1$, but $u(z) = 1 \ \forall z \in F$. Hence (ii) is also satisfied.

Sufficiency : Suppose the diffusion is not transient. Hence by part (a) it is recurrent. Therefore by (i) and (ii) above

$$\begin{aligned} u(y) &\geq E_y^P[1_{\{\sigma_F<\infty\}}u(X(\sigma_F))] = E_y^P[u(X(\sigma_F))] \\ &\geq \inf_{z\in F} u(z) > u(y) \end{aligned} \tag{2.1.15}$$

and hence a contradiction. Therefore the diffusion is transient. □

We now derive some corollaries which are analogues of Proposition 3.1 and 3.2 of Pinsky (1987).

Corollary 2.1.5 : If there exist $r_0 > 0$ and $u \in C^2(\mathbb{R}^d \backslash B(0 : \frac{r_0}{2}))$ such that

(i) $u(x) \uparrow \infty$ as $|x| \uparrow \infty$, (ii) $Lu(x) \leq 0$, $\{|x| \geq r_0\} \cap \overline{\mathcal{H}}$, (iii) $Ju(x) \leq 0$, $\{|x| \geq r_0\} \cap \partial\mathcal{H}$,

then (L, J) diffusion is recurrent.

Proof : By Ito's lemma, optional sampling theorem and by conditions (ii), (iii) in the hypothesis we have

$$E_x^P[u(X(t \wedge \sigma_K))] \leq u(x), \qquad (2.1.16)$$

where $K = \overline{B(0:r_0)}$. Let $A = \{\sigma_k < \infty\}$. If $P_x(A^c) > 0$ then by dichotomy (Theorem 2.1.4 (a)) the diffusion is transient and hence $\lim_{t\to\infty} |X(t,w)| = \infty$, for $w \in A^c$. Hence as u can be taken to be nonnegative without loss of generality,

$$\lim_{t\to\infty} E_x^P[u(X(t \wedge \sigma_K))] \geq \lim_{t\to\infty} E_x^P[1_{A^c} u(X(t \wedge \sigma_K))] = \infty. \qquad (2.1.17)$$

Note that (2.1.17) contradicts (2.1.16). Hence $P_x(A^c) = 0$.

Now letting $t \to \infty$ we have

$$E_x^P[u(X(\sigma_K))] \leq u(x). \qquad (2.1.18)$$

By part (b) of Theorem 2.1.4 we have that the diffusion is recurrent. □

Corollary 2.1.6 : If there exist $r_0 > 0$ and a function $u \in C_b^2(R^d \backslash (B(0:\frac{r_0}{2}))$ such that

(i) $Lu(x) \leq 0$, $\{|x| \geq r_0\} \cap \overline{\mathcal{H}}$, (ii) $Ju(x) \leq 0$, $\{|x| \geq r_0\} \cap \partial\mathcal{H}$, (iii) there is a point x_0 such that $|x_0| > r_0$ and $u(x_0) < \inf_{\{|x|=r_0\}\cap\overline{\mathcal{H}}} u(x)$,

then the diffusion is transient.

Proof : Let $K = \overline{B(0:r_0)}$. Without loss of generality let us take $u \geq 0$. By Ito's lemma, optional sampling theorem and by the conditions (i), (ii) of the hypothesis we have

$$E_{x_0}^P[u(X(t \wedge \sigma_K))] \leq u(x_0). \qquad (2.1.19)$$

Now

$$\begin{aligned} E_{x_0}^P[1_{\{\sigma_K<\infty\}} u(X(\sigma_K))] &= \lim_{t\uparrow\infty} E_{x_0}^P[1_{\{\sigma_K\leq t\}} u(X(\sigma_K))] \\ \leq \lim_{t\uparrow\infty} \Big\{ E_{x_0}^P[1_{\{\sigma_K\leq t\}} u(X(\sigma_K))] &+ E_{x_0}^P[1_{\{\sigma_K>t\}} u(X(t))] \Big\} \\ &= \lim_{t\to\infty} E_{x_0}^P[u(X(t \wedge \sigma_K))] \leq u(x_0). \end{aligned} \qquad (2.1.20)$$

Hence by part (c) of Theorem 2.1.4 we have transience. □

Now let us give some criteria for recurrence and transience of diffusions in terms of the coefficients of L and J. These are analogues to the criteria in Bhattacharya (1978).

Let L, J be defined as in (2.1.1), (2.1.2).

Define

$$A(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{x_i x_j}{|x|^2}, \quad B(x) = \sum_{i=1}^{d} a_{ii}(x), \quad C(x) = 2\sum_{i=1}^{d} x_i b_i(x)$$

Put

$$\begin{aligned}
\overline{\beta}(r) &= \sup_{|x|=r} \frac{B(x) - A(x) + C(x)}{A(x)} \\
\underline{\beta}(r) &= \inf_{|x|=r} \frac{B(x) - A(x) + C(x)}{A(x)} \\
\overline{I}(r) &= \int_1^r \frac{\overline{\beta}(u)}{u} du \ ; \ \underline{I}(r) = \int_1^r \frac{\underline{\beta}(u) du}{u}
\end{aligned}$$

Proposition 2.1.7 : Assume (A2.1) - (A2.3).

(a) Let $u(x) = \int_1^{|x|} \exp(-\overline{I}(r))dr$. If $u(x) \to \infty$ as $|x| \to \infty$ and $Ju(x) \leq 0$ for $x \in \partial\mathcal{H}$, $|x| > 1$, then the diffusion is recurrent.

(b) Let $v(x) = \int_1^{|x|} \exp(-\underline{I}(r))dr$. If $\lim_{x\to\infty} v(x) < \infty$, and $Jv(x) \geq 0$ for $x \in \partial\mathcal{H}, |x| > 1$, then the diffusion is transient.

Proof : Easily follows from Corollories 2.1.5 and 2.1.6 and the proof of Theorem 3.3 in Bhattacharya (1978). □

Remark 2.1.8 Note that (L, J) diffusion can be transformed to $(\tilde{L}, \partial/\partial x_1)$ diffusion through a C^2-diffeomorphism of $\overline{\mathcal{H}}$, (see Ramasubramanian(1986)). Let $\tilde{a}, \tilde{b}$ denote the coefficients of $\tilde{L}$. Define $\tilde{A}, \tilde{B}, \tilde{C}$ analogous to A, B, C above with a, b replaced by $\tilde{a}, \tilde{b}$.

Define

$$\tilde{\beta}(r) = \sup_{|x|=r} \frac{\tilde{B}(x) - \tilde{A}(x) + \tilde{C}(x)}{\tilde{A}(x)}$$

$$\tilde{u}(x) = \int_1^{|x|} \exp\left(-\int_1^r \frac{\tilde{\beta}(u)}{u} du\right) dr$$

Note that $\partial \tilde{u}/\partial x_1 = 0$ on $\partial \mathcal{H}$. Thus if $\tilde{u}(x) \to \infty$ as $|x| \to \infty$, we have that (L, J) diffusion is recurrent. Similarly we can have a condition for transience also.

Remark 2.1.9 : The boundedness assumptions in (A2.1),(A2.2) can be relaxed to linear growth conditions on a, b. Under such conditions the (L,J) diffusion does not explode. As in Lemma 2.5 of Bhattacharya(1978) the strong Feller property can be established. It is now clear that the analysis of this section can be carried through under the relaxed assumptions. We omit the details.

Remark 2.1.10 : The condition (A2.3) can be relaxed to γ_i being just bounded and Lipschitz continuous. In such a case the strong Feller property can be proved as in §3.2 of Chapter 3.

2.2 RBM in the Upper half plane with variable skew reflection

In this section we will deal with recurrence and transience of reflecting Brownian motion (RBM for short) with variable oblique reflection in the upper half plane. Rogers (1991) has dealt with this problem but has used complex analytic techniques to get the results. Here we will give a real-variable proof of these results using in particular Corollaries 2.1.5 and 2.1.6 of §2.1.

In this case it is convenient to deal with the problem in polar coordinates. Therefore we shall describe the setup in cartesian coordinates as well as in polar coordinates.

Let

$\boldsymbol{H} = \{(x_1, x_2) : x_2 > 0, -\infty < x_1 < \infty\} = \{(r, \theta) : r > 0, \theta \in (0, \pi)\}$

$\partial_1 \boldsymbol{H} = \{(x_1, 0) : x_1 > 0\} = \{(r, \theta) : r > 0, \theta = 0\}$

$\partial_2 \boldsymbol{H} = \{(x_1, 0) : x_1 < 0\} = \{(r, \theta) : r > 0, \theta = \pi\}$

$\partial \boldsymbol{H} = \partial_1 \boldsymbol{H} \cup \partial_2 \boldsymbol{H} \cup \{(0,0)\}; \ \overline{\boldsymbol{H}} = \boldsymbol{H} \cup \partial \boldsymbol{H}$

Here the generator L is the Laplacian, viz.,

$L = \frac{1}{2}\Delta = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$

For $x \in \partial \boldsymbol{H}$, let $\gamma(x)$ be the direction of reflection and let $\eta(x)$ be the angle that $\gamma(x)$ makes with the normal at x, the clockwise direction being taken to be positive.

As in §2.1 we will assume that the normal component of γ is bounded away from 0; hence w.l.g, we may take normal component to be 1. So we may write $\gamma(x) = (\gamma_1(x), 1) = (\tan\ \eta(x), 1)$. As γ is bounded, and bounded away from tangential direction note that there exists $\beta >\ 0$ such that

$$-\pi/2 + \beta \leq \eta(x) \leq \pi/2 - \beta$$

Also we assume that $\eta(x) \in C_b^3(\partial \boldsymbol{H})$

Now the boundary operator J can be written as

$$\begin{aligned} Jf(x) &= \gamma_1(x)\frac{\partial f(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_2} \\ &= \tan\ \eta(\cdot)\frac{\partial f(\cdot)}{\partial r} + \frac{1}{r}\frac{\partial f(\cdot)}{\partial \theta}, \ on \ \ \partial_1 \boldsymbol{H} \\ &= -\tan\ \eta(\cdot)\frac{\partial f(\cdot)}{\partial r} - \frac{1}{r}\frac{\partial f(\cdot)}{\partial \theta}, \ on \ \ \partial_2 \boldsymbol{H}. \end{aligned} \qquad (2.2.1)$$

For $x \in \partial_1 \boldsymbol{H}$, note that $\eta(x) = \eta(|x|, 0)$ and for $x \in \partial_2 \boldsymbol{H}$, we have $\eta(x) = \eta(|x|, \pi)$. We will use this notation in the sequel.

Theorem 2.2.1 (a) If $\limsup_{r\to\infty} \eta(r, 0) < \liminf_{r\to\infty} \eta(r, \pi)$, then the RBM is recurrent.

(b) If $\liminf_{r\to\infty} \eta(r, 0) > \limsup_{r\to\infty} \eta(r, \pi)$, then the RBM is transient.

(c) If $\eta(r, 0)$ is nondecreasing and $\eta(r, \pi)$ is nonincreasing and if

$$\lim_{r\to\infty} \eta(r, 0) = \lim_{r\to\infty} \eta(r. \pi)$$

then the RBM is recurrent.

Proof : (a) Let $\limsup_{r\to\infty}\eta(r,0) = -\xi_1$ and $\liminf_{r\to\infty}\eta(r,\pi) = \xi_2$ By hypothesis $-\xi_1 < \xi_2$; hence $\xi_1+\xi_2 = \epsilon > 0$.Put $-\xi_1+\epsilon/4 = -\theta_1$, $\xi_2-\epsilon/4 = \theta_2$ and $\alpha = (\theta_1+\theta_2)/\pi$ .Note that $\theta_1+\theta_2 > 0$ and hence $\alpha > 0$. Now define the function u on the set $\overline{B(0:1)}^c \cap \overline{\boldsymbol{H}}$ in terms of polar coordinates as follows

$$u(r,\theta) = r^\alpha \cos(\alpha\theta - \theta_1) \qquad (2.2.2)$$

Clearly $\frac{1}{2}\Delta u = 0$. Note that $-\pi/2 < -\theta_1 \le \alpha\theta - \theta_1 \le \theta_2 < \pi/2, \quad \forall\ \theta \in [0,\pi]$. Hence $\{\cos(\alpha\theta-\theta_1) : \theta\in[0,\pi]\}$ is bounded away from 0. Therefore $u(r,\theta)\to\infty$ as $r\to\infty$, since $\alpha > 0$. On $\partial_1\boldsymbol{H}$ we have,

$$Ju(r,\theta) = \alpha r^{\alpha-1}\cos\theta_1 \tan\eta(r,0) + \alpha r^{\alpha-1}\sin\theta_1 \qquad (2.2.3)$$

But since $\limsup_{r\to\infty}\eta(r,0) = -\xi_1$, there exists s_1 such that $\forall r \ge s_1$, $\eta(r,0) \le -\xi_1+\epsilon/4 = -\theta_1$. Hence $\tan\eta(r,0) \le \tan(-\theta_1)$.Consequently as $\alpha > 0$, by (2.2.3), we have on $\partial_1\boldsymbol{H}\cap\{r > s_1\}$

$$Ju(r,0) \le 0 \qquad (2.2.4)$$

Similarly on $\partial_2\boldsymbol{H}$

$$Ju(r,\pi) = -\alpha r^{\alpha-1}\cos\theta_2 \tan\eta(r,\pi) + \alpha r^{\alpha-1}\sin\theta_2 \qquad (2.2.5)$$

But as $\liminf_{r\to\infty}\eta(r,\pi) = \xi_2$, we have for some $s_2 > 0$, $\forall r\ge s_2$, $\eta(r,\pi) \ge \xi_2 - \epsilon/4 = \theta_2$. Substituting in (2.2.5) we see that, on $\partial_2\boldsymbol{H}\cap\{r\ge s_2\}$

$$Ju(r,\pi) \le 0 \qquad (2.2.6)$$

Hence by (2.2.4) and (2.2.6), we have on $[\partial_1\boldsymbol{H}\cap\{r\ge s_0\}]\cup[\partial_2\boldsymbol{H}\cap\{r\ge s_0\}]$

$$Ju \le 0$$

where $s_0 = \max\{s_1,s_2\}$

Now by Corollary 2.1.5 the process in recurrent. □

(b): Let $\liminf_{r\to\infty}\eta(r,0) = -\xi_1$ and $\limsup_{r\to\infty}\eta(r,\pi) = \xi_2$. By hypothesis $-\xi_1 > \xi_2$. Let $\xi_1+\xi_2 = -\epsilon < 0$ Put $-\xi_1 - \epsilon/4 = -\theta_1$ and $\xi_2 + \epsilon/4 = \theta_2$. Note that $-\theta_1 > \theta_2$ and let $\alpha = (\theta_1+\theta_2)/\pi < 0$. Define

$$u(r,\theta) = r^\alpha \cos(\alpha\theta - \theta_1)$$

Clearly $\frac{1}{2}\Delta u = 0$ and we have

$$Ju(r,\theta) = \alpha r^{\alpha-1}\cos\theta_1 \tan\eta(r,0) + \alpha r^{\alpha-1}\sin\theta_1 \qquad (2.2.7)$$

As $\liminf_{r\to\infty}\eta(r,0) = -\xi_1$, there exists $s_1 > 0$ such that $\forall r \geq s_1$, $\eta(r,0) \geq -\xi_1 - \epsilon/4 = -\theta_1$.

Therefore $\tan\eta(r,0) \geq \tan(-\theta_1)$. Substituting this in equation (2.2.7) we have, on $\partial_1 \boldsymbol{H} \cap \{r \geq s_1\}$

$$Ju \leq 0 \qquad (2.2.8)$$

Similarly for some $s_2 > 0$, on $\partial_2\boldsymbol{H} \cap \{r \geq s_2\}$,we have

$$Ju \leq 0 \qquad (2.2.9)$$

Combining (2.2.8) and (2.2.9), we have, on $[\partial_1\boldsymbol{H} \cap \{r \geq s_0\}] \cup [\partial_2\boldsymbol{H} \cap \{r \geq s_0\}]$

$$Ju \leq 0$$

where $s_0 = \max\{s_1, s_2\}$

Further since $0 < \cos(\alpha\theta - \theta_1) < 1$, note that we can find $r_0 > s_0$ and $\theta_0 \in [0,\pi]$ such that

$$u(r_0,\theta_0) = r_0^\alpha \cos(\alpha\theta_0 - \theta_1) < \inf_{\theta\in[0,\pi]} r^\alpha \cos(\alpha\theta - \theta_1) \ , \text{ as } \alpha < 0$$

Hence by Corollary 2.1.6, the RBM is transient.

(c): Let $\lim_{r\to\infty}\eta(r,0) = -\theta_1$ and $\lim_{r\to\infty}\eta(r,\pi) = \theta_2$ By assumption $\eta(r,0)\uparrow -\theta_1$ and $\eta(r,\pi)\downarrow\theta_2$. Note that

$$\alpha := (\theta_1+\theta_2)/\pi = 0$$

Define

$$u(r,\theta) = \log r + \theta \tan\theta_1$$

Clearly

$$\frac{1}{2}\Delta u(r,\theta) = 0 \qquad (2.2.10)$$

By (2.2.1) we have on $\partial_1 \boldsymbol{H}$.

$$Ju(r,\theta) = \frac{1}{r}\tan\eta(r,0) + \frac{\tan\theta_1}{r}$$

As $\lim_{r\to\infty} \eta(r,0) = -\theta_1$, we have

$$\tan\eta(r,0) \le \tan(-\theta_1)$$

Hence on $\partial_1 \boldsymbol{H} \cap \{r \ge s_1\}$

$$Ju \le 0$$

for some $s_1 > 0$.

Similarly we have on $\partial_2 \boldsymbol{H} \cap \{r \ge s_2\}$

$$Ju \le 0$$

for some $s_2 > 0$.

Combining we get on $[\partial_1 \boldsymbol{H} \cap \{r \ge s_0\}] \cup [\partial_2 \boldsymbol{H} \cap \{r \ge s_0\}]$

$$Ju \le 0 \qquad (2.2.11)$$

where $s_0 = \max\{s_1, s_2\}$

Also note that

$$u(r,\theta) \to \infty \text{ as } r \to \infty \qquad (2.2.12)$$

By (2.2.10), (2.2.11), (2.2.12) applying Corollary 2.1.5 we see that the RBM is recurrent. □

Remark 2.2.2 : Parts (a) and (b) of Theorem 2.2.1 have been proved by Rogers (1991) using complex analytic methods. He assumes that $\eta(\cdot)$ is a C^1 function posessing bounded derivative and further having linear growth. This is weaker than our assumption. But using our analysis we can prove Theorem 2.2.1 under this weaker assumption as well. Observe that the reflecting Brownian motion can be constructed quite easily in this case, (see 2(ii), page 228 of Rogers (1991)) and it has the semimartingale property. By invoking the necessary results from §3.2 we can prove the strong Feller property of the RBM. Now the above proof of Theorem 2.2.1 can be used without changes, to prove the required result.

A particular case of part (c) viz., when $\gamma(x) \equiv$ constant, has been dealt with by Williams (1985). In fact our choice of the function u in the above proofs was inspired by Varadhan and Williams (1985).

Remark 2.2.3 : Rogers obtains other results as well concerning reflecting Brownian motion with variable reflection field, using Pick functions of complex analysis; one may see Rogers (1990). Since 2- dimensional Brownian motion is well behaved under conformal mappings, complex analytic approach as considered by Rogers is a natural tool to use. Observe that Brownian motion in $\mathbb{R}^2$ is a critical case as far as recurrence/transience is concerned; that is Brownian motion in $\mathbb{R}^2$ just fails to be transient! This aspect is also manifest in part(b) of Theorem 2.2.1, in the sense that a mild perturbation by a suitable "reflection field" is enough to make the process transient. Our "real variable" approach enables us to consider also other critical cases like Example 5 of Section 2.5.

Proposition 2.2.4 Let $\eta(x)$ be the angle of reflection on the boundary ∂H such that it satisfies $\eta(x_1) = \eta(x_1 + 1)$, that is, we consider periodic reflecting conditions. Now let

$$\gamma_1 = \inf\{\eta(x_1) : x_1 \in [0,1]\},$$

$$\gamma_2 = \sup\{\eta(x_1) : x_1 \in [0,1]\} .$$

Then if $\gamma_1 = 0$ and $\gamma_1 < \gamma_2$, then the RBM is transient.

Proof: Put $u(r,\theta) = r^{\alpha}cos(\alpha\theta - \eta) - \theta \tan\gamma_2$, where $\alpha = -\gamma_2/2$ and $\eta = \gamma_2/2$. Then along the same lines as the proof of Theorem 2.2.1(b) we have transience of the process.

Note: The condition in Proposition 2.2.4 above is not covered by the inequalities in parts (a)and (b) of Theorem 2.2.1.

Remark 2.2.5 : Consider the generator and boundary operator as follows :

$$\begin{aligned} \tilde{L}f(x) &= (m^2+1)\frac{\partial^2 f(x)}{\partial x_1^2} + 2m\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} + \frac{\partial^2 f(x)}{\partial x_2^2}, \quad x \in \mathcal{H}, \\ \tilde{J}f(x) &= \frac{\partial f(x)}{\partial x_2}, \quad x \in \partial\mathcal{H}. \end{aligned}$$

where m is a positive constant, that is, we consider diffusion with generator $\tilde{L}$ and normal reflection at the boundary. By a transformation of the upper half plane as in Ramasubramanian (1988), we see that $(\tilde{L}, \tilde{J})$ diffusion is transformed to (Δ, J) diffusion where

$$Jf(x) = m\frac{\partial f(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_2}.$$

By part (c) of Theorem 2.2.1 we see that (Δ, J) diffusion is recurrent. Hence $(\tilde{L}, \tilde{J})$ diffusion is recurrent. It is interesting to note that Proposition 2.1.7 does not yield any information concerning the recurrence of $(\tilde{L}, \tilde{J})$ diffusion. This is not altogether very surprising because both Theorem 3.3 of Bhattacharya (1978), and Proposition 2.1.7 work well when the generator and the boundary operator preserve the class of radial functions. □

Our proof of Theorem 2.2.1(a) and a theorem of Menshikov and Williams immediately suggest the following result concerning passage-time moments.

Proposition 2.2.6: Suppose hypothesis of Theorem 2.2.1(a) holds. Let α be as in the proof of Theorem 2.2.1(a). Then there exists a positive constant $c < 1$ such that for $r > 0$,

$$(i) E_z^p(\sigma_r^p) < \infty, \quad \text{for } p < \frac{\alpha}{2}, \quad |z| > r, \quad (ii) E_z^p(\sigma_r^p) = \infty, \quad \text{for } p > \frac{\alpha}{2}, \quad |z| > rc.$$

where $\sigma_r = \inf\{t \geq 0 : |X(t)| = r\}$.

Proof: Let u be the function as in the proof of Theorem 2.2.1(a). Then the proposition follows by applying Theorem 4.1 of Menshikov and Williams (1995) to the function u. As $Ju \leq 0$, the proof of Theorem 4.1 essentially goes through, with minor changes. □

2.3 Transience down a side in half plane

In this section we revert back to the notation of §2.1. Let $\mathcal{H}^2 = \{x \in \boldsymbol{R}^2 : x_1 > 0\}$. Define

$$Lf(x) = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial f(x)}{\partial x_i}, x \in \mathcal{H}^2, \qquad (2.3.1)$$

$$Jf(x) = \frac{\partial f(x)}{\partial x_1} + \gamma_2(x)\frac{\partial f(x)}{\partial x_2}, x \in \partial\mathcal{H}^2. \qquad (2.3.2)$$

be respectively the generator and boundary operator.

Let $(X_1(t), X_2(t))$ denote the (L, J) diffusion on $\overline{\mathcal{H}^2}$. Suppose $\{X_1(t)\}$ is recurrent (that is, for any open set U in $[0, \infty)$ and any $x \in \overline{\mathcal{H}^2}$,

$P_x(X_1(t) \in U$ for a sequence of t's $\uparrow \infty) = 1)$. We give conditions for $\{X_2(t)\}$ to go to $-\infty$ a.s. Similar conditions can be given for $\{X_2(t)\}$ to go to $+\infty$. In this regard let us prove the following proposition.

Proposition 2.3.1 : Let there be a function $u \subset C^2(\boldsymbol{R}^2)$ with the properties

(i) $u \geq 0$ and $u(x) = \hat{u}(x_2)$, (ii) $\hat{u}(x_2)$ decreases as $x_2 \downarrow -\infty$, (iii) $\hat{u}(x_2)$ increases to ∞ as $x_2 \uparrow \infty$, (iv) $Lu \leq 0$ on $\mathcal{H}^2$ and $Ju \leq 0$ on $\partial\mathcal{H}^2$.

Then the diffusion is transient and further

$$P_x(\lim_{t\to\infty} X_2(t) = -\infty) = 1, \quad \forall x \in \overline{\mathcal{H}^2}$$

Proof : Let $r \in \boldsymbol{R}$ be arbitrary but fixed. Let

$\tau_r = \inf\{t \geq 0 : X_2(t) = r\}$, $S_1 = \{x \in \overline{\mathcal{H}}^2 : x_2 > r\}$ and $S_2 = \{x \in \overline{\mathcal{H}}^2 : x_2 < r\}$.

Step 1 : We will show that

$$\sup_{x \in l} P_x(\tau_r < \infty) < 1, \tag{2.3.3}$$

for any horizontal line $l \subseteq S_2$. This in particular implies that the process is transient.

Proof : Suppose not; then $\sup_{x \in l} P_x(\tau_r < \infty) = 1$. So given $\epsilon > 0$, there exists $x^{(0)} \in l$ such that

$$P_{x^{(0)}}(\tau_r < \infty) > 1 - \epsilon/2. \tag{2.3.4}$$

Note that

$$\begin{aligned} u(x^{(0)}) &\geq E_{x^{(0)}}[u(X(t \wedge \tau_r))] \\ &= E_{x^{(0)}}[1_{\{\tau_r \leq t\}} u(X(\tau_r))] + E_{x^{(0)}}[1_{\{\tau_r > t\}} u(X(t))]. \end{aligned} \tag{2.3.5}$$

Let $A = \{\tau_r < \infty\}$ and $A_T = \{\tau_r \leq T\}$

Choose T such that $P_{x^{(0)}}(A_T) > 1 - \epsilon$. This is possible as $A_T \uparrow A$. Consequently as $u \geq 0$, we get

$$\begin{aligned} u(x^{(0)}) &\geq E_{x^{(0)}}[1_{\{\tau_r \leq T\}} \hat{u}(r)] + E_{x^{(0)}}[1_{\{\tau_r > T\}} u(X(T))] \\ &= \hat{u}(r) P_{x^{(0)}}(A_T) + E_{x^{(0)}}[1_{A_T^c} u(X_T)] \\ &> \hat{u}(r)(1 - \epsilon). \end{aligned} \tag{2.3.6}$$

But this is a contradiction to (ii) above. Hence the claim in Step 1.

Step 2 : We will show that for all $x \in S_1$

$$P_x(\tau_r < \infty) = 1. \tag{2.3.7}$$

Proof : Put

$$\eta_{2k} = \inf\{t > \eta_{2k-1} : X_1(t) = 1\}; \quad \eta_{2k+1} = \inf\{t > \eta_{2k} : X_1(t) = 2\}.$$

Since the process $\{X_1(t)\}$ is recurrent note that $P_x(\eta_k < \infty) = 1$, $\forall k$. Hence by condition (iv), Ito's lemma and optional sampling theorem we have,

$$E_x^P[u(X(\eta_k \wedge \tau_r))] \leq u(x), \tag{2.3.8}$$

for all k. Let $A = \{\tau_r < \infty\}$ and $B = \{\eta_k < \infty \text{ for all } k\}$. Suppose (2.3.7) does not hold. Then $P_x(A^c) > 0$, and hence by recurrence of $\{X_1(t)\}$ we have $P_x(A^c \cap B) > 0$. By transience of the process (by Step 1) and as X_1 and X_2 are bounded below by $r \wedge 0$ $(= 0)$ on A^c, note that for a.a.$w \in A^c$.

$$X_1(t, w) + X_2(t, w) \to \infty \text{ as } t \to \infty. \tag{2.3.9}$$

This implies that, for a.a. $w \in A^c \cap B$

$$X_2(\eta_k(w), w) \to \infty \text{ as } k \to \infty. \tag{2.3.10}$$

Now by (2.3.8) and condition (iii)

$$u(x) \geq \lim_{k\to\infty} E_x^P[u(X(\eta_k \wedge \tau_r))] \geq \lim_{k\to\infty} E_x^P[1_{A^c \cap B} u(X(\tau_r \wedge \eta_k))] = \infty,$$

which is a contradiction. Hence equation (2.3.7) holds. Therefore the claim of Step 2 is proved.

Step 3 : Let $x = (x_1, x_2)$ be arbitrarily chosen. Choose r, r_1 such that $r_1 < r < x_2$. Define

$$\delta_{r_1} = \sup\{P_y(\tau_{r_1} < \infty) : y \text{ such that } y_2 = r\}.$$

By Step 1, $\delta_{r_1} < 1$. By strong Markov property,
$P_x(X_2(t) = r \text{ for a sequence of } t's \uparrow \infty) \leq P_x(\zeta_{2i+1} < \infty)$

$$= E_x^P[1_{\zeta_{2i-1}<\infty} P_{X(\zeta_{2i-1})}(\tau_r < \infty)] \leq \delta_{r_1} P_x(\zeta_{2i-1} < \infty) \leq \cdots \leq \delta_{r_1}^i, \tag{2.3.11}$$

where $\zeta_{2i} = \inf\{t > \zeta_{2i-1} : X_2(t) = r\}$ and $\zeta_{2i+1} = \inf\{t > \zeta_{2i} : X_2(t) = r_1\}$, $i = 1, 2, \cdots$

As $\delta_{r_1} < 1$ note that $\delta_{r_1}^i \to 0$ Hence $P_x(\limsup_{t\to\infty} X_2(t) < r) = 1$. As $r < 0$ is arbitrary, the proposition is proved. □

Example : Consider the function $u(x) = e^{x_2}$. Let L, J be defined as in (2.3.1), (2.3.2). $Lu(x) = (a_{22}(x) + b_2(x))e^{x_2}$, $Ju(x) = \gamma_2(x_2)e^{x_2}$.

Hence, if (i) $a_{22}(x) + b_2(x) \leq 0$ and (ii) $\gamma_2(x_2) \leq 0$

we have on assuming the recurrence of X_1, that the process is transient down to $-\infty$.

Note : Conditions for recurrence of X_1 are being investigated. In this connection one may see Ramasubramanian (1983), for conditions for recurrence of projections of diffusions in $\mathbb{R}^d$ (that is, without boundary conditions). Such conditions (together with appropriate modifications required to ensure that the derivatives along the reflecting directions are negative) in the present context are not difficult to prove.

2.4 Positive recurrence of diffusions in the half space

In this section we will deal with positive recurrence of diffusions. First let us define some stopping times which will be used in the sequel. We consider diffusions in the half space $\overline{\mathcal{H}}$, where $\mathcal{H}$ is as in §2.1. For $c > 0$, define

$$\sigma_c = \inf\{t \geq 0 : |X(t)| = c\}.$$

Definition : A point $x \in \overline{\mathcal{H}}$ is said to be *positive recurrent* if there exist bounded open sets U_1, U_2 such that $x \in U_1 \subseteq \bar{U}_1 \subseteq U_2$ and

$$\sup\{E_z^P(\sigma_{U_1}) : z \in \partial U_2\} < \infty. \qquad (2.4.1)$$

The diffusion is said to be positive recurrent if all points are positive recurrent.

Lemma 2.4.1 : Let x be a positive recurrent point; let U_1, U_2 be open balls such that (2.4.1) hold. Let U, V be balls such that $\overline{U}_1 \subseteq U \subseteq \overline{U} \subseteq U_2 \subseteq \overline{U}_2 \subseteq V$. Then

$$(i) \sup\{E_z^P(\sigma_{U_1}) : z \in \partial U\} < \infty. \qquad (2.4.2)$$

$$(ii) \sup\{E_z^P(\sigma_{U_1}) : z \in \partial V\} < \infty. \qquad (2.4.3)$$

Proof : By strong Markov property, Lemma 3 of Ramasubramanian(1986) and by positive recurrence of x, we have

$$\begin{aligned}
\sup_{y\in\partial U} E_y^P(\sigma_{U_1}) &\leq \sup_{y\in\partial U} E_y^P\left[1_{\sigma_{U_1}<\tau_{U_2}}\cdot\sigma_{U_1}\right] + \sup_{y\in\partial U} E_y^P\left[1_{\sigma_{U_1}>\tau_{U_2}}\cdot\sigma_{U_1}\right] \\
&\leq \sup_{y\in\partial U} E_y^P(\tau_{U_2}) + \sup_{y\in\partial U} E_y^P\left[E_{X(\tau_{U_2})}(\sigma_{U_1})\right] \\
&< \infty. \qquad (2.4.4)
\end{aligned}$$

(ii) By Proposition 2.1.3, existence of positive recurrent point implies that the diffusion is recurrent and hence we have $\sigma_{U_2} < \infty$ a.s. P_z for $z \in \overline{U}_2^c$. Therefore we have

$$\begin{aligned}
\sup_{z\in\partial V} E_z^P(\sigma_{U_1}) &= \sup_{z\in\partial V} E_z^P\left[E_z^P(\sigma_{U_1}|\mathcal{B}_{\sigma_{U_2}})\right] \\
&= \sup_{z\in\partial V} E_z^P(E_{X(\sigma_{U_2})}(\sigma_{U_1}) \\
&\leq \sup_{z\in\partial U_2} E_z^P(\sigma_{U_1}) < \infty. \qquad (2.4.5)
\end{aligned}$$

Proposition 2.4.2 : If there exists one positive recurrent point, then the diffusion itself is positive recurrent.

Proof : Let x_0 be a positive recurrent point and let y be an arbitrary point. We will show that y is positive recurrent. Since x_0 is a positive recurrent point, we can find two balls U_1, U_2 such that equation (2.4.1) holds. Let U_3, U_4 be balls such that $\overline{U}_2 \subseteq U_3 \subseteq \overline{U}_3 \subseteq U_4$ and $y \in U_3$. By Lemma 2.4.1,

$$\sup E_z^P\{(\sigma_{U_1}) : z \in \partial U_4\} < \infty. \qquad (2.4.6)$$

Since $\sigma_{U_3} < \sigma_{U_1}$ a.s. P_z for $z \in \partial U_4$, we have

$$\sup\{E_z^P(\sigma_{U_3}) : z \in \partial U_4\} < \infty. \qquad (2.4.7)$$

Combining equations (2.4.6) and (2.4.7) we see that y is a positive recurrent point. As y was chosen to be arbitrary we have the diffusion to be positive recurrent. □

Our next objective is to get an upper bound for the expected hitting time of a bounded open set. For this we need the following lemma.

Lemma 2.4.3 : Let A be a bounded open set in $\overline{\mathcal{H}}$, and let $r > 0$ be such that $\overline{A} \subseteq B(0:r)$. Then there exist $M > r, 0 < p_A < 1$, such that for all $x \in \overline{B(0:r)}$,

$$P_x(X(1) \in A \text{ and } |X(t)| \leq M \quad \forall t \in [0,1]) \geq p_A. \tag{2.4.8}$$

Proof : Since the diffusion has a continuous positive density and A is an open set, note that

$$p_o \equiv \inf\{P_x(X(1) \in A) : x \in \overline{B(0:r)}\} > 0. \tag{2.4.9}$$

Let $\epsilon = p_0/4$. By tightness of measures $\{P_x : x \in \overline{B(0:r)}\}$ on $C([0,1] : \mathbb{R}^d)$, we can find a compact set $K_\epsilon \subseteq C([0,1] : \mathbb{R}^d)$ such that for all $x \in \overline{B(0:r)}$,

$$P_x(K_\epsilon) > (1-\epsilon).$$

By Arzela-Ascoli's theorem, there exists M > 0 such that $|w(t)| \leq M, \quad \forall t \in [0,1]$, for all $w \in K_\epsilon$. Hence for all $x \in \overline{B(0:r)}$, by (2.4.9) we have

$$\begin{aligned} P_x(X_1^{-1}(A) \cap K_\epsilon) &= P_x(X_1^{-1}(A)) - P_x(X_1^{-1}(A) \backslash K_\epsilon) \\ &> p_0 - \epsilon = \frac{3}{4}p_0 =: p_A \end{aligned} \tag{2.4.10}$$

whence the lemma follows . □

Now with M, r as in the preceding lemma put, $\eta_0 = 0$; and for $i = 1, 2, \cdots$ define

$$\begin{aligned} \eta_1 &= \inf\{t \geq 0 : |X(t)| = r\} \quad \vee (1 \wedge \sigma_M) \\ \eta_{2i} &= \inf\{t > \eta_{2i-1} : X(t) \notin B(0:M)\} \wedge (\eta_{2i-1} + 1) \\ \eta_{2i+1} &= \inf\{t > \eta_{2i} : X(t) \in \partial B(0:r)\} \end{aligned}$$

Let $F = \{\sigma_A \leq 1, \sigma_M > 1\}$ where A,M are as in the preceding lemma.

By equation (2.4.10) note that, for any $x \in \overline{B}(0:r)$,

$$P_x(F) > p_A. \tag{2.4.11}$$

Proposition 2.4.4 : Let A, r, M, p_A be as in Lemma 2.4.3. For any $x \in \overline{B(0:r)}$,

$$E_x^P(\sigma_A) \leq \frac{1}{p_A}\left[2 + \sup_{|z|\leq M} E_z^P(\sigma_r)\right] + E_x^P(\sigma_r). \qquad (2.4.12)$$

Proof : If $\sup_{|z|\leq M} E_z^P(\sigma_r) = \infty$, note that equation (2.4.12) trivially holds. So assume $\sup_{|z|\leq M} E_z^P(\sigma_r) < \infty$. Then note that any point in $B(0:r)$ is positive recurrent and hence the diffusion itself is positive recurrent. This in particular implies that the diffusion is recurrent. Therefore all the stopping times involved in the proof are well defined.

$$\begin{aligned} E_x^P(\sigma_A) &= E_x^P\left[\int_0^{\sigma_A\wedge\eta_1} ds\right] + E_x^P\left[\sum_{i\geq 1}\int_{\sigma_A\wedge\eta_{2i-1}}^{\sigma_A\wedge\eta_{2i}} ds\right] \\ &+ E_x^P\left[\sum_{i\geq 1}\int_{\sigma_A\wedge\eta_{2i}}^{\sigma_A\wedge\eta_{2i+1}} ds\right]. \end{aligned} \qquad (2.4.13)$$

Clearly,

$$\begin{aligned} E_x^P\left[\int_0^{\sigma_A\wedge\eta_1} ds\right] \leq E_x^P(\eta_1) &= E_x^P(\sigma_r \vee (1\wedge\sigma_M)) \\ &\leq 1 + E_x^P(\sigma_r). \end{aligned} \qquad (2.4.14)$$

Next by the strong Markov property for $i \geq 1$,

$$\begin{aligned} E_x^P\left[\int_{\sigma_A\wedge\eta_{2i-1}}^{\sigma_A\wedge\eta_{2i}} ds\right] &= E_x^P\left[1_{\{\sigma_A>\eta_{2i-1}\}} E_{X(\eta_{2i-1})}\int_0^{\sigma_A\wedge\sigma_M\wedge 1} ds\right] \\ &\leq P_x(\sigma_A > \eta_{2i-1}). \end{aligned} \qquad (2.4.15)$$

As $\{\sigma_A > \eta_1\} \subseteq F^c$, we have

$$P_x(\sigma_A > \eta_1) \leq (1 - p_A). \qquad (2.4.16)$$

Now observe that for $i \geq 3$,

$$\begin{aligned} P_x(\sigma_A > \eta_{2i-1}) &= E_x^P\left[1_{\{\sigma_A>\eta_{2i-1}\}}1_{\{\sigma_A>\eta_{2i-3}\}}\right] \\ &\leq E_x^P\left[1_{\{\sigma_A>\eta_{2i-3}\}}E_{X(\eta_{2i-3})}(1_{\{\sigma_A>(\sigma_M\wedge 1)\}})\right] \\ &\leq P_x(\sigma_A > \eta_{2i-3})\sup_{|z|\leq r} P_z(\sigma_A > (\sigma_M\wedge 1)) \\ &\leq (1-p_A)P_x(\sigma_A > \eta_{2i-3}). \end{aligned} \qquad (2.4.17)$$

As $\eta_3 > \eta_1$ a.s., by equation (2.4.16) we have

$$P_x(\sigma_A > \eta_3) \leq (1 - p_A). \qquad (2.4.18)$$

By (2.4.17), (2.4.18) we have

$$P_x(\sigma_A > \eta_{2i-1}) \leq (1 - p_A)^{i-1}. \qquad (2.4.19)$$

Combining (2.4.15), (2.4.16) and (2.4.19) we have,

$$E_x^P\left[\sum_{i\geq 1} \int_{\sigma_A\wedge\eta_{2i-1}}^{\sigma_A\wedge\eta_{2i}} ds\right] \leq \frac{1}{p_A} + (1 - p_A). \qquad (2.4.20)$$

By strong Markov property and the fact that $X(\eta_{2i}) \in \overline{B(0:M)}$ we have,

$$\begin{aligned} E_x^P\left[\int_{\sigma_A\wedge\eta_{2i}}^{\sigma_A\wedge\eta_{2i+1}} ds\right] &= E_x^P\left[1_{\{\sigma_A>\eta_{2i}\}} E_{X(\eta_{2i})} \int_0^{\sigma_A\wedge\sigma_r} ds\right] \\ &\leq P_x(\sigma_A > \eta_{2i}) \sup_{|z|\leq M} E_z^P(\sigma_r). \qquad (2.4.21) \end{aligned}$$

As $\eta_{2i} > \eta_{2i-1}$ a.s., we have

$$P_x(\sigma_A > \eta_{2i}) \leq P_x(\sigma_A > \eta_{2i-1}) \leq (1 - p_A)^{i-1}. \qquad (2.4.22)$$

Combining (2.4.21) and (2.4.22) we get,

$$E_x^P\left[\sum_{i>1} \int_{\sigma_A\wedge\eta_{2i}}^{\sigma_A\wedge\eta_{2i+1}} ds\right] \leq \frac{1}{p_A}\left[\sup_{|z|\leq M} E_z^P(\sigma_r)\right]. \qquad (2.4.23)$$

Now combining equations (2.4.13), (2.4.14), (2.4.20) and (2.4.23) we have the proposition.
□

Corollary 2.4.5 : If the diffusion is positive recurrent then $E_y^P(\sigma_A) < \infty$, for any $y \in \overline{\mathcal{H}}$ and for any nonempty open set $A \subset \overline{\mathcal{H}}$.

Proof : Without loss of generality take A to be bounded open. Let $y \in \overline{\mathcal{H}}$ be arbitrary but fixed. Let $x \in A$ be arbitrary. By positive recurrence there exist open balls U_1, U_2 such that $x \in U_1 \subseteq \overline{U}_1 \subseteq U_2$ and (2.4.1) holds. Now choose $r > |y|$ such that $\overline{A} \cup \overline{U}_1 \subseteq B(0:r)$. For any M such that $\overline{B(0:r)} \cup \overline{U}_2 \subset B(0:M)$, $\sup_{|z|\leq M} E_z^P(\sigma_{U_1}) < \infty$ by Lemma 2.4.1. Hence we have $\sup_{|z|<M} E_z^P(\sigma_r) < \infty$, by continuity of sample paths. Choose a suitable M, such that Lemma 2.4.3 and Proposition 2.4.4 hold. Now the corollary follows. □.

Proposition 2.4.6 : Let $r_0 > 0,\ \epsilon > 0,\ \ u \in C^2(\mathbb{R}^d \backslash B(0 : \frac{r_0}{2}))$ be such that

(i) $Lu(x) \leq -\epsilon, \quad \{|x| \geq r_0\} \cap \overline{\mathcal{H}}$; (ii) $Ju(x) \leq 0, \quad \{|x| \geq r_0\} \cap \partial\mathcal{H}$; (iii) $u(x) \geq 0$ for all x such that $\{|x| \geq r_0\} \cap \overline{\mathcal{H}}$.

Then the diffusion is positive recurrent.

Proof : Let $\sigma_{r_0}^n = \inf\{t \geq 0 : |X(t)| \notin (r_0, n)\}$ Then by Ito's formula,

$$E_x^P\left[u(X(t \wedge \sigma_{r_0}^n))\right] - u(x) \leq -\epsilon E_x^P(t \wedge \sigma_{r_0}^n). \tag{2.4.24}$$

So $E_x^P(t \wedge \sigma_{r_0}^n) \leq \frac{1}{\epsilon}\left[u(x) - E_x^P(u(X(t \wedge \sigma_{r_0}^n)))\right]$.

Hence $E_x^P(\sigma_{r_0}^n) \leq \frac{u(x)}{\epsilon}$ as $t \to \infty$. But since $\sigma_{r_0}^n \uparrow \sigma_{r_0}$ as $n \to \infty$, we have

$$E_x^P(\sigma_{r_0}) \leq \frac{u(x)}{\epsilon} < \infty \tag{2.4.25}$$

From (2.4.25) it follows that $\sup_{|z|=r} E_z^P(\sigma_{r_0}) < \infty$ for any $r > r_0$.

Hence we have that the process is positive recurrent. □

Remarks :

(1) If the diffusion is recurrent then by the argument and results of Maruyama and Tanaka(1959), there exists a unique (upto scalar multiplicity) σ-finite invariant measure. Further if the diffusion is positive recurrent then by the same arguments, the invariant measure is a probability measure. (Note that the condition 6 of Maruyama and Tanaka (1959) is needed just for open balls.)

(2) In Bhattacharya(1978) a point x is said to be positive recurrent if for all $0 < r_0 < r_1$, we have $E_z^P(\sigma_{B(x:r_0)}) < \infty$, for all $z \in \partial B(x : r_1)$. However to prove the existence of an invariant probability measure a condition similar to (2.4.1) above is needed.

(3) Estimate (2.4.12) is stated (with a brief indication of proof) in Dupuis and Williams (1994), in the context of semimartingale reflecting Brownian motions (SRBM's)

in the orthant. As this estimate is likely to be very useful we thought it fit to write up a proof.

Note: Our analysis concerning recurrence, transience and positive recurrence can easily be extended to unbounded domains that are C^2- diffeomorphic to the half space.

2.5 Further comments and examples

(1) Let $\mathcal{H}^2 : \{(x_1, x_2) = x_1 > 0\}$ and let $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a positive definite matrix.

Define

$$\begin{aligned} Lf(x) &= \frac{1}{2}\left[a\frac{\partial^2 f(x)}{\partial x_1^2} + 2b\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} + c\frac{\partial^2 f(x)}{\partial x_2^2}\right], x \in \mathcal{H}^2 \\ Jf(x) &= \frac{\partial f(x)}{\partial x_1} + \gamma_2(x_2)\frac{\partial f(x)}{\partial x_2}, x \in \partial\mathcal{H}^2 \end{aligned}$$

where γ_2 is a C_b^3 function such that , for some $r_0 > 0$,

$$\begin{aligned} \gamma_2(x_2) &\leq \tfrac{b}{a} \quad for \quad x_2 \geq r_0 \\ &\geq \tfrac{b}{a} \quad for \quad x_2 < r_0 \end{aligned}$$

Then with the function $u(x) = \log(cx_1^2 + ax_2^2 - 2bx_1x_2)$ we see on applying Corollary 2.1.5 that (L, J) diffusion is recurrent. By a transformation note that this example can also be brought to the setup of §2.2.

2) Let $\mathcal{H}^3 = \{(x_1, x_2, x_3) : x_1 > 0\}$. Let

$$\begin{aligned} Lf(x) &= \frac{1}{2}\Delta f(x), \quad x \in \mathcal{H}^3 \\ Jf(x) &= \frac{\partial f(x)}{\partial x_1} + \gamma_2(x_2, x_3)\frac{\partial f(x)}{\partial x_2} + \gamma_3(x_2, x_3)\frac{\partial f(x)}{\partial x_3}, \quad x \in \partial\mathcal{H}^3 \end{aligned}$$

that is we consider Brownian motion in $\overline{\mathcal{H}}^3$ with reflection field $(1, \gamma_2(.), \gamma_3(.))$. Take $f(x) = \frac{1}{|x|} - 1, |x| > 1$, and f is a smooth function throughout. Then $\Delta f(x) = 0$ and

$$Jf(x) = -\frac{x_2\gamma_2(x_2, x_3)}{|x|^3} - \frac{x_3\gamma_3(x_2, x_3)}{|x|^3}$$

So on applying Corollary 2.1.6 we have transience of the process if the γ_2 and γ_3 are chosen such that

$$x_2\gamma_2(x_2, x_3) + x_3\gamma_3(x_2, x_3) \geq 0$$

(3) Let $\mathcal{H}^2 = \{(x_1, x_2) : x_1 > 0\}$. Put $L = \frac{1}{2}\Delta - \mu(\partial/\partial x_1)$, $\mu > 0$. Note that in $\mathbb{R}^2$, L-diffusion is transient as the diffusion is $(B_1(t) - \mu t, B_2(t))$ where B_1 and B_2 are independent Brownian motions.

But let us consider (L, J) diffusion where $J = \frac{\partial}{\partial x_1}$

Now by taking $u(x) = \log |x|$, we see that

$$Lu \leq 0 \quad on\ \mathcal{H}^2; \quad Ju \leq 0 \quad on\ \partial\mathcal{H}^2$$

Further $u(x) \to \infty$ as $|x| \uparrow \infty$. Hence the process is recurrent, by Corollary 2.1.5.

(4) Now consider

$$\boldsymbol{H} = \{(x_1, x_2) : x_2 > 0\}$$

. Let β_1 , β_2 be negative constants, and

$$\begin{aligned} Lf(x) &= \Delta f(x) + \beta_1 x_1 \frac{\partial f(x)}{\partial x_1} + \beta_2 x_2 \frac{\partial f(x)}{\partial x_2} \\ Jf(x) &= \gamma_1(x_1)\frac{\partial f(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_2} \end{aligned}$$

that is we consider Ornstein-Uhlenbeck process in the upper half plane with reflection field $(\gamma_1(x_1), 1)$. We can have positive recurrence of the process in the following cases.

Case(i) Let $\gamma_1(x_1) \leq 0$, for $x_1 \geq 1$ and $\gamma_1(x_1) \geq 0$, for $x_1 \leq -1$.

Then with the function $f(x) = \log |x|$, applying Proposition 2.4.6 we can see that the process is positive recurrent.

Case(ii) Let $\beta_1 = \beta_2 = \beta$. Now consider the upper half plane in polar coordinate form ,that is

$$\boldsymbol{H} = \{(r,\theta) : r > 0, \theta \in (0,\pi)\}$$

.Then the diffusion and the boundary operators transform to

$$\begin{aligned} L_{r,\theta} &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \beta r\frac{\partial}{\partial r} \\ J &= tan\ \eta(.)\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} \qquad on\ \theta = 0 \\ &= -tan\ \eta(.)\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \theta} \qquad on\ \theta = \pi. \end{aligned}$$

Now if the reflection field satisfies the condition in part(a) of Theorem 2.2.1, then by Proposition 2.4.6 applied to the function $u(r,\theta)$ as in the corresponding proof, we have positive recurrence of the Ornstein-Uhlenbeck process.

Similarly if the reflection field satisfies the condition in part(c) of Theorem 2.2.1, we have positive recurrence of the Ornstein-Uhlenbeck process.In particular we see that Ornstein-Uhlenbeck process with constant angles of reflection is positive recurrent.

(5) Let $\boldsymbol{H}$ be the upper half plane as in §2.2.

Let

$$Lf(x) = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

where $a_{ij}(x) = \delta_{ij} + (g(|x|)/|x|^2)x_i x_j$, $g(r)$ is a bounded Lipschitz continuous function.

Note that $\boldsymbol{H} = \{(r,\theta) : r > 0, \theta \in (0,\pi)\}$. Let J be given by (2.2.1). In polar coordinates L above gets transformed to,

$$L = (1+g(r))\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Case (i) : Let the reflection field satisfy the condition in part (a) of Theorem 2.2.1. Assume $g(r) \geq 0$. Take α, s_0 and u as in the proof of part (a) of Theorem 2.2.1. Note that $0 < \alpha < 1$.

Then on $[\partial_1 \mathbf{H} \cap \{r \geq s_0\}] \cup [\partial_2 \mathbf{H} \cap \{r \geq s_0\}]$ we have

$$Ju \leq 0. \tag{2.5.1}$$

As $g(r) \geq 0$ we have

$$Lu \leq g(r)\alpha(\alpha-1)r^{\alpha-2}\cos(\alpha\theta-\theta_1) \leq 0.$$

and in this case the process will be recurrent.

Case (ii) : Let the reflection field satisfy the condition in part (b) of Theorem 2.2.1. Assume $g(r) < 0$. Take α, u as in the proof of part (b) of Theorem 2.2.1. Since $\alpha < 0$ we choose $s_0 > 0$ such that on $[\partial_1 \mathbf{H} \cap \{r \geq s_0\}] \cup [\partial_2 \mathbf{H} \cap \{r \geq s_0\}]$ we have

$$Ju \leq 0. \tag{2.5.2}$$

As $\alpha < 0$ and $g(r) < 0$, note that

$$Lu \leq 0.$$

Hence the process is transient.

In particular if $g(r) = -1/(1+\log r)$, we have recurrence in the unrestricted case (see Page 202 of Friedman (1975)). But in the upper half plane with the reflection as above the process is transient.

(6) Let $\mathcal{H}^4 = \{(x_1, x_2, x_3, x_4) : x_1 > 0\}$ and let $(1, \gamma_2, \gamma_3, \gamma_4)$ be the reflection field on $\partial\mathcal{H}^4$, where γ_2, γ_3 and γ_4 are constants. Consider Brownian motion in $\overline{\mathcal{H}^4}$ with reflection field as above. The equation can be explicitly written for reflecting Brownian motion in $\overline{\mathcal{H}^4}$,

$$\begin{aligned} Z_1(t) &= B_1(t) + \xi(t) \\ Z_2(t) &= B_2(t) + \gamma_2\xi(t) \\ Z_3(t) &= B_3(t) + \gamma_3\xi(t) \\ Z_4(t) &= B_4(t) + \gamma_4\xi(t) \end{aligned}$$

where $\xi(t)$ is the local time at 0 for the Brownian motion $B_1(t)$, and the Brownian motions $B_1(t), B_2(t), B_3(t)$ and $B_4(t)$ are independent. Without loss of generality assume that $\gamma_2^2+\gamma_3^2+\gamma_4^2=1$. Let O be the orthogonal transformation in $\mathbb{R}^3(\cong \partial D)$ taking $(\gamma_2,\gamma_3,\gamma_4)$ to (1,0,0). Hence

$$\begin{aligned} O(Z_2,\ Z_3,\ Z_4)' &= O(B_2,\ B_3,\ B_4)'+O(\gamma_2\ ,\gamma_3,\ \gamma_4)'\xi(t) \\ &= (\tilde{B}_2,\ \tilde{B}_3\ ,\tilde{B}_4)'+(\xi(t),\ 0,\ 0)^t \end{aligned}$$

where $(\tilde{B}_2,\tilde{B}_3,\tilde{B}_4)$ is again a 3-dimensional Brownian motion. Consider the transformation $T:\mathbb{R}^4\to\mathbb{R}^4$ such that

$T(x_1,x_2,x_3,x_4)=(x_1,y_2,y_3,y_4)$ where $(y_2,y_3,y_4)^t=O(x_2,x_3,x_4)^t$.

As T is a smooth transformation, it would preserve recurrence and transience. Let $T(Z_1,Z_2,Z_3,Z_4)=(\tilde{Z}_1,\tilde{Z}_2,\tilde{Z}_3,\tilde{Z}_4)$,where

$$\begin{aligned} \tilde{Z}_1(t) &= B_1(t)+\xi(t) \\ \tilde{Z}_2(t) &= \tilde{B}_2(t)+\xi(t) \\ \tilde{Z}_3(t) &= \tilde{B}_3(t) \\ \tilde{Z}_4(t) &= \tilde{B}_4(t) \end{aligned}$$

Now note that $(\tilde{Z}_1,\tilde{Z}_3,\tilde{Z}_4)$ is a 3-dimensional reflecting Brownian motion with normal reflection in the space $\overline{E}=\{(x_1,x_3,x_4)\ :\ x_1\geq 0\}$ and is transient.Hence the diffusion $(\tilde{Z}_1,\tilde{Z}_2,\tilde{Z}_3,\tilde{Z}_4)$ is transient. In general the result is true for dimensions greater than 4.

(7) Further comments :

In the following situations the asymptotic behaviour of the diffusion is not clear.

(a)Brownian motion in 3 dimensions with reflection field $\gamma(x)$, where γ is bounded smooth. One would expect this process to be transient; however even when $\gamma\equiv$ constant, we don't know the result.

(b) For Ornstein-Uhlenbeck process in the half plane with drift coefficients β_1x_1,β_2x_2, $(\beta_1<0,\beta_2<0)$ we don't know about recurrence of the process. In particular we don't know the behaviour of the process when γ is such that $\langle x, \gamma(x)\rangle > 0, \forall x \in \partial D$. Also we are not able to say anything about positive recurrence.

Chapter 3

Reflecting Diffusions in the Orthant and their Asymptotics

In this chapter before studying the asymptotics we will first prove the existence of a unique solution to an appropriate stochastic differential equation in the orthant. In this regard we will deal with the Skorohod problem with a specific boundedness assumption on the reflection field.

3.1 Path dependent Skorohod Problem

Let $\mathcal{O}^d = \{x \in \mathbb{R}^d : x_i > 0, i = 1, 2 \cdots d\}$ be the d-dimensional orthant, where $d \geq 2$. For notational convenience we denote $\mathcal{O}^d$ by $\mathcal{O}$ in the sequel.
$\Omega^{\mathcal{O}} = \{w \in C([0,\infty) : \mathbb{R}^d) : w(0) \in \overline{\mathcal{O}}\}$ and $\tilde{\Omega}^{\mathcal{O}} = C([0,\infty) : \overline{\mathcal{O}})$. Denote the t^{th} coordinate projection on $\Omega^{\mathcal{O}}(resp. \tilde{\Omega}^{\mathcal{O}})$ by $\pi_t(resp. \tilde{\pi}_t)$. Now set $\mathcal{B}_t = \sigma\{\pi_r : r \leq t\}, \tilde{\mathcal{B}}_t = \sigma\{\tilde{\pi}_r : r \leq t\}$. Denote $\|w - w'\|_T = \sup\{|w(s) - w'(s)| : 0 \leq s \leq T\}$.

Definition : Let $g : [0,\infty) \times \Omega^{\mathcal{O}} \to \mathbb{R}$ be a progressively measurable function with respect to the filtration $\{\mathcal{B}_t\}$.

(a) It is said to be a *simple functional* if there exist $0 = t_0 < t_1 < t_2 \cdots < t_n \cdots$, with $t_n \to \infty$, and measurable functions g_i on Ω^C such that

(i) g_i is $\mathcal{B}_{t_i}$-measurable

(ii) $\sup\{g_i(w) : w \in \Omega^C, i = 0, 1, 2, \cdots\} < \infty$

(iii) $g(s, w) = \sum_i g_i(w) 1_{[t_i, t_{i+1})}(s)$

(b) It is said to be a *Lipschitz functional* if for $T > 0$, there exists constant $K_T > 0$ such that $|g(s, w) - g(s, w')| \leq K_T \|w - w'\|_T$, for all $w, w' \in \Omega^C$, $0 \leq s \leq T$

(c) It is said to be a *locally Lipschitz functional*, if for $T > 0$, $N > 0$, there exists $K_{T,N} > 0$ such that $|g(s, w) - g(s, w')| \leq K_{T,N} \|w - w'\|_T$, for all $s \leq T$ and w, w' with $\|w\| \leq N$, $\|w'\| \leq N$.

As $g(s, \cdot)$ is $\mathcal{B}_s$- measurable, if $w, w' \in \Omega^C$ are such that $w(r) = w'(r)$, $\forall r \leq s$, then $g(r, w) = g(r, w')$ for $r \leq s$. Hence g is a Lipschitz functional if and only if for all $T > 0$, there exists $K_T > 0$ such that $|g(s, w) - g(s, w')| \leq K_T \|w - w'\|_s$. $\forall\ 0 \leq s \leq T$, $w, w' \in \Omega^C$

We can define similar functionals on $[0, \infty) \times \tilde{\Omega}^C$ with the filtration $\{\tilde{\mathcal{B}}_t\}$ in place of $\{\mathcal{B}_t\}$.

Remark : Let $g : [0, \infty) \times \tilde{\Omega}^C \to \mathbb{R}$ be a progressively measurable function with respect to $\{\tilde{\mathcal{B}}_t\}$. For $w \in \Omega^C$, define $\tilde{w} \in \tilde{\Omega}^C$ by $\tilde{w}(t) = (|w_1(t)|, |w_2(t)|, \cdots, |w_d(t)|), t \geq 0$. Now extend g to $[0, \infty) \times \Omega^C$ by putting $g(s, w) = g(s, \tilde{w})$ for $s \geq 0$, $w \in \Omega^C$. Then the extension is progressively measurable with respect to $\{\mathcal{B}_t\}$. If the original g is a simple functional (resp-Lipschitz functional) then the extension is also a simple functional (resp. Lipschitz functional), (Of course such an extension need not be unique.) So for our purposes we may assume that the functionals concerned are adapted with respect to $\{\mathcal{B}_t\}$.

Skorohod problem (SP):

Let $b_i : [0, \infty) \times \Omega^C \to \mathbb{R}$ and $\gamma_{ki} : [0, \infty) \times \Omega^C \to \mathbb{R}$ be progressively measurable functions for $1 \leq i, k \leq d$. We assume that $\gamma_{ii} \equiv 1$. We seek a pair of maps $Z : \Omega^C \to \Omega^C$.

$Y : \Omega^{\mathcal{O}} \to \tilde{\Omega}^{\mathcal{O}}$ such that given $w \in \Omega^{\mathcal{O}}$ the following hold :

$$\text{(a)} \quad (Zw)_i(t) = w_i(t) + \int_0^t b_i(s, Zw)ds + (Yw)_i(t) + \sum_{k \neq i} \int_0^t \gamma_{ki}(s, Zw)d(Yw)_k(s) \tag{3.1.1}$$

(b) $(Yw)_i(0) = 0, \quad (Yw)_i(t)$ is nondecreasing in t.

(c) $(Yw)_i(\cdot)$ increases only when $(Zw)_i(\cdot) = 0, \quad i = 1, 2, \cdots, d.$

We will make the following assumption on the reflection field. We denote it by (BC) to mean boundary condition.

Assumption (BC)**:** Let $\gamma_{ii} \equiv 1, \quad i = 1, 2, \cdots, d$. There exist positive constants $a_1, a_2, \cdots, a_d$ and $0 < \alpha < 1$ such that

$$\sum_{i \neq k} a_i |\gamma_{ki}(s, w)| \leq \alpha a_k \tag{3.1.2}$$

for all $s \geq 0, w \in \Omega^{\mathcal{O}}, k = 1, 2, \cdots, d$. Note that the above condition is the pathwise analogue of condition (3) on p.166 of Shashiashvili (1994) (see also the note following equation (3.2.1)).

Remark 3.1.1 : For $0 \leq s < t < \infty$, let $\Omega^{\mathcal{O}}_{s,t} = \{w \in C([s,t] : \mathbb{R}^d) : w(0) \in \overline{\mathcal{O}}\}$; $\Omega^{\mathcal{O}}_{s,\infty} = \{w \in C([s,\infty) : \mathbb{R}^d) : w(0) \in \overline{\mathcal{O}}\}$. Similarly we define $\tilde{\Omega}^{\mathcal{O}}_{s,t} = C([s,t] : \overline{\mathcal{O}})$ and $\tilde{\Omega}^{\mathcal{O}}_{s,\infty} = C([s,\infty) : \overline{\mathcal{O}})$. Let $0 \leq T_1 < T_2 \leq \infty$; $r_{ki} : \Omega^{\mathcal{O}}{}_{0,T_1} \to \mathbb{R}$ be $\mathcal{B}_{T_1}-$ measurable functions satisfying (BC), viz., there exist positive constants $a_1, a_2, \cdots, a_d$ and $0 < \alpha < 1$ such that

(i) $r_{ii} \equiv 1$; (ii) $\sum_{i \neq k} a_i |r_{ki}(w)| \leq \alpha a_k, \qquad \forall w \in \Omega^{\mathcal{O}}_{0,T_1}, \ i = 1, 2, \cdots, d.$

Now consider the following problem.

Let $w^{(1)} \in \Omega^C_{0,T_1}$, $\overline{w} \in \Omega^C_{T_1,T_2}$ be fixed with $\overline{w}(T_1) \in \overline{\mathcal{O}}$. We then seek functions $z \in C([T_1,T_2] : \overline{\mathcal{O}})$, $y \in C([T_1,T_2] : \overline{\mathcal{O}})$ such that

$$\begin{aligned}
&(a) \quad z_i(t) = \overline{w}_i(t) + y_i(t) + \sum_{k \neq i} r_{ki}(w^{(1)}) y_k(t), \text{ for } T_1 \le t \le T_2 \\
&(b) \quad y_i(T_1) = 0, \quad y_i(\cdot) \text{ is nondecreasing.} \\
&(c) \quad y_i(\cdot) \text{ increases only when } z_i(\cdot) = 0, \quad i = 1, 2, \cdots, d.
\end{aligned} \tag{3.1.3}$$

By Theorem 7 of Shashiashvili(1994), unique solution (z, y) exists for the above problem. Now denote for $T_1 \le t \le T_2$,

$$\left.\begin{aligned}
Z(\overline{w}, w^{(1)}, ((r_{ki})), T_1, T_2)(t) &= z(t) \\
Y(\overline{w}, w^{(1)}, ((r_{ki})), T_1, T_2)(t) &= y(t)
\end{aligned}\right\} \tag{3.1.4}$$

Lemma 3.1.2 : Let $b_i : [0, \infty) \times \Omega^C \to \mathbb{R}$, $\gamma_{ki} : [0,\infty) \times \Omega^C \to \mathbb{R}$ be simple functionals $(1 \le i, k \le d)$ with $((\gamma_{ki}))$ satisfying (BC). Then there exists a unique solution for the Skorohod problem (SP).

Proof : *Step 1 :* Let $b_i(s, w) = \overline{b}_i(w)$, $\gamma_{ki}(s,w) = \overline{\gamma}_{ki}(w), 1 \le i, k \le d$, for all $s \ge 0$, $w \in \Omega^C$ and further $\overline{b_i}$ and $\overline{\gamma}_{ki}$ be $\mathcal{B}_0$-measurable and uniformly bounded.

For any solution (Z, Y) of the problem (SP) we should have $(Zw)(0) = w(0)$; and as $\overline{b_i}$ and $\overline{\gamma_{ki}}$ are $\mathcal{B}_0$-measurable we should have $\overline{b_i}(Zw) = \overline{b_i}(w)$, $\overline{\gamma_{ki}}(Zw) = \overline{\gamma_{ki}}(w)$. Let $w \in \Omega^C$. Take $T_1 = 0$, $T_2 = \infty$, $w^{(1)}(t) = w(0)$, $\gamma_{ki}(w^{(1)}) = \overline{\gamma_{ki}}(w)$ and $\overline{w}(t) = w(t) + t\overline{b}(w)$ Then we note from Remark 2.1 that

$$\begin{aligned}
(Zw)(t) &= Z(\overline{w}, w^{(1)}, ((\gamma_{ki})), 0, \infty)(t) \\
(Yw)(t) &= Y(\overline{w}, w^{(1)}, ((\gamma_{ki})), 0, \infty)(t), \quad T_1 \le t \le T_2
\end{aligned}$$

solves the Skorohod problem.

Step 2 : Let b_i, $((\gamma_{ki})), 1 \le i, k \le d$, be simple functionals with γ satisfying (BC). Suppose $0 = t_0 < t_1 < \cdots < t_n < \cdots, t_n \uparrow \infty$ and

$$b_i(s, w) = \sum_{j=0}^{\infty} b_{i,j}(w) 1_{[t_j, t_{j+1})}(s), \quad \gamma_{ki}(s, w) = \sum_{j=0}^{\infty} \gamma_{ki,j}(w) 1_{[t_j, t_{j+1})}(s)$$

where $b_{i,j}, \gamma_{ki,j}$ are $\mathcal{B}_{t_j}$-measurable.

The Skorohod problem becomes

$$
\begin{aligned}
(Zw)_i(t) &= w_i(t) + \sum_{j=0}^{n-1} b_{i,j}(Zw)(t_{j+1} - t_j) + b_{i,n}(Zw)(t - t_n) \\
&\quad + (Yw)_i(t) + \sum_{k \neq i} \sum_{j=0}^{n-1} \gamma_{ki,j}(Zw)((Yw)_k(t_{j+1}) - (Yw)_k(t_j)) \\
&\quad + \sum_{k \neq i} \gamma_{ki,n}(Zw)((Yw)_k(t) - (Yw)_k(t_n)) \\
&= (Zw)_i(t_n) + w_i(t) - w_i(t_n) + b_{i,n}(Zw)(t - t_n) \\
&\quad + (Yw)_i(t) - (Yw)_i(t_n) + \sum_{k \neq i} \gamma_{ki,n}[(Yw)_k(t) - (Yw)_k(t_n)]
\end{aligned}
\tag{3.1.5}
$$

for $t_n \leq t \leq t_{n+1}$, by adding and subtracting appropriate terms.

Now let us apply induction on n. By step 1, $(Z^{(1)}w, Y^{(1)}w)$ are uniquely defined on $[0, t_1]$. Now let us assume that $(Z^{(n)}w, Y^{(n)}w)$ are defined on $[0, t_n]$ solving the Skorohod problem on $[0, t_n]$. Put $T_1 = t_n, T_2 = t_{n+1}$; $w^{(1)}(t) = (Z^{(n)})w(t), t \leq t_n$. Note that $w^{(1)} \in C[0, t_n]$. Now consider the function $\overline{w}(t) = (Z^{(n)}w)(t_n) + w(t) - w(t_n) + b_{\cdot,n}(Z^{(n)}w)(t - t_n),\ \ t_n \leq t \leq t_{n+1}$. Note that $\overline{w}(t_n) = (Z^{(n)}w)(t_n) \in \overline{\mathcal{O}}$. Take $\gamma_{ki}(w^{(1)}) = ((\gamma_{ki,n}(w^{(1)})) = ((\gamma_{ki,n}(Z^{(n)}w))$. Now define $Z^{(n+1)}w, Y^{(n+1)}w$ on $[0, t_{n+1}]$ using Remark 3.1.1 as follows

$$
\begin{aligned}
(Z^{(n+1)}w)(t) &= \begin{cases} (Z^{(n)}w)(t), & 0 \leq t \leq t_n \\ Z(\overline{w}, Z^{(n)}w, ((\gamma_{ki,n})), t_n, t_{n+1})(t), & t_n \leq t \leq t_{n+1} \end{cases} \\
(Y^{(n+1)}w)(t) &= \begin{cases} (Y^{(n)}w)(t), & t \leq t_n \\ (Y^{(n)}w)(t_n) + Y(\overline{w}, Z^{(n)}w, ((\gamma_{ki,n})), t_n, t_{n+1})(t), & t_n \leq t \leq t_{n+1} \end{cases}
\end{aligned}
\tag{3.1.6}
$$

Note that $Z^{(n+1)}, Y^{(n+1)}w$ are continuous functions on $[0, t_{n+1}]$ and they solve the Skorohod problem (SP) on $[0, t_{n+1}]$. Proceeding thus we can get a solution on $[0, \infty)$. Uniqueness follows from the uniqueness result in Shashiashvili (1994). □

Note that to solve the Skorohod problem, it is enough to solve the problem on $[0, T]$ for an arbitrarily fixed $T > 0$.

Lemma 3.1.3 : Let $T > 0$, and $b_i^{(n)}, \gamma_{ki}^{(n)}, 1 \leq i, k \leq d,\ n = 1, 2, \cdots$, be progressively measurable functionals such that $b_i^{(n)}$ are uniformly bounded on $[0, T] \times \Omega_{0,T}^{\mathcal{C}}$ and $((\gamma_{ki}^{(n)}))$ satisfy (BC) with $\alpha, a_1, a_2, \cdots a_d$ independent of n. For $w \in \Omega_{0,T}^{\mathcal{C}}$ let $Z^{(n)}w, Y^{(n)}w$ denote the solution of the Skorohod problem (SP) on $[0, T]$ corresponding to $b^{(n)}, \gamma^{(n)}$. If $H \subseteq \Omega_{0,T}^{\mathcal{C}}$ is a relatively compact set then $\{(w,\ Y^{(n)}w,\ Z^{(n)}w)\ :\ n = 1,\ 2,\ \cdots,\ w \in H\}$ is relatively compact in $\Omega_{0,T}^{\mathcal{C}} \times \tilde{\Omega}^{\mathcal{C}}{}_{0,T} \times \tilde{\Omega}^{\mathcal{C}}{}_{0,T}$.

Proof : Let $w \in H$. As $(Z^{(n)}w, Y^{(n)}w)$ is the solution of (SP) corresponding to $b^{(n)}, \gamma^{(n)}$ we see that

$$
\begin{aligned}
(Z^{(n)}w)_i(t) &\triangleq x_i^{(n)}(t) + (Y^{(n)}w)_i(t) \\
&= w_i(t) + \int_0^t b_i^{(n)}(s, Z^{(n)}w)ds + (Y^{(n)}w)_i(t) \\
&\quad + \sum_{k \neq i} \int_0^t \gamma_{ki}(s, Z^{(n)}w)d(Y^{(n)}w)_k(s)
\end{aligned}
\tag{3.1.7}
$$

Writing $(Y^{(n)}w)$ in maximal function form (see p. 169 of Shashiashvili (1994)) we have

$$
(Y^{(n)}w)_i(t) = \sup_{s \leq t} \max\{0, -x_i^{(n)}(s)\}
$$

Hence

$$
\begin{aligned}
(Y^{(n)}w)_i(t) &\leq \sup_{s \leq t} |w_i(s)| + \int_0^t |b_i^{(n)}(s, Z^{(n)}w)|ds \\
&\quad + \sum_{k \neq i} \int_0^t |\gamma_{ki}^{(n)}(s, Z^{(n)}w)|d(Y^{(n)}w)_k(s)
\end{aligned}
\tag{3.1.8}
$$

Now as $\gamma_{ki}^{(n)}$ satisfies (BC), multiplying both sides of equation (3.1.8) by a_i, we get

$$
\begin{aligned}
\sum_i (Y^{(n)}w)_i(t) &\leq \frac{\sum_i a_i \sup_{s \leq t} |w_i(s)|}{(1-\alpha)\alpha_0} + \frac{\sum_i a_i C.t}{(1-\alpha)\alpha_0} \\
&\leq \frac{\sum_i a_i}{(1-\alpha)\alpha_0}[\|w\|_t + C.t]
\end{aligned}
\tag{3.1.9}
$$

where $\alpha_0 = \min_i a_i$ and C is the uniform bound of $b^{(n)}$ (see p. 179 of Shashiashvili (1994)).

Therefore by (3.1.9) we have

$$\sup\{|(Y^{(n)}w)(t)| : w \in H,\ 0 \leq t \leq T,\ n = 1, 2, \cdots\} < \infty \tag{3.1.10}$$

as H is uniformly bounded.

Hence $\{Y^{(n)}w : n = 1, 2, \cdots,\ w \in H\}$ is uniformly bounded. Now let us show equicontinuity of $\{Y^{(n)}w\ : n \geq 1,\ w \in H\}$. Note that

$$(Y^{(n)}w)_i(t) \;=\; \sup_{u \leq t} \max\{0, -x_i^{(n)}(u)\} = \max\{(Y^{(n)}w)_i(s), \sup_{s \leq u \leq t}(-x_i^{(n)}(u)\}$$

Hence as $b^{(n)}$ are uniformly bounded we get

$$\begin{aligned}(Y^{(n)}w)_i(t) - (Y^{(n)}w)_i(s) \;&=\; \max\{0, \sup_{s \leq u \leq t}(-x_i^{(n)}(u)) - (Y^{(n)}w)_i(s)\} \\ &\leq\; \max\{0, \sup_{s \leq u \leq t}(-x_i^{(n)}(u) + x_i^{(n)}(s)\} \\ &\leq\; \sup_{s \leq u \leq t} |w_i(u) - w_i(s)| + C(t-s) \\ &\quad + \sum_{k \neq i} \int_s^t |\gamma_{ki}^{(n)}(u, Z^{(n)}w)| d(Y^{(n)}w)_k(u)\end{aligned} \tag{3.1.11}$$

Now as $\gamma_{ki}^{(n)}$'s satisfy (BC) with a_i's and α independent of n, proceeding as in the derivation of (3.1.9) we get

$$\begin{aligned}\sum_i (Y^{(n)}w)_i(t) - (Y^{(n)}w)_i(s) \;&\leq\; \frac{1}{1-\alpha} \sum_i a_i \sup_{s \leq u \leq t} |w_i(u) - w_i(s)| \\ &\quad + \frac{C(t-s)\alpha_1}{(1-\alpha)\alpha_0}\end{aligned} \tag{3.1.12}$$

where $\alpha_1 = \max_i a_i$.

By (3.1.7) and (3.1.10) it is easily seen that

$$\sup\{|(Z^{(n)}w)(t)| : w \in H,\ 0 \leq t \leq T,\ n \geq 1\ \} < \infty \tag{3.1.13}$$

Hence $\{Z^{(n)}w : n \geq 1, w \in H\}$ is uniformly bounded. Since

$$\begin{aligned}|(Z^{(n)}w)_i(t) - (Z^{(n)}w)_i(s)| &\leq |w_i(t) - w_i(s)| + C(t-s) \\ &+ (Y^{(n)}w)_i(t) - (Y^{(n)}w)_i(s) \\ &+ \sum_{k \neq i} \int_s^t |\gamma_{ki}^{(n)}(u, Z^{(n)}w)| d(Y^{(n)}w)_k(u)\end{aligned} \quad (3.1.14)$$

Proceeding as in the derivation of (3.1.9) and using equations (3.1.2) and (3.1.12), we see that

$$\begin{aligned}\sum_i |(Z^{(n)}w)_i(t) - (Z^{(n)}w)_i(s)| &\leq \frac{2}{(1-\alpha)\alpha_0} \sum_i a_i \sup_{s \leq u \leq t} |w_i(u) - w_i(s)| \\ &+ \frac{2}{(1-\alpha)\alpha_0} \sum_i a_i C(t-s)\end{aligned} \quad (3.1.15)$$

By (3.1.10), (3.1.12), (3.1.13), (3.1.15), relative compactness of H and Ascoli's theorem, the desired conclusion follows. □

Theorem 3.1.4 : Let $b_i, \gamma_{ki}, 1 \leq i, k \leq d$ be Lipschitz functionals such that b is bounded. Further let b, γ be continuous in s. Let $((\gamma_{ki}))$ satisfy (BC). Then there exists a unique solution to the Skorohod problem corresponding to (b, γ).

Proof : As $((\gamma_{ki}))$ satisfies (BC), note that γ_{ki}'s are uniformly bounded. Note that it is enough to consider the problem on $[0, T]$ for some arbitrarily fixed $T > 0$.

Define

$$b_i^{(n)}(s, w) = b_i\left(\frac{k}{2^n}T, w\right) \quad , \quad \gamma_{ki}^{(n)}(s, w) = \gamma_{ki}\left(\frac{k}{2^n}T, w\right) \quad , \quad \frac{k}{2^n}T \leq s < \frac{k+1}{2^n}T$$

Note that by the definition of $b_i^{(n)}$ and $\gamma_{ki}^{(n)}$ we have

(i) $b_i^{(n)}(s, w) \to b_i(s, w)$, $\gamma_{ki}^{(n)}(s, w) \to \gamma_{ki}(s, w)$, as $n \to \infty$, $\forall s \geq 0$, $w \in \Omega_{0,T}^C$

(ii) $b^{(n)}, \gamma_{ki}^{(n)}$ are simple Lipschitz functionals with the Lipschitz constants as that of b, γ respectively.

(iii) $((\gamma_{ki}^{(n)}))$ satisfy (BC) with the same a_i's and α as that of $((\gamma_{ki}))$.

Now for $w \in \Omega^C_{0,T}$, arbitrarily fixed, let $(Z^{(n)}w, Y^{(n)}w)$ be the solution of the Skorohod problem (SP) corresponding to $(b^{(n)}, \gamma^{(n)})$. If (Zw, Yw) is a limit of $(Z^{(n)}w, Y^{(n)}w)$ (which exists by Lemma 3.1.3) then we can show that (Zw, Yw) solves the problem (SP) corresponding to (b, γ) Note that

$$\begin{aligned}(Z^{(n)}w)_i(t) &= w_i(t) + \int_0^t b_i^{(n)}(s, Z^{(n)}w)ds + (Y^{(n)}w)_i(t) \\ &\quad + \sum_{k \neq i} \int_0^t \gamma_{ki}^{(n)}(s, Z^{(n)}w)d(Y^{(n)}w)_k(s) \end{aligned} \qquad (3.1.16)$$

By Cauchy-Schwarz inequality and Lipschitz continuity of $b^{(n)}$ and b we see that

$$\sup_{t \leq T} \left| \int_0^t b_i^{(n)}(s, Z^{(n)}w)ds - \int_0^t b_i(s, Zw)ds \right| \to 0, \text{ as } n \to \infty \qquad (3.1.17)$$

Similarly by Lipschitz continuity of $\gamma^{(n)}, \gamma$ we have as $n \to \infty$,

$$\sup_{t \leq T} \left| \int_0^t \gamma_{ki}^{(n)}(s, Z^{(n)}w)d(Y^{(n)}w)_k(s) - \int_0^t \gamma_{ki}(s, Zw)d(Yw)_k(s) \right| \to 0 \qquad (3.1.18)$$

Hence combining (3.1.17) and (3.1.18) we have that (Zw, Yw) solves the Skorohod problem corresponding to (b, γ). Proceeding as in the proof of Proposition 1 of Bernard and El Kharroubi (1991)p.155, we can show that Y_i increases only when $Z_i = 0$.

Now for the uniqueness of the solution let

$$(Zw)_i(t) = w_i(t) + \int_0^t b_i(s, Zw)ds + (Yw)_i(t) + \sum_{k \neq i} \int_0^t \gamma_{ki}(s, Zw)d(Yw)_k(s)$$

and

$$(\tilde{Z}w)_i(t) = w_i(t) + \int_0^t b_i(s, \tilde{Z}w)ds + (\tilde{Y}w)_i(t) + \sum_{k \neq i} \int_0^t \gamma_{ki}(s, \tilde{Z}w)d(\tilde{Y}w)_k(s)$$

be two solutions corresponding to the functions (b, γ). Now by writing (Yw) and $(\tilde{Y}w)$ in the maximal function form (see p. 169 of Shashiashvili (1994)) and using the variational distance Lemma (see p. 170 of Shashiasvili(1994)) we get

$$\begin{aligned} &\sum_i a_i \int_0^t |d((Yw)_i - (\tilde{Y}w)_i)(s)| \\ &\leq \frac{\alpha_1 d}{(1-\alpha)} K_\gamma \int_0^t \sum_{i=1}^d |(Zw)_i(s) - (\tilde{Z}w)_i(s)| d(\sum_k (Yw)_k(s)) \\ &\quad + \frac{K_b}{1-\alpha} \int_0^t \sup_{u \leq s} \|Zw - \tilde{Z}(w)\|_u ds \end{aligned} \qquad (3.1.19)$$

Consequently we have

$$\begin{aligned}\sum_i a_i|(Zw)_i(t) - (\tilde{Z}w)_i(t)| &\\ \leq \sum_i a_i K_b \int_0^t \sup_{u\leq s} \|Zw - \tilde{Z}w\|_u ds &\\ + (1+d)\sum_i a_i \int_0^t |d((Yw)_i(s) - (\tilde{Y}w)_i(s))| &\\ + \alpha_1 d\, K_\gamma \int_0^t \sum_{i=1}^d |(Zw)_i(s) - (\tilde{Z}w)_i(s)| d(\sum_k (Yw)_k(s)) &\end{aligned} \tag{3.1.20}$$

Therefore by (3.1.19) we have

$$\begin{aligned}\sum_i |(Zw)_i(t) - (\tilde{Z}w)_i(t)| &\\ \leq \frac{2\alpha_1 d}{(1-\alpha)\alpha_0} K_\gamma \int_0^t \sum_i |(Zw)_i(s) - (\tilde{Z}w)_i(s)| d(\sum_k (Yw)_k(s) &\\ + \frac{2K_b}{(1-\alpha)\alpha_0} \int_0^t \sup_{u\leq s} \|Zw - \tilde{Z}w\|_u ds &\end{aligned} \tag{3.1.21}$$

Now let $\varphi(t) = \sup_{s\leq t} \sum_i |(Zw)_i(s) - (\tilde{Z}w)_i(s)|$

$\beta(t) = \frac{2\alpha_1 d}{(1-\alpha)\alpha_0} K_\gamma \sum_{k=1}^d (Yw)_k(t)$

$\tilde{\beta}(t) = t + \beta(t)$

Hence (3.1.21) can be written as $\varphi(t) \leq C \int_0^t \varphi(s) d\tilde{\beta}(s)$, where C is a constant. Therefore by Gronwall's Lemma (see pp.287-288 of Karatzas and Shreve (1988)) applied to $\varphi(t)$, we have $\varphi(t) = 0$. This proves the uniqueness; the proof is now complete. □

Theorem 3.1.5 : Let $b_i : [0,\infty) \times \Omega^{\mathcal{O}} \to \boldsymbol{R}$, $1 \leq i \leq d$ and $\gamma_{ki} : [0,\infty) \times \Omega^{\mathcal{O}} \to \boldsymbol{R}$, $1 \leq i,k \leq d$ be progressively measurable functionals such that

(i) b_i's are locally Lipschitz functionals, b_i's are continuous in s, and satisfy linear growth condition :

$$|b_i(s,w)| \leq K(1 + \|w\|_s) \qquad (3.1.22)$$

(ii) γ_{ki}'s are Lipschitz functionals, continuous in s and $((\gamma_{ki}))$ satisfies (BC).

Then there exists a unique solution to SP corresponding to (b, γ).

Proof : Note that by (BC), $\gamma_{ki}(\cdot)$ are all bounded. Define for $N \in \mathbb{N}$,

$$b^{(N)}(s,w) = b(s, w_N)$$

where $w_N(s) = w(s \wedge \tau_N(w)), \tau_N(w) = \inf\{t \geq 0 : |w(t)| \geq N\}$. Note that $b^{(N)}(s,\cdot)$ is a bounded Lipschitz functional for each N. Let $w \in \Omega^C$ be arbitrarily fixed. Let $(Z^{(N)}w, Y^{(N)}w)$ be the solution to the Skorohod problem (SP) corresponding to $(b^{(N)}, \gamma)$. Then we have

$$\begin{aligned}
&|(Z^{(N)}w)_i(t) - (Z^{(N+1)}w)_i(t)| \\
&\leq |\int_0^t b_i^{(N)}(s, Z^{(N)}w)ds - \int_0^t b_i^{(N+1)}(s, Z^{(N)}w)ds| \\
&\quad + |\int_0^t b_i^{(N+1)}(s, Z^{(N)}w)ds - \int_0^t b_i^{(N+1)}(s, Z^{(N+1)}w)ds| \\
&\quad + |(Y^{(N)}w)_i(t) - (Y^{(N+1)}w)_i(t)| \\
&\quad + |\sum_{k\neq i} \int_0^t \gamma_{ki}(s, Z^{(N)}w)d((Y^{(N)}w)_k(s) - (Y^{(N+1)}w)_k(s))| \\
&\quad + |\sum_{k\neq i} \int_0^t (\gamma_{ki}(s, Z^{(N)}w) - \gamma_{ki}(s, Z^{(N+1)}w))d(Y^{(N+1)}w)_k(s)|
\end{aligned} \qquad (3.1.23)$$

Now for $t < \tau_N(Z^{(N)}w)$, we see that the first term on the right side of (3.1.23) vanishes as $b_i^{(N)}$ and $b_i^{(N+1)}$ coincide in that case. But now as $b^{(N+1)}$ is Lipschitz continuous bounded functional and γ satisfies (BC), by the proof of uniqueness in Theorem 3.1.4 we see that

$$\left.\begin{aligned}
(Z^{(N)}w)(t) &= (Z^{(N+1)}w)(t) \ , \quad t < \tau_N(Z^{(N)}w) \\
(Y^{(N)}w)(t) &= (Y^{(N+1)}w)(t) \ , \quad t < \tau_N(Z^{(N)}w)
\end{aligned}\right\} \qquad (3.1.24)$$

Since $\tau_{N+1}(Z^{(N+1)}w) \geq \tau_N(Z^{(N+1)}w) = \tau_N(Z^{(N)}w)$ we see that $\tau_N(Z^{(N)}w)$ increases.

Now let us show that $\tau_N(Z^{(N)}w) \uparrow \infty$, which, essentially will prove the result.

We have, from equation (3.1.8) and the linear growth condition

$$\begin{aligned}(Y^{(N)}w)_i(t) &\leq \sup_{s\leq t}|w_i(s)| + C_b\int_0^t (1+\|Z^{(N)}w\|_s)ds \\ &+\sum_{k\neq i}\int_0^t |\gamma_{ki}(s, Z^{(N)}w)|d(Y^{(N)}w)_k(s)\end{aligned} \qquad (3.1.25)$$

Hence by using equation (3.1.2) we get

$$(1-\alpha)\sum_i a_i(Y^{(N)}w)_i(t) \leq \|w\|_t + C_b d\int_0^t (1+\|Z^{(n)}w\|_s)ds \qquad (3.1.26)$$

By the representation of $Z^{(N)}w$ and by (3.1.26) we have

$$\sup_{s\leq t}|(Z^{(N)}w)_i(s)| \leq k_1\|w\|_t + k_2\int_0^t (1+\|Z^{(N)}w\|_s)ds \qquad (3.1.27)$$

for some constants k_1, k_2. By (3.1.27) and Gronwall's lemma (see pp.287-288 of Karatzas and Shreve (1988)), we have

$$\sup_{s\leq t}\sum_i |(Z^{(N)}w)_i(s)| \leq k(t) \qquad (3.1.28)$$

where $k(t)$ depends on t, but not on N. Now if $\tau_N(Z^{(N)}w) \uparrow m$ (say), where $m < \infty$, we have $\sup_{s<m}|(Z^{(N)}w)_i(s)| \geq N$. That is we have $\sup_{s<t}\sum_i |(Zw)_i(s)| \geq N$, for all N. But as the right side of (3.1.28) is independent of N and is bounded, if we replace t by m, we have a contradiction. Hence $\tau_N(Z^{(N)}w) \uparrow \infty$.

Now define

$$\left.\begin{aligned}(Zw)(t) &= (Z^{(N)}w)(t) \ , \quad t < \tau_N(Z^{(N)}w) \\ (Yw)(t) &= (Y^{(N)}w)(t) \ , \quad t < \tau_N(Z^{(N)}w)\end{aligned}\right\} \qquad (3.1.29)$$

By equation (3.1..24) the definition is consistent. As $\tau_N(Z^{(N)}w) \uparrow \infty$, we see that for any $t \in [0,\infty)$ we can define (Zw, Yw) and further uniqueness follows as the solution is unique on $[0, \tau_N]$. □

Remark 3.1.6 : Consider the following path dependent Skorohod problem. To find $Z: \Omega^{\mathcal{O}} \mapsto \tilde{\Omega^{\mathcal{O}}}$, $Y: \Omega^{\mathcal{O}} \mapsto \tilde{\Omega^{\mathcal{O}}}$ such that for given $w \in \Omega^{\mathcal{O}}$,

$$\text{(a)} \quad (Zw)_i(t) = \int_0^t \sum_{j=1}^d \sigma_{ij}(s, Zw)dw_j(s) + \int_0^t b_i(s, Zw)ds + (Yw)_i(t)$$
$$+ \sum_{k \neq i} \int_0^t \gamma_{ki}(s, Zw)d(Yw)_k(s) \qquad (3.1.30)$$

(b) $(Yw)_i(0) = 0$, $(Yw)_i(t)$ is nondecreasing in t.

(c) $(Yw)_i(\cdot)$ increases only when $(Zw)_i(\cdot) = 0$, $i = 1, 2, \cdots, d$.

If σ_{ij}, b_i, γ_{ki} are simple functionals with γ satisfying (BC), the above problem can be solved uniquely, essentially following the argument of Lemma 3.1.2. Note that if σ_{ij} is a simple functional the first integral on the right side of (3.1.30) makes sense for every $w \in \Omega^{\mathcal{O}}$.

Remark 3.1.7 : We can similarly consider the Skorohod problem in troughs. Let $I \subseteq \{1, 2, \cdots, d\}$, $\mathcal{O}^I = \{x \in \mathbb{R}^d : x_j > 0, j \in I\}$; $\Omega_I^{\mathcal{O}} = \{w \in C([0, \infty) : \mathbb{R}^d) : w(0) \in \overline{\mathcal{O}}_I\}$ and $\tilde{\Omega}_I^{\mathcal{O}} = C([0, \infty) : \overline{\mathcal{O}}_I)$. With the notations and definitions analogous to the case of orthants, we can define the Skorohod problem in the trough $\overline{\mathcal{O}}_I$.

Skorohod Problem $(SP)^T$ **:** Let $b_i : [0, \infty) \times \Omega_I^{\mathcal{O}} \to \mathbb{R}$ and $\gamma_{ki} : [0, \infty) \times \Omega_I^{\mathcal{O}} \to \mathbb{R}$, $1 \leq i \leq d, k \in I$, be progressively measurable functionals; let $\gamma_{kk} \equiv 1, k \in I$. . We seek a pair of maps $Z: \Omega_I^{\mathcal{O}} \to \tilde{\Omega}_I^{\mathcal{O}}$ and $Y: \Omega_I^{\mathcal{O}} \to \tilde{\Omega}_I^{\mathcal{O}}$ such given $w \in \Omega_I^{\mathcal{O}}$, that the following hold

$$\text{(a)} \quad (Zw)_i(t) = w_i(t) + \int_0^t b_i(s, Zw)ds$$
$$+ \sum_{k \in I} \int_0^t \gamma_{ki}(s, Zw)d(Yw)_k(s), \; i = 1, 2, \cdots d. \quad (3.1.31)$$

(b) $(Yw)_j(0) = 0, (Yw)_j(t)$ is nondecreasing in t, $j \in I$.

(c) $(Yw)_j(\cdot)$ increases only when $(Zw)_j(\cdot) = 0$, $j \in I$.

(d) $(Yw)_j(\cdot) \equiv 0$ for $j \notin I$.

Further we have a similar assumption on the reflection field as in the case of orthants.

Assumption $(BC)^T$ **:** Let $\gamma_{kk} \equiv 1, k \in I$. There exist positive constants $a_1, a_2, \cdots, a_d$ and $0 < \alpha < 1$ such that

$$\sum_{i \neq k} a_i |\gamma_{ki}(s, w)| \leq \alpha a_k$$

for all $s \geq 0, w \in \Omega_I^G$ and $k \in I$.

When b and γ are Lipschitz functionals which are continuous in s, with γ satisfying $(BC)^T$ and b satisfying a linear growth condition, we can carry through all the arguments as in the orthant case to get hold of a unique solution for the Skorohod Problem $(SP)^T$.

3.2 Reflecting diffusions in the orthant

In this section we consider diffusions in an orthant and prove some useful results regarding them ; We have the following data :

Let $\mathcal{O}$ be the positive orthant, as in §3.1. Let $F_i = \{x \in \overline{\mathcal{O}} : x_i = 0\}$ and $\tilde{F}_i = F_i \backslash \bigcup_{j \neq i} F_j$

(A3.1) σ is a $(d \times d)$ real symmetric strictly positive definite matrix; denote $a = \sigma\sigma^*$

(A3.2) $b(\cdot) = (\ b_1(\cdot), b_2(\cdot), \cdots, b_d(\cdot)\ )$ is an $\mathbb{R}^d$-valued locally Lipschitz continuous function on $\overline{\mathcal{O}}$ satisfying a linear growth condition, viz., $|b(x)| \leq K(1 + |x|), x \in \overline{\mathcal{O}}$

(A3.3) For $i \in \{1, 2, \cdots, d\}$, $\gamma_{ij}(\cdot), j = 1, 2, \cdots, d,\ j \neq i$, are Lipschitz continuous functions defined on the face F_i. Though the functions $\gamma_{ij}(\cdot), j = 1, 2, \cdots, d, j \neq i$, are defined on F_i they can be extended to $\overline{\mathcal{O}}$ in an obvious way by $\gamma_{ij}(x) = \gamma_{ij}(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_d)$, $x \in \overline{\mathcal{O}}$. Assume $\gamma_{ii} \equiv 1$, $i = 1, 2, \cdots d$. Observe that the extensions are Lipschitz continuous on $\overline{\mathcal{O}}$; thus the functions $\gamma_{ij}, 1 \leq i, j \leq d$, may be taken to be Lipschitz continuous functions on $\overline{\mathcal{O}}$.

We require the functions $\gamma_{ki}(x), 1 \leq i, k \leq d, x \in \overline{\mathcal{O}}$ to satisfy a condition which is analogous to assumption (BC) of Section 3.1.

Condition $(BC)'$: There exist positive constants $a_1, a_2, \cdots, a_d$ and $0 < \alpha < 1$ such that for any $x \in \overline{\mathcal{O}}$,

$$\sum_{i \neq k} a_i |\gamma_{ki}(x)| \leq \alpha a_k \qquad (3.2.1)$$

Note : (3.2.1) is just condition (3) on p.166 of Shashiashvili (1994); this is a "generalisation" of the spectral radius condition, (see p.557 of Dupuis -Ishii(1993)) ; (however it must be noted that in Dupuis-Ishii(1993) the domain is assumed to be bounded).

Note : Observe that any Lipschitz continuous function g on $\overline{\mathcal{O}}$, can be extended to $\boldsymbol{R}^d$ as a Lipschitz continuous function by putting $g(x_1, x_2, \cdots x_d) = g(|x_1|, |x_2|, \cdots |x_d|)$. (We denote this extension also by g); if the function on $\overline{\mathcal{O}}$ satisfies a linear growth condition, the extension also satisfies the same growth condition. So without loss of generality we may take b, γ to be functions defined on $\boldsymbol{R}^d$.

Let $\Omega^{\mathcal{O}} = \{w \in C([0,\infty) : \boldsymbol{R}^d) : w(0) \in \overline{\mathcal{O}}\}$ be endowed with the topology of uniform convergence on compacta and the natural Borel structure. Let $\mathcal{B}$ denote the Borel σ-algebra on $\Omega^{\mathcal{O}}$. Let X_t denote the t^{th}-coordinate map on $\Omega^{\mathcal{O}}$, that is, $X_t(w) = X(t,w) := w(t)$; let $\mathcal{B}_t = \sigma\{X_s : s \leq t\}$. Now let $\{Q_x : x \in \bar{\mathcal{O}}\}$ be a family of probability measures on $(\Omega^{\mathcal{O}}, \mathcal{B})$ such that under Q_x, the canonical process $\{X_t\}$ is a Brownian motion starting at x, with drift 0 and dispersion matrix $a = \sigma\sigma^*$.

Given functions $b_i(\cdot), \gamma_{ki}(\cdot)$ satisfying the conditions (2), (3) and $(BC)'$ above we define the Lipschitz functionals denoted once again by b_i, γ_{ki} as follows.

For $s \geq 0, w \in \Omega^{\mathcal{O}}$, put $b_i(s,w) = b_i(w(s))$; $\gamma_{ki}(s,w) = \gamma_{ki}(w(s))$. It is easy to see that b_i(resp.γ_{ki}) are locally Lipschitz (resp.Lipschitz) functionals; also $((\gamma_{ki}(\cdot)))$ satisfies assumption (BC) of Section 3.1. Using results from Section 3.1, we can now show that there exists a unique pair of continuous processes (Z, Y) such that the following holds.

$$\text{(i)} \quad Z(t) = X(t) + \int_0^t b(Z(s))ds + \int_0^t \gamma^T(Z(s))dY(s) \tag{3.2.2}$$

(ii) $Z(t) \in \overline{\mathcal{O}}$, for $t \geq 0$ and $Q_x(Z(0) = x) = 1$.

(iii) $\int_0^t 1_{\{Z_i(s)>0\}} dY_i(s) = 0$, for $t \geq 0, i = 1, 2, \cdots, d$.

(iv) $Y_i(0) = 0$ and $Y_i(\cdot)$ is nondecreasing for all $i = 1, 2, \cdots d$.

Along the same lines as in Shashiashvili (1994) (see p.191) we can show that

(a) $Z(t)$ and $Y(t)$ are adapted to $\mathcal{B}_t$;

(b) $Z(t)$ is Feller continuous, strong Markov process under $\{Q_x : x \in \overline{\mathcal{O}}\}$.

We also denote $Z(t)$ by Z_t in the sequel.

Note : Shashiashvili (1994)(see p.193), observes that using the methods of his paper in an obvious manner, the existence and uniqueness of the above diffusion (when b is nonconstant) can be established. However, we have not been able to do this; the difficulty seems to be concerning the extension of the analogue of Theorem 1 in Shashiashvili (1994) to the case of step functions. In fact this has led to the present analysis in Section 3.1.

Remark 3.2.1 : Because of the semimartingale representation (3.2.2) for $Z(t)$, for any $f \in C_b^2(\overline{\mathcal{O}})$, by Ito's lemma we have,

$$\begin{aligned} f(Z_t) - f(Z_0) &= \sum_{i=1}^{d} \int_0^t \frac{\partial f(Z_s)}{\partial x_i} dX_i(s) + \sum_{i=1}^{d} \int_0^t \frac{\partial f(Z_s)}{\partial x_i} b_i(Z_s) ds \\ &+ \frac{1}{2} \sum_i \sum_j \int_0^t a_{ij} \frac{\partial^2 f(Z_s)}{\partial x_i \partial x_j} ds \\ &+ \sum_{i=1}^{d} \int_0^t \left(\sum_{j=1}^{d} \gamma_{ij}(Z_s) \frac{\partial f(Z_s)}{\partial x_j} \right) dY_i(s) \end{aligned} \qquad (3.2.3)$$

From (3.2.3) we see that the generator L and the boundary operator J of the diffusion $\{Z(t) : t \geq 0\}$ are given by

$$\begin{aligned} Lf(x) &= \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i}, \quad x \in \mathcal{O} \\ Jf(x) &= \sum_{j=1}^{d} \gamma_{ij}(x) \frac{\partial f(x)}{\partial x_j}, \qquad x \in \tilde{F}_i, i = 1, 2, \cdots d \end{aligned} \qquad (3.2.4)$$

Note that $(\gamma_{i1}(\cdot), \gamma_{i2}(\cdot), \cdots, \gamma_{i,i-1}(\cdot), 1, \gamma_{i,i+1}(\cdot), \cdots, \gamma_{i,d}(\cdot))$ is the reflection vector field on the regular part of the face F_i (that is $\tilde{F}_i$).

We denote by E_x^Q, the expectation with respect to Q_x.

Lemma 3.2.2 : For each $x \in \overline{\mathcal{O}}$, we have

$$E_x^Q\left[\int_0^\infty 1_{\partial\mathcal{O}}(Z_s) ds\right] = 0 \qquad (3.2.5)$$

Proof : Note that it is enough to consider the case when b is bounded. Since equation (3.2.2) gives a semimartingale representation for Z, by Ito's lemma, (3.2.5) can be established just as in Lemma 7.2 on p.100 of Harrison and Williams (1987). $\square$

Lemma 3.2.3 : Let $x \in \mathcal{O}$ and let $\eta = \inf\{r \geq 0 : Z(r) \in \partial\mathcal{O}\}$. Then we have $Q_x(\eta, Z(\eta))^{-1} << $ Lebesgue measure on $[0, \infty) \times \partial\mathcal{O}$.

Proof : Let $\{\lambda_x : x \in \mathbb{R}^d\}$ denote the L-diffusion in $\mathbb{R}^d$, that is under $\{\lambda_x : x \in \mathbb{R}^d\}$ the coordinate process $\{X(t) : t \geq 0\}$ behaves like the diffusion in $\mathbb{R}^d$ with generator L.

Now as $x \in \mathcal{O}$ and Z behaves like an L-diffusion till hitting $\partial\mathcal{O}$, it is enough to prove that

$$\lambda_x(\tilde{\eta}, X(\tilde{\eta}))^{-1} << \text{ Lebesgue measure on } [0, \infty) \times \partial\mathcal{O} \qquad (3.2.6)$$

where $\tilde{\eta}(w) = \inf\{r \geq 0 : X(r, w) \in \partial\mathcal{O}\}$

Now as $\partial\mathcal{O}$ is the union of a finite number of truncated hyperplanes, it is enough to prove (3.2.6) for $\mathcal{O}_i$ instead of $\mathcal{O}$, where $\mathcal{O}_i = \{x \in \mathbb{R}^d : x_i > 0\}$, with $i \in \{1, 2, \cdots, d\}$ and $\tilde{\eta}$ denoting the hitting time of $\partial\mathcal{O}_i$. Note that $\mathcal{O}_i$ is the half space and the boundary $\partial\mathcal{O}_i$ is smooth.

Consider the following problem for fixed $T > 0$:

$$\left.\begin{array}{lll} ((\partial/\partial s) + L)u(s, x) = & 0, & s < T, \quad x \in \mathcal{O}_i \\ \lim_{s \uparrow T} u(s, x) = & 0, & x \in \overline{\mathcal{O}}^i \\ u(s, x) = & g(x), & x \in \partial\mathcal{O}_i \end{array}\right\} \qquad (3.2.7)$$

where g is a bounded measurable function on $\partial\mathcal{O}_i$. Then the unique stochastic solution to the problem (3.2.7) is

$$u(s, x) = E_x^{\lambda}[1_{\{\tilde{\eta} \leq T - s\}} g(X(\tilde{\eta})]$$

where E_x^{λ} denotes expectation with respect to λ_x. In particular if we take $g = 1_A$, $A \subseteq \partial\mathcal{O}_i$, $s = 0$, then we have

$$u(0, x) = \lambda_x(\tilde{\eta} \leq T, \; X(\tilde{\eta}) \in A) \qquad (3.2.8)$$

Now the problem (3.2.7) can also be solved by the method of double layer potentials (see pp. 407-408 of Ladyzenskaja et. al (1968)) which gives the classical solution as an integral over $[0,T] \times \partial\mathcal{O}_i$, with respect to $ds\,d\sigma(z)$, where $d\sigma$ is the surface area measure on $\partial\mathcal{O}_i$. Hence (3.2.6) follows and hence the lemma. □

Lemma 3.2.4 : For each $x \in \overline{\mathcal{O}}, t > 0$, we have

$$Q_x(Z(t) \in \partial\mathcal{O}) = 0 \tag{3.2.9}$$

Proof : Case (i) Let $x \in \mathcal{O}$

Let $\eta = \inf\{r \geq 0 : Z(r) \in \partial\mathcal{O}\}$. By Lemma 3.2.2 we have

$$\int_{\partial\mathcal{O}} E_z[\int_0^{\infty} 1_{\partial\mathcal{O}}(Z(s))ds]d\sigma(z) = 0 \tag{3.2.10}$$

Now by the strong Markov property of Z,

$$\begin{aligned} Q_x(Z(t) \in \partial\mathcal{O}) &= E_x^Q[1_{\{\eta \leq t\}} 1_{\partial\mathcal{O}}(Z(t))] \\ &= E_x^Q[1_{\{\eta \leq t\}} E_{Z(\eta)}^Q(1_{\partial\mathcal{O}}(Z(t-\eta)))] \\ &= \int_{[0,t]\times\partial\mathcal{O}} E_z[1_{\partial\mathcal{O}}(Z(t-r))]dQ_x(\eta, Z(\eta))^{-1}(r,z) \\ &\leq \int_{[0,\infty)\times\partial\mathcal{O}} E_z(1_{\partial\mathcal{O}}(Z(r))dQ_x(\eta, Z(\eta))^{-1}(r,z) \end{aligned} \tag{3.2.11}$$

Now by equation (3.2.10) and Lemma 3.2.3 it follows that the right side of equation (3.2.11) is zero.

Case (ii) Let $x \in \partial\mathcal{O}$. Fix $t > 0$, and $0 < s < t$ (s arbitrary). By Markov property we get,

$$\begin{aligned} Q_x(Z(t) \in \partial\mathcal{O}) &= \int_{\mathcal{O}} Q_u(Z(t-s) \in \partial\mathcal{O})dQ_x Z(s)^{-1}(u) \\ &\quad + \int_{\partial\mathcal{O}} Q_u(Z(t-s) \in \partial\mathcal{O})dQ_x Z(s)^{-1}(u) \end{aligned} \tag{3.2.12}$$

By case (i) the first term in the right side of (3.2.12) becomes zero and hence we have

$$Q_x(Z(t) \in \partial\mathcal{O}) = Q_x(Z(s) \in \partial\mathcal{O}, Z(t) \in \partial\mathcal{O}) \tag{3.2.13}$$

by Markov property. Let now $0 < s_1 < s_2 < \cdots < s_n < t$. Iterating the above argument we get,

$$Q_x(Z(t) \in \partial\mathcal{O}) = Q_x(Z(s_1) \in \partial\mathcal{O}, Z(s_2) \in \partial\mathcal{O}, \cdots, Z(t) \in \partial\mathcal{O}) \qquad (3.2.14)$$

Hence as the left side of equation (3.2.14) is independent of n, we have ,

$$Q_x(Z(t) \in \partial\mathcal{O}) = Q_x(Z(s) \in \partial\mathcal{O} \quad \forall s \in [0,t] \cap \mathbb{Q})$$

By continuity of sample paths and as $\partial\mathcal{O}$ is closed we have,

$$Q_x(Z(t) \in \partial\mathcal{O}) = Q_x(Z(s) \in \partial\mathcal{O}, s \leq t) \qquad (3.2.15)$$

Now if $Q_x(Z(t) \in \partial\mathcal{O}) > 0$, then (3.2.15) will contradict Lemma 3.2.2. Hence the result.

□

Remark 3.2.5 : Before proceeding to the next lemma let us make the following observations :

(i) For every $x \in \overline{\mathcal{O}}$, the matrix $\gamma(x) = ((\gamma_{ki}(x))_{1 \leq i,k \leq d}$ satisfies the completely-$\mathcal{S}$ condition. (Recall that a matrix R is completely -$\mathcal{S}$ iff for every principal submatrix $\tilde{R}$ of R, we can find a vector $\tilde{v} > 0$ such that $\tilde{R}\tilde{v} > 0$. (see Bernard and El Kharroubi (1991)). Here the inequality is understood to be in each component.) This is because of the following :

Note that $\gamma(x)$ satisfies (3.2.1) with positive constants $a_1, a_2, \cdots, a_d$ and $0 < \alpha < 1$. For $I \subseteq \{1, 2, \cdots, d\}$, let $\gamma^{\,I}(x)$ be the submatrix obtained by deleting all rows and columns of $\gamma(x)$ that are indexed by I.

Let $x \in \overline{\mathcal{O}}$ and $j \in I^c$ be fixed. Now put $I_0^c = \{i \in I^c; i \neq j\}$, $I_1^c = \{i \in I_0^c : \gamma_{ji}(x) \geq 0\}$ and $I_2^c = \{i \in I_0^c : \gamma_{ji}(x) < 0\}$. Then by (3.2.1) we have $\sum_{i \in I_0^c} a_i |\gamma_{ji}(x)| \leq \alpha a_j$. Hence we have $\alpha a_j - \sum_{i \in I_1^c} a_i \gamma_{ji}(x) + \sum_{i \in I_2^c} a_i \gamma_{ji}(x) \geq 0$, that is,

$$(\alpha a_j - \sum_{i \in I_1^c} a_i \gamma_{ji}(x)) + \sum_{i \in I_2^c} a_i \gamma_{ji}(x) + 2 \sum_{i \in I_1^c} a_i \gamma_{ji}(x) + (1 - \alpha) a_j > 0. \qquad (3.2.16)$$

By (3.2.16) note that, $a_j + \sum_{i \in I_0^c} a_i \gamma_{ji}(x) > 0$. Now take $\tilde{v}_j = a_j$, $j \in I$. Then $\gamma^{\,I}(x)\tilde{v} > 0$. As $I \subseteq \{1, 2, \cdots, d\}$ is arbitrary, we conclude that $\gamma(x)$ is a completely -$\mathcal{S}$ matrix.

(ii) Now put $v = (a_1, a_2, \cdots, a_d)$. Then from the above $\gamma(x)v > 0$ for every $x \in \overline{\mathcal{O}}$. As $\gamma(x)v$ is continuous in x, for any compact set $K \subseteq \overline{\mathcal{O}}$, it now follows that one can choose $\hat{v}_K > 0$ (depending on K) such that $\gamma(x)\hat{v}_K > 1$, for all $x \in K$.

Lemma 3.2.6 : Let $U \subseteq \overline{\mathcal{O}}$ be a bounded open set and let $\tau_U = \inf\{t \geq 0 : Z(t) \notin U\}$. Then we have

$$\sup_{x \in U} E_x^Q(\tau_U) < \infty$$

In particular, the diffusion $\{Z(t) : t \geq 0\}$ exits out of any bounded set in finite time with probability one.

Proof : Let $h \in C_b^2(\overline{\mathcal{O}})$ be such that $h(x) = \exp(k_0 \sum_{i=1}^{d} a_i x_i)$, $x \in \overline{U}$ where $a_i, i = 1, 2, \cdots, d$ are positive constants as in (3.2.1), and $k_0 > 0$ is an appropriate constant to be suitably chosen later. Now for $x \in U$,

$$Lh(x) = \left(k_0^2 \sum_{i,j=1}^{d} a_{ij}(a_i \cdot a_j) + k_0 \sum_{i=1}^{d} a_i b_i(x) \right) h(x) \tag{3.2.17}$$

By positive definiteness of $a = ((a_{ij}))$, we have $\sum_{i,j=1}^{d} a_{ij}(a_i \cdot a_j) > 0$. Choose $k_0 > 0$ such that for all $x \in \overline{U}$,

$$\sum_{i=1}^{d} b_i(x) \geq \frac{1 - k_0^2 \sum_{i,j=1}^{d} a_{ij}(a_i \cdot a_j)}{k_0 \alpha_0} \tag{3.2.18}$$

where $\alpha_0 = \min_i a_i$. This is possible as the right side of equation (3.2.18) becomes regative for large k_0 and b is bounded (in particular bounded below) in $\overline{U}$. Hence by (3.2.18) and (3.2.17) we have for $x \in \overline{U}$,

$$Lh(x) \geq 1 \tag{3.2.19}$$

Now by equation (3.2.3) we have

$$\begin{aligned} h(Z(t \wedge \tau_U)) - h(Z(0)) &= \sum_{i=1}^{d} \int_0^{t \wedge \tau_U} \frac{\partial h(Z_s)}{\partial x_i} dX_i(s) + \int_0^{t \wedge \tau_U} Lh(Z_s) ds \\ &\quad + \sum_{i=1}^{d} \int_0^{t \wedge \tau_U} \left(\sum_{j=1}^{d} \gamma_{ij}(Z_s) \frac{\partial h(Z_s)}{\partial x_j} \right) dY_i(s) \end{aligned} \tag{3.2.20}$$

Note that $\partial h(x)/\partial x_j = k_0 a_j h(x)$. By Remark 3.2.5 observe that the third term on the right side of (3.2.20) is nonnegative. Further by (3.2.19) and as the first term on the right side of (3.2.20) is a martingale, we have for every $t > 0$,

$$E_x^Q\left[h(Z(t \wedge \tau_U)) - h(Z(0))\right] \geq E_x^Q(t \wedge \tau_U)$$

As h is bounded, by monotone convergence theorem, we have the result. □

Our next objective is to prove the strong Feller property of the diffusion $\{Z(t) : t \geq 0\}$, for which we need a couple of lemmas.

Lemma 3.2.7 : Let $G \subseteq \bar{O}$ be a bounded open set. Define $\tau_G = \inf\{r \geq 0 : Z(r) \notin G\}$. Then there exists a constant $k > 0$ such that, for all $t > 0$,

$$\sup_{x \in G} E_x^Q\left[\sum_{i=1}^{d} Y_i(t \wedge \tau_G)\right] \leq k \tag{3.2.21}$$

Proof : Note that by the maximal function representation of $Y_i(t)$, similar to equation (3.1.9) we have,

$$\sum_{i=1}^{d} Y_i(t) \leq C\left[\sup_{s \leq t}|w(s)| + K_b t\right] \tag{3.2.22}$$

where C is a constant and K_b is the bound of b on $\bar{G}$. From (3.2.22) and Lemma 3.2.6 note that (3.2.21) is immediate. □

Before going to the next step let us fix some notations.

Let $\Gamma(t, x, z)$ be the transition density function of the L-diffusion in $\mathbb{R}^d$. Let $T_0 > 0$. It is known that for all $x, z \in \mathbb{R}^d$, $0 < t < T_0$

$$|D_x^p \Gamma(t, x, z)| \leq k_3 t^{-\left(\frac{d+|p|}{2}\right)} \exp\left[\frac{-k_4|z - x|^2}{t}\right] \tag{3.2.23}$$

for any multiindex p, such that $|p| = p_1 + p_2 + \cdots + p_d$, with $0 \leq |p| \leq 2$, where D_x^p denotes differentiation in the x-variables and k_3, k_4 are constants that depend on a, b, T_0. (see p.376 of Ladyzenskaja et.al (1968)).

Let $U(t) = (U_i(t))_{i=1}^d$ where $U_i(t) = \sum_{k=1}^{d} \gamma_{ik}(Z_t)Y_k(t)$. For $G \subseteq \bar{\mathcal{O}}$, a bounded open set, $t > 0, x \in \bar{G}$ and $z \in G \cap \mathcal{O}$, define

$$\begin{aligned} p_G(t,x,z) &= \Gamma(t,x,z) - E_x^Q\left[1_{[0,t)}(\tau_G)\Gamma(t-\tau_G, Z(\tau_G), z)\right] \\ &+ E_x^Q\left[\int_0^{t\wedge\tau_G} \langle \nabla\Gamma(t-r, Z(r), z), dU(r)\rangle\right] \end{aligned} \qquad (3.2.24)$$

where $\nabla\Gamma(s,\xi,z)$ denotes the gradient of Γ in ξ.

Lemma 3.2.8 : For any Borel set $E \subseteq G \cap \mathcal{O}, t > 0, x \in \bar{G}$ we have

$$Q_x(Z(t) \in E, \tau_G \geq t) = \int_G 1_E(z)p_G(t,x,z)dz \qquad (3.2.25)$$

Moreover p_G is nonnegative.

Proof : For $x \in \partial G \setminus \partial\mathcal{O}$, both sides of (3.2.25) are zero. So we may assume $x \notin \partial G \setminus \partial\mathcal{O}$. To prove (3.2.25) it is enough to show that for any continuous function f with compact support $K \subseteq G \cap \mathcal{O}$,

$$\int_G f(z)p_G(t,x,z)dz = E_x^Q\left[1_{[t,\infty)}(\tau_G)f(Z_t)\right] \qquad (3.2.26)$$

Denote by $u(x)$ the left side of equation (3.2.26). As $K \subseteq G \cap \mathcal{O}$, we have $d(K, \partial\mathcal{O}) > 0$ and $d(K, \partial G) > 0$. Hence by the estimate (3.2.23), by the fact that $dU(\cdot)$ changes only when $Z(\cdot) \in \partial\mathcal{O}$ and as f is supported in K we can easily see that u is a bounded measurable function on $\bar{\mathcal{O}}$. Now observe that $\{u(Z(\tau_G \wedge r) : 0 \leq r < t\}$ is a Q_x-martingale with respect to $\{\mathcal{B}_{r\wedge\tau_G} : 0 \leq r < t\}$.

This is because of the following :

Let $C_1 = \mathcal{B}_{\tau_G \wedge r_1}$, $C_2 = \mathcal{B}_{\tau_G \wedge r_2}$ where $0 \leq r_1 \leq r_2 < t$

$$\begin{aligned} E_x^Q(u(Z(\tau_G \wedge r_2)|_{C_1}) &= E_x^Q\left(\int_{\mathcal{O}} f(z)\Gamma(t, Z(\tau_G \wedge r_2), z)dz|_{C_1}\right) \\ &-E_x^Q\left[\int_{\mathcal{O}} f(z)E^Q_{Z(\tau_G\wedge r_2)}(1_{[0,t)}(\tau_G)\Gamma(t-\tau_G, Z(\tau_G), z))|_{C_1}\right] \\ &+E_x^Q\left[\int_{\mathcal{O}} f(z)E^Q_{Z(\tau_G\wedge r_2)}\left[\int_0^{t\wedge\tau_G}\langle\nabla\Gamma(t-r, Z(r), z), dU(r)\rangle|_{C_1}\right]\right] \\ &= \int_{\mathcal{O}} f(z)\Gamma(t, Z(\tau_G \wedge r_1), z)dz \end{aligned}$$

$$
\begin{aligned}
&-E_x^Q\left[\int_{\mathcal{O}} f(z) E_x^Q\left(1_{[0,t)}(\tau_G)\right) \Gamma(t-\tau_G, Z(\tau_G), z)|_{C_2}|_{C_1}\right] \\
&+E_x^Q\left[\int_{\mathcal{O}} f(z) E_x^Q\left[\int_0^{t\wedge\tau_G} \langle \nabla\Gamma(t-r, Z(r), z), dU(r)\rangle|_{C_2}|_{C_1}\right]\right]
\end{aligned} \qquad (3.2.27)
$$

where we have used the facts that Z has the semimartingale representation (3.2.2), Γ satisfies the Kolmogorov backward equation, and the fact that $Z(t)$ has strong Markov property.

As $C_1 \subseteq C_2$, by combining the terms in equation (3.2.27) we see that $\{u(Z(\tau_G \wedge r)) : r < t\}$ is a Q_x - martingale with respect to $\{\mathcal{B}_{\tau_G\wedge r} : r < t\}$. Therefore for any $x \in \bar{G}$, we have

$$
\begin{aligned}
\int_{\mathcal{O}} f(z) p_G(t, x, z) dz &= \lim_{r\uparrow t} E_x^Q\left[u(Z(\tau_G \wedge r))\right] \\
&= \lim_{r\uparrow t} E_x^Q\left[\int_K f(z) p_G(t, Z(\tau_G \wedge r), z) dz\right] \\
&= \lim_{r\uparrow t} E_x^Q\left[\int_K f(z) 1_{[t,\infty)}(\tau_G) p_G(t, Z(\tau_G \wedge r), z) dz\right] \\
&\quad + \lim_{r\uparrow t} E_x^Q\left[\int_K f(z) 1_{[0,t)}(\tau_G) p_G(t, Z(\tau_G \wedge r), z) dz\right] \\
&= I_1 + I_2
\end{aligned} \qquad (3.2.28)
$$

Now observe the following fact : If $x \in \bar{G}$ and $x(s) \in \bar{G}$ be such that $x(s) \to x$ as $s \to t$, then

$$
\lim_{s\uparrow t} \int_{\mathcal{O}\cap G} f(z) p_G(t, x(s), z) dz = f(x) \qquad (3.2.29)
$$

This follows essentially because of the facts that f is supported in $K \subseteq G \cap \mathcal{O}$, $d(K, \partial\mathcal{O}) > 0$, $d(K, \partial G) > 0$, the bounds (3.2.23) on Γ and $\nabla\Gamma$, and dominated convergence theorem. By (3.2.29), as u is bounded, we have

$$
\begin{aligned}
I_1 &= E_x^Q\left[1_{[t,\infty)}(\tau_G) \lim_{r\uparrow t} \int_{\mathcal{O}} f(z) p_G(t, Z(r), z) dz\right] \\
&= E_x^Q\left[1_{[t,\infty)}(\tau_G) f(Z(t))\right]
\end{aligned} \qquad (3.2.30)
$$

Similarly, by (3.2.29) and dominated convergence theorem, as f is supported in K,

$$
\begin{aligned}
|I_2| &\leq E_x^Q\left[1_{[0,t)}(\tau_G) \lim_{r\uparrow t} \left|\int_K f(z) p_G(t, Z(\tau_G \wedge r), z) dz\right|\right] \\
&= E_x^Q\left[1_{[0,t)}(\tau_G) |f(Z(\tau_G))|\right] = 0
\end{aligned} \qquad (3.2.31)
$$

Now by equations (3.2.28), (3.2.30) and (3.2.31) we have equation (3.2.25).

Next let us prove the nonnegativity of p_G. Let K be a compact set and $K \subseteq G \cap \mathcal{O}$. Let $x \in \bar{G}$, $z \in K$ and $t > 0$. Choose nonnegative smooth functions $\{\varphi_n\}$ such that φ_n is supported in K, $\int_K \varphi_n(\xi)d\xi = 1$, $n = 1, 2, \cdots$, and $\varphi_n(\xi)d\xi \Rightarrow \delta_z$.

By the uniform bounds on Γ and $\nabla\Gamma$ on $\bar{G}$, it can be shown that p_G is continuous in z. Therefore we have by equation (3.2.26)

$$\begin{aligned} p_G(t, x, z) &= \lim_{n \to \infty} \int p_G(t, x, \xi)\varphi_n(\xi)d\xi \\ &= \lim_{n \to \infty} E_x^Q \left[1_{[t,\infty)}(\tau_G)\varphi_n(Z(t))\right] \geq 0 \end{aligned}$$

Hence the proof. □

Remark : By using the Feller continuity of $\{Z(t) : t \geq 0\}$, the bounds on Γ and $\nabla\Gamma$, and the fact that $dU(\cdot)$ changes only when $Z(\cdot) \in \partial\mathcal{O}$, it can be shown that $p_G(t, x, z)$ is jointly measurable in (x, z) on $\{x \in \bar{G}, z \in G \cap \mathcal{O}\}$. Then by defining it to be zero outside $\bar{G} \times (G \cap \mathcal{O})$ we can take $p_G(t, \cdot, \cdot)$ to be a jointly measurable function on $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$.

Now let $G_n = B(0 : n) \cap \bar{\mathcal{O}}$, $n = 1, 2, \cdots$. Let $p_n(t, x, z) = p_{G_n}(t, x, z)$, $n \geq 1$. Define

$$p(t, x, z) := \lim_{n \to \infty} p_n(t, x, z) \tag{3.2.32}$$

Proposition 3.2.9 : (i) $p(t, \cdot, \cdot)$ is a jointly measurable nonnegative function on $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$.
(ii) $Q_x Z(t)^{-1}$ is absolutely continuous with respect to Lebesgue measure with

$$\frac{dQ_x Z(t)^{-1}}{dz} = p(t, x, z) \tag{3.2.33}$$

(iii) The diffusion $\{Z(t) : t \geq 0\}$ is strong Feller under $\{Q_x : x \in \bar{\mathcal{O}}\}$.

Proof :
(i) By Lemma 3.2.8 and the Remark above note that $0 \leq p_n(t, x, z) \leq p_{n+1}(t, x, z)$, for $z \in \bar{\mathcal{O}}, t > 0$ and $n \geq 1$.

Hence p defined as in (3.2.32) is a jointly measurable extended real valued function on $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$.

(ii) For $n \geq 1$, let $\tau_n = \inf\{r \geq 0 : |Z(r)| \geq n\}$. As $\{Z(t)\}$ is a non exploding diffusion $\tau_n \uparrow \infty$ a.s. Let $A \subseteq \bar{\mathcal{O}}$ be a Borel set. Then by Lemma 3.2.8 we have

$$\begin{aligned} Q_x(Z(t) \in A \cap \mathcal{O}) &= \lim_{n\to\infty} Q_x(Z(t) \in A \cap \mathcal{O} \cap B(0:n),\ \tau_n \geq t) \\ &= \lim_{n\to\infty} \int_{\mathcal{O}} 1_{A\cap\mathcal{O}}(z) p_n(t,x,z)dz \\ &= \int_{\mathcal{O}} 1_{A\cap\mathcal{O}}(z) p(t,x,z)dz \end{aligned} \qquad (3.2.34)$$

Thus $Q_x(Z(t))^{-1}$ restricted to $\mathcal{O}$ is absolutely continuous with respect to Lebesgue measure.

In view of Lemma 3.2.4 we have the result. Further by (3.2.34), we see that p is real valued and is the Radon-Nikodym derivative.

(iii) Now let us show the strong Feller property. Let ψ be a smooth function on $\bar{\mathcal{O}}$ such that $\psi > 0$ and $\int_{\bar{\mathcal{O}}} \psi(z)dz = 1$. Put $\lambda(dz) = \psi(z)dz$ and $q(t,x,z) = \frac{1}{\psi(z)}p(t,x,z)$. Note that $q(t,\cdot,\cdot)$ is jointly measurable on $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$ and for $t > 0$, $x \in \bar{\mathcal{O}}$,

$$\int_{\mathcal{O}} q(t,x,z)\lambda(dz) = 1, \qquad (3.2.35)$$

Also by Feller continuity of Z, for any bounded continuous function f on $\bar{\mathcal{O}}$, we have, $x \mapsto \int_{\mathcal{O}} f(z)q(t,x,z)\lambda(dz)$ is bounded continuous.

Now by Lemma 11 and Remark on pp 60-61 of Skorohod (1989), for any bounded measurable function h on $\bar{\mathcal{O}}$ it follows that

$$\int_{\mathcal{O}} h(z)p(t,x,z)dz = \int_{\mathcal{O}} h(z)q(t,x,z)\lambda(dz)$$

is bounded continuous in x. Hence the proof. □

Note : The measurable transition density p_G given by (3.2.24) gives an expression for the Green's function for the operator $(\partial/\partial s) + L$ with Dirichlet boundary condition on $\partial G \backslash \partial\mathcal{O}$ and oblique boundary conditions on $\partial G \cap \partial\mathcal{O}$.

Lemma 3.2.10 : For any compact set $K \subseteq \bar{\mathcal{O}}$ and $\epsilon > 0$,

$$\lim_{t\downarrow 0} \sup_{x\in K} Q_x(|Z_t - x| > \epsilon) = 0 \qquad (3.2.36)$$

Proof : Note that it is enough to consider the case when b is bounded. By (3.1.9), (3.2.22) we have

$$\sum_{i=1}^{d} Y_i(t) \leq k\left[\sup_{s<t} |w_i(s)| + K_b t\right] \qquad (3.2.37)$$

where k is a constant depending on a_i's alone and K_b is the bound of b. Now by Burkholder-Davis-Gundy inequality observe that

$$EY_i^2(t) \leq k_5 CE\left(\int_0^t |\sigma|^2 ds\right) + k_6 t^2 \leq k_0 t \qquad (3.2.38)$$

for small t, where k_0, k_5, k_6, C are appropriate constants. By the representation of $Z_i(t)$ we have.

$$E_x^Q |Z_i(t) - x|^2 \leq k't + k''t^2 + k'''t^2 \leq \tilde{k}t \qquad (3.2.39)$$

for small t, and suitable constants $k', k'', k''', \tilde{k}$. Hence we have

$$\limsup_{t\downarrow 0} \sup_{x\in K} E_x^Q |Z_i(t) - x|^2 = 0,$$

and hence the result. □

Corollary 3.2.11 : Let $V \subseteq \bar{O}$ be a bounded open set and let g be a bounded measurable function such that for $x \in V$,

$$g(x) = E_x^Q \left[g(Z(\tau_V))\right]$$

Then g is a bounded continuous function on V.

Proof : In view of the strong Feller property, strong Markov property and Lemma 3.2.10 the corollary follows as in Lemma 2.1.1. □

3.3 Asymptotic properties

In this section we consider asymptotic behaviour like recurrence, transience and positive recurrence of diffusions in the orthant. We use the notations and hypotheses of Section

3.2. By Section 3.2, we note that the diffusion $\{Z(t) : t \geq 0\}$ given by equation (3.2.2) is a strong Markov, strong Feller process under $\{Q_x : x \in \bar{O}\}$.

For a Borel set $A \subseteq \bar{O}$, define

$$\tau_A =: \inf\{t \geq 0 : \quad Z(t) \notin A\}, \ \sigma_A =: \inf\{t \geq 0 : \quad Z(t) \in A\}$$

Further for $r > 0$, define $\sigma_r =: \inf\{t \geq 0 : |Z(t)| = r\}$

Definition : (a) A point $x \in \bar{O}$ is said to be a *recurrent* point for the diffusion $\{Z(t) : t \geq 0\}$ if for every $\epsilon > 0$,

$$Q_z(Z(t) \in B(x : \epsilon) \text{ for a sequence of } t's \uparrow \infty) = 1$$

(b) A point $x \in \bar{O}$ is a *transient* point for the diffusion $\{Z(t) : t \geq 0\}$, if $Q_x\left(\lim_{t \to \infty} |Z(t)| = \infty\right) = 1$. If all the points are recurrent (transient) then the diffusion itself is *recurrent* (*transient*).

(c) A point $x \in \bar{O}$ is said to be *positive recurrent* if there exist bounded open sets U_1, U_2 such that $x \in U_1 \subseteq \bar{U}_1 \subseteq U_2$ and $\sup\{E_z(\sigma_{U_1}) : z \in \partial U_2\} < \infty$. The diffusion is said to be *positive recurrent* if all points are positive recurrent.

We will first prove some crucial lemmas which will enable us to get criteria for recurrence and transience of diffusions in terms of the generator and the reflection field.

Lemma 3.3.1 : Let $A \subseteq \bar{O}$ be a nonempty bounded open set. Let $t > 0, M > 0$ be fixed. Then the following hold

(a) For any $x \in \bar{O}$, we have

$$Q_x(Z(t) \in A) > 0 \tag{3.3.1}$$

In particular the diffusion is irreducible.

(b) For any $x \in \bar{O}$ such that $|x| < M$,

$$Q_x(Z(t) \in A, \tau_M > t) > 0 \tag{3.3.2}$$

(c) $p_A^r := \inf\{Q_x(Z(t) \in A, \tau_M > t : x \in \overline{B(0:r)}\}) \ > 0$, for $0 < r < M$.

Proof : Note that (c) follows from (b) and the strong Feller property of $\{Z(t) : t \geq 0\}$. Hence it is enough to prove (a) and (b).

We will first prove (3.3.2) in the case $x \in \mathcal{O}$.

Let $x \in \mathcal{O}$ with $|x| < M$. Then $\delta_0 = (M - |x|) \wedge d(x, \partial\mathcal{O}) > 0$. Then we can find a nonempty bounded open set A_0 and $\delta \in (0, \delta_0)$ such that $\bar{A}_0 \subseteq Int(A \cap \mathcal{O})$ and $d(A_0, \partial(A \cap \mathcal{O})) > \delta$. Let $w_0 \in \Omega^{\mathcal{O}}$ be such that $w_0(0) = x$, $w_0(t) \in A_0$ and w_0 is defined by linear interpolation on $[0, t]$. Put

$$N(w_0) = \{w \in \Omega^{\mathcal{O}} : |w(s) - w_0(s)| < \delta/2 \text{ for all } 0 \leq s < t\}$$

Note that any $w \in N(w_0)$ stays in $\mathcal{O} \cap B(0 : M)$ upto time t and $w(t) \in A$.

Let Q_x denote the L-diffusion in $\mathbb{R}^d$, starting at x. The process $Z(\cdot)$ under Q_x behaves like the process $X(\cdot)$ under Q_x upto the time of hitting $\partial\mathcal{O}$. So by the Stroock-Varadhan support theorem we have,

$$\begin{aligned} Q_x(Z(t) \in A, \tau_M > t) &\geq Q_x(Z(t) \in A, \tau_M > t, \sigma_{\partial\mathcal{O}} > t) \\ &\geq Q_x(N(w_0)) > 0 \end{aligned}$$

Thus (3.3.2) holds in this case.

Proof of (a) : From the preceding, note that (3.3.1) holds for $x \in \mathcal{O}$. So we assume $x \in \partial\mathcal{O}$.

Let $t_1 < t$ be arbitrary. By Markov property of $\{Z(t) : t \geq 0)$ we have,

$$\begin{aligned} Q_x(Z(t) \in A) &= Q_x\left(Q_{Z(t_1)}(Z(t - t_1) \in A\right) \\ &= \int_{\mathcal{O}} Q_z(Z(t - t_1) \in A) dQ_x Z(t_1)^{-1}(z) > 0, \end{aligned} \qquad (3.3.3)$$

as $Q_x Z(t)^{-1}(\mathcal{O}) = 1$ by Proposition 3.2.9.

Proof of (b) : Note that we need to prove (3.3.2) only in the case when $x \in \partial\mathcal{O}$ with $|x| < M$.

Note first that $Q_x(\tau_M > t_0) > 0$ for some $t_0 > 0$; (for otherwise $\tau_M = 0$ a.s. Q_x and hence $|x| = |Z(0)| = M$ which is a contradiction). Consequently $Q_x(\tau_M > r) > 0$ for all $0 < r \leq t_0$.

Let $t > 0$ be fixed. Choose $0 < s < t$ such that $Q_x(\tau_M > s) > 0$. By Markov property

$$Q_x(Z(t) \in A, \tau_M > t) = E_x^Q\left[1_{\{\tau_M > s\}} E_{Z(s)}^Q\left\{1_{\{\tau_M > t-s\}} 1_A(Z(t-s))\right\}\right] \qquad (3.3.4)$$

Note that $Q_x(Z(s))^{-1}(B(0 : M)) > 0$ by (a) above; as (3.3.2) holds when $x \in \mathcal{O}$, by (3.3.4), we now see that (3.3.2) follows also for $x \in \partial\mathcal{O}$. □

Lemma 3.3.2 : (a) Let U_1, U_2 be open sets in $\bar{\mathcal{O}}$ such that U_1 is nonempty and $\bar{U}_1 \cap \bar{U}_2 = \phi$. Let $\sigma_i = \sigma_{U_i}$, $i = 1, 2$. Then $x \mapsto Q_x(\sigma_1 < \sigma_2)$ is a strictly positive continuous function on $\bar{U}_1^c \cap \bar{U}_2^c \cap \overline{\mathcal{O}}$.

(b) Let $U \subseteq \bar{\mathcal{O}}$ be a nonempty open set. Then $x \mapsto Q_x(\tau_U < \infty)$ is a strictly positive continuous function on U.

(c) Let $F \subseteq \bar{\mathcal{O}}$ be a compact set with nonempty interior, in $\bar{\mathcal{O}}$. If for some $x_0 \notin F$, $Q_{x_0}(\sigma_F < \infty) = 1$, then for all $x \in \bar{\mathcal{O}}, Q_{x_0}(\sigma_F < \infty) = 1$.

Proof : (a) Let $g(x) = Q_x(\sigma_1 < \sigma_2)$ and $x \in \bar{U}_1^c \cap \bar{U}_2^c \cap \overline{\mathcal{O}}$ be arbitrary. Let V be a neighborhood of x such that $x \in V \subseteq \bar{V} \subseteq \bar{U}_1^c \cap \bar{U}_2^c \cap \overline{\mathcal{O}}$. Then by strong Markov property and Lemma 3.2.6 we have

$$g(x) = E_x^Q\left[E_{Z(\tau_V)}^Q(1_{\{\sigma_1 < \sigma_2\}})\right] = E_x^Q[g(Z(\tau_V))] \qquad (3.3.5)$$

Hence by Corollary 3.2.11 we have that g is continuous on U. It remains to show that g is strictly positive.

Case (i) Let $x \in \mathcal{O} \cap \bar{U}_1^c \cap \bar{U}_2^c$. Since the diffusion $\{Z(t) : t \geq 0\}$ behaves like the L-diffusion till hitting the boundary $\partial\mathcal{O}$, by the support theorem of Stroock-Varadhan, it follows that $g(x)$ is strictly positive.

Case (ii) Let $x \in \partial\mathcal{O} \cap \bar{U}_1^c \cap \bar{U}_2^c$. We claim that there is a compact set $K \subseteq \mathcal{O} \cap \bar{U}_1^c \cap \bar{U}_2^c$ such that

$$Q_x(\sigma_K < (\sigma_1 \wedge \sigma_2)) > 0 \qquad (3.3.6)$$

Suppose not. Then $Q_x(\sigma_{\mathcal{O}} < (\sigma_1 \wedge \sigma_2)) = 0$. As $x \notin \bar{U}_1 \cup \bar{U}_2$, note that there exists $t_0 > 0$ such that $Q_x((\sigma_1 \wedge \sigma_2) > t_0) > 0$. Hence we have

$$\begin{aligned} Q_x(Z(s) \in \partial\mathcal{O} \text{ for all } s \leq t_0) &= Q_x(\sigma_{\mathcal{O}} > t_0) \\ &\geq Q_x((\sigma_1 \wedge \sigma_2) > t_0,\ (\sigma_1 \wedge \sigma_2) < \sigma_{\mathcal{O}}) \\ &= Q_x((\sigma_1 \wedge \sigma_2) > t_0) > 0 \end{aligned}$$

which contradicts Lemma 3.2.4. Hence the claim.

Now note that the process $Z(\cdot)$ under Q_x behaves like $X(\cdot)$ under Q_x till hitting $\partial\mathcal{O}$. So by strong Feller property of $\{Q_x\}$ and support theorem of Stroock and Varadhan we get

$$\begin{aligned} \inf_{z\in K} Q_z(\sigma_1 < \sigma_2) &\geq \inf_{z\in K} Q_z(\sigma_1 < (\sigma_2 \wedge \eta)) \\ &= \inf_{z\in K} Q_z(X(\cdot) \text{ hits } U_1 \text{ before } (U_2 \cup \partial\mathcal{O})) \\ &> 0 \end{aligned} \tag{3.3.7}$$

where η is the hitting time of $\partial\mathcal{O}$. By strong Markov property, (3.3.6) and (3.3.7), we have $g(x) > 0$.

(b) Follows directly from (a) by taking $U_1 = Int(U^c \cap \bar{\mathcal{O}})$. and $U_2 = \phi$.

(c) Let us prove this in two cases .

Case (i) Let $x_0 \in \mathcal{O}$. Let $h(x) = 1 - Q_x(\sigma_{F^c} < \infty)$, $x \in \bar{\mathcal{O}}$. By hypothesis, $h(x_0) = 0$. Note that by (b) above, $h(x)$ is a continuous function on F^c. By strong Markov property we have

$$0 = h(x_0) = E^Q_{x_0}[h(Z(\tau))] \tag{3.3.8}$$

where τ is the exit time of the process $\{Z(t) : t \geq 0\}$ from the ball $B(x_0 : \delta)$, where $\delta > 0$ is chosen such that $\overline{B(x_0 : \delta)} \cap F = \phi$, and $\overline{B(x_0 : \delta)} \subseteq \mathcal{O}$. By (3.3.8) we have $h(z) = 0, Q_x Z(\tau)^{-1}$ a.s. z. By the Stroock-Varadhan support theorem for L-diffusions and continuity of h, we have $h(z) = 0$, for all $z \in \partial B(x_0 : \delta)$. Now as in the proof of (c) $\Rightarrow$ (d) in Proposition 2.1.3 we can show that $h(x) = 0$, $\forall\, x \in \bar{\mathcal{O}}$.

Case (ii) Let $x_0 \in \partial\mathcal{O} \cap F^c$. Let $H \subseteq \mathcal{O} \cap F^c$ be a compact set with nonempty interior.

Suppose $Q_z(\sigma_F < \infty) < 1$ for all $z \in \mathcal{O} \setminus F$. Then by part(b) we have,

$$\inf_{z \in H} Q_z(\sigma_F = \infty) > 0 \qquad (3.3.9)$$

By (3.3.9) and strong Markov property,

$$\begin{aligned} Q_{x_0}(\sigma_F = \infty) &\geq E^Q_{x_0}(1_{\{\sigma_H < \infty\}} E^Q_{Z(\sigma_H)}(1_{\{\sigma_F = \infty\}})) \\ &\geq \inf_{z \in H} Q_z(\sigma_F = \infty) \cdot Q_{x_0}(\sigma_H < \infty) \\ &> 0 \end{aligned}$$

as $Q_{x_0}(\sigma_H < \infty) > 0$ by part(b), and hence is a contradiction to the hypothesis. Hence the result. □

Proposition 3.3.3 : The following statements are equivalent.

(a) $x_0 \in \bar{\mathcal{O}}$ is a recurrent point.

(b) $Q_{x_0}(Z(t) \in U$ for some $t \geq 0) = 1$ for all nonempty open sets $U \subseteq \bar{\mathcal{O}}$.

(c) There exist $z_0 \in \bar{\mathcal{O}}$, $0 < r_0 < r_1$, $y \in \partial B(z_0 : r_1)$ such that $Q_y(\sigma < \infty) = 1$, where $\sigma = \inf\{t \geq 0 : Z(t) \in \overline{B(z_0 : r_0)}\}$.

(d) There exists a compact set $K \subseteq \bar{\mathcal{O}}$ with nonempty interior such that $Q_x(Z(t) \in K$ for some $t \geq 0) = 1$, for all $x \in \bar{\mathcal{O}}$.

(e) $Q_x(Z(t) \in U$ for some $t \geq 0) = 1$, for all $x \in \bar{\mathcal{O}}$ and for all nonempty open sets $U \subseteq \bar{\mathcal{O}}$.

(f) $Q_x(Z(t) \in U$ for a sequence of t's $\uparrow \infty) = 1$, for all $x \in \bar{\mathcal{O}}$ and for all nonempty open sets $U \subseteq \bar{\mathcal{O}}$.

(g) The diffusion $\{Z(t) : t \geq 0\}$ is recurrent.

Proof : Note that in the case of half space, the proof of the above proposition (namely Proposition 2.1.3) essentially follows from the following facts.

(i) The diffusion is strong Markov and strong Feller.

(ii) The Corollary 3.2.11 and Lemma 3.3.2 above hold.

(iii) The diffusion exits from any bounded set in finite time with probability one (viz. Lemma 3.2.6).

(In the case of half space we used the fact that the diffusion does not hit any boundary point. But in the case of orthants this may not be true. We overcome this difficulty by means of Lemma (3.3.2) (c)). Hence the same proof can be carried through here and we get the result. □

In view of the semimartingale representation of $Z(t)$, Ito's formula and the above proposition, the following results can be obtained just as in Chapter 2.

Theorem 3.3.4 : (a) (Dichotomy) The diffusion $\{Z(t) : t \geq 0\}$ is not recurrent $\Leftrightarrow$ the diffusion is transient.

(b) The diffusion $\{Z(t) : t \geq 0\}$ is recurrent $\Leftrightarrow$ there exists a compact set $K \subseteq \overline{\mathcal{O}}$ with nonempty interior, a point $x \in K^c \cap \overline{\mathcal{O}}$ and a real measurable function u such that

(i) $u(z) \uparrow \infty$ as $|z| \uparrow \infty$; (ii) $E_x^Q[u(Z(\sigma_K))] \leq u(x)$

(c) The diffusion is transient $\Leftrightarrow$ there exists a compact set $F \subseteq \overline{\mathcal{O}}$ with nonempty interior, $y \in F^c \cap \overline{\mathcal{O}}$ and a real measurable function u such that

(i) $E_y^Q[1_{\{\sigma_H < \infty\}} u(Z(\sigma_H))] \leq u(y)$; (ii) $u(y) < \inf_{z \in H} u(z)$

Corollary 3.3.5 : If there exist $r_0 > 0$, $u \in C^2(\mathbf{R}^d \setminus B(0 : \frac{r_0}{2}))$ such that

(i) $Lu(x) \leq 0$, $\{|x| \geq r_0\} \cap \overline{\mathcal{O}}$; (ii) $\sum_{j=1}^d \gamma_{ij}(x) \frac{\partial u(x)}{\partial x_j} \leq 0, x \in \tilde{F}_i \cap \{|x| \geq r_0\}$, $i = 1, 2, \cdots d$.

(iii) $u(x) \uparrow \infty$ as $|x| \uparrow \infty$.

then the diffusion $\{Z(t) : t \geq 0\}$ is recurrent.

Corollary 3.3.6 : If there exists $r_0 > 0$ and a function $u \in C_b^2(\mathbf{R}^d \setminus B(0 : \frac{r_0}{2}))$ such that

(i) $Lu(x) \leq 0$, $\{|x| \geq r_0\} \cap \overline{\mathcal{O}}$
(ii) $\sum_{j=1}^d \gamma_{ij} \frac{\partial u(x)}{\partial x_j} \leq 0, x \in \tilde{F}_i \cap \{|x| \geq r_0\}$ for $i = 1, 2, \cdots, d$.
(iii) There is a point x_0 such that $|x_0| > r_0$ and $u(x_0) < \inf_{\{|x| = r_0\} \cap \overline{\mathcal{O}}} u(x)$ then the diffusion is transient.

Proposition 3.3.7 : The following hold for the diffusion :

(i) If there exists one positive recurrent point then the diffusion itself is positive recurrent.

(ii) Let $0 < r < M$, $A \subseteq \bar{\mathcal{O}}$ be a nonempty open set. Then we can find $p_A > 0$ such that for any $x \in \overline{B(0:r)}$,

$$E_x^Q(\sigma_A) \leq \frac{1}{p_A}[2 + \sup_{|z| \leq M} E_z(\sigma_r)] + E_x^Q(\sigma_r)$$

(iii) In particular if the diffusion is positive recurrent, then

$$E_y^Q(\sigma_A) < \infty$$

for any nonempty open set $A \subseteq \bar{\mathcal{O}}$, for any $y \in \bar{\mathcal{O}}$.

Proof : The proof is analogous to the proof in Chapter 2, essentially using Lemma 3.3.1.

Note : Part (ii) of the above proposition has been inspired by a corresponding result for semimartingale reflecting Brownian motion due to Dupuis and Williams (1994).

Proposition 3.3.8 : Let $r_0 > 0, \epsilon > 0, u \in C^2(\mathbb{R}^d \backslash B(0 : \frac{r_0}{2}))$ be such that

(i) $Lu(x) \leq -\epsilon$, $\{|x| \geq r_0\} \cap \bar{\mathcal{O}}$

(ii) $\sum_{j=1}^{d} \gamma_{ij}(x) \frac{\partial u(x)}{\partial x_j} \leq 0$, $x \in \bar{F}_i \cap \{|x| \geq r_0\}$, , $i = 1, 2, \cdots, d$.

(iii) $u(x) \geq 0$, on $\{|x| \geq r_0\} \cap \bar{\mathcal{O}}$. Then the process is positive recurrent.

Proof : Same as in the case of half space.

Example 3.3.9 : Let $\mathcal{O}$, a, b, γ, L, J be as in Section 3.2. Suppose the following hold :

(i) There exists a constant $k < 0$ such that for all $x \in \bar{\mathcal{O}}$, $\max_i b_i(x) < k$.

(ii) For each $i, j \in \{1, 2, \cdots, d\}, i \neq j$ we have $\gamma_{ij}(x) \leq 0$, $x \in \bar{\mathcal{O}}$

Then with the function $u(x) = |x|^2$, we see on applying Proposition 3.3.8 that the diffusion is positive recurrent.

Example 3.3.10 : (i) Let $\mathcal{O}, L, J$ be as in Example 3.3.9. Assume that the drift $b_i(x)$ is of the form

$$b_i(x) = \beta_i x_i, \quad \beta_i < 0$$

Suppose the following holds : For each $i,j \in \{1,2,\cdots,d\}$, $i \neq j$ we have $\gamma_{ij}(x) \leq 0$, $x \in \bar{\mathcal{O}}$.

Then by an application of Proposition 3.3.8 with the same u as in Example 3.3.9 we have that the diffusion is positive recurrent. Note that in this case the diffusion is the Ornstein-Uhlenbeck process.

(ii) Let $\mathcal{O}, L, J$ be as in Example 3.3.9. If the drifts are of the form $b_i(x) = \beta|x|$, $\beta < 0$, then again with the same u as in (i) above we see that the diffusion is positive recurrent. (This is also a variant of the Ornstein-Uhlenbeck process in the orthant.)

Example 3.3.11 : Let $\mathcal{O}^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ be the positive quadrant. Let $\gamma(x) = \begin{pmatrix} 1 & \gamma_{21}(x) \\ \gamma_{12}(x) & 1 \end{pmatrix}$ be the reflection matrix satisfying Assumption $(BC)'$ with positive constants a_1, a_2. Let $\beta_1 > 0$, $\beta_2 > 0$ be arbitrary positive numbers. Choose $k > 0$ such that $\beta_1 > \frac{a_1}{k}$ and $\beta_2 > \frac{a_2}{k}$. Define

$$\begin{aligned} Lf(x) &= \Delta f(x) + \beta_1 x_1 \frac{\partial f(x)}{\partial x_1} + \beta_2 x_2 \frac{\partial f(x)}{\partial x_2}, \quad x \in \mathcal{O}^2 \\ Jf(x) &= \frac{\partial f(x)}{\partial x_1} + \gamma_{12}(x)\frac{\partial f(x)}{\partial x_2}, \quad \{x_1 = 0\} \cap \partial\mathcal{O}^2 \\ &= \gamma_{21}(x)\frac{\partial f(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_2}, \quad \{x_2 = 0\} \cap \partial\mathcal{O}^2 \end{aligned}$$

Then with the function $u(x) = ke^{-a_1x_1 - a_2x_2}$ combined with Corollary 3.3.6 and Remark 3.2.5 (i) we see that the process is transient. In particular this says that the drift is too strong and the reflection field can't help in averting transience.

Example 3.3.12 : Let $\mathcal{O}^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ be as above. Let J be as in Example 3.3.11.

Define

$$Lf(x) = \Delta f(x) + b_1 \frac{\partial f(x)}{\partial x_1} + b_2 \frac{\partial f(x)}{\partial x_2}, \quad x \in \mathcal{O}^2$$

where $b_1 < 0$, $b_2 < 0$.

Let the following hold

(a) $2b_1 < b_2$ and $2b_2 < b_1$; (b) $\gamma_{12}(x) \leq 1/2$ and $\gamma_{21}(x) \leq 1/2$, $\quad \forall x \in \bar{O}^2$

Then with the function $u(x) = x_1^2 + x_2^2 - x_1 x_2$, on applying Proposition 3.3.8 we see that the diffusion is positive recurrent. Note that in this case the reflection field is allowed to point away from the origin also.

Remark 3.3.13 : When the diffusion is recurrent it can be shown using the results of Maruyama and Tanaka (1959) that there is a unique (upto scalar multiplicity) σ-finite invariant measure; in this context it may be noted that condition 6 of Maruyama and Tanaka (1959) is needed for just two open balls for their proof to go through. If the diffusion is positive recurrent, the invariant measure will be a probability measure; in such a case as the diffusion is irreducible and strong Feller, by the results of Khasminskii (1960) ergodicity can be established.

Remark 3.3.14 : Note that by Remark 3.1.6 the Skorohod problem for troughs is well defined and has a unique solution. Hence correspondingly we have a unique diffusion in the troughs, satisfying the conditions analogous to equation (3.2.1). Further the entire analysis of the orthants done in Section 3.2 can be easily carried through to the case of troughs. In a similar manner the results in Section 3.3 can also be extended to the case of troughs.

Remark 3.3.15 : When γ_{ki}'s are constants satisfying $(BC)'$ Dupuis and Ishii (1991) have obtained reflecting difusions in an orthant with nonconstant Lipschitz continuous dispersion, as strong solutions of the corresponding stochastic differential equations; note that such a diffusion is a semimartingale. If in addition, the dispersion coefficient is sufficiently smooth and the generator L is uniformly elliptic, then our proof of strong Feller property and our analysis concerning recurrence, transience and positive recurrence can be carried through.

Chapter 4

Reflecting Diffusions in the Quadrant

In this chapter we will construct a new class of diffusions in the quadrant using the submartingale problem approach. Then we prove the dichotomy between recurrence and transience by using certain auxiliary diffusions in the half planes.

4.1 A class of reflecting diffusions in the quadrant

In this section we will exhibit a new class of diffusions in the quadrant which can be realized as unique solutions of corresponding submartingale problems as defined below.

Let $\mathcal{Q} = \{(x_1, x_2) : x_1 > 0,\ x_2 > 0\}$ be the positive quadrant. (We use $\mathcal{Q}$ instead of $\mathcal{O}^2$ as in Chapter 3 for notational convenience).Let $\Omega^{\mathcal{Q}} = C([0,\ \infty) : \bar{\mathcal{Q}})$ be endowed with the topology of uniform convergence on compacts and the natural Borel structure. Let $X(t,w) := w(t)$ be the coordinate projection process on $\Omega^{\mathcal{Q}}$. We also denote $X(t)$ by X_t in the sequel. Define $\mathcal{B}_t = \sigma\{X(s) : s \leq t\}$. Let $\sigma(.), b(.),\ \gamma_{12}(.),\ \gamma_{21}(.)$ be functions to be defined later. Define the generator and the boundary operator respectively by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad x \in Q \qquad (4.1.1)$$

$$\begin{aligned} Jf(x) &= \frac{\partial f(x)}{\partial x_1} + \gamma_{12}(x)\frac{\partial f(x)}{\partial x_2}, \quad \{x_2 > 0\} \cap \partial Q \\ &= \gamma_{21}(x)\frac{\partial f(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_2}, \quad \{x_1 > 0\} \cap \partial Q \end{aligned} \qquad (4.1.2)$$

where $a = \sigma\sigma^T$. With these notations we can state the submartingale problem.

Let

$$\mathcal{T} = \left\{ \begin{array}{l} f \in C_b^2(\bar{Q}) : \; Jf \geq 0 \text{ on } \partial Q \setminus 0 \\ \text{and } f \text{ is constant in a neighborhood of origin} \end{array} \right\} \qquad (4.1.3)$$

Submartingale problem : For every $x \in \bar{Q}$, we seek a unique probability measure μ_x on Ω^Q satisfying the following:

(i) $\mu_x(X(0) = x) = 1$;

(ii) for every $f \in \mathcal{T}$,

$$\{f(X_t) - \int_0^t Lf(X_s)ds\} \text{ is a } \mu_x - \textit{submartingale} \qquad (4.1.4)$$

$$\text{(iii)} E_x^\mu[\int_0^\infty 1_{\{0\}}(X_s)ds] = 0 \qquad (4.1.5)$$

where E_x^μ denotes expectation with respect to μ_x.

The family $\{\mu_x : x \in \bar{Q}\}$ of probability measures as above will be termed as the *diffusion in the quadrant corresponding to* $(\sigma,\ b,\ \gamma)$ or simply the (L, J) *diffusion.*

Remark 4.1.1 There are two known classes of examples that satisfy the submartingale problem above.

(i) Let $\sigma \equiv I$, $b \equiv 0$ on $\bar{Q}$ and let γ_{12}, γ_{21} be real constants. This case has been dealt with in full detail by Varadhan and Williams (1985). (In fact the above formulation is inspired by that paper).

(ii) Suppose σ is a constant dispersion matrix, which is real symmetric and positive definite; $b(\cdot)$ is a Lipschitz continuous function of linear growth on $\bar{Q}$; $\gamma_{12}(\cdot)$, $\gamma_{21}(.)$

are bounded Lipschitz continuous functions satisfying Condition (A1)' of Section 3.2. It can be shown that a unique solution to the corresponding Skorohod problem exists. Since in this case we get hold of the strong solution, by Lemma 3.2.2, it solves the submartingale problem also.

We now stipulate the following conditions on the coefficients $\sigma,\ b,\ \gamma$.

(A4.1) For each $x \in \bar{Q}$, $\sigma(x) = ((\sigma_{ij}(x)))$ is a 2×2 real symmetric positive definite matrix and further σ satisfies the following

(i) $|\sigma(x) - \sigma(z)| \leq K|x-z|$; $|\sigma(x)| \leq K(1+|x|)$ for all $x,\ z \in \bar{Q}$.

(ii) $\inf\{$ eigenvalues of $\sigma(x) : x \in \bar{Q}\} > 0$;

(iii)$\sigma(x) \equiv I$, for $|x| \leq N$, for some $N > 0$.

(A4.2) For each $x \in \bar{Q}$, $b(x) = (b_1(x),\ b_2(x)) \in \mathbb{R}^2$; $b(\cdot)$ satisfies the following

(i) $|b(x) - b(z)| \leq K|x-z|$; $|b(x)| \leq K(1+|x|)$ for all $x,\ z \in \bar{Q}$.

(ii) $b(x) = 0$ for $|x| \leq N$ for some $N > 0$

(A4.3) (i)

$$\gamma(x) = \begin{cases} (1, \gamma_{12}(x)) & \text{on } \{x \in \partial Q,\ x_1 = 0\} \\ (\gamma_{21}(x), 1) & \text{on } \{x \in \partial Q,\ x_2 = 0\} \end{cases}$$

(ii) $\gamma(\cdot)$ is a bounded Lipschitz continuous function.

(iii) $\gamma_{12}(x) \equiv \gamma_{12}$, for $|x| \leq N$; $\gamma_{21}(x) \equiv \gamma_{21}$, for $|x| \leq N$, for some $N > 0$, where $\gamma_{12},\ \gamma_{21}$ are real constants.

We now indicate a construction of the diffusion process in $\bar{Q}$ corresponding to $(\sigma,\ b,\ \gamma)$ satisfying (A4.1) - (A4.3) above. For notational convenience we take $N \equiv 1$ in the sequel.

Let $\{\hat{P}_x : x \in \bar{Q}\}$ be the diffusion in the quadrant corresponding to $(I,\ 0,\ \hat{\gamma})$ with

$$\hat{\gamma}(x) = \begin{cases} (1, \gamma_{12}) & \text{on } \{x \in \partial Q, x_1 = 0\} \\ (\gamma_{21}, 1) & \text{on } \{x \in \partial Q, x_2 = 0\} \end{cases}$$

where $\gamma_{12},\ \gamma_{21}$ are constants as in (A4.3) (iii).

The existence of a unique family $\{\hat{P}_x : x \in \bar{Q}\}$ of probability measures is assured by Varadhan and Williams (1985).

Now let S be a smooth convex unbounded domain contained in $\bar{Q}$ such that $(Q \setminus S) \subseteq B(0 : \frac{1}{2})$ and $(\partial S \setminus \partial Q) \subseteq \overline{B(0 : \frac{1}{2})}$; that is the curved part of the boundary lies entirely within $\overline{B(0 : \frac{1}{2})}$. Let $\tilde{\gamma}$ be a bounded Lipschitz continuous function on ∂S such that $\tilde{\gamma}(x) = \gamma(x)$ for $x \in \partial Q \cap \partial S$ and $\langle \tilde{\gamma}(x), n(x) \rangle = 1$ for $x \in \partial S$, where $n(x)$ is the inward normal to ∂S at x. It is not difficult to see that such a $\tilde{\gamma}$ exists. Let $\{\tilde{P}_x : x \in S\}$ be the diffusion with state space $\bar{S}$, corresponding to $(\sigma,\ b,\ \tilde{\gamma})$; existence and uniqueness of such a diffusion is guaranteed by Stroock and Varadhan (1971).

Let $x \in \bar{Q}$ be fixed. Define the following stop times:

$$\begin{aligned}
\tau_1(w) &= \inf\{t \geq 0 : |w(t)| \geq \frac{3}{4}\} \\
\tau_{2i}(w) &= \inf\{t > \tau_{2i-1}(w) : |w(t)| \leq \frac{2}{3}\} \\
\tau_{2i+1}(w) &= \inf\{t > \tau_{2i}(w) : |w(t)| \geq \frac{3}{4}\},\quad i = 1,\ 2,\ \cdots
\end{aligned}$$

For $i = 1,\ 2,\ \cdots$, let $\tilde{D}_w^{2i-1}$ be the regular conditional probability distribution (r.c.p.d) corresponding to $\tilde{P}_x$ given $\mathcal{B}_{\tau_{2i-1}}$ and D_w^{2i} be the r.c.p.d corresponding to $\hat{P}_x$ given $\mathcal{B}_{\tau_{2i}}$.

Define

$$P_1^x = \hat{P}_x \bigotimes_{\tau_1(\cdot)} \tilde{D}_{(\cdot)}^1\ ,\ P_2^x = P_1^x \bigotimes_{\tau_2(\cdot)} D_{(\cdot)}^2$$
$$P_{2i}^x = P_{2i-1}^x \bigotimes_{\tau_{2i}(\cdot)} D_{(\cdot)}^{2i}\ ,\ P_{2i+1}^x = P_{2i}^x \bigotimes_{\tau_{2i+1}(\cdot)} \tilde{D}_{(\cdot)}^{2i+1}$$

(Here we use the notation as in Section 6.1 of Stroock and Varadhan (1979))

Now by Theorem 6.1.2 of Stroock and Varadhan (1979) we see that the measures $\{P_n^x\}_{n\geq 1}$ as above are well defined. Observe that $\mathcal{B} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{B}_{\tau_n})$. Since P_{n+1}^x and P_n^x agree on $\mathcal{B}_{\tau_n}$ by construction, on applying Theorem 1.3.5 of Stroock and Varadhan (1979) we see that there exists a unique probability measure μ_x consistent with $\{P_n^x\}_{n\geq 1}$.

Let $E_x^{\hat{P}}$, $E_x^{\tilde{P}}$, E_x^{μ} denote the expectations with respect to $\hat{P}_x$, $\tilde{P}_x$, and μ_x respectively.

Proposition 4.1.2 : Let (A4.1) - (A4.3) hold. Then the family

$\{\mu_x : x \in \bar{Q}\}$ of probability measures (given above) is the unique solution to the submartingale problem corresponding to $(\sigma,\ b,\ \gamma)$. Moreover, the process $\{X(t) : t \geq 0\}$ under $\{\mu_x\}$ is Feller continuous and strong Markov.

Proof: The only aspect which needs comment is (4.1.5). In view of the above construction, Theorems 6.1.2, 6.2.1, and 6.2.2 of Stroock and Varadhan (1979) all other assertions can be obtained by fairly standard arguments.

To show that (4.1.5) holds for all x, note that it is enough to prove (4.1.5) for $x \in B(0 : r_1)$. Now let $x \in B(0 : r_1)$ be fixed. As $X(t) \neq 0$ on $[\tau_{2i-1}, \tau_{2i}]$ observe that

$$E_x^{\mu}[\int_0^{\infty} 1_{\{0\}}(X(s))ds] = E_x^{\mu}[\int_0^{\tau_1} 1_{\{0\}}(X(s))ds] + \sum_{i=1}^{\infty} E_x^{\mu}[\int_{\tau_{2i}}^{\tau_{2i+1}} 1_{\{0\}}(X(s))ds] \tag{4.1.6}$$

For $i = 1, 2, \cdots,\ \int_{\tau_{2i}}^{\tau_{2i+1}} 1_{\{0\}}(X(s))ds$ is $\mathcal{B}_{\tau_{2i+1}}$- measurable and $\mu_x \equiv P_{2i}^x$ on $\mathcal{B}_{\tau_{2i+1}}$. Hence we have,

$$\begin{aligned} E_x^{\mu}[\int_{\tau_{2i}}^{\tau_{2i+1}} 1_{\{0\}}(X(s))ds] &= E_x^{P_{2i}}[\int_{\tau_{2i}}^{\tau_{2i+1}} 1_{\{0\}}(X(s))ds] \\ &= \int_{\Omega^0}(\int_{\tau_{2i}(w)}^{\tau_{2i+1}(w)} D_w^{2i}((X(s) = 0)ds)dP_{2i-1}^x(w) = 0 \end{aligned} \tag{4.1.7}$$

as $\{P_x : x \in \bar{Q}\}$ solves the submartingale problem for $(I,\ 0,\ \hat{\gamma})$ and D_w^{2i} is the r.c.p.d corresponding to $\hat{P}_x$ given $\mathcal{B}_{\tau_{2i}}$. As μ_x and $\hat{P}_x$ agree upto τ_1 and $\{\hat{P}_x : x \in \bar{Q}\}$ solves the submartingale problem, the first term in the right side of equation (4.1.6) also vanishes. Hence we get (4.1.5), by combining equations (4.1.6) and (4.1.7). □

Remark 4.1.3: For $\beta > 0$, let S_β be a smooth unbounded open convex set in $\bar{Q}$ such that $(Q \setminus S_\beta) \subseteq B(0 : \frac{\beta}{2})$ and $(\partial S_\beta \setminus \partial Q) \subseteq \overline{B(0 : \frac{\beta}{2})}$; that is, the curved part of the boundary lies entirely within $B(0 : \frac{\beta}{2})$. Let $a = \frac{2}{3}\beta$ and $b = \frac{3}{4}\beta$. Define the stopping times

$$\theta_1(w) = \inf\{t \geq 0 : |w(t)| \geq b\},\quad \theta_{2i}(w) = \inf\{t > \theta_{2i-1}(w) : |w(t)| \leq a\}$$
$$\theta_{2i+1}(w) = \inf\{t > \theta_{2i}(w) : |w(t)| \geq b\},\quad i = 1,\ 2,\ \cdots$$

Let $\tilde{\gamma}_\beta$ be a bounded Lipschitz continuous function on ∂S_β such that $\tilde{\gamma}_\beta(x) = \gamma(x)$ on $\partial Q \cap \partial S_\beta$ and $\langle \tilde{\gamma}_\beta(x), n(x) \rangle = 1$ for $x \in \partial S_\beta$, where $n(x)$ is the inward normal to ∂S at x. Let $\{\tilde{P}^\beta_x : x \in \bar{S}_\beta\}$ be the diffusion in $\bar{S}_\beta$ corresponding to $(\sigma,\ b,\ \gamma)$. Note that when $\beta = 1$, we take, $S_\beta = S$, $\tilde{P}^\beta \equiv \tilde{P}$ and $\theta_i \equiv \tau_i$, $i = 1,\ 2,\ \cdots$. Note that if $|x| > \beta$ then μ_x and $\tilde{P}^\beta_x$ agree till hitting $\overline{B(0:\beta)}$.

For $r > 0$, define

$$\sigma_r(w) = \inf\{t \geq 0 : \ |w(t)| = r\}$$

Lemma 4.1.4: For any $r > 0$, we have

$$\sup_{|x| \leq r} E^\mu_x(\sigma_r) < \infty \tag{4.1.8}$$

In particular, the process exits out of bounded sets in finite time.

Proof: *Case*(i) Let $r \leq \frac{3}{4}$. In this case (4.1.8) can be easily derived from Theorem 3.6 and Corollary 2.2 in Varadhan and Williams (1985), together with the facts that the diffusion $\{\hat{P}_x : x \in \bar{Q}\}$ is strong Markov, it solves (iii) of the submartingale problem and that the diffusion has continuous sample paths.

Case (ii) Let $r > \frac{3}{4}$. Let τ_i, $i = 1,\ 2,\ \cdots$ be as in the construction of μ_x. Let $r_1 := \frac{2}{3}$, $r_2 := \frac{3}{4}$ Observe that by Lemma 2.1.2(a) of Chapter 2 and by the strong Feller property of $\{\tilde{P}_x : x \in \bar{S}\}$ in $\bar{S}$ we see that

$$\inf_{|x| = r_2} \tilde{P}_x(\sigma_r < \sigma_{r_1}) =: \delta_1 > 0 \tag{4.1.9}$$

Observe that

$$E^\mu_x(\sigma_r) = E^\mu_x\left[\int_0^{\sigma_r \wedge \tau_1} ds\right] + E^\mu_x\left[\sum_{i \geq 1} \int_{\sigma_r \wedge \tau_{2i-1}}^{\sigma_r \wedge \tau_{2i}} ds\right] + E^\mu_x\left[\sum_{i \geq 1} \int_{\sigma_r \wedge \tau_{2i}}^{\sigma_r \wedge \tau_{2i+1}} ds\right] \tag{4.1.10}$$

Note that to prove (4.1.8), by Case (i) and strong Markov property, it is enough to show that

$$\sup_{|x| = r_2} E^\mu_x(\sigma_r) < \infty \tag{4.1.11}$$

Let x be such that $|x| = r_2$; then

$$E_x^\mu \left[\int_0^{\sigma_r \wedge \tau_1} ds \right] = E_x^\mu(\sigma_r \wedge \tau_1) = 0 \tag{4.1.12}$$

Now for i=1,2,...,

$$\begin{aligned} E_x^\mu \left[\int_{\sigma_r \wedge \tau_{2i-1}}^{\sigma_r \wedge \tau_{2i}} ds \right] &= E_x^\mu \left[1_{\{\sigma_r > \tau_{2i-1}\}} E^\mu_{X(\tau_{2i-1})} \int_0^{\sigma_r \wedge \tau_2} ds \right] \\ &\leq \mu_x(\sigma_r > \tau_{2i-1}) \sup_{|x|=r_2} E_x^\mu(\sigma_r \wedge \tau_2) \\ &= \mu_x(\sigma_r > \tau_{2i-1}) \sup_{|x|=r_2} E_x^{\tilde{P}}(\sigma_r \wedge \sigma_{r_1}) \end{aligned} \tag{4.1.13}$$

Further

$$\begin{aligned} E_x^\mu \left[\int_{\sigma_r \wedge \tau_{2i}}^{\sigma_r \wedge \tau_{2i+1}} ds \right] &= E_x^\mu \left[1_{\{\sigma_r > \tau_{2i}\}} E^\mu_{X(\tau_{2i})} \int_0^{\sigma_r \wedge \tau_1} ds \right] \\ &\leq \mu_x(\sigma_r > \tau_{2i}) \sup_{|x|=r_1} E_x^\mu(\tau_1) \\ &= \mu_x(\sigma_r > \tau_{2i}) \sup_{|x|=r_1} E_x^{\tilde{P}}(\sigma_{r_2}) \end{aligned} \tag{4.1.14}$$

Observe that by (4.1.9),

$$\begin{aligned} \mu_x(\sigma_r > \tau_{2i-1}) &= E_x^\mu \left[1_{\{\sigma_r > \tau_{2i-1}\}} 1_{\{\sigma_r > \tau_{2i-3}\}} \right] \\ &\leq E_x^\mu \left[1_{\{\sigma_r > \tau_{2i-3}\}} E^\mu_{X(\tau_{2i-3})} (1_{\{\sigma_r > \tau_2\}}) \right] \\ &\leq \mu_x(\sigma_r > \tau_{2i-3}) \sup_{|x|=r_2} \tilde{P}_x(\sigma_r > \sigma_{r_1}) \\ &\leq (1-\delta_1) \mu_x(\sigma_r > \tau_{2i-3}) \end{aligned} \tag{4.1.15}$$

Hence we have

$$\left. \begin{aligned} &\mu_x(\sigma_r > \tau_{2i-1}) \leq (1-\delta_1)^{i-1} \\ &\mu_x(\sigma_r > \tau_{2i}) \leq \mu_x(\sigma_r > \tau_{2i-1}) \leq (1-\delta_1)^{i-1} \end{aligned} \right\} \tag{4.1.16}$$

Let $C_1 = \sup_{|x|=r_2} E_x^{\tilde{P}}(\sigma_r \wedge \sigma_{r_1})$; $C_2 = \sup_{|x|=r_1} E_x^{\tilde{P}}(\sigma_{r_2})$.
Combining equations (4.1.10) with equations (4.1.12) - (4.1.16) , together with the facts that C_1 and C_2 are finite, we can obtain (4.1.11) and hence the result. □

Lemma 4.1.5: For any $x \in \mathcal{Q}$, any $t > 0$,

$$(a) E^{\mu}_x[\int_0^{\infty} 1_{\partial\mathcal{Q}}(X(s))ds] = 0 \qquad (4.1.17)$$

$$(b) \mu_x(X(t)) \in \partial\mathcal{Q}) = 0 \qquad (4.1.18)$$

Proof: Because of condition (iii) of the submartingale problem it is enough to prove that

$$E^{\mu}_x[\int_0^{\infty} 1_{\partial_k\mathcal{Q}}(X(s))ds] = 0 \qquad (4.1.19)$$

where $\partial_k\mathcal{Q} = \{x \in \partial\mathcal{Q} : x_k = 0,\ x_j > 0\}$, $j, k = 1,\ 2,\ j \neq k$. To prove (4.1.19) when $k = 1$, it is enough to show that

$$\int_0^{\infty} \mu_x(X(s)) \in A_r)ds = 0 \qquad (4.1.20)$$

for an arbitrary $r > 0$, where $A_r = \{x \in \partial\mathcal{Q} : x_1 = 0,\ x_2 > r\}$. Note that it is enough to prove (4.1.20) for $0 < r < 1$. Put $\beta := r$ in Remark 4.1.3 and let $\tilde{P}^{\beta}_x$, θ_i, $i = 1, 2, \cdots$ be as in Remark 4.1.3. Note that

$$E^{\mu}_x[\int_0^{\infty} 1_{A_r}(X_s)ds] = \sum_{i=1}^{\infty} E^{\mu}_x[\int_{\theta_{2i-1}}^{\theta_{2i}} 1_{A_r}(X_s)ds]$$

as $X_t \notin A_r$ when $t \in [\theta_{2i},\ \theta_{2i+1}]$. Now by strong Markov property,

$$\begin{aligned} E^{\mu}_x[\int_{\theta_{2i-1}}^{\theta_{2i}} 1_{A_r}(X_s)ds] &= E^{\mu}_x[E^{\mu}_x[\int_{\theta_{2i-1}}^{\theta_{2i}} 1_{A_r}(X_s)ds|\mathcal{B}_{\theta_{2i-1}}]] \\ &= E^{\mu}_x[E^{\mu}_{X(\theta_{2i-1})}\int_0^{\theta_2} 1_{A_r}(X_s)ds] = 0 \end{aligned}$$

as μ_z and $\tilde{P}^{\beta}_z$ agree on $[0, \theta_2]$ and due to the fact that $\{\tilde{P}^{\beta}_z : z \in \bar{S}_{\beta}\}$ spends zero time on the boundary. (see Stroock and Varadhan (1971)). Hence we get (4.1.20). Observe that (4.1.20) when k=2 can be proved similarly. This proves (4.1.17).

(b) Along the same lines as the proof of Proposition 3.2.9 we can derive this result. □

4.2 Asymptotic properties

In this section we will be concerned with certain asymptotic properties of the diffusions, like recurrence, transience and positive recurrence.

We will follow the notations as in Section 4.1. Let $\{\mu_x : x \in \bar{Q}\}$ be the diffusion in $\bar{Q}$ corresponding to $(\sigma,\ b,\ \gamma)$ that solves the submartingale problem. Recurrence, transience and positive recurrence are defined as in Chapter 2 with μ_x and Q replacing P_x and $\mathcal{H}$ respectively.

For $U \subseteq \bar{Q}$ define, $\tau_U = \inf\{t \geq 0 : X_t \notin U\}$ = exit time from U; $\sigma_U = \inf\{t \geq 0 : X_t \in U\}$ = entrance time into U. Put $\sigma_r = \inf\{t \geq 0 : |X_t| = r\}$, for $r > 0$.

We need some auxiliary diffusions in the half planes, which we describe below.

Let $\tilde{\sigma},\ \tilde{b}$ respectively denote Lipschitz continuous extensions of $\sigma,\ b$ to $\mathbb{R}^2$ and let $\tilde{L}$ be the corresponding generator.

Let $G_i = \{x \in \mathbb{R}^2 : x_i > 0\},\ i = 1, 2, \cdots$ (Note that $G_1 = \mathcal{H}^2$ and $G_2 = \boldsymbol{H}$ in our previous notation). Let $r > 0$, and let $0 < \epsilon < r$. The boundary operator $J_i^{(r)}$ on ∂G_i is defined by

$$J_i^{(r)} f(x) = \frac{\partial f(x)}{\partial x_i} + \gamma_{ij}^{(r)}(x)\frac{\partial f(x)}{\partial x_j}, \quad x \in \partial G_i$$

for $i,\ j = 1,\ 2,\ i \neq j$, where $\gamma_{ij}^{(r)}(\cdot)$ is a Lipschitz continuous function on ∂G_i such that

$$\gamma_{ij}^{(r)}(x) = \begin{cases} \gamma_{ij}(x) & \text{if } x \in \partial G_i \text{ with } x_j > r + \epsilon \\ 0 & \text{if } x \in \partial G_i \text{ with } x_j < r - \epsilon \end{cases}$$

Let $\{P_x^{i,r} : x \in \bar{G}_i\}$ denote the $(\tilde{L},\ J_i^{(r)})$- diffusion with state space $\bar{G}_i,\ i = 1,\ 2$.

Proposition 4.2.1 Assume (A4.1) - (A4.3). The following statements are equivalent.
(a) $x_0 \in \overline{Q}$ is a recurrent point.
(b) $\mu_{x_0}(X(t) \in U \text{ for some } t \geq 0) = 1$ for all nonempty open sets $U \subset \overline{Q}$.
(c) There exist $z_0 \in \overline{Q},\ 0 < r_0 < r_1,\ y \in \partial B(z_0 : r_1)$ such that $\mu_y(\tau < \infty) = 1$, where $\tau = \inf\{t \geq 0 : X(t) \in \overline{B(z_0 : r_0)}\}$.
(d) There exists a compact set $K \subset \overline{Q}$ such that $\mu_x(X(t) \in K \text{ for some } t \geq 0) = 1$, $\forall x \in \overline{Q}$.
(e) $\mu_x(X(t) \in U \text{ for some } t \geq 0) = 1$, for all $x \in \overline{Q}$ and for all nonempty open sets $U \subset \overline{Q}$.

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(d) There exists a compact set $K \subset \overline{Q}$ such that $\mu_x(X(t) \in K$ for some $t \geq 0) = 1$, $\forall x \in \overline{Q}$.

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