# SOME SPECTRAL PROPERTIES OF THREE AND FOUR BODY SCHRODINGER OPERATORS BY THE METHOD OF TIME DEPENDENT SCATTERING THEORY

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## A C K N O W L E D G E M E N T S

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# C O N T E N T S

## INTRODUCTION AND NOTATION

In the present work we shall deal with the asymptotic completeness problem in three and four (Quantum Mechanical) particle scattering. This thesis is divided into three chapters. In the first chapter we give an introduction to Scattering Phenomenon and give a description of the N-particle completeness problem. Then we collect some results preliminary to the later chapters and some results that would complete a discussion of the problem.

The second chapter consists of some technical results and a reduction of the asymptotic completeness problem in N-particle scattering via time dependent methods.

The last chapter has two sections. The first section deals with verifying the conditions laid down in chapter II, for completeness to follow, in the case of three particles. This we do under explicit conditions on the potentials. The potentials decay at the rate of $|x|^{-2-\varepsilon}$ in the pair directions and have some local singularity.

The second section of the last chapter has completeness for the four particle case. Here apart from some smoothness conditions on the pair potentials we impose implicit restrictions on the pair Hamiltonians.

We deal with only separable Hilbert spaces with inner products linear in the second variable and assume all the operator theory basic to our discussion. This background is well covered in [AJS, K1, RSI-II, W].

For a self-adjoint operator L on a Hilbert space $\mathcal{H}$ we use the following notation. $\sigma(L)$, $\sigma_p(L)$, $\sigma_c(L)$, $\sigma_{ac}(L)$ and $\sigma_{sc}(L)$ denote the spectrum of L, the point, continuous, absolutely continuous and singularly continuous spectra of L while $\mathcal{H}_p(L)$, $\mathcal{H}_c(L)$, $\mathcal{H}_{ac}(L)$ and $\mathcal{H}_{sc}(L)$ the corresponding spectral subspaces of L in $\mathcal{H}$. By $\sigma_{ess}(L)$ we denote the essential spectrum which is the union of continuous spectrum, eigenvalues of infinite multiplicity and the accumulation points of eigenvalues of L. By $E^L$ we denote the projection onto $\mathcal{H}_p(L)$. The range of a closed operator A will be denoted by R(A). An equation like $a^{\pm} = b^{\pm} + c^{\pm}$ will mean two separate equations one for each sign. Similarly a statement $S^{\pm}$ means two independent statements $S^+$ and $S^-$. An operator C on $\mathcal{H}$ and $C \otimes 1$ on $\mathcal{H} \otimes \mathcal{K}$ are denoted by the same letter C. For any two operators A,B, by $Ad_A^n\{B\}$ we mean the n-fold commutator $[A,\ldots [A,B]\ldots]$.

In the discussion of many particles Greek letters index pairs of particles while Roman letters index the particles themselves. The summation $\sum_{\gamma}$ in chapters I and II is over the set of $\gamma$'s given by $\{\gamma = (ij): 1 \leq i \leq j \leq N\}$.

For any real valued function $\psi: \mathbb{R}^{\pm} \to \mathbb{R}$ we define, whenever it exists,

$$\mathcal{E}\lim_{t\to\pm\infty} \psi(t) \equiv \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t ds\, \psi(s).$$

We indicate in [.], any reference to other works. All the absolute constants are denoted by the letter K.

## CHAPTER 1

## DESCRIPTION OF SCATTERING PHENOMENON

### § 1. Scattering cross-sections

Scattering is an essential part of Physics by means of which a Physicist tests his theories, discovers interactions among fundamental particles and studies the structure of matter in general. The scattering experiment consists, usually, of an incident beam of particles, the target (or the scattering centre) which scatters these particles and a detector to detect (count) the particles going out after scattering. The observable quantity in a scattering experiment is the cross-section which has an expression

$$\sigma = \int_{S^2} d\Omega(\omega) \frac{d\sigma}{d\Omega}(\omega)$$

in **terms** of the differential cross-section $\frac{d\sigma}{d\Omega}(\omega)$ defined as follows. If the number of particles incident on the target per unit area perpendicular to the direction of the beam is $N_o$ and the number of particles scattered in the solid angle $\Delta\omega$ about the direction $\omega$ is $N_{sc}(\omega,\Delta\omega)$, then the following limit, when it exists is defined to be the differential cross-section. That is

$$\lim_{|\Delta\omega| \to 0} \frac{N_{sc}(\omega,\Delta\omega)}{N_o \Delta\omega} = \frac{d\sigma}{d\Omega}(\omega).$$

The corss-section is the effective cross-sectional area the target presents to the beam of particles as an obstacle to movement.

In the case of quantum mechanical particles the differential cross-section is related to the probability $p(f \to \omega)$ of a particle starting in a state $f$ and getting scattered in a direction $\omega$ as follows.

When the scattering matrix (defined in the next section) is unitary, the probability that a particle incoming in a state $f$ gets scattered in a cone $C$ is given by the scattering into cones formula, after removing the part corresponding to absence of scattering as follows. (See [D, AJS]).

Proposition 1.1 If a particle in a state $f$ gets scattered into the cone $C$, the probability for this event is

$$p(f \to C) = \int_C |(S\hat{f}-\hat{f})(p)|^2 \, d^3p$$

where $f \in L^2(\mathbb{R}^3)$, $S$ is the scattering operator and " ^ " denotes the Fourier transform in $L^2(\mathbb{R}^3)$.

In the case when there is a beam of uncorrelated independently scattered particles each of which is in the state $f$, the formula gets modified and gives the cross section for the event viz.,

$$\sigma(f \to C) = p(f \to C) = \int d^2a \int d^3p |(\widehat{Rf_{\underline{a}}})(p)|^2$$

where $R = S-I$ and $f_{\underline{a}}$ denotes the rigid translate of $f$ in the plane orthogonal to the initial (incident) direction of the beam by an amount $\underline{a}$. Under suitable conditions on $R$ (see pp. 283-286 [AJS]) the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega}(\omega) = \int_0^\infty d\lambda \int_{S^2} d\omega' |2\pi i \lambda^{-\frac{1}{2}} R(\lambda,\omega',\omega)|^2 \, |f(\lambda,\omega)|^2$$

where $R(\lambda,\omega',\omega)$ is the integral kernal corresponding to $R$ in the spectral representation of the free Hamiltonian $H_0$.

In the case of multichannel scattering one takes into account the incoming and the outgoing channels and obtains the relevant cross-sections (See [AJS, T, RS III]).

§ 2. Abstract multichannel scattering [AJS].

The quantum states of a multichannel scattering system are unit vectors in a complex separable Hilbert space $\mathcal{H}$. The system has the associated Hamiltonians $H_o$, $\{H(D)\}_{D\in J}$ and $H$ which are self-adjoint operators with appropriate dense domains and the index set $J$ consists of the clusterings D. $H_o, \{H(D)\}_{D\in J}$ and $H$ are respectively called the free, cluster and total Hamiltonians. The system has the associated subspaces $M_o, \{M_D\}_{D\in J}$ and $M_\infty(H)$ which are, respectively, left invariant by the unitary evolutions $U_t$, $\{V_t(D)\}_{D\in J}$ and $V_t$ generated by $H_o, \{H(D)\}_{D\in J}$ and $H$ in that order. $M_\infty(H)$ is the collection of all scattering states. From the physical principles of scattering it is necessary that the scattering states evolve into free or clustered states for large times. It is also necessary that the scattering states giving rise to different cluster states are not the same and also in reasonable situations all free or cluster states come from scattering states. These requirements form the Asymptotic condition. We set $H(0) \equiv H_o$, $V_t(0) \equiv U_t$ in the following.

Asymptotic condition

(AC 1) (Existence of wave operators). For each $f \in M_\infty(H)$, there is some $f^\pm(D) \in M_D$, $D \in J \bigcup \{0\}$ such that for each D ,

$$\operatorname*{s\,lim}_{t\to\pm\infty} V_t f - V_t(D) f^\pm(D) = 0 .$$

or equivalently,

$$\underset{t \to \pm\infty}{\text{s lim}} \; V_t^* V_t(D) f = \Omega^{\pm}(D) f^{\pm}(D).$$

exists for all $f^{\pm}(D) \in M_D$.

(AC 2) (Orthogonality of the ranges) Let the ranges of $\Omega^{\pm}(D)$ be denoted by $F_{\pm}(D)$, for all $D \in J \bigcup \{0\}$. Then,

$$F(D) \perp F_{\pm}(C) \quad \text{if} \quad C \neq D .$$

(AC 3) (Asymptotic completeness)

$$\bigoplus_D F_+(D) = M_\infty(H) = \bigoplus_D F_-(D) \quad , \quad D \in J \bigcup \{0\}.$$

The asymptotic condition results in the following Proposition [AJS].

<u>Proposition 1.2</u> Let $\Omega^{\pm}(D)$ be defined by (AC 1) for all $D \in J \bigcup\{0\}$. Then,

(i) $\Omega^{\pm}(D)$ are partial isometries with initial space $M_D$.

(ii) $\Omega^{\pm}(D)$ intertwine H and H(D), that is for all $t \in \mathbb{R}$,

$V_t \, \Omega^{\pm}(D) = \Omega^{\pm}(D) V_t(D)$.

(iii) If $P_{\pm}(D)$ is the orthogonal projection onto $F_{\pm}(D)$, then for all $t \in \mathbb{R}$, $V_t \, P_{\pm}(D) = P_{\pm}(D) V_t$.

(iv) If (AC 1) - (AC 3) are verified, then

$$S_{DC} = \{\Omega^{+}(D)\}^* \, \Omega^{-}(C)$$

is defined for each $D, C \in J \bigcup \{0\}$.

$$V_t(D) \, S_{DC} = S_{DC} \, V_t(C)$$

and

$$\sum_{D \in J \cup \{0\}} S^{*}_{DC} \, S_{DC'} = \delta_{CC'} \, P(M_C)$$

$$\sum_{D \in J \cup \{0\}} S_{CD} \, S^{*}_{C'D} = \delta_{CC'} \, P(M_C),$$

where $P(M_C)$ is the orthogonal projection onto $M_C$. We note that (iv) expresses the unitarity of the matrix $((S_{DC}))$ of operators on $\bigoplus_{D \in J \cup \{0\}} M_D$.

The scattering will be called potential scattering when the Hamiltonians $\{H(D)\}_{D\in J}$ and $H$ are given as perturbations of $H_o$ by the multiplication operators $\{W^D\}$ and $W$ respectively on $L^2(\mathbb{R}^\nu)$. We deal with a model for potential scattering in the next section.

## § 3. A model for N-particle potential scattering

We refer to [AJS, Chapter 15] and [S 2, Section 4] for the details of this section. We take $L^2(X)$ as the space of quantum states (and corresponds to $H$ of section 2) on the relative configuration space

$$X = \{x \in \mathbb{R}^{N\nu} : \sum_{i=1}^{N} m_i x_i = 0, \quad \nu \geq 3\} \tag{1.1}$$

equipped with the inner product $\langle x,y\rangle = \sum_{i=1}^{N} m_i \, x_i y_i$. Such a space describes N-particles of masses $m_i$ $i = 1,\ldots,N$, interacting via the (real valued functions) pair potentials $\{W_{ij}(x_i-x_j)\}_{1 \leq i,j \leq N}$, $x_i, x_j \in \mathbb{R}^\nu$. The free and the total Hamiltonians of such an N-particle system are the following operators respectively.

$$H_o = -\frac{1}{2}\Delta_X = \sum_{i=1}^{(N-1)} \frac{1}{2} a_k p_k^2 \quad ; \quad a_k = \left(\frac{1}{m_{k+1}} + \frac{1}{M_k}\right), \; M_k = \sum_{i=1}^{k} m_i .$$

$$H = H_o + \sum_{\gamma} W_\gamma \quad , \gamma = (ij), \; 1 \leq j,i \leq N. \tag{1.2}$$

where $\Delta_X$ is the Laplacian on $L^2(X)$ and $p_K$'s are the conjugates to the operators of multiplication by (the Jacobi coordinates) $y_k$ given by,

$$y_k = x_{k+1} - M_k^{-1} \sum_{i=1}^{k} m_i x_i \quad , \quad M_N = 1$$

$$p_k = - i \frac{\partial}{\partial y_k}$$

on their maximal domain in $L^2(\mathbb{R}^\nu)$. In (1.2) $W_\gamma$ denotes the operator of multiplication by the function $W_\gamma(y_\gamma)$. To account for the clustering of particles we consider the partitions (clusterings) $D = \{D_1,\ldots,D_k\}$ of $\{1,\ldots,N\}$ into nonempty disjoint sets (clusters) $D_i$. Suppose, for convenience,

$$\begin{aligned} i(D) &= \{\gamma = (ij) : i,j \in D_\ell \text{ for the same } \ell,\ \ \ell = 1,\ldots,k\} \\ e(D) &= \{\gamma = (ij) : i \in D_\ell,\ j \in D_{\ell'},\ \ \ell \neq \ell',\ \ell,\ell' = 1,\ldots,k\}, \end{aligned} \tag{1.4}$$

then the cluster Hamiltonians are defined as

$$H(D) = H_0 + \sum_{\gamma \in i(D)} W_\gamma. \tag{1.5}$$

We note that if $\# D = 1$, then $e(D) = \emptyset$ and $H(D) = H$ and if $\# D = N$, $i(D) = \emptyset$ with $H(D) = H_0$ respectively. When dealing with clusters it is useful to write $X$ in terms of the internal and external configuration spaces as follows. Let $\# D \neq 1,N$, then

$$\begin{aligned} X^D &= \{x \in X : \sum_{i \in D_\ell} m_i x_i = 0, \text{ for all } \ell = 1,\ldots,k\} \\ X_D &= \{x \in X : x_i = x_j \text{ if } i,j \in D_\ell \text{ for the same } \ell,\ \ell=1,\ldots k\}. \end{aligned} \tag{1.6}$$

In this case in $X$,

$$X^D \perp X_D \quad \text{and} \quad X = X^D \oplus X_D. \tag{1.7}$$

A point $x^D$ in $X^D$ has (N-k) components, each of which is in $\mathbb{R}^\nu$ and the components of $x^D$ can be written by choosing some Jacobi coordinates for each cluster $D_\ell$ as in (1.3). Similarly a point $x_D$ in $X_D$ can be written in terms of the Jacobi coordinates obtained by treating the clusters $D_\ell$ as particles with masses $M_{D_\ell} = \sum_{i\in D_\ell} m_i$ and positions $M_{D_\ell}^{-1} \sum_{i\in D_\ell} m_i x_i$ in the configuration space X. When X has the structure as in (1.7), the tensor product

$$L^2(X) = L^2(X^D) \otimes L^2(X_D)$$

results and the Hamiltonians in (1.2) and (1.5) can be written as, using the Laplacians $\Delta_{X^D}$ and $\Delta_{X_D}$ respectively on $L^2(X^D)$ and $L^2(X_D)$, for suitable $a^D$, $a_D$,

$$\begin{aligned} H_o &= T^D + T_D = -\tfrac{1}{2} a^D \Delta_{X^D} - \tfrac{1}{2} a_D \Delta_{X_D} \\ H(D) &= H^D + T_D \quad \text{where} \quad H^D = T^D + \sum_{\gamma\in i(D)} W_\gamma \end{aligned} \tag{1.8}$$

The Hamiltonians $H_o$, $T^D$ and $T_D$ are self-adjoint on their maximal domains $\mathcal{D}(H_o)$, $\mathcal{D}(T^D)$ and $\mathcal{D}(T_D)$ in $L^2(X)$, $L^2(X^D)$ and $L^2(X_D)$ respectively. The following condition, which seems to be necessary for the Asymptotic condition to be true, shall be imposed on the potentials from now on.

<u>Assumption (A1)</u>. Let $\gamma$ be a pair and $W_\gamma$ the associated two body potential. Let $D(\gamma)$ be the clustering with $\# D = (N-1)$, $\gamma = D_i(\gamma)$ if $\# D_i(\gamma) = 2$. For such a case we set $\gamma \equiv D(\gamma)$. Thus $T^{D(\gamma)} \equiv T^\gamma$. Then we assume that

$$W_\gamma (T^\gamma + 1)^{-1}$$

is compact on $L^2(X^\gamma)$.

This assumption immediately implies that $\mathcal{D}(W_\gamma)$ contains $\mathcal{D}(T^\gamma)$ and $\forall\ \varepsilon > 0$, there is a $b(\varepsilon) > 0$ such that for each $f \in \mathcal{D}(T^\gamma)$,

$$||W_\gamma f|| \leq \varepsilon||T^\gamma f|| + b(\varepsilon)||f||$$

which is saying that $W_\gamma$ is $T^\gamma$ bounded with relative bound zero [K]. Thus by the Kato-Rellich Theorem and Remark 1.4 below, we immediately conclude that $H^D$ is self-adjoint on $\mathcal{D}(T^D)$ for all D and that $H_o$, $H(D)$, $H$ are all self-adjoint on $\mathcal{D}(H_o) = \mathcal{D}(H(D)) = \mathcal{D}(H)$.

<u>Proposition 1.3</u> (Kato-Rellich) Suppose A is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$ and B a symmetric operator with domain $\mathcal{D}(B)$. Suppose, further, that

(i) $\mathcal{D}(B)$ contains $\mathcal{D}(A)$ and

(ii) for some $a,b \in (0,\infty)$, $a < 1$ and all $f \in \mathcal{D}(A)$,

$$||Bf|| \leq a||Af|| + b||f|| ,$$

then A+B is self-adjoint on $\mathcal{D}(A)$ and essentially self-adjoint on any core of A and A+B is bounded below whenever A is so.

We refer to [K1,AJS, RS III] for a proof.

<u>Remark 1.4</u> In view of the above Proposition it easily follows that $H$, $H_o$, $H(D)$ are bounded with respect to one another.

Henceforth the unitary one parameter groups generated by $H_o$, $H(D)$ and H will be denoted by $U_t$, $V_t(D)$ and $V_t$ respectively. By virtue of the equation (1.8) we also have the decomposition

$$U_t = U_t^D\, U_{t,D} \quad \text{and} \quad V_t(D) = V_t^D\, U_{t,D}$$

where $U_t^D = \exp(-it\ T^D)$, $U_{t,D} = \exp(-it\ T_D)$ and $V_t^D = \exp(-it\ H^D)$.

As another consequence of the assumption (A1) we have the Hunzicker-Van Winter-Zhislin Theorem on the essential spectrum of H.

We set $\Sigma = \inf_{D:\,\# D \geq 2} \sigma(H(D))$.

Proposition 1.5 (HVZ) If H is the N-particle Hamiltonian with the pair potentials $W_\gamma$ satisfying condition (A1) then

$$\sigma_{ess}(H) = [\Sigma, \infty).$$

See [RS IV, SB 1] for a proof and further references on this theorem.

In the discussion of spectral properties and completeness for N-particle systems thresholds often play an important role, hence we define them in

Definition 1.6 (Thresholds of an N-particle Hamiltonian)

Consider $\sigma_p(H^D)$ for any D with $\# D \neq 1, N$. Then the thresholds $\Upsilon(H)$ of H are defined by

$$\Upsilon(H) = \bigcup_D \sigma_p(H^D) \cup \{0\}.$$

§ 4. Discussion of available results on completeness.

In the multichannel scattering theory the pioneering work is that of Fadeev [F] who dealt with the three body problem and proved completeness when there are finite number of channels. Completeness in N-particle scattering theory was solved by Iorio - O'Carrol [Io], Lavine [L 2] and Sigol [S 1] completely in the single channel case for a class of short range potentials. In numerous papers Sigal made a reduction of the N-body short range problem using a time independent method. All these culminated in [S2] where a discussion of (and a reference to) all his work can be found. Lavine [L1,2] used the repulsive nature of the pair potentials to conclude that $H^D$ has only continuous spectrum for all clusterings D and that H has a conjugate to arrive at the result whereas Iorio-O'Carrol [IO] prove the absence of point spectrum for all $H^D$ from smallness of the coupling of the potentials to the free Hamiltonian. In [S3] Sigal shows that if all the pair potentials $W_\gamma$ consist of a repulsive part and a weakly coupled part with a overall $(3+\varepsilon)$ decay at $\infty$ in the pair direction, then the resulting N-particle system has only one channel. Hagedorn [Ha1] has proved completeness for pair potentials with $(2+\varepsilon)$ decay at $\infty$, in the case of three and four particles systems assuming finite number of channels for the systems.

Fadeev's method was adapted by Newton [N] for the three particle case and this was followed by Ginibre-Moulin [GM], Howland [HO] and Thomas [T] who all make use of the stationary method in a Hilbert space and require more than $|x|^{-2}$ decay at $\infty$ in the pair directions for the pair potentials apart from having finite number of channels for the system. Mourre [M1] applies Ginibre-Moulin formulation and considers differentiable potentials with $(1+\varepsilon)$ decay at $\infty$ and assumes single channel for the three body system to prove completeness. In [M3] he gets weighted $L^2$ estimates for the total evolution of a single channel short range three particle system in the pair directions using which he proves completeness for such a system in [M4]. Hagedorn-Perry [HP] prove completeness for a finite channel three particle system having pair potentials with $(2+\varepsilon)$ decay at $\infty$.

Enss [E1] outlined a method for proving completeness in three particle scattering and obtained [E2] low energy estimates toward that end.

In [KKM] some of his ideas were

adapted, along with commutator methods of Jensen [J] to obtain estimates on the two particle evolutions and the theory of evolution of observables developed by Muthuramalingam and Sinha [MS], for proving completeness when $\{W_\gamma\}$, consisting of local singularities, decay faster than $|x^\gamma|^{-2}$ at $\infty$. In the present work we reduce the N-particle completeness to the verification of two conditions (local decay and the low energy decay of the N-particle evolution) on the lines of [MS] and [KKM]. We present the work of [KKM] and give a partial solution to the four particle completeness allowing for infinite number of channels for a class of pair potentials.

As for the two particle completeness the problem is completely solved even for the long range potentials and the literature is vast. The methods used fall in three catagories. The first is the abstract trace class theory of Kato-Birman [K2, B1]. Then the Agmon-Kuroda [A,Ku] theory of eigen function expansions, the smoothness theory of Kato-Lavine [K3, 4, L3]. See also Povzner [Po] and Ikebe [Ik]. Finally the time dependent theory was pioneered by Enss in [E3,4]. Davies [Da], Mourre [M4], Muthuramalingam and Sinha [MS] and Perry [P2] contributed to the development of the theory for short range potentials. See also Simon [SB2]. For further references and literature see [AJS, Am, Mu, RS III].

In the spectral properties of N-particle Hamiltonians a lot was done recently. Mourre in [M2] developed a theory of local conjugates using which he proved the absence of singular continuous spectrum and non-existence of infinite number of eigenvalues away from the thresholds of a three particle Hamiltonian, for a large class of potentials. Using his method the same results were obtained for the N-particle case by Perry, Sigal and Simon in [PSS]. Froese and Herbst in [FH1] proved the absence of positive eigenvalues for a wide class of N-particle Hamiltonians again using the Mourre inequality which they prove in [FH2]. They also show in [FH1] that the non threshold eigenvalues have eigenvectors with exponential decay at $\infty$. In [P1] for the same class of Hamiltonians, using the results of [FH1] Perry shows that non threshold eigenvalues do not accumulate at the thresholds from the positive side.

Having summarised the available results we shall now collect some of the results on the spectral properties and completeness of N-particle Hamiltonians. We give only those for which a complete proof is available. We will start with theorem on quadratic forms. We refer to [RS I, II] for relevant definitions and proofs.

Proposition 1.7 A closed semi bounded quadratic form on a Hilbert space $\mathcal{H}$ is the quadratic form of a unique self-adjoint operator.

Remark 1.8 If the potential W satisfies the condition (A1) then it has to be a function at least in $L^2_{loc}$. Then $x.\nabla W(x)$ will be a priori defined as a distribution and when we further require $(1+p^2)^{-1}\, x.\nabla W(x)(1+p^2)^{-1}$ to be compact then even $x.\nabla W$ becomes a function again.

Proposition 1.9 (Froese-Herbst) Let H be a N-particle Hamiltonian with the pair potentials satisfying (A1) together with $(T^\gamma+1)^{-1}\, x^\gamma.\nabla_\gamma W_\gamma(x^\gamma)(T^\gamma+1)^{-1}$ compact for each pair $\gamma$. Then

(i) $\sigma_p(H) \cap (0,\infty) = \emptyset$ and

(ii) $E \in \sigma_p(H) \setminus T(H)$ and $f_E$ is an eigen vector of H corresponding to E implies that $\int |\exp(\alpha|x|)f_E(x)|^2\, dx < \infty$ whenever $E+\alpha^2 < \beta$ where

$$\beta = \inf\{T(H) \cap [E,\infty)\}.$$

See [FH1] for a proof.

<u>Proposition 1.10</u> (Mourre-Perry-Sigal-Simon) Let the pair potentials of a N-particle system satisfy the conditions of Theorem 1.9 and also let

$$(T^{\gamma}+1)^{-1}(x^{\gamma}.\nabla_{\gamma}(x^{\gamma}.\nabla_{\gamma}W_{\gamma}(x^{\gamma})))(T^{\gamma}+1)^{-1}$$

be bounded. Then,

(i) $\sigma_{sc}(H) = \emptyset$.

(ii) $\mathcal{T}(H)$ is a closed countable set

(iii) $\sigma_p(H) \cap (\mathbb{R} \setminus \mathcal{T}(H))$ is finite with finite multiplicity for each.

See [PSS] for a proof.

Having described the spectral properties we will look at the completeness results.

<u>Proposition 1.11</u> (Iorio-O'Carrol) Let the pair potential $W_\gamma$ satisfy

$$W_{\gamma}(.) \in L^{\frac{1}{2}\nu+\varepsilon}(\mathbb{R}^{\nu}) \cap L^{\frac{1}{2}\nu-\varepsilon} \quad , \quad \nu \geqq 3.$$

Then the wave operators for the corresponding N-particle Hamiltonian $H_\lambda = H_o + \lambda \sum_{\gamma} W_\gamma$ exist and are unitary for sufficiently small $|\lambda|$. In particular H has only absolutely continuous spectrum.

See [Io] for a proof.

<u>Proposition 1.12</u> (Lavine) Consider a N-particle system with pair potentials $W_\gamma$ satisfying in addition to (A1) the following.

(i) $x^{\gamma}\nabla_{\gamma}W_{\gamma}(x^{\gamma}) \leqq 0$

(ii) $W_\gamma = W_\gamma^1 + W_\gamma^2$ with $W_\gamma^1(.) \in L^{\infty}(\mathbb{R}^{\nu})$

$W_\gamma^2(.) \in L^p(\mathbb{R}^{\nu})$ ($p = 2$ if $\nu \leq 3$, $p > 2$ if $\nu = 4$ and $p = \frac{1}{2}\nu$ if $\nu \geq 5$) and

(iii) $W_\gamma(x^\gamma) = W_\gamma(|x^\gamma|)$ with $|W_\gamma(x^\gamma)| \leq K(1 + |x^\gamma|)^{-(\frac{5}{2} + \varepsilon)}$

Then the wave operators exist and are complete.

See [L1,2] for a proof.

Then we give a two body result for which we take $H_{ac}(H)$ in the place of $M_\infty(H)$ of section 2.

Proposition 1.13 (Agmon -Kato-Kuroda) Consider the case N = 2 and $W(x) = (1 + |x|)^\delta \cdot W_1(x)$, for $\delta > 1$. Let $W_1(x)$ satisfy (A1). Then

(i) $\sigma_p(H) \cap (0,\infty)$ is a discrete subset of $(0,\infty)$ with each having finite multiplicity.

(ii) $\sigma_{sc}(H) = \emptyset$ and

(iii) The wave operators exist and are complete.

See [RS IV] for a proof.

The same result obtained using time dependent methods is contained in [E4, Da, M4, MS and P2].

Note : After the present work was done we came to know of the work of Kitada [Ki] on the asymptotic completeness of N-body Schrodinger operators.

# CHAPTER II

## N-PARTICLE COMPLETENESS - A REDUCTION

In this chapter we present a reduction to the proof of N-particle completeness by a time dependent method on the lines of the work in [MS]. This chapter consists of two sections, the first of which has some technical results and a theory of evolution of the observables A and $x^2$. In the second section we prove N-particle completeness under the assumption that the scattering states are not supported in any bounded region corresponding to a pair direction in the distant future or the remote past. We also assume that they do not have small pair energies asymptotically. Apart from these we make fairly general assumptions on the pair potentials.

Throughout this chapter unless specified otherwise we take a clustering to have $2 \leq \# D \leq N-1$ and when $\# D = 1,N$ are allowed, we set $H(D) \equiv H_o$ for $\# D = N$ and $H(D) \equiv H$ for $\# D = 1$.

### § 1. Asymptotic evolution of A and $x^2$

The generator A of dilations $Y_\theta$ on $L^2(X)$, $[(Y_\theta f)(x) = e^{\frac{1}{2}\nu\theta} f(e^{\frac{1}{2}\theta} x)]$ and the moment of inertia $x^2$ play an important role in the theory of observables, so we give a brief description of their properties below. Both A and $x^2$ are defined on $\mathcal{S}(X)$ and leave it invariant apart from being essentially self-adjoint there. A has the following explicit form on $\mathcal{S}(X)$.

$$A = \frac{1}{4} \sum_{k=1}^{N-1} (y_k p_k + p_k y_k) = A^D + A_D$$

$$A^D = -\frac{i}{4}\left(x^D \frac{\partial}{\partial x^D} + \frac{\partial}{\partial x^D} x^D\right), \quad A_D = -\frac{i}{4}\left(x_D \frac{\partial}{\partial x_D} + \frac{\partial}{\partial x_D} x_D\right) \tag{2.1}$$

$$x^2 = \sum_{k=1}^{N-1} a_k^{-1} y_k^2 = (a^D)^{-1}(x^D)^2 + (a_D)^{-1} (x_D)^2 \tag{2.2}$$

in the coordinate systems (1.3) and (1.7) respectively. Both A and $x^2$ are invariant under a change in the coordinate system. We note that $A^D$ and $A_D$ are essentially self-adjoint on $S(X^D)$ and $S(X_D)$ respectively. On $S(X)$, the following commutation relations are valid.

$$i[H_o,A] = H_o \quad , \quad i[H_o,x^2] = 4A \ . \tag{2.3}$$

At this point we make a remark.

Remark 2.1 Under the scale change $y_k' = \sqrt{a_k^{-1}}\, y_k$ , $p_k' = \sqrt{a_k}\, p_k$ the following simple expressions result for $H_o$ and $x^2$ while A remains invariant.

$$H_o = \frac{1}{2} \sum_{k=1}^{N-1} (p_k')^2 \quad , \quad x^2 = \sum_{k=1}^{N-1} (y_k')^2 = (x^{D'})^2 + (x_D')^2 \ .$$

The potentials also undergo a scale change. In view of this we make this transformation on the space X throughout this report.

We now make the following assumptions on the pair potentials and work only under these assumptions in the remaining part of this chapter and the later chapter.

Assumptions on the potentials : We use the notation introduced in (A1) and set $\rho_1(\lambda) = (1+\lambda^2)^{-\delta_1}$ , for some $\delta_1 > 1$. Then the potentials $\{W_\gamma\}$ satisfy, for each $\gamma$,

(A1) $W\ (T^\gamma+1)^{-1}$ is compact on $L^2(\mathbb{R}^\nu)$.

(A2) $(T^\gamma+1)^{-1}\ [i\ A^\gamma,W_\gamma]\,(T^\gamma+1)^{-1}$ is compact on $L^2(\mathbb{R}^\nu)$ and

(A3) $W_\gamma\ \rho_1(x\ )^{-1}(T^\gamma+1)^{-1}$ is bounded.

The Duhamel formula,

$$[C, \exp(-it\ B)] = i\int_0^t ds\ \exp(-i(t-s)B)[B,C]\exp(-is\ B) \quad (2.4)$$

gives an expression for the commutator of a bounded operator C, with the unitary group exp (-it B) generated by a self-adjoint operator B, the integral converging in the weak sense. Henceforth, we will use this formula, without further explanation, whenever necessary.

Theorem 2.2. Let $\mathcal{H}$ be a Hilbert space and C be a self-adjoint operator with $C_\varepsilon$ a family of bounded self-adjoint operators converging to C on $\mathcal{D}(C)$ as $\varepsilon \to 0$. If L is a bounded self-adjoint operator such that

$$||[C_\varepsilon, L]|| \leq K < \infty$$

[redacted] for all $1 \geq \varepsilon \geq 0$, then

(i) L leaves $\mathcal{D}(C)$ invariant and

(ii) $[C,L]$ is a bounded operator .

Further if $Ad^j_{C_\varepsilon}\{L\}$ is bounded for all $j = 1,\ldots,n$ then,

(iii) $Ad^j_{C_\varepsilon}\{L\}$ leaves $\mathcal{D}(C)$ invariant for all $j = 1,\ldots,n-1$

(iv) $Ad^j_C\{L\}$ is bounded for each $j = 1,\ldots,n$ and

(v) L leaves $\mathcal{D}(C^n)$ invariant.

Proof (i) For any $f,g \in \mathcal{D}(C)$ we have, by the self-adjointness of $C_\varepsilon$ and L, for each $\varepsilon > 0$,

$$|\langle C_\varepsilon f, Lg\rangle - \langle Lf, C_\varepsilon g\rangle| \leq ||[C_\varepsilon, L]||\ ||f||\ \ ||g|| \leq K||f||\ ||g||. \quad (2.5)$$

By taking $\lim \varepsilon \to 0$ in (2.5), this inequality implies

$$|\langle Cf,Lg\rangle - \langle Lf,Cg\rangle| \leq K||f|| \, ||g||$$
$$|\langle Cf,Lg\rangle| \leq K_1||f||\{||g|| + ||Cg||\}. \tag{2.6}$$

Since $C$ is a self-adjoint operator (2.6) implies that $Lg \in \mathcal{D}(C)$ and also since $g$ is arbitrary in $\mathcal{D}(C)$ that $L$ leaves $\mathcal{D}(C)$ invariant.

(ii) In view of (i) we have for any $g \in \mathcal{D}(C)$ taking $\lim \varepsilon \to 0$ in (2.5),

$$||[C,L]g|| \leq K||g||$$

which gives the boundedness

$$||[C,L]|| \leq K < \infty$$

by the density of $\mathcal{D}(C)$ in $\mathcal{H}$.

(iii) By taking $L_j = Ad_C^j\{L\}$ for $j = 1,2,\ldots,n-1$ and applying (i) by replacing $L$ by $L_j$ in (i), we obtain the result .

(iv) Given the result (iii), the proof is as that of (ii) replacing $L$ by $L_j$ in (ii).

(v) For $f,g \in \mathcal{D}(C^n)$, we have the following inequality.

$$|\langle C^n f,Lg\rangle - \langle Lf,C^n g\rangle| \leq |\langle f, \sum_{j=1}^{n} \binom{n}{j} Ad_C^j\{L\}C^{n-j} g\rangle|$$
$$\leq \sum_{i=1}^{n} \binom{n}{j} ||f|| \, ||Ad_C^j\{L\}|| \, ||C^{(n-j)}g||. \tag{2.7}$$

Now by the functional calculus for self-adjoint operators $\mathcal{D}(C^i)$ is contained in $\mathcal{D}(C^n)$ for all $1 \leq j \leq n$ and $C^{(n-j)}$ is $C^n$ bounded for any $1 \leq j \leq n$, Hence (2.7) implies that

$$|\langle C^n f, Lg\rangle| \leq ||f|| \{\sum_{j=1}^{n} \binom{n}{j} k_j\} ||C^n g|| \leq K_n ||f|| \; ||C^n g||$$

which implies the required result by the self-adjointness of $C^n$ and (2.7).

<u>Corollory 2.3</u> Let $S = H(D)$, $1 \leq \# D \leq N$, $2 \leq N < \infty$. Let $a, \mu$ be such that $a < \mu < \inf \sigma(S)$ and consider the bounded self-adjoint operator $(S-a)^{-1}$. Then for any positive integer $n$,

(i) $||Ad_x^n \{Z_t\}|| \leq K(|t|^n+1)$ when $Z_t = \exp(-it(S-a)^{-1})$.

(ii) $||Ad_x^n \{\phi(S)\}|| \leq K$ for any $\phi \in C_0^\infty(\mathbb{R})$.

(iii) $||(S-z)Ad_x^n\{(S-z)^{-1}\}|| \leq K$ for all $z \notin \sigma(S)$.

<u>Proof</u> We fix an $N$ and then a $D$. Then we consider the family $x_\varepsilon = x(1+\varepsilon x^2)^{-1}$ of bounded operators for $\varepsilon > 0$. Clearly $x_\varepsilon \to x$ on $\mathcal{D}(x)$. Since $Z_t$ is bounded for any $t$, we have,

$$Ad_{x_\varepsilon}\{Z_t\} = [x_\varepsilon, Z_t] = Z_t(Z_t^* x_\varepsilon Z_t - 1) = Z_t \int_0^t ds\, Z_s^* \, i[(S-a)^{-1}, x_\varepsilon] Z_s$$

which implies that

$$||Ad_{x_\varepsilon}\{Z_t\}|| \leq ||[(S-a)^{-1}, x_\varepsilon]|| \; (|t| + 1).$$

Repeating this procedure we see that for any $n$,

$$||Ad^n_{x_\varepsilon}\{Z_t\}|| \leq (|t|^n+1) \sum_{j=1}^{n} K_j \binom{n}{j} ||Ad^{n-j}_{x_\varepsilon}\{(S-a)^{-1}\}|| \; ||Ad^j_{x_\varepsilon}\{(S-a)^{-1}\}||$$

Therefore we obtain the required estimate using Theorem 2.2 if we show the uniform boundedness in $\varepsilon$ of $Ad^k_{x_\varepsilon}\{(S-a)^{-1}\}$ for any positive integer $k$.

Since $x_\varepsilon$ leaves $\mathcal{S}(X)$ invariant for $g \in \mathcal{S}(X)$ we have,

$$2\, Sx_\varepsilon g = 2\, x_\varepsilon Sg + [P,[P,x_\varepsilon]]g + [P,x_\varepsilon]Pg. \tag{2.8}$$

A computation shows that $[P,x_\varepsilon],[P[P,x_\varepsilon]]$ are bounded for each $\varepsilon$ and P is S bounded. Thus (2.8) implies that, for some $K_\varepsilon$,

$$||S\, x_\varepsilon g|| \leq K_\varepsilon(||g|| + ||Sg||) \tag{2.9}$$

Since $\mathcal{S}(X)$ is a core for S, this means that $x_\varepsilon$ leaves $\mathcal{D}(S)$ invariant for each $\varepsilon$. In view of this, on $\mathcal{D}(S)$,

$$[x_\varepsilon,S] = [x_\varepsilon,H_o] = [x_\varepsilon,P]P - \tfrac{1}{2}[[x_\varepsilon,P],P]$$

and (2.10)

$$[x_\varepsilon,[x_\varepsilon,S]] = [x_\varepsilon,[x_\varepsilon,H_o]] = ([x_\varepsilon,P])^2$$

are both uniformly bounded in $\varepsilon$. Since $x_\varepsilon$ leaves $\mathcal{D}(S)$ invariant we can write,

$$\mathrm{Ad}_{x_\varepsilon}\{(S-a)^{-1}\} = (S-a)^{-1}\,[S,x_\varepsilon](S-a)^{-1}\,.$$

Then repeatedly using (2.10) and computing explicitly we get the required uniform boundedness of $\mathrm{Ad}^k_{x_\varepsilon}\{(S-a)^{-1}\}$ proving (i). The proof in the remaining cases of D and H is the same.

(ii) Since $\chi(\lambda) = (\lambda-a)^{-1}$ is a bijection from $(\mu,\infty)$ to $(0,(\mu-a)^{-1})$ and induces a bijection from $C^\infty_o((\mu,\infty))$ to $C^\infty_o((0,(\mu-a)^{-1}))$, we set $\psi = \phi\circ\chi^{-1}$ and prove the uniform boundedness in $\varepsilon$ of $\mathrm{Ad}^n_{x_\varepsilon}\{\psi((S-a)^{-1})\}$ only.

$$\mathrm{Ad}^n_{x_\varepsilon}\{\psi((S-a)^{-1})\} = \int dt\, \hat{\psi}(t)\mathrm{Ad}^n_{x_\varepsilon}\{Z_t\}\quad,$$

hence by (i),

$$||\mathrm{Ad}^n_{x_\varepsilon}\{\phi(S)\}|| \leq ||\mathrm{Ad}^n_{x_\varepsilon}\{\psi((S-a)^{-1})\}||$$

$$\leq K\int dt\, |\hat{\psi}(t)|(|t|^n + 1) \leq K_1$$

Now the result follows by applying Theorem 2.2.

(iii) The proof is similar to that in (i) by noting that Range of $Ad^n_{x_\varepsilon}\{(S-z)^{-1}\}$ is contained in $\mathcal{D}(S)$, writing $(S-z)\ Ad^n_{x_\varepsilon}\{(S-z)^{-1}\}$ explicitely using (2.8) and applying Theorem 2.2.

<u>Lemma 2.4</u> Let $S$ be as in Corollory 2.3 and set $\rho(\lambda) = (1+\lambda^2)^{-\delta}$, $\delta > 0$. Then for all $z \notin \sigma(S)$ and $\phi \in C_o^\infty(\mathbb{R})$ the following are true.

(i) If $\psi(x)$ is an S bounded function with $\rho(x)^{-1}\ \psi(x)$ is also S bounded, then $\psi(x)(S-z)^{-n}\ \rho(x)^{-1}$ and $\psi(x)\phi(S)\rho(x)^{-1}$ are bounded.

(ii) If $\psi$ is an S bounded function then,

$\rho(x)^{-1}\ \psi(x)(S-z)^{-n}\ \rho(x)$ and $\rho(x)^{-1}\psi(x)\phi(S)\rho(x)$

are bounded.

(iii) For any pairs $\alpha,\gamma$, $\rho(x^\alpha)\phi(T^\gamma)\rho(x^\alpha)^{-1}$ is bounded.

(iv) $A^\theta(S-z)^{-1}\ (1+x^2)^{-\theta}$ , $A^\theta\ \phi(S)(1+x^2)^{-\theta}$

$(1+x^2)^{-\theta}(S-z)^{-1}\ A^\theta$ and $(1+x^2)^{-\theta}\phi(S)\ A^\theta$ are bounded for all $0 \leq \theta \leq 2$.

(v) $(x^2+1)^{\frac{1}{2}}\ A\ \phi(S)(1+x^2)^{-1}$ and $(1+x^2)^{-1}\ \phi(S)A(x^2+1)^{\frac{1}{2}}$ are bounded.

<u>Proof</u>. As in Corollory 2.3 we prove the result for a fixed N and D. We do not discuss the domain questions since they can be taken care of as in Corollory 2.3.

(i) Owing to the identity,

$$\psi(x)(S-Z)^{-n}\rho(x)^{-1} = \psi(x)(S-Z)^{-1}\rho(x)^{-1}\rho(x)(S-Z)^{-1}\rho(x)^{-1}\ldots\rho(x)(S-Z)^{-1}\rho(x)^{-1}$$

we prove only the boundedness of

$$\psi(x)(S-Z)^{-1}\ \rho(x)^{-1}\ .$$

Now we write $2\delta = \mu+\varepsilon$, where $\mu$ is the integer part of $2\delta$, in which case the result follows from

$$\psi(x)(S-z)^{-1}\, x^{\mu}(1+x^2)^{\frac{1}{2}\varepsilon}$$

being bounded. We have,

$$\psi(x)(S-z)^{-1}x^{\mu} = \psi(x)\sum_{k=1}^{\mu}\binom{\mu}{k}x^{\mu-k}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\}\ .$$

Therefore,

$$\psi(x)(S-z)^{-1}(1+x^2)^{\frac{1}{2}\varepsilon}\, x^{\mu}$$

$$= \psi(x)\sum_{k=1}^{\mu-1}\binom{\mu}{k}x^{\mu-k}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\}(1+x^2)^{\frac{1}{2}\varepsilon} + \psi(x)x^{\mu}(S-z)^{-1}(1+x^2)^{\frac{1}{2}\varepsilon} \tag{2.11}$$

A typical term of the above sum is

$$\psi(x)\; x^{\mu-k}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\}(1+x^2)^{\frac{1}{2}\varepsilon}$$

which is bounded whenever

$$\psi(x)x^{\mu-k}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\}x$$

is bounded. Since

$$\psi(x)x^{\mu-k}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\}$$

$$= \psi(x)x^{\mu-k+1}\,\mathrm{Ad}_{-x}^{k}\{(S-z)^{-1}\} + \psi(x)x^{\mu-k}\,\mathrm{Ad}_{-x}^{k+1}\{(S-z)^{-1}\},$$

and since $\psi(x)x^{\mu-k+1}(S-z)^{-1}$ and $(S-z)\,\mathrm{Ad}_{-x}^{k+1}\{(S-z)^{-1}\}$ are bounded for all $k \leq \mu - 1$, by Corollory 2.3 (iii) all the terms in the summation of (2.11) are finite. Now the remaining term in (2.11) is

$$\psi(x)x^{\mu}(S-Z)^{-1}(1+x^2)^{\frac{1}{2}\varepsilon}$$

$$= \psi(x)x^{\mu}(1+x^2)^{\frac{1}{2}\varepsilon}(S-z)^{-1} - \psi(x)x^{\mu}(S-z)^{-1}[S,(1+x^2)^{\frac{1}{2}\varepsilon}](S-z)^{-1}\ .$$

By assumption the first term and the factor $\psi(x)x^{\mu}(S-z)^{-1}$ of the second term of the above equality are bounded, while

$$[S,(1+x^2)^{\frac{1}{2}\varepsilon}](S-z)^{-1} = [H_o,(1+x^2)^{\frac{1}{2}\varepsilon}](S-z)^{-1}$$

$$= x(1+x^2)^{\frac{1}{2}\varepsilon-1} P(S-z)^{-1}+(1+x^2)^{\frac{1}{2}\varepsilon-1}+4(\tfrac{1}{2}\varepsilon-1)x^2(1+x^2)^{\frac{1}{2}\varepsilon-2}(S-z)^{-1}.$$

Therefore it is clearly bounded since $\varepsilon < 1$.

For the other result we have,

$$\psi(x)\phi(S)\rho(x)^{-1} = \psi(x)(S+i)^{-1}\rho(x)^{-1}\rho(x)(S+i)\phi(S)\rho(x)^{-1}.$$

By the previous result then the result follow if we show the boundedness of $\rho(x)\ (S+i)\phi(S)\rho(x)^{-1}$. Since $(\cdot + i)\phi(.) \in C_o^{\infty}(\mathbb{R})$, $\rho(x)(S+i)\phi(S)\rho(x)^{-1}$ is bounded by interpolation and Corollory 2.3 (ii), using the following identity when $\mu_1$ is the smallest integer greater than $\delta$.

$$(1+x^2)^{-\mu_1}\ \phi(S)x^{2\mu_1} = \sum_{k=1}^{2\mu_1} (1+x^2)^{-\mu_1}\ x^{(2\mu_1-k)}\ \mathrm{Ad}^k_{-x}\{\phi(S)\}.$$

(ii) The proof is similar to that in (i) except that we use, for $2\delta = \mu_1+\varepsilon$, the equality

$$x^{2\delta}\psi(x)(S-z)^{-1}\rho(x)$$

$$= \psi(x)\sum_{k=1}^{\mu-1}\binom{\mu}{k}\ \mathrm{Ad}^k_x\ \{(S-z)^{-1}\}x^{\mu-k}\ \rho(x)+\psi(x)x^{\varepsilon}(S-z)^{-1}\ x^{2\mu}\ \rho(x),$$

and Corollory 2.3 (iii).

(iii) When $\gamma \cap \alpha = \emptyset$ the result is trivial since $\rho(x^{\alpha})$ commutes with $\phi(T^{\gamma})$. Otherwise the proof is similar to that in (ii) when we note that $x^{\alpha}$ can be written in the coordinate system of $(x^{\gamma},x_{\gamma})$ as $x^{\alpha} = a(\gamma,\alpha)x^{\gamma} + b(\gamma,\alpha)x_{\gamma}$ and that

$$\mathrm{Ad}^k_{-x}\{\phi(T^{\gamma})\} = \{a(\gamma,\alpha)\}^k\ \mathrm{Ad}^k_{-x^{\gamma}}\ \{\phi(T^{\gamma})\}$$

is bounded by Corollory 2.3 (ii) for all integers $k > 0$.

(iv) We prove the boundedness of $A^2\phi(S)(1+x^2)^{-1}$ only. Then interpolation gives the boundedness of $A^\theta\,\phi(S)(1+x^2)^{-\theta}$ for any $0 \le \theta \le 2$. we have

$$A^2\,\phi(S)(1+x^2)^{-1} = A^2\,\phi(H_o)(1+x^2)^{-1} + A^2\{\phi(S)-\phi(H_o)\}(1+x^2)^{-1}.$$

We will prove the boundedness of the above two terms separately. Now

$$A^2\,\phi(H_o)(1+x^2)^{-1} = [A,[A,\phi(H_o)]](1+x^2)^{-1}+[A,\phi(H_o)]A(1+x^2)^{-1}+\phi(H_o)A^2(1+x^2)^{-1}$$

By the commutation properties of A and $H_o$ i.e. $[A,H_o] = iH_o$ we see that the factors $[A,[A,\phi(H_o)]]$ and $[A,\phi(H_o)]$ are again $C_o^\infty$ functions of $H_o$ so that writing out A and $A^2$ in terms of x and p and using the above equality we get that

$$||A^2\phi(H_o)(1+x^2)^{-1}|| \le K < \infty .$$

On the other hand we deal with the term $A^2\{\phi(S)-\phi(H_o)\}(1+x^2)^{-1}$ as follows. As in the proof of (ii) of Corollary 2.3 we have a function $\psi \in C_o^\infty((0,(\mu-a)^{-1}))$ associated to $\phi$ for some $\mu < \inf\sigma(S)$ and $a < \mu$ so that ,

$$A^2\{\phi(S)-\phi(H_o)\}(1+x^2)^{-1} = \int dt\,\hat{\psi}(t)A^2\{Z_t-Z_t^o\}(1+x^2)^{-1} \qquad (2.12)$$

where

$$Z_t = \exp(-it(S-a)^{-1}) \quad \text{and} \quad Z_t^o = \exp(-it(H_o-a)^{-1}).$$

Therefore recalling that $S = H(D)$ for some D, we have,

$$A^2\{Z_t-Z_t^o\}(1+x^2)^{-1} = \sum_{\gamma\in i(D)} i\int_0^t ds\,A^2\,Z_{t-s}^o(H_o-a)^{-1}\,W_\gamma(S-a)^{-1}\,Z_s(1+x^2)^{-1}.$$

Now

$$A^2\,Z_{t-s}^o(H_o-a)^{-1}\,W_\gamma(S-a)^{-1}\,Z_s(1+x^2)^{-1}$$

$$= \{A^2(H_o-a)^{-1}(1+x^2)^{-1}\}\{(1+x^2)Z_{t-s}^o(1+x^2)^{-1}\}\{(1+x^2)W_\gamma(S-a)^{-1}(1+x^2)^{-1}\}$$
$$\{(1+x^2)Z_s(1+x^2)^{-1}\} .$$

The first factor in the above term is bounded as before while the second and the fourth factors have the bounds $K(1+s^2)$ and $K(1+(t-s)^2)$ respectively by Corollory 2.3 (i) and the third factor is bounded by (ii). Hence

$$||A^2\{Z_t - Z_t^o\}(1+x^2)^{-1}|| \leq K \int_0^t ds\ (1+(t-s)^2)(1+s^2) \leq K(1+|t|^5).$$

Now the result follows since

$$||A^2\{\phi(S)-\phi(H_o)\}(1+x^2)^{-1}|| \leq K \int dt|\hat{\psi}(t)|(1+|t|^5) < \infty .$$

(iv) The proof is similar to that of (iii).

Next we have the following definitions and compactness results from [PSS].

<u>Definition 2.5</u> (Almost verbatim from [PSS]) We define $D \bigcup C$ for any two clusterings $D, C$, $1 \leq \# D, \# C \leq N$ by "Draw lines between each pair in $\{1,\ldots,N\}$ in a cluster of $D$ and then between each pair in a cluster of $C$. The connected components of $\{1,\ldots,N\}$ after this is done, form the clusters of $D \bigcup C$".

<u>Definition 2.6</u> Let $D$ be a clustering. Then a bounded operator is $D$-compact if

(i) $B$ commutes with $\exp(-ia\ P_D)\ \forall\ a \in X_D$ (in which case $B \equiv \{B(p_o)\}$ in the spectral representation of $P_D$ with $B(p_D)$ a bounded operator on $L^2(X^D)$ for almost every $p_D$).

(ii) $p_D \to B(p_D)$ is norm continuous and $||B(p_D)||_{L^2(X^D)} \to 0$ as $|p_D| \to \infty$ and

(iii) $B(p_D)$ is compact for each $p_D$.

Remark 2.7 We note that if $\alpha$ is any pair then $W_\alpha(H_o+1)^{-1}$ is $\alpha$-compact. Because (i) and (iii) are clear while for (ii) we write

$\{W_\alpha(H_o+1)^{-1}\}(t_\alpha) = \{W_\alpha(T^\alpha+1)^{-1}\}\{(T^\alpha+1)(T^\alpha+ t_\alpha+1)^{-1}\}$. Now $(T^\alpha+1)(T^\alpha+t_\alpha+1)^{-1}$ is continuous in $t_\alpha$, is defined for all $f \in L^2(X^\alpha)$ and goes to zero strongly as $t_\alpha \to 0$. By the compactness of $W_\alpha(T^\alpha+1)^{-1}$, (ii) now follows.

Proposition 2.8 Let the bounded operators $B_1$, $B_2$ be D and C compact respectively for two clusterings D and C. Then their product $B_1B_2$ is $D \bigcup C$ compact.

We recall the definition of $\mathcal{E}$ lim given in the introduction for the following:

Proposition 2.9 (Wiener) Let $V_t$ be a strongly continuous one parameter group of unitary operators, with generator B, on a Hilbert space $\mathcal{H}$, Then for any compact operator C,

$$\mathcal{E} \lim_{t \to \infty} ||C\, V_t\, f|| = 0$$

whenever $f \in \mathcal{H}_c(B)$.

For a proof of the above proposition we refer to [RS III].

Next we give a widely known very useful decay result on the free evolution of a N-particle system. In fact it is here that the restriction on the dimension $\nu \geq 3$, enters and the restriction is so that a large class of zero energy free states have time decay faster than the inverse power asymptotically.

Proposition 2.10 We consider $H_o$ for $2 \leq N < \infty$, $\nu \geq 3$, the pairs $\gamma, \beta$ so that $\gamma \cap \beta = \emptyset$ and set $\rho(\lambda) = (1+\lambda^2)^{-\delta}$, $\delta > \frac{1}{2}$. Then for some $\mu_1$, $1 < \mu_1 < 2\delta$,

(i) $||\rho(x^\gamma)\exp(-itH_o)\ \rho(x^\beta)|| \leq K(1+|t|)^{-\mu_1}$

and

(ii) for any clustering D, with $\gamma \in e(D)$,

$$||\rho(x^\gamma)\exp(-it\ T_D)\rho(x_D)|| \leq K(1+|t|)^{-\mu_1}$$

The proof is standard and uses the $L^1$ to $L^\infty$ decay of $\exp(-itH_o)$ in t and the translation invariance of the Lebesgue measure along with interpolation, so we omit it. However one can see [Lemma 16.3, AJS].

If the free states propagate with non zero energy then corresponding to regions expanding with non-overlapping velocities there is arbitrary decay. This is also true in some directions of phase space as we will see in the following Theorem.

Theorem 2.11 We consider $H_o$ for $2 \leq N < \infty$ and $\phi \in C_o^\infty(\mathbb{R})$ with positive constants a,b and c.

(i) If $\frac{1}{2}b^2 = \sup\ \mathrm{supp}\,\phi$, $b+c < a$, then for $|s| \leq |t|$,

$$||F(|x| > a|t|)\ U_s\ \phi(H_o)F(|x| \leq c|t|)|| \leq K_M(1+|t|)^{-M}$$

(ii) If $\frac{1}{2}b^2 = \inf\ \mathrm{supp}\,\phi$, $c+a < b$ then

$$||F(|x| \leq a|t|)\ U_t\ \phi(H_o)F(|x| \leq c|t|)|| \leq K_M(1+|t|)^{-M}$$

for any positive integer M.

(iii) Let supp $\phi$ be contained in $(0,\infty)$ and let $0 < c <$ inf supp $\phi$. Then

$$\lim_{t\to\pm\infty} ||F(|x| \leq c|t|)\, \phi(H_o)U_t\, F(A \gtrless 0)|| = 0.$$

(iv) $$\operatorname*{s\,lim}_{t\to\pm\infty} F(A \lessgtr 0)U_t f = 0.$$

<u>Proof</u> We give a heuristic argument for (i), a rigorous proof using stationary phase can be found in [Da] or [Mu]. We have

$$U_t\, F(|x| \leq c|t|)U_t^* = F(|x-Ps| < c|t|).$$

Thus the lower bound to the gap between the regions $(|x'| > a|t|)$ and $(|x| \leq c|t|)$ after evolution is

$$|x'-(x-Ps)| \geq |x'| - (|x| + |Ps|) \geq [a-(b+c)]\,|t|,$$

since $|s| \leq |t|$. So classically we expect the regions to be disjoint after evolution. However the spreading of the wave packets gives only the stated decay. The case (ii) is similar.

(iii) The idea of the proof is to write $F(A \lessgtr 0)\phi(H_o)U_t^*$ in the diagonal representation of A using the phase function

$$I(\lambda,p,x) = e^{i(t\frac{1}{2}p^2-p.x-\lambda \log |p|)}$$

and note that

$$\left|\frac{\partial}{\partial p} I(\lambda,x,p)\right| \geq |p|t - \hat{p}.x - \lambda|p|^{-1} \geq t|p| - |x|$$
$$\geq (b-c)\,|t|$$

for $\lambda \lessgtr 0$ and $t \gtrless 0$ to use stationary phase method and obtain the stated decay. See Perry [P2] for a rigorous proof using Mellin transforms.

(iv) This follows easily from (iii) by taking an $f$ with $\phi(H_o)f = f$, $\phi \in C_o^\infty((0,\infty))$ and writing

$$F(A \lessgtr 0)U_t f$$

$$= F(A \lessgtr 0)\phi(H_o)U_t\, F(|x| > c|t|)f + F(A \lessgtr 0)\phi(H_o)U_t\, F(|x| \leq c|t|)f.$$

and then using the density of $\{\phi(H_o)f : \phi \in C_o^\infty((0,\infty)),\ f \in L^2(X)\}$ to conclude the result.

We are now ready to prove the existence of wave operators and all that in the next theorem. For this theorem we set $V_t(D) \equiv U_t$ and $E^D \equiv 1$ for $\# D = N$.

Theorem 2.12 For all D such that $2 \leq \# D \leq N$, the following hold true.

(i) $\tilde{\Omega}^\pm(D) = \underset{t\to\pm\infty}{\text{s lim}}\, V_t^* V_t(D)$ exist on $L^2(X)$ and

$$\Omega^\pm(D) = \underset{t\to\pm\infty}{\text{s lim}}\, V_t^* V_t(D)E^D \equiv \tilde{\Omega}^\pm(D)E^D ,$$

(ii) For any $\phi \in L^\infty(\mathbb{R})$ and $B(D)$ = one of $\tilde{\Omega}^\pm(D)$, $\Omega^\pm(D)$,

$$\phi(H)B(D) = B(D)\phi(H(D)),\ B^*(D)\phi(H) = \phi(H(D))B^*(D).$$

(iii) If $F_\pm(D)$ denotes the range of $\Omega^\pm(D)$, then for all $D \neq C$,

$$F_\pm(D) \perp F_\pm(C).$$

Henceforth we set $\Omega^\pm(0) \equiv \Omega^\pm(D)$, if $\# D = N$ and note that $\tilde{\Omega}^\pm(0) = \Omega^\pm($

Proof. (i) Let $\# D < N$. We give a proof by Cook's method and show that $V_t^* V_t(D)f$ is Cauchy in t for f in a total set in $L^2(X)$. Therefore we consider a vector f of the form $f = (H(D)+i)^{-1} f^D g_D$ with $f^D \in S(X^D)$ and $g_D \in S(X_D)$. Then for $s \leq t$,

$$||V_t^* V_t(D)f - V_s^* V_s(D)f|| = || \int_s^t d\tau \, V_\tau^* \sum_{\gamma \in e(D)} W_\gamma \, V_\tau(D)f||$$

$$\leq \sum_{\gamma \in e(D)} \int_s^t d\tau ||W_\gamma (H(D)+i)^{-1} \rho_1(x^\gamma)^{-1}|| \; ||\rho_1(x^\gamma) V_\tau(D) f^D g_D||$$

Thus we have using Lemma 2.4 (i),

$$\lim_{s,t \to \infty} ||\{V_t^* \, V_t(D) - V_s^* \, V_s(D)\}f||$$

$$\leq \lim_{s,t \to \infty} \sum_{\gamma \in e(D)} K_\gamma \int_s^t d\tau \; ||\rho_1(x^\gamma) U_{\tau,D} \rho_1(x_D)|| \; ||V_\tau^D f^D|| \, ||\rho_1(x_D)^{-1} \ldots$$

Since $||\rho_1(x_D)^{-1} g_D||$ is finite, the result now follows from Theorem 2.10. If # D = N, the same argument goes through by taking $f = (H_o+1)^{-1} g$ with $g \in \mathcal{S}(X)$.

(ii) We prove that $V_t \, \tilde{\Omega}^\pm(D) = \tilde{\Omega}^\pm(D) V_t(D)$ for all $t \in \mathbb{R}$. The result then follows by functional calculus. So we have for $f \in L^2(X)$ and a fixed $t$ (omitting the indices $\pm$ in the following),

$$V_t \, \tilde{\Omega}(D) f = \lim_{s \to \infty} V_t \, V_s^* \, V_s(D) f = \lim_{s \to \infty} V_{s-t}^* \, V_{s-t}(D) \; V_t(D) f$$

$$= \lim_{(s-t) \to \infty} V_{s-t}^* \, V_{s-t}(D) \; V_t(D) f = \tilde{\Omega}(D) \; V_t(D) f.$$

Since f was arbitrary the operator equality results.

(iii) When # D ≠ N, by density arguments and Propositions 1.9 and 1.10 it suffices to consider the case when $E^D$ and $E^C$ are rank one projections. In that case if $\lambda^D$ and $\lambda^C$ are the corresponding eigenvalu[es] of $H^D$ and $H^C$ respectively, we have (again omitting the indices ±),

$$|\langle \Omega(D)f, \Omega(C)g\rangle| = \lim_{t\to\infty} |\langle V_t^* V_t(D)E^D f, V_t^* V_t(C)E^C g \rangle|$$

$$= \lim_{t\to\infty} |\langle f, \exp\{it\,(\lambda^D + T_D - \lambda^C - T_c)\}g\rangle|$$

$$= \lim_{t\to\infty} |\langle f, \exp\{it(T_D-T_c)\}g \rangle| = 0. \tag{2.13}$$

Since $(T_D-T_C)$ has only absolutely continuous spectrum when one of D,C has N elements the proof is again similar.

To be able to describe the evolution of A and $x^2$ under $V_t$ we need the following three technical results. All the three are from [MS] with minor modifications, hence we do not prove any of these. We denote for the next Proposition, by $C_{oo}(\mathbb{R}^k)\{C_o(\mathbb{R}^k)\}[C_b(\mathbb{R}^k)]$ the space of continuous functions, of compact support {that vanish at $\infty$} [which are bounded] respectively.

<u>Proposition 2.13</u> Fix $k \geq 1$. Suppose for each $s \in \mathbb{R}^+$, $\underline{L}_s = (L_s^1,\dots,L_s^k)$ and $\underline{L} = (L^1,\dots,L^k)$ be two families of commuting self-adjoint operators on a Hilbert space $H$ and let $f \in H$. Then, the following conditions are equivalent.

(i) $E \lim_{s\to\infty} \|\exp(-i\underline{u}\cdot\underline{L}_s)f - \exp(-i\,\underline{u}\cdot\underline{L})\| = 0, \quad \forall\, u \quad \mathbb{R}^k.$

(ii) $E \lim_{s\to\infty} \|\{\phi(\underline{L}_s) - \phi(\underline{L})\}f\| = 0,$ when $\phi$ is in any of $S(\mathbb{R}^k)$, $C_{oo}(\mathbb{R}^k)$, $C_o(\mathbb{R}^k)$ and $C_b(\mathbb{R}^k)$.

<u>Proposition 2.14</u> Let $f \in H$. Whenever $L, L_s \geq 0$ on $H$,

(i) $E \lim_{s \to \infty} ||\{\exp(-it L_s) - \exp(-it L)\}f|| = 0, \quad \forall t \in \mathbb{R}$. if and only if

$E \lim_{s \to \infty} ||\{\exp(-t L_s) - \exp(-t L)\}f|| = 0, \quad \forall t \geq 0$

and for self adjoint $L_s$.

$E \lim_{s \to \infty} ||\{\exp(-it L_s)-1\}f|| = 0 \quad \forall t \in \mathbb{R}$ if and only if

$E \lim_{s \to \infty} ||\phi(L_s)f|| = 0$, for every continuous bounded function $\phi$ vanishing in a neighbourhood of zero.

We set $W \equiv \sum_\gamma W_\gamma$, $B \equiv -(W + i[A,W]) \equiv \sum_\gamma B^\gamma$ where $B^\gamma \equiv -(W_\gamma + i[A^\gamma, W_\gamma]$ and note that $(H_o+1)^{-1} B (H_o+1)^{-1}$ is bounded.

<u>Proposition 2.15</u> Let $f,g \in S(X)$. Then,

(i) $\langle V_t f, Ag\rangle - \langle Af, V_t^* g\rangle - t\langle V_t H f, g\rangle = \int_0^t ds \ \langle B V_s f, V_{s-t} g\rangle$ .

(ii) $\langle V_t f, x^2 g\rangle - \langle x^2 f, V_t^* g\rangle - 4t \langle Af, V_t^* g\rangle$

$$- 2t^2 \langle V_t Hf, g\rangle = 4 \int_0^t ds \int_0^s d\tau \langle B V_\tau f, V_{\tau-t} g\rangle .$$

(iii) $A U_t f = U_t A f + t U_t H_o f$.

(iv) $x^2 U_t f = U_t x^2 f + 4t A U_t f - 2t^2 U_t H_o f$

and

(v) $\langle V_t f, x^2 g\rangle - \langle x^2 f, V_t^* g\rangle - 4t \langle V_t f, Ag\rangle + 2t^2 \langle V_t f, H_o g\rangle$

$$= -4 \int_0^t ds \ s \ \langle B V_s f, V_{s-t} g\rangle - 2t^2 \langle W V_t f, g\rangle$$

Remark  If we have a function $\phi \in C_o^\infty(\mathbb{R}^\nu)$, then $\phi(x^\gamma)V_t f$ can not be expected to go to zero asymptotically in any sense if f is in the range of $\Omega^\pm(\gamma)$. Because, then $V_t f$ would behave like $V_t(\gamma)g$ for some g in $R(\Omega^\pm(\gamma)^*)$ asymptotically and this range in particular contains bound states of $H^\gamma$, which always stay in bounded regions of $X^\gamma$ under evolution.

In view of this remark we can only expect the scattering states to leave, if at all, any region bounded in all pair directions only if they are orthogonal to all the cluster wave operators. Though on the ranges of $\Omega^\pm(0)$ the following condition is valid, we remove even these in the formulation of local decay for later convenience.

Since $\Sigma^\oplus F_\pm(D)$ is a subspace of $\mathcal{H}_{ac}(H)$ by Theorem 2.12, we define

$$\mathcal{H}^\pm = \mathcal{H}_c(H) \ominus \{ \sum_{D:\#D\neq 1}^\oplus F_\pm(D)\}$$

and note that $\mathcal{H}^\pm$ is a closed subspace of $\mathcal{H}_c(H)$.

Local Decay Condition : (LD) Let $f \in \mathcal{H}^\#$, then for all pairs $\gamma$ and any $\phi \in C_o(\mathbb{R}^\nu)$,

$$E \lim_{t\to\pm\infty} ||\phi(x^\gamma)V_t f|| = 0.$$

The behaviour of some observables is dealt in the following Theorem which is found to be very useful in showing completeness.

Theorem 2.16  Let $f \in \mathcal{H}^\pm$ and suppose the condition (LD) is satisfied. Then,

(i) $$E \lim_{t\to\pm\infty} ||(V_t^* Y_{u/t} V_t - V_u)f|| = 0, \quad \forall \; u \in \mathbb{R}.$$

(ii) $$E \lim_{t\to\pm\infty} ||V_t^* \psi(\tfrac{A}{t})V_t - \psi(H)f|| = 0, \quad \forall \; \psi \in C_o^\infty(\mathbb{R}).$$

(iii) $\mathcal{E}\lim_{t\to\pm\infty} ||\{V_t^* U_t \exp(-iu \frac{A}{t})U_t^* V_t\}f-f|| = 0, \quad \forall u \in \mathbb{R}.$

(iv) $\mathcal{E}\lim_{t\to\pm\infty} ||\{V_t^* U_t \exp(-u \frac{x^2}{t^2})U_t^* V_t\}f-f|| = 0, \quad \forall u \geq 0.$

(v) $\mathcal{E}\lim_{t\to\pm\infty} ||\{V_t^* U_t \psi(\left|\frac{x}{t}\right|)U_t^* V_t - \psi(0)\}f|| = 0, \quad \forall \psi \in C_b(\mathbb{R}^+)$ .

(vi) $\mathcal{E}\lim_{t\to\pm\infty} ||F(|x| > a|t|)U_t^* V_t f|| = 0 , \quad \forall a > 0$

The proof of this theorem follows very closely that in [MS]. Before we start proving this theorem a quick technical Lemma is in order. We observe that $\mathcal{D}(H) \cap \mathcal{H}^{\pm}$ is dense in $\mathcal{H}^{\pm}$ and take B as in Proposition 2.15.

<u>Lemma 2.17</u> Let $G_u = \exp(-u\, x^2)$ for $u \geq 0$ and let $f \in \mathcal{D}(H) \cap \mathcal{H}^{\pm}$.

(i) If the condition (LD) is satisfied then

$$\mathcal{E}\lim_{t\to\pm\infty} ||(H_o+1)^{-1}\, B\, V_t f|| = 0$$

(ii) $\mathcal{E}\lim_{t\to\pm\infty} \int_0^u ds\, t^{-1} \int_0^t d\tau ||(H_o+1)^{-1}\, B\, V_\tau\, Y_{s/t}\, f|| = 0.$

(iii) $\text{Max}\, \{||[(H_o+1)^{-1}\, G_u(H_o+1)-1]f||,\ ||[(H_o+1)G_u(H_o+1)^{-1} -1]f||\}$

$$\leq K\{(\sqrt{u}+u)||f|| + ||(G_u-1)f||\}, \quad \forall u \geq 0.$$

(iv) $\mathcal{E}\lim_{t\to\pm\infty} \int_0^u ds\, t^{-2} \int_0^t \tau d\tau ||(H_o+1)^{-1}\, B\, V_\tau\, G_{st^{-2}}\, f|| = 0, \quad \forall u \geq 0$ .

<u>Proof</u>. (i) We have,

$$||(H_o+1)^{-1}\, B V_\tau\, f|| \leq \sum_\gamma ||(H_o+1)^{-1}\, B^\gamma\, V_\tau f||$$

$$\leq \sum_\gamma \{2||(H_o+1)^{-1}\, B^\gamma (H_o+1)^{-1}\, V_\tau (H+i) f|| + \sum_\beta ||(H_o+1)^{-1} B^\gamma (H_o+1)^{-1}|| \, || W_\beta V_\tau f||\} \tag{2.14}$$

Now take a family $\phi_R \in C_o^\infty(\mathbb{R})$, $0 \leq \phi_R \leq 1$, supp $\phi_R$ contained in $(|x^\gamma| \leq R)$ and $\phi_R \equiv 1$ on $(|x^\gamma| \leq \frac{1}{2} R)$. Then by (2.14),

$$||(H_o+1)^{-1} B V_\tau f||$$

$$\leq \sum_\gamma \{2||(H_o+1)^{-1} B^\gamma (H_o+1)^{-1} (1-\phi_R(x\ ))V_\tau (H+i)f||$$

$$+ ||(H_o+1)^{-1} B^\gamma (H_o+1)^{-1}|| [||\phi_R(x^\gamma)V_\tau f|| + \sum_\beta ||W_\beta (H+i)^{-1} \rho_1(x^\beta)^{-1}||$$

$$||\rho_1(x^\beta)V_\tau (H+i)f||]\} . \qquad (2.15)$$

Using the ass'mption (A3) on the potentials, the condition (LD), Lemma 2.4 (i) and (2.15) we get the result by an $\frac{\varepsilon}{3}$ argument.

(ii) We have

$$\int_0^{|u|} ds\ t^{-1} \int_0^t d\tau ||(H_o+1)^{-1} B\ V_\tau\ Y_{s/t}\ f||$$

$$\leq \int_0^{|u|} ds\ t^{-1} \int_0^t d\tau \{||(H_o+1)^{-1} B(H+i)^{-1} V_\tau (H+i)f||$$

$$+ K||(H_o+1)^{-1} B(H+i)^{-1}||\ ||(H_o+1)(Y_{s/t}-1)f||\} \qquad (2.16)$$

Now (2.16) and the estimate

$$||(H_o+1)(Y_\theta-1)g|| \leq ||(e^\theta-1)H_o g|| + ||(Y_\theta^*-1)H_o g|| + ||(Y_\theta-1)g||,\ \forall\ \theta \in \mathbb{R} \qquad (2.17)$$

for $g \in S(X)$ and density arguments yield the result.

(iii) By the commutation relations between P and x we have,

$$\exp(-ux^2)H_o = H_o \exp(-ux^2) - 2\, iu\, P.x \exp(-ux^2)$$
$$+ u \exp(-ux^2) - 2u^2x^2 \exp(-ux^2) \qquad (2.18)$$

Then by the triangle inequality,

$$||\{(H_o+1)^{-1} \exp(-ux^2)(H_o+1)-1\}f||$$
$$\leq ||\{(H_o+1)^{-1} G_u(H_o+1) - G_u\}f|| + ||(G_u-1)f|| \qquad (2.19)$$

Now using (2.18) in (2.19) we obtain the inequality,

$$||\{(H_o+1)^{-1} G_u(H_o+1) - G_u\}f||$$
$$\leq \{2u||(H_o+1)^{-1} P\cdot x\, G_u|| + u||G_u|| + 2u^2||x^2G_u||\}||f||;$$

which together with the observation that

$$||\;|x|G_u|| \leq K u^{-\frac{1}{2}} \text{ and that } ||x^2 G_u|| \leq K u^{-1}$$

leads to the required result. Similarly the same estimate follows for the term $||\{(H_o+1)G_u(H_o+1)^{-1}-1\}f||$.

(iv) Upon observing that

$$||(H_o+1)^{-1} B V_\tau G_{st^{-2}}f||$$
$$\leq ||(H_o+1)^{-1} B V_\tau f|| + K||(H_o+1)^{-1} B(H+i)^{-1}||\;||(H_o+1)(G_{st^{-2}}-1)f||$$

the result comes from (i) and (iii).

<u>Proof of Theorem 2.16</u> We note that (ii) comes from (i) and Proposition 2.13 and (v) together imply (vii) so we prove only (i), (iii), (iv) and (v).

(i) Since we are considering a sequence of contractions with a unitary limit, we need only to prove the weak convergence on a total set and by similarity we prove only the + case. Thus we consider only

$$||(H+i)^{-1}(V_t^* Y_{u/t} V_t - V_u)f||$$

for $t \geq 1$. Again by density we take $f \in \mathcal{D}(H) \cap \mathcal{H}^{\pm}$ then by the triangle inequality,

$$\mathcal{E} \lim_{t \to \infty} ||(H+i)^{-1}(V_t^* Y_{u/t} V_t - V_u)f||$$

$$\leq \mathcal{E} \lim_{t \to \infty} \{||(H+i)^{-1}(H_o+1)|| \, ||(H_o+1)^{-1} Y_{u/t}(H_o+1)^{-1}\}$$

$$\{||(H_o+1)^{-1}(V_t - Y^*_{u/t} V_{u+t} Y_{u/t})f|| \tag{2.20}$$

By the action of $Y_\theta$ on $H_o$ it is easy to see that the first two factors are uniformly bounded in t. Now for $f,g \in \mathcal{S}(X)$ since $Y_{s/t}$ leaves $\mathcal{S}(X)$ invariant, and since $s \to H\,Y_{s/t}$ is strongly continuous owing to the relation $H_o Y_\theta = e^\theta Y_\theta H_o f$, we have ,

$$\langle V_t f - Y^*_{u/t} V_{u+t} Y_{u/t} f, g\rangle$$

$$= -\int_0^u ds \frac{d}{ds} \langle V_{s+t} Y_{s/t} f, Y_{s/t} g\rangle$$

$$= -i\int_0^u ds\{\langle (H+At^{-1})Y_{s/t} f, V^*_{s+t} Y_{s/t} f\rangle - \langle V_{s+t} Y_{s/t} f, At^{-1} Y_{s/t} g\rangle\} \tag{2.21}$$

Applying Proposition 2.15 now to (2.21) we obtain the identity,

$$\langle V_t f - Y^*_{u/t} V_{u+t} Y_{u/t} f, g\rangle$$

$$= i\int_0^u ds[t^{-1}\{\int_0^{t+s} d\tau \langle BV_\tau Y_{s/t} f, V_{\tau-s-t} Y_{s/t} g\rangle\} + st^{-1}\langle V_{s+t} H Y_{s/t} f, Y_{s/t} g\rangle] \tag{2.22}$$

which can be extended as a quadratic form to $\mathcal{D}(H_o) \bigcap H^+$ by the boundedness of $(H_o+1)^{-1} B(H_o+1)^{-1}$ on $S(X)$ and the estimate

$$||\{(H_o+1)Y_\theta(H_o+1)^{-1}-1\}f|| \leq K\{|(e^{-\theta}-1)|\ ||f|| + ||(Y_\theta^*-1)f||\},\ \forall\ \theta \in \mathbb{R} \tag{2.23}$$

Then we have the following estimate,

$$||(H_o+1)^{-1}(V_t f - Y_{u/t}^* V_{u+t} Y_{u/t})f||$$

$$\leq K t^{-1}\int_0^u ds\{s||f|| + \int_0^{t+s} d\tau||(H_o+1)^{-1} B V_\tau Y_{s/t} f||\} \tag{2.24}$$

from which the result follows using Lemma 2.17 (ii).

(iii) Using Proposition 2.15)(iii) and (v) we obtain for $f,g \in S(X)$ and $t \geq 1$,

$$<U_t^* V_t f - Y_{u/t}^* U_t^* V_t Y_{u/t} f, g>$$

$$= -\int_0^u ds \frac{d}{ds} <U_t^* V_t Y_{s/t} f, Y_{s/t} g>$$

$$= it^{-1}\int_0^u ds\{<V_t Y_{s/t} f, U_t A Y_{s/t} g> - <V_t A Y_{s/t} f, U_t Y_{s/t} g>\}$$

$$= it^{-1}\int_0^u ds\{\int_0^t d\tau <BV_\tau Y_{s/t} f, V_{\tau-t} U_t Y_{s/t} g>$$

$$+ t<(H-H_o)V_t Y_{s/t} f, U_t Y_{s/t} g>\} \tag{2.25}$$

As in the proof of (i) the above identity extends as a quadratic form on $\mathcal{D}(H_o)$ and one gets, by the estimate (2.23),

$$||(H_o+1)^{-1}(U_t^* V_t f - Y^*_{u/t}\ U_t^*\ V_t f)||$$

$$\leq\ K\ t^{-1}\int_0^{|u|} ds\{\int_0^t d\tau ||(H_o+1)^{-1}\ B\ V_\tau\ Y_{s/t}f|| + t||(H_o+1)^{-1} W\ V_t\ Y_{s/t}f||\}.$$

Hence for $f \in \mathcal{D}(H_o) \bigcap \mathcal{H}^+$,

$$||(H+i)^{-1}(V_t^*\ U_t\ Y_{u/t}\ U_t^*\ V_t f-f)||$$

$$\leq ||(H+i)^{-1}(Y_{u/t}f-f)||$$

$$+ K||(H_o+1)^{-1}Y_{u/t}(H_o+1)||\int_0^{|u|} ds\{t^{-1}\int_0^t d\tau||(H_o+1)^{-1}BV_\tau Y_{s/t}f||$$

$$+ ||(H_o+1)^{-1}\ W\ V_t\ Y_{s/t}f||\},$$

from which the result follows by density as in the proof of (i) using Lemma 2.17.

(iv) Applying Theorem 2.15 (iv) and (v) successively, we get, for $f,g \in S(X)$ and $t \geq 1$, that

$$\langle G_{ut^{-2}}\ U_t^*\ V_t f - U_t^* V_t\ G_{ut^{-2}} f, g\rangle$$

$$= \int_0^u ds \frac{d}{ds} \langle U_t^*\ V_t\ G_{(u-s)t^{-2}}\ f,\ G_{st^{-2}}\ g\rangle$$

$$= t^{-2}\int_0^u ds\ \{\langle V_t x^2\ G_{(u-s)t^{-2}}\ f,\ U_t\ G_{st^{-2}}\ g\rangle$$

$$- \langle V_t\ G_{(u-s)t^{-2}}\ f,\ U_t\ x^2\ G_{st^{-2}}\ g\rangle\}$$

$$= t^{-2}\int_0^u ds\{\langle x^2\ G_{(u-s)t^{-2}}\ f,\ V_t^*\ U_t\ G_{st^{-2}}\ g\rangle$$

$$- \langle V_t\ G_{(u-s)t^{-2}}\ f,\ x^2\ U_t\ G_{st^{-2}}\ g\rangle$$

$$+ 4t\langle V_t G_{(u-s)t^{-2}} f, AU_t G_{st^{-2}} g\rangle - 2t^2 \langle V_t G_{(u-s)t^{-2}} f, H_o U_t G_{st^{-2}} g\rangle\}$$

$$= 4t^{-2} \int_0^u ds \int_0^t \tau \, d\tau \langle B V_\tau G_{(u-s)t^{-2}} f, V_{\tau-t} U_t G_{st^{-2}} g\rangle$$

$$+ 2\int_0^u ds \langle W V_t G_{(u-s)t^{-2}} f, U_t G_{st^{-2}} g\rangle .$$

Extending the above identity as a quadratic form to $\mathcal{D}(H_o)$ and using Lemma 2.17 (iii), we have the following estimate for $f \in \mathcal{D}(H) \bigcap H^+$.

$$\|(H+i)^{-1} (V_t^* U_t G_{ut^{-2}} U_t^* V_t f - f)\|$$

$$\leq K\|(H_o+1)^{-1}(G_{ut^{-2}} U_t^* V_t f - U_t^* V_t G_{ut^{-2}} f)\| + \|(G_{ut^{-2}} - 1)f\|$$

$$\leq 4 K_1 t^{-2} \int_0^u ds \int_0^t \tau d\tau \|(H_o+1)^{-1} B V_\tau G_{(u-s)t^{-2}} f\|$$

$$+ 2 K_1 \int_0^u ds \|(H_o+1)^{-1} W V_t G_{(u-s)t^{-2}} f\| + \|(G_{ut^2} - 1)f\|.$$

The result now follows from Lemma 2.17 (iii) and (iv) and density arguments.

(v) Since $\lambda \to \lambda^2$ is an isomorphism of $[0,\infty)$, it is enough to prove the result for $\phi(x^2/t^2)$ which follows by a direct application of Proposition 2.14 (ii), (i) and Proposition 2.13 (i), (ii) in that order.

## § 2. N-Particle completeness

With one more result we will be ready to prove N-particle completeness under the stated assumptions.

Lemma 2.18 (i) For each pair $\gamma$, let $\phi_\gamma \in C_o^\infty(\mathbb{R})$, with $0 \leq \phi_\gamma \leq 1$ and $\inf \operatorname{supp} \phi_\gamma = \frac{1}{2} b_\gamma^2$ and $a < b \equiv \inf_\gamma b_\gamma$. Then

$$\lim_{t\to\infty} \|(\tilde{\Omega}^{\pm}(0)-1) \prod_{\gamma} \phi_\gamma(T^\gamma)U_t\, F(|x| \leq a|t|)\| = 0 .$$

Let $\phi \in C_0^\infty(\mathbb{R})$. Whenever the condition (LD) is satisfied, we have the following for every $f \in H^{\pm}$.

(ii) $$E \lim_{t\to\pm\infty} \|\{\phi(H)-\phi(H(D))\}V_t f\| = 0, \quad 2 \leq \# D \leq N .$$

(iii) $$E \lim_{t\to\pm\infty} \|\{\phi(H^D)-\phi(T^D)\}V_t f\| = 0; \quad 2 \leq \# D < N.$$

Proof: (+ case only). Since

$$(\Omega^+(0)-1) \prod_{\gamma} \phi_\gamma(T^\gamma)F(|x| \leq a|t|)$$

$$= \sum_{\alpha} \int_0^\infty ds\, V_s^* \, iW_\alpha\, U_{t+s} \prod_{\gamma} \phi_\gamma(T^\gamma)F(|x| \leq a|t|) \equiv \sum_{\alpha} \int_0^\infty I_\alpha(s,t),$$

it suffices to show for each $\alpha$, that $\int_0^\infty ds\|I_\alpha(s,t)\| \to 0$ as $t \to \infty$. Owing to the inequality

$$F(|x| \leq a|t|) \leq F(|x^\beta| \leq a|t|)$$

for any $\beta$, we have, for $\rho_1$ as in (A3),

$$\|I_\alpha(s,t)\| \leq \|W_\alpha \prod_{\gamma} \phi_\gamma(T^\gamma)\, U_{t+s}\, F(|x^\alpha| \leq a|t|)\|$$

$$\leq \|W_\alpha(T^\alpha+1)^{-1} \rho_1(x^\alpha)^{-1}\| \, \|\rho_1(x^\alpha) \prod_{\gamma\neq\alpha} \phi_\gamma(T^\gamma)\rho_1(x^\alpha)^{-1}\|$$

$$\cdot\|\rho_1(x^\alpha)(T^\alpha+1)\phi_\alpha(T^\alpha)U_{t+s}\, F(|x^\alpha| \leq a|t|)\| \quad (2.26)$$

Now by the condition on the potentials and Lemma 2.4 (i), (iii), the first two factors of (2.26) are bounded, the third factor of (2.26) is dominated by

$$||\rho_1(x^\alpha)F(|x^\alpha| > c|t+s|)|| \; ||(T^\alpha+1)\phi(T^\alpha)||$$

$$+ \; ||F(|x^\alpha| \leq c|t+s|)(T^\alpha+1)\phi(T^\alpha) \; U^\alpha_{t+s} \; F(|x^\alpha| \leq a|t+s|)||$$

$$\leq \quad K(1+|t+s|)^{-\delta_1} .$$

Since $\delta_1 > 1$ the result easily follows.

(ii) By a standard argument using the Stone-Weirstress theorem, [PSS, SB1], it is enough to show for all integers $n > 0$ and $z \notin \sigma(H)$ that

$$E \lim_{t \to \pm\infty} ||\{(H-z)^{-n} - (H_0-z)^{-n}\}V_t f|| \; = \; 0$$

which follows from showing that

$$E \lim_{t \to \pm\infty} ||\{(H_0-z)^{-1} - (H-z)^{-1}\}V_t \; f|| \; = \; 0 \; .$$

This is clear by the estimate

$$||\{(H_0-z)^{-1}-(H-z)^{-1}\}V_t f|| \; \leq \; \sum_\gamma ||\{(H_0-z)^{-1} \; W_\gamma(H-z)^{-1}\rho_1(x^\gamma)^{-1}|| \; ||\rho_1(x^\gamma)V_t f||$$

Lemma 2.4 (i) and the local decay assumption (LD).

(iii) The proof is as in the earlier one because,

$$(H^D+i)^{-1} - (T^D+i)^{-1} = - \sum_{\gamma \in i(D)} (H^D+i)^{-1} \; W_\gamma(T^D+i)^{-1}.$$

<u>Low Energy Decay Condition (LED)</u>. There exists a set $\mathcal{D}^\pm$ of vectors dense in $\mathcal{H}^\pm$ such that for each $f \in \mathcal{D}^\pm$, there exist constants $b^\pm(f)$ so that for each pair $\gamma$ and $0 < b_\gamma \leq b^\pm(f)$,

$$E \lim_{t \to \pm\infty} ||F(T^\gamma < \tfrac{1}{2} b_\gamma^2)V_t f|| \; = \; 0.$$

Then the completeness result is the following.

Theorem 2.19 Consider a N-particle system with the pair potentials satisfying (A1) - (A3) and further let the N-particle evolution satisfy the conditions (LD) and (LED). Then the scattering is complete, that is

$$\sum_{D:\#D\neq 1}^{\oplus} F_+(D) = \mathcal{H}_c(H) = \sum_{D:\#D\neq 1}^{\oplus} F_-(D) ,$$

in particular $\mathcal{H}_c(H) = \mathcal{H}_{ac}(H)$.

Proof. By the intertwining relations we have that $F_\pm(D)$ is contained in $\mathcal{H}_{ac}(H)$, for all $D : \# D \geq 2$. Thus $\sum_{D:\#D\geq 2} F_\pm(D)$ is contained in $\mathcal{H}_{ac}(H)$.

We shall show that $\mathcal{H}^\pm = \{0\}$ which implies the conclusion. We take only the + case and show that the set $\mathcal{D}^+$ of (LED) is zero thereby showing $\mathcal{H}^+ = \{0\}$ since $\mathcal{D}^+$ is dense in $\mathcal{H}^+$. To this goal we take any $f \in \mathcal{D}^+$, then for a $b_\gamma$ as in (LED),

$$||f|| \leq E \lim_{t\to\infty} \{<V_t f, \prod_\gamma F(T^\gamma > \tfrac{1}{2} b_\gamma^2)V_t f> + \sum_\gamma < V_t f, F(T^\gamma \leq \tfrac{1}{2} b_\gamma^2)V_t f>\}.$$

By (LED) all the terms in the above inequality except the first are zero on the right hand side. Also it is straightforward to verify that there is a $\phi_\gamma \in C_0^\infty((0,\infty))$, $0 \leq \phi_\gamma \leq 1$ such that

$$\prod_\gamma F(T^\gamma > \tfrac{1}{2} b_\gamma^2) \leq \prod_\gamma \phi(T^\gamma) .$$

Hence for every $c > 0$,

$$||f||^2 \leq E \lim_{t\to\infty} \{<V_t f, (-\tilde{\Omega}^+(0)+1) \prod_\gamma \phi(T^\gamma)U_t F(|x| > c|t|)U_t^* V_t f> + <V_t f, (-\tilde{\Omega}^+(0)+1) \prod_\gamma \phi(T^\gamma)U_t F(|x| \leq c|t|)U_t^* V_t f> + <V_t f, \tilde{\Omega}^+(0) \prod_\gamma \phi_\gamma(T^\gamma)V_t f>\} .$$

The first term in the above inequality is zero by Theorem 2.16 (vi), the second term by Lemma 2.18 (i) if we choose $c < \min \inf \operatorname{supp} \phi_\gamma$ and the last by the intertwining relations and the fact that ${}^{\gamma}\tilde{\Omega}^{+}(0)^{*}f = 0$ since $f \in \mathcal{H}^{+}$.

Finally we remark that eventhough the general theory of N-particle completeness is simple as we saw in this chapter the verification of the conditions (LD) and (LED) proves to be difficult in many particle situations. For the three particle case we are able to verify these under general conditions on the potentials though for the other cases the proof is still missing.

# CHAPTER III

## COMPLETENESS IN THREE AND FOUR-PARTICLE SCATTERING

In the present chapter we apply the theory developed in chapter II to three and four particle scattering. In section 1 we deal with the three particle case and assume that the pair potentials decay at the rate of $(2+\varepsilon)$ in the pair directions. We allow them to have some local singularity. In the second section we verify the conditions (LD) and (LED) for the four particle case under some smoothness and faster decay assumptions on the pair potentials then those for the three particle case. We also make strong implicit assumptions corresponding to two-particle subsystems of the four particle system. Hence ours is only a partial result for the four particle case.

We normalise the coordinates and momenta as in Remark 2.1 without further comment.

### § 1. Three-particle asymptotic completeness

This section is based on the work in [KKM]. In this section we prove the local decay and low energy decay results for the three particle system. We allow the pair potentials to be sufficiently general as far as local signularities are concerned and assume them to be of short range with $(2+\varepsilon)$ decay at $\infty$. This restriction is used only in the proof of Lemma 3.3. If this Lemma can be generalised to the potentials with $(1+\varepsilon)$ decay at $\infty$, then we can conclude completeness for such potentials also.

Throughout the section we take $N = 3$. Our assumptions on the pair potentials for this section are the following

We set $\rho_2(\lambda) = (1+\lambda^2)^{-\delta_2}$ for some $\delta_2 > 1$.

<u>Assumptions on the potentials</u> The pair potentials satisfy

(A1) $W_\gamma (T^\gamma+1)^{-1}$ is compact on $L^2(\mathbb{R}^\nu)$

(A2) $(T^\gamma+1)^{-1} x_\gamma \cdot \nabla_\gamma W_\gamma(x^\gamma)(T^\gamma+1)^{-1}$ is compact on $L^2(\mathbb{R}^\nu)$.

(A3) $W_2\ \rho_2(x\ )^{-1}\ (T^\gamma+1)^{-1}$ is bounded and

(A4) $(T^\gamma+1)^{-1}\ \rho_2(x^\gamma)^{-\frac{1}{2}}\ [A^\gamma, W_\gamma]\rho_2(x^\gamma)^{-\frac{1}{2}}\ (T^\gamma+1)^{-1}$ is bounded.

We note that (A4) implies (A3) of section II. In our proof in most Lemmas (A3) suffices instead of (A4). So we refer to (A3) instead of (A4). when a weaker condition is sufficient.

We also note that the above assumptions mean that, at least for the case $\nu = 3$, the potentials $W_\gamma \in L^{3/2}_{loc}(\mathbb{R}^3)$ and $W_\gamma(x^\gamma) \sim (1+|x^\gamma|)^{-\delta_2}$ $\delta_2 > 1$ at $\infty$ are allowed.

We remark at this point that the only nontrivial clusters in the three particle case are pairs $\gamma$. Hence we denote in this section all clusterings D, with # D = 2, by the pair contained in D.

We start with a few technical Lemmas concerning two particle Hamiltonians before we get down to proving the local decay condition for the three particle case.

We set, for any $\delta > 0$, $\rho(\lambda) = (1+\lambda^2)^{-\delta}$

<u>Lemma 3.1</u> Let the pair potentials satisfy (A1),(A2) and let $W_\gamma\ \rho(x^\gamma)^{-1}\ (T^\gamma+1)^{-1}$ be a bounded operator for any pair $\gamma$. Then for any $\phi \in C_0^\infty(\mathbb{R}^+)$,

(i) $||F(|x^\gamma| > \mu_1\ r)\{\phi(H^\gamma) - \phi(T^\gamma)\}|| \leq K(1+\mu_1 r)^{-2\delta}$, $\forall\ r \in \mathbb{R}^+$.

(ii) $||F(|x^\gamma| > \mu_1\ r)W_\gamma\{\phi(H^\gamma)-\phi(T^\gamma)\}|| \leq K(1+\mu_1 r)^{-4\delta}$, $\forall\ r \in \mathbb{R}^+$

and

(iii) for any $\chi \in C_o^\infty(\mathbb{R}^\nu)$,

$$\chi(x^\gamma)\{\phi(H^\gamma) - \phi(H)\}$$

is a compact operator.

Proof (i) We fix a pair $\gamma$ and consider as in Corollory 2.3, $\mu < \inf \sigma(H^\gamma)$ take an $a < \mu$ and set $Z_t = \exp(-i\,t\,(H^\gamma - a)^{-1})$, $Z_t^o = \exp(-it(T^\gamma - a)^{-1})$ for any real t. Then for a suitable $\psi \in C_o^\infty((0,(\mu-a)^{-1}))$

$$F(|x^\gamma| > \mu_1 r)\{\phi(H^\gamma) - \phi(T^\gamma)\}$$

$$= F(|x| > \mu_1 r)\int dt\ \hat{\psi}(t)\int_0^t ds\ Z_s^o (T^\gamma - a)^{-1}(-i\ W_\gamma)(H^\gamma - a)^{-1}\ Z_{t-s}$$

Thus

$$||F(|x^\gamma| > \mu_1 r)\{\phi(H^\gamma) - \phi(T^\gamma)\}||$$

$$\leq \int dt|\hat{\psi}(t)|\int_0^t ds||F(|x^\gamma| > \mu_1 r)\rho(x^\gamma)||\,||\rho(x^\gamma)^{-1}\ Z_s^o\rho(x^\gamma)||\circ$$

$$\circ||\rho(x^\gamma)^{-1}(T^\gamma - a)^{-1}\ \rho(x^\gamma)||\,||\rho(x^\gamma)^{-1}\ iW_\gamma(H^\gamma - a)^{-1}||$$

By Corollory 2.3 (i), Lemma 2.4 (i) and the hypothesis on $W_\gamma$ the last three factors of the integrand in the above integral are bounded by $K(1+|s|^2)$. Therefore,

$$||F(|x^\gamma| > \mu_1 r)\{\phi(H^\gamma) - \phi(T^\gamma)\}||$$

$$\leq (K\int dt\ |\hat{\psi}(t)|(1+|t|^3))(1+\mu_1 r)^{-2\delta} \leq K_1(1+\mu_1 r)^{-2\delta}.$$

(ii) The proof of this result is similar to that of (i) using the decomposition

$$F(|x^\gamma| > \mu_1 r) W_\gamma (T^\gamma - a)^{-1} Z_s^o \, i W_\gamma (H^\gamma - a)^{-1}$$

$$= \{F(|x^\gamma| > \mu_1 r)\rho(x^\gamma)^2\}\{\rho(x^\gamma)^{-2} W_\gamma (T^\gamma - a)^{-1} \rho(x^\gamma)\}$$

$$\{\rho(x^\gamma)^{-1} Z_s^o \rho(x^\gamma)\}\{\rho(x^\gamma)^{-1} W_\gamma (H^\gamma - a)^{-1}\},$$

corollory 2.3 (i), Lemma 2.4 (i) and (ii).

(iii) We set $Z_t(\gamma) = \exp(-it(H(\gamma)-a)^{-1})$. As in (i) we have for some $\psi \in C_o^\infty((0,(\mu-a)^{-1}))$, when $\mu < \inf \sigma(H)$ and $a < \mu$,

$$\chi(x^\gamma)\{\phi(H(\gamma)) - \phi(H)\}$$

$$= \sum_{\alpha\neq\gamma} \int dt \, \hat{\psi}(t) \int_0^t ds \, \chi(x^\gamma) Z_s(\gamma)(H(\gamma)-a)^{-1}(-iW_\alpha)(H-a)^{-1} Z_{t-s}$$

$$= \sum_{\alpha\neq\gamma} \int dt \, \hat{\psi}(t) \int_0^t ds \{\chi(x^\gamma)(H(\gamma)-a)^{-1} Z_s(\gamma)\}\{-iW_\alpha (H(\alpha)-a)^{-1}\}.$$

$$\{(H(\alpha)-a)(H-a)^{-1}\} Z_{t-s}\} \cdot \qquad (3.1)$$

The first two factors of the integrand of the above term are respectively $\gamma$ and $\alpha$ compact (See Remark 2.7). The remaining factors being bounded for each $s$ and $t$ the integrand is compact by Theorem 2.8 since $X^{\gamma \cup \alpha} = X$. Norm continuity of the integrand and its boundedness in $t$ and $s$ imply that the integral in (3.1) is finite since $\int dt |t| (\hat{\psi}(t)| < \infty$. Hence the result.

Our next result is a decay result on the two particle group in certain regions of the phase space. The results using the dilation generator was given by Jensen who got only $t^{-1}$ decay in [J]. His result was improved in [KKM]. The phase space decay corresponding to low particle energies was given in [E2] and a much simpler proof was provided by the referee of [KKM]. Eventhough the $(2+\varepsilon)$ decay restriction on the potentials in the following theorem is unnecessary (see [KKM] Appendix) we give only a weaker result here.

Recall that for $N = 2$, $U_s = \exp(-isH_o)$ and $V_s = \exp(-isH)$ specify the two partial free and total evolutions.

Theorem 3.3 Consider the case $N = 2$ with the potential satisfying (A1) and (A2). Let $\phi,\psi \in C_o^\infty(\mathbb{R})$ $0 \le \phi,\psi \le 1$.

(i) If the potential satisfies (A5) and if inf supp $\phi > 0$, then

$$||(1+x^2)^{-1} \phi(H)V_t (1+x^2)^{-1}|| \le K(1+|t|)^{-2} (1+ \log(1+|t|)^2).$$

(ii) Under the conditions of (i),

$$||(1+x^2)^{-\mu} \phi(H) V_t (1+x^2)^{-\mu}|| \le K(1+|t|)^{-\mu'}$$

for every $\mu' < 2\mu$, with $0 \le \mu \le 1$.

(iii) If $(1+x^2)^\delta W(H+i)^{-1}$ is bounded for any $\delta > \frac{1}{2}$, if inf supp $\psi = \frac{1}{2} b^2$ and if c and a are positive constants with $c + b < a$, then

$$||F(|x| > a|u|)V_s \phi(H)U_\tau F(|x| \le c|u|)|| \le K(1+|u|)^{-2\delta}.$$

$s,\tau$ and u with $|s| + |\tau| \le |u|$.

Proof (i) As a quadratic form on $\mathcal{D}(H) \cap \mathcal{D}(A)$, we have the equality,

$$[A,V_t] = t H V_t - \int_0^t ds\, V_{t-s} (W+i[A,W])V_s \quad . \tag{3.2}$$

By (A4) and (A5) with $\delta_1 > \frac{1}{2}$ we can apply the smoothness result of Levine [L3] to get, for $f,g \in L^2(X)$,

$$\int_{-\infty}^{\infty} ds < V_s \phi_1(H)f, (W+i[A,W])V_s \phi_2(H)g> \le K||f|| \; ||g|| \tag{3.3}$$

for any $\phi_1, \phi_2 \in C_o^\infty(\mathbb{R} \setminus \{0\})$. So applying (3.3) to the second term of (3.2) after multiplying both the sides of (3.2) by $H^{-\frac{1}{2}} \phi(H)^{\frac{1}{2}}$ we get,

$$\| (1+x^2)^{-\frac{1}{2}} V_t \, \phi(H)(1+x^2)^{-\frac{1}{2}} \| \leq K(1+|t|)^{-1} \tag{3.4}$$

Having done this, we can find $\psi, \chi \in C_o^\infty((0,\infty))$ such that $\phi(H) = H\chi(H)\psi(H)$, then from (3.2) we have,

$$t(x^2+1)^{-1} V_t \, \phi(H)(x^2+1)^{-1} = (x^2+1)^{-1} \psi(H) tHV_t \, \chi(H)(x^2+1)^{-1}$$

$$\equiv I_1 + I_2 \tag{3.5}$$

where

$$I_1 \equiv (x^2+1)^{-1} \psi(H)\{AV_t - V_t A\}\chi(H)(x^2+1)^{-1}$$

$$I_2 \equiv -\int_0^t ds \; (x^2+1)^{-1} \psi(H) \; V_{t-s}(W+i[A,W])V_s \; \chi(H)(x^2+1)^{-1}. \tag{3.6}$$

By Lemma 2.4 (iv) both $(x^2+1)^{-1} \psi(H) \; A(x^2+1)^{\frac{1}{2}}$ and $(x^2+1)^{\frac{1}{2}} A\chi(H)(x^2+1)^{-1}$ are bounded, hence using the estimate (3.4) we obtain the following estimate

$$\|I_1\| \leq K(1+|t|)^{-1} \tag{3.7}$$

On the ohter hand (A5), Lemma 2.4 (iii) and (3.4) imply that

$$\|I_2\| \leq \int_0^t ds\{\|(x^2+1)^{-1} \psi(H)(H^2+1)V_{t-s}(1+x^2)^{-\frac{1}{2}}\| .$$

$$\cdot \|(1+x^2)^{\frac{1}{2}} \; (H-i)^{-1}\rho_2(x)^{\frac{1}{2}}\| \; \|\rho_2(x)^{-\frac{1}{2}}(H+i)^{-1}(W+i[A,W])(H+i)^{-1}\rho_2(x)^{-\frac{1}{2}}\|$$

$$\cdot \|\rho_2(x)^{\frac{1}{2}} \; (H-i)^{-1}(1+x^2)^{\frac{1}{2}}\| \cdot \|(1+x^2)^{\frac{1}{2}} \; V_s(H^2+1)\chi(H)(x^2+1)^{-1}\|$$

$$\leq \int_0^t ds \; (1+|t-s|)^{-1} \; (1+|s|)^{-1} \leq K(1+|t|)^{-1} \; (1+\log(1+|t|)). \tag{3.8}$$

The result now follows from using the estimates (3.7) and (3.8) in (3.5).

(ii) This follows from (i) through interpolation .

(iii) We take $\psi_1 \in C_0^\infty(\mathbb{R})$, $0 \leq \psi_1 \leq 1$, $\psi_1 \equiv 1$ on the support of $\psi$ and sup supp $\psi_1 = \frac{1}{2} s_1^2$ such that $b_1 + c + \varepsilon < a$ for some $\varepsilon > 0$. We set for any positive M,

$$F_j \equiv F(|x| > (c + \frac{(M-j)}{M}\varepsilon)(s-t_j) + (a - \frac{j\varepsilon}{M})(t_j + |u|))$$

for any $j = 1,\dots,M-1$ and $0 \leq t_{M-1} \leq t_{M-2} \cdots \leq t_1 \leq s$. Then by Lemma 3.1 (ii) we have

$$\begin{aligned}
B(s,\tau,|u|) &\equiv ||F(|x| > a|u|)V_s\psi(H)\, U_\tau\, F(|x| \leq c|u|)|| \\
&\leq O((1+|u|)^{-2\delta}) + ||F(|x| > a|u|)\psi_1(H_o)V_s\, \psi(H)U_\tau\, F(|x| \leq c|u|)|| \\
&\leq O((1+|u|)^{-2\delta}) + ||F(|x| > a|u|)\psi_1(H_o)U_s(1-F_1)||\; ||\psi(H)U_\tau F(|x| \leq c|u|)|| \\
&+ ||F(|x| > a|u|)\psi_1(H_o)U_s||\,||F_1\{\psi(H)-\psi(H_o)\}||\; ||U_\tau\, F(|x| \leq c|u|)|| \\
&+ ||F(|x| > a|u|)\psi_1(H_o)U_s||\; ||F_1\, \psi(H_o)U_\tau\, F(|x| \leq c|u|)|| \\
&+ \int_0^s dt_1||F(|x| > a|u|)\psi_1(H_o)U_{s-t_1}(1-F_1)||\; ||W\psi(H)V_{t_1}\, U_\tau F(|x| \leq c|u|)|| \\
&+ \int_0^s dt_1||F(|x| > a|u|)U_{s-t_1}\, \psi_1(H_o)W\, F_1\, V_{t_1}\psi(H)U_\tau\, F(|x| \leq c|u|)|| \,.
\end{aligned}$$

Now using Theorem 2.11 (i), the second, fourth and the fifth terms are $O((1+|u|)^{-2\delta})$ while the third term is $O((1+|u|)^{-2\delta})$ by Lemma 3.1(i). The last term has a factor $F_1\, V_{t_1}\, \psi(H)\, U_\tau\, F(|x| \leq c|u|)$ which is of the form we started with. Hence we can repeate the whole procedure. Doing this (M-1) times we obtain the following inequality for $B(s,\tau,|u|)$.

$$B(s,\tau,|u|) \leq O((1+|u|)^{-2\delta})$$

$$+ \int_{0\leq t_{M-1}\leq\cdots\leq t_1\leq s} dt_1\ldots dt_{M-1}\,||F(|x|>a|u|)U_{s-t_1}\psi_1(H_o)WF_1\cdots WF_{M-1}$$

$$\cdot V_{t_{M-1}}\,\psi(H)U_\tau F(|x|\leq c|u|)||.$$

It is now easy to see using Lemma 3.1 (ii) that the integrand in the above term is bounded by

$$\prod_{j=1}^{M-1} K_j\,(1+|u|+s+t_j)^{-2\delta}$$

for constants $K_j$ depending upon j. Therefore when $M > 2\delta(2\delta-1)^{-1}+1$,

$$\int_{0\leq t_{M-1}\leq\cdots\leq t_1\leq s} dt_1\ldots dt_{M-1}\prod_{j=1}^{M-1} K_j\,(1+|u|+s+t_j)^{-2\delta} \leq K(1+|u|)^{-2\delta}.$$

Hence the result.

As an input to proving the local decay result for the three particle system, we have a compactness result connected to the wave operators.

We set $\rho(\lambda) = (1+\lambda^2)^{-1}$.

<u>Lemma 3.3.</u> Let $\gamma$ be a pair. If $\psi \in C_o^\infty(\mathbb{R}\setminus T(H))$, $0 \leq \psi \leq 1$ then,

$$(\tilde{\Omega}^{\pm}(\gamma)-1)\psi(H(\gamma))\rho(x^\gamma)F(A_\gamma \gtrless 0)$$

is compact.

<u>Proof</u> (+ sign only). Since $H^\gamma$ is lower bounded and $H^\gamma \leq H(\gamma)$ as operators it is clear that there is a $\phi \in C_o^\infty(\mathbb{R})$ such that $\phi(H^\gamma)\psi(H(\gamma)) = \psi(H(\gamma))$. Therefore we show that

$$(\tilde{\Omega}^{+}(\gamma)-1)\phi(H^\gamma)\psi(H(\gamma))\rho(x^\gamma)F(A_\gamma > 0)$$

is compact. We set $F_\gamma^+ \equiv F(A_\gamma > 0)$, then

$$(\tilde{\Omega}^{+}(\gamma)-1)\phi(H^{\gamma})\psi(H(\gamma))\rho(x^{\gamma})F_{\gamma}^{+}$$

$$= \int ds\ V_s^{*} \sum_{\alpha\neq\gamma} i\ W_{\alpha}\ V_s(\gamma)\phi(H^{\gamma})\psi(H(\gamma))\rho(x^{\gamma})F_{\gamma}^{+} \equiv \sum_{\alpha\neq\gamma} \int ds\ I^{\gamma}(\alpha,s) \quad (3.9)$$

By writing the integrand of (3.9) as

$$(V_s^{*})(iW_{\alpha}(H_o+1)^{-1})((H_o+1)V_s(\gamma)\phi(H^{\gamma})\psi(H(\gamma))\rho(x^{\gamma}))(F_{\gamma}^{+})$$

we see that it is a compact operator (since $\alpha \bigcup \gamma$ - compact operator is a compact operator) for each s by Proposition 2.8 since the second and the third factors are $\alpha$ and $\gamma$ - compact respectively by Remark 2.7 with the remaining factors being bounded. Also the integrand in (3.9) is norm continuous in s. Hence it is enough to show the integrability in s of $||I^{\gamma}(\alpha,s)||$ to conclude the result. Now the condition (A4) implies the condition (A3) hence by Lemma 2.4 (i), for a $\rho_1$ as in (A3),

$$||I^{\gamma}(\alpha,s)||$$

$$\leq ||W_{\alpha}(H(\gamma)+i)^{-1}\ \rho_1(x^{\alpha})^{-1}||\ ||\rho_1(x^{\alpha})V_s(\gamma)(H(\gamma)+i)^{-1}\psi(H(\gamma))\phi(H^{\gamma})\rho(x^{\gamma})F_{\gamma}^{+}||$$

$$\leq\ K||\rho_1(x^{\alpha})V_s(\gamma)(H(\gamma)+i)\psi(H(\gamma))\phi(H^{\gamma})\rho(x^{\gamma})F_{\gamma}^{+}|| \quad .$$

Since $(H(\gamma)+i)\psi(H(\gamma)) \equiv (\psi_1+\psi_2)(H(\gamma))$ for some $\psi_1,\psi_2$ having the same support properties as $\psi$, in the following estimate we continue to write $\psi$ for $\psi_1,\psi_2$ without loss of generality. Now $0 \notin \text{supp}\ \psi$, hence there is a $d > 0$, such that $\text{supp}\ \psi \bigcap (-d,d) = \emptyset$. In view of this we choose three functions $\phi_1 \in C_o^{\infty}((-\infty,0))$, $\phi_2 \in C_o^{\infty}(\mathbb{R})$ and $\phi_3 \in C_o^{\infty}((0,\infty))$, $0 \leq \phi_1,\phi_2,\phi_3 \leq 1$ such that $(\phi_1+\phi_2+\phi_3)(H^{\gamma})\phi(H^{\gamma}) = \phi(H^{\gamma})$ and supp $\phi_2$ contained in $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$ with $b^2 < d$ and $\phi_2 \equiv 1$ on $(-\frac{1}{4}b^2, \frac{1}{4}b^2)$. At this stage we only need b to be positive. Later we make a choice of this b, depending on d for some estimates to be possible. Thus we write, throwing away a factor

$\rho(x^\gamma)^{-1}\ \phi(H^\gamma)\rho(x^\gamma)$ and using Lemma 2.4 (i), that

$$\begin{aligned} ||\rho_1(x^\alpha)V_s(\gamma)\psi(H(\gamma))\phi(H^\gamma)\rho(x^\gamma)F_\gamma^+|| &\leq \sum_{i=1}^{3} I_i^\gamma(\alpha,s) \\ I_i^\gamma(\alpha,s) &\equiv ||\rho_1(x^\alpha)V_s(\gamma)\psi(H(\gamma))\phi_i(H^\gamma)\rho(x^\gamma)F_\gamma^+|| \end{aligned} \tag{3.10}$$

We estimate each of the $I_i^\gamma(\alpha,s)$ separately. By the choice of $\phi_2$, since $\phi_1 + \phi_2 + \phi_3 \equiv 1$ on the support of $\phi$, supp $\phi_1$ is in $(-\infty, -\frac{1}{4}b^2)$. Hence by Proposition 1.9 there are only finitely many eigenvalues of $H^\gamma$ in the support of $\phi_1$. Therefore without loss of generality for the purpose of the estimate, we can take the support of $\phi_1$ to contain only one eigenvalue (say) $\lambda^\gamma$ of $H^\gamma$. Then

$$V_s^\gamma\ \phi(H^\gamma) = \exp(-is\ \lambda^\gamma)\phi(H^\gamma).$$

Also since $\lambda^\gamma \in T(H)$, it is not in the support of $\psi$, thus there exists a $\psi_1 \in C_0^\infty((0,\infty))$ such that

$$\phi_1(H^\gamma)\psi(H(\gamma))\psi_1(T_\gamma) = \phi_1(H^\gamma)\psi(H(\gamma)).$$

Therefore using the decomposition $V_s(\gamma) = V_s^\gamma\ U_{s,\gamma}$ we have,

$$I_1^\gamma(\alpha,s) \leq K||\rho_1(x^\alpha)\rho(x^\gamma)U_{s,\gamma}\psi_1(T_\gamma)F_\gamma^+||\ ||\rho(x^\gamma)^{-1}\ \phi_1(H^\gamma)\psi(H(\gamma))\rho(x^\gamma)|| \tag{3.11}$$

By Lemma 2.4 (i) the second factor of (3.11) is finite. In the first factor we use the inequality that for $\alpha \neq \gamma$

$$\rho_1(x^\alpha)\ \rho(x^\gamma) \leq K\ \rho_1(x_\gamma).$$

We also take $c > 0$ such that $c <$ inf supp $\psi_1$ and use a partition of identity

$$1 = F(|x_\gamma| \leq c|s|) + F(|x_\gamma| > c|s|)$$

along with theorem 2.11 (iii), so that

$$||\rho_1(x^\gamma)F(|x_\gamma| > c|s|)|| \le K(1+|s|)^{-\delta_1}$$

and

$$||F(|x_\gamma| \le c|s|)U_{s,\gamma}\,\psi_1(T_\gamma)F_\gamma^+|| \le K(1+|s|)^{-M}$$

for arbitrary M. Then we obtain,

$$I_1^\gamma(\alpha,s) \le K||\rho_1(x_\gamma)U_{s,\gamma}\,\psi_1(T_\gamma)F_\gamma^+|| \le K(1+|s|)^{-\delta_1} \qquad (3.12)$$

The second term $I_2^\gamma(\alpha,s)$ is estimated as follows. Again from the support properties of $\phi_2$ and $\psi$ it follows that there is a $\psi_3 \in C_c^\infty((0,\infty))$, $0 \le \psi_3 \le 1$ with supp $\psi_3$ contained in $(\frac{1}{2}b^2,\infty)$ such that

$$\phi_2(H^\gamma)\psi(H(\gamma))\psi_3(H(\gamma)) = \phi_2(H^\gamma)\psi(H(\gamma)).$$

Hence

$$I_2^\gamma(\alpha,s) \le ||\rho_1(x^\alpha)\psi(H(\gamma))\rho_1(x^\alpha)^{-1}||\cdot||\rho_1(x^\alpha)V_s(\gamma)\phi_2(H^\gamma)\psi_3(T_\gamma)\rho(x^\gamma)F_\gamma^+||$$

Then by Lemma 2.4 (i)

$$I_2^\gamma(\alpha,s) \le K||\rho_1(x^\alpha)V_s(\gamma)\phi_2(H^\gamma)\psi_1(T_\gamma)\rho(x^\gamma)F_\gamma^+|| \qquad (3.13)$$

Now

$$1 = F(|x^\gamma| > a_1|s|) + F(|x^\gamma| \le a_1|s|,\ |x_\gamma| > a_2|s|) + F(|x_\gamma| \le a_2|s|)$$

for any $a_1, a_2 > 0$. Therefore using this partition of the identity in (3.13) we get

$$I_2^\gamma(\alpha,s) \le K\{||\rho_1(x^\alpha)||\ ||F(|x^\gamma|>a_1|s|)\ V_s^\gamma\ \phi_2(H^\gamma)\rho(x^\gamma)||$$

$$+ ||\rho_1(x^\alpha)F(|x^\gamma| \le a_1|s|,\ |x_\gamma| > a_2|s|)||$$

$$+ ||\rho_1(x^\alpha)||\ ||F(|x_\gamma| \le a_2|s|)U_{s,\gamma}\ \psi_3(T_\gamma)F_\gamma^+||\ ||\rho(x^\gamma)||\} \qquad (3.14)$$

We have for each $\alpha \neq \gamma$ ,

$$x^{\alpha} = c_1(\alpha,\gamma)x^{\gamma} + c_2(\alpha,\gamma)x_{\gamma} \quad . \tag{3.15}$$

Now we make a choice of the constants b, $b_1$, $a_1$, $a_2$ given d and the numbers $c_1$, $c_2$ given by

$$c_1 = \max_{\alpha \neq \gamma} c_1(\alpha,\gamma) \quad , \quad c_2 = \min_{\alpha \neq \gamma} \{c_2(\alpha,\gamma)\}.$$

Let

$$b_1 > a_2, \quad c_2 a_2 > c_1 a_1, \quad b < a_1 \quad \text{and} \quad \frac{1}{2}(b^2+b_1^2) < d \tag{3.16}$$

A simple calculation shows that such a choice is possible when $d > 0$. Then by Theorem 3.2 (iii), the first term on the right side of (3.14) has the estimate,

$$||F(|x^{\gamma}| > a_1|s|)V_s^{\gamma}\,\phi_2(H^{\gamma})\rho(x^{\gamma})|| \leq K(1+|s|^{-\delta_1}$$

The second term of (3.14) is dominated by $K(1+|s|)^{-\delta_1}$, by the choice of the constants in (3.16) since

$$|x^{\alpha}| > ||c_1(\gamma,\alpha)|\ |x^{\gamma}| - |c_2(\gamma,\alpha)|\ |x_{\gamma}|| \geq |c_2|x_{\gamma}|-c_1|x^{\gamma}|| \ .$$

The third term of (3.14) is estimated as

$$||F_{\gamma}(|x_{\gamma}| \leq a_2|s|)U_{s,\gamma}\ \phi_3(T_{\gamma})F_{\gamma}^{+}|| \leq K(1+|s|)^{-M} .$$

For arbitrary $M > 0$, again by Theorem 2.11 (iii). Hence

$$I_2^{\gamma}(\alpha,s) \leq K(1+|s|)^{-\mu} \quad \text{for some} \quad \mu_1 > 1$$

Now we turn to the last term $I_3^{\gamma}(\alpha,s)$ of (3.10)

$$I_3^{\gamma}(\alpha,s) \leq ||\rho_1(x^{\alpha})V_s(\gamma)\phi_3(H^{\gamma})\psi(H(\gamma))\rho(x^{\gamma})F_{\gamma}^{+}|| \ .$$

Decomposing $V_s(\gamma) = V_s^{\gamma} U_{s,\gamma}$, using Lemma 2.4 (i), since $U_{s,\gamma}$ commutes with $\rho(x^{\gamma})$,

$$I_3^{\gamma}(\alpha,s) \leq ||\rho_1(x^{\alpha})V_s^{\gamma}\ \phi_3(H^{\gamma})\rho(x^{\gamma})||\ ||\rho(x^{\gamma})^{-1}\ \psi(H(\gamma))\rho(x^{\gamma})||$$

$$\leq K||\rho_1(x^{\alpha})V_s^{\gamma}\ \phi_3(H^{\gamma})\rho(x^{\gamma})||$$

$$\leq K\{||\rho_1(x^{\alpha})U_s\ \rho(x^{\gamma})||\ ||\rho(x^{\gamma})^{-1}\ \phi(H^{\gamma})\rho(x^{\gamma})||$$

$$+ \quad ||\rho_1(x^{\alpha})(V_s^{\gamma}-U_s^{\gamma})\phi_3(H^{\gamma})\rho(x^{\gamma})||\} \tag{3.17}$$

By Lemma 2.4 (i), Theorem 2.10 (i) the first factor of (3.17) is bounded by $K(1+|s|)^{-\mu_1}$ for some $\mu_1 > 1$ and we use the Duhamel formula in the second to obtain

$$I_3(\alpha,s) \leq O((1+|s|)^{-\mu_1}) + K\int_0^s d\tau\ ||\rho_1(x^{\alpha})U_{s-\tau}^{\gamma}\ W_{\gamma}\ V_{\tau}^{\gamma}\ \phi_3(H^{\gamma})\rho(x^{\gamma})||$$

$$\leq O((1+|s|)^{-\mu_1}) + \int_0^s d\tau||\rho_1(x^{\alpha})U_{s-\tau}^{\gamma}\ \rho_2(x^{\gamma})^{\frac{1}{2}}||\cdot$$

$$\cdot||\rho_2(x^{\gamma})^{-\frac{1}{2}}\ W_{\gamma}(H^{\gamma}+i)^{-1}\ \rho_2(x^{\gamma})^{-\frac{1}{2}}||\cdot$$

$$\cdot||\rho_2(x^{\gamma})^{\frac{1}{2}}\ V_s(H^{\gamma}+i)\phi_3(H^{\gamma})\rho(x^{\gamma})||.$$

By (A4) and Lemma 2.4 (i), the second factor in the integrand of the above inequality is finite. Since $\rho_2(x^{\gamma})^{\frac{1}{2}} = (1+(x^{\gamma})^2)^{-\frac{1}{2}\delta_2}$ $\delta_2 > 1$, by Theorem 2.10 (i), the first factor of the integrand is bounded by $K(1+|s-\tau|)^{-\mu_1}$ for some $\mu_1 > 1$. On the other hand, $(H^{\gamma}+i)\phi_3(H^{\gamma})$ has the same support properties as $\phi_3(H^{\gamma})$, therefore applying Theorem 3.2 (ii) it is bounded by $K(1+|\tau|)^{-\mu_1}$, $\mu_1 > 1$. Thus by a simple integeration,

$$I_3^{\gamma}(\alpha,s) \le O((1+|s|)^{-\mu_1}) + K\int_0^s d\tau\ (1+|s-\tau|)^{-\mu_1}(1+|\tau|)^{-\mu_1}$$

$$\le K((1+|s|)^{-\mu_1}).$$

Collecting the estimates for $I_i^{\gamma}(\alpha,s)$, $i = 1,2,3$ in (3.10) we get the required integrability.

We had defined earlier two sets of wave operators namely $\tilde{\Omega}^{\pm}(D)$ and $\Omega^{\pm}(D)$. The latter are the physical wave operators while the former serve as auxiliary objects useful for the proof of completeness by the method we present here. We now give a Lemma on the ranges of $\tilde{\Omega}^{\pm}(D)$, a result which depends on the completeness of two particle scattering.

Lemma 3.4 The ranges of $\tilde{\Omega}^{\pm}(\gamma)$ for all pairs $\gamma$ is exactly $F_{\pm}(\gamma) \oplus F_{\pm}(0)$.

Proof (+ case only). We have, by the definition of

$\tilde{\Omega}^{+}(\gamma)$ and $\Omega^{+}(\gamma)$, for any $f \in L^2(X)$,

$$\tilde{\Omega}^{+}(\gamma)f = \Omega^{+}(\gamma)f + \tilde{\Omega}^{+}(\gamma)(1-E^{\gamma})f. \tag{3.18}$$

Now by the completeness of two particle scattering,

$$\tilde{\Omega}^{+}(\gamma)(1-E^{\gamma})f = s\lim_{t\to\infty} (V_t^{*}\,U_t)(U_t^{*}\,V_t(\gamma)(1-E^{\gamma})f)$$

$$= \Omega^{+}(0)(\omega^{+}(\gamma))^{*}\ f.$$

Since $(\omega^{+}(\gamma))^{*}$ is an isometry from $(1-E^{\gamma})L^2(X)$ onto $L^2(X)$ and since $\Omega^{+}(0)$ maps $L^2(X)$ isometrically onto $F_{+}(0)$, the range of $\tilde{\Omega}^{+}(\gamma)(1-E^{\gamma})$ is precisely $F_{+}(0)$. Now the orthogonality of the closed subspaces $F_{+}(\gamma)$ and $F_{+}(0)$ for any pair $\gamma$ implies the result. #

We are ready to prove the local decay result now.

Theorem 3.5 The three particle system with the pair potentials satisfying (A1) - (A3) verifies the local decay condition (LD).

Proof (+ case only). Since $\Upsilon(H)$ is a closed countable set, $\{f \in H^+ : \psi(H)f = f \text{ for some } \psi \in C_0^\infty(\mathbb{R} \setminus \Upsilon(H))\}$ is dense in $H^+$. By density we prove the result only for f's in the above set. That is for $\phi \in C_0(\mathbb{R}^\nu)$,

$$E \lim_{t \to \infty} ||\phi(x^\gamma)U_t f|| = 0$$

for all pairs $\gamma$. Now by Stone-Weirstrass Theorem, any $\phi \in C_0(\mathbb{R}^\gamma)$ can be approximated, in norm, by a sequence $\phi_n$ of functions in $C_0^\infty(\mathbb{R}^\nu)$ and hence it is enough to take $\phi$ to be in $C_0^\infty(\mathbb{R}^\nu)$ for proving the above relation. Now for $\phi \in C_0^\infty(\mathbb{R}^\nu)$, $(1+(x^\gamma)^2)^{\frac{1}{2}} \phi(x^\gamma)$ is bounded therefore if we prove

$$E \lim_{t \to \infty} ||(1+(x^\gamma)^2)^{\frac{1}{2}} V_t f|| = 0 ,$$

the result would follow and this equality itself comes by proving

$$E \lim_{t \to \infty} ||(1+(x^\gamma)^2)^{-\frac{1}{2}} V_t f||^2 = 0 .$$

Thus we set $\rho(x^\gamma) = (1+(x^\gamma)^2)^{-1}$ in the following and consider

$$E \lim_{t \to \infty} \langle V_t f, \rho(x^\gamma) V_t f\rangle$$

$$= E \lim_{t \to \infty} \{\langle V_t f, [\psi(H)-\psi(H(\gamma))]\rho(x^\gamma)V_t f\rangle$$

$$+ \langle V_t f, \psi(H(\gamma))\rho(x^\gamma)V_t f\rangle\}$$

By Lemma 3.1 (iii) and Proposition 2.9 the first factor is zero. Now

$$\mathcal{E}\lim_{t\to\infty} ||\rho(x^{\gamma})^{\frac{1}{2}} V_t f||^2 = \mathcal{E}\lim_{t\to\infty} \{<V_t f, (-\tilde{\Omega}^{+}(\gamma)+1)\psi(H(\gamma))\rho(x^{\gamma})F(A_{\gamma} > 0)V_t f>$$

$$+ <V_t f, \tilde{\Omega}^{+}(\gamma)\psi(H(\gamma))\rho(x^{\gamma})F(A_{\gamma} > 0)V_t f>$$

$$+ <V_t f, (-\tilde{\Omega}^{-}(\gamma)+1)\psi(H(\gamma))\rho(x^{\gamma})F(A_{\gamma} < 0)V_t f>$$

$$+ <V_t f, \tilde{\Omega}^{-}(\gamma)\psi(H(\gamma))\rho(x^{\gamma})F(A_{\gamma} < 0)V_t f>\} .$$

Again by Lemma 3.3 and Proposition 2.9 the first and the third terms are zero in the above equality. The second term is zero by the intertwining relations and the fact that $f \perp F_{+}(\gamma)$. The last term is equal to

$$\mathcal{E}\lim_{t\to\infty} <F(A_{\gamma} < 0)\ V_t(\gamma)(\Omega^{-}(\gamma))^{*} f,\ \phi(H^{\gamma})\rho(x^{\gamma})V_t f>$$

and is dominated by

$$\mathcal{E}\lim_{t\to\infty} ||F(A_{\gamma} < 0)\ U_{t,\gamma}g||$$

for some $g \in L^2(X)$ and is zero by Theorem 2.11 (iv).

With the local decay condition verified for the three particle system, we are only left to prove the low energy decay restriction which will imply completeness by applying the theory of chapter II. So we start with a norm estimate corresponding to small pair energies.

<u>Lemma 3.6</u> Let $\psi \in C_0^{\infty}((0,\infty))$, $0 \leq \psi \leq 1$ and $\inf \operatorname{supp} \psi = \frac{1}{2} b^2$. Then there exists $b_1$, $0 < b_1 < b$, $\phi \in C_0^{\infty}(\mathbb{R})$, $0 \leq \phi \leq 1$, $\sup \operatorname{supp} \phi = \frac{1}{2} b_1^2$ such that for some $c > 0$ and all pairs $\gamma$,

$$\lim_{t\to\infty} ||(\tilde{\Omega}^{\pm}(\gamma)-1)\phi(H^{\gamma})\psi(H(\gamma))U_t\ F(|x| \leq c|t|)|| = 0 .$$

Proof. (+ case only) We have,

$$(\tilde{\Omega}^{+}(\gamma)-1)\phi(H^{\gamma})\psi(H(\gamma))U_t\ F(|x| \leq c|t|)$$

$$= \int ds\ V_s^* \sum_{\alpha\neq\gamma} W_\alpha\ V_s(\gamma)\phi(H^{\gamma})\psi(H(\gamma))U_t\ F(|x| \leq c|t|)$$

$$= \sum_{\alpha\neq\gamma} \int_0^\infty ds\ I^{\gamma}(\alpha,s,t).$$

Then for each $\alpha$ we show that

$$\int ds||I^{\gamma}(\alpha,s,t)|| = 0 \quad \text{as} \quad t \to \infty .$$

Now we have,

$$||I^{\gamma}(\alpha,s,t)|| \leq$$

$$||W_\alpha(H(\gamma)+i)^{-1}\rho_1(x^{\alpha})^{-1}||\,||\rho_1(x^{\alpha})\phi(H^{\gamma})\psi(H(\gamma))(H(\gamma)+i)V_s(\gamma)U_tF(|x|\leq c|t|)||$$

and we note that

$$\psi_1(H(\gamma)) \equiv (H(\gamma)+i)\psi(H(\gamma))$$

has the same support properties as $\psi$, hence by Lemma 2.4 (i) we have,

$$||I^{\gamma}(\alpha,s,t)|| \lesssim ||\rho_1(x^{\alpha})\phi(H^{\gamma})\psi_1(H(\gamma))V_s(\gamma)U_tF(|x| \leq c|t|)|| \quad (3.19)$$

To estimate the right side of (3.19) we make use of the partition of identity

$$1 = F(|x^{\gamma}| > a_1|t+s|)+F(|x^{\gamma}| \leq a_1|t+s|,|x_{\gamma}|>a_2|t+s|)+F(|x_{\gamma}|\leq a_2|t+s|) \quad (3.20)$$

for some $a_1,a_2 > 0$. Then using the relation (3.15) and $c_1$, $c_2$ defined thereof, we choose the constants $a_1,a_2$ to satisfy $a_1 > b_1$, $c_2a_2 > c_1a_1$, $b_2 > a_2$ and $\frac{1}{4}(b_1^2 + b_2^2) < b^2$. With this choice of the constants (depending only on b), it is clear that there is a $\psi_2 \in C_0^\infty((0,\infty))$, $0 \le \psi_2$, inf supp $\psi_2 = \frac{1}{2} b^2$ and

$$\phi(H^\gamma)\psi_1(H(\gamma))\psi_2(T_\gamma) = \phi(H^\gamma)\psi_1(H(\gamma)).$$

Using this relation and (3.20) in (3.19) we obtain the inequality,

$$||I^\gamma(\alpha,s,t)|| \le K||\rho_1(x^\alpha)\psi_1(H(\gamma)\rho_1(x^\alpha)^{-1}||\cdot$$

$$\cdot\{||\rho_1(x^\alpha)|| \; ||F(|x^\gamma|>a_1|t+s|)V_s(\gamma)\phi(H^\gamma)\psi_2(T_\gamma)U_tF(|x| \le c|t+s|)||$$

$$+ ||\rho_1(x^\alpha)F(|x^\gamma| \le a_1|t+s|, |x_\gamma| > a_2|t+s|)|| \cdot$$

$$+ ||\rho_1(x^\alpha)||\,||F(|x_\gamma|\le a_2|t+s|)V_s(\gamma)\phi(H^\gamma)\psi_2(T_\gamma)U_tF(|x|\le c|t+s|)||\} \tag{3.21}$$

Now we choose $c < \min\{(a_1-b_1), (b_2-a_2)\}$ and note that

$$F(|x| \le c|t+s|) \le F(|x^\gamma| \le c|t+s|, |x_\gamma| \le c|t+s|).$$

Then the first term of (3.21) has the estimate, by Theorem 3.2 (iii), for some $\mu_1 > 1$,

$$F(|x^\gamma| > a_1|t+s|)V_s(\gamma)\phi(H^\gamma)\psi_2(T_\gamma)U_t\, F(|x| \le c|t+s|)||$$

$$\le ||\psi_2(T_\gamma)||\;||F(|x^\gamma|>a_1|t+s|)V_s^\gamma\,\phi(H^\gamma)U_t^\gamma\,F(|x^\gamma| \le c|t+s|)||$$

$$\le K(1+|s+t|)^{-\mu_1} . \tag{3.22}$$

The second term has the estimate, by the choice of $a_1, a_2$,

$$||\rho_1(x^\alpha)F(|x^\gamma| \le a_1|t+s|, |x_\gamma| > a_2|t+s|)|| \le K(1+|t+s|)^{-2\delta_1}. \tag{3.23}$$

The last term of (3.21) has the following estimate by Theorem 2.11 (ii).

$$||F(|x_\gamma| \leq a_2|t+s|)V_s(\gamma)\phi(H^\gamma)\psi_2(T_\gamma)U_t\ F(|x_\gamma| \leq c|t+s|)||$$

$$\leq \ ||F(|x_\gamma| \leq c|t+s|)U_{s+t,\gamma}\ \psi_2(T_\gamma)F(|x_\gamma| \leq c|t+s|)||$$

$$\leq \ K(1+|t+s|)^{-M} \qquad (3.24)$$

for arbitrary $M > 0$, Now using Lemma 2.4 (i), to bound the first factor in (3.21), and the estimates (3.22) to (3.24) in (3.21) we get the result.

We are now all set for proving the low energy decay condition (LED). We take $\mathcal{D}^{\pm} = \{f \in \mathcal{H}^{\pm} : \psi(H)f = f,\ \psi \in C_0^\infty\ (\mathbb{R} \setminus \{0\})\}$. Clearly $\mathcal{D}^{\pm}$ is dense in $\mathcal{H}^{\pm}$ and serves as the set mentioned in the condition (LED). With this we have the

<u>Theorem 3.7</u>. Let the three particle system have pair potentials satisfying (A1), (A3), (A4) and (A5). Then for each $f^{\pm} \in \mathcal{D}^{\pm}$, there is a $b_{\pm}(f^{\pm}) \equiv b_{\pm}$ such that for each pair $\gamma$ and $b_\gamma \leq b_{\pm}$,

$$\mathcal{E} \lim_{t \to \pm\infty} ||F(T^\gamma \ \leq \ \tfrac{1}{2}\, b_\gamma^2)V_t\ f^{\pm}|| = 0\ .$$

<u>Proof</u>. (+ case only) We drop the superscript on $f^+$ in the following and take a $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that $\psi(H)f = f$, supp $\psi \cap (-\frac{1}{2}\, b^2, \frac{1}{2}\, b^2) = \emptyset$ for some $b > 0$. It suffices to prove that for each pair, there is a $b_+$ such that

$$\mathcal{E} \lim_{t \to \infty} ||F(T^\gamma \leq \tfrac{1}{2}\, b_+^2)V_t f||^2 = 0$$

to conclude the result. This follows if we prove the same with $\phi(T^\gamma)$ replacing $F(T^\gamma < \frac{1}{2}\, b_+^2)$ for a function $\phi \in C_0^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $[-\frac{1}{4}\, b_1^2, \frac{1}{4}\, b_1^2]$. Thus we choose a $b_1$, $(0 < b_1 < b)$ as in Lemma 3.6 set $b_+^2 = \frac{1}{2}\, b_1^2$ and take $\phi$ to have support in $(-\frac{1}{4}\, b_1^2, \frac{1}{4}\, b_1^2)$. Then we have to show only that

$$E \lim_{t\to\infty} \langle V_t f, \phi(T^\gamma)V_t f\rangle = 0$$

since

$$||F(T^\gamma < \tfrac{1}{2} b_+^2)V_t f||^2 \leq \langle V_t f, \phi(T^\gamma)V_t f\rangle.$$

Now we have the following inequality for any $c > 0$.

$$\begin{aligned}
\langle V_t f, \phi(T^\gamma)V_t f\rangle \leq \\
&\langle V_t f, \{\phi(T^\gamma)-\phi(H^\gamma)\}V_t f\rangle + \langle V_t f, \phi(H^\gamma)\{\psi(H)-\psi(H(\gamma))\}V_t f\rangle \\
&+ \langle V_t f, \phi(H^\gamma)\psi(H(\gamma))U_t F(|x| > c|t|)U_t^* V_t f\rangle \\
&+ \langle V_t f, (-\tilde{\Omega}^+(\gamma)+1)\phi(H^\gamma)\psi(H(\gamma))U_t F(|x| \leq c|t|)U_t^* V_t f\rangle \\
&+ \langle V_t f, \tilde{\Omega}^+(\gamma)\phi(H^\gamma)\psi(H(\gamma))U_t F(|x| \leq c|t|)U_t^* V_t f\rangle . \qquad (3.25)
\end{aligned}$$

Then by using the inter-twinning relations in the fifth term of (3.25) we have,

$$\begin{aligned}
\langle V_t f, \phi(T^\gamma)V_t f \rangle \leq \\
&||\{\phi(T^\gamma)-\phi(H^\gamma)\}V_t f||\ ||\{\psi(H)-\psi(H(\gamma))\}V_t f||+||F(|x|>c|t|)U_t^* V_t f|| \\
&+ ||(\tilde{\Omega}(\gamma)-1)\phi(H^\gamma)\psi(H(\gamma))U_t F(|x| \leq c|t|)||+||V_t(\gamma)(\tilde{\Omega}^+(\gamma))^* f||
\end{aligned}$$

(3.26)

When we take $E \lim_{t\to\infty}$ on both the sides of (3.26), the first and the second terms are zero by Lemma 2.18 (iii) and (ii) respectively. We choose some $c > 0$ as in Theorem 3.7 so that the fourth term is zero. Now the third term is zero by Theorem 2.16 (vi). Finally the last term is zero since by choice $f \in (F_+(\gamma))^\perp$.

Since we verified the local decay and the low energy decay conditions, by the theory of chapter II we now have a theorem.

Theorem 3.8 (Completeness of three particle scattering). Let the pair potentials of a three particle system satisfy the conditions (A1) - (A5). Then,

(i) $\mathcal{H}_{sc}(H) = \{0\}$

and

(ii) $\sum_{D}^{\oplus} F_{+}(D) = \mathcal{H}_{ac}(H) = \sum_{D}^{\oplus} F_{-}(D)$ .

Remark 3.9 (i) Except for Theorem 3.2 (i), (ii) we do not require the condition (A4) that is $(2+\varepsilon)$ decay on the pair potentials at $\infty$, $(1+\varepsilon)$ decay suffices.

(ii) We admit in principle the possibility of having infinite number of eigenvalues for the two particle subsystems, though technically the condition (A 4) implies finiteness of such eigenvalues.

(iii) We also do not make use of the decay of eigen functions of the two particle subsystems.

(iv) However we make use of the absence of positive eigenvalues.

## § 2. Four particle scattering

In this section we verify the local decay and the low energy decay conditions for the four particle system under more stringent conditions on the pair potentials than those assumed in section 1. We also have some implicit conditions on the spectral properties of the two particle system. We list our assumptions on the potentials below.

Since we verified the local decay and the low energy decay conditions, by the theory of chapter II we now have a theorem.

Theorem 3.8 (Completeness of three particle scattering). Let the pair potentials of a three particle system satisfy the conditions (A1) - (A5). Then,

(i) $\mathcal{H}_{sc}(H) = \{0\}$

and

(ii) $\sum_{D}^{\oplus} F_+(D) = \mathcal{H}_{ac}(H) = \sum_{D}^{\oplus} F_-(D)$ .

Remark 3.9 (i) Except for Theorem 3.2 (i), (ii) we do not require the condition (A4) that is $(2+\varepsilon)$ decay on the pair potentials at $\infty$, $(1+\varepsilon)$ decay suffices.

(ii) We admit in principle the possibility of having infinite number of eigenvalues for the two particle subsystems, though technically the condition (A 4 ) implies finiteness of such eigenvalues.

(iii) We also do not make use of the decay of eigen functions of the two particle subsystems.

(iv) However we make use of the absence of positive eigenvalues.

§ 2. Four particle scattering

In this section we verify the local decay and the low energy decay conditions for the four particle system under more stringent conditions on the pair potentials than those assumed in section 1. We also have some implicit conditions on the spectral properties of the two particle system. We list our assumptions on the potentials below.

Throughout this section unless specified otherwise, we take N = 4 and assume that the pair potentials satisfy the following. We set $\rho_3(\lambda) \equiv (1+\lambda^2)^{-\delta_3}, \delta_3 > \frac{5}{4}$.

(A1) $W_\gamma (T^\gamma+1)^{-1}$ is a compact operator on $L^2(\mathbb{R}^\nu)$.

(A2) $(T^\gamma+1)^{-1} (x^\gamma \nabla_\gamma W_\gamma(x^\gamma))(T^\gamma+1)^{-1}$ is compact on $L^2(\mathbb{R}^\nu)$.

(A6) $W_\gamma \, \rho_3(x^\gamma)^{-1} (T^\gamma+1)^{-1}$ is a bounded operator.

(A7) $[T^\gamma, W_\gamma](T^\gamma+1)^{-1}$, $\mathrm{Ad}^k_{A^\gamma}\{W_\gamma\} \; (T^\gamma+1)^{-1}$ are bounded for k = 1,2,3 for all pairs $\gamma$.

(A8) The two body Hamiltonians $H^\gamma$ do not have any quasi boundstates at zero (that is functions f such that $\frac{1}{2}\Delta_\gamma f = W_\gamma f$, $f \notin L^2(\mathbb{R}^\nu)$) and $\sigma_p(H^\gamma) = \emptyset$ for all pairs $\gamma$.

<u>Remark</u> The asymptotic completeness statement (AC3) under assumption (A8) reduces to

$$\sum_D^{\oplus} F_+(D) = H_{ac}(H) = \sum_D^{\oplus} F_-(D)$$

the direct sum being taken over only a class of two cluster clusterings corresponding to three particle boundstates and the clustering consisting of three particle boundstates and the clustering consisting of four free particles. All the pair clusters will be absent.

In the sequel we employ the following notation. If D is a clustering with # D = 2, then $L^2(X^D)$ corresponds to a three particle Hilbert space and $H^D$ a three particle Hamiltonian. In such a case for any pair $\gamma \in i(D)$, we write

$$L^2(X^D) = L^2(X^\gamma) \otimes L^2(X^D_\gamma)$$

where we note that it is possible to decompose

$$X^D = X^\gamma \oplus X_\gamma^D$$

and we write

$$H^D(\gamma) = T^D + W_\gamma, \quad T^D = T^\gamma + T_\gamma^D, \quad x^D = (x^\gamma, x_\gamma^D) \quad \text{and} \quad P^D = (P^\gamma, P_\gamma^D).$$

An operator $B_\gamma^D$ is to be understood as an operator on $L^2(X_\gamma^D)$ or as $1 \otimes B_\gamma^D$ on $L^2(X^D)$.

In the case of clusterings with # D = 2, in the sequel, we often have to distinguish two cases. So we have a definition.

__Definition 3.10__ Let D be a clustering with # D = 2. If $i(D) = \{\alpha, \beta\}$ for disjoint pairs $\alpha, \beta$ then we call D as a __disconnected__ clustering. Otherwise it will be called a __connected__ clustering.

We have a few technical results to start with as in Section 1. We recall that

$$\rho_3(\lambda) = (1+\lambda^2)^{-\delta_3}, \quad \delta_3 > \frac{5}{4}.$$

__Theorem 3.11__ Let D be a clustering with # D = 2, $\psi \in C_0^\infty(\mathbb{R})$ and $a > 0$.

(i) $$||F(\bigwedge_{\alpha \in i(D)} |x^\alpha| > a\,r)\{\psi(H^D) - \psi(T^D)\}|| \leq K(1+ar)^{-2\delta_3}$$

(ii) If D is connected, then for any distinct pairs $\alpha, \beta, \gamma \in i(D)$,

$$||F(|x^\gamma| \wedge |x^\beta| > ar)\{\psi(H^D(\alpha)) - \psi(H^D)\}|| \leq K(1+ar)^{-2\delta_3}$$

(iii) For any $\beta \in i(D)$,

$$||F(|x^\beta| > ar)\{\psi(H^D(\beta)) - \psi(T^D)\}|| \leq K(1+ar)^{-2\delta_3}$$

(iv) For any $\phi \in C_o(\mathbb{R}^\nu)$,

$$\phi(x^D)\{\psi(H) - \psi(H(D))\}$$

is a compact operator.

(v) $\rho_3(x^\beta)\{\psi(H^D)-\psi(H^D(\beta))\}\ \rho_3(x^D)^{-1}$ is a bounded operator for any $\beta \in i(D)$.

We note that $H^D - T^D = W_\alpha + W_\beta$ in the case of disconnected D and $H^D - H^D(\beta) = W_\alpha + W_\gamma$ in the case of connected D. Also $H^D(\beta) - T^D = W_\beta$, $H - H(D) = \sum_{\gamma \varepsilon (D)} W_\gamma$ for every D and $\inf \sigma(H) > -\infty$. Using these facts and the decay properties of the potentials (condition (A6)), the proof of this Theorem proceeds exactly as that of Lemma 3.1, using Lemma 2.4. Hence we omit the proof.

Now we state two results useful for proving the local decay condition. The first of these, a two body result, is due to Hagadorn and it was successfully exploited in [HP] for proving completeness in three particle scattering. It is this result that the assumption (A8) about the absence of boundstates or quasi boundstates at 0 energy of the pair Hamiltonians is required. This condition is necessitated by a counter example of Jensen-Kato given in [JK]. The second result is on the N-particle total evolution of [JMP] who use the conjugate operator methods developed by Mourre. There is a technical flaw in [JMP], however this can be corrected, [SKB].

<u>Proposition 3.12</u> Consider the two body Hamiltonian $H^\alpha$ for any pair $\alpha$ with $E^\alpha$ its point spectral projection. If $W_\alpha\ \rho(x^\alpha)^{-1}$ is $T^\alpha$ compact for $\rho(\lambda) = (1+\lambda^2)^{-\delta}$, $\delta > 1$, then as functions of t,

(i) $||\rho(x^{\alpha})^{\frac{1}{2}} \exp(-itH^{\alpha})(1-E^{\alpha})\rho(x^{\alpha})^{\frac{1}{2}}|| \in L^1(\mathbb{R})$

(ii) If $\gamma$ is a pair such that $\gamma \cap \alpha = \emptyset$, then

$$||\rho(x^{\gamma})^{\frac{1}{2}} \exp(-it\, H^{\alpha})\ (1-E^{\alpha})\rho(x^{\alpha})^{\frac{1}{2}}|| \in L^1(\mathbb{R}).$$

A proof of the above result can be found in [Ha] and [HP].

Since in our case by (A8), $E^{\alpha} \equiv 0$, we have the above result with 1 replacing $(1-E^{\alpha})$.

For the following Theorems we set $\rho_4(\lambda) = (1+\lambda^2)^{-\delta_4}$, $\delta_4 > \frac{3}{4}$.

<u>Theorem 3.13</u> Let $H$ be the N-particle Hamiltonian with the pair potentials satisfying (A1), (A2) and (A6). Then for $\psi \in C_0^{\infty}(\mathbb{R} \setminus T(H))$,

$$||\rho_4(A)V_t\ \psi(H)\rho_4(A)|| \leq K(1+|t|)^{-\mu_1}, \text{ for some } \mu_1 > 1.$$

This result is a straightforward corollory of the main results of [JMP]. Hence we refer to this work for a proof.

<u>Remark</u>. The result of [JMP] implies that if $\delta_4 > \frac{5}{8}$, we have $(1+\varepsilon)$ decay when the potentials are 4 times differentiable. However if one admits a large number of differentiations, then $\delta_4 > \frac{1}{2}$ will also lead to $(1+\varepsilon)$ decay in time. In the latter case the assumption on $\rho_3$ of (A6) can be relaxed to $\delta_3 > 1$ in the proof of Theorem 3.15 and hence in the final result.

<u>Corollory 3.14</u> Let $D$ be a clustering with # D = 2. If the potentials satisfy the conditions of Theorem 3.13 then for any $\psi \in C_0^{\infty}(\mathbb{R} \setminus T(H))$ and $\mu_1 > 1$,

$$||\rho_4(x^D)V_t^D\ \psi(H^D)\rho_4(x^D)|| \leq K(1+|t|)^{-\mu_1} .$$

Proof Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi\,\psi = \psi$. Then we write

$$||\rho_4(x^D)V_t^D\,\psi(H^D)\;\rho_4(x^D)|| \;\leq\; ||\rho_4(x^D)\chi(H^D)\rho_4(A^D)^{-1}||.$$

$$\cdot||\rho_4(A^D)\psi(H^D)V_t^D\;\rho_4(A^D)||\cdot||\rho_4(A^D)^{-1}\;\chi(H^D)\rho_4(x^D)||.$$

The first and the third factors on the right hand side of the above inequality are finite by Lemma 2.4 (iv). Now the result follows by using Theorem 3.13 to estimate $||\rho_4(A^D)V_t^D\psi(H^D)\rho_4(A^D)||$.

Using the above corollory we derive some useful $L^1$ estimates on some three particle total evolutions weighted in the two particle sectors. We make use of the method of Hagedorn and Perry for this purpose. Our estimate is an extention of their result in [HP].

Recall that $\rho_4(\lambda) = (1+\lambda^2)^{-\delta_4}$, $\delta_4 > \frac{3}{4}$.

Theorem 3.15 Let D be a connected clustering with $\#\, D = 2$ and $\alpha \in i(D)$. Then for $\psi \in C_0^\infty\;(\mathbb{R} \setminus \Upsilon(H^D))$ we have for some $\mu_1 > 1$,

$$||\rho_4(x^\alpha)V_t^D\;\psi(H^D)\rho_4(x^D)|| \;\leq K(1+|t|)^{-\mu_1}.$$

Proof Let $\phi \in C_0^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on the support of $\psi$ and set,

$$I^D(\alpha,t) \equiv \rho_4(x^\alpha)V_t^D\psi(H^D)\rho_4(x^D) = \rho_4(x^\alpha)\phi(H^D)V_t^D\;\psi(H^D)\rho_4(x^D)$$

Then,

$$I^D(\alpha,t) = \rho_4(x^\alpha)\{\phi(H^D)-\phi(H^D(\alpha))V_t^D\;\psi(H^D)\rho_4(x^D)$$

$$+\;\rho_4(x^\alpha)\phi(H^D(\alpha))V_t^D(\alpha)\psi(H^D)\rho_4(x^D)$$

$$+\;\rho_4(x^\alpha)\phi(H^D(\alpha))\{V_t^D-V_t^D(\alpha)\}\psi(H^D)\rho_4(x^D) \qquad (3.27)$$

By Theorem 3.10 (v) and Corollory 3.14, the first term of (3.27) has the following estimate for some $\mu_1 > 1$.

$$||\rho_4(x^\alpha)\{\phi(H^D)-\phi(H^D(\alpha))\}V_t^D\ \psi(H^D)\rho_4(x^D)||$$

$$\leq ||\rho_4(x^\alpha)\{\phi(H^D)-\phi(H^D(\alpha))\}\ \rho_4(x^D)^{-1}||\ ||\rho_4(x^D)V_t^D\ \psi(H^D)\rho_4(x^D)||$$

$$\leq K(1+|t|)^{-\mu_1} \tag{3.28}$$

The second term of (3.27) is estimated using Lemma 2.4 (i) and Proposition 3.12. Since $E^\alpha \equiv 0$ by assumption (A8), we have,

$$||\rho_4(x^\alpha)\phi(H^D(\alpha))V_t^D(\alpha)\psi(H^D)\rho_4(x^D)||$$

$$\leq ||\rho_4(x^\alpha)\phi(H^D(\alpha))\rho_4(x^\alpha)^{-1}||\ ||\rho_4(x^\alpha)V_t^\alpha\rho_4(x^\alpha)||\,||\rho_4(x^\alpha)^{-1}\psi(H^D)\rho_4(x^D)||$$

$$\leq\ K\ h(t)$$

where $h \in (L^1\cap L^\infty)(\mathbb{R})$. Using (3.28) and (3.29) for some $h_1 \in (L^1\cap L^\infty)(\mathbb{R})$, we obtain the following estimate

$$||\ I^D(\alpha,t)||\ \leq\ h_1(t)+||\rho_4(x^\alpha)\phi(H^D(\alpha))(V_t^D-V_t^D(\alpha))\psi(H^D)\rho_4(x^D)||$$

$$\leq h_1(t)+\int_0^t ds\sum_{\substack{\gamma\varepsilon i(D)\\ \gamma\neq\alpha}}||\rho_4(x^\alpha)\phi(H^D(\alpha))V_{t-s}^D(\alpha)W_\gamma V_s^D\psi(H^D)\rho_4(x^D)|| \tag{3.30}$$

Now for any $R > 0$, taking the sum over $\gamma \varepsilon i(D)$, $\gamma \neq \alpha$,

$$||I^D(\alpha,t)||$$

$$\leq h_1(t)+\sum_{\gamma\neq\alpha}\{\int_0^t ds||\rho_4(x^\alpha)\phi(H^D(\alpha))V_{t-s}^D(\alpha)W_\gamma F(|x^D|\leq R)V_s^D\psi(H^D)\rho_4(x^D)||$$

$$+\int_0^t ds||\rho_4(x^\alpha)\phi(H^D(\alpha))V_{t-s}^D(\alpha)W_\gamma\rho_4(x^\gamma)^{-1}\ F(|x^D|>R)||\cdot$$

$$||\rho_4(x^\gamma)V_s^D\psi(H^D)\rho_4(x^D)||\} \tag{3.31}$$

By Proposition 3.12 and Corollory 3.14 the estimate that

$$\sum_{\gamma\neq\alpha} \int_0^t ds \| \rho_4(x^\alpha)\phi(H^D(\alpha))V^D_{t-s}(\alpha)W\ F(|x^D|\leq R)V^D_s\psi(H^D)\rho_4(x^D)\|$$

$$\leq \sum_{\gamma\neq\alpha} \int_0^t ds \|\rho_4(x^\alpha)\phi(H^D(\alpha))(H^D(\alpha)+i)\rho_4(x^\alpha)^{-1}\| \|\rho_4(x^\alpha)V^\alpha_{t-s}\rho_4(x^\gamma)\| \cdot$$

$$\cdot \|\rho_4(x^\gamma)(H^D(\alpha)+i)^{-1}\ W_\gamma\|\ \|F(|x^D| \leq R)V^D_s\ \psi(H^D)\rho_4(x^D)\| \cdot$$

$$\leq h_2(t,R)$$

follows with $h_2(.,R) \in (L^1 \quad L^\infty)(\mathbb{R})$ for each fixed R. Hence the inequality (3.31) becomes

$$\|I^D(\alpha,t)\|$$

$$\leq h_1(t)+h_2(t,R)+\sum_{\gamma\neq\alpha} \int_0^t ds\|\rho_4(x^\alpha)V^D_{t-s}(\alpha)\phi(H^D(\alpha))W_\gamma\rho_4(x^\gamma)^{-1}F(|x^D|>R)\| \cdot$$

$$\|I^D(\gamma,s)\| \quad (3.32)$$

Now we set,

$$K^D(\alpha,t) \equiv \|I^D(\alpha,t)\|$$

and

$$J_R(t,\gamma,\alpha) \equiv \|\rho_4(x^\alpha)V^D_t(\alpha)\phi(H^D(\alpha))W_\gamma\ \rho_4(x^\gamma)^{-1}\ F(|x^D| > R)\| \qquad (3.33)$$

We claim that

$$\|J_R(.,\gamma,\alpha)\|_1 \to 0$$

as $R \to \infty$. We take $\chi_T$ to be the characteristic function of $[0,T]$. Then the inequality (3.32) we write for $t \in [0,T]$ as

$$K^D(\alpha,t) \leq h_R(t) + \sum_{\gamma\neq\alpha} \int_0^t ds\ J_R(t-s,\gamma,\alpha)K^D(\gamma,s)$$

$$\chi_T\ K^D(\alpha,t) \equiv h_R(t) + \sum_{\gamma\neq\alpha} (J_R(.,\gamma,\alpha) * K^D(\gamma,.)\chi_T)(t) \ ,$$

so that

$$||\chi_T\, K^D(\alpha,.)||_1 \le ||h_R||_1 + \sum_{\gamma\neq\alpha} ||J_R(.,\gamma,\alpha)||_1\; ||\chi_T\, K^D(\gamma,.)||_1 .$$

Therefore

$$\sum_\alpha ||\chi_T K^D(\alpha,.)||_1 \le 3||h_R||_1 + (2\max_{\gamma\neq\alpha}||J_R(.,\gamma,\alpha)||_1) \sum_\alpha ||\chi_T K^D(\alpha,.)||_1$$

which implies, by choosing R large enough, that, for all $\alpha \in i(D)$,

$$||\chi_T\, K^D(\alpha,.)||_1 \le K < \infty ,$$

for $K$ independent of $T$ for all $T > 0$. Now we show our claim. We note that pointwise in t, $J_R(t,\gamma,\alpha) \to 0$ as $R \to \infty$, by the compactness of

$$\rho_4(x^\alpha) V_t^D(\alpha)\phi(H^D(\alpha)) W_\gamma\; \rho_4(x^\gamma)^{-1}$$

for each fixed t, since $F(|x^D| > R) \to 0$ strongly as $R \to \infty$. Therefore the claim follows by Lebesgue dominated convergence theorem if we show that $J_R(t,\gamma,\alpha)$ is integrable in t, uniformly in R. For this we choose a $\delta_4 > \frac{3}{4}$ such that $\delta_3 - \delta_4 > \frac{1}{2}$ and set $\rho_5(\lambda) = (1+\lambda^2)^{\delta_4 - \delta_3}$. Then by (3.33), Lemma 2.4 (i) and Corollory (3.14) we have,

$$\int_0^\infty dt\; J_R(t,\gamma,\alpha)$$

$$\le \int_0^\infty dt||\rho_4(x^\alpha)V_t^D(\alpha)\rho_5(x\;)||\cdot||\rho_5(x^\gamma)^{-1}\;\phi(H^D(\alpha))W_\gamma\rho_4(x^\gamma)^{-1}||$$

$$\le\; K\int_0^\infty dt||\rho_4(x^\alpha)V_t^\alpha\;\rho_5(x^\gamma)|| < \infty .$$

Hence the result. Finally we note that the method of proof is not applicable for disconnected clusterings D.

We shall now fix a few constants coming from the relations between the coordinates. Consider any connected clustering with # D = 2. Then for any distinct pairs $\gamma, \alpha \varepsilon i(D)$, $x^{\gamma} = c(\alpha,\gamma)x^{\alpha} + d(\alpha,\gamma)\, x_{\alpha}^{D}$. Then we set,

$$c_1(D) = \max_{\gamma \neq \alpha} c(\alpha,\gamma), \quad c_2(D) = \max_{\gamma \neq \alpha} d(\alpha,\gamma)$$
$$d_1(D) = \min_{\gamma \neq \alpha} c(\alpha,\gamma), \quad d_2(D) = \min_{\gamma \neq \alpha} d(\alpha,\gamma) \tag{3.35}$$

Lemma 3.16  Let D be a connected clustering with # D = 2. Given any positive number d, there exist positive constants b, $a_3$ and $a_4$, smaller then d and depending only on d such that for $\phi \in C_o^{\infty}(\mathbb{R})$, $0 \leq \phi \leq 1$, supp $\phi$ contained in $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$ and for pairs $\gamma, \alpha \in i(D)$ the following estimates are valid.

(i) $$||F(|x^{\alpha}| \leq a_3|t|)F(|x_{\alpha}^{D}| > a_4|t|)V_{\tau}^{D}(\alpha)\phi(H^D(\alpha))\rho_3(x^{\gamma})|| \leq K(1+|t|)^{-\mu_2}$$

for some $\mu_2 > 2$ and all $\tau \in (0,t)$.

(ii) There is a $c > 0$ such that for some $\mu_1 > 1$ and all $\tau \in (0,s)$,

$$||F(|x_{\alpha}^{D}| > a_4|t+s|)V_{\tau}^{D}(\alpha)\phi(H^D(\alpha))U_t^D\, F(|x^D| \leq c|t+s|)|| \leq K(1+|t+s|)^{-\mu_1}$$

Proof (i) We consider the partition of the identity,

$$1 = F(|x^{\alpha}| > a_6|t|)+F(|x^{\alpha}|\leq a_6|t|, |x_{\alpha}^{D}|> a_5|t|)+F(|x_{\alpha}^{D}| \leq a_5|t|),$$

and $a_6 < d$, $a_6 > b + a_3$, $d_2(D)a_5 > c_1(D)a_6$ and $a_4 > b+a_5$. (3.36)

Then we consider the inequality that

$$\begin{aligned} &||F(|x^{\alpha}| < a_3|t|)F(|x_{\alpha}^{D}| > a_4|t|)V_{\tau}^{D}(\alpha)\phi(H^D(\alpha))\rho_3(x^{\gamma})|| \\ &\leq \{||\rho_3(x^{\gamma})||\ ||F(|x^{\alpha}| \leq a_3|t|)V_{\tau}^{\alpha}\ \phi(H^D(a))F(|x^{\alpha}| > a_6|t|)|| \\ &+ ||F(|x^{\alpha}| \leq a_6|t|,\ |x_{\alpha}^{D}| > a_5|t|)\rho_3(x^{\gamma})|| \\ &+ ||\rho_3(x^{\gamma})||\ ||F(|x_{\alpha}^{D}|>a_4|t|)U_{\tau,\alpha}^{D}\ \phi(H^D(\alpha))F(|x_{\alpha}^{D}|\leq a_5|t|)||\} \end{aligned} \tag{3.37}$$

We recall that $\rho_3(\lambda) = (1+\lambda^2)^{-\delta_3}$, $\delta_3 > \frac{5}{4}$, hence the estimate for the second term of (3.37) is clear by the third of the inequalities in (3.36). For the first term we have an explanation to offer. We see that except for the factor $\phi(H^D(\alpha))$ which is an operator on $L^2(X^D)$, the remaining factors are operators on $L^2(X^\alpha)$ and $\phi(H^D(\alpha))$ commutes with $\exp(-i\, t'\, T^D_\alpha)$ for each $t' \in \mathbb{R}$. Thus we can write $\phi(H^D(\alpha))$ in terms of operator valued functions $\phi_{t^D_\alpha}(H^\alpha)$ with each of $\phi_{t^D_\alpha}(.)$ having the upper bound of support atmost the upper bound for the support of $\phi$ by virtue of the non-negativity of $T^D_\alpha$. Also owing to the non-negativity of $H^\alpha$ (By (A8)), $t^D_\alpha$ is restricted to a compact set. Hence we can estimate the first term of (3.37) with $\phi_{t^D_\alpha}(H^\alpha)$ replacing $\phi(H^D(\alpha))$ and then take a supremum over $t^D_\alpha$ to get a final estimate. Hence using the inequality (3.36), Theorem 3.3 (iii) (with $\tau = 0$ there) we get that for some $\mu_2 > 2$,

$$||F(|x^\alpha| \leq a_3|t|)V^\alpha_\tau\, \phi(H^D(\alpha))F(|x^\alpha| > a_6|t|)|$$

$$\leq \sup_{0 \leq t^D_\alpha \leq K_1} ||F(|x^\alpha| \leq a_3|t|)V^\alpha_\tau\, \phi_{t^D_\alpha}(H^\alpha)F(|x^\alpha| > a_6|t|)||$$

$$\leq K(1+|t|)^{-\mu_2}.$$

The last term of (3.37) is similarly estimated using Theorem 2.11 (i) and the inequality (3.36). In fact it is for this term that we crucially need the condition (A8) that $\sigma_{pp}(H\ ) = \emptyset$ so that $H^\alpha + T^D_\alpha < \frac{1}{2}b^2$ implies $T^D_\alpha \leq \frac{1}{2}b^2$, which is not true if $H^\alpha < 0$ is allowed. Hence we have the result.

(ii) The proof is similar to that of (i) if we note that

$$||F(|x^D_\alpha| > a_4|t+s|)V^D_\tau(\alpha)\phi(H^D(\alpha))U^D_t\, F(|x^D| \leq c|t+s|)||$$

$$\leq ||F(|x^D_\alpha| > a_4|t+s|)U^D_{t+s,\alpha}\, \phi(H^D(\alpha))F(|x_\alpha| \leq c|t+s|)||,$$

choose $b+c < a_4$ and use Theorem 2.11 (i).

<u>Theorem 3.17</u> Let $D$ be a clustering as in Theorem 3.15. Given a positive number $d$, there exist positive $a_1$, $b$ and $c$, less than $d$, such that for any $\phi \in C_0^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, $\text{supp}\,\phi \subseteq (-\frac{1}{2}b^2, \frac{1}{2}b^2)$ and some $\mu_1 > 1$, $\tau \in [0,s]$

$$||F(|x^D| > a_1|t+s|)\phi(H^D)V_\tau^D\, U_t^D\, F(|x^D| \leq c|t+s|)|| \leq K(1+|t+s|)^{-\mu_1}$$

<u>Proof</u> We take some $a_1 > 0$ and $a_2 = \dfrac{a_1}{2\sqrt{3}}$ and note that since there is a partition of the identity (see [RS IV, p. 134], or [SB1]) $\{F_\alpha\}$ with

$$1 = \sum_{\alpha \in i(D)} F_\alpha\Big(\bigwedge_{\substack{\gamma \neq \alpha \\ \gamma \in i(D)}} |x^\gamma| > \frac{1}{2\sqrt{3}}|x^D|\Big) \equiv \sum_{\alpha \in i(D)} F_\alpha\Big(\frac{1}{2\sqrt{3}}|x^D|\Big)$$

and since

$$||F(|x^D|>a_1|t+s|)F_\alpha\Big(\bigwedge_{\gamma\neq\alpha}|x^\alpha| > \frac{1}{2\sqrt{3}}|x^D|\Big)\phi(H^D)V_\tau^D\, U_t^D\, F(|x^D|\leq c|t+s|)||$$

$$\leq ||F_\alpha(a_2|t+s|)\phi(H^D)V_\tau^D\, U_t^D\, F(|x^D| \leq c|t+s|)||\,,$$

it suffices to show that there exist positive $a_2$, $b$, $c$, less than $d$, such that,

$$||F_\alpha(a_2|t+s|)\phi(H^D)V_\tau^D\, U_t^D\, F(|x^D| \leq c|t+s|)|| \leq K(1+|t+s|)^{-\mu_1}\,.$$

Now for any $a_3$, $b$, $c > 0$, we have,

$$||F_\alpha(a_2|t+s|)\phi(H^D)V_\tau^D\, U_t^D\, F(|x^D| \leq c|t+s|)||$$

$$\leq ||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)\{\phi(H^D)-\phi(T^D)\}||$$

$$+||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)\phi(T^D)U_{\tau+t}^D\, F(|x^D| \leq c|t+s|)||$$

$$+||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)\phi(T^D)(V_\tau^D-U_\tau^D)U_t^D F(|x^D| \leq c|t+s|)||$$

$$+||F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|)\{\phi(H^D)-\phi(H^D(\alpha))\}||$$

$$+ \| F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|)\phi(H^D(\alpha))V_\tau^D(\alpha)U_t^D\, F(|x^D| \leq c|t+s|)\|$$

$$+ \| F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|)\phi(H^D(\alpha))(V_\tau^D - V_\tau^D(\alpha))U_t^D F(|x^D| \leq |t+s|)\| \tag{3.38}$$

By Theorem 3.11 (i) the first term of (3.38) has the estimate that

$$\| F_\alpha(a_2|t+s|)F(|x^\alpha| > a_3|t+s|)\{\phi(H^D) - \phi(T^D)\}\| \leq K(1+|t+s|)^{-\mu_1},\ \mu_1 > 1. \tag{3.39}$$

We choose the constants $a_2$, $a_3$ to satisfy

$$a_2 \wedge a_3 > b+c \tag{3.40}$$

Then the second term of (3.38) has the following estimate using Theorem 2.11(i), for every $M > 0$.

$$\| F_\alpha(a_2|t+s|)F(|x^\alpha| > a_3|t+s|)U_{t+\tau}^D\ \phi(T^D)F(|x^D| \leq c|t+s|)\| \leq K(1+|t+s|)^{-M}. \tag{3.41}$$

By Theorem 3.11 (ii), the fourth term of (3.38) is estimated as, for $\mu_1 > 1$,

$$\| F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|)\{\phi(H^D) - \phi(H^D(\alpha))\}\| \leq K(1+|t+s|)^{-\mu_1},$$

Using $c_2(D)$, $c_1(D)$ as in (3.35), we choose $a_4$ and $a_2$ and $a_3$ are further required to satisfy,

$$a_4 > c_2(D)^{-1}\{a_2 - c_1(D)a_3\} \tag{3.42}$$

In that case it is clear that

$$F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|) \leq F(|x^\alpha| \leq a_3|t+s|)F(|x_\alpha^D| > a_4|t+s|).$$

Then we estimate the fifth term of (3.38) by, for any $M > 0$,

$$||F_\alpha(a_2|t+s|)F(|x^\alpha| \leq a_3|t+s|)V^D_\tau(\alpha)\phi(H^D(\alpha))U^D_t F(|x^D| \leq c|t+s|)||$$

$$\leq ||F(|x^D_\alpha| > a_4|t+s|)V^D_{\tau+t,\alpha}\,\phi(H^D(\alpha))F(|x^D| \leq c|t+s|)||$$

$$\leq K(1+|t+s|)^{-M}. \tag{3.43}$$

as in Lemma 3.16 using Theorem 2.11 (ii) after choosing

$$a_4 > b + c. \tag{3.44}$$

The third term is estimated as follows.

$$||F_\alpha(a_2|t+s|)F(|x^\alpha| > a_3|t+s|)\phi(T^D)(V^D_\tau - U^D_\tau)U^D_t\, F(|x^D| \leq c|t+s|)||$$

$$\leq \sum_{\gamma \notin (D)} \int_0^\tau ds_1 ||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)U^D_{s_1}\phi(T^D)W_\gamma V^D_{\tau-s_1}U_t\, F(|x^D|\leq c|t+s|)||$$

$$\leq \sum_{\gamma \in i(D)} \int_0^\tau ds_1 ||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)U^D_{s_1}\phi(T^D)(T^D+1)\rho_3(x^\gamma)|| \cdot$$

$$\cdot ||\rho_3(x^\gamma)^{-1}(T^D+1)^{-1}\, W_\gamma||\cdot$$

By Lemma 2.4 (i), making use of the partition

$$F(|x^\gamma| > c|t+s|) + F(|x^\gamma| \leq c|t+s|)$$

of the identity, the above term is dominated by,

$$\sum_{\gamma \in i(D)} \int_0^\tau ds_1 \{||F_\alpha(a_2|t+s|)F(|x^\alpha|>a_3|t+s|)U^D_{s_1}\phi(T^D)F(|x^\gamma|\leq c|t+s|)||\,||\rho_3(x^\gamma)||$$

$$+ ||F(|x^\gamma|>c|t+s|)\rho_3(x^\gamma)||\}\cdot$$

Now using the arbitrary decay in (t+s) of the first term of the above integrand via Theorem 2.11 (i) and the $(1+|t+s|)^{-5/2}$ decay of the second term the bound $K(1+|t+s|)^{-\mu_1}$, $\mu_1 > 1$ is clear for the third term of

(3.38). The sixth term in (3.38) is estimated using Lemma 3.16 after choosing $a_3$, $a_4$ as in Lemma 3.16.

The last term in (3.38) is dominated by

$$\sum_{\gamma\neq\alpha} \int_0^\tau ds_1 \| F(|x^\alpha| \leq a_3|t+s|) F(|x_\alpha^D| > a_4|t+s|) V_{s_1}^D(\alpha)\phi(H^D(\alpha))(H^D(\alpha)+i)\rho_3(x^\gamma)\| \cdot$$

$$\cdot \| \rho_3(x^\gamma)^{-1}(H^D(\alpha)+i)^{-1} W_\gamma V_{\tau-s_1}^D F(|x^D| \leq c|t+s|)\| \cdot$$

By Lemma 3.16 it then follows that, for some $\mu_2 > 2$, this expression is bounded by

$$K\int_0^\tau ds_1 (1+|t+s|)^{-\mu_2} \leq K(1 + |t+s|)^{-\mu_1} \quad , \quad \mu_1 > 1.$$

Finally one can easily verify that the constants $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$, b and c can be chosen satisfying the conditions (3.36), (3.40), (3.42) and (3.44) for any given $d > 0$.

Our next result is a compactness statement connected with the wave operators of a class of clusterings.

We set $\rho(\lambda) = (1+\lambda^2)^{-1}$.

<u>Lemma 3.18</u> Let D be a clustering as in Theorem 3.15. Then for any $\psi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}(H))$,

$$(\tilde{\Omega}^\pm(D)-1)\psi(H(D))\rho(x^D)F(A_D \gtrless 0)$$

is compact.

<u>Proof</u>. (+ case only). We set $F_D^+ \equiv F(A_D > 0)$. We consider $b > 0$, small enough so that $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$ supp $\psi = \emptyset$ and take functions $\phi_1, \phi_2, \phi_3 \in C_0^\infty(\mathbb{R})$, $0 \leq \phi_1, \phi_2, \phi_3 \leq 1$, with their supports respectively contained in $(-\infty, 0)$, $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$ and $(0,\infty)$ so that

$(\phi_1+\phi_2+\phi_3)(H^D)\psi(H(D)) = \psi(H(D))$ and $\phi_2 \equiv 1$ on $(-\tfrac{1}{4}b^2, \tfrac{1}{4}b^2)$. Now,

$$(\tilde{\Omega}^+(D)-1)\psi(H(D))\rho(x^D)F_D^+$$

$$= \sum_{\gamma\varepsilon e(D)} \int_0^\infty ds\ V_s^* \ iW_\gamma \ V_s(D)\psi(H(D))\rho(x^D)F_D^+ .$$

By writing the integrand, in the above integral, as

$$(V_s)(W_\gamma(H_o+1)^{-1})((H_o+1)V_s(D)\psi(H(D))\rho(x^D))F_D^+ ,$$

we see that it is a product of a $\gamma$-compact and D-compact operators (by Remark 2.7) with $\gamma \varepsilon e(D)$ and $\# D = 2$. Hence by Proposition 2.8 the compactness of the product is clear. Now as in Lemma 3.3 it is enough to show, using (A6) and Lemma 2.4 (i), the integrability of

$$||\rho_3(x^\gamma)V_s(D)\psi_1(H(D))\rho(x^D)F_D^+||$$

for each $\gamma \varepsilon e(D)$ where $\psi_1(H(D)) \equiv \psi(H(D))(H(D)+i)$. This $\psi_1$, clearly, has the same support properties as $\psi$. We have

$$||\rho_3(x^\gamma)V_s(D)\psi_1(H(D))\rho(x^D)F_D^+||$$

$$\leq \sum_{i=1}^{3} ||\rho_3(x^\gamma)V_s(D)\phi_i(H^D)\psi_1(H(D))\rho(x^D)F_D^+||$$

$$\equiv \sum_{i=1}^{3} I_i^D(\gamma,s) \qquad (3.45)$$

We estimate the terms of (3.45) one by one.

By the choice of $\phi_2$, the support of $\phi_1$ is in $(-\infty, \tfrac{1}{4}b^2)$ and by Proposition 1.9, the eigenvalues of $H^D$ in $(-\infty,c]$, $c < 0$ are finite in number. Hence the support of $\phi_1$ has only finitely many eigenvalues. We take therefore, without loss of generality the support of $\phi_1$ to contain only one eigenvalue (say) $\lambda^D$ of $H^D$. Then,

$$I_1^D(\gamma,s) \leq ||\rho_3(x^\gamma)(1+x_D^2)^{\mu}\rho(x^D)||\ ||\rho(x^D)^{-1}\phi_1(H^D)\rho(x^D)||.$$

$$\cdot ||(1+x_D^2)^{-\mu} U_{t,D}\psi_1(\lambda^D+T_D)F_D^+|| \cdot \qquad (3.46)$$

for any $\mu > 0$. Since $\rho_3(x^\gamma) = (1+(x^\gamma)^2)^{-\delta_3}$, $\delta_3 > \frac{5}{4}$ and $\rho(x^D)=(1+(x^D)^2)^{-1}$,

$$\rho_3(x^\gamma)\rho(x^D) \leq K(1+x_D^2)^{-\mu}$$

for every $0 \leq \mu \leq 1$. Also since $\lambda^D \in T(H)$ and since $\lambda^D \notin \text{supp}\,\psi_1$, by choice, the function

$$\psi_{\lambda^D}(.) \equiv \psi_1(\lambda^D + .)$$

has support in $(0,\infty)$. Thus using Lemma 2.4 (i), Theorem 2.11 (iii) in (3.46) we get, by choosing $\mu > \frac{1}{2}$, that

$$I_1^D(\gamma,s) \leq K(1+|s|)^{-\mu_1}, \quad \mu_1 > 1 . \qquad (3.47)$$

Till now b was some positive number. Now we choose b depending upon $\psi$ for estimating the second term of (3.45). We take $b_1 > 0$ with $(-\frac{1}{2}(b_1^2 + b^2), \frac{1}{2}(b_1^2+b^2)) \cap \text{supp}\,\psi = \emptyset$ and also take a function $\psi_2 \in C_0^\infty(0,\infty)$, $0 \leq \psi_2 \leq 1$ satisfying

$$\psi_2(T_D)\phi_2(H^D)\psi_1(H(D)) = \phi_2(H^D)\psi_1(H(D)).$$

Then

$$I_2^D(\gamma,s) \leq ||\rho_3(x^\gamma)\psi_1(H(D))\rho_3(x^\gamma)^{-1}||.$$

$$\cdot\{||\rho_3(x^\gamma)||\ ||F(|x^D|>a_1|s|)V_s(D)\phi_2(D)\psi_2(T_D)\rho(x^D)F_D^+||$$

$$+ ||\rho_3(x^\gamma)F(|x^D| \leq a_1|s|, |x_D| > a_8|s|)||$$

$$+ ||\rho_3(x^\gamma)||\ ||F(|x_D| \leq a_8|s|)V_s(D)\phi_2(H^D)\psi_2(T_D)\rho(x^D)F_D^+||\}$$

(3.48)

We have $x^{\gamma} = c(\gamma,D)x^{D} + d(\gamma,D)x_{D}$. Therefore we set

$$\max_{\gamma \in e(D)} c(\gamma,D) = c_3(D) \text{ and } \min_{\gamma \in e(D)} d(\gamma,D) = c_4(D). \tag{3.49}$$

and take $a_1, b$ to satisfy the conditions of Lemma 3.16, we also take

$$c_4(D)a_8 > c_3(D)a_1 \quad , \quad b_1 > a_8 \tag{3.50}$$

With this choice of the constants, for some $c > 0$ as in Lemma 3.4 (ii), we see that the second term of (3.48) is dominated by $K(1+|s|)^{-\mu_1}$, for some $\mu_1 > 1$, by (A6), Lemma 2.4(i) and (3.50). The first term is dominated by, using Lemma 2.4 (i) and Theorem 3.17 (with $\tau = 0$ there),

$$K||F(|x^D| > a_1|s|)V_s^D \phi_2(H^D)\rho(x^D)||$$

$$\leq K||F(|x^D| > a_1|s|)V_s^D \phi_2(H^D)F(|x^D| \leq c|s|)||$$

$$+ ||F(|x^D| > c|s|)\rho(x^D)||$$

$$\leq K(1+|s|)^{-\mu_1} \quad , \quad \mu_1 > 1 . \tag{3.51}$$

The last term of (3.48) is estimated using Lemma 2.4 (i) and Theorem 2.11 (iii), since $b_1 > a_8$, as

$$||F(|x_D| \leq a_8|s|)U_{s,D}\, \psi_2(T_D)F_D^+|| \leq K(1+|s|)^{-\mu_1} , \mu_1 > 1 . \tag{3.52}$$

Thus the estimate

$$I_2^D(\gamma,s) \leq K(1+|s|)^{-\mu_1} \quad , \quad \mu_1 > 1 \quad , \tag{3.53}$$

follows from (3.51), (3.52) and the inequality (3.50).

The third term of (3.45) is dealt with as follows:

$$I_3^D(\gamma,s) \leq ||\rho_3(x^\gamma)\psi_1(H(D))\rho_3(x^\gamma)^{-1}|| \ ||\rho_3(x^\gamma)V_s^D \ \phi_3(H^D)\rho(x^D)|| \ \cdot$$

$$\leq \quad K||\rho_3(x^\gamma)V_s^D \ \phi_3(H^D)\rho(x^D)|| \tag{3.54}$$

Using Lemma 2.4 (i). Now we take a pair $\alpha \varepsilon i(D)$ with $\alpha \cap \gamma = \emptyset$. It is easy to see that such a pair exists. Then adding and subtracting a $V_s^D(\alpha)$ to $V_s^D$ in $I_3^D(\gamma,s)$, we have

$$I_3^D(\gamma,s) \leq K\{||\rho_3(x^\gamma) \ V_s^D(\alpha)\phi_3(H^D)\rho(x^D)|| + \sum_{\substack{\beta\varepsilon i(D)\\ \beta\neq\alpha}} \int_0^s ds_1||\rho_3(x^\gamma)V_{s-s_1}^D(\alpha)W_\beta \ V_s^D \ \phi_3(H^D)\rho(x^D)||\} \tag{3.55}$$

The first term of (3.55) is estimated using Proposition 2.10 (i) and Lemma 2.4 (i) as

$$||\rho_3(x^\gamma)V_s^D(\alpha)\phi_3(H^D)\rho(x^D)|| \leq ||\rho_3(x^\gamma)V_s^D(\alpha)\rho(x^D)|| \ ||\rho(x^D)^{-1} \ \phi_3(H^D)\rho(x^D)||$$

$$\leq \quad K||\rho_3(x^\gamma)U_{s,\alpha}^D \ \rho(x_\alpha^D)|| \quad \leq \quad K(1+|s|)^{-\mu_1}, \ \mu_1 \ > 1 \ . \tag{3.56}$$

In the estimate of the second term of (3.55), we use Proposition 2.10 (i), Lemma 2.4 (i), Theorem 3.15 and (A6). Recall that $\rho_4(\lambda) = (1+\lambda^2)^{-\delta_4}$, $\delta_4 > \frac{3}{4}$. We take $\rho_5(\lambda) = (1+\lambda^2)^{-\delta_5}$, where $\delta_5 = \delta_3-\delta_4$. Then from (3.55) and (3.56) we see that

$$I_3^D(\gamma,s) \leq K(1+|s|)^{-\mu_1} + \int_0^s ds_1||\rho_3(x^\gamma)V_{s-s_1}^D(\alpha)\rho_5(x^\beta)||.$$

$$||\rho_5(x^\beta)^{-1} \ W_\beta(H^D+i)^{-1}\rho_4(x^\beta)^{-1}|| \ ||\rho_4(x^\beta)V_{s_1}^D\phi_3(H^D)(H^D+i)\rho(x^D)||$$

$$\leq \quad K(1+|s|)^{-\mu_1} + K\int_0^s ds_1||\rho_3(x^\gamma)U_{s-s_1,\alpha}^D \ \rho_5(x^\beta)|| \ \cdot$$

$$\cdot \ ||\rho_4(x^\beta)V_{s_1}^D \ \phi_3(H^D)(H^D+i)\rho(x^D)||$$

$$\leq \quad K(1+|s|)^{-\mu_1} + K\int_0^s ds_1 \ (1+|s-s_1|)^{-\mu_1} \ (1+|s_1|)^{-\mu_1} \tag{3.57}$$

Thus,

$$I_3^D(\gamma,s) \leq K(1 + |s|)^{-\mu_1} . \tag{3.58}$$

Using the estimates (3.47), (3.54) and (3.58) in (3.45) we get the required result.

Lemma 3.19 Let D be a disconnected clustering with # D = 2. Then for $\psi \in C_o^\infty(R \setminus T(H))$,

$$(\tilde{\Omega}^{\pm}(D)-1)\psi(H(D))\rho(x^D)F(A_D \gtrless 0)$$

is compact.

Proof (+ case only) We have

$$(\tilde{\Omega}^{+}(D)-1)\psi(H(D))\rho(x^D)F(A_D > 0)$$

$$= \sum_{\gamma\epsilon e(D)} \int_0^\infty ds\, V_s^* \, iW_\gamma \, V_s(D)\psi(H(D))\rho(x^D)F(A_D > 0)$$

By an argument as in Lemma 3.18 it is enough to show the integrability of

$$I_\gamma^D(s) \equiv ||\rho_3(x^\gamma)V_s(D)\psi_1(H(D))\rho(x^s)||$$

for

$$\psi_1(H(D)) = (H(D)+i)\psi(H(D)) .$$

$$I_\gamma^D(s) \leq ||\rho_3(x^\gamma)V_s^\alpha \, \rho(x^\alpha)|| \; ||\rho(x^\alpha)^{-1} \, \psi_1(H(D))\rho(x^\alpha)|| .$$

Now using Lemma 2.4 (i) and Proposition 3.12 (ii) we get that

$$I_\gamma^D(s) \leq K(1+|s|)^{-\mu_1} , \mu_1 > 1 .$$

Hence the result.

Remark We remark that in the above Lemma the condition (A8) on the potentials is really not necessary. The result could have been proved on the lines of Lemma 3.3 if the condition (A8) is not used.

We collect the results of Lemmas 3.18 and 3.19 in the following

Theorem 3.20. Let D be any clustering with # D = 2. If $\psi \in C_o^\infty(\mathbb{R} \setminus T(H))$ and $\rho$ as in Lemma 3.18 then,

$$(\tilde{\Omega}^{\pm}(D)-1)\psi(H(D))\rho(x^D)F(A_D \gtrless 0).$$

is a compact operator.

As in the three particle scattering we have a Lemma on the ranges of the wave operators.

Lemma 3.21 Let D be any clustering. If the two and three particle scattering is complete, then

(i) Range $(\tilde{\Omega}^{\pm}(D)) = F_{\pm}(0) \oplus F_{\pm}(D)$

(ii) If $D = \{\alpha,\beta\}$ is a disconnected clustering, then

Range $(\tilde{\Omega}^{\pm}(D)) = F_{\pm}(0)$.

The proof of this Lemma proceeds exactly as that of Lemma 3.4 hence we omit the proof.

Our next Theorem is the first step towards proving the local decay result. The idea of the theorem is one of iteration and involves in showing that if $B^D$ is a D compact operator with # D = 2 and if

$$E \lim_{t \to \infty} ||B^D V_t f|| = 0$$

then it is true for $B^D$ with # D = 3 and so on.

We recall the definition of $H^{\pm}$ and $\rho$ that

$$H^{\pm} = H_c(H) \ominus \sum_{D:\, 2\leq \# D\leq N}^{\oplus} F_{\pm}(D) \text{ and } \rho(\lambda) = (1+\lambda^2)^{-1}.$$

Theorem 3.22 Let $\phi \in C_0^{\infty}(\mathbb{R}^{2\nu})$. For any clustering with $\# D = 2$ and $f \in H^{\pm}$,

(i) $E \lim_{t\to\pm\infty} ||\phi(x^D)V_t f|| = 0$

(ii) $E \lim_{t\to\pm\infty} ||\rho(x^{\alpha})\rho(x^{\gamma})V_t f|| = 0$ for $\alpha \neq \gamma$.

Proof (+ case only) We note that (ii) easily follows from (i). We take $f$ such that $\psi(H)f = f$ with $\psi \in C_0^{\infty}(\mathbb{R}\setminus T(H))$ by density. As in Theorem 3.5 we consider only

$$E \lim_{t\to\infty} \langle V_t f, \rho(x^D)V_t f\rangle$$

and then we have the equality,

$$E \lim_{t\to\infty} \langle V_t f, \rho(x^D)V_t f\rangle$$

$$\begin{aligned} = E \lim_{t\to\infty} \{ & \langle V_t, (\psi(H)-\psi(H(D)))\rho(x^D)V_t f\rangle \\ & + \langle V_t f, \tilde{\Omega}^{+}(D)\psi(H(D))\rho(x^D)F(A_D > 0)V_t f\rangle \\ & + \langle V_t f, (-\tilde{\Omega}^{+}(D)+1)\psi(H(D))\rho(x^D)F(A_D > 0)V_t f\rangle \\ & + \langle V_t f, \tilde{\Omega}^{-}(D)\psi(H(D))\rho(x^D)F(A_D < 0)V_t f\rangle \\ & \langle V_t f, (-\tilde{\Omega}^{-}(D)+1)\psi(H(D))\rho(x^D)F(A_D < 0)V_t f\rangle\} \end{aligned} \qquad (3.59)$$

The first, third and the fifth terms of (3.59) are zero by Theorem 3.11 (iv), Theorem 3.20 and Proposition 2.9. The second term is zero by the intertwining relations and the fact that $(\tilde{\Omega}^{+}(D))^{*}f = 0$. Using the intertwining relations again we see that the fourth term of (3.59) is dominated by,

$$E \lim_{t \to \infty} ||F(A_D < 0)V_t(D)\psi(H(D))(\tilde{\Omega}^-(D))^* f|| \qquad (3.60)$$

which is zero by Theorem 2.11 (iv).

Now we can prove the local decay result for the four particle scattering.

<u>Theorem 3.23</u> (Local Decay). Let $\gamma$ be any pair and let $\phi \in C_0(\mathbb{R}^\nu)$. Then for any $f \in \mathcal{H}^\pm$ we have

$$E \lim_{t \to \pm\infty} ||\phi(x^\gamma)V_t f|| = 0.$$

<u>Proof</u> (+ case). As before we show, for any $f$ with $\psi(H)f = f$, $\psi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}(H))$ and $\rho(\lambda) = (1+\lambda^2)^{-1}$, that

$$E \lim_{t \to \infty} \langle V_t f, \rho(x^\gamma)V_t f\rangle = 0.$$

Now we take a clustering $C$ with $C = \{\gamma, \beta\}$, $\gamma \cap \beta = \emptyset$. Then,

$$E \lim_{t \to \infty} \langle V_t f, \rho(x^\gamma)V_t f\rangle$$

$$= E \lim_{t \to \infty} \{\langle V_t f, [\psi(H)-\psi(H(C))]\rho(x^\gamma)V_t f\rangle$$

$$+ \langle V_t f, \tilde{\Omega}^+(C)\psi(H(C))\rho(x^\gamma)V_t f\rangle$$

$$+ \langle V_t f, (-\tilde{\Omega}^+(C)+1)\psi(H(C))\rho(x^\gamma)V_t f\rangle\} \qquad (3.61)$$

The first term of (3.61) is dominated by

$$E \lim_{t \to \infty} ||[\psi(H) - \psi(H(C))]\rho(x^\gamma)V_t f||$$

Here the factor $[\psi(H)-\psi(H(C))]$ is approximated in norm by linear combinations of polynomials $(H+i)^{-M}-(H(C)+i)^{-M}$ by Stone-Weistrase theorem. Also since,

$$-(H+i)^{-M}+(H(C)+i)^{-M} = \sum_{\ell=1}^{M} (H(C)+i)^{-\ell}[(H(C)+i)^{-1}-(H+i)^{-1}](H+i)^{-M+\ell-1}$$

we need to consider only

$$E\lim_{t\to\infty} ||[(H(C)+i)^{-1}-(H+i)^{-1}](H+i)^{-M}\rho(x^{\gamma})V_t f||$$

for any $M > 0$. But we have,

$$E\lim_{t\to\infty} ||[(H(C)+i)^{-1}-(H+c)^{-1}](H+i)^{-M}\rho(x^{\gamma})V_t f||$$

$$\leq E\lim_{t\to\infty} \sum_{\alpha\in e(D)} ||(H(C)+i)^{-1}W_{\alpha}\rho_3(x^{\alpha})^{-1}||\ ||\rho_3(x^{\alpha})(H+i)^{-M-1}\rho_3(x^{\gamma})^{-1}||$$

$$||\rho_3(x^{\alpha})\rho(x^{\gamma})V_t f|| .$$

This is zero by Theorem 3.22. Thus the first term of (3.61) is zero. The second term is zero by the intertwining relations since $(\tilde{\Omega}^{+}(C))^{*}f = 0$.

We consider the last term of (3.61) now. We have

$$(\tilde{\Omega}^{+}(C)-1)\psi(H(C))\rho(x^{\gamma})V_t f$$

$$= \sum_{\alpha\in e(C)} \int_0^{\infty} ds\ I^{\alpha}(C,s)V_t f . \qquad (3.62)$$

Pointwise in $s$ the integrand of (3.62) is dominated by

$$||W_{\alpha}(H(C)+i)^{-1}\rho_3(x^{\alpha})^{-1}||\ ||\rho_3(x^{\alpha})\psi(H(C))(H(C)+i)V_s(C)\rho_3(x^{\alpha})^{-1}||$$

$$||\rho_3(x^{\alpha})\rho_2(x^{\gamma})V_t f||$$

so that by (A 6) and Lemma 2.4(i), pointwise in s, the $E\lim_{t\to\infty}$ of the norm of the integrand of (3.62) is zero. Then by showing the integrability in s of $||I^{\alpha}(C,s)||$ we can conclude that the second term of (3.61) is zero using Lebesgue dominated convergence theorem. We have by (A6), Lemma 2.4(i) and Proposition 3.12 (ii) that,

$$||I^{\alpha}(C,s)|| \leq ||W_{\alpha}\ \psi(H(C))\rho_3(x^{\alpha})^{-1}||\,||\rho_3(x^{\alpha})V_s(C)\rho(x^{\gamma})||$$

$$\leq K||\rho_3(x^{\alpha})V_s^{\gamma}\ \rho(x^{\gamma})|| \leq K(1+|s|)^{-\mu_1}\ ,\ \mu_1 > 1.$$

Hence the last term of (3.61) is also zero.

With the local decay result in our hand we have to show only the low energy decay condition to apply the theory of chapter II and conclude the completeness result. We warm up to this task by proving a few Lemmas first.

Lemma 3.24 Let $\psi \in C_o^{\infty}(\mathbb{R})$ with supp $\psi \cap (-\frac{1}{2}d^2, \frac{1}{2}d^2) = \emptyset$. Then for any clustering D with # D = 2, there exist constants b,c, less than d, such that for any $\phi \in C_o^{\infty}(\mathbb{R})$, $0 \leq \phi \leq 1$ and supp $\phi$ contained in $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$,

$$\lim_{t\to\infty} ||(\tilde{\Omega}^{\pm}(D)-1)\phi(H^D)\psi(H(D))U_t\ F(|x| \leq c|t|)|| = 0.$$

Proof (+ case only). We consider two cases.

Case 1. D is a disconnected clustering $D = \{\alpha,\beta\}$. In this case we have for every b,c > 0,

$$(\tilde{\Omega}^{+}(D)-1)\phi(H^D)\psi(H(D))U_t\ F(|x| \leq c|t|)$$

$$= \sum_{\gamma\in e(D)} \int_0^{\infty} ds\ V_s^{*}\ iW_{\gamma}\ V_s(D)\phi(H^D)\psi(H(D))U_t\ F(|x| \leq c|t|)$$

$$\equiv \sum_{\gamma\in e(D)} \int_0^{\infty} ds\ I_1^{\gamma}(D,s,t).$$

By the hypothesis on $H^{\alpha}$ it is clear that there are functions $\phi_1, \phi_2$ and $\phi_3, \phi_1, \phi_2$ having their support in $(-\frac{1}{2}b^2, \frac{1}{2}b^2)$ and $\phi_3$ having its support in $(-\frac{1}{2}b_1^2, \infty)$ respectively for some $b_1$ chosen so that

$$\phi_1(H^{\alpha})\phi_2(H^{\beta})\phi(H^{D}) = \phi(H^{D})$$

and

$$\phi(H^{D})\phi_3(T_D)\psi(H(D)) = \psi(H(D))$$

and this implies that

$$\phi(H^{D})\psi(H(D)) = \phi(H^{D})\psi(H(D))\phi_1(H^{\alpha})\phi_2(H^{\beta})\phi_3(T_D). \tag{3.63}$$

Then

$$\|I_1^{\gamma}(D,s,t)\| \le \|W_{\gamma}\ \phi(H^{D})\psi(H(D))\rho_3(x^{\gamma})^{-1}\| \circ$$

$$\circ \|\rho_3(x^{\gamma})V_s(D)\phi_1(H^{\alpha})\phi_2(H^{\beta})\phi_3(T_D)U_t\ F(|x| \le c|t+s|)\|. \tag{3.64}$$

To estimate the right side of the relation (3.64) we need a relation between the coordinates, which is,

$$x^{\gamma} = c_1(\gamma,D)x^{\alpha} + c_2(\gamma,D)x^{\beta} + d(\gamma,D)x_D$$

and then we take,

$$\max_{\gamma\epsilon e(D)} \{c_2(\gamma,D),\ c_1(\gamma,D)\} = c_3(D), \quad \min_{\gamma\epsilon e(D)} d(\gamma,D) = d_3(D). \tag{3.65}$$

If we choose the constants $d_1$, $d_2$, b, $b_1$ and c to satisfy the inequalities,

$$(b_1^2+b^2) \le d^2,\ d_1 > b+c,\ d_3(D)d_2 > c_3(D)d_1 \ \text{and}\ d_2 + c < b_1, \tag{3.66}$$

then we can estimate the quantity in (3.64) making use of the partition of identity

$$1 = F(|x^{\alpha}| > d_1|t+s|) + F(|x^{\alpha}| \le d_1|t+s|,\ |x^{\beta}| > d_1|t+s|)$$
$$+ F(|x^{\alpha}| \le d_1|t+s|,\ |x^{\beta}| \le d_1|t+s|,\ |x_D| > d_2|t+s|)$$
$$+ F(|x^{\alpha}| \le d_1|t+s|,\ |x^{\beta}| \le d_1|t+s|,\ |x_D| \le d_2|t+s|)$$

in the following way.

$$
\begin{aligned}
&||I_1^\gamma(D,s,t)|| \\
&\leq K\{||\rho_3(x^\gamma)||\ ||F(|x^\alpha|>d_1|t+s|)V_s^\alpha\ \phi_1(H^\alpha)U_t^\alpha\ F(|x^\alpha|\leq c|t+s|)|| \\
&\quad + ||\rho_3(x^\gamma)F(|x^\alpha| + |x^\beta| \leq 2\,d_1|t+s|,\ |x_D| > d_2|t+s|)|| \\
&\quad + ||\rho_3(x^\gamma)||\ ||F(|x^\beta| > d_1|t+s|)V_s^\beta\ \phi_2(H^\beta)U_t^\beta\ F(|x^\beta| \leq c|t+s|)|| \\
&\quad + ||\rho_3(x^\gamma)||\ ||F(|x_D| \leq d_2|t+s|)U_{t+s,D}\ \phi_3(T_D)F(|x_D| \leq c|t+s|\ ||\}
\end{aligned}
\tag{3.67}
$$

By Theorem 3.2 (iii), the first and the third terms and by Theorem 2.11 (i) the fourth term are bounded by $K(1+|t+s|)^{-\mu_2}$, for some $\mu_1 > 1$. The same bound follows for the second term from the definition of $\rho_3$ and the inequalities (3.65) and (3.66). Now that we have,

$$||I_1^\gamma(D,s,t)|| \leq K(1+|t+s|)^{-\mu_1}, \quad \mu_1 > 1$$

the result follows easily.

<u>Case 2</u>. When $D$ is a connected clustering, we have again

$$
\begin{aligned}
&(\tilde{\Omega}^+(D)-1)\phi(H^D)\psi(H(D))U_t\ F(|x| \leq c|t|) \\
&\quad = \sum_{\gamma\in e(D)} \int_0^\infty ds\ V_s^*\ iW_\gamma\ V_s(D)\phi(H^D)\psi(H(D))U_t\ F(|x| \leq c|t|) \\
&\quad = \sum_{\gamma\in e(D)} \int_0^\infty ds\ I_2^\gamma(D,s,t).
\end{aligned}
$$

As before it is enough to prove the estimate

$$||I_2^\gamma\ (D,s,t)|| \leq K(1+|t+s|)^{-\mu_1}, \quad \mu_1 > 1$$

to conclude the result. We choose a function $\psi_2 \in C_0^\infty((0,\infty))$ with supp $\psi_2$ contained in $(\frac{1}{2}b_2^2,\infty)$ satisfying,

$$\phi(H^D)\psi_2(T_D)\psi(H(D)) = \phi(H^D)\psi(H(D)) \quad .$$

Now using Lemma 2.4 (i), the partition of identity,

$$1=F(|x^D| > a_1|t+s|)+F(|x^D| \leq a_1|t+s|, |x_D|>d_3|t+s|)+F(|x_D|\leq d_3|t+s|)$$

we have,

$$||I_2^\gamma(D,s,t)||$$

$$\leq ||W_\gamma \ \psi(H(D))\rho_3(x^\gamma)^{-1}||\circ||\rho_3(x^\gamma)V_s(D)\phi(H^D)\psi_2(T_D)U_t \ F(|x|\leq c|t+s|)||$$

$$\leq K\{||\rho_3(x^\gamma)||\,||F(|x^D| > a_1|t+s|)V_s^D \ \phi(H^D)U_t^D \ F(|x^D| \leq c|t+s|)||$$

$$+ ||\rho_3(x^\gamma)F(|x^D| \leq a_1|t+s|, |x_D| > d_3|t+s|)||$$

$$+ ||\rho_3(x^\gamma)|| \ ||F(|x_D| \leq d_3|t+s|)U_{s+t,D} \ \psi_2(T_D)F(|x_D|\leq c|t+s|)||\} \tag{3.68}$$

Now we can write $x^\gamma$ in terms of the coordinates $x^\alpha$, $x_\alpha^D$ and $x_D$ as

$$x^\gamma = c_4(\gamma,D)x^\alpha \pm \ c_5(\gamma,D)x_\alpha^D + d(\gamma,D)x_D$$

and set

$$c_6(D) = \max_{\gamma\epsilon e(D)} \{c_4(\gamma,D), \ c_5(\gamma,D)\}, \ d_6(D) = \min_{\gamma\epsilon e(D)} \{d(\gamma,D)\} \tag{3.69}$$

and choose the constants $a_1$, b and c depending on d as in Theorem 3.17. We also choose $b_2$, $d_3$ satisfying

$$(b^2+b_2^2) \leq d^2, \ \ d_6(D)d_3 > \sqrt{2} \quad c_6(D)a_1, \ b_2 > d_3 + c \tag{3.70}$$

We note that all these constants depend only upon d and a calculation shows that it is possible to make a choice of these constants. By (3.70) then

$$||I_2^{\gamma}(D,s,t)||$$

$$\le K\{||F(|x^D| > a_1|t+s|)V_s^D \, \phi(H^D)U_t^D \, F(|x^D| \le c|t+s|)||$$

$$+ \ ||F(|x_D| \le d_3|t+s|)U_{s+t,D} \, \psi_2(T_D)F(|x_D| \le c|t+s|)||\}$$

$$+ \ K(1+|t+s|)^{-\mu_1} \ , \ \mu_1 > 1 \ .$$

Now applying Theorem 3.17 and Theorem 2.11 (ii) respectively to the right side of the above inequality, we obtain the estimate that

$$||I_2^{\gamma}(D,s,t)|| \le K(1+|t+s|)^{-\mu_2} \ , \ \mu_1 > 1.$$

Thus the required result follows.

For the next Lemma we recall the notation introduced in the condition (A1) of Chapter I, section 4 that if $\gamma,\alpha$ are pairs they also denote the corresponding (three cluster) clusterings. Then we can define their union $\alpha \bigcup \gamma$ if they are distinct, through definition 2.5. Let $D^{\alpha\gamma} \equiv \alpha \bigcup \gamma$. Then clearly # $D^{\alpha\gamma} = 2$. Given any pair $\gamma$ there four possibilities for a distinct $\alpha$ satisfying $\alpha \bigcap \gamma \neq \emptyset$. However (it is easy to see that) these four $\alpha$'s will give rise to only two distinct configurations for $D^{\alpha\gamma}$. In view of this we define a product $\prod\limits_{K}^{\gamma}$ to mean

$\prod\limits_{K}^{\gamma} \equiv$ Product over $K = \alpha,\beta$ where $\alpha,\beta$ are some distinct pairs such that $\gamma \bigcap \alpha = \emptyset$, $\gamma \bigcap \beta = \emptyset$ $\gamma \neq \alpha$, $\gamma \neq \beta$ and $D^{\alpha\gamma} \neq D^{\beta\gamma}$ .

<u>Lemma 3.25</u> For any positive number $d$ and a pair $\gamma$, there exist positive constants $b'$, $b_1$, $c$, $b_1 < b' < d$, $c < b' < d$ and $\phi_\gamma \in C_0^\infty(\mathbb{R})$, $0 \le \phi_\gamma \le 1$, supp $\phi_\gamma$ contained in $(-\frac{1}{2}b_1^2, \frac{1}{2}b_1^2)$ such that for $\beta$ with $\beta \bigcap \gamma = \emptyset$ and any $\phi \in C_0^\infty(\mathbb{R})$, $0 \le \phi \le 1$ with supp $\phi$ contained in $(\frac{1}{2}b^2, \infty)$ ,

$$\lim_{t\to\pm\infty} \|(\tilde{\Omega}^{\pm}(\gamma)-1)\{\prod_{\kappa}^{\gamma} \phi(T_{\gamma}^{D^{\kappa\gamma}})\}\phi_{\gamma}(H^{\gamma})\phi(T^{\beta})F(|x| \leq c|t|)\| = 0 \ .$$

Proof (+ case only). As usual we have,

$$(\tilde{\Omega}^{+}(\gamma)-1)\phi_{\gamma}(H^{\gamma})\{\prod_{\kappa}^{\gamma} \phi(T_{\gamma}^{D^{\kappa\gamma}})\}\phi(T^{\beta})U_t\, F(|x| \leq c|t|)$$

$$= \sum_{\alpha\in e(\gamma)} \int_0^{\infty} ds\, V_s^{*}\, iW_{\alpha}\, V_s(\gamma)\phi_{\gamma}(H^{\gamma})\{\prod_{\kappa}^{\gamma} \phi(T_{\gamma}^{D^{\kappa\gamma}})\}\phi(T^{\beta})F(|x|\leq c|t|)$$

$$\equiv \sum_{\alpha\in e(\gamma)} \int_0^{\infty} I_{\beta}^{\gamma}(\alpha,s,t).$$

We will conclude the result by showing the estimate

$$\|I_{\beta}^{\gamma}(\alpha,s,t)\| \leq K(1+|t+s|)^{-\mu_1}, \ \mu_1 > 1 \ .$$

We divide the proof of this estimate into two cases.

Case 1 $\beta = \alpha$. By writing,

$$V_s(\gamma) = \exp(-is(H^{\gamma}+T_{\beta\cup\gamma}))\cdot\exp(-isT^{\beta}) \equiv V_{s,\beta}(\gamma)U_{s,\beta} \ ,$$

we have

$$\|I_{\beta}^{\gamma}(\beta,s,t)\|$$

$$\leq \|W_{\beta}(T^{\beta}+1)^{-1}\rho_3(x^{\beta})^{-1}\| \ \|\rho_3(x^{\beta})U_{s+t}^{\beta}\ \phi(T^{\beta})F(|x^{\beta}| \leq c|t+s|)\|.$$

Now we choose $c$ such that $2c < b'$, then using Lemma 2.4 (i) and Theorem 2.11 (ii) it follows that,

$$\|I_{\beta}^{\gamma}(\beta,s,t)\| \leq K(1+|t+s|)^{-\mu_1} \quad \mu_1 > 1.$$

Case 2 $\beta \neq \alpha$. Note that $\phi(T_\gamma^{D^{\kappa\gamma}})$ and $\phi(T^\beta)$ commute with $H^\gamma$. Thus,

$$||I_\beta^\gamma(\alpha,s,t)||$$
$$\leq ||W_\alpha\{\prod_{\kappa\neq\alpha}^{\gamma} \phi(T_\gamma^{D^{\kappa\gamma}})\}\phi(T^\beta)(H^\gamma+i)^{-1}\rho_3(x^\alpha)^{-1}||.$$
$$||\rho_3(x^\alpha)V_s(\gamma)\phi(T_\gamma^{D^{\alpha\gamma}})(H^\gamma+i)\phi_\gamma(H^\gamma)U_t\, F(|x| \leq c|t+s|)||.$$

Now using Lemma 2.4 (i) to bound the first factor and making use of the partition of the identity,

$$1 = F(|x^\gamma| > \ell_1|t+s| + F(|x^\gamma| \leq \ell_1|t+s|,\ |x_\gamma^{D^{\alpha\gamma}}| > \ell_2|t+s|)$$
$$+ F(|x_\gamma^{D^{\alpha\gamma}}| \leq \ell_2|t+s|)\ ,$$

we estimat $||I_\beta^\gamma(\alpha,s,t)||$ to get,

$$||I_\beta^\gamma(\alpha,s,t)||$$
$$\leq K\{||\rho_3(x^\alpha)||\ ||F(|x^\gamma| > \ell_1|t+s|)V_s^\gamma(H^\gamma+i)\phi(H^\gamma)U_t\, F(|x^\gamma|\leq c|t+s|)||$$
$$+ ||\rho_3(x^\gamma)F(|x^\gamma| \leq \ell_1|t+s|,\ |x_\gamma^{D^{\alpha\gamma}}| > \ell_2|t+s|)||$$
$$+ ||\rho_3(x^\gamma)||\,||F(|x_\gamma^{D^{\alpha\gamma}}| \leq \ell_2|t+s|)\, U_{s+t,\gamma}^{D^{\alpha\gamma}}\, \phi(T_\gamma^{D^{\alpha\gamma}})F(|x_\gamma^{D^{\alpha\gamma}}| \leq |t+s|)||\}. \tag{3.71}$$

Now we have the following relation between the coordinates

$$x^\alpha = c(\gamma,D^{\alpha\gamma})x^\gamma + d(\gamma,D^{\alpha\gamma})x_\gamma^{D^{\alpha\gamma}}$$

we then take

$$c_1' = \max_\gamma\{\max_\alpha c(\gamma,D^{\alpha\gamma})\}$$

and $\qquad\qquad\qquad\qquad$ (3.72)

$$c_2' = \min_\gamma\{\min_\alpha d(\gamma,D^{\alpha\gamma})\}$$

and choose the constants to satisfy the following inequalities.

$$(b_1^2+5(b')) \leq d^2,\ \ell_1 > b_1 + c,\ \ell_2 < b'+c,\ c_2'\ell_2 > c_1'\ell_1. \tag{3.73}$$

Then using Theorem 3.2 (iii) and Theorem 2.11 (i) respectively for the first and the third terms of (3.71) along with (3.73) and (3.72) for the second term of (3.71) we obtain the estimate that

$$||I_\beta^\gamma(\alpha,s,t)|| \leq K(1+|t+s|)^{-\mu_1}, \quad \mu_1 > 1 .$$

Hence the result.

Now we are ready to prove the low energy decay condition for the four particle scattering. We define the sets

$$\mathcal{D}^{\pm} = \{f \in \mathcal{H}^{\pm} : \psi(H)f = f, \ \text{for some } \psi \in C_0^\infty(\mathbb{R} \setminus T(H)),\ 0 \leq \psi \leq 1\}.$$

We see that $\mathcal{D}^{\pm}$ is dense in $\mathcal{H}^{\pm}$. We also write for any two functions $h(t)$ and $g(t)$,

$$h = o_\varepsilon(g) \quad \text{iff} \quad \mathcal{E}\lim_{t\to\infty} \frac{h(t)}{g(t)} = 0 .$$

Thus $h = o_\varepsilon(1)$ means that $\mathcal{E}\lim_{t\to\infty} h(t) = 0$.

<u>Theorem 3.26</u> (Low Energy Decay) Let H be the four particle Hamiltonian with the pair potentials satisfying (A1), (A2) and (A5) - (A8). Then for any $f \in \mathcal{D}^{\pm}$ and any pair $\gamma$, there is a constant $b^{\pm}(f) > 0$ such that for any $c_\gamma \leq b^{\pm}(f)$,

$$\mathcal{E}\lim_{t\to\pm\infty} ||F(T^\gamma < \tfrac{1}{2} c_\gamma^2)V_t f|| = 0 .$$

<u>Proof</u> (+ sign only) Since $f \in \mathcal{D}^+$, there is a $\psi \in C_0^\infty(\mathbb{R} \setminus T(H))$, $0 \leq \psi \leq 1$, $\psi(H)f = f$ and for some $d > 0$, $\operatorname{supp}\psi \cap (-\frac{1}{4}d^2, \frac{1}{4}d^2) = \emptyset$. We fix this $d$ and consider the constants $b$, $b_1$, $b'$ and $c$ coming from

Lemmas 3.24 and 3.25 with a further restriction that

$$b_1^2 + 4(b')^2 < b^2, \tag{3.74}$$

which is always possible. Then we have the following inequality, for any pairs $\gamma,\beta$ with $\gamma \cap \beta = \emptyset$ and constants $b_6$, $b_7$ and $b_8$ (recalling the definition of $\overset{\gamma}{\underset{\kappa}{\Pi}}$ from Lemma 3.25).

$$\begin{aligned} F(T^\gamma < \tfrac{1}{2}b_6^2)\psi(H_o) \\ \leq \{F(T^\gamma < \tfrac{1}{2}b_6^2) \prod_\kappa^\gamma F(T_\gamma^{D^{\kappa\gamma}} > \tfrac{1}{2}\, b_7^2) F(T^\beta > \tfrac{1}{2}\, b_8^2) \\ + \sum_\kappa F(T^\gamma \leq \tfrac{1}{2}\, b_6^2)\ F(T_\gamma^{D^{\kappa\gamma}} \leq \tfrac{1}{2}\, b_7^2) \\ + F(T^\gamma \leq \tfrac{1}{2}\, b_6^2) F(T^\beta \leq \tfrac{1}{2}\, b_8^2)\}\psi(H_o) \end{aligned} \tag{3.75}$$

Now we choose the constants $b_6$, $b_7$ and $b_8$ to satisfy,

$$\begin{aligned} b_6 < \tfrac{1}{4}\, b_1,\ \tfrac{1}{4}\, b' < \min\,(b_7,b_8),\ \min\,(b_6,b_7,b_8) < \tfrac{1}{16}\, b \\ (b_6^2+b_7^2+b_8^2) < \tfrac{1}{8}\, d^2 \end{aligned} \tag{3.76}$$

With this choice of the constants and the inequality (3.75) we see that there are functions $\phi_1,\phi,\phi_2 \in C_o^\infty(\mathbb{R})$, $0 \leq \phi_1,\phi_2,\phi \leq 1$, supp $\phi$ contained in $(\tfrac{1}{2}(b')^2,\infty)$, supp $\phi_1$ contained in $(-\tfrac{1}{2}b^2,\ \tfrac{1}{2}b^2)$ and supp $\phi_2$ contained in $(-\tfrac{1}{2}\, b_1^2,\ \tfrac{1}{2}\, b_1^2)$ so that by setting $C^{\gamma\beta} = \gamma \cup \beta$, for $\beta : \beta \cap \gamma = \emptyset$, the inequality (3.75) becomes,

$$\begin{aligned} F(T^\gamma < \tfrac{1}{2}\, b_6^2)\psi(H_o) \\ \leq \{\phi_2(T^\gamma) \prod_\kappa^\gamma \phi(T_\gamma^{D^{\kappa\gamma}})\ \phi(T^\beta) + \sum_\alpha \phi_1(T^{D^{\kappa\gamma}}) + \phi_1(T^{C^{\gamma\beta}})\}\psi(H_o). \end{aligned} \tag{3.77}$$

Using (3.77) we then write,

$$<V_t f,\ F(T^{\gamma} < \tfrac{1}{2} b_6^2) V_t f> = <V_t f,\ F(T^{\gamma} < \tfrac{1}{2} b_6^2)\psi(H) V_t f>$$

$$\leq \ ||\{\psi(H) - \psi(H_o)\} V_t \ f||$$

$$+ \sum_{K} <V_t f,\ \{\phi_1(T^{D^{K\gamma}}) - \phi_1(H^{D^{K\gamma}})\} V_t \ f>$$

$$+ \sum_{K} <V_t f,\ \phi_1(H^{D^{K\gamma}})\ \psi(H_o) V_t f>$$

$$+ <V_t f, \{\phi_2(T^{\gamma}) - \phi_2(H^{\gamma})\} \prod_{K}^{\gamma} \phi(T_{\gamma}^{D^{K\gamma}}) \phi(T^{\beta}) \psi(H_o) V_t \ f>$$

$$+ <V_t f, \phi_2(H^{\gamma}) \prod_{K}^{\gamma} \phi(T_{\gamma}^{D^{K\gamma}})\ \phi(T^{\beta}) \psi(H_o) V_t f>$$

$$+ <V_t f,\ \{\phi_1(T^{C^{\gamma\beta}}) - \phi_1(H^{C^{\gamma\beta}})\} \psi(H_o) V_t f>$$

$$+ <V_t f,\ \phi_1(H^{C^{\gamma\beta}}) \psi(H_o) V_t f>$$

$$\leq \ \{||\{\psi(H) - \psi(H_o)\} V_t f|| + \sum_{K} ||\{\phi_1(T^{D^{K\gamma}}) - \phi_1(H^{D^{K\gamma}})\}\ V_t f||$$

$$+ ||\{\phi_2(T^{\gamma}) - \phi_2(H^{\gamma})\} V_t f|| + ||\{\phi_1(T^{C^{\gamma\beta}}) - \phi_1(H^{C^{\gamma\beta}})\} V_t f||\}$$

$$+ \sum_{K} \ <V_t f,\ \phi_1(H^{D^{K\gamma}})\ \psi(H_o) V_t f>$$

$$+ <V_t f, \phi_2(H^{\gamma}) \prod_{K}^{\gamma} \ \phi(T_{\gamma}^{D^{K\gamma}}) \phi(T^{\beta}) \psi(H_o) V_t f>$$

$$+ <V_t f,\ \phi_1(H^{C^{\gamma\beta}}) \psi(H_o) V_t f> \tag{3.78}$$

$E$ lim of the terms in the braces are zero by Lemma 2.18 (ii). We note that $\psi(H)V_t f$ can be substituted for $\psi(H_o)V_t f$ or $\psi(H(D))V_t f$, for any D, (whenever necessary) in the $E$ lim . Thus replacing $\psi(H_o)V_t f$ by

$U_t F(|x| \leq c|t|) U_t^* V_t f + U_t F(|x| > c|t|) U_t^* V_t f$ and using Theorem 2.16 (vi) we have,

$$\langle V_t f, F(T^\gamma < \tfrac{1}{b} b_6^2) V_t f\rangle$$

$$\leq o_E(1) + \sum_K \langle V_t f\ (-\tilde{\Omega}^+(D^{K\gamma})+1)\phi_1(H^{D^{K\gamma}})\psi(H(D^{K\gamma}))U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle$$

$$+ \sum_K \langle V_t f, \tilde{\Omega}^+(D^{K\gamma})\phi_1(H^{D^{K\gamma}})\psi(H(D^{K\gamma}))U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle$$

$$+ \langle V_t f, (-\tilde{\Omega}^+(\gamma)+1)\phi_2(H^\gamma) \prod_K^\gamma \phi(T_\gamma^{D^{K\gamma}})\phi(T^\beta)U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle$$

$$+ \langle V_t f, \tilde{\Omega}^+(\gamma)\phi_2(H^\gamma) \prod_K^\gamma \phi(T_\gamma^{D^{K\gamma}})\phi(T^\beta)U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle$$

$$+ \langle V_t f, (-\tilde{\Omega}^+(C^{\beta\gamma})+1)\phi_1(H^{C^{\beta\gamma}})\psi(H(C^{\beta\gamma}))U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle$$

$$+ \langle V_t f, \tilde{\Omega}^+(C^{\beta\gamma})\phi_1(H^{C^{\beta\gamma}})\psi(H(C^{\beta\gamma}))U_t\ F(|x| \leq c|t|)U_t^* V_t f\rangle \qquad (3.79)$$

The third, fifth and the seventh terms in (3.79) are identically zero since $f \in H^\pm$, while the second and sixth terms in (3.79) are $o_E(1)$ by Lemma 3.24, the fourth is also $o_E(1)$ by Lemma 3.25.

Since we have local decay and low energy decay results we can now state the completeness.

<u>Theorem 3.27</u> (4-particle completeness). Let H be a four particle Hamiltonian with the pair potentials satisfying the conditions (A1), (A2), (A5) - (A8). Then for such a Hamiltonian,

(i) $$\sum_D^{\oplus} F_+(D) = H_c(H) = \sum_D^{\oplus} F_-(D)$$

(ii) $$H_{sc}(H) = \{0\}.$$

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