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CURVATURES OF LEFT INVARIANT METRICS ON LIE GROUPS
AND
PARAMETRIC HOMOTOPY PRINCIPLE FOR A CLASS OF PARTIAL DIFFERENTIAL RELATIONS ON CLOSED MANIFOLDS

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CONTENTS

Preface

The title of this thesis refers to two problems of different nature having no connection between them. These problems are presented in two parts. In Part I we have settled two conjectures of Milnor, on Lie groups with left-invariant metrics, in the affirmative and in Part II we have obtained a new Smale - Hirsch - Gromov - type theorem on the homotopy classification of a class of partial differential relations on closed manifolds. More detailed introductions to these parts are given at the beginning of each part. It has been our attempt to make the presentations as self-contained as possible. References to a part appears at the end of that part. Results like theorem, proposition, and lemma are numbered consecutively throughout Part II

Part I

CURVATURES OF LEFT-INVARIANT METRICS ON LIE GROUPS

0. Introduction :

In this part we settle affirmatively two conjectures proposed by Milnor [4]. By putting a left-invariant Riemannian metric on a Lie group, Milnor had obtained some interesting results and posed questions which provide relevant examples and counter-examples in the context of the general theory of Riemannian manifolds. We deal here with two of his several questions. The following are the descriptions of the conjectures.

Let G be a Lie group with a left-invariant metric, and $\mathfrak{g}$ be its associated Lie algebra with (biquadratic) curvature function $k : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ corresponding to the metric. Milnor proved ([4], Corollary 1.3) that if u belongs to the centre of $\mathfrak{g}$, then, for any left-invariant metric,

$$k(u,v) \geq 0 \quad \text{for all } v \text{ in } \mathfrak{g} .$$

He conjectured that the central elements of $\mathfrak{g}$ are the only ones with this property. More precisely, if u in $\mathfrak{g}$ satisfies the inequality $k(u,v) \geq 0$ for all metrics and for all v in $\mathfrak{g}$, then u belongs to the centre of $\mathfrak{g}$. Thus this characterises the central elements of a Lie algebra in terms of the curvature function of all metrics.

The second conjecture concerns a result of Nolan Wallach (See Milnor [4], Theorem 3.4) which states that if the universal covering of G is not homeomorphic to Euclidean space, or equivalently, if G contains a compact non-commutative subgroup, then G admits a left-invariant metric of strictly positive scalar curvature. It was conjectured that these are the only groups which admit a left-invariant metric of positive scalar curvature. More explicitly, if G is a connected Lie group whose universal covering space is homeomorphic to Euclidean space, then any left-invariant metric on G is either flat or has strictly negative scalar curvature. This completes a characterisation of Lie groups which admit a left-invariant metric of strictly positive scalar curvature in terms of the topology of the universal covering space of the Lie group.

It should be mentioned that after proving the above results, I came to know that Leite and Dotti de Miatello [3] had proved the first conjecture, and Uesu [5] had proved the second conjecture. However our proofs of the results are different and more direct in approach. I also understood that Bergery [1] also obtained some more general results in the present context. I have no access to the work of Bergery.

1. A characterization of the central elements of a Lie algebra :

We begin by recalling some definitions and results we shall use in our proofs. Let G be an n-dimensional connected Lie group and $\mathfrak{g}$ be its associated Lie algebra consisting of smooth left-invariant vector fields on G. If e denotes the identity of G and $T_e(G)$ the tangent space to G at e then the bijection $\mathfrak{g} \longrightarrow T_e(G)$, given by $X \longrightarrow X(e)$, induces a Lie algebra structure on $T_e(G)$. In what follows we shall not distinguish between $\mathfrak{g}$ and $T_e(G)$.

Giving G a left-invariant Riemannian metric $<\,,\,>$ is the same as choosing an inner product on $\mathfrak{g}$. Thus, if $e_1,\ldots,e_n$ be a basis for $\mathfrak{g}$, this amounts to choosing a positive definite symmetric nxn matrix $((g_{ij}))$ and putting $<e_i,e_j> = g_{ij}$, $1 \leq i,\ j \leq n$. Then for all $x, y \in \mathfrak{g}$, $<x,y>$ becomes constant on G, and G with such a metric becomes a complete, homogeneous Riemannian manifold. Consequently, in order to study the curvature properties of G it is enough to study the curvature properties at e.

Recall that if ∇ denotes the associated Levi-Civita connection and

$$R_{xy} = \nabla_{[x,y]} - \nabla_x \nabla_y + \nabla_y \nabla_x , \quad x, y \in \mathfrak{g} ,$$

is the curvature tensor, then the biquadratic curvature function is given by $k(x,y) = <R_{xy}(x), y>$, and that

$$K(x,y) = \frac{k(x,y)}{<x,x><y,y> - <x,y>^2}$$

is the sectional curvature associated to the linear subspace spanned by x and y in $T_e(G)$. Observe that if $x, y, z \in \mathfrak{g}$ then $\nabla_x y \in \mathfrak{g}$, and hence $R_{xy}(z) \in \mathfrak{g}$. Thus $<R_{xy}(x), y>$ is constant on G, and hence can be identified with its valu at e. Now let $e_1,\ldots,e_n$ be an orthonormal basis of $\mathfrak{g}$. Then we may write

$$[e_i,e_j] = \sum_k \alpha_{ijk} e_k ,$$

or equivalently , $\alpha_{ijk} = \langle [e_i,e_j] , e_k \rangle$.

Clearly these structural constants satisfy $\alpha_{ijk} = -\alpha_{ikj}$. As proved by Milnor [4], we have the following formula for k

$$k(e_1,e_2) = \sum_m \Big(\tfrac{1}{2}\alpha_{12m}(-\alpha_{12m}+\alpha_{2m1}+\alpha_{m12}) - \tfrac{1}{4}(\alpha_{12m}-\alpha_{2m1}+\alpha_{m12})(\alpha_{12m}+\alpha_{2m1}-\alpha_{m12}) - \alpha_{m11}\alpha_{m22}\Big) \qquad \ldots (1)$$

The main result of this section is

Theorem 1 : If u in $\mathfrak{g}$ satisfies $k(u,v) \geq 0$ for all metrics on G and for all v in $\mathfrak{g}$, then u belongs to the centre of $\mathfrak{g}$.

The proof of the theorem will follow from a more general result which is stated in the proposition below.

Let us choose a basis $e_1,\ldots,e_n$ of $\mathfrak{g}$ so that $e_1 = u$ and $e_2 = v$, and restrict attention to left-invariant metrics given by matrices which are of the form $\mathrm{diag}(\sigma_1,\ldots,\sigma_n)$ with respect to the above basis , and examine the hypothesis $k(u,v) \geq 0$.

Let $[e_i,e_j] = \sum \alpha_{ijk} e_k$ and $\nabla_{e_i} e_j = \sum \Gamma^k_{ij} e_k$. If $x, y, z \in \mathfrak{g}$ then $\langle x,y\rangle$ is constant and hence we have

$$< \nabla_x y, z > + < y, \nabla_x z > = x < y,z > = 0 .$$

Moreover, $\nabla_x y - \nabla_y x = [x,y]$.

Combining these identities one gets

$$< \nabla_x y,z > = \frac{1}{2} (< [x,y] ,z > - < [y,z] ,x > + < [z,x] , y >)$$

Hence

$$\Gamma^k_{ij} = \frac{1}{\sigma_k} < \nabla_{e_i} e_j, e_k >$$

$$= \frac{1}{2\sigma_k} (\sigma_k \alpha_{ijk} - \sigma_i \alpha_{jki} + \sigma_j \alpha_{kij}) \quad 1 \leq i, j, k \leq n .$$

Then

$$k(e_1,e_2) = < \nabla_{[e_1,e_2]} e_1,e_2 > - < \nabla_{e_1} \nabla_{e_2} e_1,e_2 > + < \nabla_{e_2} \nabla_{e_1} e_1,e_2 >$$

$$= \Sigma \alpha_{12m} \Gamma^2_{m1} \sigma_2 - \Sigma \Gamma^m_{21} \Gamma^2_{1m} \sigma_2 + \Sigma \Gamma^m_{11} \Gamma^2_{2m} \sigma_2$$

$$= \frac{\sigma_2}{2\sigma_2} \Sigma \alpha_{12m}(\sigma_2 \alpha_{m12} - \sigma_m \alpha_{12m} + \sigma_1 \alpha_{2m1})$$

$$- \frac{\sigma_2}{4\sigma_2} \Sigma \frac{1}{\sigma_m} (\sigma_m \alpha_{21m} - \sigma_2 \alpha_{1m2} + \sigma_1 \alpha_{m21})$$

$$(\sigma_2 \alpha_{1m2} - \sigma_1 \alpha_{m21} + \sigma_m \alpha_{21m})$$

$$+ \frac{\sigma_2}{4\sigma_2} \Sigma \frac{1}{\sigma_m} (\sigma_m \alpha_{11m} - \sigma_1 \alpha_{1m1} + \sigma_1 \alpha_{m11})$$

$$(\sigma_2 \alpha_{2m2} - \sigma_2 \alpha_{m22} + \sigma_m \alpha_{22m})$$

$$= \left\{ -\frac{1}{2} \Sigma\, \alpha_{12m}^2\, \sigma_m + \frac{1}{2} \Sigma\, \alpha_{12m}(\sigma_2\, \alpha_{m12} + \sigma_1\, \alpha_{2m1}) \right\}$$

$$+ \left\{ -\frac{1}{4} \Sigma\, \alpha_{12m}^2\, \sigma_m + \frac{1}{4} \Sigma\, \frac{1}{\sigma_m} (\sigma_2\, \alpha_{1m2} - \sigma_1\, \alpha_{m21})^2 \right\}$$

$$- \Sigma\, \frac{1}{\sigma_m}\, \sigma_1\, \sigma_2\, \alpha_{m11}\, \alpha_{m22} \qquad \ldots\ldots (2)$$

Note that the formula (1) may be obtained from the second equality above simply by putting $\sigma_1 = \ldots = \sigma_n = 1$.

The formula (2) suggests the following proposition.

Proposition : Let $e_1,\ldots,e_n$ be a basis of $\mathfrak{g}$. Suppose for all metrics of the form $< e_1,e_1 > = < e_2,e_2 > = 1$, $< e_3,e_3 > = \ldots = < e_n,e_n > = \sigma > 0$, $< e_i,e_j > = 0$ for $i \neq j$ and $1 \leq i, j \leq n$, we have $k(e_1,e_2) \geq 0$, then $[e_1,e_2] = 0$.

Proof : In the formula (2) above, we put $\sigma_1 = \sigma_2 = 1$ and $\sigma_i = \sigma$ for $i \geq 3$. Then splitting each summation over m into sum over $m \leq 2$ and sum over $m > 2$, and collecting the sums over $m \leq 2$ together and sums over $m > 2$ together, we get

$$k(e_1,e_2) = \left\{ -\frac{1}{2} (\alpha_{121}^2 + \alpha_{122}^2) + \frac{1}{2} (\alpha_{121}\, \alpha_{211} + \alpha_{122}\, \alpha_{212}) \right.$$

$$\left. - \frac{1}{4} (\alpha_{121}^2 + \alpha_{122}^2) + \frac{1}{4} (\alpha_{121}^2 + \alpha_{122}^2) \right\}$$

$$+ \left\{ -\frac{1}{2} \sigma \sum_{m\geq 3} \alpha_{12m}^2 + \frac{1}{2} \sum_{m\geq 3} \alpha_{12m}(\alpha_{m12} + \alpha_{2m1}) \right.$$

$$- \frac{1}{4} \sigma \sum_{m\geq 3} \alpha_{12m}^2 + \frac{1}{4\sigma} \sum_{m\geq 3} (\alpha_{1m2} - \alpha_{m21})^2$$

$$\left. - \frac{1}{\sigma} \sum_{m\geq 3} \alpha_{m11}\, \alpha_{m22} \right\}$$

$$= -(\alpha_{121}^2 + \alpha_{122}^2) - \frac{3}{4}\sigma \sum_{m\geq 3} \alpha_{12m}^2 + \frac{1}{2}\sum_{m\geq 3} \alpha_{12m}(\alpha_{m12} + \alpha_{2m1})$$

$$+ \frac{1}{4\sigma}\sum_{m\geq 3}(\alpha_{1m2} - \alpha_{m21})^2 - \frac{1}{\sigma}\sum_{m\geq 3} \alpha_{m11}\,\alpha_{m22}$$

Now it is clear that if this expression has to be ≥ 0 for all $\sigma > 0$ then α_{12m} must be zero for all m. For, if $\alpha_{12m} \neq 0$ for some $m \geq 3$, then by taking σ to be large enough we can make the second term sufficiently large compared to the other terms so that the whole expression becomes negative. Thus $\alpha_{12m} = 0$ for all $m \geq 3$. But then we are left with four terms of which the last two can again be made sufficiently small by choosing σ suitably so that, if one of α_{121} and α_{122} is non-zero, the expression that remains becomes negative. Thus $\alpha_{121} = \alpha_{122} = 0$ also. Thus we have $[e_1, e_2] = 0$.

Proof of theorem 1 : The proof follows immediately from the above proposition. Given any $v \in \mathfrak{G}$, which is not a scalar multiple of u, we take $e_1 = u$, $e_2 = v$ and the metrics as in the proposition to conclude that $[e_1, e_2] = [u, v] = 0$. So u must be central in $\mathfrak{G}$.

2. Left-invariant metrics of non-positive scalar curvature :

Recall that with respect to an orthonormal basis $e_1, \ldots, e_n$ for the tangent space at a point of a Riemannian manifold, the scalar curvature at the point is given by

$$\rho = \sum_{i,j} K(e_i, e_j)$$

and it is independent of the orthogonal basis chosen.

The main result of the present section is

Theorem 2 : If G is a connected Lie group whose universal covering space is homeomorphic to Euclidean space, then any left-invariant metric on G is either flat or has strictly negative scalar curvature.

Remark 1 : It is already known that if the Lie group G is solvable, then any left-invariant metric on G is either flat or else has strictly negative scalar curvature, [4, Theorem 3.1]. Moreover, a connected Lie group with left-invariant metric is flat if and only if the associated Lie algebra splits as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is a commutative subalgebra, $\mathfrak{u}$ is a commutative ideal, and where the linear transformation ad b is skew-adjoint for every $b \in \mathfrak{b}$, [4, Theorem 1.5]. Thus, in particular, a Lie group G with a flat left-invariant metric must be solvable, because $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{b} \oplus \mathfrak{u}, \mathfrak{b} \oplus \mathfrak{u}] \subset \mathfrak{u}$ and hence $[\mathfrak{g}^1, \mathfrak{g}^1] = 0$ (A Lie group is solvable if its Lie algebra is solvable).

Thus what remains to be shown is that if G satisfies the hypothesis of Theorem and if in addition G is not solvable, then any left-invariant metric on G has strictly negative scalar curvature.

Remark 2 : Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ (Vector space direct sum) be the Levi-decomposition of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{h}$ is a solvable ideal and $\mathfrak{t}$ is a semi-simple subalgebra. Therefore G is not solvable if and only if $\mathfrak{t} \neq 0$ in the above decomposition.

Let $\mathfrak{t} = \mathfrak{s}_1 + \ldots + \mathfrak{s}_r$

be the Lie algebra direct sum of simple ideals $\mathfrak{s}_i$ of $\mathfrak{t}$. A theorem of Iwasawa (See [4, p. 327]) states that the universal covering of G is homeomorphic to Euclidean space if and only if every compact subgroup of G is commutative. Now, looking at the classification of simple Real Lie groups as given in [2, Chapter X], we find that $SL(2,\mathbb{R})$ is the only group that does not have non-commutative compact subgroups. Thus if the universal covering of G is homeomorphic to Euclidean space then each $\mathfrak{s}_i$ in the above decomposition of $\mathfrak{t}$ must be $\mathfrak{sl}(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$.

Now we prove an important inequality which will be used in proof of Theorem 2.

Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ (Vector space direct sum)
where $\mathfrak{h}$ is an ideal, and $\mathfrak{s}$ is a unimodular subalgebra of $\mathfrak{g}$ (This means that, for all $x \in \mathfrak{s}$ ad x as a linear transformation on $\mathfrak{s}$ has trace 0). Thus if $x, y \in \mathfrak{s}$ then $[x,y] \in \mathfrak{s}$, and if $x \in \mathfrak{g}$, $y \in \mathfrak{h}$ then $[x,y] \in \mathfrak{h}$.

Given a left-invariant metric on G, with Lie algebra $\mathfrak{g}$ as above, we choose an orthonormal basis $e_{r+1},\ldots,e_n$ with respect to the metric restricted to $\mathfrak{h}$, and extend it to an orthonormal basis $e_1,\ldots,e_n$ of $\mathfrak{g}$.

For $1 \leq i \leq r$, let

$$e_i = x_i + h_i \qquad \text{..... (1)}$$

where $x_i \in \mathfrak{s}$ and $h_i \in \mathfrak{h}$.

Let $\rho(\mathfrak{g})$ denote the scalar curvature of the metric on $\mathfrak{g}$, $\rho(\mathfrak{h})$ scalar curvature of the metric on $\mathfrak{g}$ restricted to $\mathfrak{h}$, $\rho'(\mathfrak{s})$ the scalar curvature of the metric on $\mathfrak{s}$ with respect to which $x_1,\ldots,x_r$ are orthonormal (Notice that $x_1,\ldots,x_r$ form a basis of $\mathfrak{s}$).

Proposition : $\rho(\mathfrak{g}) \leq \rho'(\mathfrak{s}) + \rho(\mathfrak{h})$

Proof : Let $[e_i, e_j] = \sum_k \alpha_{ijk}\, e_k$, $1 \leq i, j \leq n$.

We first note that if $1 \leq i, j \leq r$,

$$[x_i, x_j] = \sum_{k=1}^{r} \alpha_{ijk}\, x_k \qquad \text{..... (2)}$$

This is obtained by equating the $\mathfrak{s}$ - components of the two sides of

$$[e_i, e_j] = [x_i + h_i, x_j + h_j].$$

On one hand, we have

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{ijk}\, e_k$$

$$= \sum_{k\leq r} \alpha_{ijk}\, x_k + \sum_{k\leq r} \alpha_{ijk}\, h_k + \sum_{k>r} \alpha_{ijk}\, e_k$$

showing that the $\mathfrak{s}$ - component of $[e_i, e_j]$ is $\sum_{k\leq r} \alpha_{ijk}\, x_k$.

On the other hand,

$$[x_i + h_i, x_j + h_j] = [x_i, x_j] + [x_i, h_j] + [h_i, x_j + h_j],$$

and therefore its $\mathfrak{g}$ - component is $[x_i, x_j]$. Note that (2) just says that α_{ijk}, $i, j, k \leq r$ are also the structural constants of the Lie algebra $\mathfrak{g}$ with respect to the basis $x_1, \ldots, x_r$. Another important observation that will be used in the ensuing computations is that if either i or $j > r$, then $\alpha_{ijk} = 0$ for $k \leq r$. This is obvious because $\mathfrak{h}$ is an ideal.

We write

$$\rho(\mathfrak{G}) = \sum_{i>r,\ j>r} K(e_i, e_j) + 2 \sum_{i\leq r,\ j>r} K(e_i, e_j)$$

$$+ \sum_{i\leq r,\ j\leq r} K(e_i, e_j) ,$$

and then evaluate, using formula (1) of Section 1, the three component summations in terms of α_{ijk}'s as follows :

$$\sum_{i\leq r,\ j\leq r} K(e_i, e_j) = \sum_{\substack{i\leq r \\ j\leq r, k\leq r}} \Big(\tfrac{1}{2}\alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij})$$

$$- \tfrac{1}{4}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})(\alpha_{ijk} + \alpha_{jki} - \alpha_{kij}) - \alpha_{kii}\alpha_{kjj}\Big)$$

$$+ \sum_{\substack{i\leq r \\ j\leq r, k>r}} \Big(-\tfrac{1}{2}\alpha_{ijk}^2 - \tfrac{1}{4}\alpha_{ijk}^2\Big)$$

$$= \rho'(\mathfrak{g}) - \tfrac{3}{4} \sum_{\substack{i\leq r \\ j\leq r, k>r}} \alpha_{ijk}^2$$

$$\sum_{i>r, j>r} K(e_i, e_j) = \sum_{\substack{i>r \\ j>r, k>r}} \Big(\tfrac{1}{2}\alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij})$$

$$- \tfrac{1}{4}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})(\alpha_{ijk} + \alpha_{jki} - \alpha_{kij}) - \alpha_{kii}\alpha_{kjj}\Big)$$

$$+ \sum_{j>r,k\leq r} \left(-\frac{1}{4}(-\alpha_{jki}+\alpha_{kij})(\alpha_{jki}-\alpha_{kij}) - \alpha_{kii}\alpha_{kjj}\right)$$

$$= \rho(\mathfrak{h}) + \sum_{\substack{i>r \\ j>r,k\leq r}} \left(\frac{1}{4}(\alpha_{kji}+\alpha_{kij})^2 - \alpha_{kii}\alpha_{kjj}\right)$$

Next, if $A = 2 \sum_{i\leq r,j>r} \alpha_{kii}\alpha_{kjj} = \sum_{\substack{i\leq r,j>r \\ i>r,j\leq r}} \alpha_{kii}\alpha_{kjj}$, then

$$2 \sum_{i\leq r,j>r} K(e_i,e_j) = \sum_{i\leq r,j>r} (\alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij})$$

$$-\frac{1}{2}(\alpha_{ijk} - \alpha_{jki}+\alpha_{kij})(\alpha_{ijk}+\alpha_{jki}-\alpha_{kij})) - A$$

$$= \sum_{\substack{i\leq r \\ j>r,k>r}} \alpha_{ijk}(-\alpha_{ijk} + \alpha_{kij})$$

$$-\frac{1}{2} \sum_{i\leq r,j>r} (\alpha_{ijk}+\alpha_{kij})(\alpha_{ijk}-\alpha_{kij}) - A$$

$$= \sum_{\substack{i\leq r \\ j>r,k>r}} (-\alpha_{ijk}^2-\alpha_{ijk}\alpha_{ikj}) - \frac{1}{2} \sum_{\substack{i\leq r \\ j>r,k>r}} \alpha_{ijk}^2$$

$$+ \frac{1}{2} \sum_{i\leq r,j>r} \alpha_{kij}^2 - A$$

$$= -\frac{1}{2} \sum_{\substack{i\leq r \\ j>r,k>r}} (\alpha_{ijk}+\alpha_{ikj})^2 + \frac{1}{2} \sum_{\substack{i\leq r \\ j>r,k\leq r}} \alpha_{ikj}^2$$

$$- \sum_{\substack{i\leq r,j>r \\ i>r,j\leq r}} \alpha_{kii}\alpha_{kjj}$$

Thus we have

$$\rho(\mathfrak{g}) = \rho'(\mathfrak{s}) + \rho(\mathfrak{h}) - \frac{3}{4} \sum_{\substack{i\leq r \\ j\leq r, k>r}} \alpha_{ijk}^2 + \frac{1}{4} \sum_{\substack{i>r \\ j>r, k\leq r}} (\alpha_{kij} + \alpha_{kji})^2$$

$$- \frac{1}{2} \sum_{\substack{i\leq r \\ j>r, k>r}} (\alpha_{ijk} + \alpha_{ikj})^2 + \frac{1}{2} \sum_{\substack{i\leq r \\ j>r, k\leq r}} \alpha_{ikj}^2$$

$$- \left[\sum_{\substack{i>r \\ j>r, k\leq r}} \alpha_{kii}\alpha_{kjj} + \sum_{\substack{i\leq r, j>r \\ i>r, j\leq r \\ k\leq r}} \alpha_{kii}\alpha_{kjj} + \sum_{\substack{i\leq r, j\leq r \\ k\leq r}} \alpha_{kii}\alpha_{kjj} \right] \qquad \ldots (3)$$

Note that the last term in the square brackets, which has been added to facilitate a compact form for the expression, is actually zero , because it is equal to $\sum_{k\leq r} \left(\sum_{i\leq r} \alpha_{kii} \right)^2 = \sum_{k\leq r} (\text{trace of ad } e_k \text{ on } \mathfrak{s})^2 = 0$, since $\mathfrak{s}$ is a unimodular subalgebra. The expression in the square brackets similarly becomes $\sum_{k\leq r} \left(\sum_{i=1}^{n} \alpha_{kii} \right)^2 = \sum_{k\leq r} (\text{tr ad } e_k)^2$. The third and sixth, as also the fourth and fifth, terms in the last expression (3) for $\rho(\mathfrak{g})$ are similar except for the permutation of indices.

Thus we have, after observing these facts,

$$\rho(\mathfrak{g}) = \rho'(\mathfrak{s}) + \rho(\mathfrak{h}) - \frac{1}{4} \sum_{\substack{i\leq r \\ j\leq r, k>r}} \alpha_{ijk}^2 - \frac{1}{4} \sum_{\substack{i\leq r \\ j>r, k>r}} (\alpha_{ijk} + \alpha_{ikj})^2$$

$$- \sum_{k\leq r} (\text{tr ad } e_k)^2$$

So clearly $\rho(\mathfrak{g}) \leq \rho'(\mathfrak{s}) + \rho(\mathfrak{h})$

and the proposition is proved.

Proof of Theorem 2 : As already observed if G is not solvable and its universal covering space is homeomorphic to Euclidean space then its Lie algebra

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} ,$$

where $\mathfrak{p}$ is a solvable ideal and $\mathfrak{k}$ is a semi-simple subalgebra. Further $\mathfrak{k} = \mathfrak{s}_1 + \ldots + \mathfrak{s}_m$ (Lie algebra direct sum) where each $\mathfrak{s}_i$ is a simple ideal of $\mathfrak{k}$ and is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, the Lie algebra of $SL(2, \mathbb{R})$. $\mathfrak{sl}(2, \mathbb{R})$ is a vector space spanned by v_1, v_2, v_3 with product being given by

$$[v_2, v_3] = v_1, \quad [v_3, v_1] = v_2, \quad [v_1, v_2] = - v_3$$

In particular $\mathfrak{sl}(2, \mathbb{R})$ is unimodular.

Given a left-invariant metric on G, in order to apply the preceding proposition, we take $\mathfrak{s} = \mathfrak{s}_1$ and $\mathfrak{h} = \mathfrak{s}_2 + \ldots + \mathfrak{s}_m + \mathfrak{p}$. Clearly $\mathfrak{h}$ is an ideal and $\mathfrak{s}$ is unimodular and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$

Thus $\rho(\mathfrak{g}) \leq \rho_1(\mathfrak{s}_1) + \rho(\mathfrak{s}_2 + \ldots + \mathfrak{s}_m + \mathfrak{p})$

where $\rho_1(\mathfrak{s}_1)$ is scalar curvature of some left-invariant metric on $SL(2, \mathbb{R})$.

Again we split $\mathfrak{s}_2 + \ldots + \mathfrak{s}_m + \mathfrak{p}$ as the direct sum of $\mathfrak{s} = \mathfrak{s}_2$ and $\mathfrak{h} = \mathfrak{s}_3 + \ldots + \mathfrak{s}_m + \mathfrak{p}$. Hence by successive applications of the Proposition, we get

$$\rho(\mathfrak{g}) \leq \sum_{i=1}^{m} \rho_i(\mathfrak{s}_i) + \rho(\mathfrak{p})$$

where $\rho_i(\mathfrak{S}_i)$ is the scalar curvature of some left-invariant metric on $SL(2,\mathbb{R})$; $\rho(\mathfrak{P})$ is the scalar curvature of the given metric restricted to $\mathfrak{P}$ (i.e., scalar curvature of subgroup P of G, with Lie algebra $\mathfrak{P}$, and the left-invariant metric on G restricted to P).

As remarked earlier we already know $\rho(\mathfrak{P}) \leq 0$, since $\mathfrak{P}$ is solvable. And due to [4, Corollary 4.7], scalar curvature of a left-invariant metric on $SL(2,\mathbb{R})$ is always strictly negative.

Thus clearly $\rho(\mathfrak{G}) < 0$ which is what we set out to prove if G is not solvable. This completes the proof of Theorem 2.

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Part II

PARAMETRIC HOMOTOPY PRINCIPLE FOR A CLASS OF PARTIAL DIFFERENTIAL RELATIONS ON A CLOSED MANIFOLD

0. Introduction

In this part we use a Smale-Hirsch-Gromov technique to obtain a new Gromov-type theorem for a class of differential 1-forms on a closed manifold. Let M be a manifold of dimension n, and k be a positive integer such that $2k+1 \leq n$. Then we say that a smooth 1-form ω on M is of type k if $\omega \wedge (d\omega)^k \neq 0$ everywhere on M. Let $\Omega_k(M)$ be the space of all 1-forms of type k on M, endowed with the subspace topology of the C^∞ - topology on $\bigwedge^1(M)$, the space of 1-forms on M. On the other hand, let $J^1(T^* M)$ denote the bundle over M of 1-jets of local 1-forms on M, and $O_k(M)$ be the subset of $J^1(T^* M)$ that is obtained by taking 1-jets of local 1-forms of type k (on the domain of definition). Then $O_k(M)$ is an open subbundle of $J^1(T^* M)$, and is invariant under the action of the pseudogroup of local diffeomorphisms of M (see below for specific definition). Let $\Gamma(O_k(M))$ denote the space of continuous sections of $O_k(M)$ with the compact-open topology. Then we prove

Theorem 1. If M is a closed manifold, and $2k+1 < n$, then the 1-jet map $j^1 : \Omega_k(M) \longrightarrow \Gamma(O_k(M))$ is a weak homotopy equivalence (abbreviated w.h.e.).

Thus the existence and classification of a certain subset of $\wedge^1(M)$, characterised by certain differential inequalities, is reduced to the classical problem of existence and classification of sections of a bundle. This result extends and strengthens the scope of Smale-Hirsch-Gromov techniques as a tool of solving problems of this nature.

To put this result into its proper perspective, let us look at the earlier developments in this particular field which has variously been called integrability results, homotopy classification of stable or regular sections, parametric homotopy principle for obtaining solutions to partial differential relations. In 1969, Gromov (see Poenaru [6] or Haefliger [3]) obtained the following theorem.

Consider the pseudogroup of local diffeomorphisms of M. Suppose that for each U open in M there corresponds a bundle $E(U)$ over U satisfying the condition that, for $U' \subset U$, $E(U') = E(U)|_{U'}$, and that if $\varphi : U \longrightarrow V$ be a diffeomorphism between open subsets of M then there exists a bundle isomorphism $\bar{\varphi} : E(U) \longrightarrow E(V)$ projecting to φ, and satisfying the condition that the map from Diffeo (U) to Diffeo $E(U)$ sending φ to $\bar{\varphi}$ is continuous. Moreover, assume that if $\varphi : U \longrightarrow V$ and $\psi : V \longrightarrow W$ be local diffeomorphisms then $\overline{\psi \circ \varphi} = \bar{\psi} \circ \bar{\varphi}$. Let $J^r(E)$ be the bundle of r-jets of local sections of E, and $O(E)$ be an open subbundle of $J^r(E)$ invariant under the action induced by pseudogroup of local diffeomorphisms in the sense that if $f : U \longrightarrow E(U)$ represents an element of $O(E)$ at $u \in U$, and $\varphi : U \longrightarrow V$ is diffeomorphism between open subsets of M, then $j^r(\bar{\varphi} \circ f \circ \varphi^{-1})(\varphi(u)) \in O(E)$.

Let $\Gamma(O(E))$ be the space of continuous sections of $O(E)$ with the compact-open topology. Let Ω be the space of smooth sections of E over M whose r-jets lie in $\Gamma(O(E))$, with the C^{∞} - topology. Say that M is closed if it is compact and without boundary, and that M is open if no connected component of $M \setminus \partial M$ is closed. Then Gromov's Theorem states that if M is open, then the r-jet map $j^r : \Omega \longrightarrow \Gamma(O(E))$ is a w.h.e.

In a later development [2], Gromov called $O(E)$ a partial differential relation and elements of Ω solutions of $O(E)$. Then the conclusion of the theorem is that the solutions of $O(E)$ satisfy the parametric homotopy principle.

It is readily seen first of all that our theorem follows from this if M is open and $2k+1 \leq \dim M$, $k > 0$. One simply takes $E = T^* M$, $r = 1$ and $O(E) = O_k(M)$, in which case $\Omega = \Omega_k(M)$.

It is wel-known that Gromov's Theorem is not true in general for closed manifolds. For example, let $M = S^6$, $E = T^* M$, $r = 1$ and $O(E)$ be the space of 1-jets of local 1-forms of rank 3, that is, satisfying $(da)^3 \neq 0$. Then Ω is the space of symplectic 1-forms on S^6 which is easily seen to be empty. But $\Gamma(O(E))$ is not empty since it has the homotopy type of the space of almost complex structures on S^6.

In 1976, Du Plessis [1] extended Gromov's theorem for closed manifolds. He called the open subbundle $O(E)$ a regularity condition, and elements of Ω regular sections. He defined $O(E)$ to be integrable if $j^r : \Omega \longrightarrow \Gamma(O(E))$ is a w.h.e., and $O(E)$ to be extensible if there exists (i) a C^∞ - bundle E' over $M \times \mathbb{R}$ (ii) a C^∞ - bundle map $\pi : i^* E' \longrightarrow E$, where i is the inclusion map of M into $M \times \mathbb{R}$, (iii) a regularity condition $O'(E') \subset J^r(E')$ over $M \times \mathbb{R}$ such that $\pi^r(i^* O'(E')) = O(E)$, where $\pi^r : i^* J^r(E') \longrightarrow J^r(E)$ is the map induced by π. Du Plessis proved that if $O(E)$ is extensible, then it is integrable.

It may be noted that the proof of our theorem does not follow from Du Plessis' theorem. However, it depends on certain modification of the arguments of Du Plessis using the Main Flexibility Theorem of Gromov [2]. We shall explain these points in details in (5.5).

We conclude this section with a brief review which preceded Gromov's theorem. The earliest result in these directions was proved by Whitney in 1937. Known as the Whitney-Graustein theorem, it classifies immersions of S^1 in $\mathbb{R}^2$. During 1958-59, Smale extended this result first to immersions of S^1 in a Riemannian manifold, and then to immersions of S^n into $\mathbb{R}^p$ for $n < p$. Smale proved that the derivative map from the space of immersions of S^n into $\mathbb{R}^p$ to the space of injective tangent bundle maps $T S^n \longrightarrow T \mathbb{R}^p$ is a w.h.e. The result of Smale, besides being a generalisation of Whitney's result, was also the first important step towards resolution of a conjecture of Ehresmann (1948). It contains the ideas that led to, and it forms the basis of, the technique of all the subsequent works that appeared later. Smale's result was extended by Hirsch in 1959 to immersions of a manifold M into another manifold W with $\dim M < \dim W$. These ideas were clarified subsequently by Thom, Poenaru, Poenaru-Haefliger, Hirsch-Palais. Then in 1967, Phillips [5] proved that the derivative map is also a w.h.e. from the space of submersions of M into W to the space of surjective tangent bundle maps $TM \longrightarrow TW$. When M is closed a similar result is also available for k-mersions of M into W which are maps of rank $\geq k$ everywhere. In 1969 Feit proved that the derivative map from the space of k-mersions of M into W to the space of tangent bundle maps from TM to TW of rank $\geq k$ is a w.h.e. provided M is closed and $k < p$. The references to all these works may be found either in Phillips [5] or in Du Plessis [1].

1. Some definitions, notations and remarks

(1.1) Let X and Y be topological spaces and let $[X; Y]$ denote the homotopy classes of continuous maps from X to Y. Then a map $f : X \longrightarrow Y$ is said to be a *weak homotopy equivalence* (abbreviated w.h.e.) if for all compact polyhedra Q, the induced map

$$f_* : [Q ; X] \longrightarrow [Q ; Y] ,$$

defined by $[g] \longmapsto [f \circ g]$, is a bijection. This is equivalent to the fact that f induces a bijection between path components of X and Y and isomorphisms between homotopy groups of the spaces X and Y.

(1.2) The map $f : X \longrightarrow Y$ is said to be a Serre-fibration if given a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & X \\ \Big\downarrow{\scriptstyle id \times 0} & & \Big\downarrow{\scriptstyle f} \\ Q \times I & \xrightarrow{G} & Y \end{array}$$

(i.e., $f \circ G_0(q) = G(q,o)$ for all q in Q, a compact polyhedron and $I = [0,1]$)

there exists a map $\bar{G} : Q \times I \longrightarrow X$ such that $\bar{G}(q,0) = G_0(q)$ for all q in Q, and $f \circ \bar{G} = G$. This should hold for all Q. This diagram is called a lifting problem (G_0, G) and $\bar{G}$ is called a solution to the lifting problem or simply a lifting.

(1.3) Now we state an important and easy lemma

Lemma 2. Suppose we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{g}} & E' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

where p and p' are Serre-fibrations and g w.h.e. Then $\bar{g}$ is w.h.e. if and only if its restriction to each fibre of E is a w.h.e.

This follows by considering the homotopy exact sequences of the fibrations p and p' and the maps induced by g and $\bar{g}$ between these sequences and applying the wel-known 5-Lemma.

(1.4) If M_0 be a manifold of dimension n with boundary, and $\varphi : S^{\lambda-1} \times D^{n-\lambda} \longrightarrow M_0$ be an embedding. Then a manifold M is said to be obtained from M_0 by attaching a handle of index λ and M_1 is obtained from the disjoint union $M_0 \cup D^{\lambda} \times D^{n-\lambda}$ by identifying $S^{\lambda-1} \times D^{n-\lambda} \subset D^{\lambda} \times D^{n-\lambda}$ with its image under φ.

By an appropriate choice of a Morse function on a manifold M of dimension n, one obtains a handle-body decomposition of M

$$M_0 \subset M_0' \subset M_1 \subset M_1' \subset \dots \subset M_{L-1} \subset M_{L-1}' \subset M_L = M \qquad \dots. (1)$$

where M_0 is an n-ball D^n, M_i is M_{i-1}' with a handle of index λ_i attached, and M_i' is a collar-like neighbourhood of M_i (that is, $M_i' = M_i \cup \partial M_i \times [0,1]$). We may suppose that all the handles are attached in some coordinate neighbourhood, and $\lambda_i \leq \lambda_{i+1}$, so that $\lambda_L = n$ if M is closed. Also note that if M is closed, the number of handles attached is finite, and that there could be more than one handle of index n. For an open manifold, the number of handles attached could be infinite but all the handles have indices strictly less than n so that there is always a transverse disc $D^{n-\lambda_i}$ of positive dimension in the handle.

(1.5) We collect here some of the notations which has been introduced in Section 0 for ready reference.

If U is open in M, then

$\Lambda^1(U)$ = space of smooth 1-forms on U,

$\Omega_k(U) = \{ w \in \Lambda^1(U) : w \wedge (dw)^k \neq 0$ everywhere on $U \}$.

Note that if U is not open, the above spaces still make sense. For in this case a smooth 1-form w on U is just the restriction of a smooth 1-form w on some open neighbourhood of U in M. Also if this w is of type k on U, then it will be so on an open neighbourhood containing U.

$T^* U$ = cotangent bundle of U .

$J^1(T^* U)$ = bundle of 1-jets of local 1-forms on U.

$O_k(U)$ = bundle of 1-jets of local 1-forms of type k on U.

$\Gamma(O_k(U))$ = space of continuous sections of $O_k(U)$.

Our Theorem 1 says that the map $j^1 : \Omega_k(M) \longrightarrow \Gamma(O_k(M))$ is a w.h.e. if M is closed, and $2k+1 < n$. A remark on the theorem is that, since M is compact, the above w.h.e. is a homotopy equivalence by a result of Palais [4] .

2. The idea of the proof of Theorem 1.

The proof is based on an induction scheme with respect to a handle body decomposition of the closed manifold M

$$M_0 \subset M_0' \subset M_1 \subset M_1' \subset \dots \subset M_i \subset M_i' \subset M_{i+1} \subset M_{i+1}' \subset \dots \subset M_{L-1} \subset M_{L-1}' \subset M_L = M ,$$

where M_i and M_i' , $0 \leq i \leq L-1$, are as described in (1.4) :

$$M_i = M'_{i-1} + \text{a handle of index } \lambda_i ,$$

$$M'_i = M_i \cup \partial M_i \times [0,1] .$$

The decomposition gives rise to the following commutative diagram , where the vertical maps are the restriction maps and the horizontal maps are the 1-jet maps j^1 .

$$
\begin{array}{ccc}
\dot{\Omega}_k(M) & \xrightarrow{\;j_L\;} & \Gamma(O_k(M)) \\
r'_{L-1}\downarrow & & \downarrow s'_{L-1} \\
\dot{\Omega}_k(M'_{L-1}) & \xrightarrow{\;j'_{L-1}\;} & \Gamma(O_k \mid M'_{L-1}) \\
r_{L-1}\downarrow & & \downarrow s_{L-1} \\
\vdots & & \vdots \\
r_{i+1}\downarrow & & \downarrow s_{i+1} \\
\dot{\Omega}_k(M_{i+1}) & \xrightarrow{\;j_{i+1}\;} & \Gamma(O_k \mid M_{i+1}) \\
r'_i\downarrow & & \downarrow s'_i \\
\dot{\Omega}_k(M'_i) & \xrightarrow{\;j'_i\;} & \Gamma(O_k \mid M'_i) \\
r_i\downarrow & & \downarrow s_i \\
\dot{\Omega}_k(M_i) & \xrightarrow{\;j_i\;} & \Gamma(O_k \mid M_i) \\
r'_{i-1}\downarrow & & \downarrow s'_{i-1} \\
\vdots & & \vdots \\
r'_0\downarrow & & \downarrow s'_0 \\
\dot{\Omega}_k(M'_0) & \xrightarrow{\;j'_0\;} & \Gamma(O_k \mid M'_0) \\
r_0\downarrow & & \downarrow s_0 \\
\dot{\Omega}_k(M_0) & \xrightarrow{\;j_0\;} & \Gamma(O_k \mid M_0)
\end{array}
$$

Note that J_L is j^1 according to our previous notation.

Then the proof of Theorem 1 follows by induction from the following propositions, using Lemma 2 of Section 1.

Proposition 3. The map J_0 is a w.h.e.

This is the starting point of the induction.

Proposition 4. Each of the maps s_i, s_i', and r_i' are Serre fibrations, for $0 \leq i \leq L-1$.

Proposition 5. For each $1 \leq i \leq L$, $r_i : \Omega_k(M_i) \longrightarrow \Omega_k(M_{i-1}')$ is a Serre fibration (recall that $M_i = M_{i-1}'$ + a handle of index λ_i), when (a) $\lambda_i < n$, and also when (b) $\lambda_i = n$, where $n = \dim M_i$.

Note that in this proposition we have stated the two cases separately, because their proofs are different.

Proposition 6. The restriction of J_i (respectively J_i') to each fibre of r_i (respectively r_i') is a w.h.e., for $0 \leq i \leq L-1$.

The proofs of these propositions, except for Proposition 5(b), follow from the proofs of the corresponding facts of Gromov theory (see Haefliger [3], or Poenaru [6], or Du Plessis [1]). Therefore the proof of Theorem 1 will be complete if we establish Proposition 5(b), or equivalently, the following Proposition 7(b).

Let $A = 2\,D^{\lambda} \times D^{n-\lambda}$ and $B = S^{\lambda-1} \times [1,2] \times D^{n-\lambda}$ ($2\,D^{\lambda}$ being disc of radius 2) so that A is B with a handle of index λ attached.

Proposition 7. The restriction map $r : \Omega_k(A) \longrightarrow \Omega_k(B)$ is a Serre fibration when (a) $\lambda < n$, and when (b) $\lambda = n$.

Note that we want to prove the case (a) also because it will be needed in the proof of the case (b).

3. Preliminary results

The following facts, which will be used without mention, allow us to make some constructions needed in the proof of Proposition 7. Although these facts are stated for general fibre bundle, we need them only in the context of Proposition 7, when the base space is considered to be within some coordinate neighbourhood, so that the bundle is trivial.

Fact 1. Let E be a bundle over a manifold X, and Y be a closed subset of X. Then, if $f : X \longrightarrow E$ be a continuous section which is differentiable on Y, there exists a differentiable section $g : X \longrightarrow E$ such that $g = f$ on Y.

Fact 2. Let $Y \subset X$ be a relative C-W pair, and E be a bundle over $X \times I$. Then any partial section

$$f : Y \times I \cup X \times 0 \longrightarrow E$$

can be extended to whole of $X \times I$.

Let us explain how these facts are used here. In our case, Y will be a submanifold of the same dimension as X, and E will be cotangent bundle T^*X, so that $E|_Y = T^*Y$. Let $p : E \longrightarrow X$ be the projection map, and $\Gamma(E)$ denote the space of continuous sections of E. Let

$$\begin{array}{ccc} Q & \xrightarrow{F} & \Gamma(E) \\ \downarrow & & \downarrow r \\ Q \times I & \xrightarrow{f} & \Gamma(E|Y) \end{array}$$

be a commutative diagram, where r is the restriction map. Consider the bundle $E \times Q \times I$ over $X \times Q \times I$ with the projection map $p \times id \times id$. We have then a partial section

$$g : Y \times Q \times I \cup X \times Q \times 0 \longrightarrow E \times Q \times I$$

given by $g(y, q, t) = (f(q,t)(y), q, t)$

and $g(x, q, 0) = (F(q)(x), q, 0)$

$x \in X, \quad y \in Y, \quad q \in Q, \; t \in I$.

By Fact 2, there exists an extension G of g to whole of $X \times Q \times I$. This G gives rise to a lifting of the lifting problem (F, f). Further, if F and f take values in the space of smooth sections, our G above can be chosen so that $G | X \times \{q\} \times \{t\}$ is smooth for each $(q,t) \in Q \times I$.

In our set up a smooth section $f : Y \longrightarrow E$ would mean restriction of a smooth section on an open set in X containing Y. But suppose we have a continuous family of sections $g : Q \longrightarrow \Gamma(E|Y)$. Then we can get $\tilde{g} : Q \longrightarrow \Gamma(E|U)$ for some U open in X containing Y such that $\tilde{g} = g$ on Y. This can be seen from the above consequence of Facts 1 and 2, and some standard arguments using a triangulation of Q (start with some extension of $f(q_o)$ for some vertex $q_o \varepsilon Q$, and then extend it over 1-simplices, and then over 2-simplices, and so on).

Now we recall a standard notation. If $f : M \longrightarrow N$ be a smooth map and $\eta \varepsilon \bigwedge^1(N)$, then pull-back $f^* \eta$ of η is a 1-*form on* M *defined by*

$$f^* \eta(v) = \eta(df_*(v)) , \quad v \varepsilon (TM)_x .$$

Products of differential forms will always be exterior multiplication.

4. Proof of Proposition 7(a)

This result is due to Gromov (see Poenaru [5]). A careful consideration of these techniques will be needed in the case $\lambda = n$. Consider the following lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_o} & \bar{\Omega}_k(A) \\ \Big\downarrow{\scriptstyle id \times 0} & & \Big\downarrow{\scriptstyle r} \\ Q \times I & \xrightarrow{G} & \bar{\Omega}_k(B) \end{array}$$

Then, by results of the last section, there exists

$$\tilde{G} : Q \times I \longrightarrow \bigwedge^1(2D^\lambda \times (1+d)\, D^{n-\lambda})$$

for some $d > 0$ such that

$$\tilde{G}(q,0) = G_0(q) \quad \text{on } A ,$$

and $$\tilde{G}(q,t) = G(q,t) \quad \text{on } B .$$

This is done by first getting a lift $F : Q \times I \longrightarrow \Lambda^1(A)$ of the pair (G, G_0), and then getting an extension $\tilde{F}$ of F over an open set U containing A. There exists $d > 0$ such that $2D^\lambda \times (1+d)D^{n-\lambda} \subset U$. $\tilde{G}$ is the restriction of $\tilde{F}$. Now by a compactness argument, there exists $\eta > 0$ and $0 < b < 1$ such that

$$\tilde{G}(q,0)\Big|_{2D^\lambda \times (1+\eta)D^{n-\lambda}} \varepsilon \; \bar{\Omega}_k(2D^\lambda \times (1+\eta)\, D^{n-\lambda}) ,$$

and $\tilde{G}(q,t)\Big|_{S^{\lambda-1} \times [b,2] \times (1+\eta)D^{n-\lambda}} \varepsilon \; \bar{\Omega}_k(S^{\lambda-1} \times [b,2] \times (1+\eta)\, D^{n-\lambda})$.

This argument may be described briefly as follows. Let $q \varepsilon Q$. Then $\tilde{G}(q,0)|_A \; \varepsilon \; \bar{\Omega}_k(A)$, and hence there exists an open set $U_q \supset A$ such that $\tilde{G}(q,0)|_{U_q} \; \varepsilon \; \bar{\Omega}_k(U_q)$. The assignment $q' \longrightarrow \tilde{G}(q', 0)|_{U_q}$ is a continuous map $Q \longrightarrow \Lambda^1(U_q)$. So by continuity, there exists a neighbourhood of q in Q which is mapped into $\bar{\Omega}_k(U_q)$. By compactness of Q there exists an open set U containing A such that $\tilde{G}(q,0)|_U \; \varepsilon \; \bar{\Omega}_k(U)$ for all q in Q. Similarly using the fact that $\tilde{G}(q,t)|_B \; \varepsilon \; \bar{\Omega}_k(B)$ and compactness of $Q \times I$, analogous arguments give an open set V containing B such that $\tilde{G}(q,t)|_V \; \varepsilon \; \bar{\Omega}_k(V)$ for all $(q,t) \; \varepsilon \; Q \times I$. Thus η and b can be chosen as above.

Notations. We will denote the extension $\tilde{G}(q,0)\Big|_{2D^\lambda \times (1+\eta)D^{n-\lambda}}$ of $G_0(q)|_A$ by $G_0(q)$ also.

Let $B_x^y = S^{\lambda-1} \times [x,2] \times (1+y) D^{n-\lambda}$ and $A^y = 2D^\lambda \times (1+y) D^{n-\lambda}$, $0 < x \leq 1$ and $y > 0$. Define

$$G' : Q \times I \longrightarrow \Omega_k(B_b^\eta)$$

by
$$G'(q,t) = \tilde{G}(q,t)\big|_{B_b^\eta} .$$

Lemma 8. Let a and c be numbers such that $b < a < c < 1$. Then there exists a partition $t_o = 0 < t_1 < \dots < t_N = 1$ of I, and maps $\xi_i : Q \times [t_i, t_{i+1}] \longrightarrow \Omega_k(B_b^\eta)$, $0 \leq i \leq N-1$, such that

$$\xi_i(q,t) = G'(q,t) \text{ on } B_c^\eta$$

$$\xi_i(q,t_i) = G'(q,t_i)$$

$$\xi_i(q,t) = G'(q,t_i) \text{ on } S^{\lambda-1} \times [b,a] \times (1+\eta)D^{n-\lambda} .$$

Proof. Find a smooth function $\alpha : [b, 2] \longrightarrow [0,1]$ such that $\alpha|_{[b,a]} \equiv 0$ and $\alpha|_{[c,2]} \equiv 1$ and let $L : Q \times I \times I \longrightarrow \Omega_k(B_b^\eta)$ be the map defined by $L(q,t,s) = G'(q, \min(t+s,1))$,

$$(q,t,s) \in Q \times I \times I .$$

Then we have $L(q,t,0) = G'(q,t)$.

Now define

$$L^\alpha : Q \times I \times I \longrightarrow \Lambda^1(B_b^\eta) \text{ by}$$

$$L^\alpha(q,t,s)(x \times u,y) = \left[(1-\alpha(u))L(q,t,0)+\alpha(u)L(q,t,s)\right](x \times u,y) ,$$

where $(x \times u,\ y) \in B_b^{\eta} = S^{\lambda-1} \times [b,2] \times (1+\eta)\, D^{n-\lambda}$. Then

$$L^{\alpha}(q,t,0) = L(q,t,0) = G'(q,t) \in \Omega_k(B_b^{\eta})$$

Now, $\Omega_k(B_b^{\eta})$ being an open subset of $\Lambda^1(B_b^{\eta})$, by continuity of L^{α} and compactness of $Q \times I$, there exists $\varepsilon > 0$ such that

$$L^{\alpha}(q,t,s) \in \Omega_k(B_b^{\eta}) \quad \text{for all } (q,t) \in Q \times I,\ s \leq \varepsilon .$$

Now choose a partition $0 = t_o < t_1 < \ldots < t_N = 1$ of I such that $t_{i+1} - t_i \leq \varepsilon$ for $0 \leq i \leq N-1$. Then put

$$\xi_i(q,t) = L^{\alpha}(q,t_i,t-t_i) \quad \text{for } t_i \leq t \leq t_{i+1} .$$

It is easily checked that ξ_i satisfies the requirements. This completes the proof of the lemma.

Now choose $\eta = \eta_1 > \eta_2 > \ldots > \eta_N > 0$ and $c = c_1 < c_2 < \ldots < c_N < 1$. Define $\beta_1 : Q \times [0,t_1] \longrightarrow \Omega_k(A^{\eta})$

by $$\beta_1(q,t) = \begin{cases} \xi_o(q,t) & \text{on } B_b^{\eta} \\ G_o(q) & \text{otherwise} \end{cases}$$

(where $G_o(q)$ is assumed extended over A^{η} as indicated earlier).

Suppose we have defined

$$\beta_i : Q \times [0,\ t_i] \longrightarrow \Omega_k(A^{\eta_i})$$

such that

$$\beta_i(q,t) = G'(q,t) \quad \text{on } B_{c_i}^{\eta_i}$$

and $$\beta_i(q,0) = G_o(q) .$$

We propose to construct

$$\beta_{i+1} : Q \times [0, t_{i+1}] \longrightarrow \Omega_k(A^{\eta_{i+1}})$$

satisfying $\beta_{i+1}(q,t) = G'(q,t)$ on $B_{c_{i+1}}^{\eta_{i+1}}$

and $\beta_{i+1}(q,0) = G_o(q)$.

Let m_i be chosen so that $c_i < m_i < c_{i+1}$. Consider an isotopy

$$\varphi_t : A^{\eta_i} \longrightarrow A^{\eta_i}, \; 0 \leq t \leq t_i ,$$

satisfying

(1) $\varphi_o = id$

(2) $\varphi_t = id$ on $B_{c_{i+1}}^{\eta_i}$ for all t

(3) There exists a neighbourhood V of

$S^{\lambda-1} \times \{c_i\} \times (1 + \eta_{i+1}) D^{n-\lambda}$ such that

$\varphi_t = id$ on V for ··ll t

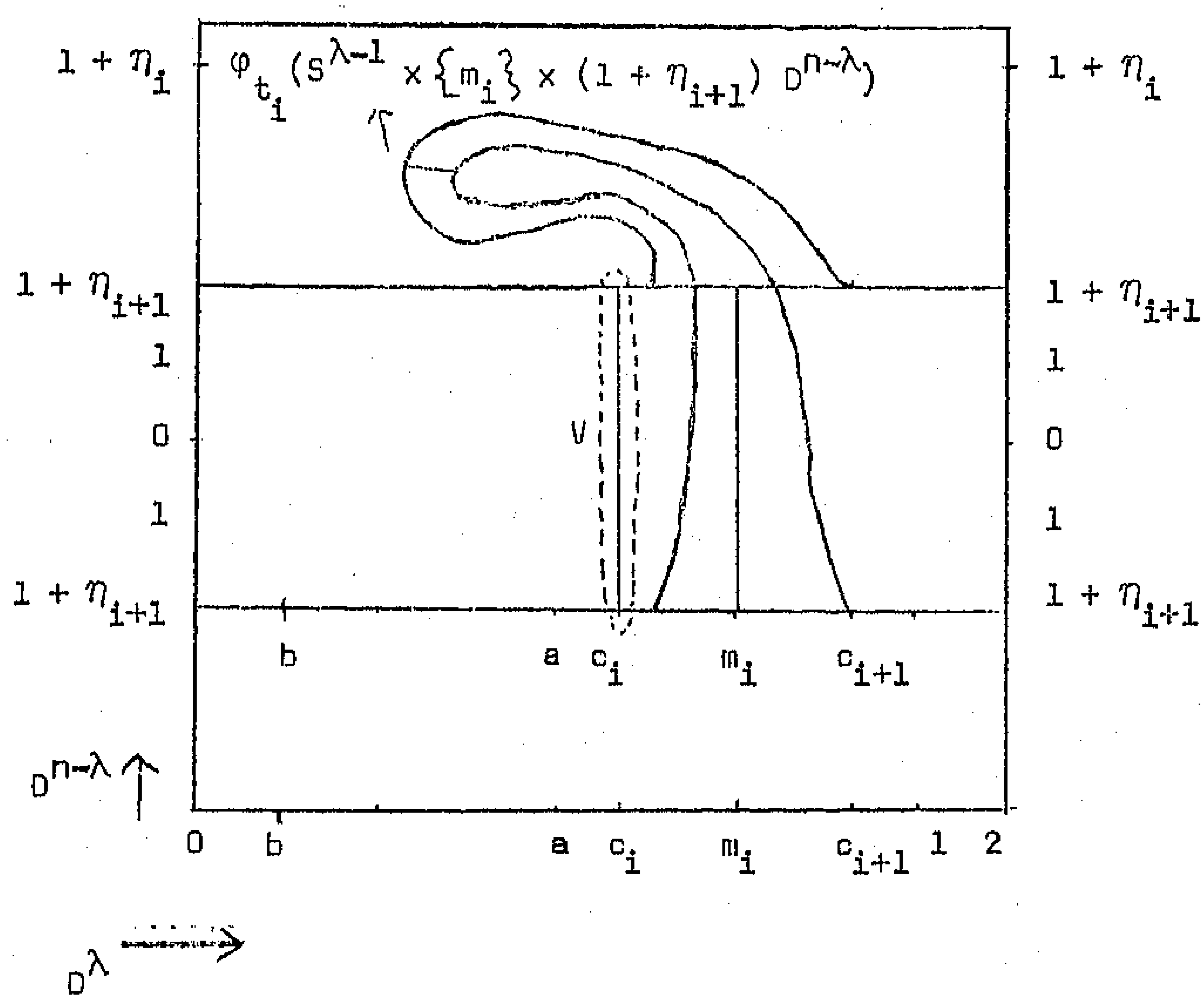
(4) $\varphi_t(S^{\lambda-1} \times [c_i, 2] \times (1 + \eta_{i+1}) D^{n-\lambda}) \subset B_b^{\eta_i}$

(5) $\varphi_{t_i}(S^{\lambda-1} \times \{m_i\} \times (1+\eta_{i+1})D^{n-\lambda}) \subset Int(S^{\lambda-1} \times [b,a] \times (1+\eta_i)D^{n-\lambda})$

(As a consequence a neighbourhood of $S^{\lambda-1} \times \{m_i\} \times (1+\eta_{i+1})D^{n-\lambda}$ is mapped by φ_{t_i} into $int(S^{\lambda-1} \times [b,a] \times (1+\eta_i) D^{n-\lambda})$.

Figure 1 on next page gives an idea of this isotopy.

Figure 1



Dotted lines show neighbourhood V of $S^{\lambda-1} \times \{c_i\} \times (1 + \eta_{i+1})D^{n-\lambda}$

Curved portion shows the position of φ_{t_i} over the domain $S^{\lambda-1} \times [c_i, c_{i+1}] \times (1 + \eta_{i+1})\, D^{n-\lambda}$.

Then define

$$\beta_{i+1} : Q \times [0, t_{i+1}] \longrightarrow \Omega_k(A^{\eta_{i+1}})$$

as follows:

For $t \leq t_i$

$$\beta_{i+1}(q,t) = \begin{cases} \varphi_t^* \beta_i(q,t) & \text{on } B_{c_i}^{\eta_{i+1}} \\ \beta_i(q,t) & \text{on } c_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \end{cases}$$

For $t_i \leq t \leq t_{i+1}$

$$\beta_{i+1}(q,t) = \begin{cases} \varphi_{t_i}^* \xi_i(q,t) & \text{on } B_{m_i}^{\eta_{i+1}} \\ \beta_{i+1}(q,t_i) & \text{on } m_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \end{cases}$$

Finally β_N gives the desired lift to the problem (G_o, G). This completes the proof of Proposition 7(a).

5. <u>Proof of Proposition 7(b)</u>

(5.1) We want to prove that the restriction map $r : \Omega_k(2D^n) \longrightarrow \Omega_k(S^{n-1} \times [1,2])$ is a Serre-fibration, that is, it satisfies polyhedral covering homotopy property. It is sufficient to prove that r has local polyhedral covering homotopy property. This means that, for every $w \in \Omega_k(S^{n-1} \times [1,2])$, there exists a neighbourhood $\mathcal{U}$ of w in $\Omega_k(S^{n-1} \times [1,2])$ such that $r : r^{-1}\mathcal{U} \longrightarrow \mathcal{U}$ is a Serre-fibration.

(5.2) We will consider subsets of $2D^n \times \mathbb{R}$ of the form $W \times J$, where W is an open subset or n-submanifold of $2D^n$, and J an interval in $\mathbb{R}$, and define spaces Ω'_{k+1} over these domains as follows.

$$\Omega'_{k+1}(W \times J) = \left\{ \eta \in \Omega_k(W \times J) : \eta(w,u) \in T^* W_w ,\ (w,u) \in W \times J \right\} .$$

Recall that by our earlier definition

$$\Omega_{k+1}(W \times J) = \left\{ \eta \in \Lambda^1(W \times J) : \eta \wedge (d\eta)^{k+1} \neq 0 \text{ everywhere on } W \times J \right\} .$$

The definition of Ω'_{k+1} makes sense, since $T^* W$ over $W \times J$ is a subbundle of $T^*(W \times J)$.

It may be noted that Ω'_{k+1} is not invariant under the action of all local diffeomorphisms. However, it is invariant under a subclass of fibre-preserving diffeomorphisms as shown below in (5.3). This is the main point of difference of our theorem with that of Du Plessis [1] . We shall discuss this point in details in (5.5) below

A diffeomorphism $\varphi : W \times J \longrightarrow W \times J$ is said to be fibre-preserving if it is of the form

$$\varphi(w,u) = (w, g(w,u)) ,\quad (w,u) \in W \times J ,$$

for some smooth map $g : W \times J \longrightarrow \mathbb{R}$.

(5.3) We now show that Ω'_{k+1} is invariant under the action of the fibre-preserving diffeomorphisms. If $y_1, \ldots, y_n$ is a coordinate system on W induced from a coordinate system on $2D^n$, then any $f \in \Omega'_{k+1}(W \times J)$ has the form

$$f(w,u) = \sum_{i=1}^{n} f_i(w,u)dy_i \quad (w,u) \in W \times J$$

where f_i's are smooth functions. Therefore if φ is a fibre-preserving diffeomorphism

$$\varphi^* f = (d\varphi)^T f \circ \varphi = \begin{pmatrix} & & \frac{\partial g}{\partial y_1} \\ I_n & & \vdots \\ & & \frac{\partial g}{\partial y_n} \\ 0 \; 0 & \ldots \; 0 & \frac{\partial g}{\partial u} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \end{pmatrix} \circ \varphi$$

$= f \circ \varphi$, where I_n is the nxn identity matrix,

or $\varphi^* f = \sum_{i=1}^{n} f_i \circ \varphi \, dy_i$. Thus $\varphi^* f \in \Omega'_{k+1}$.

Lemma 9. Let U be open in $2D^n$, $w \in \Omega_k(U)$, and $x \in U$. Then there exists a neighbourhood V_x of x such that, for any n-submanifold Z of V_x, there exists an open neighbourhood H_Z of $w|_Z$, and a continuous map

$$\rho_Z : H_Z \longrightarrow \Omega'_{k+1}(Z \times \mathbb{R})$$

satisfying

(1) $\rho_Z(h) \circ i = h$ for all $h \in H_Z$

(2) $\rho_Z(h|_{Z'}) = \rho_Z(h)|_{Z' \times \mathbb{R}}$ for Z' n-submanifold of Z.

Here i indicates the inclusion map of Z in $Z \times \mathbb{R}$.

Proof. In terms of a coordinate system $y_1, \ldots, y_n$ on $2D^n$, we have

$$\omega \wedge (d\omega)^k = \Sigma\, \omega_{i_1 \ldots i_{2k+1}}\, dy_{i_1} \wedge \ldots \wedge dy_{i_{2k+1}},$$

where, for $f \in \bigwedge^1(U)$, $f_{i_1 \ldots i_{2k+1}}$ denotes the coefficient of $dy_{i_1} \wedge \ldots \wedge dy_{i_{2k+1}}$ in the expansion of $f \wedge (df)^k$. Since $\omega \wedge (d\omega)^k(x) \neq 0$, $\exists\, i_1^o, \ldots, i_{2k+1}^o$ such that $\omega_{i_1^o \ldots i_{2k+1}^o}(x) \neq 0$ and hence $\omega_{i_1^o \ldots i_{2k+1}^o} \neq 0$ in a neighbourhood V_x of x. Let τ be an index different from $i_1^o, \ldots, i_{2k+1}^o$. Note that τ exists, since $2k+1 < n$. Let

$$\eta(\underline{y}, u) = \omega(\underline{y}) + u\, d y_\tau, \quad (y,u) \in V_x \times \mathbb{R}.$$

Then

$$\eta \wedge (d\eta)^{k+1} = (\omega + u dy_\tau) \wedge (d\omega + du\ dy_\tau)^{k+1}$$

$$= \omega \wedge (d\omega)^{k+1} + (k+1)\, \omega \wedge (d\omega)^k \wedge du \wedge dy_\tau$$

$$+ u dy_\tau \wedge (d\omega)^{k+1}$$

Notice that the second term is the only term involving du. Thus the coefficient of $dy_{i_1^o} \wedge \ldots \wedge dy_{i_{2k+1}^o} \wedge du \wedge dy_\tau$ in $\omega \wedge (d\omega)^k \wedge du \wedge dy_\tau$, and hence in $\eta \wedge (d\eta)^{k+1}$, is $\omega_{i_1^o \ldots i_{2k+1}^o}$ which is nonzero on V_x.

So $\eta \wedge (d\eta)^{k+1} \neq 0$ on $V_x \times \mathbb{R}$

Now let $H_Z = \left\{ f \in \Omega_k(Z) \; ; \; f_{i_1^0 \dots i_{2k+1}^0} \neq 0 \text{ everywhere on } Z \right\}$,

and $\rho_Z(f) = f + u \, d y_\tau$ for $f \in H_Z$.

Clearly these satisfy the conditions (1) and (2), and the proof of the lemma is complete.

(5.4) Now coming back to the proof of Proposition 7(b) , let $w \in \Omega_k(S^{n-1} \times [1,2])$, and $w' \in \Omega_k(S^{n-1} \times [b,2])$ be an extension of w for some $b < 1$. Then, for all $x \in S^{n-1} = S^{n-1} \times \{1\}$, there exists open neighbourhood V_x of x in $S^{n-1} \times [b,2]$ and map ρ as given by Lemma 9. We may choose V_x to be of the form $U_x \times [1-c_x, 1+c_x]$, $1-c_x > b$. Since S^{n-1} is compact, there exists a finite subcover U_i and a $c > 0$ such that $1-c > b$ and $U_i \times [1-c, 1+c] \subset V_x$ for some x .

Now let $\mathcal{U}$ be a neighbourhood of w in $\Omega_k(S^{n-1} \times [1,2])$ consisting of all $f \in \Omega_k(S^{n-1} \times [1,2])$ with

$$f|_{U_i \times [1, 1+c]} \in H_{U_i \times [1, 1+c]}$$

We consider then a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\;G_0\;} & r^{-1}\mathcal{U} \\ \Big\downarrow{\scriptstyle \mathrm{id} \times 0} & & \Big\downarrow{\scriptstyle r} \\ Q \times I & \xrightarrow{\;G\;} & \mathcal{U} \end{array}$$

Since Q is compact there exists $a \in (0, \frac{1}{2}c]$ such that, for each $q \in Q$,

$$G_0(q)\,|_{U_i \times [1-2a,\, 1+c]} \in H_{U_i \times [1-2a,\, 1+c]} .$$

Define

$$G'(q,t) = \begin{cases} G(q,t) & \text{on } S^{n-1} \times [1,2] \\ G_0(q) & \text{on } (1-a)\, D^n \end{cases}$$

Let $X = S^{n-1} \times [1-2a, 1-a] \cup [1, 1+c]$

$Y = S^{n-1} \times [1-2a, 1+c]$

Then it remains to lift $(G'|_X , G_0|_Y)$.

(5.5) Remark. Before entering into the main body of the proof, perhaps this is the right place to look at the idea of Du Plessis and to point out the differences between our approach and his. Recalling the terminology of Du Plessis as given in Section 0, $\Omega'(U)$ is the space of sections of E', over $(n+1)$ submanifold U of $M \times \mathbb{R}$, which are mapped by the r-jet map j^r into $\Gamma(0'(U))$. Note that $\Omega'(U)$ is invariant under the action of diffeomorphisms of U because $0'(E')$ is assumed invariant under the action induced by local diffeomorphisms of $M \times \mathbb{R}$. The key idea of extensibility was to get a lemma of which our Lemma 9 is an analogue. This lemma may be stated roughly as follows.

Lemma 10 (Du Plessis). Let U be open in $2D^n$, $f \in \Omega(U)$, and $x \in U$. Then there exists a neighbourhood V_x of x, and a number $\varepsilon > 0$ such that, for any n-submanifold Z of V_x, there exists a continuous map

$$\rho_Z : H_Z \longrightarrow \Omega'(Z \times [-\varepsilon, \varepsilon]),$$

where H_Z is a neighbourhood of $f|_Z$ in $\Omega(Z)$, satisfying

$$\text{(i)} \quad \pi \circ \rho_Z(h) \circ i = h \qquad \text{for all } h \in H_Z$$

$$\text{(ii)} \quad \rho_{Z'}(h|_{Z'}) = \rho_Z(h)\,|_{Z' \times [-\varepsilon, \varepsilon]},$$

$h \in H_Z$ and Z' any n-submanifold of Z.

The essential points of Du Plessis' proof may now be described briefly as follows. First one follows, for a given $f \in \Omega(S^{n-1} \times [1,2])$, an argument analogous to that given in (5.4) to get a neighbourhood $\mathcal{U}$ of f in $\Omega(S^{n-1} \times [1,2])$, reducing the lifting problem (G, G_0) to $(G'|_X, G_0|_Y)$ with appropriate changes in notations. Then there remains to be constructed a lifting over the annulus $S^{n-1} \times [1-a, 1]$. This may be done by breaking the annulus into small enough handles so that each handle lies in one of $U_i \times [1-2a, 1+c]$. Then, starting with 1-handles, a lifting problem over $X+T$ (T being a 1-handle)

$$\begin{array}{ccc}
Q & \xrightarrow{\;G_0|\;} & \Omega(T) \\
\Big\downarrow{\scriptstyle id \times 0} & & \Big\downarrow{\scriptstyle r} \\
Q \times I & \xrightarrow{\;G'|\;} & \Omega(T \cap X)
\end{array}$$

is converted into a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\;H_o\;} & \overline{\Omega}'(T \times [-\varepsilon,\varepsilon]) \\ \big\downarrow & & \big\downarrow r \\ Q\times I & \xrightarrow{\;H\;} & \overline{\Omega}'((T \cap X) \times [-\varepsilon,\varepsilon]) \end{array}$$

by defining $H_o(q) = \rho_T(G_o(q))$

and $H(q,t) = \rho_{T \cap X}(G'(q,t))$.

This transformed lifting problem may be solved using Gromov's Theorem (Proposition 7(a)) with $O'(E')$ as a regularity condition over $T \times [-\varepsilon, \varepsilon]$. Therefore, if

$$\bar{H} : Q\times I \longrightarrow \overline{\Omega}'(T \times [-\varepsilon,\varepsilon])$$

is a lifting of the transformed problem, then a lifting over T is obtained by

$$\bar{G}(q,t) = \pi \circ \bar{H}(q,t) \circ i \ .$$

It is a consequence of the condition (iii) of extensibility (see Section 0) that $\bar{G}(q,t) \in \overline{\Omega}(T)$. Let $U'(1)$ denote X + all handles of indices 1 and G_1 denote a lifting over $U'(1)$. Then, if T is a 2-handle attached to $U'(1)$, we find that we again have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\;G_o|\;} & \overline{\Omega}(T) \\ \big\downarrow & & \big\downarrow r \\ Q\times I & \xrightarrow{\;G_1|\;} & \overline{\Omega}(U'(1) \cap T) \end{array}$$

and this problem may again be transformed to a lifting problem (H, H_0) over $T \times [-\varepsilon, \varepsilon]$, and so on.

This setting is simple and elegant in the sense that it reduces closed manifold case to Gromov's Theorem (open manifold case) by going over to some higher dimensional manifold. But this cannot be applied in our case. In Du Plessis case the requirement that Ω' be invariant under diffeomorphisms of the extended domain is used in an essential manner at the n-handle case, where the transverse disc $[-\varepsilon, \varepsilon]$ allows construction of deformations of Gromov, which are required to get a lifting of the transformed problem (H, H_0) over $T \times [-\varepsilon, \varepsilon]$ (T being a n-handle). It can be seen that as $T \times 0$ bends around in $T \times [-\varepsilon, \varepsilon]$ (see Figure 1, page 34), the pull-back of a 1-form on $T \times [-\varepsilon, \varepsilon]$ to T may vanish at some points, and hence it will not be in Ω_k. And such deformations have to be used a number of times. Looking at the Main Flexibility Theorem (Gromov [2], pp.78), we find that one need not verify that the sheaf Ω_{k+1}' on $M \times \mathbb{R}$ satisfy $\Omega'_{k+1}|_{M\times 0} = \Omega_k$. It is enough to have $\Omega'_{k+1}|_{M\times 0} \subset \Omega_k$, and a local result of the kind of Lemma 10, where the space Ω' is invariant only under fibre-preserving diffeomorphisms, diffeomorphisms that move only up and down in the additional dimension. For Ω'_{k+1}, the argument of Du Plessis goes through alright up to the liftings over handles T of indices $< n$, as in these cases, to lift the transformed lifting problem (H, H_0) in Ω'_{k+1}, we only need deformations of the form $\varphi \times id$, where φ is a local diffeomorphism of $2D^n$ and id is an appropriate restriction of the identity map from $\mathbb{R}$ to $\mathbb{R}$.

At the n-handle stage, we shall use the idea of Main Flexibility Theorem mentioned above, and we shall need a covering lifting problem (H, H_o) in Ω'_{k+1} as described in Proposition 13 below.

(5.6) Now we propose to break up the annulus $S^{n-1} \times [1-a, 1]$ into handles such that handles of indices n are pushed to within one of $U_i \times [1 - 2a, 1+c]$.

Let K be a triangulation of S^{n-1} each of whose $(n-1)$ simplices $|A|$ lies in one of the neighbourhoods U_i say U_A. Then for each simplex C in K, $|C| \subset U_A$ for each $(n-1)$ simplex A such that $C < A$. For some open neighbourhood $U(C)$ of $|C| \times [1-2a, 1+c]$ in Y, we define

$$^{A}H^{C}_{o} : Q \longrightarrow \Omega'_{k+1}(U(C) \times D^1) \text{ by } {}^{A}H^{C}_{o}(q) = \rho_{U(C)}(G_o(q))$$

$$^{A}H^{C} : Q\times I \longrightarrow \Omega'_{k+1}((U(C) \cap X) \times D^1) \text{ by}$$

$$^{A}H^{C}(q,t) = \rho_{U(C)\cap X}(G'(q,t))$$

for each simplex C and each $(n-1)$ simplex A such that $C < A$.

Let K^{ℓ} be the ℓ-skeleton of K and suppose that inductively we have constructed the following:

(1) a neighbourhood $\bar{U}(C)$ of $|C| \times [1-2a, 1+c]$ in $U(C)$ for each $(j-1)$ simplex C such that

$U(K^{j-1}) = X \cup \bigcup_{C} \bar{U}(C)$ is X plus a neighbourhood of $|K^{j-1}| \times [1-2a, 1+c]$ in Y

(2) a lifting $G^{j-1} : Q\times I \longrightarrow \bar{\Omega}_k(U(K^{j-1}))$ of

$$(G', G_o|_{U(K^{j-1})}) \text{ , and}$$

(3) a lifting ${}^AH^C : Q\times I \longrightarrow \bar{\Omega}'_{k+1}(\bar{U}(C)\times D^1)$ of

$$({}^AH^C|\bar{U}(C)\cap X\ ,\ {}^AH^C_o|\bar{U}(C))$$

such that, for each $j-1$ simplex C and each $(n-1)$ simplex A with $C < A$,

$$\begin{aligned} {}^AH^C(q,t)\, i &= G^{j-1}(q,t) \\ {}^AH^C(q,t) &= {}^AH^{C'}(q,t) \text{ on } (\bar{U}(C)\cap\bar{U}(C'))\times D^1 , \end{aligned}$$

where C, C' are $j-1$ simplices facing A

Let $U'(K^{j-1})$ be a neighbourhood of $X\cup|K^{j-1}|\times[1-2a, 1+a]$ in $U(K^{j-1})$, and, for each j-simplex E, let $\bar{U}(E)$ be a neighbourhood of $E\times[1-2a, 1+a]$ in $U(E)$ such that

$$\bar{U}(E)\cap U'(K^{j-1})\subset\cup\{\bar{U}(C)\,|\,C \text{ a } j-1 \text{ simplex facing } E\} ,$$

and such that there is a diffeomorphism

$$(\bar{U}(E)\ ,\ \bar{U}(E)\cap U'(K^{j-1}))\ \cong\ (2D^{j+1}\times D^{n-j-1},\ S^j\times[1,2]\times D^{n-j-1}) .$$

Then for each E we have a $(j+1)$ - handle lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\ G_o|\ } & \bar{\Omega}_k(\bar{U}(E)) \\ \big\downarrow & & \big\downarrow r \\ Q\times I & \xrightarrow{\ G^{j-1}|\ } & \bar{\Omega}_k(\bar{U}(E)\cap U'(K^{j-1})) \end{array}$$

and, for each E and each $(n-1)$ simplex A such that $E < A$, a $(j+1)$ - handle lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\;{}^{A}H_{o}^{E}\;} & \Omega'_{k+1}(\bar{U}(E) \times D^1) \\ \downarrow & & \downarrow \\ Q\times I & \xrightarrow{\;{}^{A}\bar{H}^{E}\;} & \Omega'_{k+1}((\bar{U}(E) \cap U'(K^{j-1})) \times D^1) \end{array}$$

(where ${}^{A}\bar{H}^{E}(q,t) = {}^{A}H^{C}(q,t)$ in $\bar{U}(C) \times D^1$)

such that ${}^{A}H_{o}^{E}(q)\, i = G_o(q)|$ and ${}^{A}\bar{H}^{E}(q,t)\, i = G^{j-1}(q,t)|$

In order to make the induction step complete, we distinguish two cases :

(1) If $j+1 < n$, use Proposition 11 below to find liftings ${}^{A}H^{E}$ for $({}^{A}\bar{H}^{E}, {}^{A}H_{o}^{E}|)$ such that ${}^{A}H^{E}(q,t)\, i = {}^{A'}H^{E}(q,t)\, i = {}^{A'}H^{E}(q,t)\, i$, for $(n-1)$ simplices A and A' with E as a common face. Then define G^j on $U(K^j) = X \cup \bigcup_E \bar{U}(E)$ by

$$G^j(q,t) = \begin{cases} {}^{A}H^{E}(q,t)\, i & \text{on } \bar{U}(E) \\ G^{j-1}(q,t) & \text{on } U'(K^{j-1}) \end{cases}$$

(2) If $j+1 = n$, use Proposition 13 below to find liftings $\bar{G}$ for $(G^{n-2}|_{\bar{U}(A) \cap U'(K^{n-2})}, G_o|_{\bar{U}(A)})$. Then define

$$G^{n-1}(q,t) = \begin{cases} \bar{G}(q,t) & \text{on } \bar{U}(A) \\ G^{n-2}(q,t) & \text{on } U'(K^{n-2}) \end{cases}$$

Then G^{n-1} is the required lift to the problem $(G'|_X, G_o|_Y)$. This completes the proof of Proposition 7b.

(5.7) We now prove the propositions referred above.

Proposition 11 : Let $A = 2D^{\lambda} \times D^{n-\lambda}$, $B = S^{\lambda-1} \times [1,2] \times D^{n-\lambda}$ with $\lambda < n$. Suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{\quad G_o \quad} & \Omega_k(A) \\ \downarrow & & \downarrow r \\ Q\times I & \xrightarrow{\quad G \quad} & \Omega_k(B) \end{array}$$

and lifting problems

$$\begin{array}{ccc} Q & \xrightarrow{\quad H_o^j \quad} & \Omega'_{k+1}(A \times D') \\ \downarrow & & \downarrow r \qquad j = 1,\dots,m \\ Q\times I & \xrightarrow{\quad H^j \quad} & \Omega'_{k+1}(B \times D') \end{array}$$

such that

$$H_o^j(q)\, i = G_o(q)$$
$$H_o^j(q,t)\, i = G(q,t)$$

for all $q \in Q$, $t \in I$, $j = 1,\dots,m$

Then there xists a lifting $\bar{G}$ of (G, G_o), and liftings $\bar{H}^j$ of (H^j, H^j_o) such that $\bar{H}^j(q,t)\, i = \bar{G}(q,t)$, $(q,t) \in Q\times I$, $j = 1,\dots,m$.

Proof. Recall that $B^y_x = S^{\lambda-1} \times [x, 2] \times (1+y) D^{n-\lambda}$, $A^y = 2D^{\lambda} \times (1+y) D^{n-\lambda}$, where $0 < x < 1$ and $y > 0$.

As in Section 4 , there exists $d > 0$, and

$$\tilde{G} : Q\times I \longrightarrow \Lambda^1(A^d)$$

satisfying

$$\tilde{G}(q,0) = G_o(q) \quad \text{on} \quad A$$

$$\tilde{G}(q,t) = G(q,t) \quad \text{on} \quad B$$

Further there exist, for each $j = 1,\dots,m$,

$$\tilde{H}^j : Q\times I \longrightarrow \Lambda^1(A^d \times D^1)$$

such that

$\tilde{H}^j(q,t)$ lies in the subbundle T^*A^d of $T^*(A^d \times D^1)$, and

such that

$$\tilde{H}^j(q,0) = H^j_o(q) \quad \text{on} \quad A \times D^1 ,$$

$$\tilde{H}^j(q,t) = H^j(q,t) \quad \text{on} \quad B \times D^1 ,$$

and $\quad \tilde{H}^j(q,t) \circ i = \tilde{G}(q,t) \quad \text{on} \quad A^d ,$

for all $q \in Q$, $t \in I$.

We shall indicate the steps for getting $\tilde{H}^j$

Consider

$$\begin{array}{ccc} Q & \xrightarrow{\;F_j\;} & \Lambda^1(A\times D^1) \\ \big\downarrow & & \big\downarrow r \\ Q\times I & \xrightarrow{\;f_j\;} & \Lambda^1(B\times D^1 \cup A\times 0) \end{array}$$

where $F_j(q) = H_o^j(q)$, and

$$f_j(q,t)(x,u) = \begin{cases} H^j(q,t)(x,u) \ , & (x,u) \in B\times D^1 \\ \widetilde{G}(q,t)(x) \ , & (x,u) \in A\times 0 \end{cases}$$

There exists a lifting $\overline{F}_j : Q \times I \longrightarrow \Lambda^1(A\times D^1)$ to the pair (f_j, F_j), which can be chosen to lie in $T^* A$, as F_j and f_j lie in $T^* A$. Next let

$$\widetilde{H}^j : Q\times I \longrightarrow \Lambda^1(A\times D^1 \cup A^d \times 0) \text{ be defined by}$$

$$\widetilde{H}^j(q,t)(x,u) = \begin{cases} \overline{F}_j(q,t)(x,u) \ , & (x,u) \in A\times D^1 \\ \widetilde{G}(q,t)(x) \ , & (x,u) \in A^d \times 0 \end{cases}$$

There exists an extension of $\widetilde{H}^j$ to an open set U containing $A\times D^1 \cup A^d \times 0$, which can be chosen to lie in $T^* A^d$ We shall denote this extension again by $\widetilde{H}^j$. There exists $\delta > 0$ such that $A^\delta \times D^1 \subset U$. Our final d (for $\widetilde{G}$ and $\widetilde{H}^j$) can be chosen to be smaller of this δ

and d, this latter d has been chosen earlier for $\widetilde{G}$. Now, since, as before in Section 4,

$$\widetilde{G}(q,0)\,|_A \;\varepsilon\; \Omega_k$$

$$\widetilde{G}(q,t)\,|_B \;\varepsilon\; \Omega_k$$

$$\widetilde{H}^j(q,0)\,|_{A\times D^1} \;\varepsilon\; \Omega'_{k+1}\,,$$

$$\widetilde{H}^j(q,t)\,|_{B\times D^1} \;\varepsilon\; \Omega'_{k+1}\,,$$

there exist $0 < b < 1$ and $0 < \eta < d$ such that

$$\widetilde{G}(q,0)\,|_{A^\eta} \;\varepsilon\; \Omega_k\,,$$

$$\widetilde{G}(q,t)\,|_{B_b^\eta} \;\varepsilon\; \Omega_k\,,$$

$$\widetilde{H}^j(q,0)\,|_{A^\eta\times D^1} \;\varepsilon\; \Omega'_{k+1}\,,$$

$$\widetilde{H}^j(q,t)\,|_{B_b^\eta\times D^1} \;\varepsilon\; \Omega'_{k+1}$$

The arguments here for $\widetilde{H}^j$ are analogous to the arguments for $\widetilde{G}$ in Section 4. One has only to note that $\Omega'_{k+1}(W) \subset \Omega_{k+1}(W)$ for W an open subset of $A^d \times D^1$. Then one gets η and b such that

$$\widetilde{H}^j(q,0)\,|_{A^\eta\times D^1} \;\varepsilon\; \Omega_{k+1}(A^\eta \times D^1)$$

and $$\widetilde{H}^j(q,t)\,|_{B_b^\eta\times D^1} \;\varepsilon\; \Omega_{k+1}(B_b^\eta \times D^1)\,,$$

and then one uses the fact that $\widetilde{H}^j(q,t)$ lies in the subbundle $T^* A^\eta$ of $T^*(A^\eta \times D^1)$, and, hence by definition, lies in Ω'_{k+1} over appropriate domains as above.

Let $H'^j : Q \times I \longrightarrow \Omega'_{k+1}(B_b^{\eta} \times D^1)$ be defined as the restriction of $\widetilde{H}^j$ to $B_b^{\eta} \times D^1$. Notice that $\widetilde{H}^j(q,0)|_{A^{\eta} \times D^1}$ is an extension of $H_o^j(q)$. We shall denote this extension again by $H_o^j(q)$.

Let $G' : Q \times I \longrightarrow \Omega_k(B_b^{\eta})$ be the restriction of $\widetilde{G}$. Again $\widetilde{G}(q,0)|_{A^{\eta}}$ is an extension of $G_o(q)$, which we shall denote by $G_o(q)$ also.

$$H'^j(q,0) = H_o^j(q) \quad \text{on} \quad B_b \times D^1$$

$$H'^j(q,t) = H^j(q,t) \text{ on } B \times D^1$$

$$\text{and} \qquad H'^j \circ i = G' \quad \text{on} \quad B_b^{\eta} .$$

As before, choose a and c such that $b < a < c < 1$.

Lemma 12. There exists a partition $0 = t_o < t_1 < \ldots < t_N = 1$ of $[0,1]$ and maps

$$\xi_i : Q \times [t_i, t_{i+1}] \longrightarrow \Omega_k(B_b^{\eta})$$

such that

$$\xi_i(q,t) = G'(q,t) \quad \text{on} \quad B_c^{\eta}$$

$$\xi_i(q,t_i) = G'(q,t_i)$$

$$\xi_i(q,t) = G'(q,t_i) \quad \text{on} \quad S^{\lambda-1} \times [b,a] \times (1+\eta)D^{n-\lambda} ,$$

and maps $\quad \xi_i^j : Q \times [t_i, t_{i+1}] \longrightarrow \Omega'_{k+1}(B_b^{\eta} \times D^1)$,

such that $\quad \xi_i^j = H'^j$ on $B_c^{\eta} \times D^1$

$$\xi_i^j(q,t_i) = H'^j(q,t_i)$$

$$\xi_i^j(q,t) = H'^j(q,t_i) \text{ on } S^{\lambda-1} \times [b,a] \times (1+\eta)D^{n-\lambda} \times D^1 .$$

Further $\xi_i^j \circ i = \xi_i$ on B_b^η.

Proof. The proof is similar to that of Lemma 8. We define

$$L^j : Q \times I \times I \longrightarrow \Omega'_{k+1}(B_b^\eta \times D^1) \quad \text{as}$$

$$L^j(q,t,s) = H'^j(q, \min(t+s, 1)) .$$

Then, using the map α of Lemma 8, we define

$$L^{j\alpha} : Q \times I \times I \longrightarrow \Lambda^1(B_b^\eta \times D^1) \quad j = 1,\dots,m,$$

$$L^{j\alpha}(q,t,s)(x \times u, y, z) = \Big[(1-\alpha(u))L^j(q,t,0) + \alpha(u)L^j(q,t,s)\Big](x \times u, y, z)$$

$$x \times u \in S^{\lambda-1} \times [b,2] , \; y \in (1+\eta)D^{n-\lambda} , \; z \in D^1$$

For each $(q,t) \in Q \times I$, $L^{j\alpha}(q,t,0) = L^j(q,t,0) = H'^j(q,t) \in \Omega'_{k+1}(B_b^\eta \times D^1) \subset \Omega_{k+1}(B_b^\eta \times D^1)$ which is an open subset of $\Lambda^1(B_b^\eta \times D^1)$. Thus there exists ε_j such that $L^{j\alpha}(q,t,s) \in \Omega_{k+1}(B_b^\eta \times D^1)$ for all $(q,t) \in Q \times I$ and $s \leq \varepsilon_j$. Notice that since $L^j(q,t,0)$ and $L^j(q,t,s)$ both lie in the subbundle $T^* B_b^\eta$ of $T^*(B_b^\eta \times D^1)$, $L^{j\alpha}(q,t,s)$ being a convex combination of the two, also lies in the same bundle. Thus by definition, $L^{j\alpha}(q,t,s) \in \Omega'_{k+1}(B_b^\eta \times D^1)$ for all $(q,t) \in Q \times I$, $s \leq \varepsilon_j$ and for each $j = 1,\dots,m$. Again, we define L^α exactly as in Lemma 8 and get $\varepsilon_0 > 0$ such that $L^\alpha(q,t,s) \in \Omega_k(B_b^\eta)$ for all $(q,t) \in Q \times I$ and $s \leq \varepsilon_0$. Let $\varepsilon = \min(\varepsilon_0, \varepsilon_1,\dots,\varepsilon_m)$. Then choose $0 = t_0 < t_1 < \dots < t_N = 1$ such that $t_{i+1} - t_i \leq \varepsilon$ for $0 \leq i \leq N-1$. Now for t such that $t_i \leq t \leq t_{i+1}$, define

$$\xi_i(q,t) = L^{\alpha}(q,t_i,t-t_i)$$

$$\xi_i^j(q,t) = L^{j\alpha}(q,t_i,t-t_i)$$

These ξ_i's and ξ_i^j's satisfy the requirements of Lemma 12. This completes the proof.

Next one proposes to construct

$$\beta_i : Q \times [0,t_i] \longrightarrow \Omega_k(A^{\eta_i})$$

such that $\beta_i = G'$ on $B_{c_i}^{\eta_i}$

and $\beta_i(q,0) = G_o(q)$ on A^{η_i}

$$\beta_i^j : Q \times [0,t_i] \longrightarrow \Omega'_{k+1}(A^{\eta_i} \times D^1)$$

such that $\beta_i^j = H'^j$ on $B_{c_i}^{\eta_i} \times D^1$

$$\beta_i^j(q,0) = H_o^j(q) \text{ on } A^{\eta_i} \times D^1$$

$$\beta_i^j \circ i = \beta_i \text{ on } A^{\eta_i}$$

(Recall $G_o(q)$ is assumed extended over A^{η} and $H_o^j(q)$ is assumed extended over $A^{\eta} \times D^1$ as indicated earlier).

The rest of the proof resulting in the construction of the liftings β_N and β_N^j by induction is the same save one small but important difference. At the $(i+1)^{th}$ stage we considered as isotopy $\varphi_t : A^{\eta_i} \longrightarrow A^{\eta_i}$, $0 \leq t \leq t_i$, to construct β_{i+1} from β_i. In the case of β^j's, we consider $\varphi_t \times \mathrm{id} : A^{\eta_i} \times D^1 \longrightarrow A^{\eta_i} \times D^1$. One should note here that Ω'_{k+1} remains invariant under such transformations. So recalling the construction of β_{i+1}, we have

$$\beta_{i+1}(q,t) = \begin{cases} \text{When } t \leq t_i & \\ \varphi_t^* \beta_i(q,t) & \text{on } B_{c_i}^{\eta_{i+1}} \\ \beta_i(q,t) & \text{on } c_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \\ \text{When } t_i \leq t \leq t_{i+1} & \\ \varphi_{t_i}^* \xi_i(q,t) & \text{on } B_{m_i}^{\eta_{i+1}} \\ \beta_{i+1}(q,t_i) & \text{on } m_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \\ \text{(as defined earlier).} & \end{cases}$$

$$\beta_{i+1}^j(q,t) = \begin{cases} \text{When } t \leq t_i & \\ (\varphi_t \times id)^* \beta_i^j(q,t) & \text{on } B_{c_i}^{\eta_{i+1}} \times D^1 \\ \beta_i^j(q,t) & \text{on } c_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \times D^1 \\ \text{When } t_i \leq t \leq t_{i+1} & \\ (\varphi_{t_i} \times id)^* \xi_i^j(q,t) & \text{on } B_{m_i}^{\eta_{i+1}} \times D^1 \\ \beta_{i+1}^j(q,t_i) & \text{on } m_i D^\lambda \times (1+\eta_{i+1}) D^{n-\lambda} \times D^1 \\ \text{(as defined earlier).} & \end{cases}$$

It is easy to see from these constructions that

$$\beta_{i+1}^j (q,t)(x,y,0) = \beta_{i+1}(q,t)(x,y) .$$

Proposition 13. Let $A = 2D^n$, $B = S^{n-1} \times [1,2]$, $D^1 = [-1,1]$. Suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_o} & \bar{\Omega}_k(A) \\ \downarrow & & \downarrow r \\ Q\times I & \xrightarrow{G} & \bar{\Omega}_k(B) \end{array}$$

and a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{H_o} & \bar{\Omega}'_{k+1}(A\times D^1) \\ \downarrow & & \downarrow r \\ Q\times I & \xrightarrow{H} & \bar{\Omega}'_{k+1}(B\times D^1) \end{array}$$

satisfying

$$H_o(q) \circ i = G_o(q) \quad \text{and} \quad H(q,t) \circ i = G(q,t)$$

Then there exists a lift $\bar{G}$ of (G, G_o)

Proof. The proof follows from an adaptation of the arguments of the Main Flexibility Theorem of Gromov [2, p.78].

Let $B_b = S^{n-1} \times [b,2]$ $0 < b \leq 1$. First we shall prove an easy lemma.

Lemma 14. Let $i : B_b \longrightarrow B_b \times D^1$ be the inclusion map. Then, if $\eta \in \Omega'_{k+1}(B_b \times D^1)$, $i^* \eta = \eta \circ i \in \Omega_k(B_b)$.

Proof. Let $\eta(y,u) = \Sigma f_j(y,u)dy_j$, $(y,u) \in B_b \times D^1$. Then

$$d\eta = \sum_{\ell,j} \frac{\partial f_j}{\partial y_\ell} dy_\ell \wedge dy_j + \sum_j \frac{\partial f_j}{\partial u} du \wedge dy_j .$$

If $w = i^* \eta$, we have $w(y) = \Sigma f_j(i(y))dy_j$

and $dw(y) = d\, i^* \eta(y)$

$$= i^* d\eta(y)$$

$$= \Sigma \frac{\partial f_j}{\partial y_\ell} (i(y))dy_\ell \wedge dy_j$$

Now $$\eta \wedge (d\eta)^{k+1} = (\Sigma f_j dy_j) \wedge (\Sigma \frac{\partial f_j}{\partial y_\ell} dy_\ell \wedge dy_j)^{k+1}$$

$$+ (k+1)(\Sigma f_j dy_j) \wedge (\Sigma \frac{\partial f_j}{\partial y_\ell} dy_\ell \wedge dy_j)^k \wedge (\Sigma \frac{\partial f_j}{\partial u} du \wedge dy_j) .$$

Therefore, since $\eta \in \Omega'_{k+1}$, $\eta \wedge (d\eta)^{k+1}(i(y)) \neq 0$ for all $y \in B_b$. This means that if $y \in B_b$, then either the first term above is not zero at $i(y)$ which implies $w \wedge (dw)^{k+1} \neq 0$ at y, or the second term is not zero at $i(y)$ which implies $w \wedge (dw)^k \neq 0$ at y. Thus we have $w = i^* \eta \in \Omega_k(B_b)$. That completes the *proof* of Lemma.

Now coming back to proof of Proposition 13, we extend

$$H \text{ to } H' : Q \times I \longrightarrow \Omega'_{k+1}(B_b \times D^1) \text{ for some } b : 0 < b < 1$$

such that $H'(q,0) = H_o(q)$ on $B_b \times D^1$.

Let $G' = H' \circ i$, where i is the inclusion map of B_b into $B_b \times D^1$. Then $G' : Q \times I \longrightarrow \Omega_k(B_b)$ is an extension of G which satisfies $G'(q,0) = G_o(q)$ on B_b. Also, by Lemma 13, $G'(q,t) \in \Omega_k(B_b)$, since $H' \circ i = i^* H' = G'$. We choose a, c such that $0 < b < a < c < 1$. As before we obtain a partition $t_o = 0 < t_1 < \ldots < t_N = 1$ of I, and maps

$$\xi_i : Q \times [t_i, t_{i+1}] \longrightarrow \Omega'_{k+1}(B_b \times D^1)$$

satisfying

$$\xi_i(q,t) = \begin{cases} H(q,t_i) & \text{if } t = t_i \\ H(q,t_i) & \text{outside } S^{n-1} \times (a,2] \times (-\frac{1}{2}, \frac{1}{2}) \\ H(q,t) & \text{on } B_c \times [-\frac{1}{4}, \frac{1}{4}]. \end{cases}$$

The arguments for the construction of ξ_i are again similar to those given in Lemma 8. This time we define

$$L : Q \times I \times I \longrightarrow \Omega'_{k+1}(B_b \times D^1) \text{ by}$$

$$L(q,t,s) = H'(q, \min(t+s, 1))$$

Then we choose a function $\alpha : B_b \times D^1 \longrightarrow I$ such that

$$\alpha \equiv 0 \text{ outside } S^{n-1} \times (a,2] \times (-\frac{1}{2}, \frac{1}{2})$$

and $$\alpha \equiv 1 \text{ on } B_c \times [-\frac{1}{4}, \frac{1}{4}].$$

If $x \in B_b$, $v \in D^1$, let

$L^{\alpha}(q,t,s)(x,v) = \left[(1-\alpha)L(q,t,0) + \alpha\, L(q,t,s)\right](x,v)$. Then $L^{\alpha}(q,t,0) = L(q,t,0) = H'(q,t) \in \Omega'_{k+1}(B_b \times D^1) \subset \Omega_{k+1}(B_b \times D^1)$ which is an open subset of $\Lambda^1(B_b \times D^1)$. Then, by continuity, there exists $\varepsilon > 0$ such that $L^{\alpha}(q,t,s) \in \Omega_{k+1}(B_b \times D^1)$ for all $(q,t) \in Q{\times}I$ and $s \leq \varepsilon$. All the $L^{\alpha}(q,t,s)$ lie in the subbundle $T^* B_b$ of $T^*(B_b \times D^1)$. Thus by definition $L^{\alpha}(q,t,s) \in \Omega'_{k+1}(B_b \times D^1)$ for all $(q,t) \in Q{\times}I$, $s \leq \varepsilon$.

Now choose $0 = t_o < t_1 < \dots < t_N = 1$ of I such that $t_{i+1}-t_i \leq \varepsilon$ for all $i = 0,1,\dots,N-1$.

Finally let

$$\xi_i(q,t) = L^{\alpha}(q,\ t_i,\ t - t_i)\ ,\quad t \in \left[t_i,\ t_{i+1}\right].$$

This completes the construction of ξ_i .

Now we shall construct the lift $\bar{G}$ by induction.

Let $\bar{G}_1 : Q \times \left[0,t_1\right] \longrightarrow \Omega_k(A)$ be defined as

$$\begin{aligned} \bar{G}_1(q,t) &= \xi_o(q,t) \circ i \qquad \text{on } B_b \\ &= G_o(q) \qquad\qquad \text{outside } B_b . \end{aligned}$$

Then $\bar{G}_1(q,0) = G_o(q)$ on A

and $\bar{G}_1(q,t) = G'(q,t)$ on B_c .

Having constructed

$$\bar{G}_i : Q \times \left[0,\ t_i\right] \longrightarrow \Omega_k(A)$$

satisfying

$$\bar{G}_i(q,0) = G_o(q)$$

$$\bar{G}_i(q,t) = G'(q,t) \text{ on } B_{c_i} \text{ for some } c_i,\ c < c_i < 1,$$

We propose to define

$$\bar{G}_{i+1} : Q \times [0, t_{i+1}] \longrightarrow \Omega_k(A) \text{ as follows :}$$

Let ρ and $\tau > 0$ be such that $c_i < \rho < 1$ and $c_i < \rho-\tau \leq \rho+\tau < 1$. Considering ξ_i we find $\bar{\varepsilon} > 0$ and

$$\bar{\xi}_i : Q \times [t_i, t_i + \bar{\varepsilon}] \longrightarrow \Omega'_{k+1}(B_b \times D^1)$$

satisfying

$$(1)\quad \bar{\xi}_i(q,t_i) = \xi_i(q,t_i)$$

$$(2)\quad \bar{\xi}_i(q,t_i) = \begin{cases} \xi_i(q,t_i) & \text{outside } B_{\rho-\tau} \times D^1 \\ \xi_i(q,t) & \text{on } B_\rho \times D^1 \end{cases}$$

This is obtained by a repetition of the arguments for constructing ξ_o's from H', which will now be replaced by $\bar{\xi}_i$ and ξ_i respectively, and the α is changed to $\bar{\alpha}$ which is 0 outside $B_{\rho-\tau} \times D^1$ and 1 on $B_\rho \times D^1$. We note here that if, for some $(x,v) \in B_b \times D^1$, and some $q \in Q$,

$$\xi_i(q,t)(x,v) = \xi_i(q,t_i)(x,v) \text{ for all } t \in [t_i, t_{i+1}],$$

then $\bar{\xi}_i$ thus constructed also has the same property, that is,

$$\bar{\xi}_i(q,t)(x,v) = \bar{\xi}_i(q,t_i)(x,v) \text{ for all } t \in [t_i, t_i + \bar{\varepsilon}].$$

In other words, if $\xi_i(q,t)$ is time-independent at (x,v) , so is $\bar{\xi}_i(q,t)$.

Now construct an isotopy δ_t , $t_i \leq t \leq t_{i+1}$, of B_b into $B_b \times D^1$ satisfying

(1) $\delta_t(x)$ is of the form $(x,v(x,t))$, $x \in B_b$, $t \in [t_i, t_{i+1}]$.

(2) $\delta_t(x) = (x,0)$ if $x \notin S^{n-1} \times (\rho - \tau, \rho+\tau)$, for all $t \in [t_i, t_{i+1}]$

(3) $\delta_{t_i}(x) = (x,0)$ on B_b

(4) $\delta_t = \delta_{t_i + \bar{\varepsilon}}$ if $t \geq t_i + \bar{\varepsilon}$

(5) $\delta_{t_i + \bar{\varepsilon}}$ moves $S^{n-1} \times \rho \times 0$ at a distance of at least $\frac{1}{2}$ away from $S^{n-1} \times \rho \times 0$ in $B_b \times D^1$.

(An idea of (δ_t) is given in Figure 2 on the next page).

We point out here that in view of Lemma 13 proved earlier $\delta_t^* \Omega'_{k+1}(B_b \times D^1) \subset \Omega_k(B_b)$. This is because $\delta_t = \Psi_t \circ i$ for some fibre preserving diffeomorphism Ψ_t of $B_b \times D^1$. So if $\eta \in \Omega'_{k+1}(B_b \times D^1)$, then $\delta_t^* \eta = i^*(\Psi_t^* \eta) \in \Omega_k(B_b)$ by Lemma 14.

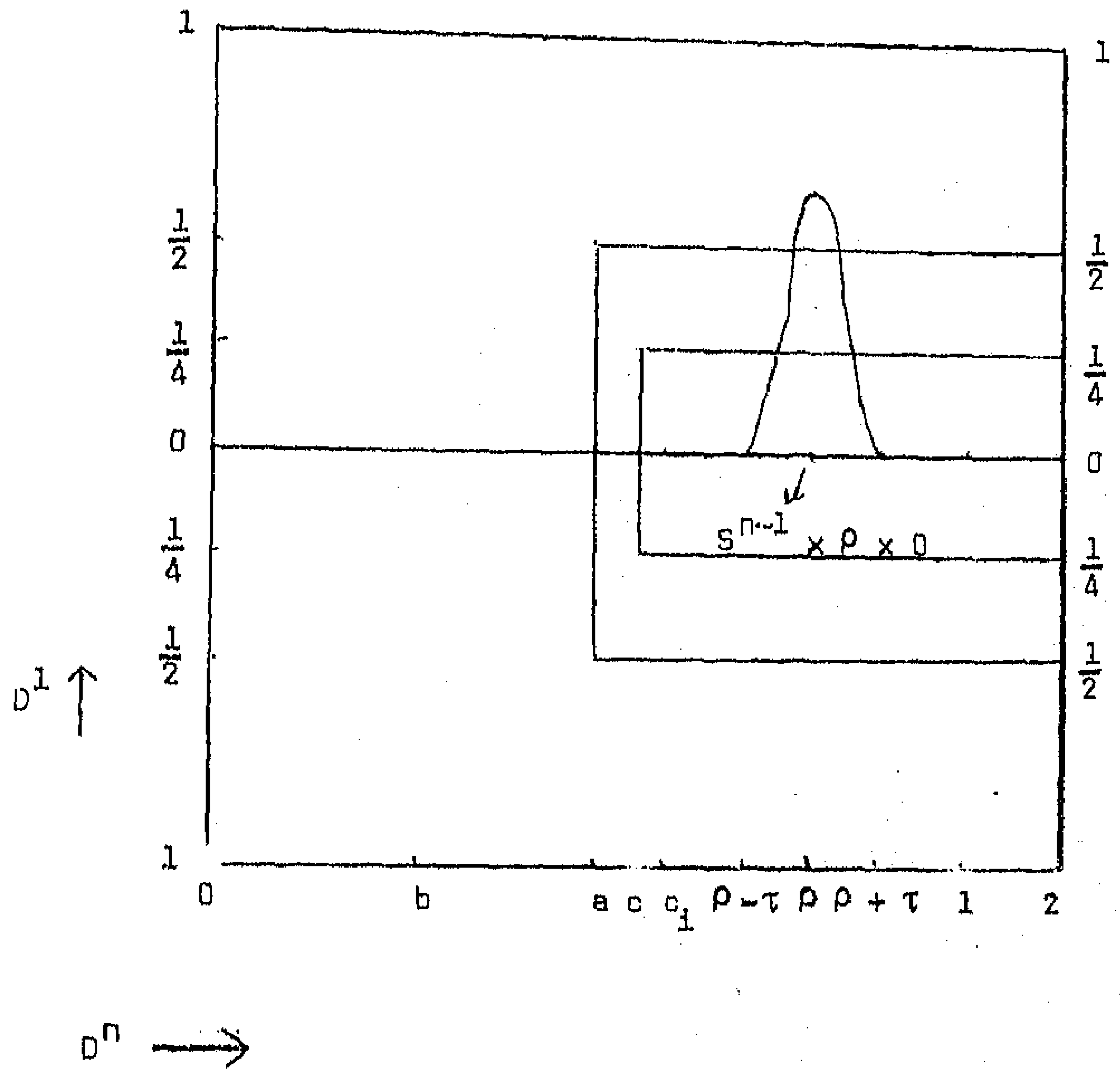
Next construct

$$\tilde{\xi}_i : Q \times [t_i, t_{i+1}] \longrightarrow \Omega_k(B_{\rho_i})$$

by

$$\tilde{\xi}_i(q,t) = \begin{cases} \delta_t^* \xi_i(q,t) & \text{on } B_\rho \\ \delta_t^* \bar{\xi}_i(q, \min(t, t_i + \bar{\varepsilon})) & \text{outside } B_\rho . \end{cases}$$

<u>Figure 2</u>



The bump over $S^{n-1} \times [\rho - \tau, \rho + \tau]$ indicates the position of $\delta_{t_i} + \bar{\varepsilon}$ over this domain.

We verify now that $\widetilde{\xi}_i$ is wel-defined. For $t_i \leq t \leq t_i + \bar{\varepsilon}$,

$$\widetilde{\xi}_i(q,t) = \delta_t^* \bar{\xi}_i(q,t) \quad \text{on } B_{c_i},$$

because of condition (2) for $\bar{\xi}_i$. For $t \geq t_i + \bar{\varepsilon}$, as δ_t satisfies properties (4) and (5), it takes a neighbourhood of $S^{n-1} \times \rho \times 0$ outside $B_a \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ where $\xi_i(q,t)$ is independent of time t.

We note that $\widetilde{\xi}_i(q,t)$ becomes time-independent outside $B_{\rho-\tau}$ in B_{c_i} and on $B_{\rho+\tau}$

$$\begin{aligned} \widetilde{\xi}_i(q,t) &= \xi_i(q,t) \circ i \\ &= H'(q,t) \circ i = G'(q,t) \end{aligned}$$

Further $$\begin{aligned} \widetilde{\xi}_i(q,t_i) &= \xi_i(q,t) \circ i \quad \text{on } B_{c_i} \\ &= G'(q,t_i) \quad \text{on } B_{c_i} \end{aligned}$$

Now we define

$$\bar{G}_{i+1} : Q \times [0, t_{i+1}] \longrightarrow \Omega_k(A)$$

by

$$\bar{G}_{i+1}(q,t)(x) = \begin{cases} \bar{G}_i(q,t)(x) & \text{if } t \leq t_i,\ x \in A \\ \widetilde{\xi}_i(q,t)(x) & \text{if } t \geq t_i,\ x \in B_{c_i} \\ \bar{G}_i(q,t_i)(x) & \text{if } t \geq t_i,\ x \notin B_{c_i} \end{cases}$$

Let $c_{i+1} = \rho + \tau$. We observe that

$$\bar{G}_{i+1}(q,0) = G_0(q)$$

and $$\bar{G}_{i+1}(q,t) = G'(q,t) \quad \text{on } B_{c_{i+1}}$$

Thus by induction we get the desired lift $\bar{G}$ to the lifting problem (G, G_0).

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