The Asymptotic Norming Properties And Related Themes In Banach Spaces

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Contents

Chapter 1

Introduction

In the first part of this chapter, we explain the main theme of this thesis. The second part consists of some of the notions and results used in subsequent discussions.

It is a very familiar fact that a point outside a (bounded) closed convex set in a Banach space can be separated from the latter by a hyperplane. One can ask whether the separation can be effected by disjoint balls. This is a typical example of a ball separation property, study of which has become important in Banach space theory. In this thesis, we study several such properties along with some other related notions like the Asymptotic Norming Property for which, however, a ball-separation characterization is not available at present.

We begin our discussion with (see the end of the section for the relevant definitions) the Asymptotic Norming Properties (ANP) of Banach spaces. The ANP was first introduced by James and Ho [JH] to show that the class of separable Banach spaces with the Radon-Nikodým Property (RNP) is larger than those isomorphic to separable duals. Three different ANP's were introduced and proved to be equivalent in separable Banach spaces. Later Ghoussoub and Maurey [GM] proved that in separable Banach spaces ANP's are equivalent to the RNP. Whether the two properties are equivalent in general is still an open question. Subsequently, Hu and Lin [HL1] obtained some isometric characterization of ANP's and showed that the three ANP's are equivalent in Banach spaces admitting a locally uniformly convex renorming, a class larger than separable Banach spaces. In dual spaces, they introduced a stronger notion called the w*-ANP, which turned out to have some nice geometric equivalents. In fact, they showed that w*-ANP-III and w*-ANP-II are respectively equivalent to Namioka-Phelps Property (referred to as the Property (**) in [NP]) and Hahn-Banach smoothness considered by Sullivan in [Su]. The latter property in turn grew out of the concept of U-subspaces introduced by Phelps [P1]. More recently, Chen and Lin [CL] have obtained some ball separation characterization of w*-ANP's which suggest similar characterization can be obtained for ANP's too. Both ANP and w*-ANP's are hereditary in nature.

In the non-hereditary class of ball separation properties, i.e., properties which are not inherited by subspaces, we study nicely smooth spaces, Property (II) and the Ball Generated Property (BGP). Nicely smooth spaces were first introduced by Godefroy [G3] while the BGP by Godefroy and Kalton [GK]. Property (II), which is a natural weakening of both w*-ANP-II as well as the Mazur Intersection Property (MIP) was introduced by Chen and Lin [CL].

In Chapter 2, we introduce a new ANP which lies between the strongest and the weakest ones. We denote it by ANP-II$'$. This new ANP has nice geometric properties and we give an example to show that it is clearly distinct from the other ANP's. We also introduce a w*-version of ANP-II$'$. This gives a very elegant characterization of Property (V) introduced by Sullivan in [Su]. We also study stability of this new ANP along with its w*-version. In particular, we prove that it is hereditary and that it can be lifted from a Banach space X to the Bochner spaces $L_p(\mu, X)$ $(1 < p < \infty)$. We also give a ball separation characterization for w*-ANP-II$'$. Some of the proofs in this chapter are mainly modifications, refinements and adaptations from [HL1], [HL2], [HL3] and [CL].

In Chapter 3, we study some stability properties of w*-ANP's. We show that these are all separably determined properties and some of them are stable under c_0-sums. They all fail to have the "three space property". That Hahn-Banach smoothness is a separably-determined property was recently proved by Oja and Põldvere [OP] using different techniques. In this chapter, we also study the stability of Property (II) and see that it is stable under c_0-sum, ℓ_p-sum $(1 < p < \infty)$ and can be lifted to Bochner L_p-spaces $(1 < p < \infty)$ for the Lebesgue measure on $[0,1]$. Property (II) also fails to be a three space property. We conclude this section by proving that L_1-preduals with Property (II) are essentially $\ell_1(\Gamma)$ for some Γ. This leads to the interesting problem of classification of Banach spaces with Property (II) among L_1-preduals. However, this is not considered here.

In Chapter 4, we study nicely smooth Banach spaces and related ideas. We obtain some necessary and some sufficient conditions for a space to be nicely smooth and show that they are equivalent for separable and Asplund spaces. We obtain a sufficient condition for the BGP and show that Property (II) implies BGP which in turn implies nicely smooth. We show that nicely smooth spaces are stable under ℓ_p, c_0 and finite ℓ_1 sums. Also it can be lifted from a Banach space X to Bochner L_p-spaces for Lebesgue measure $[0,1]$. It is not a three space property. We show that every equivalent renorming of a Banach space is nicely smooth if and only if it is reflexive. Similar results were earlier observed in Chapter 3 for Property (II) and Hahn-Banach smoothness.

Chapter 5 is devoted to the study of all these ball separation properties in the context of $C(K,X)$ spaces where K is a compact Hausdorff space. We prove that for any of these properties (let us denote it by P), $C(K,X)$ has P if and only if X has P and K is finite. We also show that under a compact approximation of identity on X, if $\mathcal{L}(X)$ has any of these properties then X is finite dimensional. We conclude this chapter by showing that for a compact Hausdorff space K, $\mathcal{L}(X, C(K))$ has P if and only if X^* has P and K is finite.

In Chapter 6 we discuss ball separation properties in tensor product spaces. This short chapter raises more questions than it answers. We show that ANP-I is not stable under either the injective or the projective tensor product; ANP-II′

and w*-ANP-I and w*-ANP-II$'$ are not stable under the injective tensor product. Analogous questions for the other ANP's remains unanswered at present. It was already proved in [GS], that if X and Y are Asplund and nicely smooth, then $X \otimes_\varepsilon Y$ is also so. We show that for any two Banach spaces X and Y with $X \otimes_\varepsilon Y$ nicely smooth, forces both X and Y to be nicely smooth. We conclude with a simple result that connects the nice smoothness of the space of compact operators from X to Y^* with the reflexivity of spaces X, Y and their projective tensor product $X \otimes_\pi Y$ (when X^* has the approximation property and any bounded linear transformation from X to Y^* is compact).

Of the two classes of ball separation properties studied, the hereditary and non-hereditary, we have among them the following relations, some of which had been established earlier and some in the course of this thesis :

$$\begin{tikzcd}
 & w^*\text{-ANP-I} \arrow[r, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \text{MIP} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
w^*\text{-ANP-II}' \arrow[dr, Rightarrow] & w^*\text{-ANP-II} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Prop } (II) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{BGP} \arrow[dl, Rightarrow] \\
 & w^*\text{-ANP-III} \arrow[r, Rightarrow] & \text{NS} &
\end{tikzcd}$$

It is perhaps interesting to investigate under what conditions these implications can be reversed.

Notations, Conventions and General Preliminaries

General reference to this work are the monographs [Bo], [DU] and [P2]. We work with only *real* Banach spaces. Unless otherwise mentioned, by a subspace we mean a closed linear subspace. We will identify any $x \in X$ with its canonical image $\hat{x} \in X^{**}$. For any $A \subseteq X$, let $co(A)$, $aco(A)$ and $span(A)$ respectively denote the convex hull, the absolutely convex hull (i.e., $co(A \cup -A)$) and the linear space generated by A.

Definition 1.0.1 For a Banach space X, let $S_X = \{x : \|x\| = 1\}$, $B_X = \{x : \|x\| \leq 1\}$, $B[a,r] = \{x : \|x - a\| \leq r\}$ and $B(a,r) = \{x : \|x - a\| < r\}$

(a) Let Φ be a subset of B_{X^*}, where X^* is the dual of X. Φ is called a norming set for X if $\|x\| = \sup_{x^* \in \Phi} x^*(x)$, for all $x \in X$. A subspace F of X^* is said to be a norming subspace of X^* if the unit ball B_F is a norming set for X.

(b) A sequence $\{x_n\}$ in S_X is said to be asymptotically normed by Φ if for any $\varepsilon > 0$, there exists $x^* \in \Phi$ and $N \in \mathbb{N}$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geq N$.

(c) For $\kappa =$ I, II or III, a sequence $\{x_n\}$ in X is said to have the property κ if

I. $\{x_n\}$ is convergent.

II. $\{x_n\}$ has a convergent subsequence.

III. $\bigcap_{n=1}^{\infty} \overline{co}\{x_k : k \geq n\} \neq \emptyset$.

(d) Let Φ be a norming set for X. X is said to have the asymptotic norming property κ with respect to Φ (Φ-ANP-κ), $\kappa =$ I, II or III, if every sequence in S_X that is asymptotically normed by Φ has property κ.

(e) X is said to have the asymptotic norming property κ (ANP-κ), $\kappa =$ I, II or III, if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ for $(X, \|\cdot\|)$ such that X has Φ-ANP-κ.

Lemma 1.0.2 [G4, Lemma 1.1] *Let X be a Banach space and F a subspace of X. Then the following are equivalent :*

(*a*) *F is a norming subspace of X.*

(*b*) *B_F is w^*-dense in B_{X^*}.*

Theorem 1.0.3 [HL1, Theorem 2.3]

(*a*) *X has Φ-ANP-III*

(*b*) $X^{**} \setminus X = \{x^{**} : \|x^{**}\| > \sup_{\phi \in \Phi} x^{**}(\phi)\}$

(*c*) *Every sequence in S_X that is asymptotically normed by Φ has a weakly convergent subsequence.*

Remark 1.0.4 It follows that a reflexive space has Φ-ANP-III for any norming set Φ. And that a space X has B_{X^*}-ANP-III if and only if X is reflexive.

In the sequel, we will say a sequence $\{x_n\}$ has property III if it has a weakly convergent subsequence. By the above Theorem, this does not alter the definition of Φ-ANP-III.

Definition 1.0.5 Let X be a Banach space and X^* its dual. Let $C \subseteq X$ and $D \subseteq X^*$.

(a) Let $f \in X^*$ and $\alpha > 0$. Then the set $S(C, f, \alpha) = \{x \in C : f(x) > \sup f(C) - \alpha\}$ is called the open slice of C determined by f and α.

(b) A point $x \in C$ is called a denting point of C if the family of open slices containing x forms a local base for the norm topology at x (relative to C).

(c) A point $x \in C$ is called an exposed point of C if there exists $f \in X^*$ such that $f(x) > f(y)$ whenever $x \neq y$ and $y \in C$ and f is said to expose x. The point x is called strongly exposed if there exists f which exposes x such that $\{S(C, f, \alpha) : \alpha > 0\}$ is a neighbourhood base for x in C for the norm topology.

(d) If $D \subseteq X^*$ and the slices are determined by functionals from X, we get the corresponding definitions of w*-slices, w*-denting points, w*-exposed points and w*-strongly exposed points respectively.

(e) A point $x^* \in D$ is said to be a weak*-weak point of continuity (w*-w PC) of D if x^* is a point of continuity of the identity map from (D, w^*) to (D, w).

(f) A point $x^* \in D$ is said to be a weak*-norm point of continuity (w*-PC) of D if x^* is a point of continuity of the identity map from (D, w^*) to $(D, \|\cdot\|)$.

(g) A Banach space X is said to have the Kadec Property (K) if the weak and the norm topologies coincide on the unit sphere, i.e., $(S_X, w) = (S_X, \|\cdot\|)$.

(h) X is said to have Kadec-Klee Property (KK) if for any sequence $\{x_n\}$ and $\{x\}$ in B_X such that $\lim_n \|x_n\| = \|x\| = 1$ and w-$\lim_n x_n = x$, then $\lim_n \|x_n - x\| = 0$.

Remark 1.0.6 Denting points can be defined in an alternative but equivalent way. One can refer to [Bo] for this. We take the one convenient for our purpose. It is well-known that a $x^* \in B_{X^*}$ is w*-denting if and only if it is extreme and w*-PC.

Theorem 1.0.7 [HL1, Theorem 2.4] *Let Φ be a norming set for a Banach space X. The following are equivalent :*

(a) X has Φ-ANP-II.

(b) X has Φ-ANP-III and X has (K).

(c) X has Φ-ANP-III and X has (KK).

Theorem 1.0.8 [HL1, Theorem 2.5] *Let Φ be a norming set for a Banach space X. The following are equivalent :*

(a) X has Φ-ANP-I.

(b) X has Φ-ANP-II and X is strictly convex.

(c) X has Φ-ANP-III and all points of S_X are denting points of B_X.

We quickly note the following Corollary :

Corollary 1.0.9 *Let Φ be a norming set for X. Then X has Φ-ANP-κ if and only if X has Ψ-ANP-κ for some (and hence, all) $\Phi \subseteq \Psi \subseteq \overline{aco}(\Phi)$.*

Definition 1.0.10 Let X^* be a dual Banach space. X is said to have w*-ANP-κ ($\kappa =$ I, II or III) if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ in $(B_X, \|\cdot\|)$ such that $(X^*, \|\cdot\|)$ has Φ-ANP-κ.

Remark 1.0.11 If $\Phi \subseteq B_X$ is a norming set for X^*, we necessarily have $\overline{aco}(\Phi) = B_X$. Hence, by the above Corollary, we can always take $\Phi = B_X$, and in the sequel, we indeed do so.

Theorem 1.0.12 [HL1, Theorem 3.1] *For a dual Banach space X^*,*

(a) *X has w^*-ANP-I if and only if every point of S_{X^*} is a w^*-denting point of B_{X^*}.*

(b) *X has w^*-ANP-II if and only if $(S_{X^*}, w^*) = (S_{X^*}, \|\cdot\|)$.*

(c) *X has w^*-ANP-III if and only if $(S_{X^*}, w^*) = (S_{X^*}, w)$.*

Definition 1.0.13 (a) The duality mapping D for a Banach space X is the set-valued function from S_X to S_{X^*} defined by

$$D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}, \quad x \in S_X.$$

Any selection of D is called a support mapping.

(b) Let $F : X \longrightarrow \mathbb{R}$ be a function. Then F is said to be Fréchet differentiable at $x \in X$ if there exists an $f \in X^*$ such that

$$\lim_{\lambda \longrightarrow 0+} \left| \frac{F(x+\lambda y) - F(x)}{\lambda} - f(y) \right| = 0$$

uniformly for $y \in S_X$. It is well-known that the norm of X is Fréchet differentiable at x if and only if the duality mapping is single-valued and norm-norm continuous at x. It is also known that $x^* \in S_{X^*}$ is w*-strongly exposed by x if and only if the norm is Fréchet differentiable at x with $D(x) = x^*$ (see [P2]).

(c) A Banach space X is said to be smooth if for all $x \in S_X$, the duality mapping is single-valued.

(d) A Banach space X is said to be very smooth if every $x \in S_X$ has a unique norming element in X^{***}. It is known that X is very smooth if and only if the duality mapping D is single-valued and is norm-weak continuous.

(e) A Banach space X is said to be Asplund if each continuous convex function $F : X \longrightarrow \mathbb{R}$ is Fréchet differentiable on a dense G_δ subset of X. It is well-known that X is Asplund if and only if X^* has RNP if and only if every separable subspace of X has a separable dual (See [Bo]).

(f) For a Banach space X, let $X^\perp = \{x^\perp \in X^{***} : x^\perp(x) = 0 \text{ for all } x \in X\}$. X is said to be Hahn-Banach smooth if for all $x^* \in X^*$, $\|x^* + x^\perp\| = \|x^*\| = 1$ implies $x^\perp = 0$.

In other words, $x^* \in X^{***}$ is the unique Hahn-Banach extension of $x^*|_X$. Obviously X is Hahn-Banach smooth if and only if

$$X^* = \{x^{***} \in X^{***} : \|x^{***}\| = \sup_{x \in B_X} x^{***}(x)\}.$$

(g) X is said to have Namioka-Phelps Property if all points of S_{X^*} are w*-PC's of B_{X^*}.

(h) X is said to have the Mazur Intersection Property (MIP) if the w*-denting points of B_{X^*} are dense in S_{X^*}, or equivalently, every closed bounded convex set is the intersection of closed balls containing it.

(i) X is said to have Property (II) if the w*-PC's of B_{X^*} are dense in S_{X^*}, or equivalently, every closed bounded convex set is the intersection of closed convex hull of finite union of balls.

(j) X is said to have the Ball Generated Property (BGP) if every closed bounded convex set is ball-generated, i.e., it can be realized as the intersection of finite unions of balls.

(k) X is said to be nicely smooth if X^* has no proper norming subspace.

Remark 1.0.14 (i) Property (II) for a Banach space X defined above, should not be confused with property II defined for a sequence $\{x_n\}$ earlier. To avoid any such ambiguity, let us always denote the former (i.e., for X) by (II) and the latter (i.e., for $\{x_n\}$) just by II.

(ii) As just noted, all these properties deal with ball separation. We use some of these properties in our analysis of the w*-ANP in Chapter 2. For a detailed discussion one should refer to [G3], [CL] [B1] and [GK].

Theorem 1.0.15 [HL2, Theorem 1] *Let X be a Banach space. X is Hahn-Banach smooth if and only if X has w^*-ANP-III if and only if the weak and w^*-topologies coincide on S_{X^*}.*

It is clear from above that X has w*-ANP-II if and only if X has the Namioka-Phelps Property.

For a Banach space X, let

$$C_X = \{x^{**} \in X^{**} : \|x^{**} + \hat{x}\| \geq \|x\|\}$$

for all $x \in X$.

Lemma 1.0.16 [GK, Lemma 2.3] *Let X be a Banach space and $x^{**} \in X^{**}$. Then the following are equivalent :*

(*a*) $x^{**} \in C_X$.

(*b*) $ker x^{**} \bigcap B_{X^*}$ *is w^*-dense in* B_{X^*}.

Thus we immediately have

Lemma 1.0.17 *Let X be a Banach space. Then $x^{**} \in C_X$ if and only if $ker x^{**}$ is a norming subspace of X.*

Definition 1.0.18 A Banach space X is said to satisfy the finite intersection property (FIP) if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection.

It is well-known that any dual space and its 1-complemented subspaces have FIP.

Theorem 1.0.19 [GK, Theorem 2.8] *Let X be a Banach space. Then the following are equivalent :*

(i) X has FIP

(ii) $X^{**} = X + C_X$.

Let X be a Banach space, and (Ω, Σ, μ) be a measure space. Let $L_p(\mu, X)$ denote the Lebesgue-Bochner function of p-integrable X-valued functions defined on Ω $(1 < p, q < \infty)$. Recall from [DU] that if $1/p + 1/q = 1$, $1 < p, q < \infty$, the space $L_q(\mu, X^*)$ is isometrically isomorphic to a norming subspace of $L_p(\mu, X)^*$ and that they coincide if and only if X^* has the RNP with respect to μ.

$\mathcal{L}(X, Y)$ (resp. $\mathcal{K}(X, Y)$) denotes the Banach space of bounded linear operators (resp. compact linear operators) from X to Y. [LC] and [DU, Chapter VIII] contains all the necessary information on tensor product spaces. We just recall here that for the projective tensor product of X and Y (denoted by $X \otimes_\pi Y$), $\mathcal{L}(X, Y^*)$ with the usual operator norm can be identified with $(X \otimes_\pi Y)^*$ while for the injective tensor product (denoted by $X \otimes_\varepsilon Y$), the space of integral operators $\mathcal{I}(X, Y^*)$ from X to Y^* with the integral norm can be identified with $(X \otimes_\varepsilon Y)^*$ (see [DU] for details).

Chapter 2

On a New Asymptotic Norming Property

2.1 The New ANP

The following lemma will be useful in our subsequent discussions.

Lemma 2.1.1 [HL1, Lemma 2.2] *Let $\{x_n^*\}$ be a sequence in S_{X^*} and let Φ be a subset of B_X. If Φ is a norming set of $\overline{span}\{x_n^* : n \in I\!N\}$, and $\{x_n^*\}$ is asymptotically normed by Φ, then*

$$\|x^*\| = \sup_{\phi\in\Phi} x^*(\phi) = 1$$

for all x^* in the w^*-closure of $\{x_n^*\}$.

Definition 2.1.2 A sequence $\{x_n\}$ in X is said to have the property II′ if $\{x_n\}$ is weakly convergent.

Let X be a Banach space and let $\Phi \subseteq B_{X^*}$ be a norming set for X. X is said to have Φ-ANP-II′ if any sequence $\{x_n\}$ in S_X which is asymptotically normed by Φ has property II′. X is said to have ANP-II′ if there exists an equivalent norm $\|\cdot\|$ and a norming set Φ such that X has Φ-ANP-II′.

Remark 2.1.3 Clearly, Φ-ANP-I $\Rightarrow$ Φ-ANP-II$'$ $\Rightarrow$ Φ-ANP-III, so that every result on the equivalence of ANP-I and ANP-III also yields their equivalence with ANP-II$'$.

Theorem 2.1.4 *Let Φ be a norming set for a Banach space X. The following are equivalent :*

(a) X has Φ-ANP-I

(b) X has Φ-ANP-II$'$ and X has (K)

(c) X has Φ-ANP-II$'$ and X has (KK)

Proof : Since Φ-ANP-I implies both ANP-II and ANP-II$'$, (a) $\Rightarrow$ (b) follows from (a) $\Rightarrow$ (b) of Theorem 1.0.7, and (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Let $\{x_n\}$ be a sequence in S_X which is asymptotically normed by Φ. Since X has Φ-ANP-II$'$, $\{x_n\}$ is weakly convergent to some $x \in X$. Then applying Lemma 2.1.1 for $\{x_n\}$ in $X \hookrightarrow X^{**}$ we have $\|x\| = 1$ and hence by (KK) we have $x_n \to x$ in norm. ∎

Theorem 2.1.5 *Let Φ be a norming set for a Banach space X. The following are equivalent :*

(a) X has Φ-ANP-II$'$.

(b) X has Φ-ANP-III and X is strictly convex.

Proof : (a) $\Rightarrow$ (b) To show that X is strictly convex, let $x, y \in S_X$ and suppose $z = (x+y)/2 \in S_X$. For each $n \in \mathbb{N}$, let $x_n^* \in \Phi$ be such that $x_n^*(z) > 1 - 1/n$. Then $x_n^*(x) > 1 - 2/n$. Hence the sequence $\{x_n\}$ with $x_{2n-1} = z$, $x_{2n} = x$, $n \in \mathbb{N}$ is asymptotically normed by Φ. Since X has Φ-ANP-II$'$, $x = z$.

(b) $\Rightarrow$ (a) Let $\{x_n\}$ be a sequence in S_X which is asymptotically normed by Φ. Since X has Φ-ANP-III, $D = \bigcap \overline{co}\{x_k \ : k \geq n\} \neq \emptyset$. Now X has Φ-ANP-III implies $\{x_n\}$ has weak cluster points and all of them must be in D. Since $D \subseteq S_X$ is convex and X is strictly convex, D is a singleton. Moreover, since every subsequence of $\{x_n\}$ is also asymptotically normed by Φ, that singleton is the weak limit of $\{x_n\}$. Hence X has Φ-ANP-II'. ■

Remark 2.1.6 In the proof of $(a) \Longrightarrow (b)$ above, we use the same technique as in Theorem 2.5 in [HL1].

Lemma 2.1.7 [HL1, Lemma 2.6] *Let $\|\cdot\|_i$, $i = 1, 2$, be equivalent norms on a Banach space X and let Φ_i be a norming set of $(X, \|\cdot\|_i)$, $i = 1, 2$. For all $x \in X$ define*

$$|||x||| = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}.$$

and

$$\Phi = \{\lambda_1 x_1^* + \lambda_2 x_2^* : x_i^* \in \Phi_i,\ \lambda_i \geq 0,\ (i = 1, 2),\ \lambda_1^2 + \lambda_2^2 = 1\}.$$

Then $|||\cdot|||$ is an equivalent norm on X and Φ is a norming set of $(X, |||\cdot|||)$ with the following properties :

(*i*) *If one of $(X, \|\cdot\|_i)$, $i = 1, 2$ is strictly convex, then so is $(X, |||\cdot|||)$.*

(*ii*) *If one of $(X, \|\cdot\|_i)$, $i = 1, 2$ has (KK), then so does $(X, |||\cdot|||)$.*

(*iii*) *If one of $(X, \|\cdot\|_i)$, $i = 1, 2$ has Φ_i-ANP-III, then $(X, |||\cdot|||)$ has Φ-ANP-III.*

Theorem 2.1.8 *For a Banach space X, the following are equivalent :*

(*a*) *X has ANP-I*

(*b*) *X has ANP-II' and there exists an equivalent norm on X such that X has (KK).*

Proof : The proof is immediate from Theorem 2.1.4 and Lemma 2.1.7. ■

Similarly, using Theorem 2.1.5 and Lemma 2.1.7, one has

Theorem 2.1.9 *For a Banach space X, the following are equivalent :*

(*a*) *X has ANP-II'*

(*b*) *X has ANP-III and X is strictly convexifiable, i.e., admits an equivalent strictly convex norm.*

Definition 2.1.10 Let X^* be a dual Banach space. Then X has w*-ANP-II$'$ if there is an equivalent norm $\|\cdot\|$ on X such that $(X^*, \|\cdot\|)$ has B_X-ANP-II$'$.

Remark 2.1.11 As before we could have taken some $\Phi \subseteq B_X$ that is norming for X^*, but that does not give us anything new.

It is well-known that a dual space X^* is strictly convex if and only if every two dimensional quotient space of X is smooth [D1]. L. P. Vlasov [V] transformed this fact into the following equivalent form :

Theorem 2.1.12 [V] *X^* is rotund if and only if for every increasing sequence $\{B_n\}$ of open balls in X with radii increasing and unbounded, the set $\overline{(\cup B_n)}$ is either all of X or a half space.*

Sullivan [Su] generalized this to the following stronger property :

Definition 2.1.13 A Banach space X is said to have the Property (V), if there do not exist an increasing sequence $\{B_n\}$ of open balls with radii increasing and unbounded, and norm one functionals x^* and y_k^* such that for some constant c,

$x^*(b) > c$ for all $b \in \cup B_n$,

$y_k^*(b) > c$ for all $b \in B_n$, $n \leq k$ and

$\mathrm{dist}(co(y_1^*, y_2^*, \ldots), x^*) > 0$.

Theorem 2.1.14 [Su] *A Banach space X has Property (V) if and only if X is Hahn-Banach smooth and X^* is strictly convex.*

We need the following well-known fact :

Lemma 2.1.15 [C, Proposition 25.13] *Let E be a locally convex space, K a compact convex set in E and $x \in K$. Then the following are equivalent :*

(a) *x is an extreme point of K.*

(b) *The family of open slices containing x forms a local base for the topology of E at x (relative to K).*

Definition 2.1.16 Let X be a Banach space X and let X^* be its dual. Let $W \subseteq X^*$ be a closed bounded convex set. A point $x^* \in W$ is said to be a w*-strongly extreme point of W if the family of w*-slices containing x^* forms a local base for the weak topology of X^* at x^* (relative to W).

Remark 2.1.17 As the name suggests, a w*-strongly extreme point is necessarily an extreme point. This is also immediate from Lemma 2.1.15.

Theorem 2.1.18 *Let X be a Banach space with dual X^*. The following are equivalent :*

(a) *X has w*-ANP-II'.*

(b) *X^* is strictly convex and X is Hahn-Banach smooth.*

(c) *X has Property (V).*

(d) *All points of S_{X^*} are w*-strongly extreme points of B_{X^*}.*

Proof : $(a) \Leftrightarrow (b)$ is immediate from Theorem 1.0.15 and Theorem 2.1.5, while $(b) \Leftrightarrow (c)$ is just Theorem 2.1.14.

$(b) \Rightarrow (d)$. Since Hahn-Banach smoothness implies $(S_{X^*}, w^*) = (S_{X^*}, w)$, and the norm is lower semi-continuous with respect to weak and w*-topology of X^*, any $x^* \in S_{X^*}$ is a w*-w PC of B_{X^*}. Now, since X^* is strictly convex, every $x^* \in S_{X^*}$ is an extreme point of B_{X^*}. By Lemma 2.1.15, for any point $x^* \in S_{X^*}$, the family of w*-open slices containing x^* forms a local base for the w*-topology, and therefore the weak topology of X^* relative to B_{X^*}.

$(d) \Rightarrow (b)$. From (d) and Remark 2.1.17 it is immediate that X^* is strictly convex and $(S_{X^*}, w^*) = (S_{X^*}, w)$, whence by Theorem 1.0.15, we get (b). ∎

Definition 2.1.19 [HL2] A Banach space X is said to be Quasi-Fréchet differentiable if for any convergent sequence $\{x_n\}$ in S_X and any $x_n^* \in D(x_n)$, $n \in I\!N$, the sequence $\{x_n^*\}$ has a norm convergent subsequence.

From [HL2, Theorem 4], it is known that if X has w*-ANP-I (resp. w*-ANP-II) then X is Hahn-Banach smooth and Fréchet differentiable (resp. Quasi-Fréchet differentiable). The following question posed in [HL2] still seems to be open.

Question 2.1.20 *Let X be a Banach space which is Hahn-Banach smooth and Fréchet differentiable (resp. Quasi-Fréchet differentiable). Does it follow that X has w*-ANP-I (resp. w*-ANP-II)?*

Theorem 2.1.21 *If X has w*-ANP-II', then X is very smooth.*

Proof : By Theorem 2.1.18, X^* is strictly convex and X is Hahn-Banach smooth. Thus, X is smooth and Hahn-Banach smooth and as noted in [Su], X is very smooth. We, however, prefer the following direct and ANP-like argument.

As before, X is smooth. Now let $\{x_n\}$ be a sequence in S_X such that $x_n \to x$. Let $D(x_n)$ be the singleton $\{x_n^*\}$. We have $|x_n^*(x) - 1| \le |x_n^*(x) - x_n^*(x_n)| \le \|x_n^*\|\|x - x_n\| \le \|x - x_n\| \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} x_n^*(x) = 1$. So $\{x_n^*\}$ is asymptotically normed by B_X, and hence, is weakly convergent to x^*, say. Now, $|x^*(x) - 1| = |x^*(x) - x_n^*(x) + x_n^*(x) - 1| \le |x^*(x) - x_n^*(x)| + |x_n^*(x) - 1| \longrightarrow 0$, as $n \to \infty$. Hence, $x^* \in D(x)$ and since X is smooth, $\{x^*\} = D(x)$. Hence X is very smooth. ∎

Analogous to Question 2.1.20, we have the following question :

Question 2.1.22 *Let X be a Banach space which is Hahn-Banach smooth and very smooth. Does this imply X has w*-ANP-II'?*

Definition 2.1.23 X is said to be weakly Hahn-Banach smooth, if for all $x \in S_X$, and $x_n^* \in S_{X^*}$, $\lim_{n\to\infty} x_n^*(x) = 1$ implies that $\{x_n^*\}$ has a weakly convergent subsequence.

Remark 2.1.24 Let us say that a Banach space X has property P_1^*-κ ($\kappa =$ I, II, II$'$ or III), if for any convergent sequence $\{x_n\}$ in S_X, and any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$, the sequence $\{x_n^*\}$ has property κ (recall that property III means having a weakly convergent subsequence). Then clearly, w*-ANP-$\kappa \Rightarrow P_1^*$-κ and

$$
\begin{array}{lcl}
X \text{ has property } P_1^*\text{-III} & \iff & X \text{ is weakly Hahn-Banach smooth.} \\
X \text{ has property } P_1^*\text{-II} & \iff & X \text{ is Quasi-Fréchet differentiable.} \\
X \text{ has property } P_1^*\text{-II}' & \iff & X \text{ is very smooth.} \\
X \text{ has property } P_1^*\text{-I} & \iff & X \text{ is Fréchet differentiable.}
\end{array}
$$

Thus Questions 2.1.20 and 2.1.22 are essentially whether the implication w*-ANP-$\kappa \Rightarrow P_1^*$-κ ($\kappa =$ I, II and II$'$) can be reversed under Hahn-Banach smoothness.

Observe that if we can reverse the implications for $\kappa =$ II and II$'$, the result for $\kappa =$ I would follow. Observe also that the Question 2.1.22 (i.e., $\kappa =$ II$'$) actually boils down to

Question 2.1.25 *If X is smooth and Hahn-Banach smooth, is X^* strictly convex ?*

As for $\kappa =$ II, observe that since $D(S_X)$ is dense in S_{X^*}, w*-ANP-II is equivalent to the apparently weaker property that any sequence $\{x_n^*\}$ in $D(S_X)$ that is asymptotically normed by B_X has a convergent subsequence. Now, if $\{x_n\} \subseteq S_X$ and $x_n^* \in D(x_n)$ is such that the sequence $\{x_n^*\}$ is asymptotically normed by B_X then, under Hahn-Banach smoothness, must $\{x_n\}$ have a convergent subsequence ?

Example 2.1.26 *In general, for a Banach space X, the properties ANP-I, II, II$'$ and III are all distinct, i.e., except for the obvious implications Φ-ANP-I $\Rightarrow$ Φ-ANP-II $\Rightarrow$ Φ-ANP-III and Φ-ANP-I $\Rightarrow$ Φ-ANP-II$'$ $\Rightarrow$ Φ-ANP-III, no other implication is true.*

Proof : (1) Let $X = c_0$, $X^* = \ell_1$. Since $(S_{X^*}, w) = (S_{X^*}, \|\cdot\|)$ on ℓ_1, by Theorem 1.0.12, X^* has B_X-ANP-II. But X^* is not strictly convex. Hence, X^* has neither B_X-ANP-I nor B_X-ANP-II$'$.

(2) Let $X = \ell_2$. Let $\|\cdot\|_0$ be defined as $\|x\|_0 = \max\{1/2(\|x\|_2), \|x\|_\infty\}$. Clearly this norm is equivalent to $\|\cdot\|_2$. For $(\alpha_k) \in \ell_2$, let $T(\alpha_k) = \alpha_k/k$. Then $T : \ell_2 \to \ell_2$ is 1-1 continuous linear map. Hence the equivalent dual norm $\|x\|_3 = \|x\|_0 + \|Tx\|_2$ is strictly convex [D1]. Also since ℓ_2 is reflexive, it has B_X-ANP-III with respect to $\|\cdot\|_3$ (Remark 1.0.4). Thus by Theorem 2.1.5, $(\ell_2, \|\cdot\|_3)$ has B_X-ANP-II$'$. We claim that $(\ell_2, \|\cdot\|_3)$ lacks (KK). Then from Theorem 1.0.7, $(\ell_2, \|\cdot\|_3)$ lacks B_X-ANP-II, and hence, B_X-ANP-I. Indeed, let $x = (1, 0, 0, \ldots)$ and for each k let $x_k = (1, 0, \ldots, 0, 1, 0, \ldots)$ (1 in the kth place). Then $\|x_k\|_3 = 1 + (1 + 1/k^2)^{1/2}$ and $\|x\|_3 = 2$ so that $\|x_k\|_3 \to \|x\|_3$, also $x_k \to x$ weakly. However, for each x_k, $\|x - x_k\|_3 = 1 + 1/k$ which shows $(\ell_2, \|\cdot\|_3)$ does not have (KK).

The above two examples show that a space may have ANP-III, but may lack either ANP-II or II$'$. The following is an example of a Banach space which has ANP-III but lacks both ANP-II and II$'$.

(3) Let $X^* = \ell_2 \oplus_\infty \mathbb{R}$. It is clear that X^* is reflexive, and hence, has B_X-ANP-III. However X^* is not strictly convex, and hence cannot have B_X-ANP-II$'$. Also the weak and the norm topologies do not coincide on S_{X^*}. Indeed, ℓ_2 being infinite dimensional, by Riesz's Lemma (see [Di]), there exists a sequence $\{x_n\}$ in S_{ℓ_2} such that $\|x_n - x_m\|_2 \geq 1$, $n \neq m$. Let $z_n = (x_n, 1)$. So $\|z_n\|_\infty = 1$ and $\|z_n - z_m\|_\infty \geq 1$. Clearly, $\{z_n\}$ cannot have any norm convergent subsequence. Now ℓ_2 being reflexive, B_{ℓ_2} is weakly compact. Hence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging weakly to some $x \in B_{\ell_2}$ (say). Then obviously $(x_{n_k}, 1) = z_{n_k}$ converges weakly to $(x, 1) = z$ (say) and $\|z\|_\infty = 1$. ∎

Remark 2.1.27 (2) of Example 2.1.26 is due to M. A. Smith which occurred in [Su] as an example of a very smooth Banach space whose norm is not Fréchet differentiable.

It is known that if X^* is separable, then X^* admits a locally uniformly convex dual norm [DGZ]. And in that case, all points of S_{X^*} are w*-denting points of B_{X^*}. Thus we have the following :

Theorem 2.1.28 *For a separable Banach space X, the following are equivalent :*

(*i*) *X^* admits a locally uniformly convex dual norm.*

(*ii*) *X has w*-ANP-I*

(*iii*) *X has w*-ANP-II*

(*iv*) *X has w*-ANP-II'*

(*v*) *X has w*-ANP-III*

(*vi*) *X^* has RNP.*

Remark 2.1.29 In [HL1] it was asked whether a dual space X^* having w*-ANP-I admits a locally uniformly convex dual norm. For separable Banach spaces, Theorem 2.1.28 gives an affirmative answer to this question. However the question is still open for non-separable spaces. We also note that there exist Banach spaces whose dual has RNP but lacks w*-ANP-III [HL1].

2.2 Stability Results

Theorem 2.2.1 *Let X be a Banach space with Φ-ANP-κ, $\kappa =$ I, II, II' or III. Then for any closed subspace Y of X, Y has $\Phi|_Y$-ANP-κ where $\Phi|_Y = \{y^* : y^* = x^*|_Y,\ x^* \in \Phi\}$.*

Theorem 2.2.2 *Let X be a Banach space such that X has w*-ANP-κ, $\kappa =$ I, II, II' or III. Then for any closed subspace Y of X, Y has w*-ANP-κ.*

Proof : Let $\{y_n^*\} \subseteq S_{Y^*}$ be asymptotically normed by B_Y. For every $n \geq 1$, let x_n^* be a norm preserving extension of y_n^* to X. Then $\{x_n^*\}$ is asymptotically normed by B_X, and hence has property κ. Now the restriction map $x^* \to x^*|_Y$ brings property κ back to $\{y_n^*\}$. ■

Corollary 2.2.3 *Hahn-Banach smoothness and Property* (V) *are hereditary.*

Remark 2.2.4 This observation appears to be new. Note that we do not need the stability of the ANPs under quotients to prove the above theorem. In fact, it is not clear whether the ANPs are indeed stable under quotients.

Let X be a Banach space, $1 < p, q < \infty$ with $1/p + 1/q = 1$ and (Ω, Σ, μ) be a positive measure space so that Σ contains an element with finite positive measure. Let Φ be a norming set for X. Then define $\Phi_1 = co(\Phi \cup \{0\}) \setminus S_X$ and

$$\Delta_n = \left\{ \sum_{i=1}^{m} \lambda_i x_i^* \chi_{E_i} \; : x_i^* \in \Phi_1, \frac{(n-1)}{n} \leq \|x_i^*\| \leq \frac{n}{(n+1)}, E_i \in \Sigma, \right.$$
$$\left. E_i \cap E_j = \emptyset, \text{ for } i \neq j, \lambda_i > 0 \text{ with } \sum_{i=1}^{m} \lambda_i^q \mu(E_i) = 1 \right\}$$

Then $\Delta(\Phi, \mu, q) = \bigcup_{n \geq 1} \Delta_n$ is a norming set for $L_p(\mu, X)$ [HL5].

Theorem 2.2.5 [HL5, Theorem 6] *Let X be a Banach space, $\Phi \subseteq B_{X^*}$ be a norming set for X. X has Φ-ANP-III if and only if $L_p(\mu, X)$ has $\Delta(\Phi, \mu, q)$-ANP-III.*

Thus we have

Theorem 2.2.6 *Let X be a Banach space, $\Phi \subseteq B_{X^*}$ be a norming set for X. X has Φ-ANP-II′ if and only if $L_p(\mu, X)$ has $\Delta(\Phi, \mu, q)$-ANP-II′.*

Proof : The result follows from Theorem 2.1.5, Theorem 2.2.5 and the fact that X is strictly convex if and only if $L_p(\mu, X)$ is strictly convex [D2]. ∎

Remark 2.2.7 Let X be a Banach space. If $(X, \|\cdot\|)$ has ANP-II, the space $(L_p(\mu, X), \|\cdot\|)$ may not have ANP-II. For an example, see [HL5]. Thus we have nicer stability results for ANP-II′ than for ANP-II.

Theorem 2.2.8 *Let X be a Banach space. X has Property (V) if and only if $L_p(\mu, X)$ has Property (V) $(1 < p < \infty)$.*

Proof : By Corollary 2.2.3, X inherits Property (V) from $L_p(\mu, X)$.

Conversely, if X has Property (V), by Theorem 2.1.14, X is Hahn-Banach smooth. Hence X is an Asplund space. Thus, $L_p(\mu, X)^* = L_q(\mu, X^*)$, where $1/p + 1/q = 1$. From Theorem 2.2.5, $L_p(\mu, X)$ is Hahn-Banach smooth. The result now follows from Theorem 2.1.5 and the fact that X^* strictly convex implies $L_q(\mu, X^*)$ is strictly convex. ∎

2.3 Ball-Separation Properties

In a recent work, Chen and Lin [CL] have obtained certain ball-separation properties of Banach spaces, which can be used to obtain ball-separation characterizations of w*-ANP-κ, (κ = I, II, III). We obtain a similar characterization for w*-ANP-II$'$.

The following notions will be useful for our future course of discussions.

Definition 2.3.1 [CL] Let $\mathcal{A}$ be a collection of bounded subsets of a Banach space X. Then $f \in S_{X^*}$ is said to be a $\mathcal{A}$-denting point (resp. $\mathcal{A}$-PC) of B_{X^*}, if for each $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists a w*-slice S of B_{X^*} (resp. w*-neighbourhood S) such that $f \in S$ and $\text{dia}_A S < \varepsilon$, where $\text{dia}_A S = \sup\{\|f - g\|_A : f, g \in S\}$ and $\|f - g\|_A = \sup\{|f(x) - g(x)| : x \in A\}$

Remark 2.3.2 If $\mathcal{A}$ consists of all bounded subsets of X, then it is easy to see that a $\mathcal{A}$-denting(resp. $\mathcal{A}$-PC) of B_{X^*} is a w*-denting (resp. w*-PC) point of B_{X^*}.

Definition 2.3.3 [CL] We say that $\mathcal{A}$ is a compatible collection of bounded subsets of X if

(a) If $A \in \mathcal{A}$ and $C \subseteq A$, then $C \in \mathcal{A}$.

(b) For each $A \in \mathcal{A}$, $x \in X$, $A + x \in \mathcal{A}$ and $A \bigcup \{x\} \in \mathcal{A}$.

(c) For each $A \in \mathcal{A}$, $\overline{aco}(A) \in \mathcal{A}$.

Theorem 2.3.4 [CL, Theorem 1.3] *Let X be a normed linear space, and $\mathcal{A}$ be a compatible collection of bounded subsets of X. If $f_0 \in S_{X^*}$, then the following are equivalent :*

(i) f_0 is an $\mathcal{A}$-denting point of B_{X^*}.

(ii) For all $A \in \mathcal{A}$, if $\inf f_0(A) > 0$, then there exists a ball B in X such that $A \subseteq B$, and $\inf f_0(B) > 0$.

(iii) For all $A \in \mathcal{A}$, if $\inf f_0(A) > \alpha$, then there exists a ball B in X such that $A \subseteq B$, and $\inf f_0(B) > \alpha$.

Theorem 2.3.5 [CL, Theorem 4.3] *Let X be Banach space, and $\mathcal{A}$ be a compatible collection of bounded subsets of X. If $f_0 \in S_{X^*}$, then the following are equivalent :*

(i) f_0 is an $\mathcal{A}$-PC of B_{X^*}.

(ii) For all $A \in \mathcal{A}$, if $\inf f_0(A) > 0$, then there exist finitely many balls $B_1, B_2, \ldots B_n$ in X such that $A \subseteq \overline{co}\{\bigcup_{i=1}^n B_i\}$ and $\inf f_0(\overline{co}\{\bigcup_{i=1}^n B_i\}) > 0$.

(iii) For all $A \in \mathcal{A}$, if $\inf f_0(A) > \alpha$, then there exist finitely many balls $B_1, B_2, \ldots B_n$ in X such that $A \subseteq \overline{co}\{\bigcup_{i=1}^n B_i\}$ and $\inf f_0(\overline{co}\{\bigcup_{i=1}^n B_i\}) > \alpha$.

Remark 2.3.6 One can see from the proof of the above theorem in [CL], that it works even if we redefine compatible collection of sets by changing the first criterion in the following manner :

If $A \in \mathcal{A}$ and $C \subseteq A$, C closed, then $C \in \mathcal{A}$.

In fact this seems to be the justification of Corollary 1.11 in [CL] where $\mathcal{A} = \{$all compact subsets of $X\}$ is taken to be compatible.

Theorem 2.3.7 [CL, Theorem 3.1] *Let X be a Banach space and let $f_0 \in S_{X^*}$, then the following are equivalent :*

(a) f_0 is a w^*-w PC of B_{X^*}.

(*b*) *For any* $x_0^{**} \in X^{**}$, *if* $x_0^{**} \notin f_0^{-1}(0) = \{x^{**} \in X^{**} : f_0(x^{**}) = 0\}$, *then there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $x_0^{**} \in B^{**}$ *and* $B^{**} \bigcap f_0^{-1}\{0\} = \emptyset$.

And we immediately have,

Theorem 2.3.8 *Let* X *be a Banach space and let* $f_0 \in S_{X^*}$, *then the following are equivalent :*

(*i*) f_0 *is a* w^*-w *PC of* B_{X^*}.

(*ii*) *for any* $x_0^{**} \in X^{**}$ *and* $\alpha \in I\!R$, *if* $f_o(x_0^{**}) > \alpha$, *then there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $x_0^{**} \in B^{**}$ *and* $\inf f_0(B^{**}) > \alpha$.

The following ball-separation characterizations of w*-ANP's can be obtained using Theorem 2.3.4, Theorem 2.3.5 and Theorem 2.3.7. We have stated it in a slightly modified form.

Theorem 2.3.9 *For a Banach space* X,

(*i*) X *has w*-ANP-I if and only if for any w*-closed hyperplane* H *in* X^{**}, *and any bounded convex set* A *in* X^{**} *with* $dist(A, H) > 0$, *there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $A \subseteq B^{**}$ *and* $B^{**} \cap H = \emptyset$.

(*ii*) X *has w*-ANP-II if and only if for any w*-closed hyperplane* H *in* X^{**}, *and any bounded convex set* A *in* X^{**} *with* $dist(A, H) > 0$, *there exist finitely many balls* $B_1^{**}, B_2^{**}, \ldots, B_n^{**}$ *in* X^{**} *with centres in* X *such that* $A \subseteq \overline{co}(\cup_{k=1}^n B_k^{**})$ *and* $\overline{co}(\cup_{k=1}^n B_k^{**}) \cap H = \emptyset$.

(*iii*) X *has w*-ANP-III if and only if for any w*-closed hyperplane* H *in* X^{**}, *and any* $x^{**} \in X^{**} \setminus H$, *there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $x^{**} \in B^{**}$ *and* $B^{**} \cap H = \emptyset$.

Now let us obtain a similar characterization of w*-ANP-II′. In fact, we characterize w*-strongly extreme points of B_{X^*}.

In view of Remark 2.3.6 we get,

Theorem 2.3.10 *Let X be a Banach space and let $f_0 \in S_{X^*}$, then the following are equivalent :*

(a) *f_0 is an extreme point of B_{X^*}.*

(b) *for any compact set $A \subseteq X$ if $\inf f_0(A) > 0$, then there exists a ball B in X such that $A \subseteq B$ and $\inf f_0(B) > 0$.*

(c) *for any finite set $A \subseteq X$ if $\inf f_0(A) > 0$, then there exists a ball B in X such that $A \subseteq B$ and $\inf f_0(B) > 0$.*

Proof : (a) $\Leftrightarrow$ (b). Let $\mathcal{A} = \{$all compact subsets of $X\}$. Now B_{X^*} is w*-compact and the restricted b-w* topology on B_{X^*} (i.e., the topology of uniform convergence on compact subsets of X, see [DS] for more on this) coincides with the restricted w*-topology. Thus it follows from Lemma 2.1.15 that f_0 is an extreme point of B_{X^*} if and only if the w*-slices form a neighbourhood base of f_0 in the restricted b-w*-topology, i.e., f_0 is an extreme point if and only if it is an $\mathcal{A}$-denting point. The rest of the proof follows from Theorem 2.3.6.

(a) $\Leftrightarrow$ (c). It suffices to repeat the above argument with the compatible collection $\mathcal{A} = \{$all compact sets with finite affine dimension$\}$, after observing that $co\{x_1, x_2, \ldots, x_n\} \in \mathcal{A}$ for any finite number of points $x_1, x_2, \ldots, x_n \in X$. ■

Remark 2.3.11 The idea of the proof of the above theorem has been adapted from [WZ, Lemma 2] and [B2, Corollary 2].

Theorem 2.3.12 *Let X be a Banach space and $f_0 \in S_{X^*}$, then the following are equivalent:*

(a) *f_0 is a w*-strongly extreme point of B_{X^*}.*

(b) *f_0 is a w*-w PC and an extreme point of B_{X^*}.*

(c) *for any compact set $A \subseteq X^{**}$, if $\inf f_0(A) > 0$, then there exists a ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and $\inf f_0(B^{**}) > 0$.*

(d) *for any finite set $A \subseteq X^{**}$, if $\inf f_0(A) > 0$, then there exists a ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and $\inf f_0(B^{**}) > 0$.*

Proof : $(a) \Leftrightarrow (b)$ follows from Lemma 2.1.15 and the proof of Theorem 2.1.18.

$(b) \Rightarrow (c)$. Since f_0 is a w*-strongly extreme point of B_{X^*}, it is easily seen that it is an extreme point of $B_{X^{***}}$. Thus by Theorem 2.3.10, for any compact set A in X^{**} with $\inf f_0(A) > 0$, there exists a ball in $B^{**} = B^{**}(x_0^{**}, r) \subseteq X^{**}$ such that $A \subseteq B^{**}$ and $\inf f_0(B^{**}) > 0$. Now, $\inf f_0(B^{**}(x_0^{**}, r)) > 0$ implies $f_0(x_0^{**}) > r$. Since f_0 is a w*-w PC, by Theorem 2.3.8, there exists a ball $B^{**}(x, t) \subseteq X^{**}$ such that $x_0^{**} \in B^{**}(x, t)$ and $\inf f_0(B^{**}(x, t)) > r$. This implies $f_0(x) > r + t$. Also, $A \subseteq B^{**}(x_0^{**}, r) \subseteq B^{**}(x, r + t)$ and $\inf f_0(B^{**}(x, r + t) > 0$.

$(c) \Rightarrow (d)$ is trivial.

$(d) \Rightarrow (b)$. Taking $A \subseteq X$, it follows from Theorem 2.3.10 that f_0 is extreme in B_{X^*}. And taking A to be a singleton, it follows from Theorem 2.3.8 that f_0 is a w*-w PC. ∎

Corollary 2.3.13 *Let X be a Banach space. Then the following are equivalent :*

(*i*) *X has w*-ANP-II′.*

(*ii*) *for any w*-closed hyperplane H in X^{**}, and any compact set A in X^{**} with $A \cap H = \emptyset$, there exists a ball B^{**} in X^{**} with centre in X such that $A \subseteq B^{**}$ and $B^{**} \cap H = \emptyset$.*

(*iii*) *for any w*-closed hyperplane H in X^{**}, and any finite set A in X^{**} with $A \cap H = \emptyset$, there exists a ball B^{**} in X^{**} with centre in X such that $A \subseteq B^{**}$ and $B^{**} \cap H = \emptyset$.*

Chapter 3

Some Stability results on Weak*-Asymptotic Norming Properties

3.1 Hahn-Banach Smoothness, U-subspaces and their Permanence Properties

Our first result gives a simpler proof of the following theorem by E. Oja and M. Põldvere [OP]. But let us first quote the following definition and result which we need for proving the theorem and for subsequent discussions.

Definition 3.1.1 [OP] Let X be a Banach space, Y a subspace of X. Y is said to have property U in X, if for any $y^* \in Y^*$ there exists a unique norm preserving extension of y^* in X^*. Henceforth we will refer to such a subspace as a U-subspace of X.

Remark 3.1.2 Notice that, X is Hahn-Banach smooth if X is a U-subspace of X^{**} under the canonical embedding of X in X^{**}.

Proposition 3.1.3 [SY] *If X is any Banach space, N is a separable subspace of X, and F is a separable subspace of X^*, then X has a separable subspace M containing N which admits a linear extension operator, i.e., a linear mapping $T : M^* \longrightarrow X^*$, such that for each $f \in M^*$, Tf is a norm preserving extension of f and $T(M^*) \supseteq F$.*

Theorem 3.1.4 *X is Hahn-Banach smooth if and only if every separable subspace of X is Hahn-Banach smooth.*

Proof : We have already observed in Theorem 2.2.2 that Hahn-Banach smoothness is hereditary.

Conversely, let X be such that all its separable subspaces are Hahn-Banach smooth. We will show that X is Hahn-Banach smooth, i.e., X has w*-ANP-III. Let $\{x_n^*\}$ be a sequence in S_{X^*} which is asymptotically normed by B_X. In view of Theorem 1.0.3, it is enough to show that $\{x_n^*\}$ has a weakly convergent subsequence. Since $\|x_n^*\| = 1$ for all n, for $m, n \in I\!N$, select $x_{nm} \in B_X$ such that $x_n^*(x_{nm}) \geq 1 - 1/m$. Also since $\{x_n^*\}$ is asymptotically normed by B_X, for each $k \in I\!N$, there exists $n_k \in I\!N$ and $x_k \in B_X$ such that $x_n^*(x_k) > 1 - 1/k$ for all $n \geq n_k$. Let $Y = \overline{span}[\{x_{nm}\} \bigcup \{x_k\}]$. Clearly $\{x_n^*\}$ is asymptotically normed by B_Y. By Proposition 3.1.3, there exists a separable $Y' \supset Y$ and a linear extension operator $T : Y'^* \longrightarrow X^*$ such that $TY'^* \supset \overline{span}\{x_n^*\}$. Hence there exists $y_n^* \in Y'^*$ such that $T(y_n^*) = x_n^*$ and $\|y_n^*\| = \|x_n^*\|$. Since Y' is separable, Y' has w*-ANP-III. Let $\{y_{nl}^*\}$ be the subsequence of $\{y_n^*\}$ weakly converging to y^*. Let $T(y^*) = x^*$, then $\|y^*\| = \|x^*\|$. Also let $x^{**} \in X^{**}$, then $T^*(x^{**}) \in (Y')^{**}$, hence $T^*(x^{**})(y_{nl}^*) \longrightarrow T^*(x^{**})(y^*)$ which implies $x^{**}T(y_{nl}^*) \longrightarrow x^{**}T(y^*)$ which in turn implies $x^{**}(x_{nl}^*) \longrightarrow x^{**}(x^*)$, and this is true for all $x^{**} \in X^{**}$. Hence $\{x_{nl}^*\}$ is a subsequence of $\{x_n^*\}$ weakly converging to x^*. This completes the proof. ■

We next consider the stability of being a U-subspace under ℓ_1 sums.

Proposition 3.1.5 *Let $Y \subsetneq X$ be a proper subspace of X and let Z be any nonzero Banach space, then the ℓ_1-direct sum $Y \bigoplus_1 Z$ is not a U-subspace of $X \bigoplus_1 Z$.*

Proof : Let $y^* \in Y^*$, $0 < \|y^*\| < 1$ and let $z^* \in S_{Z^*}$. Let $x^* \in X^*$ be a norm preserving extension of y^*. Since $\|x^*\| < 1$ and Y is a proper subspace of X, choose $\tau \in Y^\perp$ such that $\tau \neq 0$ and $\|x^* \pm \tau\| \leq \|x^*\| + \|\tau\| \leq 1$. Now $\|(x^* \pm \tau, z^*)\| = \max(\|x^* \pm \tau\|, \|z^*\|) = 1$. Thus $(x^* \pm \tau, z^*)$ are two distinct norm preserving extensions of (y^*, z^*). ∎

Before our next result, let us recall the following definitions :

Definition 3.1.6 [HWW] Let X be a Banach space.

(i) A linear projection P is called an L-projection if

$$\|x\| = \|Px\| + \|x - Px\|,$$

for all $x \in X$. If a closed subspace Y of X is the range of an L-projection, it is called an L-summand of X.

(ii) A linear projection P is called an M-projection if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\},$$

for all $x \in X$. If a closed subspace Y of X is the range of an M-projection, it is called an M-summand of X. A closed subspace $Y \subseteq X$ is said to be an M-ideal if $Y^\perp$ is an L-summand of X^*.

Corollary 3.1.7 *If X is non-reflexive and Hahn-Banach smooth, then X has no non-trivial L-projections.*

Proof : Suppose $X = Y \bigoplus_1 Z$ is a non-trivial L-decomposition. Since X is not reflexive, assume without loss of generality, Y is non-reflexive. Since $X = Y \bigoplus_1 Z$ is a U-subspace of $X^{**} = Y^{**} \bigoplus_1 Z^{**}$, it is a U-subspace of $Y^{**} \bigoplus_1 Z$ as well. By Proposition 3.1.5, this is a contradiction. Hence there are no non-trivial L-projections on X. ∎

Remark 3.1.8 It was observed by Sullivan in [Su] that if a dual space is Hahn-Banach smooth, then it is reflexive.

Corollary 3.1.9 *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces. Then $\bigoplus_{\ell_1(\Gamma)} X_i$ is Hahn-Banach smooth if and only if all but finitely many X_i's are trivial, i.e., $\{0\}$, and the remaining are reflexive.*

Proof : First, suppose only finitely many X_i's are non-trivial and reflexive. Then obviously $\bigoplus_{i=1}^{n} X_i$ is reflexive and hence Hahn-Banach smooth.

Conversely, suppose $\bigoplus_{\ell_1(\Gamma)} X_i$ is Hahn-Banach smooth. Suppose X_m is not reflexive for some $m \in \Gamma$. Hence $\bigoplus_{\ell_1(\Gamma)} X_i$ is not reflexive. In this case X_m is an L-summand of $\bigoplus_{\ell_1(\Gamma)} X_i$ contradicting Corollary 3.1.7. So X_i's are reflexive for all i. Hence $X_i = Y_i^*$ for dual space Y_i^*. This implies

$$X = \bigoplus_{\ell_1(\Gamma)} Y_i^* = Y^*.$$

Hence X is a dual space and by Remark 3.1.8 it follows that X is reflexive. This in turn implies that

$$\bigoplus_{c_0(\Gamma)} Y_i = Y = Y^{**} = X^* = \bigoplus_{\ell_\infty(\Gamma)} Y_i$$

and the first and the last spaces are equal only when Y_i's are zero for all but finitely many i's. ∎

Corollary 3.1.10 *If, for a Banach space X, every equivalent renorming is Hahn-Banach smooth, then X is reflexive.*

Proof : Let $X = Y \bigoplus Z$ be a nontrivial direct sum. Clearly such a decomposition is always possible by taking Y to be finite dimensional and Z its complement. Now define $\|x\|_1 = \|y\| + \|z\|$ where $x = y + z$, $y \in Y$, $z \in Z$. Applying the open mapping theorem, it readily follows that $\|\cdot\|_1$ is an equivalent norm on X and this new norm has nontrivial L-projection. Therefore every non-reflexive space can be renormed to fail Hahn-Banach smoothness. Hence the result. ∎

Remark 3.1.11 In [HL2] the authors showed that X is reflexive if and only if for any equivalent norm on X, X is Hahn-Banach smooth and has ANP-III. The corollary above is a much stronger result with a simpler proof.

Corollary 3.1.12 *Hahn-Banach smoothness is not a three space property.*

Proof : Let M be Hahn-Banach smooth and non-reflexive, e.g., c_0. Let $X = M \bigoplus_1 M$. Then X/M is isometrically isomorphic to M, hence Hahn-Banach smooth. From Corollary 3.1.7, it follows that X is not Hahn-Banach smooth. ∎

Theorem 3.1.13 *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces. For each $i \in \Gamma$, let Y_i be a U-subspace of X_i. Then the c_0-direct sum $\bigoplus_{c_0(\Gamma)} Y_i$ is a U-subspace of $\bigoplus_{c_0(\Gamma)} X_i$.*

Proof : Let $X = \bigoplus_{c_0(\Gamma)} X_i$, then $X^* = \bigoplus_{\ell_1(\Gamma)} X_i^*$. Similarly, $Y = \bigoplus_{c_0(\Gamma)} Y_i$, $Y^* = \bigoplus_{\ell_1(\Gamma)} Y_i^*$. Let $y^* \in Y^*$. Let $x^* = (x_i^*)_{i\in\Gamma}$ and $z^* = (z_i^*)_{i\in\Gamma}$ be norm-preserving extensions of $y^* = (y_i^*)_{i\in\Gamma}$. Clearly $x_i^* \neq 0$ if and only if $y_i^* \neq 0$ if and only if $z_i^* \neq 0$. Thus $x_i^* = y_i^* = z_i^*$ on Y_i^* for all i. Now $\|x^*\| = \|y^*\|$ implies $\sum(\|x_i^*\| - \|y_i^*\|) = 0$. Since $\|x_i^*\| \geq \|y_i^*\|$, we have $\|x_i^*\| = \|y_i^*\|$ for all i. Similarly for z_i^*. Thus $\|z_i^*\| = \|x_i^*\|$ for all i. Since each Y_i is a U-subspace of X_i, it follows that $x_i^* = z_i^*$ for all i. Hence $z^* = x^*$. ∎

Before proceeding to our next result, we need to prove a simple lemma for which the following "three ball characterization" of M-ideals is needed.

Theorem 3.1.14 [HWW, Theorem I.2.2] *A closed subspace Y of a Banach space X is an M-ideal if and only if for all $y_1, y_2, y_3 \in B_Y$, all $x \in B_X$ and all $\varepsilon > 0$, there is $y \in Y$ such that*

$$\|x + y_i - y\| < 1 + \varepsilon \quad (i = 1, 2, 3).$$

Lemma 3.1.15 *If $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces then $\bigoplus_{c_0} X_i$ is an M-ideal in $\bigoplus_{\ell_\infty} X_i$.*

Proof : Let $Y = \bigoplus_{c_0} X_i$, $X = \bigoplus_{\ell_\infty} X_i$. Let $y_1, y_2, y_3 \in B_Y$, $x \in B_X$ and $\varepsilon > 0$. Since $y_i \in \bigoplus_{c_0(\Gamma)} X_i$, $y_i = y_i^{(n)}$, we have $\|y_i^{(n)}\| \longrightarrow 0$, as $n \longrightarrow \infty$. Thus, there exists a $N \in \mathbb{N}$ such that $\|y_i^{(n)}\| < \varepsilon$ for all $n \geq N$, $i = 1,2,3$. Define $y = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(N)}, 0, \ldots)$. Clearly $y \in B_Y$. Also for any $i = 1,2,3$,

$$\|x^{(n)} + y_i^{(n)} - y^{(n)}\| = \|y_i^{(n)}\| \leq 1 \quad \text{if} \quad n \leq N$$

whereas

$$\|x^{(n)} + y_i^{(n)} - y^{(n)}\| = \|x^{(n)} + y_i^{(n)}\| \leq \|x^{(n)}\| + \|y_i^{(n)}\| < 1 + \varepsilon \quad \text{if} \quad n > N.$$

Therefore,

$$\|x + y_i - y\| < 1 + \varepsilon$$

Hence by the Theorem 3.1.14, Y is an M-ideal. ■

Remark 3.1.16 As remarked in [HWW], any M-ideal is a U-subspace. Given a family $\{X_i\}_{i\in\Gamma}$ of Banach spaces, the above lemma gives an easy way of generating M-ideals.

Corollary 3.1.17 *If $\{X_i\}_{i\in\Gamma}$ is a family of Hahn-Banach smooth spaces, then so is $\bigoplus_{c_0} X_i$.*

Proof : Since X_i is Hahn-Banach smooth for all i, X_i is a U-subspace of X_i^{**} for all i. So by Theorem 3.1.13, this implies $\bigoplus_{c_0} X_i$ is a U-subspace of $\bigoplus_{c_0} X_i^{**}$ which, by Lemma 3.1.15, is an M-ideal in $\bigoplus_{\ell_\infty} X_i^{**} = (\bigoplus_{c_0} X_i)^{**}$. Thus $\bigoplus_{c_0} X_i$ is a U-subspace of $(\bigoplus_{c_0} X_i)^{**}$. Hence $\bigoplus_{c_0} X_i$ is Hahn-Banach smooth. ■

Theorem 3.1.18 *Let X be a Banach space. Let $x_0 \in S_X$. Suppose $\bigoplus_{\ell_\infty} span\{x_0\}$ is a U-subspace of $\bigoplus_{\ell_\infty} X$ (countable sum). Then x_0 is a smooth point of X and the unique norming functional $x_0^* \in X^*$ is w^*-strongly exposed by x_0.*

Proof : To show that x_0 is a smooth point, suppose

$$\|x^*\| = \|y^*\| = x^*(x_0) = y^*(x_0) = 1.$$

Fix a Banach limit L on $\ell_\infty, \|L\| = 1$. Define $\Phi, \Psi : \bigoplus_\infty X \longrightarrow I\!R$ by $\Phi(\{x_n\}) = L(\{x^*(x_n)\})$ and $\Psi(\{x_n\}) = L(\{y^*(x_n)\})$. Since $x^*(x_0) = y^*(x_0) = 1$, clearly $\|\Phi\| = \|\Psi\| = 1$. For $\{\alpha_n x_0\} \in \bigoplus_\infty span\{x_0\}$, $\Phi(\{\alpha_n x_0\}) = L(\{\alpha_n\}) = \Psi(\{\alpha_n x_0\})$. Also clearly $\|\Phi\| = \|\Psi\| = 1$ on $\bigoplus_\infty span\{x_0\}$. Therefore by hypothesis, $\Phi = \Psi$. Treating $x \in X$ as a constant sequence, we thus get $x^*(x) = y^*(x)$ for all $x \in X$.

To show that x_0^* is strongly exposed by x_0, let $\{x_n^*\} \subseteq B_{X^*}$ and $x_n^*(x_0) \longrightarrow 1 = x_0^*(x_0)$.

Claim : $x_n^* \longrightarrow x_0^*$ in norm.

Define $\delta : \bigoplus_\infty X \longrightarrow I\!R$ by $\delta(\{x_n\}) = L(\{x_n^*(x_n)\})$. By our assumption, $\|\delta\| = 1$ and $\delta(\{\alpha_n x_0\}) = L(\{\alpha_n\})$ and $\|\delta\| = 1$ on $\bigoplus_\infty span\{x_0\}$. Therefore by the uniqueness of extension, $\delta = \Phi$ (Φ as above), i.e.,

$$L(\{x_n^*(x_n)\}) = L(\{x_0^*(x_n)\}) \quad \text{for all} \quad \{x_n\} \in \bigoplus_\infty X \tag{3.1}$$

Suppose $x_n^* \not\rightarrow x^*$ in norm. By passing through a subsequence if necessary, we may assume that there exists $\varepsilon > 0$ such that $\|x_n^* - x_0^*\| \geq \varepsilon$. Choose $x_n \in S_X$ such that $x_n^*(x_n) - x_0^*(x_n) \geq \varepsilon/2$. So for this choice of $\{x_n\}$, $L(\{x_n^*(x_n) - x_0^*(x_n)\}) \geq \varepsilon/2$. But this contradicts (3.1). Hence the claim. ∎

Example 3.1.19 *We now use the above theorem to show that being a U-subspace is not preserved under ℓ_∞-direct sums.*

Proof : Suppose X is a reflexive Banach space that is strictly convex (see following remark) but fails the property H (i.e., there exists a sequence $\{x_n\} \subseteq X$ such that $x_n \longrightarrow x$ weakly, $\|x_n\| \longrightarrow \|x\|$, but $x_n \not\rightarrow x$ in norm). Then in such a space X, there are $x_0 \in S_X$, $\{x_n\} \subseteq S_X$, $x_n \longrightarrow x_0$ weakly, but not in the norm. Fix $x_0^* \in S_{X^*}$, $x_0^*(x_0) = 1$. Since X is strictly convex, $span\{x_0^*\}$ is a U-subspace of X^*. However x_0^* does not strongly expose x_0. Therefore $\bigoplus_\infty span\{x_0^*\}$ is not a U-subspace of $\bigoplus_\infty X^*$.

Remark 3.1.20 One such example due to M. A. Smith (discussed earlier in Example 2.1.26) given in [Su] is the following renorming of ℓ_2 :

Let $\|x\|_0 = \max\{\frac{1}{2}\|x\|_2, \|x\|_\infty\}$. Define $T : \ell_2 \longrightarrow \ell_2$ by $T(\{\alpha_k\}) = \{\alpha_k/k\}$. Finally $|||x||| = \|x\|_0 + \|Tx\|_2$ is an equivalent norm with the required property.

Example 3.1.21 *By considering $I\!R$ as a U-subspace of the Euclidean $I\!R^2$ and taking a non-atomic measure λ, we shall show that $L_1(\lambda)$ is not a U-subspace of $L_1(\lambda, I\!R^2)$.*

Proof : Let $(\Sigma, \mathcal{M}, \lambda)$ be a non-atomic probability measure space. So there exists $A \in \mathcal{M}$ such that $0 < \lambda(A) < 1$. Define $\phi : L_1(\lambda) \longrightarrow I\!R$ by $\phi(f) = \int_A f d\lambda$. Hence

$$\|\phi\| = \|\chi_A\|_\infty = 1.$$

Note that $L_1(\lambda, I\!R^2) = \{(f_1, f_2) : f_i \in L_1(\lambda), i = 1, 2\}$, where $\|(f_1, f_2)\|_1 = \int_\Sigma (|f_1(w)|^2 + |f_2(w)|^2)^{\frac{1}{2}} d\lambda(w)$. With this identification, we have

$$L_1(\lambda, I\!R^2)^* = L_\infty(\lambda, I\!R^2).$$

where $(g_1, g_2) \in L_\infty(\lambda, I\!R^2)$ and $\|(g_1, g_2)\|_\infty = ess.sup\{(|g_1(w)|^2 + |g_2(w)|^2)^{\frac{1}{2}} : w \in I\!R\}$. Then $\Phi = (\chi_A, 0) \in L_\infty(\lambda, I\!R^2)$ is a norm preserving extension of ϕ. Let $\Psi = (\chi_A, \chi_{A^c}) \in L_\infty(\lambda, I\!R^2)$. Then $\|\Psi\| = 1$. Also

$$\Psi(f, 0) = \int_A f d\lambda = \Phi(f, 0).$$

Thus Ψ is a norm-preserving extension of ϕ different from Φ. ∎

Remark 3.1.22 Suppose Y is a U-subspace of X. Then is the analogue of Example 3.1.21 true? Or in other words, under what condition on X does $L_1(\lambda, Y)$ become a U-subspace of $L_1(\lambda, X)$?

Theorem 3.1.23 *If X is Hahn-Banach smooth and has finite intersection property (FIP), then X is reflexive.*

Proof : Since X has FIP, it follows from Theorem 1.0.19 that $X^{**} = X + C_X$ where $C_X = \{F \in X^{**} : \|F + \hat{x}\| \geq \|x\| \text{ for all } x \in X\}$. Let $\Lambda \in C_X$ and $\|\Lambda\| = 1$. Then by Lemma 1.0.16, $\overline{B_{ker\Lambda}}^{*} = B_{X^*}$. Let $\|x^*\| = 1$ and $x^*_\alpha \in B_{ker\Lambda}$ such that $x^*_\alpha \xrightarrow{w^*} x^*$. By w*-lower semicontinuity of the norm it follows that $\|x^*_\alpha\| \longrightarrow 1$. Since X is Hahn-Banach smooth, the weak and w*-topologies coincide on S_{X^*}. So, $x^*_\alpha \longrightarrow x^*$ weakly. In particular, $\Lambda(x^*_\alpha) \longrightarrow \Lambda(x^*)$. Thus $\Lambda(x^*) = 0$ for all $x^* \in S_{X^*}$, a contradiction. Hence $C_X = \{0\}$ and consequently X is reflexive. ∎

Remark 3.1.24 That Hahn-Banach smoothness for a dual space implies reflexivity was first remarked by Sullivan [Su]. The same result for 1-complemented subspaces of a dual space was noted by Lima [L2].

3.2 Spaces with Property (II) and their Permanence Properties under Various Conditions

In this section we study w*-ANP-II and related properties. Proceeding similarly as in Theorem 3.1.4, we conclude that

Theorem 3.2.1 *The w*-ANP-κ (κ = I, II, II') is a separably determined property.*

One can easily prove

Theorem 3.2.2 *Suppose X is a Banach space, Φ a norming set for X. If for all separable subspace Y, Y has $\Phi|_Y$-ANP-κ (κ = I, II, II', III) then X also has Φ-ANP-κ (κ = I, II, II', III).*

Remark 3.2.3 One can look at Theorem 3.2.2 in a more general setup in the context of equivalent renormings. So we can ask

Question 3.2.4 *Suppose X is such that all its separable subspaces have ANP-κ (κ = I, II, II', III), then does X also have ANP-κ (κ = I, II, II', III)?*

Let us look into this question a little more closely. It has already been remarked earlier that for separable Banach spaces ANP $\Longleftrightarrow$ RNP. It is still an open question whether this is true in general. However it is shown in [JH] that even for non-separable spaces, ANP $\Longrightarrow$ RNP. Since it is known that RNP is a separably determined hereditary property, the affirmative answer to the above question will therefore show that RNP $\Longrightarrow$ ANP.

Analogous to Theorem 3.1.23, we have the following result.

Theorem 3.2.5 *If X has Property (II) and has FIP, then X is reflexive. In particular, any dual space with Property (II) is reflexive.*

Proof : As before, we will show that $C_X = \{0\}$. Let $\Lambda \in C_X$. Since X has Property (II), the w*-PC's of B_{X^*} are dense in S_{X^*}. Hence it suffices to show $\Lambda(x^*) = 0$ for any w*-PC $x^* \in S_{X^*}$. But this follows from arguments similar to the proof of Theorem 3.1.23. ■

We next look at the stability results for Property (II). The following lemma will be useful for our subsequent discussions.

Lemma 3.2.6 *If X has Property (II) and $A \subseteq B_{X^*}$ is such that $B_{X^*} = \overline{co}^{w^*}(A)$, then $B_{X^*} = \overline{co}(A)$.*

Proof : Let $y^* \in B_{X^*}$. Since X has Property (II), for any $\varepsilon > 0$ there exists x^*, a w*-PC of B_{X^*}, such that, $\|x^* - y^*\| < \varepsilon/2$. Since $B_{X^*} = \overline{co}^{w^*}(A)$, there exists $\{x^*_\alpha\} \subseteq co(A)$ such that $x^*_\alpha \longrightarrow x^*$ in the w*-topology, and hence $x^*_\alpha \longrightarrow x^*$ in norm. So there exists α_0 such that $\|x^* - x^*_\alpha\| < \varepsilon/2$ for all $\alpha \geq \alpha_0$. This implies $\|y^* - x^*_\alpha\| < \varepsilon$ for all $\alpha \geq \alpha_0$. Hence $B_{X^*} = \overline{co}(A)$. ■

Corollary 3.2.7 (*i*) *If X is separable and has Property (II), then X^* is separable.*

(*ii*) *If X^* has Property (II), then X is reflexive.*

Proof : (i) Let $\{x_n\}$ be a countable dense subset of S_X. Let $\{x_n^*\}$ denote the corresponding norming functionals. Then $B_{X^*} = \overline{co}^{w^*}\{x_n^*\}$. Since X has Property (II), $B_{X^*} = \overline{co}\{x_n^*\}$. Hence, X^* is separable.

(ii) We simply observe that if X^* has Property (II), then $B_{X^{**}} = \overline{co}(B_X) = B_X$. ∎

Remark 3.2.8 In a recent work Moreno and Sevilla [SM] have given examples of non-Asplund spaces which have MIP (hence Property (II)). So it follows that Property (II) cannot be hereditary. Indeed, if it were hereditary, then all its separable subspaces will also have Property (II). Thus by Corollary 3.2.7, it follows that these separable subspaces will have separable duals, or in other words, X will be Asplund, a contradiction.

Now, let us look at Property (II) for c_0 direct sums. For this we prove the following useful characterization of w*-PC's for c_0 sums of Banach spaces.

Proposition 3.2.9 *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces and $X = \bigoplus_{c_0(\Gamma)} X_i$. Then $x^* = (x^*(i))_{i\in\Gamma} \in S_{X^*}$ is a w*-PC of B_{X^*} if and only if for each $i \in \Gamma$, either $x^*(i) = 0$ or $x^*(i)/\|x^*(i)\|$ is a w*-PC of $B_{X_i^*}$.*

Proof : First suppose $x^* = (x^*(i))_{i\in\Gamma}$ is a w*-PC of B_{X^*}. Let $x^*(i_0) \neq 0$ and $x_\alpha^*(i_0) \xrightarrow{w^*} x^*(i_0)/\|x^*(i_0)\|$, where $x_\alpha^*(i_0) \in B_{X_{i_0}^*}$. Define y_α^* as

$$y_\alpha^*(i) = \begin{cases} x_\alpha^*(i)\|x^*(i)\| & \text{if} \quad i = i_0 \\ x^*(i) & \text{if} \quad i \neq i_0 \end{cases}$$

Then,

$$\begin{aligned} \|y_\alpha^*\| &= \sum_{i\in\Gamma} \|y_\alpha^*(i)\| \\ &= \sum_{i\neq i_0} \|y_\alpha^*(i)\| + \|x_\alpha^*(i_0)\|\|x^*(i_0)\| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i \neq i_0} \|x^*(i)\| + \|x^*(i_0)\| \\ &= \|x^*\| = 1. \end{aligned}$$

Hence $y^*_\alpha \in B_{X^*}$, and we have $y^*_\alpha \xrightarrow{w^*} x^*$, which implies $y^*_\alpha \longrightarrow x^*$ in norm, which in turn implies that $x^*_\alpha(i_0) \longrightarrow x^*(i_0)/\|x^*(i_0)\|$ in norm. Hence $x^*(i_0)/\|x^*(i_0)\|$ is a w*-PC of $B_{X^*_{i_0}}$.

Conversely, let $x^* = (x^*(i))_{i\in\Gamma} \in S_{X^*}$. Let Γ_0 be the countable set such that for $i \in \Gamma_0$, $x^*(i) \neq 0$, and $x^*(i)/\|x^*(i)\|$ is a w*-PC of $B_{X^*_i}$. Let $x^*_\alpha \xrightarrow{w^*} x^*$, $x^*_\alpha \in B_{X^*}$. This implies $x^*_\alpha(i) \xrightarrow{w^*} x^*(i)$ in X^*_i, for all $i \in \Gamma$.

Claim:

$$\lim_\alpha \|x^*_\alpha(i)\| = \|x^*(i)\| \quad \text{for all } i \in \Gamma$$

Proof of the claim:

Since norm is a w*-lower semi-continuous function, it follows that

$$1 = \|x^*\| \leq \liminf_\alpha \|x^*_\alpha\| \leq \limsup_\alpha \|x^*_\alpha\| \leq 1.$$

This implies $\lim_\alpha \|x^*_\alpha\| = 1$.

Similarly,

$$\|x^*(i)\| \leq \liminf_\alpha \|x^*_\alpha(i)\| \quad \text{for all } i \in \Gamma.$$

We first observe that

$$\|x^*(i)\| = \liminf_\alpha \|x^*_\alpha(i)\| \quad \text{for all } i \in \Gamma. \tag{3.2}$$

Indeed, if for some $i_0 \in \Gamma$, $\|x^*(i_0)\| < \liminf_\alpha \|x^*_\alpha(i_0)\|$. Then

$$1 \leq \sum_{i\in\Gamma} \|x^*(i)\| < \sum_{i\in\Gamma} \liminf_\alpha \|x^*_\alpha(i)\| \leq \liminf_\alpha \sum_{i\in\Gamma} \|x^*_\alpha(i)\| \leq 1,$$

a contradiction. Now suppose that for some $i_0 \in \Gamma$,

$$\|x^*(i_0)\| = \liminf_\alpha \|x^*_\alpha(i_0)\| < \limsup_\alpha \|x^*_\alpha(i_0)\|.$$

Then

$$\begin{aligned}
\|x^*(i_0)\| &< \limsup_\alpha \|x^*_\alpha(i_0)\| \\
&= \limsup_\alpha \left[1 - \sum_{i\neq i_0} \|x^*_\alpha(i)\|\right] \\
&= 1 - \liminf_\alpha \sum_{i\neq i_0} \|x^*_\alpha(i)\| \\
&\leq 1 - \sum_{i\neq i_0} \liminf_\alpha \|x^*_\alpha(i)\| \\
&= 1 - \sum_{i\neq i_0} \|x^*(i)\| \quad \text{[by (3.2)]}
\end{aligned}$$

Thus, $1 = \|x^*\| = \sum_{i\in\Gamma} \|x^*(i)\| < 1$, a contradiction. Hence the claim is proved.

Since $x^*(i)/\|x^*(i)\|$ is a w^*-PC of $B_{X_i^*}$, $i \in \Gamma_0$, it now follows that $x^*_\alpha(i) \longrightarrow x^*(i)$ in norm for all $i \in \Gamma$. Now, for $\varepsilon > 0$, there exists a finite set $A \subseteq \Gamma_0$ with N_0 elements such that $\sum_{i\notin A} \|x^*(i)\| \leq \varepsilon/4$. Also since $x^*_\alpha(i) \longrightarrow x^*(i)$ in norm for all i, there exists α_0 such that $\|x^*_\alpha(i) - x^*(i)\| < \varepsilon/4N_0$ for all $i \in A$, for all $\alpha \geq \alpha_0$. Now, for all $\alpha \geq \alpha_0$,

$$\begin{aligned}
&\Big|\|x^*_\alpha - x^*\| + \|x^*\| - \|x^*_\alpha\|\Big| \\
&= \Big|\sum_{i\in A} \|(x^*_\alpha - x^*)(i)\| + \sum_{i\notin A} \|(x^*_\alpha - x^*)(i)\| \\
&\qquad + \sum_{i\in A} \|x^*(i)\| + \sum_{i\notin A} \|x^*(i)\| - \sum_{i\in A} \|(x^*_\alpha(i)\| - \sum_{i\notin A} \|(x^*_\alpha(i)\|\Big|. \\
&= \Big|(\sum_{i\notin A} \|(x^*_\alpha - x^*)(i)\| - \sum_{i\notin A} \|(x^*_\alpha(i)\|) \\
&\qquad + (\sum_{i\in A} \|x^*(i)\| - \sum_{i\in A} \|(x^*_\alpha(i)\|) + \sum_{i\in A} \|(x^*_\alpha - x^*)(i)\| + \sum_{i\notin A} \|x^*(i)\|\Big| \\
&< \sum_{i\in A} 2\|(x^*_\alpha - x^*)(i)\| + \sum_{i\notin A} 2\|x^*(i)\| \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon
\end{aligned}$$

Hence $x_\alpha^* \longrightarrow x^*$ in norm and consequently x^* is a w*-PC. ∎

Remark 3.2.10 The last part of the proof of the above proposition is adapted from Yost's arguments in [Y, Lemma 9].

Thus we have the following theorem which we prove using the same technique as in [BRo]. However, we include the proof for completeness.

Theorem 3.2.11 *Let $(X_i)_{i\in\Gamma}$ be a family of Banach spaces and $X = \bigoplus_{c_0(\Gamma)} X_i$. Then X has Property (II) (respectively w*-ANP-II) if and only if each X_i has Property (II) (respectively w*-ANP-II.)*

Proof : For Property (II).

First suppose X_i has Property (II) for all $i \in \Gamma$. Let $x^* = (x^*(i))_{i\in\Gamma} \in S_{X^*}$. Fix $\varepsilon > 0$. Let $\Gamma_0 = \{i \in \Gamma : x^*(i) \neq 0\}$. Then, for $i \in \Gamma_0$, $x^*(i)/\|x^*(i)\| \in S_{X_i^*}$. Since each X_i has Property (II), there exists $y^*(i)$ a w*-PC of $B_{X_i^*}$ such that $\|x^*(i)/\|x^*(i)\| - y^*(i)\| < \varepsilon$.

Define $z^* = (z^*(i))_{i\in\Gamma}$ by

$$z^*(i) = \begin{cases} \|x^*(i)\| y^*(i) & \text{if} \quad i \in \Gamma_0 \\ 0 & \text{if} \quad i \notin \Gamma_0 \end{cases}.$$

Then by Proposition 3.2.9, it follows that z^* is a w*-PC of B_{X^*} and $\|x^* - z^*\| < \varepsilon$.

Conversely, suppose X has Property (II). Let $i_0 \in \Gamma$ and $x^*(i_0) \in S_{X_{i_0}^*}$. Fix $0 < \varepsilon < 1$. Define $x^* = (x^*(i))$ by

$$x^*(i) = \begin{cases} x^*(i_0) & \text{if} \quad i = i_0 \\ 0 & \text{if} \quad i \neq i_0 \end{cases}.$$

Then $x^* \in S_{X^*}$ and hence there exists a w*-PC of B_{X^*} such that $\|x^* - y^*\| < \varepsilon$. Hence, $\|x^*(i_o) - y^*(i_0)\| < \varepsilon$, so, $y^*(i_0) \neq 0$. Again by Proposition 3.2.9,

$y^*(i_0)/\|y^*(i_0)\|$ is a w*-PC of $B_{X_{i_0}^*}$. Now,

$$\begin{aligned} \|x^*(i_o) - y^*(i_0)/\|y^*(i_0)\|\| &< \|x^*(i_o) - y^*(i_0)\| + |1 - \|y^*(i_0)\|| \\ &< \varepsilon + \big|\|x^*(i_0)\| - \|y^*(i_0)\|\big| \\ &\leq \varepsilon + \|x^*(i_0) - y^*(i_0)\| \\ &< 2\varepsilon \end{aligned}$$

Hence X_{i_0} has Property (II).

For w*-ANP-II.

Suppose X has w*-ANP-II. Then the latter being hereditary, it follows that X_i has w*-ANP-II for all $i \in \Gamma$.

Conversely, suppose X_i has w*-ANP-II. Let $x^* = (x^*(i)) \in S_{X^*}$. Since X_i has w*-ANP-II, it follows that either $x^*(i) = 0$ or $x^*(i)/\|x^*(i)\|$ is a w*-PC of $B_{X_i^*}$. Consequently it follows from Proposition 3.2.9 that x^* is a w*-PC of B_{X^*}. Hence X has w*-ANP-II. ■

Remark 3.2.12 (a) Since ℓ_1 (resp. ℓ_∞) is not strictly convex, it clearly follows from Theorem 1.0.12 and Theorem 2.1.18 that w*-ANP-I and w*-ANP-II$'$ are not stable under c_0-sum (resp. ℓ_1-sum).

(b) As noted in [Ra], the ℓ_1-direct sum of spaces with Namioka-Phelps Property always fails Namioka-Phelps Property.

(c) Also, Property (II) is not stable under ℓ_∞ sum. In fact, ℓ_∞ does not have Property (II) since it is a non-reflexive dual space.

(d) It follows from [BRo], that the MIP is not stable under ℓ_1 or c_0 sums.

An argument similar to Corollary 3.1.12 shows that

Corollary 3.2.13 *The w*-ANP-κ (κ = I, II, II$'$) is not a three space property.*

Remark 3.2.14 (a) It has been already noted in [B1], that MIP is not a three space property.

(b) In their paper [HL4], Hu and Lin showed that if Y is a subspace of X such that X/Y has ANP-I, then X has ANP-κ (κ = I, II, III) if and only if and Y has ANP-κ (κ = I, II, III). Similar conclusion follows for ANP-II$'$ also.

One can ask the following :

Question 3.2.15 *Suppose Y and X/Y have ANP-κ (κ = I, II, II$'$, III), then does X have ANP-κ (κ = I, II, II$'$, III)?*

We now consider Property (II) for ℓ_p $(1 < p < \infty)$ direct sums.

Proposition 3.2.16 *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces. Then $X = \bigoplus_{\ell_p} X_i$ $(1 < p < \infty)$ has Property (II) if and only if for each $i \in I$, X_i has Property (II).*

Proof : Since $X^* = \bigoplus_{\ell_q} X_i^*$, where $1/p + 1/q = 1$, and $x^* \in S_{X^*}$ is a w*-PC of B_{X^*} if and only if for each $i \in I$, either $x_i^* = 0$ or $x_i^*/\|x_i^*\|$ is a w*-PC of $B_{X_i^*}$ [HL3, Proposition 2.14], the proof follows similarly as the proof of Theorem 3.2.11. ∎

It is known that if (Ω, Σ, μ) is a non-atomic measure space, then $f \in S_{L_p(\mu,X)^*}$ is w*-PC if and only if it is a w*-denting point of $B_{L_p(\mu,X)^*}$ [HL3]. We use the following proposition to deduce our next result.

Proposition 3.2.17 [BRo, Corollary 12] *Let X be a Banach space, λ denote the Lebesgue measure on $[0,1]$ and $1 < p < \infty$. The space $L_p(\lambda, X)$ has the MIP if and only if X has the MIP and is Asplund.*

And we immediately have

Corollary 3.2.18 *Let X be a Banach space, λ denote the Lebesgue measure on $[0,1]$ and $1 < p < \infty$. The space $L_p(\lambda, X)$ has Property (II) if and only if it has the MIP if and only if X has the MIP and is Asplund.*

Remark 3.2.19 It follows that there exists space with Property (II) such that $L_p(\lambda, X)$ does not have Property (II). Clearly any finite dimensional space which does not have MIP (e.g., $\mathbb{R}^n$ with ℓ_1 or sup norm) serves as an example.

Proposition 3.2.20 *Let X, Y, Z be infinite dimensional Banach spaces such that $X = Y \bigoplus_1 Z$. $(y^*, z^*) \in S_{X^*}$ is a w*-PC if and only if $\|y^*\| = 1$, $\|z^*\| = 1$ and y^*, z^* are w*-PC's of B_{Y^*} and B_{Z^*} respectively.*

Proof : First, let $\|y^*\| = \|z^*\| = 1$, and y^*, z^* are w*-PC's. Then obviously (y^*, z^*) is a w*-PC.

Conversely, suppose (y^*, z^*) is a w*-PC of B_{X^*}. Let $\{y^*_\alpha\}$ be a net in S_{Y^*} such that $y^*_\alpha \longrightarrow y^*$ in the w*-topology. Thus $\|(y^*_\alpha, z^*)\| = 1$ and $(y^*_\alpha, z^*) \longrightarrow (y^*, z^*)$ in the w*-topology. This implies $(y^*_\alpha, z^*) \longrightarrow (y^*, z^*)$ in norm. This in turn implies that $y^*_\alpha \longrightarrow y^*$ in norm. This also implies $\|y^*\| = 1$. Similarly for z^*. ∎

Now we readily have

Corollary 3.2.21 *Let X be an infinite dimensional Banach space. Then the following are true :*

(a) *If X has Property (II), then X has no non-trivial L-projections.*

(b) *If every equivalent renorming of X has Property (II), then X is reflexive.*

(c) *Property (II) is not a three space property.*

Let us now look at Banach spaces which are L_1-preduals with Property (II).

We need the following characterization of L_1-preduals due to Lima [L1]. This result is also noted in [HWW].

Theorem 3.2.22 [L1, Theorem 5.8] *A real or a complex Banach space is an L_1-predual, if and only if for all $f \in extB_{X^*}$, $span\{f\}$ is an L-summand in X^*.*

Theorem 3.2.23 *If X is an L_1-predual with Property (II) then X^* is isometric to $\ell_1(\Gamma)$ for some Γ.*

Proof : Let $x^* \in extB_{X^*}$, then $-x^* \in extB_{X^*}$ also. Hence there exists $A \subseteq extB_{X^*}$ such that $A \bigcap -A = \emptyset$ and $A \bigcup -A = extB_{X^*}$. Now $B_{X^*} = \overline{aco}^{w^*}(A)$. By Lemma 3.2.6, $B_{X^*} = \overline{aco}(A)$.

Since X is an L_1-predual, it follows from Theorem 3.2.22, that for each $f \in A$, $span\{f\}$ is an L-summand. Thus, for any $f_1, f_2, \ldots, f_n \in A$, $B_{(span\{f_1, f_2, \ldots, f_n\})} = co\{\pm f_i : i = 1, \ldots, n\}$. Thus $\Phi : \ell_1(A) \longrightarrow X^*$ defined by $\Phi(\alpha) = \sum \alpha(f) \cdot f$ is a linear isometry.

We shall show that Φ is onto.

Since Φ is an isometry, $\Phi(\ell_1(A)$ is norm closed in X^*. Clearly, elements of X^* of the form $\sum_{i=1}^{n} \lambda_i f_i$ (with $\sum |\lambda_i| = 1, f_i \in A, i = 1 \ldots n$) are in $\Phi(\ell_1(A))$, and since $B_{X^*} = \overline{aco}(A)$, such elements are dense in B_{X^*}. Thus $\Phi(\ell_1(A))$ must contain B_{X^*}, and hence, X^*. ∎

Remark 3.2.24 One can perhaps try to classify Property (II) among L_1-preduals. It is known that if X is an L_1-predual and $Y \subseteq X$ is a separable subspace of X, then there exists Z a separable L_1-predual such that $Y \subseteq Z \subseteq X$ [L1]. A similar result is true for Property (II) also [CL]. So one can ask the following

Question 3.2.25 *Given X is an L_1-predual with Property (II) and $Y \subseteq X$, Y separable, does there exist a separable Z which is an L_1-predual with Property (II), such that $Y \subseteq Z \subseteq X$?*

Chapter 4

On Nicely Smooth Spaces

4.1 The Relations between Property (II), BGP and Nice Smoothness in Banach spaces

The following lemma is well-known and will be used in the subsequent discussions.

Lemma 4.1.1 [GS, Lemma 2.4] *Let X be a Banach space. Then the following are equivalent :*

(a) *For all $x^{**} \in X^{**}$,*

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] = \{x^{**}\}$$

(b) *For all $x^{**} \in X^{**} \setminus X$,*

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] \bigcap X = \emptyset$$

(c) *X^* contains no proper norming subspace of X.*

Proposition 4.1.2 *For a Banach space X, the following are equivalent :*

(a) *X is nicely smooth.*

(*b*) $C_X = \{0\}$.

(*c*) *For all* $x^{**} \in X^{**}$,

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] = \{x^{**}\}$$

(*d*) *Every norming set* $A \subseteq B(X^*)$ *separates points of* X^{**}.

Proof : Equivalence of (a) and (c) follows from the definition of nicely smooth spaces and the equivalence of (a) and (c) in Lemma 4.1.1.

$(a) \Rightarrow (b)$. Let $x^{**} \in C_X$. Then by Lemma 1.0.17, it follows that $ker x^{**}$ is a norming subspace of X^*. Since X is nicely smooth it has no proper norming subspace. Hence $X^{**} = ker x^{**}$ which implies $x^{**} = 0$.

$(b) \Rightarrow (c)$. In view of $(a) \Rightarrow (b)$ in Lemma 4.1.1, it is enough to prove for all $x^{**} \in X^{**} \setminus X$,

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] \bigcap X = \emptyset$$

Let $y \in \bigcap_{x \in X} B[x, \|x^{**} - x\|] \bigcap X$. This implies

$$\begin{aligned} \|y - x\| &\leq \|x^{**} - x\| \quad \text{for all} \quad x \in X \\ \text{i.e.,} \quad \|x\| &\leq \|x^{**} - x - y\| \quad \text{for all} \quad x \in X \end{aligned}$$

which implies $x^{**} - y \in C_X = \{0\}$, or in other words, $x^{**} = y \in X^{**} \setminus X$, a contradiction.

$(a) \Rightarrow (d)$. Let A be a norming set for X. Suppose there exists $x^{**} \neq 0$ such that $x^{**}|_A = 0$. This implies $x^{**}|_{span(A)} = 0$, hence $x^{**}|_{X^*} = 0$, a contradiction.

$(d) \Rightarrow (a)$. Suppose F is a proper norming subspace. Then there exists a non-zero $x^{**} \in X^{**}$ such that $x^{**}|_F = 0$. But this implies B_F, a norming set, does not separate points of X^{**}, a contradiction. ∎

Remark 4.1.3 Godefroy observed in [G3] that if a separable space is nicely smooth, then it has a separable dual. And a dual space is nicely smooth if and only if it is reflexive.

We now identify some necessary and some sufficient conditions for a space to be nicely smooth.

Proposition 4.1.4 *For a Banach space X, consider the following statements :*

(a) X^* is the closed linear span of the w^*-weak PCs of B_{X^*}.

(b) Any two distinct points in X^{**} are separated by disjoint closed balls having centres in X.

(c) X is nicely smooth.

(d) For every norm dense set $A \subseteq S_X$ and every support mapping ϕ, the set $\phi(A)$ separates points of X^{**}.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Proof : $(a) \Rightarrow (b)$. Let $x_0^{**} \neq y_0^{**}$. By (a), there exists a w*-w PC in B_{X^*}, such that $(x_0^{**} - y_0^{**})(x_0^*) > 0$. Let m be such that

$$x_0^{**}(x_0^*) > m > y_0^{**}(x_0^*)$$

Hence $x_0^{**} \notin {x_0^*}^{-1}(m) = \{x^{**} \in X^{**} : x_0^*(x^{**}) = m\} = H$ (say). Now applying Theorem 2.3.7, it follows that there exists a ball B_1^{**} with centre in X with $x_0^{**} \in B_1^{**}$ and $B_1^{**} \bigcap H = \emptyset$. Similarly there exists a ball B_2^{**} with centre in X such that $y_0^{**} \in B_2^{**}$ and $B_2^{**} \bigcap H = \emptyset$. It is now clear that $B_1^{**} \bigcap B_2^{**} = \emptyset$.

$(b) \Rightarrow (c)$. By Proposition 4.1.2, it suffices to show that for all $x^{**} \in X^{**}$,

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] = \{x^{**}\}$$

If possible, let $y^{**} \neq x^{**}$ and $y^{**} \in \bigcap_{x \in X} B[x, \|x^{**} - x\|]$. This implies

$$\|y^{**} - x\| \leq \|x^{**} - x\| \text{ for all } x \in X \tag{4.1}$$

By (b), it follows that there exists balls B_1 and B_2 with centres in X such that $x^{**} \in B_1$, $y^{**} \in B_2$ and $B_1 \bigcap B_2 = \emptyset$. Without loss of generality we can choose

$$B_1 = B(x_1, \|x_1 - x^{**}\|).$$

Clearly $y^{**} \notin B_1$, but this contradicts (4.1).

$(c) \Rightarrow (d)$. We only need to observe that $\phi(A)$ is a norming set for X. The result now follows from Proposition 4.1.2 (d). ■

Corollary 4.1.5 *If in the setup of Proposition 4.1.4, we have in addition that w^*-w PC's of B_{X^*} form a norming set, then (a), (b), (c) are equivalent. And under the even stronger assumption,*

$$\{x \in S_X : D(x) \text{ intersects } w^*\text{-}w \text{ PC's of } B_{X^*}\}$$

is dense in S_X, all the statements in Proposition 4.1.4 are equivalent.

Proof : The first statement being easy, we need only to prove $(d) \Rightarrow (a)$ in Proposition 4.1.4 in the second case. Let

$$\begin{aligned} A &= \{\text{w}^*\text{-w PC's of } B_{X^*}\} \\ \text{and} \qquad B &= \{x \in S_X : D(x) \bigcap A \neq \emptyset\} \end{aligned}$$

Then B is dense in S_X and there is a support mapping ϕ, such that for each $x \in B$, $\phi(x) \in A$. By Proposition 4.1.4 (d), $\phi(B)$ separates points of X^{**}. This implies $\overline{span}[\phi(B)] = X^*$ and hence, $\overline{span} A = X^*$. ■

Remark 4.1.6 We do not know whether the implications of Proposition 4.1.4, can be reversed. It seems to be an interesting question to investigate the class of Banach spaces satisfying the conditions in Corollary 4.1.5.

In the next few results, we use the terminology of [GK] relating to the ball topology, i.e., the weakest topology making all closed balls closed. We have the following characterization of the BGP.

Proposition 4.1.7 *X has the BGP if and only if every $x^* \in X^*$ is ball-continuous on B_X.*

Proof : Suppose X has the BGP. Let $f \in X^*$ and I be any closed interval in $\mathbb{R}$, then $f^{-1}(I) \bigcap B_X$ is ball generated. Hence f is ball continuous on B_X.

For the converse, we simply observe that in this case, the ball topology coincides with the weak topology on B_X. ∎

Remark 4.1.8 The above proof is adapted from [GK, Theorem 8.3].

The following result is a slight alteration of [CL1, Theorem 1].

Theorem 4.1.9 *X has the BGP if and only if for every $x^* \in B_{X^*}$ and $\varepsilon > 0$, there exists w^*-slices $S_1, S_2, \ldots, S_n$ of B_{X^*} such that for any $(x_1^*, x_2^*, \ldots, x_n^*) \in \prod_{i=1}^n S_i$, there are scalars $a_1, a_2, \ldots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x_i^*\| \leq \varepsilon$.*

Definition 4.1.10 A point x_o^* in a convex set $K \subseteq X^*$ is called a w*-small combination of slices (SCS) point of K, if for every $\varepsilon > 0$, there exist w*-slices $S_1, S_2, \ldots, S_n$ of K, and a *convex* combination $S = \sum_{i=1}^n \lambda_i S_i$ such that $x_o^* \in S$ and $\operatorname{diam}(S) < \varepsilon$.

Proposition 4.1.11 *If X^* is the closed linear span of the w^*-SCS points of B_{X^*}, then X has the BGP.*

Proof : Let $x^* \in X^*$ and $\varepsilon > 0$. Since the set of w*-SCS points of B_{X^*} is symmetric and spans X^*, there exist w*-SCS points $x_1^*, x_2^*, \ldots, x_n^*$ of B_{X^*}, and positive scalars $a_1, a_2, \ldots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x_i^*\| \leq \varepsilon/2$. By definition of w*-SCS points, for each $i = 1, 2, \ldots, n$, there exist w*-slices $S_{i1}, S_{i2}, \ldots, S_{im_i}$ of B_{X^*}, and a convex combination $S_i = \sum_{k=1}^{m_i} \lambda_{ik} S_{ik}$ such that $x_i^* \in S_i$ and $\operatorname{diam}(S_i) < \varepsilon/(2\sum_{i=1}^n a_i)$. Now, for any $(x_{ik}^*)_{1\leq i\leq n, 1\leq k\leq m_i} \in \prod_{i=1}^n \prod_{k=1}^{m_i} S_{ik}$,

$$\begin{aligned} \|x^* - \sum_{i=1}^n \sum_{k=1}^{m_i} a_i \lambda_{ik} x_{ik}^*\| &\leq \|x^* - \sum_{i=1}^n a_i x_i^*\| + \sum_{i=1}^n a_i \|x_i^* - \sum_{k=1}^{n} \lambda_{ik} x_{ik}^*\| \\ &\leq \varepsilon/2 + \sum_{i=1}^n a_i \operatorname{diam}(S_i) \leq \varepsilon. \end{aligned}$$

Hence by Theorem 4.1.9, X has the BGP. ∎

Remark 4.1.12 This gives a weaker sufficient condition for the BGP than the ones discussed in [CHL, Theorem 7]. See Corollary 4.1.15 below.

We need the following lemma due to J. Bourgain for our next result.

Lemma 4.1.13 [Ro, Lemma 1.5] *Let E be a locally convex space, K a bounded convex subset of E, and W a weak neighbourhood in K. Then W contains a convex combination of slices of K. That is, there exist $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum \lambda_i = 1$ and $\lambda_i > 0$ for all i, and slices $S_1, \ldots, S_n$ of K, so that $\sum \lambda_i S_i \subseteq W$.*

Corollary 4.1.14 *Property (II) implies the BGP which, in turn, implies nicely smooth.*

Proof : Recall that X has Property (II) if and only if w*-PCs of B_{X^*} are norm dense in S_{X^*}, and that a w*-PC is necessarily a w*-SCS point (this follows from Lemma 4.1.13). Thus, Property (II) implies the BGP.

That the BGP implies nicely smooth is proved in [GK]. But here is an elementary proof.

Let F be a norming subspace of X^*. Then B_X is $\sigma(X, F)$-closed, so that every ball-generated set is also $\sigma(X, F)$-closed. But if every closed bounded convex set is $\sigma(X, F)$-closed, then $F = X^*$. ∎

Corollary 4.1.15 *If X is an Asplund space (or, separable), the following are equivalent :*

(a) X^* is the closed linear span of the w-strongly exposed points of B_{X^*}.*

(b) X^* is the closed linear span of the w-denting points of B_{X^*}.*

(c) X^* is the closed linear span of the w-SCS points of B_{X^*}.*

(d) X has the BGP.

(e) X^* is the closed linear span of the w-weak PCs of B_{X^*}.*

(f) Any two distinct points in X^{**} are separated by disjoint closed balls having centres in X.

(*g*) *X is nicely smooth.*

(*h*) *For every norm dense set $A \subseteq S_X$ and every support mapping ϕ, the set $\phi(A)$ separates points of X^{**}.*

Proof : Clearly, $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (g)$, and $(b) \Rightarrow (e)$. And from Proposition 4.1.4, $(e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h)$.

Now if X is Asplund (if X is separable, (h) implies X^* is separable), then for $A = \{x \in S_X : \text{the norm is Fréchet differentiable at } x\}$, and any support mapping ϕ, $\phi(A) = \{\text{w}^*\text{-strongly exposed points of } B(X^*)\}$. Hence, $(h) \Rightarrow (a)$. ∎

Remark 4.1.16 In this case, the conditions in Corollary 4.1.5 are satisfied.

Theorem 4.1.17 [CHL, Theorem 12] *Let X be an infinite dimensional Banach space. If all the separable infinite dimensional subspaces of X have the BGP then X has the BGP.*

And we immediately have

Proposition 4.1.18 *If every separable subspace of X is nicely smooth, then X has the BGP, and hence, is nicely smooth.*

Theorem 4.1.19 *X is nicely smooth with FIP if and only if X is reflexive.*

Proof : Sufficiency is obvious from weak compactness of closed balls in reflexive spaces.

For necessity, recall from Theorem 1.0.19 that X has FIP if and only if $X^{**} = X + C_X$. Since X is nicely smooth, $C_X = \{0\}$, and consequently, X is reflexive. ∎

Remark 4.1.20 Since Hahn-Banach smooth spaces (resp. spaces with Property (II)) are nicely smooth, Theorem 3.1.23 (resp. Theorem 3.2.5) of Chapter 3 follows as a corollary.

Theorem 4.1.21 *A Banach space X is reflexive if and only if every equivalent renorming is nicely smooth.*

Proof : The converse being trivial, suppose X is not reflexive. Let $x^{**} \in X^{**} \setminus X$ and let $F = \{x^* \in X^* : x^{**}(x^*) = 0\}$. Define a new norm on X by

$$\|x\|_1 = \sup\{x^*(x) : x^* \in B_F\} \quad \text{for } x \in X$$

Then $\|\cdot\|_1$ is a norm on X with F as a proper norming subspace.

Claim: $\|\cdot\|_1$ is an equivalent norm on X.

Clearly $\|\cdot\|_1 \leq \|\cdot\|$.

Conversely, by standard duality relations, for every $x \in X$, $\|x\|_1 = \|\hat{x}|_F\| = d(x, E)$, where $E = span\{x^{**}\}$. Let $Y = X^{**}/E$. The map $T : X \longrightarrow Y$ defined by $Tx = [x] = x + E$ is a one-one continuous linear map.

We check that $T(X)$ is a closed, and hence complete, subspace of Y. Let $x_n + E \longrightarrow x_0^{**} + E$. We will show that there exists $x_0 \in X$ such that $x_0^{**} + E = x_0 + E$.

Since $\|x_n - x_0^{**} + E\| \longrightarrow 0$, we can find scalars λ_n such that $\|x_n - x_0^{**} + \lambda_n x^{**}\| \longrightarrow 0$. If $\{\lambda_n\}$ is unbounded, passing through a subsequence if necessary, $\|x_n/\lambda_n + x^{**}\| \longrightarrow 0$, i.e., $x^{**} \in X$, a contradiction. And if $\{\lambda_n\}$ is bounded, again passing through a subsequence if necessary, $\lambda_n \longrightarrow \lambda$. Then $x_n \longrightarrow x_0^{**} - \lambda x^{**} = x_0 \in X$ (say). It is clear that this is the required x_0.

It now follows from the Open Mapping Theorem that T is an open map. Thus there exists a constant M such that

$$\|x\| \leq M\|[x]\| = M\|x\|_1.$$

Hence the claim. ∎

Remark 4.1.22 The main idea of the proof of the above theorem has been adapted from the proof of [GK, Theorem 8.2]. The details are supplied by us.

4.2 Stability Results

Now we obtain some stability results for nicely smooth spaces.

Theorem 4.2.1 *Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a family of Banach spaces. Then $X = \bigoplus_{\ell_p} X_\alpha$ $(1 < p < \infty)$ is nicely smooth if and only if for each $\alpha \in \Gamma$, X_α is nicely smooth.*

Proof : We will show that $C_X = \{0\}$ if and only if for every $\alpha \in \Gamma$, $C_{X_\alpha} = \{0\}$. And the rest follows from Proposition 4.1.2. Now, $X = \bigoplus_{\ell_p} X_\alpha$ implies $X^{**} = \bigoplus_{\ell_p} X_\alpha^{**}$, and $x^{**} \in C_X$ if and only if

$$\begin{aligned} \|x^{**} + \hat{x}\|_p &\geq \|x\|_p \text{ for all } x \in X \\ \Longleftrightarrow \sum_{\alpha\in\Gamma} \|x_\alpha^{**} + \hat{x}_\alpha\|^p &\geq \sum_{\alpha\in\Gamma} \|x_\alpha\|^p \text{ for all } x \in X \end{aligned}$$

It is immediate that if for every $\alpha \in \Gamma$, $x_\alpha^{**} \in C_{X_\alpha}$, then $x^{**} \in C_X$. And hence, $C_X = \{0\}$ implies for every $\alpha \in \Gamma$, $C_{X_\alpha} = \{0\}$.

Conversely, suppose for every $\alpha \in \Gamma$, $C_{X_\alpha} = \{0\}$. Let $x^{**} \in X^{**} \setminus \{0\}$. Let $\alpha_o \in \Gamma$ be such that $x_{\alpha_o}^{**} \neq 0$. Then $x_{\alpha_o}^{**} \notin C_{X_{\alpha_o}}$. Hence, there exists $x_{\alpha_o} \in X_{\alpha_o}$ such that $\|x_{\alpha_o}^{**} + \hat{x}_{\alpha_o}\| < \|x_{\alpha_o}\|$. Choose $\varepsilon > 0$ such that $\|x_{\alpha_o}^{**} + \hat{x}_{\alpha_o}\|^p + \varepsilon < \|x_{\alpha_o}\|^p$. Now for this $\varepsilon > 0$, there exists a finite $\Gamma_o \subseteq \{\alpha \in \Gamma : x_\alpha^{**} \neq 0\}$ such that $\alpha_o \in \Gamma_o$ (if $\alpha_0 \notin \Gamma_0$, replace Γ_0 by $\Gamma_0 \bigcup \{\alpha_0\}$). $\sum_{\alpha\notin\Gamma_o} \|x_\alpha^{**}\|^p < \varepsilon$. If $\alpha \in \Gamma_o$, then $x_\alpha^{**} \notin C_{X_\alpha}$. Hence, there exists $x_\alpha \in X_\alpha$ such that $\|x_\alpha^{**} + \hat{x}_\alpha\| < \|x_\alpha\|$. Define $y \in X$ by

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \quad \alpha \in \Gamma_o \\ 0 & \text{otherwise} \end{cases}.$$

Then we have,

$$\begin{aligned} \|x^{**} + \hat{y}\|_p^p &= \sum_{\alpha\in\Gamma} \|x_\alpha^{**} + \hat{y}_a\|^p \\ &= \sum_{\substack{\alpha\in\Gamma_o \\ \alpha\neq\alpha_0}} \|x_\alpha^{**} + \hat{x}_a\|^p + \|x_{\alpha_o}^{**} + \hat{x}_{\alpha_o}\|^p + \sum_{\alpha\notin\Gamma_o} \|x_\alpha^{**}\|^p \end{aligned}$$

$$< \sum_{\substack{\alpha\in\Gamma_o \\ \alpha\neq\alpha_0}} \|x_\alpha\|^p + \|x^{**}_{\alpha_o} + \hat{x}_{\alpha_o}\|^p + \varepsilon$$

$$< \sum_{\alpha\in\Gamma_o} \|x_\alpha\|^p = \|y\|_p^p$$

which shows that $x^{**} \notin C_X$. ∎

Remark 4.2.2 (a) The above argument also works for finite ℓ_1 (or ℓ_∞) sums and shows that if X is the ℓ_1 (or ℓ_∞) sum of $X_1, X_2, \ldots, X_n$, then X is nicely smooth if and only if for every coordinate space X_i is so.

However, if Γ is infinite, $X = \bigoplus_{\ell_1} X_\alpha$ is never nicely smooth as $\bigoplus_{c_o} X_\alpha^*$ is a proper norming subspace of $X^* = \bigoplus_{\ell_\infty} X_\alpha^*$.

A similar argument also shows that being nicely smooth is not stable under infinite ℓ_∞ sums.

(b) Since Property (II) is not preserved under finite ℓ_1 sums, the space $c_0 \oplus_1 c_0$ produces an example of a nicely smooth space, which being Asplund has BGP, but lacks Property (II).

We now show that being nicely smooth is stable under c_o sums.

Theorem 4.2.3 *Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a family of Banach spaces. Then $X = \bigoplus_{c_o} X_\alpha$ is nicely smooth if and only if for each $\alpha \in \Gamma$, X_α is nicely smooth.*

Proof : As before, we will show that $C_X = \{0\}$ if and only if for every $\alpha \in \Gamma$, $C_{X_\alpha} = \{0\}$.

Necessity is similar to that in Theorem 4.2.1.

Conversely, suppose for every $\alpha \in \Gamma$, $C_{X_\alpha} = \{0\}$. And let $x^{**} \in X^{**} \setminus \{0\}$. Let $\alpha_o \in \Gamma$ be such that $x^{**}_{\alpha_o} \neq 0$. Then $x^{**}_{\alpha_o} \notin C_{X_{\alpha_o}}$. Hence, there exists $x_{\alpha_o} \in X_{\alpha_o}$ such that $\|x^{**}_{\alpha_o} + \hat{x}_{\alpha_o}\| < \|x_{\alpha_o}\|$. Let

$$\|x^{**}\|_\infty = \sup_{\alpha\in\Gamma} \|x^{**}_\alpha\| = M \quad \text{(say)}.$$

Case 1. If $\|x_{\alpha_o}\| > M$, then define, $y \in X$ by

$$y_\alpha = \begin{cases} x_{\alpha_o} & \text{if} \quad \alpha = \alpha_o \\ 0 & \text{otherwise} \end{cases}$$

We have,

$$\|x^{**} + \hat{y}\|_\infty = \max\{\sup\{\|x^{**}_\alpha\|_{\alpha \neq \alpha_o}\}, \|x^{**}_{\alpha_o} + \hat{x}_{\alpha_o}\|\} < \|x_{\alpha_0}\|.$$

Hence $x^{**} \notin C_X$.

Case 2. If $\|x_{\alpha_0}\| < M$, there exists $\lambda > 1$ such that $\lambda\|x_{\alpha_0}\| > M$. Now define

$$y_\alpha = \begin{cases} \lambda x_{\alpha_o} & \text{if} \quad \alpha = \alpha_o \\ 0 & \text{otherwise} \end{cases}$$

then,

$$\|x^{**} + \hat{y}\|_\infty = \max\{\sup\{\|x^{**}_\alpha\|_{\alpha \neq \alpha_o}\}, \|x^{**}_{\alpha_o} + \lambda\hat{x}_{\alpha_o}\|\}$$

Now, $\|x^{**}_{\alpha_o} + \lambda\hat{x}_{\alpha_o}\| < \|x^{**}_{\alpha_o} + \hat{x}_{\alpha_o}\| + \| - \hat{x}_{\alpha_o} + \lambda\hat{x}_{\alpha_o}\| < (1 + \lambda - 1)\|\hat{x}_{\alpha_o}\|$. This implies $\|x^{**} + y\|_\infty < \lambda\|x_{\alpha_0}\| = \|y\|$, which again shows $x^{**} \notin C_X$. ∎

Corollary 4.2.4 *Nice smoothness is not a three space property.*

Proof : Let $X = c$, the space of all convergent sequences with the sup norm. Recall that $c^* = \ell_1$ and that ℓ_1 acts on c as

$$\langle a, x\rangle = a_0 \lim x_n + \sum_{n=0}^{\infty} a_{n+1}x_n, \quad a = \{a_n\}_{n=0}^{\infty} \in \ell_1,\ x = \{x_n\}_{n=0}^{\infty} \in c$$

Then $\{a \in \ell_1 : a_0 = 0\}$ is a proper norming subspace for c. Put $Y = c_o$. Then Y is nicely smooth and $\dim(X/Y) = 1$, so that X/Y is also nicely smooth. But, by above, X is not nicely smooth. ∎

Following the arguments of [BRo, Proposition 2], we now show that the BGP is inherited by M-summands (cf. Definition 3.1.6).

Proposition 4.2.5 *If Y is an M-summand in X and X has the BGP, then so does Y.*

Proof : Let K be a closed bounded convex set in Y. Since X has the BGP,

$$K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}],$$

where for each i and k, $K \cap B[x_{ik}, r_{ik}] \neq \emptyset$

Given i and k, let $x \in K \cap B[x_{ik}, r_{ik}] \subseteq Y$, then $\|x - x_{ik}\| \leq r_{ik}$, so that $\|x_{ik} - Px_{ik}\| = \|(x - x_{ik}) - P(x - x_{ik})\| \leq \|x - x_{ik}\| \leq r_{ik}$.

CLAIM : $K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B_Y[Px_{ik}, r_{ik}]$ ... $(*)$.

Since $\|P\| = 1$, we have

$$K = P(K) \subseteq P(\bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}]) \subseteq \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B_Y[Px_{ik}, r_{ik}].$$

Conversely, if x is in the RHS of $(*)$, for all $i \in I$, there exists k such that $\|x - x_{ik}\| = \max\{\|x - Px_{ik}\|, \|x_{ik} - Px_{ik}\|\} \leq r_{ik}$, as $\|x_{ik} - Px_{ik}\| \leq r_{ik}$. Thus, $x \in \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}] = K$. ■

We need the following lemma for our next result.

Lemma 4.2.6 [BRo, Lemma 10] *Let X be a Banach space, (Ω, Σ, μ) be a measure space and $1 < p < \infty$. Let $1/p + 1/q = 1$. A simple function of the form $g = \sum_{i=1}^{\infty} x_i^* \chi_{E_i} \in S_{L_q(\mu, X^*)}$ is a w^*-denting point of $B_{L_p(\mu,X)^*}$ if and only if for $i = 1, 2, \ldots$, $x_i^*/\|x_i^*\|$ is a w^*-denting point of B_{X^*}.*

Theorem 4.2.7 *Let X be a Banach space, μ denote the Lebesgue measure on $[0,1]$ and $1 < p < \infty$. Then the following are equivalent :*

(*a*) *$L_p(\mu, X)$ has BGP.*

(*b*) *$L_p(\mu, X)$ is nicely smooth.*

(c) *X is nicely smooth and Asplund.*

Proof : Clearly $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$. Since $L_q(\mu, X^*)$ is always a norming subspace of $L_p(\mu, X)^*$, $1/p + 1/q = 1$, and they coincide if and only if X^* has the RNP, (b) implies X^* has the RNP, or, X is Asplund. Also for any norming subspace $F \subseteq X^*$, $L_q(\mu, F)$ is a norming subspace of $L_p(\mu, X)^*$. Hence, (b) also implies X is nicely smooth.

$(c) \Rightarrow (a)$. If X is nicely smooth and Asplund, by Corollary 4.1.15, X^* is the closed linear span of the w*-denting points of B_{X^*}. And it suffices to show that $L_p(\mu, X)^* = L_q(\mu, X^*)$ is the closed linear span of the w*-denting points of $B_{L_q(\mu, X^*)}$.

Let $F = \sum_{i=1}^n \alpha_i x_i^* \chi_{A_i}$ with $x_i^* \in S_{X^*}$ for all $i = 1, 2, \ldots, n$ be a simple function in $S_{L_q(\mu, X^*)}$. Let $\varepsilon > 0$. Now, for each $i = 1, 2, \ldots, n$, there exists $\lambda_{ik} \in \mathbb{R}$, and x_{ik}^*, w*-denting points of B_{X^*}, $k = 1, 2, \ldots, N$, such that $\|x_i^* - \sum_{k=1}^N \lambda_{ik} x_{ik}^*\| < \varepsilon$. For $k = 1, 2, \ldots, N$. Define

$$F_k = \sum_{i=1}^n \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i}$$

Since each x_{ik}^* is a w*-denting points of B_{X^*}, for each k, it follows from Lemma 4.2.6 that $F_k / \|F_k\|$ is a w*-denting point of $B_{L_q(\mu, X^*)}$. And,

$$\begin{aligned}
\|F - \sum_{k=1}^N F_k\|_q^q &= \|\sum_{i=1}^n \alpha_i x_i^* \chi_{A_i} - \sum_{k=1}^N \sum_{i=1}^n \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i}\|_q^q \\
&= \sum_{i=1}^n |\alpha_i|^q \|x_i^* - \sum_{k=1}^N \lambda_{ik} x_{ik}^*\|^q \mu(A_i) \\
&< \sum_{i=1}^n \varepsilon^q |\alpha_i|^q \mu(A_i) \leq \varepsilon^q \|F\|_q^q \leq \varepsilon \qquad \blacksquare
\end{aligned}$$

Chapter 5

Ball Separation Properties in Spaces of Operators

5.1 Generalities on Vector Measures

Let K be a compact Hausdorff space and X a Banach space. Let $C(K,X)$ denote the set of all continuous functions defined on K taking values in X, where continuity is defined as follows :

f is continuous at $k_0 \in K$ if

$$\lim_{k \longrightarrow k_0} \|f(k) - f(k_0)\| = 0.$$

The norm on $C(K,X)$ is defined as

$$\|f\| = \sup\{\|f(k)\| : k \in K\}, \quad f \in C(K,X).$$

It is well-known that $C(K,X)$ is a Banach space with respect to this norm. In the particular case when $\dim(X) = 1$, we get the space $C(K)$. For a detailed discussion on $C(K,X)$ spaces one can refer to [Si].

Now we briefly discuss the analogue of Riesz's Representation Theorem in this general setup. Let $\mathcal{B}(K)$ denote the Borel subsets of K. We recall that a set function $\mu : \mathcal{B}(K) \longrightarrow X^*$ is called countably additive if for every pairwise disjoint Borel sets $\{K_n\} \subseteq K$, $\mu(\bigcup_{n=1}^{\infty} K_n) = \sum_{n=1}^{\infty} \mu(K_n)$, in the sense of convergence in the norm topology of X^*. Let $x \in X$. We define $\mu_x : \mathcal{B}(K) \longrightarrow I\!R$ as follows

$$\mu_x(A) = \mu(A)(x).$$

Then μ_x is a scalar measure defined on $\mathcal{B}(K)$. μ is said to be (weakly) regular if for each $x \in X$, μ_x is regular in the usual sense. One can similarly define weakly countably additive X^*-valued measures.

Let μ be a countably additive measure on $\mathcal{B}(K)$ taking values in X^*. Let $A \subseteq \mathcal{B}(K)$. Define

$$\operatorname*{Var}_{E \subseteq A}(\mu) = \sup \| \sum_{i=1}^{n} \mu(E_n) \|$$

where supremum is taken over all finite disjoint partitions $\{E_n\}$ of A and let $\|\mu\| = \operatorname*{Var}_{E \subseteq K}(\mu)$. μ is said to be of bounded variation if $\|\mu\| < \infty$. It is well-known that all countably additive X^*-valued measures form a Banach space with the variation norm. Now let us define the Gowurin integral for a function $f \in C(K, X)$.

Let $z = \sum_{i=1}^{n} x_i \chi_{E_i}$, $x_i \in X$, E_i's disjoint Borel subsets of K (χ_{E_i} is the characteristic function of E_i), $i = 1, \ldots, n$. z is called a simple function taking values in X. Define

$$\int_K \langle z(k), d\mu(k) \rangle = \sum_{i=1}^{n} \mu(E_i)(x_i)$$

It can be easily seen that any $z \in C(K, X)$ is the uniform limit of sequence $\{z_n\}$ of simple functions. So one can define

$$\int_K \langle z(k), d\mu(k) \rangle = \lim_n \int_K \langle z_n(k), d\mu(k) \rangle$$

and verify that the limit is independent of the choice of the sequence $\{z_n\}$. From this definition it immediately follows that

$$\left| \int_K \langle z(k), d\mu(k) \rangle \right| \leq \|z\| \|\mu\|.$$

The following representation theorem is well-known :

Theorem 5.1.1 *Let K be a compact Hausdorff space and X a Banach space. Then the space $C(K,X)^*$ is isometrically isomorphic to the space $M(K,X^*)$ of all X^*-valued countably additive (weakly) regular Borel measures μ, of bounded variation and endowed with the norm*

$$\|\mu\| = \underset{E \subseteq K}{Var}\, \mu$$

The correspondence is given by $\Phi \in C(K,X)^* \longleftrightarrow \mu \in M(K,X^*)$,

$$\Phi(z) = \int_K \langle z(k), d\mu(k)\rangle, \quad z \in C(K,X).$$

We add some remarks concerning supports of the vector measures described in the above theorem. Let μ be such a measure. If $\{E_i\}_{i\in I}$ is a collection of open sets with $\mu(E_i) = 0$ for all i, then it follows from the regularity of scalar measures μ_x that $\mu_x(E) = 0$, where $E = \bigcup_{i\in I} E_i$, for all $x \in X$, and hence $\mu(E) = 0$. Thus, it makes sense to define the (closed) support $S(\mu)$ of μ by

$$S(\mu) = K \setminus \bigcup\{E : E \text{ open}, \mu(E) = 0\}.$$

As in the scalar case $S(\mu)$ is the smallest closed set in K for which $\mu[S(\mu)] = \mu(K)$ and it is also characterized by the property :

$$\text{if } E \text{ is open in } K, \text{ and } E \cap S(\mu) \neq \emptyset \text{ then } \mu(E) \neq 0.$$

We define an *atom* $E \in \mathcal{B}(K)$ for μ in the usual way, i.e., $\mu(E) \neq 0$ and if $E' \subseteq E$, $E' \in \mathcal{B}(K)$ then either $\mu(E') = 0$ or $\mu(E') = \mu(E)$. It follows immediately that an atom for μ is also an atom for each μ_x, $x \in X$. We note that if μ is non-atomic, i.e., μ has no atoms, then $S(\mu)$ is a perfect set.

Now suppose that μ vanishes on singletons, i.e., $\mu(\{k\}) = 0$ for each $k \in K$. Then we claim that the vector measure μ has no atoms. For if not, suppose E is such an atom. Then by the previous remark, E is an atom for μ_x for each $x \in X$, and by a well-known result for scalar measures, $\mu_x(E) = \mu_x(p_x)$ for some $p_x \in E$. But then,

$$\langle x, \mu(E)\rangle = \mu_x(E) = \mu_x(p_x) = \langle x, \mu(\{p_x\})\rangle = 0$$

for all $x \in X$, whence $\mu(E) = 0$, a contradiction.

Definition 5.1.2 A topological space is said to be scattered if it has no perfect subsets.

For a detailed discussion on scattered spaces one can refer to [La].

We first prove the following useful representation of $C(K,X)^*$, when K is scattered.

Lemma 5.1.3 *Let K be a scattered compact Hausdorff space. Then for any Banach space X,*

$$C(K,X)^* = \oplus_{\ell_1(|K|)} X^*.$$

Proof : Let $\Phi \in C(K,X)^*$. Then by Theorem 5.1.1, there exists $\mu \in M(K,X^*)$, such that $\|\Phi\| = \|\mu\|$

Clearly, $\mu \in M(K,X^*)$ is purely atomic.

For, if not, μ will have a non-atomic part whose support will be a perfect set, a contradiction. Also, $\mu \in M(K,X^*)$ can take non-zero values for at most countably many points of K. Indeed, $\{k : \|\mu(\{k\})\| \geq 1/n\}$ has to be finite for each n, since $\|\mu\| < \infty$. Hence

$$\{k : \|\mu(\{k\})\| > 0\} = \bigcup_{n=1}^{\infty} \{k : \|\mu(\{k\})\| > \frac{1}{n}\}$$

is countable. Clearly, $\mu = \sum_{i=1}^{\infty} \mu(\{k_i\}) \otimes \delta(k_i)$, where $\mu(\{k_i\}) \otimes \delta(k_i)$ is given by

$$\mu(\{k_i\}) \otimes \delta(k_i)(x) = \mu(\{k_i\})(x(k_i)), \quad x \in C(K,X)$$

Hence $\mu \in \oplus_{\ell_1(|K|)} X^*$.

Conversely, any $(x_i^*) \in \oplus_{\ell_1(|K|)} X^*$ can be identified with some $\mu \in M(K,X^*)$ by defining $\mu(\{k_i\}) = x_i^*$ for some $k_i \in K$, the non-zero x_i^*'s being at most countable. Hence the lemma. ∎

5.2 Property (II) in $C(K, X)$ and $\mathcal{L}(X)$

The following result shows that for a scattered compact space K and a Banach space X, U-subspaces can be "lifted" from X to $C(K, X)$.

Proposition 5.2.1 *Let K be a scattered compact space and suppose Y is a U-subspace of X. Then $C(K,Y)$ is a U-subspace of $C(K, X)$.*

Proof : We only need to observe that if K is a scattered compact space, then by Lemma 5.1.3, $C(K, X)^* = \bigoplus_{\ell_1(\Gamma)} X^*$ for some discrete set Γ. The conclusion then follows from arguments identical to the proof of Theorem 3.1.13. ■

Remark 5.2.2 Unlike the situation for ℓ_1 direct sums considered in Proposition 3.1.5, in the case of the space $C(K, X)$, $C(K,Y)$ may be a U-subspace of $C(K, X)$ for some U-subspace Y of X (without any extra topological assumptions on the compact set K).

Example 5.2.3 *Let $Y \subsetneq X$ be a proper M-ideal (for example consider $X = \ell_\infty$ and $Y = c_0$). Then, for any compact Hausdorff space K, it is known [HWW, Proposition VI.3.1] that $C(K,Y)$ is an M-ideal in $C(K, X)$ and is thus a U-subspace.*

Now we look at Property (II) for $C(K, X)$ spaces. First we have the following result for the special case $C(K)$.

Proposition 5.2.4 *Let K be a compact Hausdorff space. Then $C(K)$ has Property (II) if and only if K is finite.*

Proof : Suppose $C(K)$ has Property (II). Now $\partial_e B_{C(K)^*} = \{\pm\delta(k) : k \in K\}$, then by Krein-Milman theorem, $B_{C(K)^*} = \overline{co}^{w^*}\{\pm\delta(k) : k \in K\}$. However since $C(K)$ has Property (II), it follows from Lemma 3.2.6 that this w*-closure is same as the norm closure. Now arguing similarly as in Theorem 3.2.23, it follows that

$C(K)^*$ is isometric to $\ell_1(|K|)$. Thus K does not support a non-atomic measure. Hence K is scattered and K', the set of isolated points of K, is dense in K.

Claim : $B_{C(K)^*} = \overline{co}^{w^*}\{\pm\delta(k') : k' \in K'\}$.

Indeed, let $k \in K$, there exists $k_\alpha \in K'$ such that $k_\alpha \longrightarrow k$ which implies $f(k_\alpha) \longrightarrow f(k)$ for all $f \in C(K)$, i.e., $\delta(k_\alpha) \xrightarrow{w^*} \delta(k)$. Hence

$$\overline{co}^{w^*}\{\pm\delta(k') : k' \in K'\} = \overline{co}^{w^*}\{\pm\delta(k) : k \in K\}.$$

Hence the claim. Since $C(K)$ has Property (II), by Lemma 3.2.6, we have $B_{C(K)^*} = \overline{co}\{\pm\delta(k') : k' \in K'\}$. Now, for an accumulation point $k \in K$,

$$\|\delta(k) - \sum_{i=1}^{n} \alpha_i \delta(k_i)\| = 1 + \sum_{i=1}^{n} |\alpha_i|, \quad k_i \in K'.$$

Thus $\delta(k)$ cannot be approximated in the norm by a sequence from $co\{\pm\delta(k') : k' \in K'\}$. This shows that $K' = K$ and hence K is finite.

Conversely, if K is finite, say $|K| = n$. Then $C(K)$ is isometric to ℓ_∞^n, so has Property (II). ∎

Proposition 5.2.5 *$C(K,X)$ has Property (II) if and only if X has Property (II) and K is finite.*

Proof : Suppose $C(K,X)$ has Property (II). Proceeding similarly as in Proposition 5.2.4, it follows that K is finite. Hence, $C(K,X) = \bigoplus_{c_0(|K|)} X$. Thus By Theorem 3.2.11, it follows that X has Property (II).

Conversely, suppose X has Property (II) and K is finite. Then $C(K,X) = \bigoplus_{c_0(|K|)} X$. Since X has Property (II), it follows from Theorem 3.2.11 that $C(K,X)$ has Property (II). ∎

We similarly have

Corollary 5.2.6 *$C(K,X)$ has w^*-ANP-II if and only if K is finite and X has w^*-ANP-II.*

Remark 5.2.7 (a) Since ℓ_1^n is not strictly convex, it follows that $C(K,X)$ does not have w*-ANP-κ ($\kappa = \mathrm{I}, \mathrm{II}'$), even if K is finite and X has w*-ANP-κ ($\kappa = \mathrm{I}, \mathrm{II}'$).

(b) Similarly, it follows that $C(K,X)$ does not have the MIP even if K is finite and X has the MIP.

We now consider Property (II) for spaces of operators $\mathcal{L}(X)$ on a Banach space X. Since this is not a hereditary property, it is not clear if $\mathcal{L}(X)$ has Property (II), then X and X^* should also have it (which in turn will force X to be reflexive). Our first result shows that under a mild approximation condition, finite dimensional spaces are the only ones for which $\mathcal{L}(X)$ has Property (II).

Theorem 5.2.8 *Let X be a Banach space such that there exists a bounded net $\{K_\alpha\}$ of compact operators such that $K_\alpha(x) \longrightarrow x$ weakly, for all $x \in X$. If $\mathcal{L}(X)$ has Property (II), then X is finite dimensional.*

Proof : For any $x \in X$, $x^* \in X^*$, if $x \otimes x^*$ denotes the functional defined on $\mathcal{L}(X)$ by $x \otimes x^*(T) = x^*(T(x))$, then $\|x^* \otimes x^*\| = \|x\|\|x^*\|$. Indeed,

$$\begin{aligned} \|x \otimes x^*\| &= \sup_{\|T\|=1} x^*(Tx) \\ &\leq \sup_{\|T\|=1} \|x^*\|\|Tx\| \\ &\leq \|x^*\|\|x\| \end{aligned}$$

Conversely, suppose $\|x_0^*\| = 1 = \|x_0\|$. Given, $\varepsilon > 0$, choose $\phi \in X^*$, $\|\phi\| = 1$, $\phi(x_0) > 1 - \varepsilon$ and choose $y_0 \in X$, $\|y_0\| = 1$, $x_0^*(y_0) > 1 - \varepsilon$. Define $T \in \mathcal{L}(X)$ by $Tx = \phi(x)y_0$. Then $\|T\| = \sup_{\|x\|\leq 1} |\phi(x)| = \|\phi\| = 1$ and $x_0^*(Tx_0) = x_0^*[\phi(x_0)y_0] = \phi(x_0)x_0^*(y_0) > (1-\varepsilon)^2$. And this implies $\|x_0 \otimes x_0^*\| \geq 1$. Hence

$$\|x \otimes x^*\| = \|x^*\|\|x\|.$$

Since $\|T\| = \sup_{\|x^*\|=1, \|x\|=1}(x^*(T(x)) = \sup_{\|x^*\|=1, \|x\|=1} x \otimes x^*(T)$, it follows that $A = \{x \otimes x^* : \|x^*\| = 1, \|x\| = 1\}$ determines the norm on $\mathcal{L}(X)$. Therefore by an application of the separation theorem, $B_{\mathcal{L}(X)^*} = \overline{co}^{w^*}(A)$. Since $\mathcal{L}(X)$ has Property (II), it follows from Lemma 3.2.6, $B_{\mathcal{L}(X)^*} = \overline{co}(A)$

Claim : $K_\alpha \longrightarrow I$ weakly.

Since, $B_{\mathcal{L}(X)^*} = \overline{co}(A)$, for any $\Phi \in B_{\mathcal{L}(X)^*}$ and any $\varepsilon > 0$, we have

$$\|\Phi - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*\| < \varepsilon, \quad 0 \le \alpha_i \le 1, \quad \sum_{i=1}^{n} \alpha_i = 1, \quad x_i \otimes x_i^* \in A.$$

i.e.,

$$\left|\Phi(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha)\right| < \varepsilon \|K_\alpha\|.$$

$$\begin{aligned}
\text{So,}\quad |\Phi(K_\alpha) - \Phi(I)| &= \Bigg|\Phi(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha) \\
&\quad + \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(I) \\
&\quad + \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(I) - \Phi(I)\Bigg| \\
&\le \left|\Phi(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha)\right| \\
&\quad + \left|\sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(I)\right| \\
&\quad + \left|\sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(I) - \Phi(I)\right| \\
&\le \varepsilon \|K_\alpha\| + \left|\sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(K_\alpha) - \sum_{i=1}^{n} \alpha_i x_i \otimes x_i^*(I)\right| + \varepsilon \\
&= \varepsilon \|K_\alpha\| + \left|\sum_{i=1}^{n} \alpha_i x_i^*(K_\alpha x_i) - \sum_{i=1}^{n} \alpha_i x_i^*(x_i)\right| + \varepsilon \\
&\le \varepsilon \sup_\alpha \|K_\alpha\| + \varepsilon + \varepsilon
\end{aligned}$$

for all $\alpha \geq \alpha_0$ for some α_0 (since $K_\alpha x \longrightarrow x$ weakly). Hence $\Phi(K_\alpha) \longrightarrow \Phi(I)$, i.e., the claim follows.

Now, $K_\alpha \longrightarrow I$ weakly. The space of compact operators is a closed, hence weakly closed, subspace. Thus I is a compact operator. Hence, X is finite dimensional. ∎

For our next result we need the following theorem.

Theorem 5.2.9 [Ba, Theorem 1] *For reflexive Banach spaces E and F, $\mathcal{L}(E,F)$ is reflexive if and only if $\mathcal{L}(E,F) = E^* \otimes_\varepsilon F$.*

Corollary 5.2.10 *$E \otimes_\pi F^*$ is reflexive if and only if E and F are reflexive and*

$$\mathcal{L}(E,F) = E^* \otimes_\varepsilon F.$$

Consequently, if $E \otimes_\pi E^*$ is reflexive, then E is finite dimensional.

Theorem 5.2.11 *Let X^* be a dual Banach space such that $\mathcal{L}(X^*)$ has Property (II). Then X is finite dimensional.*

Proof : It is known that $\mathcal{L}(X^*) = (X \bigotimes_\pi X^*)^*$. Since $\mathcal{L}(X^*)$ is now a dual space with Property (II), it is reflexive. But this implies X^* is reflexive, which in turn implies X is also such. Since X is reflexive, $T \longrightarrow T^*$ becomes an isometric isomorphism of $\mathcal{L}(X)$ onto $\mathcal{L}(X^*)$. This implies $\mathcal{L}(X)$ is also reflexive. Hence by Corollary 5.2.10, it follows that X is finite dimensional. ∎

Remark 5.2.12 The same argument also shows that if $\mathcal{L}(X^*)$ is Hahn-Banach smooth, then X is finite dimensional.

In the case of $\mathcal{L}(X,Y)$, we have some partial results.

Theorem 5.2.13 *Let $Y = C(K)$. Then $\mathcal{L}(X,Y)$ has Property (II) if and only if X^* has Property (II) and K is finite.*

Proof : Suppose $\mathcal{L}(X,Y)$ has Property (II). One checks as in the proof of Theorem 5.2.8, that $\{\delta(k)\otimes x : x \in B_X,\, k \in K\}$ determines the norm of $\mathcal{L}(X,Y)$ and since the latter has Property (II), it follows that

$$B_{\mathcal{L}(X,Y)^*} = \overline{co}\{\delta(k)\otimes x : x \in B_X,\, k \in K\}.$$

Claim : $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.

Let $\Phi \in B_{\mathcal{L}(X,Y)^*}$ such that $\Phi|_{\mathcal{K}(X,Y)} = 0$. For any $\varepsilon > 0$, one has

$$\|\Phi - \sum_{1=1}^{n} \alpha_i \delta(k_i)\otimes x_i\| < \varepsilon,\ 0 \le \alpha_i \le 1,\ \sum_{i=1}^{n} \alpha_i = 1,\ \delta(k_i)\otimes x_i \in B_{\mathcal{L}(X,Y)^*}.$$

Since K is a compact Hausdorff space, there exists open neighborhoods U_i containing k_i such that for $i \neq j$, $U_i \bigcap U_j = \emptyset$. Let $U_{n+1} = K \setminus \{k_1, k_2 \dots k_n\}$. Thus $\{U_i\}_{i=1}^{n+1}$ is an open cover for K. Hence there exists a partition of unity $\{f_i : i = 1, 2, \dots, n+1\}$ such that $f_i|_{K\setminus U_i} = 0$. Thus

$$f_i(k_i) = 1, \quad f_i(k_j) = 0, \quad i \neq j.$$

Define $T : X \longrightarrow C(K)$ as follows

$$T(x)(k) = \sum_{i=1}^{n+1} \phi_i(x) f_i(k), \quad \phi_i \in S_{X^*}.$$

Clearly, T is finite dimensional, hence $T \in \mathcal{K}(X,Y)$. Also,

$$|T(x)(k)| \le \sum_{i=1}^{n} |\phi_i(x)| f_i(k) \le \|x\| \sum_{i=1}^{n} f_i(k) = \|x\| \quad \text{for all } k \in K.$$

This implies

$$\|T\| \le 1. \tag{5.1}$$

We also have, for $i \le n$,

$$\begin{aligned} \delta(k_i \otimes x_i)(T) &= T(x_i)(k_i) = \sum_{j=1}^{n+1} \phi_j(x_i) f_j(k_i) \\ &= \phi_i(x_i) f_i(k_i) = \phi_i(x_i). \end{aligned} \tag{5.2}$$

Thus from (5.1) and (5.2) we have the following :

$$\left\| \Phi(T) - \sum_{i=1}^{n} \alpha_i \delta(k_i) \otimes x_i(T) \right\| \leq \varepsilon \|T\| \leq \varepsilon.$$

i.e.,

$$\left\| \sum_{i=1}^{n} \alpha_i \delta(k_i) \otimes x_i(T) \right\| \leq \varepsilon \quad (\text{since } \Phi(T) = 0.)$$

i.e.,

$$\left| \sum_{i=1}^{n} \alpha_i \phi_i(x_i) \right| \leq \varepsilon.$$

Now, for each x_i there exists $\phi_i \in S_{X^*}$ such that $\phi_i(x_i) > \|x_i\| - \varepsilon$ which implies

$$\sum_{i=1}^{n} \alpha_i (\|x_i\| - \varepsilon) < \sum_{i=1}^{n} \alpha_i \phi_i(x_i) < \varepsilon.$$

Hence, $\sum_{i=1}^{n} \alpha_i \|x_i\| < 2\varepsilon$. Thus,

$$\|\Phi\| \leq \|\Phi - \sum_{i=1}^{n} \alpha_i \delta(k_i) \otimes x_i\| + \|\sum_{i=1}^{n} \alpha_i \delta(k_i) \otimes x_i\| \leq \varepsilon + 2\varepsilon,$$

which implies $\Phi = 0$. Hence the claim.

Because of the isometric identification $\mathcal{K}(X, C(K)) = C(K, X^*)$ (see [DS, Theorem 1, pg-490]), $C(K, X^*)$ has Property (II), and it follows from Proposition 5.2.5 that K is finite and X^* has Property (II).

Conversely, if X^* has Property (II) and K is finite, it follows that any operator $T : X \longrightarrow C(K)$ is finite dimensional, hence compact. Hence

$$\mathcal{L}(X, Y) = \mathcal{K}(X, Y).$$

and again using $\mathcal{K}(X, C(K)) = C(K, X^*)$, it follows from Proposition 5.2.5 that $\mathcal{L}(X, Y)$ has Property (II). ∎

We similarly have,

Corollary 5.2.14 *Let $Y = C(K)$. Then $\mathcal{L}(X,Y)$ has w*-ANP-II if and only if X^* has w*-ANP-II and K is finite.*

Remark 5.2.15 (a) Since ℓ_1^n is not strictly convex, it follows that $\mathcal{L}(X, C(K))$ does not have w*-ANP-κ ($\kappa =$ I, II$'$), even if K is finite and X^* has w*-ANP-κ ($\kappa =$ I, II$'$).

(b) Similarly, it follows that $\mathcal{L}(X, C(K))$ does not have the MIP even if K is finite and X^* has the MIP.

5.3 Nice smoothness in $C(K, X)$ and $\mathcal{L}(X)$

The following results for nicely smooth Banach spaces closely parallel the corresponding results for Property (II). However, we include the proofs for completeness.

Proposition 5.3.1 *Let K be a compact Hausdorff space. Then $C(K, X)$ is nicely smooth if and only if K is finite and X is nicely smooth.*

Proof : For a compact Hausdorff space K and a Banach space X, the set

$$A = \{\delta(k) \otimes x^* : k \in K,\ x^* \in S_{X^*}\} \subseteq B_{C(K,X)^*}$$

is a norming set for $C(K, X)$. So, if $C(K, X)$ is nicely smooth, $C(K, X)^* = \overline{span}(A)$. It follows that K admits no nonatomic measure, whence K is scattered. Now, let K' denote the set of isolated points of K. Then K' is dense in K, so, the set

$$A' = \{\delta(k) \otimes x^* : k \in K',\ x^* \in S_{X^*}\}$$

is also norming. Thus, $C(K, X)^* = \overline{span}(A')$. But if $k \in K \setminus K'$, then for any $x^* \in S_{X^*}$, $\delta(k) \otimes x^* \notin \overline{span}(A')$. Hence, $K = K'$, whence K must be finite. Hence,

$$C(K, X) = \bigoplus_{c_0(|K|)} X.$$

Thus by Theorem 4.2.3 it follows that X is nicely smooth.

Conversely suppose X is nicely smooth and K is finite. Hence,

$$C(K, X) = \bigoplus_{c_0(|K|)} X.$$

Again applying Theorem 4.2.3, it follows that and $C(K, X)$ is nicely smooth. ■

We similarly have

Corollary 5.3.2 *$C(K, X)$ is Hahn-Banach smooth if and only if X is Hahn-Banach smooth and K is finite.*

Remark 5.3.3 It is immediate that for $C(K)$ spaces Property (II), the BGP and being nicely smooth (indeed, any of the conditions of Proposition 4.1.4) are equivalent, and are equivalent to reflexivity.

Proposition 5.3.4 *Let X be a Banach space such that there exists a bounded net $\{K_\alpha\}$ of compact operators such that $K_\alpha x \longrightarrow x$ weakly for all $x \in X$. If $\mathcal{L}(X)$ is nicely smooth, then X is finite dimensional.*

Proof : As in the proof of Theorem 5.2.8, it follows that $A = \{x \otimes x^* : \|x^*\| = 1, \|x\| = 1\}$ is a norming set, and hence, $\mathcal{L}(X)^* = \overline{span}(A)$.

Claim : $K_\alpha \longrightarrow I$ weakly.

Since $\{K_\alpha\}$ is bounded, it suffices to check that $K_\alpha \longrightarrow I$ on A, i.e., to check $x^*(K_\alpha(x)) \longrightarrow x^*(x)$ for all $\|x\| = 1$, $\|x^*\| = 1$. But, $K_\alpha(x) \longrightarrow x$ weakly, hence the claim.

Thus, I is a compact operator, so that X is finite dimensional. ■

Proposition 5.3.5 *If $\mathcal{L}(X^*)$ is nicely smooth, then X is finite dimensional.*

Proof : We need only to observe that $\mathcal{L}(X^*) = (X \otimes_\pi X^*)^*$ is a nicely smooth dual space, hence reflexive. The rest of the proof follows as in Theorem 5.2.11. ■

Proposition 5.3.6 *For a compact Hausdorff space K, $\mathcal{L}(X, C(K))$ is nicely smooth if and only if X is reflexive and K is finite.*

Proof : Suppose $\mathcal{L}(X, C(K))$ is nicely smooth. By definition of the norm, $A = \{\delta(k) \otimes x : x \in B(X),\ k \in K\}$ is a norming set for $\mathcal{L}(X, C(K))$, and hence, $\mathcal{L}(X, C(K))^* = \overline{span}(A)$. Arguing similarly as in Theorem 5.2.13, it follows that $\mathcal{L}(X, C(K)) = \mathcal{K}(X, C(K)) = C(K, X^*)$. Since $C(K, X^*)$ is nicely smooth, it follows from Proposition 5.3.1 that X^* is nicely smooth and K is finite. Now by Remark 4.1.3, it follows that X is reflexive.

For the converse part, X being reflexive, it follows from Remark 4.1.3 that X^* is nicely smooth. We argue similarly as in Theorem 5.2.11 and get

$$\mathcal{L}(X, C(K)) = \mathcal{K}(X, C(K)) = C(K, X^*).$$

Again applying Proposition 5.3.1 it follows that $\mathcal{L}(X, C(K))$ is nicely smooth. ∎

We similarly have

Corollary 5.3.7 *For a compact Hausdorff space $\mathcal{L}(X, C(K)$ is Hahn-Banach smooth if and only if X is Hahn-Banach smooth and K is finite.*

Chapter 6

Ball Separation Properties in Tensor Product spaces

In this chapter we investigate several ball separation properties in tensor product spaces. In our discussion we consider the injective and projective tensor products only.

6.1 Asymptotic Norming Properties in Tensor Product Spaces

Lemma 6.1.1 *Let X, Y be two Banach spaces. Suppose $\Phi \subseteq B_{X^*}$, $\Psi \subseteq B_{Y^*}$ be norming sets for X and Y respectively. Then $\Phi \otimes \Psi$ is a norming set for $X \otimes_\varepsilon Y$.*

Proof : Since Φ, Ψ are norming sets for X and Y respectively, $\overline{co}^{w^*}(\Phi) = B_{X^*}$ and $\overline{co}^{w^*}(\Psi) = B_{Y^*}$. Thus

$$B_{X^*} \otimes B_{Y^*} = \overline{co}^{w^*}(\Phi) \otimes \overline{co}^{w^*}(\Psi)$$

But $B_{X^*} \otimes B_{Y^*}$ is a norming set for $X \otimes_\varepsilon Y$ where the norm is given by

$$\|\sum_{i=1}^{n} x_i \otimes y_i\| = \sup\{\sum_{i=1}^{n} x^*(x_i)y^*(y_i) : x^* \in B_{X^*}, y^* \in B_{Y^*}\}.$$

Thus it follows that $co(\Phi) \otimes co(\Psi)$ is a norming set for $X \otimes_\varepsilon Y$. Since $co(\Phi \otimes \Psi) \supset co(\Phi) \otimes co(\Psi)$, it follows that $co(\Phi \otimes \Psi)$ and hence $\Phi \otimes \Psi$ is a norming set for $X \otimes_\varepsilon Y$. ∎

The following results will be used in subsequent discussions.

Theorem 6.1.2 [Gr] *For Hilbert spaces E and F, $T \in \mathcal{L}(E, F)$ is an extreme contraction if and only if T or T^* is an isometry.*

Definition 6.1.3 A Banach space X is said to have the approximation property if for each compact set $K \subseteq X$ and $\varepsilon > 0$ there is a continuous finite rank operator $T : X \longrightarrow X$ such that for all $x \in K$, $\|Tx - x\| \leq \varepsilon$.

Theorem 6.1.4 [DU] Let X, Y be Banach spaces. Suppose X^* has approximation property. Then

$$X^* \otimes_\varepsilon Y^* = (X \otimes_\pi Y)^*$$

if and only if $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$.

Suppose H_1 is any Hilbert space, and H_2 a finite dimensional Hilbert space. Then using the above theorem it follows that

$$H_1 \otimes_\varepsilon H_2 = (H_1 \otimes_\pi H_2)^*.$$

Now, H_1, H_2 have ANP-I. If possible, let $H_1 \otimes_\varepsilon H_2$ also have ANP-I. Then by Theorem 1.0.8, $H_1 \otimes_\varepsilon H_2$ is strictly convex. This implies $S_{H_1 \otimes_\varepsilon H_2} = extB_{H_1 \otimes_\varepsilon H_2} = extB_{(H_1 \otimes_\pi H_2)^*} = extB_{\mathcal{L}(H_1, H_2)}$, and by Theorem 6.1.2, it follows that $extB_{\mathcal{L}(H_1, H_2)}$ is a proper subset of $S_{\mathcal{L}(H_1, H_2)} = S_{H_1 \otimes_\varepsilon H_2}$, a contradiction. Since spaces having ANP-II$'$ are strictly convex, similar results follow for ANP-II$'$ also.

Thus we have

Theorem 6.1.5 *ANP-I and ANP-II$'$ are not stable under injective tensor product.*

Remark 6.1.6 Similar result was observed for MIP in [RS1].

The following theorem is well-known.

Theorem 6.1.7 [W]

$$dentB_{(X\otimes_\pi Y)} = dentB_X \otimes dentB_Y.$$

Suppose X, Y has ANP-I. If possible, let $X \otimes_\pi Y$ have ANP-I. Then from Theorem 1.0.3 and Theorem 6.1.7, it follows that

$$S_{(X\otimes_\pi Y)} = dentB_{(X\otimes_\pi Y)} = dentB_X \otimes dentB_Y = S_X \otimes S_Y.$$

which is not possible if $\dim(X)$ or $\dim(Y) \geq 2$.

Thus we have

Theorem 6.1.8 *ANP-I is not stable under projective tensor product.*

Remark 6.1.9 (a) Similar results for MIP was observed in [B1] and also in [BRo1]

(b) In view of Theorem 6.1.5 and Theorem 6.1.8, one can ask similar questions for other ANP's too.

The following well-known results characterise the w*-denting points and extreme point of the dual unit ball of $(X \otimes_\varepsilon Y)^*$.

Theorem 6.1.10 (*a*) [RS1]

$$w^*\text{-}dentB_{(X\otimes_\varepsilon Y)^*} = w^*\text{-}dentB_{X^*} \otimes w^*\text{-}dentB_{Y^*}.$$

(*b*) [RS]

$$extB_{(X\otimes_\varepsilon Y)^*} = extB_{X^*} \otimes extB_{Y^*}.$$

Thus we have the following :

Theorem 6.1.11 *For any two Banach spaces, X, Y with $\dim X, Y \geq 2$, $X \otimes_\varepsilon Y$ never has w^*-ANP-κ ($\kappa =$ I, II').*

Proof : For w*-ANP-I

If possible, let $X \otimes_\varepsilon Y$ have w*-ANP-I From Theorem 1.0.12 and Theorem 6.1.10, it follows that

$$\begin{aligned} S_{(X\otimes_\varepsilon Y)^*} &= \text{w*-dent} B_{(X\otimes_\varepsilon Y)^*} \\ &= \text{w*-dent} B_{X^*} \otimes \text{w*-dent} B_{Y^*} \\ &\subseteq S_{X^*} \otimes S_{Y^*} \end{aligned}$$

a contradiction, as not all integral operators have such a simple description (see [DS, p 231]). Similarly for w*-ANP-II′. ∎

Remark 6.1.12 We can perhaps look at the analogues of the above theorem for w*-ANP-κ (κ = II, III). But there seems to be no characterisation of w*-w PC's and w*-PC's of $B_{(X\otimes_\varepsilon Y)^*}$. In a recent work, Rao [Ra1] has exhibited some of the w*-PC's of $B_{(X\otimes_\varepsilon Y)^*}$. However a complete description is yet to be established.

6.2 Nice smoothness in Tensor Product spaces

Theorem 6.2.1 [GS] *If X, Y are nicely smooth Asplund spaces, then $X\otimes_\varepsilon Y$ is nicely smooth.*

We prove the converse in a more general set-up.

Theorem 6.2.2 *Let X, Y be Banach spaces such that $X \otimes_\varepsilon Y$ is nicely smooth. Then both X and Y are nicely smooth.*

Proof : Let M and N be norming subspaces of X^* and Y^* respectively. Then B_M and B_N are norming sets for X and Y respectively, and so, by Lemma 6.1.1, $B_M \otimes B_N$ is a norming set for $X \otimes_\varepsilon Y$. And since this space is nicely smooth,

$$(X \otimes_\varepsilon Y)^* = \overline{span}^{\|\cdot\|}(B_M \otimes B_N)$$

(dual norm). Suppose $x^* \in X^*$, then for any $y^* \in S_{Y^*}$ and $\varepsilon > 0$, there exist $f_i \in B_M$, $e_i \in B_N$ and $\lambda_i \in I\!R$ such that

$$\|x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i\| < \varepsilon.$$

Applying to elementary tensors, this implies

$$\begin{aligned}
\left|(x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i)(x \otimes y)\right| &< \varepsilon\|x\|\|y\| \quad \text{for all } x \in X,\, y \in Y \\
\Longrightarrow \left|x^*(x)y^*(y) - \sum_{i=1}^{n} \lambda_i f_i(x) e_i(y)\right| &< \varepsilon\|x\|\|y\| \quad \text{for all } x \in X,\, y \in Y \\
\Longrightarrow \left\|x^*(x)y^* - \sum_{i=1}^{n} \lambda_i f_i(x) e_i\right\| &< \varepsilon\|x\| \quad \text{for all } x \in X
\end{aligned}$$

Choose x such that $f_i(x) = 0$ for all $i = 1, 2, \ldots, n$.

Then, $\|x^*(x)y^*\| < \varepsilon\|x\|$, i.e., $|x^*(x)| < \varepsilon\|x\|$. That is, if $E = \bigcap ker f_i$, then $E^* = X^*/E^\perp = X^*/span\{f_i\}$ and $\|x^*|_E\| < \varepsilon$. This happens if and only if $d(x^*, span\{f_i\}) < \varepsilon$.

It follows that $x^* \in M$ and hence X is nicely smooth. Similarly for Y. Hence the theorem. ∎

It seems difficult to obtain analogues of Theorems 6.2.1 and 6.2.2 for the projective tensor product. However, we have the following

Proposition 6.2.3 *Suppose X, Y are Banach spaces such that X^* has the approximation property and $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$, i.e., any bounded linear operator from X to Y^* is compact. Then the following are equivalent :*

(a) $\mathcal{K}(X, Y^*)$ is nicely smooth.

(b) X, Y are reflexive (and hence nicely smooth).

(c) $X \otimes_\pi Y$ is reflexive (and hence nicely smooth).

Proof : $(a) \Rightarrow (b)$. Since X^* has the approximation property,

$$\mathcal{K}(X, Y^*) = X^* \otimes_\varepsilon Y^*$$

and it follows from Theorem 6.2.2 that X^* and Y^* are nicely smooth, and therefore, X and Y are reflexive.

$(b) \Rightarrow (c)$. This is a well-known result of Holub (see [DU] and [Ba]).

$(c) \Rightarrow (a)$. X and Y being closed subspaces of the reflexive space $X \otimes_\pi Y$ are themselves reflexive and from

$$\mathcal{K}(X, Y^*)^* = (X \otimes_\pi Y)^{**} = X \otimes_\pi Y$$

it follows that $\mathcal{K}(X, Y^*)$ is reflexive, and hence, nicely smooth. ∎

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