The Mazur Intersection Property In Banach Spaces And Related Topics

Pradipta Bandyopadhyaya

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It appears to be customary to begin by thanking one's supervisor. But I find it superfluous, particularly since Professor A. K. Roy means much more to me than merely my supervisor. After all, one normally doesn't thank his parents for holding his hands when he is taking his first faltering steps, for teaching him to walk.

Though I consider this entire thesis as a kind of joint venture, Prof. Roy has allowed me to pass much of it as my own. For the records, only the materials of Chapter 3 have so far appeared as a joint paper, which too he has permitted me to include in this thesis without any questions asked.

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Contents

Chapter 1

Introduction

In the first part of this chapter we explain in general terms the main theme of this thesis and provide a chapterwise summary of its principal results. The second part recapitulates some of the known notions and results used in the subsequent chapters. The numbers given in parentheses correspond to those in the list of references on page 70.

S. Mazur [40] was the first to consider the following smoothness property in normed linear spaces, called the Mazur Intersection Property (MIP), or, more briefly, the Property (I) :

> *Every closed bounded convex set is the intersection of closed balls containing it.*

He showed that any reflexive Banach space with a Fréchet-differentiable norm has this property.

Later, R. R. Phelps [42] provided a more geometric insight into this property by showing that

(*a*) *A normed linear space X has the MIP if the w^*-strongly exposed points of the unit ball $B(X^*)$ of the dual X^* are norm dense in the unit sphere $S(X^*)$.*

(*b*) *If a normed linear space X has the MIP, every support mapping on X maps norm dense subsets of $S(X)$ to norm dense subsets of $S(X^*)$.*

(*c*) *A finite dimensional normed linear space X has the MIP if and only if the extreme points of $B(X^*)$ are norm dense in $S(X^*)$.*

He also asked whether the sufficient condition (a) is also necessary. To date, this remains an open question.

Nearly two decades later, Phelps' characterisation (c) was extended by J. R. Giles, D. A. Gregory and B. Sims [21] to general normed linear spaces, developing an idea due to F. Sullivan [51], and they proved, *inter alia*,

Theorem 1.1 *For a normed linear space X, the following are equivalent :*

(*a*) *The w^*-denting points of $B(X^*)$ are norm dense in $S(X^*)$.*

(*b*) *X has the MIP.*

(*c*) *Every support mapping on X maps norm dense subsets of $S(X)$ to norm dense subsets of $S(X^*)$.*

They also showed that in dual Banach spaces, the MIP implies reflexivity and considered the weaker property that every weak* compact convex set in a dual space is the intersection of balls (w*-MIP). Investigating the necessity of Phelps' condition (a), they showed that it is indeed necessary if, in addition, the dual X^* has the w*-MIP, or, X is an Asplund space. They now asked whether the MIP necessarily implies Asplund. To date, this also remains open.

Notice that if X is separable and has the MIP, Phelps' condition (c) (or, Theorem 1.1(c) above) implies that it has a separable dual and hence is Asplund. So one asks, is the MIP hereditary, i.e., inherited by subspaces ? The answer, unfortunately, is no. In Chapter 4, we give an example and discuss the subspace question in more detail.

However, since the Asplund Property is invariant under equivalent renorming, a more pertinent question is whether the existence of an equivalent norm with the MIP is hereditary. Some discussions on MIP-related renorming questions may be found in [9], [47], [54] and [57]. However, in this work, we do not discuss any renorming problem but concentrate instead on some of the isometric questions that arise.

Recently, there have appeared several papers dealing with similar intersection properties for compact convex sets [54,47] (called the Property CI), weakly compact convex sets [57] and compact convex sets with finite affine dimension [49].

In Chapter 2, we give a unified treatment of the intersection properties for these diverse classes of sets by considering the MIP for the members of a general family — subject to some mild restrictions — of closed bounded convex sets in a Banach space and recapture all the known results as special cases. We also introduce a new condition of separation of convex sets which is a variant of the following :

Disjoint bounded convex sets are contained in disjoint balls

and this apparently stronger condition turns out to be equivalent to the intersection property in all known cases. This strengthens the results of Zizler [56]. We should point out that our proofs in this chapter are usually modifications, refinements and adaptations to our very general set-up of arguments for particular cases to be found in [21], [47] and [54]. This chapter, which is essentially contained in [3], also provides much of the background for what follows in the later chapters.

Whitfield and Zizler [55] have also defined the Uniform Mazur Intersection Property, a property somewhat stronger than the MIP. But in this thesis, we only briefly touch upon their work.

In Chapter 3, we discuss the question of lifting the MIP and the CI from a Banach space X to its associated Bochner L^p space and their stability under ℓ^p sums, $1 < p < \infty$. In particular, we prove that the ℓ^p sum of a family of Banach spaces has the MIP (or, the CI) if and only if each coordinate space has it; that the Bochner L^p space for the Lebesgue measure on $[0, 1]$ always has the CI, while the MIP in X is equivalent to a weaker intersection property in the Bochner L^p space which turns out to be equivalent to the MIP if and only if X is Asplund. Most of these results have already appeared in print in [4].

In Chapter 4, we present a collection of partial results relating to various aspects of the MIP that raise more question than they answer. Apart from the subspace question mentioned above, in this chapter we discuss the relation between the MIP and a farthest point phenomenon that was observed by K. S. Lau [35], but seems to have passed largely unnoticed since then. Lau had shown that in a reflexive space the MIP is equivalent to the following :

> *Every closed bounded convex set is the closed convex hull of its farthest points.*

We extend this result to characterise the w*-MIP in w*-Asplund dual spaces using a result of Deville and Zizler [10]. As far as we know, besides the work of Tsarkov [53] — who has characterised the MIP in finite-dimensional spaces in terms of convexity of bounded Chebyshev sets — this is the only attempt at an intrinsic characterisation of the MIP. We also point out some new direction of investigation that is indicated by our result.

In this chapter, we also discuss the MIP in projective tensor product spaces. Ruess and Stegall [46] have shown that the injective tensor product of two Banach spaces of dimension ≥ 2 never has the MIP. And Sersouri [48] has shown that in fact there is a two-dimensional compact convex set in $X \otimes_\epsilon Y$ that is not intersection of balls. The situation appears to be much more difficult for projective tensor product spaces. Here we show that the projective tensor product of two Banach spaces X and Y never has the MIP if X and Y are Hilbert spaces or are two-dimensional ℓ^p spaces for a large range of values of p. For this purpose, we characterise the extreme contractions from ℓ_2^p to ℓ_2^q and obtain their closure. The technique used is similar to [34]. In the process, we reestablish relevant special cases of the results obtained in [25,26,28].

Chapter 5 is somewhat independent of the rest. Here we discuss the following phenomenon : For a closed bounded convex set K in a Banach space X and a point $x_o \in K$, the following implications are easy to establish :

$$\begin{array}{ccc} x_o \text{ strongly exposed} & \Longrightarrow & x_o \text{ denting point} \\ \Downarrow & & \Downarrow \\ x_o \text{ exposed and PC} & \Longrightarrow & x_o \text{ extreme and PC} \end{array}$$

where by PC we mean a point of continuity (defined more precisely later). A recent result (see Theorem 1.5 below) is that x_o is denting if and only if x_o is extreme and PC, i.e., the implication down the right hand side above is reversible. This naturally leads to the conjecture that the implication down the left hand side above, too, can be reversed, i.e., if x_o is exposed and PC (or, equivalently, x_o is an exposed denting point) then x_o is strongly exposed.

We show that the conjecture is false by constructing a counterexample in the Banach space ℓ^1 and provide a characterisation of strongly exposed points among points of continuity of a closed bounded convex set. As a corollary, we deduce that the conjecture is true for the points of weakly compact sets. We also show that the counterexample, in some sense, is actually typical of ℓ^1, i.e., we characterise Banach spaces containing ℓ^1 in terms of the validity of the above conjecture. We also briefly touch upon the necessity of Phelps' condition (a) in the light of our characterisation of (w*-) strongly exposed points. This chapter is a revised version of [2].

Notations, Conventions and General Preliminaries

General reference to this work are the monographs [7], [13] and [20]. We work only with *real* Banach spaces. Unless otherwise mentioned, by a subspace we always mean a *norm closed linear* subspace. The closed unit ball and the unit sphere of a Banach space X will be denoted by $B(X)$ and $S(X)$ respectively. For $z \in X$ and $r > 0$, we denote by $B_r[z]$ (resp. $B_r(z)$) the closed (resp. open) ball of radius r and centre z.

For $x \in S(X)$, $D(x) = \{f \in S(X^*) : f(x) = 1\}$. The set valued map D is called the *duality map* and any selection of D is called a *support mapping*. For $f \in S(X^*)$, we similarly define the *inverse duality map* as $D^{-1}(f) = \{x \in S(X) : f(x) = 1\}$.

For a bounded set $K \subseteq X$, denote $[K] = \cap\{B : B$ closed ball containing $K\}$ and following Franchetti [19], call a bounded set *admissible* if $K = [K]$.

In this terminology, a Banach space X has the MIP (resp. the CI) if every norm closed bounded (resp. norm compact) convex set is admissible.

For $K \subseteq X$, $f \in X^*$ and $\alpha > 0$, the set $S(K, f, \alpha) = \{x \in K : f(x) > \sup f(K) - \alpha\}$ is called the open *slice* of K determined by f and α. For $A \subseteq X$, denote by $\text{co}(A)$ (resp. $\text{aco}(A)$) the convex (resp. absolutely convex) hull of A and by $\text{int}(A)$, the norm interior of A. For $A \subseteq X$, $f \in X^*$, $\|f\|_A = \sup\{|f(x)| : x \in A\}$, $A^o = \{f \in X^* : \|f\|_A \leq 1\}$ and for $B \subseteq X^*$, $A\text{-dia}(B) = \sup\{\|b_1 - b_2\|_A : b_1, b_2 \in B\}$. For $A_1, A_2 \subseteq X$, $\text{dist}(A_1, A_2) = \inf\{\|x_1 - x_2\| : x_i \in A_i,\ i = 1, 2\}$. For $A \subseteq X$, $\overline{A}^{\sigma}$ denotes the closure of A for the topology σ. Whenever the topology is not specified, we mean the *norm* topology.

We identify an element $x \in X$ with its canonical image $\hat{x}$ in X^{**}. Let $\widehat{K}$ be the image of a closed bounded convex $K \subseteq X$ under this canonical embedding of X in X^{**}. We denote by $\widetilde{K}$ the w*-closure of $\widehat{K}$ in X^{**}. $\widetilde{K}$ is of course a w*-compact convex set.

For a measure space (Ω, Σ, μ) and $A \in \Sigma$, we denote by χ_A, the indicator function of A, and by $\mu|_A$ the restriction of the measure μ to the σ-field of measurable subsets of A.

Let X be a real Banach space, let F be a total subspace of X^* (i.e., F separates points of X). Let σ denote the $\sigma(X, F)$ topology on X. For a closed bounded convex set $K \subseteq X$, recall that $x_o \in K$ is called

(a) a *σ-denting point of* K if, for each $\varepsilon > 0$, $x_o \notin \overline{\text{co}}^{\sigma}(K \setminus B_\varepsilon(x_o))$.

(b) a *σ-point of continuity (σ-PC) for* K if the identity map, $id : (K, \sigma) \to (K, \text{norm})$ is continuous at x_o.

(c) a *very strong extreme point of* K if, for every sequence $\{x_n\}$ of K-valued Bochner integrable functions on $[0, 1]$ with respect to the Lebesgue measure, the condition

$$\lim_{n\to\infty} \left\| \int_o^1 x_n(t)dt - x_o \right\| = 0 \text{ implies } \lim_{n\to\infty} \int_o^1 \|x_n(t) - x_o\| dt = 0.$$

(d) an *σ-exposed point of* K if there exists $f \in F$ such that f exposes x_o, i.e., $f(x) < f(x_o)$ for all $x \in K \setminus \{x_o\}$.

(e) a *σ-strongly exposed point of K* if there exists $f \in F$ such that f exposes x_o and for any sequence $\{x_n\} \subseteq K$, $\lim_n f(x_n) = f(x_o)$ implies that $\lim_n \|x_n - x_o\| = 0$.

In the sequel, whenever σ is the *weak* topology, i.e., $F = X^*$, we omit the prefix σ while referring to the above concepts.

The following lemma is very well-known and can be found in [8, Proposition 25.13] or more recently in [45, Lemma 1.3] :

Lemma 1.2 *Let E be a locally convex space, K be a compact convex set in E and $x \in K$. The following are equivalent :*

(a) x is an extreme point of K.

(b) The family of open slices containing x forms a local base for the topology of E at x (relative to K).

In Chapter 5, we note an analogue of this lemma for exposed points.

The following characterisation of a σ-denting point is an immediate consequence of the Hahn-Banach Theorem :

Lemma 1.3 *Let X be a Banach space, F be a total subspace of X^*, K be a closed bounded convex set in X and $x \in K$. The following are equivalent :*

(a) x is a σ-denting point of K.

(b) The family of σ-open slices containing x forms a local base for the norm topology at x (relative to K).

The following lemma is also well-known :

Lemma 1.4 *Let K be a closed bounded convex set of a Banach space X. Then $x^{**} \in \widetilde{K}$ is a w^*-PC if and only if $x^{**} = \hat{x}$ for some $x \in K$ and x is a PC.*

The following characterisation of a denting point of a closed bounded convex set in a Banach space is a consequence of the two lemmas above and is essentially contained in [38] and [39] :

Theorem 1.5 *Let x be an element in a closed bounded convex set K of a Banach space. Then the following are equivalent :*

(a) *x is a denting point of K.*

(b) *$\hat{x}$ is a w^*-denting point of $\widetilde{K}$.*

(c) *x is a very strong extreme point of K.*

(d) *For any probability space (Ω, Σ, μ) and any net $\{x_\alpha\}$ of K-valued Bochner integrable functions on Ω, the condition*
$$\lim_\alpha \| \int_\Omega x_\alpha(\omega) d\mu - x \| = 0 \text{ implies } \lim_\alpha \int_\Omega \|x_\alpha(\omega) - x\| d\mu = 0.$$

(e) *x is an extreme point of K which is also a PC.*

Let X be a real Banach space, let F be a norming subspace of X^* (i.e., $\|\hat{x}\|_{B(F)} = \|x\|$, for all $x \in X$). Then $B(X)$ is $\sigma(X, F)$-closed and $B(F)$ is w*-dense in $B(X^*)$.

We will need the following generalisation of Lemma 1.4 and Theorem 1.5 (a) $\Longleftrightarrow$ (e) in the sequel.

Lemma 1.6 *Let X be a Banach space, F be a norming subspace of X^*.*

(a) *A point $x_o^* \in B(X^*)$ is a w^*-PC of $B(X^*)$ if and only if x_o^* is a w^*-PC of $B(F)$.*

(b) *A point $x_o^* \in B(X^*)$ is a w^*-denting point of $B(X^*)$ if and only if x_o^* is an extreme point and a w^*-PC of $B(F)$.*

We will also use the following well-known results :

Lemma 1.7 [42, Lemma 3.1] *Let X be a normed linear space and $\varepsilon > 0$. If $f, g \in S(X^*)$ are such that $x \in B(X) \cap f^{-1}(0)$ implies $|g(x)| \leq \varepsilon/2$, then either $\|f - g\| \leq \varepsilon$ or $\|f + g\| \leq \varepsilon$.*

Lemma 1.8 [20, p 205] *Let K be a closed bounded convex subset of a Banach space X. Given $f \in X^* \setminus \{0\}$ and x_o contained in a slice S of K determined by f, there exists $\varepsilon > 0$ such that whenever $\|f - g\| < \varepsilon$, there is a slice S' of K determined by g such that $x_o \in S' \subseteq S$.*

For a Banach space X, a function $F : X \longrightarrow I\!R$ is said to be

(a) *Gateaux differentiable* at a point $x \in X$ if there exists an $f \in X^*$ such that

$$\lim_{\lambda \to 0+} \left| \frac{F(x + \lambda y) - F(x)}{\lambda} - f(y) \right| = 0 \text{ for all } y \in B(X)$$

and (b) *Fréchet differentiable* at x if the convergence is uniform over $y \in B(X)$.

In particular, if F is the norm, we have the following duality result :

Theorem 1.9 [20, Theorem 3.5.4] *In a Banach space X*

(*a*) *the norm is Gateaux differentiable (or smooth) at a point $x \in S(X)$ if and only if $D(x)$ is single-valued and in that case, x w^*-exposes $D(x)$ in $B(X^*)$.*

(*b*) *the norm is Fréchet differentiable at $x \in S(X)$ if and only if $D(x)$ is single-valued and x w^*-strongly exposes $D(x)$ in $B(X^*)$.*

Recall that a Banach space X is *Asplund* if every continuous convex function $F : X \to I\!R$ is Fréchet differentiable on a dense G_δ subset of X, and a dual space X^* is *w^*-Asplund* if every continuous w*-lower semicontinuous convex function $F : X^* \to I\!R$ is Fréchet differentiable on a dense G_δ subset of X^*.

The following characterisation theorem is a classic [7]:

Theorem 1.10 *For a Banach space X,*

(*i*) *the following are equivalent :*

(*a*) *X is Asplund.*

(*b*) *Separable subspaces of X have separable dual.*

(*c*) *X^* has the Radon-Nikodým Property (RNP).*

(*d*) *X^* has the RNP with respect to the Lebesgue measure on* $[0, 1]$.

(*ii*) *the following are equivalent :*

(*a*) *X^* is w^*-Asplund.*

(*b*) *X has the RNP.*

If X is a Banach space and (Ω, Σ, μ) a measure space, let $L^p(\mu, X)$ denote the Lebesgue-Bochner function space of p-integrable X-valued functions defined on Ω, $1 \leq p < \infty$ (see [14]). Recall (from [13]) that if $\frac{1}{p} + \frac{1}{q} = 1$ $(1 < p < \infty)$, the space $L^q(\mu, X^*)$ is isometrically isomorphic to a subspace of $L^p(\mu, X)^*$ and that they coincide if and only if X^* has the RNP with respect to μ. Also recall (from [14] and [31]) that $L^p(\mu, X)^*$ is isometrically isomorphic to the following two spaces :

(1) $V_q(\mu, X^*)$ = the space of all vector measures $F : \Sigma \longrightarrow X^*$ such that the q-variation of F is finite, i.e.,

$$\|F\|_q = \sup\left\{\sum_{E\in\pi} \frac{\|F(E)\|^q}{\mu(E)^q}\mu(E) : \pi \text{ is a finite partition of } \Omega\right\}^{1/q} < \infty,$$

and (2) $L^q(\mu, X^*; X)$ = the space of all w*-equivalence classes of X^*-valued w*-measurable functions h, such that the real-valued function $\|h(\cdot)\| \in L^q(\mu, I\!R)$. Only the representation given by (1) will be used by us.

[13, Chapter VIII] contains all necessary information on tensor product spaces. We just recall here that the dual of the projective tensor product, $X \otimes_\pi Y$, of two Banach spaces X and Y, is $\mathcal{L}(X, Y^*)$, the space of continuous linear operators from X to Y^*.

Chapter 2

The Mazur Intersection Property for a Family of Closed Bounded Convex Sets

2.1. The Set-up and Main Result

Let X be a real Banach space, let F be a norming subspace of X^* and let

$$\mathcal{A} = \{K \subseteq X : K \text{ is admissible}\}$$

Then $\mathcal{A}$ is a family of norm bounded, $\sigma(X, F)$-closed convex sets that is closed under arbitrary intersection, translation and scalar multiplication. Naturally, $\mathcal{A}$ contains all singletons and all closed balls. In this chapter, we denote the $\sigma(X, F)$ topology on X simply by σ.

Taking $\mathcal{A}$ as our model, we let $\mathcal{C}$ be a family of norm bounded, σ-closed convex sets with the following properties :

(A1) $\mathcal{C}$ is closed under arbitrary intersection, translation and scalar multiplication.

(A2) $C_1, C_2 \in \mathcal{C} \Longrightarrow \overline{\text{aco}}^\sigma(C_1 \cup C_2) \in \mathcal{C}$.

(A3) $C \in \mathcal{C}$, C absolutely convex and $f \in F \Longrightarrow C \cap f^{-1}(0) \in \mathcal{C}$.

Note that (A1) implies that $\mathcal{C}$ contains all singletons.

EXAMPLES : (i) $\mathcal{C} = \{$all closed bounded convex sets in $X\}$, $F = X^*$

(ii) $X = Y^*$, $\mathcal{C} = \{$all w*-compact convex sets in $X\}$, $F = \widehat{Y}$

(iii) $\mathcal{C} = \{$all compact convex sets in $X\}$, $F =$ any norming subspace

(iv) $\mathcal{C} = \{$all compact convex sets in X with finite affine dimension$\}$, $F =$ any norming subspace

(v) $\mathcal{C}$ = {all weakly compact convex set in X}, F = any norming subspace.

Let $\mathcal{F} = \{C^o : C \in \mathcal{C}\}$. Then $\mathcal{F}$ is a local base for a locally convex Hausdorff vector topology τ on X^*, the topology of uniform convergence on elements of $\mathcal{C}$. Clearly, τ is stronger than the w*-topology on X^* and weaker than the norm topology.

DEFINITIONS : (1) Denote by E_τ, the set of all extreme points of $B(X^*)$ which are also points of continuity of the identity map, $id : (B(X^*),\text{w}^*) \longrightarrow (B(X^*), \tau)$.

(2) For $C \in \mathcal{C}$, C absolutely convex, $\varepsilon > 0$ and $\delta > 0$, we say that a point $x \in S(X)$ belongs to the set $M_{C,\varepsilon,\delta}(X)$ if

$$\sup_{y \in C, 0<\lambda\leq\delta} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} < \varepsilon$$

For $C = B(X)$, we write $M_{C,\varepsilon,\delta}(X)$ simply as $M_{\varepsilon,\delta}(X)$. Let $M_{C,\varepsilon}(X) = \cup_{\delta>0} M_{C,\varepsilon,\delta}(X)$ and for $C = B(X)$ write $M_{C,\varepsilon}(X)$ simply as $M_\varepsilon(X)$, or even as M_ε whenever there is no scope of confusion. Similarly, $M_{C,\varepsilon}(X)$ will often be abbreviated as $M_{C,\varepsilon}$. Notice that $\cap_{\varepsilon>0} M_\varepsilon(X)$ gives the set of points of $S(X)$ at which the norm is Fréchet differentiable.

(3) $H_\tau = \cap\{\overline{D(M_{C,\varepsilon}(X))}^\tau : C \in \mathcal{C}, C$ absolutely convex, $C \subseteq B(X)$ and $\varepsilon > 0\}$.

Lemma 2.1 *For any absolutely convex $C \in \mathcal{C}$, $C \subseteq B(X)$, $x \in S(X)$ and $\delta > 0$, let*

$$\begin{aligned} d_1(C, x, \delta) &= \sup_{y \in C, 0<\lambda\leq\delta} \frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} \\ d_2(C, x, \delta) &= C\text{-}dia\ S(B(X^*), \hat{x}, \delta) \\ d_3(C, x, \delta) &= C\text{-}dia\ [\cup\{D(y) : y \in S(X) \cap B_\delta(x)\}] \end{aligned}$$

Then for any α, $\delta > 0$, we have,

(*i*) $d_2(C, x, \alpha) \leq d_1(C, x, \delta) + \dfrac{2\alpha}{\delta}$

(*ii*) $d_3(C, x, \delta) \leq d_2(C, x, \delta)$

(*iii*) $d_1(C, x, \delta) \leq d_3(C, x, 2\delta)$

Proof : (i) Fix α, $\delta > 0$. Put $d_1 = d_1(C, x, \delta)$ and $d_2 = d_2(C, x, \alpha)$.

For each $n \geq 1$, we can choose $f_n, g_n \in S(B(X^*), \hat{x}, \alpha)$ such that $\|f_n - g_n\|_C > d_2 - \frac{1}{n}$. Choose $y_n \in C$ such that $(f_n - g_n)(y_n) > d_2 - \frac{1}{n}$. Then

$$\frac{\|x+\delta y_n\| + \|x - \delta y_n\| - 2}{\delta} \geq \frac{f_n(x+\delta y_n) + g_n(x-\delta y_n) - 2}{\delta} \geq d_2 - \frac{1}{n} - \frac{2\alpha}{\delta}$$

Thus, $d_1 \geq d_2 - \frac{2\alpha}{\delta}$.

(ii) immediately follows from the observation that $\cup\{D(y) : y \in S(X) \cap B_\delta(x)\} \subseteq S(B(X^*), \hat{x}, \delta)$.

(iii) Let $\lambda \leq \delta$. Observe that for any $y \in C \subseteq B(X)$,

$$\begin{aligned}\left\|\frac{x \pm \lambda y}{\|x \pm \lambda y\|} - x\right\| &\leq \left\|\frac{x \pm \lambda y}{\|x \pm \lambda y\|} - (x \pm \lambda y)\right\| + \lambda = |1 - \|x \pm \lambda y\|\,| + \lambda \\ &= |\,\|x\| - \|x \pm \lambda y\|\,| + \lambda \leq 2\lambda\end{aligned}$$

Let $f \in D(\frac{x+\lambda y}{\|x+\lambda y\|})$ and $g \in D(\frac{x-\lambda y}{\|x-\lambda y\|})$. Then

$$\begin{aligned}\frac{\|x+\lambda y\| + \|x-\lambda y\| - 2}{\lambda} &= \frac{f(x+\lambda y) + g(x-\lambda y) - 2}{\lambda} \\ &\leq (f-g)(y) \leq \|f-g\|_C\end{aligned}$$

■

And we immediately have :

Lemma 2.2 *For any absolutely convex $C \in \mathcal{C}$, $C \subseteq B(X)$, $x \in S(X)$ and $\varepsilon > 0$, the following are equivalent :*

(*i*) $x \in M_{C,\varepsilon}(X)$

(*ii*) *There is a $\delta \in (0,1)$ such that the C-dia $S(B(X^*), \hat{x}, \delta) < \varepsilon$*

(*iii*) *There is a $\delta \in (0,1)$ such that C-dia $[\cup\{D(y) : y \in S(X) \cap B_\delta(x)\}] < \varepsilon$.*

Corollary 2.2.1 *For any absolutely convex $C \in \mathcal{C}$, $C \subseteq B(X)$ and $\varepsilon > 0$, $M_{C,\varepsilon}(X)$ is an open subset of $S(X)$.*

Proof : Follows immediately from the Lemma above and Lemma 1.8. ■

REMARK : In the case of $C = B(X)$, Lemma 2.1 is quantitatively more precise than Lemma 1 of [55].

Now we have our main result :

Theorem 2.3 *If X, F and $\mathcal{C}$ are as above, consider the following statements :*

(a) $F \subseteq \overline{\mathbb{R}^+ E_\tau}^\tau$

(b) $F \subseteq \overline{\mathbb{R}^+ H_\tau}^\tau$

(c) *If $C_1, C_2 \in \mathcal{C}$ are such that there exists $f \in F$ with $\sup f(C_1) < \inf f(C_2)$ then there exist disjoint closed balls B_1, B_2 such that $C_i \subseteq B_i$, $i = 1, 2$.*

(d) *Every $C \in \mathcal{C}$ is admissible.*

(e) *For every norm dense subset A of $S(X)$ and every support mapping $\phi : S(X) \longrightarrow S(X^*)$, $F \subseteq \overline{\mathbb{R}^+ \phi(A)}^\tau$.*

Then we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

(For the reverse implications, see corollaries and remarks at the end of this section.)

Proof : (a) $\Rightarrow$ (b) Enough to prove $E_\tau \subseteq H_\tau$.

Let $f \in E_\tau$. Let $C \in \mathcal{C}$, C absolutely convex, $C \subseteq B(X)$ and $\varepsilon > 0$. We want to prove $f \in \overline{D(M_{C,\varepsilon})}^\tau$. Let $K \in \mathcal{C}$ and $0 < \eta < \varepsilon$. We may assume $K \subseteq B(X)$. Let $K_o = \overline{\text{aco}}^\sigma(K \cup C)$. Then $K_o \subseteq B(X)$ as $B(X)$ is σ-closed. Note that $K_o \in \mathcal{C}$ by (A2).

Since f is an extreme point of the w*-compact convex set $B(X^*)$ and $id : (B(X^*), \text{w}^*) \to (B(X^*), \tau)$ is continuous at f, it follows from Lemma 1.2 that w*-slices of $B(X^*)$ containing f form a base for the relative τ-topology at f. Thus there exist $x \in S(X)$ and $0 < \delta < 1$ such that $f \in S = S(B(X^*), \hat{x}, \delta)$ and K_o-dia$(S) < \eta$.

Now, by Lemma 2.2, $x \in M_{K_o,\eta} \subseteq M_{C,\varepsilon}$ and for any $f_x \in D(x)$, $f_x \in D(M_{C,\varepsilon})$ and $f_x \in S$, so, $\|f - f_x\|_K \leq \|f - f_x\|_{K_o} < \eta$.

(b) $\Rightarrow$ (c) Let $C_1, C_2 \in \mathcal{C}$ and $f \in S(F)$ be such that $\sup f(C_1) < \inf f(C_2)$. Let $z \in X$ be such that $f(z) = \frac{1}{2}(\sup f(C_1) + \inf f(C_2))$ and put $\varepsilon = \frac{1}{12}(\inf f(C_2) - \sup f(C_1)) > 0$. Then, $\inf f(C_2 - z) > 5\varepsilon$ and $\inf(-f)(C_1 - z) > 5\varepsilon$. We may assume without loss of generality that

$z = 0$, $C_i \subseteq B(X)$, $i = 1,2$ and $\|f\| = 1$. Let $K = \overline{\text{aco}}^\sigma(C_1 \cup C_2)$, then $K \in \mathcal{C}$, K is absolutely convex and $K \subseteq B(X)$.

By (b), there is $\lambda \geq 0$ and $g \in H_\tau$ such that $\|f - \lambda g\|_K < \varepsilon$. If $\lambda = 0$, we have $\|f\|_K < \varepsilon$ and hence, $\inf_{C_2} f < \varepsilon$, a contradiction. Thus, $\lambda > 0$.

Now, $g \in H_\tau \subseteq \overline{D(M_{K,\varepsilon/\lambda})}^\tau$. So, we can find $x \in M_{K,\varepsilon/\lambda}$ and $h \in D(x)$ such that $\|g - h\|_K < \varepsilon/\lambda$. By definition, there is a $\delta > 0$ such that

$$\sup_{y \in K, 0<\alpha\leq\delta} \frac{\|x + \alpha y\| + \|x - \alpha y\| - 2}{\alpha} < \frac{\varepsilon}{\lambda}$$

Choose an integer $n > \frac{\lambda}{\varepsilon\delta}$. The proof will be complete once we show that $B_1 = B_{(n-1)\varepsilon/\lambda}[-n\varepsilon x/\lambda]$ and $B_2 = B_{(n-1)\varepsilon/\lambda}[n\varepsilon x/\lambda]$ work.

Clearly, B_1 and B_2 are disjoint. Suppose, if possible, $y \in C_2$ and $y \notin B_2$. Then $y \in K$. Take $\alpha = \frac{\lambda}{n\varepsilon} < \delta$ and observe that

$$\begin{aligned}
\frac{\|x + \alpha y\| + \|x - \alpha y\| - 2}{\alpha} &= \frac{\|x + \alpha y\| - \|x\|}{\alpha} + \|\frac{x}{\alpha} - y\| - \frac{1}{\alpha} \\
&\geq h(y) + \frac{(n-1)\varepsilon}{\lambda} - \frac{n\varepsilon}{\lambda} = h(y) - \frac{\varepsilon}{\lambda} \\
&\geq g(y) - \frac{2\varepsilon}{\lambda} = \frac{1}{\lambda}[\lambda g(y) - 2\varepsilon] \\
&> \frac{1}{\lambda}[f(y) - 3\varepsilon] \geq \frac{1}{\lambda}[5\varepsilon - 3\varepsilon] = \frac{2\varepsilon}{\lambda}
\end{aligned}$$

This contradicts the fact that $x \in M_{K,\varepsilon/\lambda}$. The other inclusion follows similarly once we note that K, and hence $M_{K,\varepsilon/\lambda}$, is symmetric and $h \in D(x)$ implies $(-h) \in D(-x)$.

(c) $\Rightarrow$ (d) Since singletons are in $\mathcal{C}$ and every $C \in \mathcal{C}$ is σ-closed, (d) follows from (c).

(d) $\Rightarrow$ (e). (We adapt Phelps' [42] arguments) Let A be a norm dense subset of $S(X)$ and ϕ be a support mapping. Let $f \in S(F)$, $K \in \mathcal{C}$ and $0 < \varepsilon < 1$. We may assume $K \subseteq B(X)$ and further that K is absolutely convex and $\|f\|_K > 1 - \varepsilon/2$ (Let $x \in B(X)$ such that $f(x) > 1 - \varepsilon/2$. Let $L = \overline{\text{aco}}^\sigma[\{x\} \cup K]$. Then $L \subseteq B(X)$, $L \in \mathcal{C}$ and $\|\cdot\|_L \geq \|\cdot\|_K$). Let $u \in K$ be such that $f(u) > 1 - \varepsilon/2$. Put $u' = \varepsilon u/4$ and $D = K \cap f^{-1}(0)$. Then $D \in \mathcal{C}$ [by (A3)] and $u' \notin D$. By (c), there exists $z \in X$ and $r > 0$ such that $D \subseteq B_r[z]$ and $\|u' - z\| > r$.

Let $\mu = \|u' - z\| - r > 0$. Put $w = \frac{1}{r+\mu}(ru' + \mu z)$. Then $\|w - z\| = r$. Put $x = \frac{1}{r}(w - z) \in S(X)$. Let $C = \overline{\text{co}}[\{u'\} \cup B_r[z]]$. Let $0 < \delta < \frac{\mu}{(r+\mu)}$. If $p \in B_{r\delta}[w]$, then $\|p - w\| < \frac{r\mu}{(r+\mu)}$, so, $p = w + \frac{r\mu}{r+\mu} \cdot y$ for some $y \in X$, $\|y\| < 1$. Thus, $p = \frac{r}{r+\mu}u' + \frac{\mu}{r+\mu}(z + ry)$. Now, $z + ry \in B_r(z)$ and hence, $p \in \text{int}(C)$. So, $B_{r\delta}[w] \subseteq \text{int}(C)$.

Let $y \in B_\delta[x] \cap A$ and $g = \phi(y)$. Put $v = ry + z$. Clearly, $\|v - w\| \leq r\delta$, hence $v \in \text{int}(C)$ and $g(v) = \sup g(B_r[z])$. Now, $v \in \text{int}(C) \Rightarrow$ there exists $t \in (0,1)$ and $v' \in B_r(z)$ such that $v = tu' + (1-t)v'$. Thus, $g(v) = tg(u') + (1-t)g(v') < tg(u') + (1-t)g(v)$. Also, $0 \in D \subseteq B_r[z] \Longrightarrow 0 \leq g(v) < g(u') = \frac{1}{4}\varepsilon g(u) \leq \frac{1}{4}\varepsilon \cdot \|g\|_K$. So, $0 < \|g\|_K \leq \|g\| = 1$. Put $\lambda = 1/\|g\|_K$. Then $\sup \lambda g(D) \leq \sup \lambda g(B_r[z]) = \lambda g(v) < \frac{1}{4}\varepsilon \cdot \|\lambda g\|_K = \varepsilon/4$. By symmetry of D, $\|\lambda g\|_D \leq \varepsilon/4$.

Now, by Lemma 1.7 applied to the linear space $\text{sp}(K)$, spanned by K, equipped with μ_K, the gauge or Minkowski functional of K, we have

$$\left\| \frac{f}{\|f\|_K} + \lambda g \right\|_K \leq \frac{\varepsilon}{2} \text{ or } \left\| \frac{f}{\|f\|_K} - \lambda g \right\|_K \leq \frac{\varepsilon}{2}$$

But $u \in K$ and $\varepsilon < 1$ implies $f(u)/\|f\|_K \geq f(u) > 1 - \varepsilon/2 > \varepsilon/2$ and $\lambda g(u) > 0$. Thus, $\|\frac{f}{\|f\|_K} - \lambda g\|_K \leq \varepsilon/2$. Then we have

$$\|f - \lambda g\|_K \leq \frac{\varepsilon}{2} + \left\| \frac{f}{\|f\|_K} - f \right\|_K = \frac{\varepsilon}{2} + (1 - \|f\|_K) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

Corollary 2.3.1 *If in the set-up of Theorem 2.3, we have that the set*

$$A = \{x \in S(X) : D(x) \cap E_r \neq \emptyset\}$$

is norm dense in $S(X)$, then all the statement in Theorem 2.3 are equivalent.

Proof : We simply note that in this case there is a support mapping that maps A into E_r and hence, $(e) \Rightarrow (a)$. ■

Corollary 2.3.2 [54,47,49] *In the case of examples (iii), i.e.,* C = {*all compact convex sets in* X}, F = *any norming subspace, and (iv), i.e.,* C = {*all compact convex sets in* X *with finite affine dimension*}, F = *any norming subspace, all the statements in Theorem* 2.3 *are equivalent and* (c) *can be reformulated as*

(c′) *Disjoint members of* C *can be separated by disjoint closed balls.*

Proof : In example (iii), τ is the bw* topology (see [16] for more on bw* topology) and in example (iv), τ is the w*-topology on X^* and in both cases $\overline{F}^{\tau} = X^*$, so we may as well take $F = X^*$. Further, as the bw* topology agrees with the w*-topology on bounded sets, in both the cases, E_τ = {extreme points of $B(X^*)$}. Clearly, in both cases A as in Corollary 2.3.1 is $S(X)$ and so, in Theorem 2.3 all the statements are equivalent.

Since members of C in both cases are σ-compact, (c) $\Longleftrightarrow$ (c′). ■

REMARKS : 1. In example (v), i.e., C = {all weakly compact convex set in X}, F = any norming subspace, τ is the Mackey topology on X^* (see [8] for further information) and again, $\overline{F}^{\tau} = X^*$.

In this case, we *do not know* whether any of the implications in Theorem 2.3 can be reversed. However, we note that $(a) \Rightarrow (d)$ in Theorem 2.3 gives a weaker sufficiency condition for MIP for weakly compact sets than the one used in [57]. And in this case, too, (c) and (c′) of Corollary 2.3.2 are equivalent.

2. It seems unlikely that, in general, the implications in Theorem 2.3 can be reversed. Nevertheless, it appears to be an interesting and difficult problem to find conditions on X, F, and C under which this can be done. In particular, when is the assumption of Corollary 2.3.1 satisfied ? In this context, the work of Namioka [41] on neighbourhoods of extreme points may conceivably be relevant.

However, there is yet another situation when the statements can actually be shown to be equivalent. And particular cases of this yield the characterisations of MIP and w*-MIP, i.e., examples (i) and (ii). This we take up in the next section.

3. Note that the subspace $F \subseteq X^*$ was assumed to be norming in order to ensure that σ-closure of norm bounded sets remain norm bounded which is implicit in (A2). However, if (A2) is satisfied, as in examples (iii), (iv) and (v), for any total subspace F, our results easily carry through with only minor technical modifications in the proofs.

2.2. The MIP with Respect to a Norming Subspace F

Our standing assumption in this section is that F is a subspace of X^* such that

the set $T_F = \{x \in S(X) : D(x) \cap S(F) \neq \emptyset\}$ *is a norm dense subset of* $S(X)$

(we shall write simply T when there is no confusion). Then F is necessarily norming. However, one can give examples (see below) of norming subspaces where this property does not hold. Let $\mathcal{C} = \{$all norm bounded, σ-closed convex sets in $X\}$. We say that X has F-MIP if every $C \in \mathcal{C}$ is admissible.

EXAMPLES : (i) $F = X^*$, $T = S(X)$ and we have the MIP.

(ii) $X = Y^*$, $F = \widehat{Y}$, $T = D(S(Y))$, which is dense by Bishop-Phelps Theorem [5], and we have the w*-MIP.

Now, since $B(X) \in \mathcal{C}$, τ is the norm topology on X^* and $E_\tau = \{$w*-denting points of $B(X^*)\}$. We need the following reformulation of Lemma 2.2 :

Lemma 2.4 *For $x \in S(X)$, F, T as above and $\varepsilon > 0$, the following are equivalent :*

(*i*) $x \in M_\varepsilon$

(*ii*) *x determines a slice of $B(F)$ of diameter less than ε*

(*iii*) *There exists $\delta > 0$ such that*

$$\operatorname{dia}[\cup\{D(y) \cap S(F) : y \in T \cap B_\delta(x)\}] < \varepsilon.$$

Proof : $(i) \Rightarrow (ii) \Rightarrow (iii)$ follows as easy adjustments of Lemma 2.1.

$(iii) \Rightarrow (i)$ Let $\delta > 0$ be as in (iii). Let $d_o = \text{dia}[\cup\{D(y) \cap S(F) : y \in T \cap B_\delta(x)\}] < \varepsilon$. Choose $\delta_o > 0$ such that $\delta_o^2 + 2\delta_o < \delta$ and $\delta_o^2 + 2\delta_o/\varepsilon < (1 - d_o/\varepsilon)$. Let $y \in S(X)$, $0 < \lambda < \delta_o$. Then as in the proof of Lemma 2.1, $\left\| \frac{x \pm \lambda y}{\|x + \lambda y\|} - x \right\| \leq 2\lambda$.

Find $x_1, x_2 \in T$ such that $\left\| \frac{x+\lambda y}{\|x+\lambda y\|} - x_1 \right\| \leq \lambda^2$ and $\left\| \frac{x-\lambda y}{\|x-\lambda y\|} - x_2 \right\| \leq \lambda^2$. Let $f_1, f_2 \in S(F)$ such that $f_i \in D(x_i)$, $i = 1, 2$. Observe that $\|x_i - x\| \leq \lambda^2 + 2\lambda \leq \delta_o^2 + 2\delta_o < \delta$, i.e., $x_1, x_2 \in T \cap B_\delta(x)$. Thus $\|f_1 - f_2\| \leq d_o$. Now,

$$0 \leq 1 - f_1\left(\frac{x + \lambda y}{\|x + \lambda y\|}\right) = f_1\left(x_1 - \frac{x + \lambda y}{\|x + \lambda y\|}\right) \leq \left\| \frac{x + \lambda y}{\|x + \lambda y\|} - x_1 \right\| \leq \lambda^2.$$

So, $f_1(x + \lambda y) \geq (1 - \lambda^2)\|x + \lambda y\|$. Similarly, $f_2(x - \lambda y) \geq (1 - \lambda^2)\|x - \lambda y\|$. So, we have

$$\begin{aligned}
\frac{\|x + \lambda y\| + \|x - \lambda y\| - 2}{\lambda} &\leq \frac{f_1(x + \lambda y) + f_2(x - \lambda y) - 2(1 - \lambda^2)}{\lambda(1 - \lambda^2)} \\
&= \frac{(f_1 + f_2)(x) - 2 + \lambda(f_1 - f_2)(y) + 2\lambda^2}{\lambda(1 - \lambda^2)} \\
&\leq \frac{\|f_1 - f_2\| + 2\lambda}{1 - \lambda^2} \leq \frac{d_o + 2\lambda}{1 - \lambda^2} \leq \frac{d_o + 2\delta_o}{1 - \delta_o^2}
\end{aligned}$$

(since $\dfrac{d_o + 2\lambda}{1 - \lambda^2}$ is increasing in λ). Thus,

$$\sup_{y \in S(X), 0 < \lambda \leq \delta_o} \frac{\|x + \lambda y\| + \|x - \lambda y\| - 2}{\lambda} \leq \frac{d_o + 2\delta_o}{1 - \delta_o^2} < \varepsilon$$

by the choice of δ_o. ■

DEFINITION : Let $D_F : T \to S(F)$ be defined by $D_F(x) = D(x) \cap S(F)$. We say that D_F is quasi-continuous if for every $f \in S(F)$ and $\varepsilon > 0$, there exists $x \in T$ and $\delta > 0$ such that $y \in B_\delta(x) \cap T$ implies $D_F(y) \subseteq B_\varepsilon[f]$. Notice that for $X = Y^*$ and $F = \widehat{Y}$, D_F is the inverse duality map.

Now, we are in a position to prove :

Theorem 2.5 *If X, F and T are as above, the following are equivalent :*

(a) *The w^*-denting points of $B(X^*)$ are norm dense in $S(F)$.*

(b) *For every $\varepsilon > 0$, $D(M_\varepsilon) \cap S(F)$ is norm dense in $S(F)$.*

(c) *If $C_1, C_2 \in C$ are such that there exists $f \in F$ with $\sup f(C_1) < \inf f(C_2)$ then there exist disjoint closed balls B_1, B_2 such that $C_i \subseteq B_i$, $i = 1, 2$.*

(d) *X has the F-MIP.*

(e) *D_F is quasi-continuous.*

(f) *For every support mapping ϕ that maps T into $S(F)$ and for every norm dense subset A of T, $\phi(A)$ is norm dense in $S(F)$.*

(Observe that since M_ε is open and T is dense, $D(M_\varepsilon) \cap S(F) \neq \emptyset$ whenever $M_\varepsilon \neq \emptyset$.)

Proof : (a) $\Rightarrow$ (b) Let f be a w*-denting point of $B(X^*)$ and $\varepsilon > 0$. As noted above $f \in S(F)$. Proceeding as in Theorem 2.3 (we may take $K = B(X)$ and so, $K_o = B(X)$), for any $0 < \eta < \varepsilon$, there is $x \in S(X)$ and $\alpha > 0$ such that $f \in S = S(B(X^*), \hat{x}, \alpha)$ and $\text{dia}(S) < \eta$. Since T is dense in $S(X)$, by Lemma 1.8, there is $y \in T$ and $\delta > 0$ such that $f \in S' = S(B(X^*), \hat{y}, \delta)$ and $S' \subseteq S$. Again as in Theorem 2.3, $y \in M_\varepsilon \cap T$ and for any $f_y \in D(y) \cap S(F)$, $\|f_y - f\| < \eta$.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (f) Just a simplified version of the implication (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) in Theorem 2.3 where we replace K by $B(X)$.

(e) $\Rightarrow$ (f) Follows from definitions.

(f) $\Rightarrow$ (e) Suppose D_F is not quasi-continuous. Then there exist $f \in S(F)$ and $\varepsilon > 0$ such that for all $x \in T$ and $\delta > 0$, there exists $y(x, \delta) \in B_\delta(x) \cap T$ such that $D_F(y) \not\subseteq B_\varepsilon[f]$.

Let $A = \{y(x, \delta) : x \in T,\ \delta > 0\}$. Then A is dense in T and we can get a support mapping ϕ that maps T into $S(F)$ and $\phi(A) \cap B_\varepsilon[f] = \emptyset$.

(e) $\Rightarrow$ (b) Let $f \in S(F)$ and $\varepsilon > 0$. Let $0 < \eta < \varepsilon/2$. By (e), there exists $x \in T$ and $\delta > 0$ such that $y \in T \cap B_\delta(x)$ implies $D(y) \cap S(F) \subseteq B_\eta[f]$.

But then $\text{dia}[\cup\{D(y)\cap S(F) : y \in T\cap B_\delta(x)\}] \le 2\eta < \varepsilon$. So by Lemma 2.4, $x \in M_\varepsilon \cap T$ and $f_x \in D(x)\cap S(F)$ implies $\|f_x - f\| < \eta$.

(b) $\Rightarrow$ (a) For $n \ge 1$, let $D_n = \{f \in S(F) : f$ is contained in a w*-open slice of $B(F)$ of diameter $< \frac{1}{n}\}$. By Lemma 2.4, $D(M_{1/n}) \cap S(F) \subseteq D_n$. Thus, for all $n \ge 1$, D_n is norm open dense subset of $S(F)$ and by Baire Category Theorem, $\cap D_n$ is dense in $S(F)$. But, it is easy to see that $\cap D_n =$ {w*-denting points of $B(X^*)$}. ■

For completeness, we record in full the following immediate consequences, most of which are well-known results in the MIP literature.

Theorem 2.6 [21, Theorem 2.1 and 3.1] *For a Banach space X*

(*1*) *the following are equivalent :*

(*a*) *The w*-denting points of $B(X^*)$ are norm dense in $S(X^*)$.*

(*b*) *For every $\varepsilon > 0$, $D(M_\varepsilon(X))$ is norm dense in $S(X^*)$.*

(*c*) *Whenever C_1, C_2 are closed bounded convex sets in X with $\text{dist}(C_1, C_2) > 0$, there exist disjoint closed balls B_1, B_2 such that $C_i \subseteq B_i$, $i = 1, 2$.*

(*d*) *X has the MIP.*

(*e*) *The duality map is quasi-continuous.*

(*f*) *Every support mapping maps norm dense subsets of $S(X)$ to norm dense subsets of $S(X^*)$.*

(*2*) *the following are equivalent :*

(*a*) *The denting points of $B(X)$ are norm dense in $S(X)$.*

(*b*) *For every $\varepsilon > 0$, $D^{-1}(M_\varepsilon(X^*))$ is norm dense in $S(X)$.*

(*c*) *Disjoint w*-compact convex sets in X^* can be separated by disjoint closed balls.*

(*d*) *X^* has the w*-MIP.*

(*e*) *The inverse duality map is quasi-continuous.*

(*f*) *Every support mapping that maps $D(S(X))$ into $S(X)$ maps norm dense subsets of $D(S(X))$ to norm dense subsets of $S(X)$.*

Proof : (1) Let $\text{dist}(C_1, C_2) = \delta > 0$. Let $K_2 = \overline{C_2 + B_{\delta/2}(0)}$. Then C_1 and K_2 are disjoint closed convex sets and K_2 has non-empty interior. Now, $f \in X^*$ that separates C_1 and K_2 strictly separates C_1 and C_2. Thus (1) follows from Theorem 2.5.

(2) Observe that $\{\hat{x} : x \text{ is a denting point of } B(X)\} = \{\text{w*-denting points of } B(X^{**})\}$. Now (2) is immediate from Theorem 2.5. ∎

REMARK : Since disjoint closed balls always have positive distance, this result cannot be strengthened. Note that this Theorem and Corllary 2.3.2 considerably strengthen Corollaries on p 341 and p 343 respectively of [56].

In the following proposition, we collect without proof some observations regarding the MIP. Recall (from [22]) that a norming subspace of X^* is *minimal* if it is contained in all norming subspaces.

Proposition 2.7 *For a Banach space X*

(*a*) *If the norm is Fréchet differentiable at all $x \in S(X)$, then X has the MIP and is Asplund. If the norm is smooth at all $x \in S(X)$, then X has the CI.*

(*b*) [42] *If X is finite dimensional, then X has the MIP (equivalently, the CI) if and only if the extreme points of $B(X^*)$ are dense in $S(X^*)$. In particular, a 2-dimensional space has the MIP if and only if it is smooth.*

(*c*) [42] *If the w*-strongly exposed points of $B(X^*)$ are dense in $S(X^*)$, X has the MIP.*

(*d*) [21] *If X has the MIP and the norm is Fréchet differentiable on a dense subset of $S(X)$, then the w*-strongly exposed points of $B(X^*)$ are dense in $S(X^*)$. In particular, this happens when X has the MIP and either X^* has the w*-MIP or X is Asplund.*

(*e*) [51] *If X is separable and has the MIP, then X is Asplund.*

(*f*) *If X has the F-MIP, then F is the minimal norming subspace of X^*. Hence, if X has the MIP, then X^* contains no proper norming subspace. In particular, X^* has the MIP implies X is reflexive.*

(*g*) *X has the MIP implies X^{**} has the w*-MIP.*

REMARKS : 1. Apropos (d) above, *does X has the MIP imply X is Asplund ?* In this context, Sullivan [51] has shown that if for some $\varepsilon > 0$, $M_\varepsilon = S(X)$, then X is Asplund.

2. As a strengthening of (f) above, he has also proved that if X is the range of a norm one projection in a dual space and for some $0 < \varepsilon < 1$, $D(M_\varepsilon)$ is dense in $S(X^*)$, then X is reflexive.

3. On the other hand, Godefroy and Kalton [22] have proved that if X is the range of a norm one projection in a dual space and every closed bounded convex set in X is the intersection of *finite union* of balls, then X is reflexive.

4. Does the minimal norming subspace of the dual, if it exists, always satisfy our standing assumption of this section ? Does there exist a subspace F of X^* satisfying our standing assumption that is minimal with respect to this property ?

5. Is the converse of (g) above true ?

2.3. A Digression

In [21], a version of Lemma 2.4 was proved for $X = Y^*$, $F = \widehat{Y}$ using the Bollobás' estimates for the Bishop-Phelps Theorem (see [5] and [6]). Specifically, the authors of [21] used the fact that in this case the following holds :

$$(*)\begin{cases} \textit{For every } x \in S(X) \textit{ and every sequence } \{f_n\} \subseteq S(F) \textit{ such} \\ \textit{that } f_n(x) \longrightarrow 1, \textit{ there exists a sequence } \{x_n\} \subseteq T \textit{ and} \\ f_{x_n} \in D(x_n) \cap S(F) \textit{ such that } \|x_n - x\| \to 0 \textit{ and } \|f_{x_n} - f_n\| \to 0. \end{cases}$$

In fact, one can show that in this situation, the following stronger property holds (see [20]) :

$$(**)\begin{cases} \textit{For every } x \in S(X),\ f \in S(F) \textit{ and } \varepsilon > 0 \textit{ with } f(x) > 1 - \varepsilon^2 \\ \textit{there exists } y \in T \textit{ and } f_y \in D(y) \cap S(F) \textit{ such that } \|x - y\| \le \varepsilon \\ \textit{and } \|f - f_y\| \le \varepsilon. \end{cases}$$

Using the fact that $(**)$ holds for $F = X^*$, one can show that $(**)$ also holds if F is an L-summand in X^*, i.e., there is a projection P on X^* with $P(X^*) = F$ such that for any $f \in X^*$, $\|f\| = \|Pf\| + \|f - Pf\|$. Here is a quick proof.

Let $x \in S(X)$, $f \in S(F)$ and $0 < \varepsilon < 1$ be given such that $f(x) > 1-\varepsilon^2$. Then there exists $y \in S(X)$ and $y^* \in D(y)$ such that $\|x - y\| \leq \varepsilon$ and $\|f - y^*\| \leq \varepsilon$. Note that $\|f - Py^*\| \leq \|f - y^*\| \leq \varepsilon < 1$, so that $Py^* \neq 0$. If $y^* = Py^*$, we are done. Otherwise observe that

$$\begin{aligned} 1 &= \|y^*\| = \|Py^*\| + \|y^* - Py^*\| \quad \text{and} \\ 1 &= y^*(y) = Py^*(y) + (y^* - Py^*)(y) \\ &= \|Py^*\| \left(\frac{Py^*}{\|Py^*\|}\right)(y) + \|y^* - Py^*\| \left(\frac{y^* - Py^*}{\|y^* - Py^*\|}\right)(y) \leq 1 \end{aligned}$$

Thus, $Py^*/\|Py^*\| \in D(y) \cap S(F)$, whence $y \in T$. Further

$$\begin{aligned} \left\| f - \frac{Py^*}{\|Py^*\|} \right\| &\leq \|f - Py^*\| + (1 - \|Py^*\|) \\ &= \|f - Py^*\| + \|y^* - Py^*\| = \|f - y^*\| \leq \varepsilon \end{aligned}$$

Clearly, if $(**)$ holds for the pair (X, F), it also holds for the pair $(F, \widehat{X})$. In particular, $(**)$ holds for each of the following :

(1) $X = C[0,1]$, $F = \{$discrete measures on $[0, 1]\}$,

(2) $X = C[0,1]$, $F = \{$absolutely continuous measures on $[0, 1]\}$,

(3) $X = L^1[0,1]$, $F = C[0,1]$.

So, in these cases, $(*)$ also holds and the proof of [21] can be used to prove Lemma 2.4.

Another example of a situation in which $(**)$ is satisfied is furnished by Proposition 3.4 in the next chapter.

However, one can construct examples (see below) to show that $(**)$ does not, in general, follow from the density of T in $S(X)$. It would be interesting to know whether $(*)$ does. Also it would be interesting to find general sufficiency conditions for $(**)$ to hold which would cover at least the case $X = Y^*$ and $F = \widehat{Y}$. In particular, is the following obviously necessary condition also sufficient for $(**)$ to hold :

T is dense in $S(X)$ and $D(T) \cap S(F)$ is dense in $S(F)$?

But these may be difficult problems.

EXAMPLE : Let X be a non-reflexive Banach space. Let $F \subseteq X^*$ be a norming subspace which is an L-summand in X^*. Let P be the corresponding L-projection. Let $f_o \in (I-P)(X^*)$ such that $\|f_o\| = 1$ and f_o does not attain its norm on $B(X)$. Let $F_1 = F \oplus_1 \mathbb{R} f_o$. Then F_1 is a norming subspace of X^* and $f_o \in S(F_1)$. Let $0 < \varepsilon < \frac{1}{2}$. Suppose there exists $x \in S(X)$, $g \in S(F_1)$ such that $\|f_o - g\| < \varepsilon$ and $g(x) = 1$.

Now, $g = f + \alpha f_o$ for some $f \in F$, $\alpha \in \mathbb{R}$. We have

$$1 = \|g\| = \|f\| + \|\alpha f_o\| = \|f\| + |\alpha|$$

If $\alpha = 0$, $g = f$ and we have

$$\varepsilon > \|f_o - g\| \geq \|P f_o - P g\| = \|g\| = 1$$

So, $\alpha \neq 0$. Also, $f = 0$ implies $g = \alpha f_o$ and so $f_o(x) = \pm 1$, a contradiction as f_o does not attain its norm. Thus, $f \neq 0$.

But then,

$$1 = g(x) = f(x) + \alpha f_o(x) = \|f\| \cdot \frac{f}{\|f\|}(x) + |\alpha| f_o\left(\frac{\alpha x}{|\alpha|}\right) \leq \|f\| + |\alpha| = 1$$

This implies $f_o\left(\frac{\alpha x}{|\alpha|}\right) = 1$. Again, a contradiction.

As noted earlier, the pair (X, F) satisfies $(**)$, so T_F is dense in $S(X)$ and $D(T_F) \cap S(F)$ is dense in $S(F)$. Now, clearly $T_{F_1} \supseteq T_F$, but the above shows $D(T_{F_1}) \cap S(F_1)$ is not dense in $S(F_1)$. Consequently, though T_{F_1} is dense in $S(X)$, $(**)$ is not satisfied.

Also, interchanging the roles of X and F_1, the above shows that though $\widehat{X}$ is a norming subspace of F_1^*, $T_{\widehat{X}} = D(T_{F_1}) \cap S(F_1)$ is not dense, i.e., our standing assumption is not satisfied.

Finally, we note that $X = C[0,1]$, $F = \{$discrete measures on $[0, 1]\}$ and $f_o = \lambda|_{[0,1/2]} - \lambda|_{[1/2,1]}$ satisfies the hypothesis of the above example, where λ denotes the Lebesgue measure.

Chapter 3

Bochner L^p Spaces and The ℓ^p Sums of Banach Spaces with the MIP

3.1. The ℓ^p Sums

That the existence of an equivalent renorming with the MIP or the CI is stable under c_o or ℓ^p sums for $1 < p < \infty$ was established by A. Sersouri [47] using a renorming result for the MIP by R. Deville [9] and for the CI by himself [47].

We can, however, prove the following direct result for the ℓ^p sums, $1 < p < \infty$.

Theorem 3.1 *Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of Banach spaces. Then the space $X = (\oplus_{\alpha\in\Gamma} X_\alpha)_{\ell^p}$ with the usual ℓ^p-norm $(1 < p < \infty)$ has the MIP (resp. the CI) if and only if for all $\alpha \in \Gamma$, the space X_α has the MIP (resp. the CI).*

Recall that if X is as above, $X^* = (\oplus_{\alpha\in\Gamma} X_\alpha^*)_{\ell^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. The following lemma is well-known :

Lemma 3.2 [50, p 154] *Let $\{X_\alpha\}$ and X be as above. A point $x = (x_\alpha) \in S(X)$ is an extreme point of $B(X)$ if and only if for all $\alpha \in \Gamma$, either $x_\alpha = 0$ or $x_\alpha/\|x_\alpha\|$ is an extreme point of $B(X_\alpha)$.*

[50] also contains similar characterisations of various other types of extreme points of $B(X)$. In the dual situation, we can prove

Lemma 3.3 *Let $\{X_\alpha\}$ and X be as above. A point $x^* = (x^*_\alpha) \in S(X^*)$ is a w*-denting point of $B(X^*)$ if and only if for every α, either $x^*_\alpha = 0$ or $x^*_\alpha/\|x^*_\alpha\|$ is a w*-denting point of $B(X^*_\alpha)$.*

We postpone the proof of Lemma 3.3 till the next section where similar results will be proved in a more general setting.

Proof of Theorem 3.1 : (*The MIP*) : Let for all $\alpha \in \Gamma, X_\alpha$ have the MIP. Let $x^* = (x^*_\alpha) \in S(X^*)$. Fix $\varepsilon > 0$. Let $\Gamma_o = \{\alpha \in \Gamma : x^*_\alpha \neq 0\}$. For $\alpha \in \Gamma_o$, $x^*_\alpha/\|x^*_\alpha\| \in S(X^*_\alpha)$ and since X_α has the MIP, there exists y^*_α, a w*-denting point of $B(X^*_\alpha)$, such that $\left\|y^*_\alpha - \frac{x^*_\alpha}{\|x^*_\alpha\|}\right\| < \varepsilon$. Define $z^* = (z^*_\alpha)$ by

$$z^*_\alpha = \begin{cases} \|x^*_\alpha\| y^*_\alpha & \text{if } \alpha \in \Gamma_o \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 3.3, z^* is a w*-denting point of $B(X^*)$ and $\|x^* - z^*\|_q < \varepsilon$.

Conversely, if X has the MIP, let $\alpha_o \in \Gamma$ and $x^*_{\alpha_o} \in S(X^*_{\alpha_o})$. Fix $0 < \varepsilon < 1$. The point $x^* = (x^*_\alpha)$ defined by

$$x^*_\alpha = \begin{cases} x^*_{\alpha_o} & \text{if } \alpha = \alpha_o \\ 0 & \text{otherwise} \end{cases}$$

is in $S(X^*)$ and hence there exists y^*, a w*-denting point of $B(X^*)$, such that $\|x^* - y^*\|_q < \varepsilon$.

Clearly, $\|x^*_{\alpha_o} - y^*_{\alpha_o}\| < \varepsilon < 1$ and so, $y^*_{\alpha_o} \neq 0$. Also, by Lemma 3.3, $y^*_{\alpha_o}/\|y^*_{\alpha_o}\|$ is a w*-denting point of $B(X^*_{\alpha_o})$. Now,

$$\begin{aligned} \left\| x^*_{\alpha_o} - \frac{y^*_{\alpha_o}}{\|y^*_{\alpha_o}\|} \right\| &\leq \|x^*_{\alpha_o} - y^*_{\alpha_o}\| + |1 - \|y^*_{\alpha_o}\|\,| < \varepsilon + |\,\|x^*_{\alpha_o}\| - \|y^*_{\alpha_o}\|\,| \\ &< \varepsilon + \|x^*_{\alpha_o} - y^*_{\alpha_o}\| < 2\varepsilon \end{aligned}$$

Hence, X_{α_o} has the MIP.

(*The CI*) : Let for all $\alpha \in \Gamma$, X_α have the CI.

Let $x^* = (x^*_\alpha) \in S(X^*)$. Let K be a compact set in X. We may assume $K \subseteq B(X)$.

Let $\pi_\alpha : X \longrightarrow X_\alpha$ be the co-ordinate projection. Then $K_\alpha = \pi_\alpha(K)$ is a compact set in X_α, for all $\alpha \in \Gamma$. Fix $\varepsilon > 0$. Choose $\Gamma_o \subseteq \Gamma$, Γ_o finite, such that $\|x^*\chi_{\Gamma\setminus\Gamma_o}\|_q < \varepsilon$ and for all $\alpha \in \Gamma_o$, $x^*_\alpha \neq 0$. For $\alpha \in \Gamma_o$, there exists y^*_α, an extreme point of $B(X^*_\alpha)$ and $a_\alpha \geq 0$ such that $\|x^*_\alpha - a_\alpha y^*_\alpha\|_{K_\alpha} < \varepsilon\|x^*_\alpha\|^q$, where $\|z^*_\alpha\|_{K_\alpha} = \sup\{|z^*_\alpha(k_\alpha)| : k_\alpha \in K_\alpha\}$. Define $u^* = (u^*_\alpha)$ by

$$u^*_\alpha = \begin{cases} a_\alpha y^*_\alpha & \text{if } \alpha \in \Gamma_o \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $u^*_\alpha/\|u^*_\alpha\| = y^*_\alpha$ is an extreme point of $B(X^*_\alpha)$. So, by Lemma 3.2, $u^*/\|u^*\|_q$ is an extreme point of $B(X^*)$ and for any $k = (k_\alpha) \in K$, we have

$$\begin{aligned} |(x^* - u^*)(k)| &= |\sum_{\alpha\in\Gamma}(x^*_\alpha - u^*_\alpha)(k_\alpha)| \\ &\leq \sum_{\alpha\in\Gamma_o}|(x^*_\alpha - a_\alpha y^*_\alpha)(k_\alpha)| + \sum_{\alpha\notin\Gamma_o}|x^*_\alpha(k_\alpha)| \\ &\leq \sum_{\alpha\in\Gamma_o}\|x^*_\alpha - a_\alpha y^*_\alpha\|_{K_\alpha} + (\sum_{\alpha\notin\Gamma_o}\|x^*_\alpha\|^q)^{1/q}(\sum_{\alpha\notin\Gamma_o}\|k_\alpha\|^p)^{1/p} \\ &\leq \varepsilon\sum_{\alpha\in\Gamma_o}\|x^*_\alpha\|^q + \varepsilon(\sum_{\alpha\notin\Gamma_o}\|k_\alpha\|^p)^{1/p} \leq \varepsilon[\|x^*\|^q_q + \|k\|_p] \leq 2\varepsilon \end{aligned}$$

since $x^* \in S(X^*)$ and $k \in K \subseteq B(X)$. Hence $\|x^* - u^*\|_K \leq 2\varepsilon$.

Conversely, let X have the CI. Let $\alpha_o \in \Gamma$, $x^*_{\alpha_o} \in S(X^*_{\alpha_o})$, $K_{\alpha_o} \subseteq X_{\alpha_o}$ compact. Define $x^* \in S(X^*)$ as in the case of the MIP. Define $K = \{(x_\alpha) : x_{\alpha_o} \in K_{\alpha_o}$ and $x_\alpha = 0$ for $\alpha \neq \alpha_o\}$. Clearly, K is compact and for any $z^* \in X^*$, $\|z^*\|_K = \|z^*_{\alpha_o}\|_{K_{\alpha_o}}$.

Now, the CI in X_{α_o} clearly follows from that in X. ■

REMARK : It follows from Proposition 4.1 of the next chapter that if the c_o or the ℓ^∞ sum of a family of Banach spaces has the MIP (or the CI), then each of them has the MIP (the CI). The converse is not true in general as the finite-dimensional spaces $\ell^\infty_n, n \geq 1$, fail the CI.

See also Proposition 4.2 below.

3.2. Bochner L^p Spaces

Let X be a Banach space, (Ω, Σ, μ) a measure space, $1 < r < \infty$. Let $A \subseteq S(X)$. Define

$$\mathcal{M}_r(A) = \{f \in S(L^r(\mu, X)) : f \text{ is of the form } f = \sum_{i=1}^{n} x_i \chi_{E_i} \text{ with } n \geq 1 \text{ arbitrary, } E_i \in \Sigma \text{ and } x_i/\|x_i\| \in A \text{ for all } i = 1, 2, \ldots, n\}$$

$$\mathcal{P}_r(A) = \{f \in S(L^r(\mu, X)) : f \text{ is of the form } f = \sum_{i=1}^{\infty} x_i \chi_{E_i} \text{ with } E_i \in \Sigma \text{ and } x_i = 0 \text{ or } x_i/\|x_i\| \in A \text{ for all } i \geq 1\} \text{ and,}$$

$$\mathcal{N}_r(A) = \{f \in S(L^r(\mu, X)) : \text{ for almost all } t \in \Omega, \text{ either } f(t) = 0 \text{ or } f(t)/\|f(t)\| \in A\}$$

Note that $\mathcal{M}_r(A) \subseteq \mathcal{P}_r(A) \subseteq \mathcal{N}_r(A) \subseteq S(L^r(\mu, X))$.

The following results are well-known :

(1) If $A = \{$extreme points of $B(X)\}$ then $\mathcal{N}_r(A) \subseteq \{$extreme points of $B(L^r(\mu, X))\}$. (See, e.g., [52]).

(2) If $A = \{$strongly exposed points of $B(X)\}$ then $\mathcal{N}_r(A) \supseteq \{$strongly exposed points of $B(L^r(\mu, X))\} \supseteq \mathcal{M}_r(A)$ ([32,23,24]).

(3) If $A = \{$denting points of $B(X)\}$ then $\mathcal{N}_r(A) = \{$denting points of $B(L^r(\mu, X))\}$ [37].

A survey of similar results may be found in [50]. In the dual situation, we note that :

(1) If $A = \{$extreme points of $B(X^*)\}$ then $\mathcal{N}_q(A) \subseteq \{$extreme points of $B(L^q(\mu, X^*))\}$. In the following we prove a stronger assertion.

(2) Any w*-denting point or w*-strongly exposed point of $B(V_q(\mu, X^*))$, being a w*-PC, necessarily belongs to $L^q(\mu, X^*)$) (see Proposition 3.4 below). And it is implicit in [32] that if $A = \{$w*-strongly exposed points of $B(X^*)\}$ then $\mathcal{M}_q(A) \subseteq \{$w*-strongly exposed points of $B(L^q(\mu, X^*))\}$. Below, we prove by different methods, a similar result for w*-denting points of $B(V_q(\mu, X^*))$, the proof being more involved than that for w*-strongly exposed points.

But, first we note the following :

Proposition 3.4 *Let $Y = L^p(\mu, X)$ and $F = L^q(\mu, X^*)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then every simple function in $S(Y)$ is in T, where $T = \{x \in Y : D(x) \cap S(F) \neq \emptyset\}$. So, T is dense in $S(Y)$, F is norming and we are in* *the set-up of section 2.2. Moreover, the pair (Y, F) satisfies $(**)$ as defined in section 2.3.*

Proof : Let $x = \sum_{i=1}^{n} x_i \chi_{E_i}$ be any simple function in $S(Y)$ and $\phi : S(X) \longrightarrow S(X^*)$ be any support mapping. Define

$$\Phi(x) = \sum_{i=1}^{n} \|x_i\|^{p-1} \phi\left(\frac{x_i^*}{\|x_i^*\|}\right) \chi_{E_i}$$

Then, $\Phi(x) \in S(L^q(\mu, X^*)) \subseteq S(V_q(\mu, X^*))$ and $\langle x, \Phi(x) \rangle = 1$. Thus, $\Phi(x) \in D(x) \cap S(F)$. This proves the first part of the Proposition.

Now, let $x \in S(L^p(X))$, $f \in S(L^q(X^*))$ and $\varepsilon > 0$ be such that $f(x) > 1 - \varepsilon^2$. Choose $0 < \eta < \varepsilon$ such that $0 < \eta[2(\varepsilon+1) - \eta] < f(x) - (1 - \varepsilon^2)$. Let z and g be simple functions in $S(L^p(X))$ and $S(L^q(X^*))$ respectively such that $\|x - z\|_p < \eta$ and $\|f - g\|_q < \eta$. Refining the partitions if necessary, we may assume that there is a finite partition $\{E_1, E_2, \ldots, E_n\}$ of Ω such that $z = \sum_{i=1}^{n} z_i \chi_{E_i}$ and $g = \sum_{i=1}^{n} z_i^* \chi_{E_i}$ where $z_i \in X$, $z_i^* \in X^*$. Now, $g(z) = \sum_{i=1}^{n} z_i^*(z_i)\mu(E_i) > f(x) - 2\eta > 1 - (\varepsilon - \eta)^2$, by the choice of η. Now, consider the discrete measure space $\Omega' = \{1, 2, \ldots, n\}$ with measure P, where $P(i) = \mu(E_i)$. Then z and g can be isometrically identified with elements of $S(L^p(P, X))$ and $S(L^q(P, X^*))$ respectively. But as P is discrete, $L^p(P, X)^* = L^q(P, X^*)$ and so $(**)$ is satisfied, i.e., there exists vectors $(y_1, y_2, \ldots, y_n)$ and $(y_1^*, y_2^*, \ldots, y_n^*)$ in $S(L^p(P, X))$ and $S(L^q(P, X^*))$ respectively such that $\sum_{i=1}^{n} y_i^*(y_i)P(i) = 1$ and $[\sum_{i=1}^{n} \|z_i - y_i\|^p P(i)]^{1/p} \leq (\varepsilon - \eta)$ and $[\sum_{i=1}^{n} \|z_i^* - y_i^*\|^q P(i)]^{1/q} \leq (\varepsilon - \eta)$. Put $y = \sum_{i=1}^{n} y_i \chi_{E_i}$ and $f_y = \sum_{i=1}^{n} y_i^* \chi_{E_i}$. Then $y \in S(L^p(\mu, X))$, $f_y \in S(L^q(\mu, X^*))$ and $f_y(y) = 1$. Further $\|x - y\|_p \leq (\varepsilon - \eta) + \eta = \varepsilon$ and $\|f - f_y\|_q \leq \varepsilon$. ■

The CI

Lemma 3.5 *Let X be a Banach space, (Ω, Σ, μ) a measure space and $1 < q < \infty$. Let $A = \{$extreme points of $B(X^*)\} \subseteq S(X^*)$. Then*

(a) $\mathcal{M}_q(A) = \mathcal{M}_q(S(X^*)) \cap \{$*extreme points of* $B(V_q(\mu, X^*))\}$.

(b) $\mathcal{P}_q(A) = \mathcal{P}_q(S(X^*)) \cap \{$*extreme points of* $B(V_q(\mu, X^*))\}$.

Proof : We prove (a), the proof of (b) being an easy adjustment.

Let $F = \sum_{i=1}^{n} x_i^* \chi_{A_i} \in \mathcal{M}_q(A)$. Suppose that there exist $G_1, G_2 \in B(V_q(\mu, X^*))$ with $F = \frac{1}{2}(G_1 + G_2)$.

We will prove that $F = G_1 = G_2$, i.e., for any $E \in \Sigma$ with $\mu(E) > 0$, $F(E) = G_1(E) = G_2(E)$. Fix any such $E \in \Sigma$. Put $A_o = \Omega \setminus (\cup_{i=1}^{n} A_i)$. Then $\pi = \{A_o, A_1, \dots, A_n\}$ is a partition of Ω. For $i = 0, 1, \dots, n$, define $E_{i1} = E \cap A_i$ and $E_{i2} = E^c \cap A_i$. Put $x_o^* = 0$.

Now, for any $i = 0, 1, \dots, n$ and $j = 1, 2$ with $\mu(E_{ij}) \neq 0$, we have $x_i^* \mu(E_{ij}) = F(E_{ij}) = \frac{1}{2}[G_1(E_{ij}) + G_2(E_{ij})]$ or,

$$x_i^* = \frac{1}{2}\left[\frac{G_1(E_{ij}) + G_2(E_{ij})}{\mu(E_{ij})}\right]$$

So,

$$\|x_i^*\| \leq \frac{1}{2}\left[\frac{\|G_1(E_{ij})\| + \|G_2(E_{ij})\|}{\mu(E_{ij})}\right]$$

and hence

$$\|x_i^*\|^q \leq \frac{1}{2}\left[\frac{\|G_1(E_{ij})\|^q + \|G_2(E_{ij})\|^q}{\mu(E_{ij})^q}\right]$$

by the convexity of the map $t \longrightarrow t^q$ $(q > 1)$. But then

$$\begin{aligned} 1 &= \|F\|_q^q = \sum_{i=0}^{n} \|x_i^*\|^q \mu(A_i) \\ &= \sum_{i=0}^{n}\sum_{j=1}^{2} \|x_i^*\|^q \mu(E_{ij}) = \sum_{\{(i,j):\mu(E_{ij})\neq 0\}} \|x_i^*\|^q \mu(E_{ij}) \\ &\leq \frac{1}{2}\left[\sum_{\mu(E_{ij})\neq 0} \frac{\|G_1(E_{ij})\|^q}{\mu(E_{ij})^q}\mu(E_{ij}) + \sum_{\mu(E_{ij})\neq 0} \frac{\|G_2(E_{ij})\|^q}{\mu(E_{ij})^q}\mu(E_{ij})\right] \\ &\leq \frac{1}{2}[\|G_1\|_q^q + \|G_2\|_q^q] \leq 1 \end{aligned}$$

So, we must have

$$\|x_i^*\|^q = \frac{1}{2}\left[\frac{\|G_1(E_{ij})\|^q + \|G_2(E_{ij})\|^q}{\mu(E_{ij})^q}\right] \text{ for all } i, j \text{ with } \mu(E_{ij}) \neq 0$$

Then the strict convexity of the map $t \longrightarrow t^q$ gives

$$\|x_i^*\| = \frac{\|G_1(E_{ij})\|}{\mu(E_{ij})} = \frac{\|G_2(E_{ij})\|}{\mu(E_{ij})} \text{ whenever } \mu(E_{ij}) \neq 0$$

Thus,

$$\frac{x_i^*}{\|x_i^*\|} = \frac{1}{2}\left[\frac{G_1(E_{ij})}{\|G_1(E_{ij})\|} + \frac{G_2(E_{ij})}{\|G_2(E_{ij})\|}\right] \text{ whenever } \mu(E_{ij}) \neq 0$$

Now, the extremality of $x_i^*/\|x_i^*\|$ implies

$$\frac{x_i^*}{\|x_i^*\|} = \frac{G_1(E_{ij})}{\|G_1(E_{ij})\|} = \frac{G_2(E_{ij})}{\|G_2(E_{ij})\|}$$

whence

$$x_i^* = \frac{G_1(E_{ij})}{\mu(E_{ij})} = \frac{G_2(E_{ij})}{\mu(E_{ij})} \text{ whenever } \mu(E_{ij}) \neq 0$$

So,

$$F(E) = \sum_{i=0}^{n} x_i^* \mu(E_{i1}) = \sum_{i=0}^{n} G_1(E_{i1}) = G_1(E)$$

Similarly, $F(E) = G_2(E)$.

The reverse inclusion being obvious, this completes the proof. ■

Theorem 3.6 *For any Banach space X, any finite non-atomic measure space (Ω, Σ, μ) and $1 < p < \infty$, the space $L^p(\mu, X)$ has the CI.*

Proof : Let τ denote the topology on $V_q(\mu, X^*)$ of uniform convergence on compact subsets of $L^p(\mu, X)$. By Proposition 3.4, $B(L^q(\mu, X^*))$ is w*-dense in $B(V_q(\mu, X^*))$, so $L^q(\mu, X^*)$ is τ-dense in $V_q(\mu, X^*)$, and simple functions are norm dense in $L^q(\mu, X^*)$. Thus it suffices, by Lemma 3.5(a), to show that given any simple function $F \in S(L^q(\mu, X^*))$, any compact subset K of $L^p(\mu, X)$ and any $\varepsilon > 0$, there exists a function $F_1 \in \mathcal{M}_q(A)$ such that $\|F - F_1\|_K < \varepsilon$, where $A = \{\text{extreme points of } B(X^*)\} \subseteq S(X^*)$.

Now, since K is compact and simple functions are dense in $L^p(\mu, X)$, there exists simple functions $\{g_1, g_2, \ldots, g_m\}$ such that $K \subseteq \cup_{i=1}^m B_{\varepsilon/4}(g_i)$. Suppose we have found a function $F_1 \in \mathcal{M}_q(A)$ such that $|(F - F_1)(g_j)| < \varepsilon/2$ for all $j = 1, 2, \ldots, m$. Then for any $g \in K$, there exists g_j such that $\|g - g_j\|_p < \varepsilon/4$. So, $|(F - F_1)(g)| \leq |(F - F_1)(g_j)| + |(F - F_1)(g - g_j)| \leq \frac{1}{2}\varepsilon + (\|F\|_q + \|F_1\|_q) \cdot \|g - g_j\|_p \leq \frac{1}{2}\varepsilon + 2 \cdot \frac{1}{4}\varepsilon = \varepsilon$, since $\|F\|_q = \|F_1\|_q = 1$.

Therefore we may as well assume K is a finite set $\{g_1, g_2, \ldots, g_m\}$. Further, passing to finer partitions if necessary, we may assume there exists a partition $A_1, A_2, \ldots, A_n$ of Ω such that $\mu(A_i) > 0$ for all $i = 1, 2, \ldots, n$ and each of the functions $g_1, g_2, \ldots, g_m$ and F take constant values on each A_i, i.e., the functions $g_1, g_2, \ldots, g_m$ and F have the form $F = \sum_{i=1}^{n} \alpha_i x_i^* \chi_{A_i}$ with $x_i^* \in S(X^*)$ for all $i = 1, 2, \ldots, n$ and $g_j = \sum_{i=1}^{n} x_{ij} \chi_{A_i}$ for all $j = 1, 2, \ldots, m$, where some of the α_i's and x_{ij}'s may be zero and the $\alpha_i x_i^*$'s and x_{ij}'s need not all be distinct. Then,

$$F(g_j) = \sum_{i=1}^{n} \alpha_i x_i^*(x_{ij}) \mu(A_i) \text{ for all } j = 1, 2, \ldots, m$$

Now, μ is finite implies $\|F\|_1 = \sum_{i=1}^{n} |\alpha_i| \mu(A_i) < \infty$. Fix $0 < \eta < \varepsilon / \|F\|_1$. Since $B(X^*)$ is w*-compact and convex, by the Krein-Milman Theorem we have :

> for each $i = 1, 2, \ldots, n$, there exists $\lambda_{ik} \geq 0$, $k = 1, 2, \ldots, N$ with $\sum_{k=1}^{N} \lambda_{ik} = 1$ and $x_{i1}^*, x_{i2}^*, \ldots, x_{iN}^*$, extreme points of $B(X^*)$ such that
>
> $$|(x_i^* - \sum_{k=1}^{N} \lambda_{ik} x_{ik}^*)(x_{rs})| < \eta \text{ for all } r = 1, 2, \ldots, n, \ s = 1, 2, \ldots, m.$$

Since μ is non-atomic, for each $i = 1, 2, \ldots, n$, there is a partition $\{A_{i1}, A_{i2}, \ldots, A_{iN}\}$ of A_i such that $\mu(A_{ik}) = \lambda_{ik} \mu(A_i)$ for all $k = 1, 2, \ldots, N$. Define

$$F_1 = \sum_{i=1}^{n} \sum_{k=1}^{N} \alpha_i x_{ik}^* \chi_{A_{ik}}$$

Then

$$\|F_1\|_q^q = \sum_{i=1}^{n} |\alpha_i|^q \sum_{k=1}^{N} \mu(A_{ik}) = \sum_{i=1}^{n} |\alpha_i|^q \mu(A_i) = \|F\|_q^q = 1$$

and since each $x_{ik}^* \in A$, $F_1 \in \mathcal{M}_q(A)$. Further, for all $j = 1, 2, \ldots, m$

$$\begin{aligned} F_1(g_j) &= \sum_{i=1}^{n} \sum_{k=1}^{N} \alpha_i x_{ik}^*(x_{ij}) \mu(A_{ik}) = \sum_{i=1}^{n} \sum_{k=1}^{N} \alpha_i x_{ik}^*(x_{ij}) \lambda_{ik} \mu(A_i) \\ &= \sum_{i=1}^{n} \alpha_i \mu(A_i) (\sum_{k=1}^{N} \lambda_{ik} x_{ik}^*)(x_{ij}) \end{aligned}$$

and so,

$$\begin{aligned}|(F-F_1)(g_j)| &= |\sum_{i=1}^{n}\alpha_i\mu(A_i)[x_i^* - \sum_{k=1}^{N}\lambda_{ik}x_{ik}^*](x_{ij})| \\ &\leq \sum_{i=1}^{n}|\alpha_i|\cdot\mu(A_i)\cdot|[x_i^* - \sum_{k=1}^{N}\lambda_{ik}x_{ik}^*](x_{ij})| \\ &\leq \eta\sum_{i=1}^{n}|\alpha_i|\mu(A_i) = \eta\|F\|_1 < \varepsilon\end{aligned}$$

■

REMARKS : 1. This shows that the CI is indeed much weaker than smoothness, as $L^p(\mu, X)$ is smooth if and only if X is (see [36]).

2. If the existence of an equivalent CI renorming is hereditary, it would follow from Theorem 3.6 that every Banach space admits an equivalent CI renorming. This was also noted by Sersouri [47], although in a different context.

The MIP

Coming to the question of the MIP, we need the following well-known result :

Lemma 3.7 [50, p 158] *Let (Ω, Σ, μ) be a measure space, X a Banach space and $1 \leq p < \infty$. For $\{f_n\}$ and f in $L^p(\mu, X)$, if $\|f_n\|_p \longrightarrow \|f\|_p$ and $f_n \longrightarrow f$ a.e. $[\mu]$, then $\|f_n - f\|_p \longrightarrow 0$.*

Lemma 3.8 *Let X be a Banach space, (Ω, Σ, μ) a measure space and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $A = \{w^*$-denting points of $B(X^*)\} \subseteq S(X^*)$. Then $\mathcal{P}_q(A) = \mathcal{P}_q(S(X^*)) \cap \{w^*$-denting points of $B(V_q(\mu, X^*))\}$.*

Proof : Let $g = \sum_{i=1}^{\infty} x_i^* \chi_{E_i} \in \mathcal{P}_q(A)$. Then by Lemma 3.5(b), g is an extreme point of $B(V_q(\mu, X^*))$. So, by Lemma 1.6, it suffices to show that g is a w*-PC of $B(L^q(\mu, X^*))$, i.e., to prove that if $\{g_\alpha\} \subseteq B(L^q(\mu, X^*))$ is a net such that $g_\alpha \xrightarrow{w^*} g$ then $\|g_\alpha - g\|_q \longrightarrow 0$.

Put $E_o = \Omega \setminus \cup_{i=1}^{\infty} E_i$. Then $g_\alpha \chi_{E_o} \xrightarrow{w^*} g\chi_{E_o} = 0$ and $g_\alpha \chi_{\Omega\setminus E_o} \xrightarrow{w^*} g\chi_{\Omega\setminus E_o} = g$. Now, by the w*-lower semicontinuity of the norm we have :

$$1 = \|g\|_q \leq \liminf_\alpha \|g_\alpha\|_q \leq \limsup_\alpha \|g_\alpha\|_q \leq 1$$

and hence

$$\lim_\alpha \|g_\alpha\|_q = 1 = \|g\|_q$$

Similarly, we have

$$\lim_\alpha \|g_\alpha \chi_{\Omega\setminus E_o}\|_q = 1$$

Now,

$$\|g_\alpha\|_q^q = \|g_\alpha \chi_{E_o}\|_q^q + \|g_\alpha \chi_{\Omega\setminus E_o}\|_q^q$$

so that

$$\limsup_\alpha \|g_\alpha \chi_{E_o}\|_q^q = \lim_\alpha (\|g_\alpha\|_q^q - \|g_\alpha \chi_{\Omega\setminus E_o}\|_q^q) = 0$$

Thus, it suffices to show that $\|g_\alpha \chi_{\Omega\setminus E_o} - g\|_q = 0$.

So, we may assume without loss of generality that (Ω, Σ, μ) is σ-finite, and for almost all t, $g(t) \neq 0$.

Fix $\varepsilon > 0$, choose $N \geq 1$ such that $\sum_{i=N+1}^{\infty} \|x_i^*\|^q \mu(E_i) < \varepsilon$. Put $F = \cup_{i=1}^{N} E_i$. We have,

$$\|g\chi_F\|_q^q \leq \liminf_\alpha \|g_\alpha \chi_F\|_q^q \leq \limsup_\alpha \|g_\alpha \chi_F\|_q^q$$

and,

$$\|g\chi_{\Omega\setminus F}\|_q^q \leq \liminf_\alpha \|g_\alpha \chi_{\Omega\setminus F}\|_q^q \leq \limsup_\alpha \|g_\alpha \chi_{\Omega\setminus F}\|_q^q$$

Suppose

$$\|g\chi_F\|_q^q < \limsup_\alpha \|g_\alpha \chi_F\|_q^q$$

Then there exists a subnet $\{g_{\beta_\alpha}\}$ of $\{g_\alpha\}$ such that

$$\lim_\alpha \|g_{\beta_\alpha} \chi_F\|_q^q = \limsup_\alpha \|g_\alpha \chi_F\|_q^q \text{ and } \lim_\alpha \|g_{\beta_\alpha} \chi_{\Omega\setminus F}\|_q^q \text{ exists.}$$

Then

$$1 = \|g_\alpha \chi_F\|_q^q + \|g_\alpha \chi_{\Omega\setminus F}\|_q^q < \lim_\alpha \|g_{\beta_\alpha} \chi_F\|_q^q + \lim_\alpha \|g_{\beta_\alpha} \chi_{\Omega\setminus F}\|_q^q = \lim_\alpha \|g_{\beta_\alpha}\|_q^q \leq 1$$

a contradiction. So, we have,

$$\|g\chi_F\|_q^q = \lim_\alpha \|g_\alpha \chi_F\|_q^q$$

Similarly,

$$\|g\chi_{\Omega\setminus F}\|_q^q = \lim_\alpha \|g_\alpha \chi_{\Omega\setminus F}\|_q^q$$

Now,

$$\begin{aligned}
\|g_\alpha - g\|_q^q &= \|(g_\alpha - g)\chi_F\|_q^q + \|(g_\alpha - g)\chi_{\Omega\setminus F}\|_q^q \\
&\leq \|(g_\alpha - g)\chi_F\|_q^q + [\|g_\alpha \chi_{\Omega\setminus F}\|_q + \|g\chi_{\Omega\setminus F}\|_q]^q \\
&\leq \|(g_\alpha - g)\chi_F\|_q^q + 2^{q-1}[\|g_\alpha \chi_{\Omega\setminus F}\|_q^q + \|g\chi_{\Omega\setminus F}\|_q^q]
\end{aligned}$$

(by convexity of the map $t \to t^q$).

So, if $\|(g_\alpha - g)\chi_F\|_q \longrightarrow 0$, and $\varepsilon > 0$ is given, we can choose α_o such that for all $\alpha \geq \alpha_o$,

$$\|(g_\alpha - g)\chi_F\|_q^q < \varepsilon \text{ and } \|g_\alpha \chi_{\Omega\setminus F}\|_q^q < \|g\chi_{\Omega\setminus F}\|_q^q + \varepsilon$$

i.e., we have for $\alpha \geq \alpha_o$, $\|g_\alpha - g\|_q^q \leq \varepsilon + 2^{q-1} \cdot 3\varepsilon$.

Thus, it suffices to prove $\|(g_\alpha - g)\chi_F\|_q \longrightarrow 0$. So, again, without loss of generality, we may assume μ is finite and g is of the form $\sum_{i=1}^n x_i^* \chi_{E_i} \in S(L^q(\mu, X^*))$ with $\{E_1, E_2, \ldots, E_n\}$ a partition of Ω.

Suppose now, that $\|g_\alpha - g\|_q \not\to 0$. Then there exists $\varepsilon_o > 0$ and a subnet $\{g_{\beta_\alpha}\}$ with $\beta_\alpha \geq \alpha$ for all α, such that $\|g_{\beta_\alpha} - g\|_q \geq \varepsilon_o$ for all α. For notational simplicity, put $G_\alpha = g_{\beta_\alpha}$. Then $G_\alpha \in B(L^q(\mu, X^*))$ for all α, $G_\alpha \xrightarrow{w^*} g$ and $\|G_\alpha - g\|_q \geq \varepsilon_o$ for all α.

As noted earlier,

$$\lim_\alpha \|G_\alpha\|_q = 1 = \|g\|_q \tag{1}$$

Fix $A \in \Sigma$, $\mu(A) > 0$. Put $A_{i1} = A \cap E_i$ and $A_{i2} = E_i \setminus A_{i1}$. Fix A_{ij} such that $\mu(A_{ij}) > 0$. For any $x \in X$, $x\chi_{A_{ij}} \in L^p(\mu, X)$, so, $\langle x\chi_{A_{ij}}, G_\alpha\rangle \longrightarrow \langle x\chi_{A_{ij}}, g\rangle$, i.e., $\langle x, \int_{A_{ij}} G_\alpha d\mu\rangle \longrightarrow \langle x, x_i^*\rangle \mu(A_{ij})$. This implies

$$\frac{1}{\mu(A_{ij})} \int_{A_{ij}} G_\alpha d\mu \xrightarrow{w^*} x_i^*$$

whence

$$\begin{aligned}\|x_i^*\| &\le \liminf_\alpha \frac{1}{\mu(A_{ij})}\|\int_{A_{ij}} G_\alpha d\mu\| \le \liminf_\alpha \frac{1}{\mu(A_{ij})}\int_{A_{ij}}\|G_\alpha\| d\mu \\ &\le \liminf_\alpha \left[\frac{1}{\mu(A_{ij})}\int_{A_{ij}}\|G_\alpha\|^q d\mu\right]^{1/q} \qquad (2)\end{aligned}$$

Thus, we have

$$\|x_i^*\|^q \mu(A_{ij}) \le \liminf_\alpha \int_{A_{ij}} \|G_\alpha\|^q d\mu \text{ for all } i,j$$

Adding, we get

$$\begin{aligned}1 &= \|g\|_q^q = \sum_{i=1}^n\sum_{j=1}^2 \|x_i^*\|^q\mu(A_{ij}) \le \sum_{i=1}^n\sum_{j=1}^2 \liminf_\alpha \int_{A_{ij}}\|G_\alpha\|^q d\mu \\ &\le \liminf_\alpha \sum_{i=1}^n\sum_{j=1}^2 \int_{A_{ij}}\|G_\alpha\|^q d\mu = \liminf_\alpha \|G_\alpha\|_q^q \le 1\end{aligned}$$

So,

$$\|x_i^*\|^q \mu(A_{ij}) = \liminf_\alpha \int_{A_{ij}} \|G_\alpha\|^q d\mu \text{ for all } i,j$$

Suppose, for some i_o, j_o, we have

$$\|x_{i_o}^*\|^q \mu(A_{i_oj_o}) < \limsup_\alpha \int_{A_{i_oj_o}} \|G_\alpha\|^q d\mu$$

Then,

$$\begin{aligned}\|x_{i_o}^*\|^q\mu(A_{i_oj_o}) &< \limsup_\alpha \left[\int_\Omega \|G_\alpha\|^q d\mu - \sum_{(i,j)\ne(i_o,j_o)}\int_{A_{ij}}\|G_\alpha\|^q d\mu\right] \\ &= 1 - \liminf_\alpha \sum_{(i,j)\ne(i_o,j_o)}\int_{A_{ij}}\|G_\alpha\|^q d\mu \\ &\le 1 - \sum_{(i,j)\ne(i_o,j_o)} \liminf_\alpha \int_{A_{ij}}\|G_\alpha\|^q d\mu \\ &= 1 - \sum_{(i,j)\ne(i_o,j_o)} \|x_i^*\|^q\mu(A_{ij})\end{aligned}$$

making

$$1 = \|g\|_q^q = \sum_{i=1}^n\sum_{j=1}^2 \|x_i^*\|^q\mu(A_{ij}) < 1$$

So, we must have

$$\|x_i^*\|^q \mu(A_{ij}) = \lim_\alpha \int_{A_{ij}} \|G_\alpha\|^q d\mu \text{ for all } i,j$$

Now, by (2) above and by the limsup-version of (2) we have that for all i, j,

$$\|x_i^*\|\mu(A_{ij}) = \lim_\alpha \int_{A_{ij}} \|G_\alpha\| d\mu$$

Thus,

$$\int_A \|g(t)\| d\mu = \sum_{i=1}^n \|x_i^*\|\mu(A_{i1}) = \lim_\alpha \sum_{i=1}^n \int_{A_{i1}} \|G_\alpha(t)\| d\mu = \lim_\alpha \int_A \|G_\alpha(t)\| d\mu$$

Since $A \in \Sigma$ was arbitrary with $\mu(A) > 0$, we have that for all $A \in \Sigma$ with $\mu(A) > 0$,

$$\lim_\alpha \int_A \|G_\alpha(t)\| d\mu = \int_A \|g(t)\| d\mu$$

whence, in $L^q(\mu, I\!R)$, $\|G_\alpha(\cdot)\| \xrightarrow{w^*} \|g(\cdot)\|$. Now, since the space $L^q(\mu, I\!R)$ is uniformly convex and $\|g(\cdot)\| \in S(L^q(\mu, I\!R))$, $\|g(\cdot)\|$ is a w*-denting point of $B(L^q(\mu, I\!R))$ and hence a w*-PC. So, we have

$$\int_\Omega | \|G_\alpha(t) - \|g(t)\| \, |^q d\mu \longrightarrow 0 \tag{3}$$

Again, fix A_{ij} such that $\mu(A_{ij}) > 0$. Then

$$\frac{1}{\mu(A_{ij})} \int_{A_{ij}} G_\alpha d\mu \xrightarrow{w^*} x_i^*$$

and

$$\frac{1}{\mu(A_{ij})} \int_{A_{ij}} \|G_\alpha\| d\mu \longrightarrow \|x_i^*\|$$

so

$$\frac{\int_{A_{ij}} G_\alpha d\mu}{\int_{A_{ij}} \|G_\alpha\| d\mu} \xrightarrow{w^*} \frac{x_i^*}{\|x_i^*\|}$$

Since $x_i^*/\|x_i^*\|$ is a w*-PC of $B(X^*)$, this means

$$\left\| \frac{x_i^*}{\|x_i^*\|} - \frac{\int_{A_{ij}} G_\alpha d\mu}{\int_{A_{ij}} \|G_\alpha\| d\mu} \right\| \longrightarrow 0 \text{ or } \left\| \frac{x_i^*}{\|x_i^*\|} - \frac{\int_{A_{ij}} G_\alpha d\mu}{\mu(A_{ij})\|x_i^*\|} \right\| \longrightarrow 0 \text{ (as } \alpha \longrightarrow \infty)$$

Choose a net $\{\varepsilon_\alpha\}$ such that $\varepsilon_\alpha > 0$ for all α and $\varepsilon_\alpha \longrightarrow 0$. We have

$$\begin{aligned}
& \left\| \int_{A_{ij}} \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} d\mu - \int_{A_{ij}} \frac{G_\alpha(t)}{\|x_i^*\|} d\mu \right\| \\
= & \left\| \int_{A_{ij}} G_\alpha(t) \left[\frac{\|x_i^*\| - \|G_\alpha(t)\| - \varepsilon_\alpha}{\|x_i^*\|(\|G_\alpha(t)\| + \varepsilon_\alpha)} \right] d\mu \right\| \\
\leq & \frac{1}{\|x_i^*\|} \int_{A_{ij}} | \|x_i^*\| - \|G_\alpha(t)\| - \varepsilon_\alpha | d\mu \\
\leq & \frac{1}{\|x_i^*\|} \left[\mu(A_{ij}) \left\{ \frac{1}{\mu(A_{ij})} \int_{A_{ij}} | \|g(t)\| - \|G_\alpha(t)\| |^q d\mu \right\}^{1/q} + \mu(A_{ij})\varepsilon_\alpha \right] \\
\leq & \frac{\mu(A_{ij})}{\|x_i^*\|} \left[\left\{ \frac{1}{\mu(A_{ij})} \int_{A_{ij}} | \|g(t)\| - \|G_\alpha(t)\| |^q d\mu \right\}^{1/q} + \varepsilon_\alpha \right]
\end{aligned}$$

Now, by (3) and as $\varepsilon_\alpha \longrightarrow 0$, we conclude that

$$\left\| \int_{A_{ij}} \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} d\mu - \int_{A_{ij}} \frac{G_\alpha(t)}{\|x_i^*\|} d\mu \right\| \longrightarrow 0$$

whence

$$\left\| \frac{x_i^*}{\|x_i^*\|} - \frac{1}{\mu(A_{ij})} \int_{A_{ij}} \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} d\mu \right\| \longrightarrow 0$$

Now, $x_i^*/\|x_i^*\|$ is a w*-denting, and hence denting, point of $B(X^*)$, so by Theorem 1.5(d), we get

$$\frac{1}{\mu(A_{ij})} \int_{A_{ij}} \left\| \frac{x_i^*}{\|x_i^*\|} - \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} d\mu \right\| \longrightarrow 0 \tag{4}$$

Now,

$$\begin{aligned}
& \int_{A_{ij}} \|x_i^* - G_\alpha(t)\| d\mu \\
= & \|x_i^*\| \int_{A_{ij}} \left\| \frac{x_i^*}{\|x_i^*\|} - \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} + \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} - \frac{G_\alpha(t)}{\|x_i^*\|} \right\| d\mu \\
\leq & \|x_i^*\| \left[\int_{A_{ij}} \left\| \frac{x_i^*}{\|x_i^*\|} - \frac{G_\alpha(t)}{\|G_\alpha(t)\| + \varepsilon_\alpha} \right\| d\mu \right. \\
& \left. + \int_{A_{ij}} \frac{\|G_\alpha(t)\|}{\|G_\alpha(t)\| + \varepsilon_\alpha} \cdot \frac{| \|x_i^*\| - \|G_\alpha(t)\| | + \varepsilon_\alpha}{\|x_i^*\|} d\mu \right]
\end{aligned}$$

The first term of the sum goes to zero by (4) and the second by (3). Hence we have

$$\int_{A_{ij}} \|x_i^* - G_\alpha(t)\| d\mu \longrightarrow 0 \text{ or } \int_{A_{ij}} \|g(t) - G_\alpha(t)\| d\mu \longrightarrow 0$$

Now, since A_{ij} and A were arbitrary, we have actually proved that

$$\int_\Omega \|g(t) - G_\alpha(t)\| d\mu \longrightarrow 0$$

Using this fact and (1), we can choose $\{\alpha_n\}$ such that both $G_{\alpha_n}(t) \to g(t)$ a.e. $[\mu]$ and $\|G_{\alpha_n}\|_q \to \|g\|_q$. Hence, we have, by Lemma 3.7, $\|G_{\alpha_n} - g\|_q \to 0$. But this contradicts the choice of $\{G_\alpha\}$.

The converse follows immediately from Lemma 1.6, Lemma 3.5 and the observation that if $g = \sum_{i=1}^\infty x_i \chi_{E_i}$ and $\{y_\alpha^*\} \subseteq B(X^*)$, $y_\alpha^* \xrightarrow{w^*} x_k^*/\|x_k^*\|$ for some k then $g_\alpha \stackrel{\text{def}}{=} \sum_{i \neq k} x_i^* \chi_{E_i} + \|x_k^*\| y_\alpha^* \chi_{E_k} \in B(V_q(\mu, X^*))$ and $g_\alpha \xrightarrow{w^*} g$. ■

Proof of Lemma 3.3 : In the set-up of Lemma 3.8, put $X = (\oplus_{\alpha\in\Gamma} X_\alpha)_{\ell^p}$, $\Omega = \Gamma$, $\Sigma =$ Power set of Γ and $\mu =$ counting measure. Then $L^p(\mu, X) = \ell^p(X)$ and $L^p(\mu, X)^* = \ell^q(X^*) = L^q(\mu, X^*)$. Now,

$$\ell^p(X) = \{((x_{\alpha\beta})_{\alpha\in\Gamma})_{\beta\in\Gamma} : \sum_{\alpha\in\Gamma} \sum_{\beta\in\Gamma} \|x_{\alpha\beta}\|^p < \infty\}$$

Identify X with the subspace $\{((x_\alpha \delta_{\alpha\beta})_{\alpha\in\Gamma})_{\beta\in\Gamma}\} \subseteq \ell^p(X)$, where

$$\delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

Then X^* gets identified with the subspace $\{((x_\alpha^* \delta_{\alpha\beta})_{\alpha\in\Gamma})_{\beta\in\Gamma}\} \subseteq \ell^q(X^*)$. Observe that with this identification, a net $\{x_\rho^*\} \subseteq B(X^*)$ is w*-convergent to a point $x_o^* \in S(X^*)$ if and only if the net $\{x_\rho^*\} \subseteq B(\ell^q(X^*))$ is w*-convergent to $x_o^* \in S(\ell^q(X^*))$.

Now, Lemma 3.3 follows immediately from Lemma 3.8. ■

Lemma 3.9 *Let X be a Banach space, (Ω, Σ, μ) a probability space and $1 < r < \infty$. Let $A \subseteq S(X)$. The following are equivalent :*

(a) *A is norm dense in $S(X)$.*

(b) *$\mathcal{M}_r(A)$ is norm dense in $S(L^r(\mu, X))$.*

(c) *$\mathcal{P}_r(A)$ is norm dense in $S(L^r(\mu, X))$.*

(d) *$\mathcal{N}_r(A)$ is norm dense in $S(L^r(\mu, X))$.*

Proof : Since simple functions in $S(L^r(\mu, X))$ are norm dense (a) implies (b), and since $\mathcal{M}_r(A) \subseteq \mathcal{P}_r(A) \subseteq \mathcal{N}_r(A)$, $(b) \Longrightarrow (c) \Longrightarrow (d)$.

Now, if $x \in S(X)$, $x\chi_\Omega \in S(L^r(\mu, X))$. So, by (*d*), there exists a sequence $\{f_n\} \subseteq \mathcal{N}_r(A)$ such that $\|f_n - x\chi_\Omega\|_r \longrightarrow 0$. But then some subsequence $\{f_{n_k}\}$ converges to $x\chi_\Omega$ a.e. $[\mu]$. Now, if $t \in \Omega$ is such that $\|f_{n_k}(t) - x\chi_\Omega(t)\| \longrightarrow 0$, then $\|f_{n_k}(t) - x\| \longrightarrow 0$. Since $\|x\| = 1$, for all sufficiently large k, $\|f_{n_k}(t)\| \neq 0$ and

$$\left\| \frac{f_{n_k}(t)}{\|f_{n_k}(t)\|} - x \right\| \longrightarrow 0$$

As $f_{n_k} \in \mathcal{N}_r(A)$, all such $f_{n_k}(t)/\|f_{n_k}(t)\| \in A$. Hence (*a*) follows. ■

Theorem 3.10 *For any Banach space X, finite measure space (Ω, Σ, μ) and $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following are equivalent :*

(*i*) *X has the MIP*

(*ii*) *$L^p(\mu, X)$ has the $L^q(\mu, X^*)$-MIP.*

Proof : Define Y, F and T as in Proposition 3.4. Suppose $L^p(\mu, X)$ has the F-MIP. Let A be a norm dense subset of $S(X)$ and $\phi : S(X) \longrightarrow S(X^*)$ be a support mapping. By Lemma 3.9, $\mathcal{M}_p(A)$ is norm dense in $S(L^p(\mu, X))$. Let $x = \sum_{i=1}^n x_i \chi_{E_i} \in \mathcal{M}_p(A)$, define $\Phi(x)$ as in the proof of Proposition 3.4. Then, Φ can be extended to a support mapping $\tilde{\Phi} : S(L^p(\mu, X)) \longrightarrow S(V_q(\mu, X^*))$. Now, $\tilde{\Phi}$ maps T into $S(F)$, $\mathcal{M}_p(A)$ is norm dense in $S(L^p(\mu, X))$ and $L^p(\mu, X)$ has the F-MIP, so by Theorem 2.5, $\tilde{\Phi}(\mathcal{M}_p(A)) = \Phi(\mathcal{M}_p(A))$ is norm dense in $S(F)$. Observe that $\mathcal{M}_q(\phi(A)) \supseteq \Phi(\mathcal{M}_p(A))$, so that $\mathcal{M}_q(\phi(A))$ is norm dense in $S(L^q(\mu, X^*))$. Now, Lemma 3.9 implies that $\phi(A)$ is norm dense in $S(X^*)$.

Since A was an arbitrary dense subset of $S(X)$ and ϕ an arbitrary support mapping on X, Theorem 2.6 implies X has the MIP.

Conversely, let X have the MIP. Let $A = \{$w*-denting points of $B(X^*)\}$. By Theorem 2.6, A is norm dense in $S(X^*)$. By Lemma 3.9, $\mathcal{M}_q(A)$ is norm dense in $S(L^q(\mu, X^*))$. And by Lemma 3.8, $\mathcal{M}_q(A) \subseteq \{$w*-denting points of $V_q(\mu, X^*)\}$. Again by Theorem 2.5, $L^p(\mu, X)$ has the F-MIP. ■

Theorem 3.11 *For any Banach space X, any finite measure space (Ω, Σ, μ) and $1 < p < \infty$, the space $L^p(\mu, X)$ has the MIP if and only if X has the MIP and X^* has the RNP with respect to μ.*

Theorem 3.12 *For any Banach space X, any finite measure space (Ω, Σ, μ) and $1 < p < \infty$, X^* has the w^*-MIP if and only if $L^p(\mu, X)^*$ has the w^*-MIP.*

Proof: As observed earlier, if $A = \{$denting points of $B(X)\}$ then $\mathcal{N}_p(A) = \{$denting points of $B(L^p(\mu, X))\}$. Now, the result is immediate from Theorem 2.6 and Lemma 3.9. ■

Corollary 3.12.1 *Let X be a Banach space, let λ denote the Lebesgue measure on $[0, 1]$, let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q}$. Then*

(a) *$L^p(\lambda, X)$ has the CI.*

(b) *X has the MIP if and only if $L^p(\lambda, X)$ has the $L^q(\lambda, X^*)$-MIP.*

(c) *$L^p(\lambda, X)$ has the MIP if and only if X has the MIP and X is Asplund.*

(d) *X^* has the w^*-MIP if and only if $L^p(\lambda, X)^*$ has the w^*-MIP.*

REMARK : If X has the MIP implies X is Asplund — which is a long-standing conjecture — then we must have $L^p(\lambda, X)$ has the MIP. However, in the light of our results, this would be tantamount to showing that if $L^q(\lambda, X^*)$ is the minimal norming subspace of $V_q(\lambda, X^*)$, then it must be the whole of the latter space. But that seems unlikely.

Chapter 4

Miscellaneous Results

4.1. The Subspace Question

Phelps [42] noted that a two-dimensional space has the MIP if and only if it is smooth. Now, if the MIP were hereditary, a space X having the MIP would then have all its two-dimensional subspaces smooth and hence X itself would be smooth. However, Deville [9] sketches an argument to show that it is possible to construct a non-smooth norm on $\mathbb{R}^3$ with the MIP and Tsarkov [53] has produced one such norm. Below, we produce a non-smooth norm on $\mathbb{R}^3$ with the MIP that is much simpler than the one given by Tsarkov.

EXAMPLE : In $\mathbb{R}^3$, consider the following set

$$U = \{(x, y, z) : |x| \leq 1, |z| \leq 1, y^2 \leq (1 - x^2)(1 - z^2)\}$$

U is a closed bounded symmetric convex set with non-empty interior. Hence the Minkowski functional of U defines a norm on $\mathbb{R}^3$ equivalent to the Euclidean norm. The norm turns out to be

$$\|(x, y, z)\| = \frac{1}{2}[\{(|x| + |z|)^2 + y^2\}^{1/2} + \{(|x| - |z|)^2 + y^2\}^{1/2}]$$

The corresponding pre-dual norm on $\mathbb{R}^3$ with the above as the dual norm is given by

$$\|(a_1, a_2, a_3)\| = \begin{cases} |a_1| + |a_2| & \text{if } a_2^2 \leq |a_1 a_3| \\ \left[\dfrac{(a_1^2 + a_2^2)(a_2^2 + a_3^2)}{a_2^2}\right]^{1/2} & \text{otherwise} \end{cases}$$

Observe that the surface of U is given by all those points where one of the defining inequalities is an equality, and that except for points on the plane $y = 0$ (where the restricted norm is the ℓ^∞ norm on $\mathbb{R}^2$) all the points on the surface are exposed and hence extreme points of U. So, by Phelps' characterisation, the pre-dual norm indeed has the MIP. Moreover, the part of the sphere lying on the plane $y = 0$ is a square and hence, the pre-dual norm is not smooth.

And if we consider the subspace $a_2 = 0$, the inherited norm is the ℓ^1 norm on $\mathbb{R}^2$, which clearly lacks the MIP.

Observe further that the usual projection onto this subspace is a norm-1 projection. So, neither complemented subspaces nor even those complemented by norm-1 projections necessarily inherit the MIP.

We, however, have the following :

Proposition 4.1 *If X has the MIP (resp. the CI) and a subspace Y of X is the range of an M-projection P, i.e., $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ for all $x \in X$, then Y has the MIP (resp. the CI).*

Proof : Let K be a closed bounded (resp. compact) convex set in Y. Since X has the MIP (resp. the CI), there exist $\{x_i\}_{i\in I} \subseteq X$ and $\{r_i\}_{i\in I}$ with $r_i > 0$ for all $i \in I$, such that $K = \cap_{i\in I} B_{r_i}[x_i]$, where $B_r[x] = \{z \in X : \|z - x\| \leq r\}$. Let $x \in K \subseteq Y$, then for all $i \in I, \|x - x_i\| \leq r_i$ so that $\|x_i - Px_i\| = \|(x - x_i) - P(x - x_i)\| \leq \|x - x_i\| \leq r_i$.

CLAIM : $K = \cap_{i\in I}\{y \in Y : \|y - Px_i\| \leq r_i\}$.

Firstly, since $\|P\| = 1$, we have $K = P(K) \subseteq P[\cap_{i\in I} B_{r_i}[x_i] \subseteq \cap_{i\in I}\{y \in Y : \|y - Px_i\| \leq r_i\} = RHS$. Conversely, if $x \in RHS$, for all $i \in I$, $\|x - x_i\| = \max\{\|x - Px_i\|, \|x_i - Px_i\|\} \leq r_i$, as $\|x_i - Px_i\| \leq r_i$. Thus $x \in \cap_{i\in I} B_{r_i}[x_i] = K$. ■

REMARK : All the known examples where the heredity of the MIP fails are non-smooth spaces. So, it is pertinent to ask :

Is the property of being a smooth space with the MIP hereditary ?

We note that no counterexample is possible in finite dimensions as smoothness is hereditary and implies the MIP in finite dimensions.

Recall that a property (say, P) is called *three space property* if for a Banach space X and a subspace Y of X, Y has P and the quotient space X/Y has P together implies X has P.

Proposition 4.2 *For any two Banach spaces X_1 and X_2, the space $X = X_1 \oplus_{\ell^1} X_2$ fails to have the MIP. And hence, the MIP is not a three space property.*

Proof : The dual of X is given by $X^* = X_1^* \oplus_{\ell^\infty} X_2^*$, the extreme points of $B(X^*)$ are of the form (x_1^*, x_2^*), where x_i^* is extreme in $B(X_i^*)$, which are 'far enough' from norm 1 elements of the form $(x^*, 0)$.

Observe that in this case, for $Y = X_1$, $X/Y = X_2$ and the above shows that even if both X_1 and X_2 have the MIP, X does not. ∎

4.2. The MIP and Farthest Points

For a closed and bounded set $K \subseteq X$, define

(i) $r_K(x) = \sup\{\|x - y\| : y \in K\}$, $x \in X$. r_K is called the *farthest distance map*,

(ii) $Q_K(x) = \{y \in K : \|x - y\| = r_K(x)\}$, $x \in X$. Q_K is called the *anti-metric projection*,

(iii) $D(K) = \{x \in X : Q_K(x) \neq \emptyset\}$, and

(iv) $b(K) = \cup\{Q_K(x) : x \in D(K)\}$, i.e., $b(K)$ is the set of all *farthest points* of K.

Call a closed and bounded set K *densely remotal* if $D(K)$ is norm dense in X. We note the following :

Lemma 4.3 [35,10] *A closed and bounded set K is densely remotal with respect to any equivalent norm if and only if K is weakly compact.*

Lemma 4.4 [10, Proposition 3] *If a dual space X^* is w^*-Asplund, then every w^*-compact set in X^* is densely remotal with respect to any equivalent dual norm.*

The proofs of the following Lemma and Theorem are essentially already contained in [35] and [17]. For the sake of completeness, however, we include the proofs which have been recast using our terminology.

Lemma 4.5 *Let X be a Banach space and F be a norming subspace of X^*. If there exists a $\sigma(X,F)$-closed, norm bounded convex set $K \subseteq X$ that is not admissible, then there exists a $\sigma(X,F)$-closed, norm bounded convex set $K_1 \subseteq X$ with non-empty norm interior (i.e., $\text{int}(K_1) \neq \emptyset$) that is not admissible.*

Proof : Let K be as above. Let $x_o \in [K] \setminus K$. Since K is $\sigma(X,F)$-closed, there exists $f \in F$ such that $\sup f(K) < f(x_o)$.

Let B be a closed ball such that $\sup f(B) < \sup f(K)$. Let $K_1 = \overline{\text{co}}^{\sigma(X,F)}(K \cup B)$. Then, K_1 is $\sigma(X,F)$-closed, norm bounded convex set with $\text{int}(K_1) \neq \emptyset$ and $K \subseteq K_1$. Now, $x_o \in [K] \subseteq [K_1]$ and $\sup f(K_1) \leq \sup f(K) < f(x_o)$, i.e., $x_o \notin K_1$. ■

Theorem 4.6 *Let X be a Banach space and F be a norming subspace of X^*. Consider the following statements :*

(*a*) *Any $\sigma(X,F)$-closed, norm bounded convex set in X is the $\sigma(X,F)$-closed convex hull of its farthest points.*

(*b*) *X has the F-MIP.*

(*c*) *Any $\sigma(X,F)$-closed, norm bounded,* densely remotal *convex set in X is the $\sigma(X,F)$-closed convex hull of its farthest points.*

Then (*a*) $\Longrightarrow$ (*b*) $\Longrightarrow$ (*c*).

(The statements can be shown to be equivalent in some special cases. See corollaries below.)

Proof : (*a*) $\Longrightarrow$ (*b*) Suppose there exists a $\sigma(X,F)$-closed, norm bounded convex set $K \subseteq X$ that is not admissible. By Lemma 4.5, we may assume $\text{int}(K) \neq \emptyset$.

Let $x_o \in [K] \setminus K$ and $y_o \in \text{int}(K) \subseteq \text{int}([K])$. Since $[K] \setminus K$ is norm open in $[K]$, there exists $0 < \lambda < 1$ such that $z_o = \lambda x_o + (1-\lambda) y_o \in [K] \setminus K$.

Note that $z_o \in \text{int}([K])$, and hence, so is any point of the form

$$(*) \qquad \alpha z_o + (1-\alpha)x,\ \alpha \in (0,1],\ x \in K.$$

Let $K_1 = \text{co}(K \cup \{z_o\})$. Then K_1 is $\sigma(X,F)$-closed, norm bounded and convex. The proof will be complete once we show that $b(K_1) \subseteq K$.

Let $x \in X$. Then $B = B_{r_K(x)}[x]$ is a closed ball containing K, and so, contains $[K]$. Since each point of the form $(*)$ is in $\text{int}([K])$, it is in $\text{int}(B)$, i.e., its distance from x is strictly less than $r_K(x)$. Note that $r_K(x) \le r_{K_1}(x) \le r_{[K]}(x) = r_K(x)$. Thus, $Q_{K_1}(x) \subseteq K$. Since $x \in X$ was arbitrary, $b(K_1) \subseteq K$.

$(b) \implies (c)$ Let K be a $\sigma(X,F)$-closed, norm bounded, densely remotal convex set in X. Clearly, $\overline{\text{co}}^{\sigma(X,F)}(b(K)) \subseteq K$. Suppose there exists $x \in K \setminus \overline{\text{co}}^{\sigma(X,F)}(b(K))$. Since X has the F-MIP, there exists $y \in X$ and $r > 0$ such that $\overline{\text{co}}^{\sigma(X,F)}(b(K)) \subseteq B_r[y]$ and $\|x-y\| > r$. As K is densely remotal, there exists $z \in D(K)$ such that $\|y-z\| < \frac{1}{3}(\|x-y\|-r)$. Let $x_1 \in Q_K(z)$. Then $x_1 \in b(K)$ and hence, $\|x_1 - y\| \le r$. Now, since $x \in K$, we have

$$\begin{aligned}\|x-y\| &\le \|x-z\| + \|z-y\| \le r_K(z) + \|z-y\| = \|x_1 - z\| + \|z-y\| \\ &\le \|x_1 - y\| + 2\|z-y\| < r + \frac{2}{3}(\|x-y\| - r) = \frac{2}{3}\|x-y\| + \frac{1}{3}r\end{aligned}$$

But this implies $\|x-y\| < r$, a contradiction that completes the proof. ■

Corollary 4.6.1 *If X^* is a w^*-Asplund dual space, then X^* has the w^*-MIP if and only if every w^*-compact convex set in X^* is the w^*-closed convex hull of its farthest points.*

Proof : Since X^* is w*-Asplund, by Lemma 4.4, every w*-compact convex set in X^* is densely remotal and the result follows from Theorem 4.6. ■

Corollary 4.6.2 [35] *If X is a reflexive Banach space, then X has the MIP if and only if every closed bounded convex set in X is the closed convex hull of its farthest points.*

REMARKS : 1. In studying the farthest points, the emphasis so far seems to be on finding conditions *on the sets* to ensure existence of farthest points. In view of Theorem 4.6, a natural question is can one give conditions *on the norm* to ensure that a 'reasonably large' family of sets admit farthest points ? As far as we know, this line of investigation has rarely (see e.g., [18]) been pursued in the past. In particular, does the F-MIP imply *every* $\sigma(X, F)$-closed, norm bounded, *convex* set in X is densely remotal ? Is the condition X^* is w*-Asplund necessary in Corollary 4.6.1 ?

In this context, notice that if the norm is strictly convex (respectively, locally uniformly convex), any farthest point of a closed bounded convex set is also an extreme (resp. denting) point. So, if the MIP implies every closed bounded convex set admits farthest points, then strictly (resp. locally uniformly) convex spaces with the MIP must necessarily have the Krein-Milman Property (KMP) (resp. the RNP) (see [7] or [13] for details). However, the space c_o, which does not have the KMP, admits a strictly convex Fréchet differentiable norm (see e.g., [11]). Hence, the answer must generally be no. Now, is the answer yes, if in addition, the space has the RNP ?

2. As noted before, X has the MIP implies X^{**} has the w*-MIP. And if it also implies X is Asplund, then by Corollary 4.6.1, every w*-compact convex set in X^{**} is the w*-closed convex hull of its farthest points. Can this be proved directly ?

4.3. The MIP in Projective Tensor Product Spaces

For two Banach spaces X and Y, the dual of $X \otimes_\pi Y$ is $\mathcal{L}(X, Y^*)$. Now, the extremal structure of such operator spaces are known only in some very special cases. See [29] or [34] for a survey. Here we only recall that the extreme contractions from a Hilbert space to another has been characterised, by Kadison [33] in the complex case (see also [30]) and by Grzaslewicz [26] in the real case, as :

Theorem 4.7 *For Hilbert spaces E and F, $T \in \mathcal{L}(E, F)$ is an extreme contraction if and only if T or T^* is an isometry.*

As a corollary we deduce :

Theorem 4.8 *For real Hilbert spaces E and F of dimension ≥ 2, $E \otimes_\pi F$ never has the MIP.*

Proof : We simply note that the set of extreme contractions is closed and is not the whole of $S(\mathcal{L}(E, F))$. ■

Coming to 2-dimensional ℓ^p-spaces, some notations and preliminaries first. For $1 < p < \infty$, $x = (x_1, x_2) \in \ell_2^p$ with $\|x\| = 1$, define $x^{p-1} = (sgn(x_1)|x_1|^{p-1}, sgn(x_2)|x_2|^{p-1})$ and $x^o = (-x_2, x_1)$. Notice that, in general, x^{p-1} is the unique norming functional of x and $\{x, (x^o)^{p-1}\}$ is a basis for ℓ_2^p, and if $p = 2$, $x^{p-1} = x$ and $\{x, x^o\}$ is orthonormal. Assume $x_1 \geq x_2 \geq 0$. Now, for $r \in \mathbb{R}$, denote by $f_p(x, r) = x + r(x^o)^{p-1}$ and $F_p(x, r) = \|f_p(x, r)\|^p$.

Then $F_p(x, r) = |x_1 - r x_2^{p-1}|^p + |x_2 + r x_1^{p-1}|^p$. Clearly, $F_p(x, r) = 1 + |r|^p$ if $p = 2$ or $x_2 = 0$. Otherwise,

$$\begin{aligned} F_p(x, r) &= x_2^{-p} \cdot |r x_2^p - x_1 x_2|^p + x_1^{-p} \cdot |r x_1^p + x_1 x_2|^p \\ &= x_2^{-p} G(r x_2^p - x_1 x_2) + x_1^{-p} G(r x_1^p + x_1 x_2) \end{aligned}$$

where $G(u) = |u|^p$. Thus

$$\frac{\partial}{\partial r} F_p(x, r) = G'(r x_2^p - x_1 x_2) + G'(r x_1^p + x_1 x_2) \qquad (1)$$

and $G'(0) = 0$, $G'(u) = p \cdot sgn(u) \cdot |u|^{p-1}$. Clearly, (1) also holds if $p = 2$ or $x_2 = 0$.

Now, $G'(u)$ is an odd function, positive and strictly increasing for $u > 0$. Since the two arguments of G' in (1) add up to r, we have if $r > 0$ (resp. $r < 0$), the one larger in absolute value is positive (resp. negative), and so, $\frac{\partial}{\partial r} F_p(x, r)$ is positive (resp. negative), i.e., $F_p(x, r)$ is strictly increasing (resp. decreasing) in $r > 0$ (resp. $r < 0$).

Further, if $p \neq 2$ and $x_2 \neq 0$

$$\left.\begin{array}{rcl} F_p(x,0) & = & 1 \\ \frac{\partial}{\partial r}F_p(x,0) & = & 0 \\ \frac{\partial^2}{\partial r^2}F_p(x,0) & = & p(p-1)(x_1x_2)^{p-2} \\ \frac{\partial^3}{\partial r^3}F_p(x,0) & = & p(p-1)(p-2)(x_1x_2)^{p-3}[x_1^p - x_2^p] \\ \frac{\partial^4}{\partial r^4}F_p(x,0) & = & p(p-1)(p-2)(p-3)(x_1x_2)^{p-4}[1-3(x_1x_2)^p] \end{array}\right\} \quad (2)$$

and so, if $1 < q < \infty$, for $H_{pq}(x,r) = [F_p(x,r)]^{q/p}$, we have,

$$\left.\begin{array}{rcl} H_{pq}(x,0) & = & 1 \\ \frac{\partial}{\partial r}H_{pq}(x,0) & = & 0 \\ \frac{\partial^2}{\partial r^2}H_{pq}(x,0) & = & q(p-1)(x_1x_2)^{p-2} \\ \frac{\partial^3}{\partial r^3}H_{pq}(x,0) & = & q(p-1)(p-2)(x_1x_2)^{p-3}[x_1^p - x_2^p] \\ \frac{\partial^4}{\partial r^4}H_{pq}(x,0) & = & q(p-1)(x_1x_2)^{p-4}[(p-2)(p-3) \\ & & -3(x_1x_2)^p\{(p-2)(p-3)+(p-q)(p-1)\}] \end{array}\right\} \quad (3)$$

Now, let $1 < p, q < \infty$. Let $T : \ell_2^p \longrightarrow \ell_2^q$, $\|T\| = 1$. Then there exists $x = (x_1, x_2) \in \ell_2^p$ such that $\|x\| = 1 = \|Tx\|$. Let $Tx = y = (y_1, y_2)$. Let $I_{xy} = \{T : \|T\| \leq 1, Tx = y\}$. Then for any $T \in I_{xy}$, $(T - x^{p-1} \otimes y)$ annihilates x and so is of rank ≤ 1, whence $(T - x^{p-1} \otimes y) = x^o \otimes u$, for some $u \in \ell_2^p$. Further, $T^*(y^{q-1}) = x^{p-1}$, that is $(T^* - y \otimes x^{p-1})$ annihilates y^{q-1}, whence $(T^* - y \otimes x^{p-1}) = (y^o)^{q-1} \otimes v$, for some $v \in \ell_2^q$. Combining, T must be of the form

$$T_s = x^{p-1} \otimes y + s x^o \otimes (y^o)^{q-1}, \text{ for some } s \in I\!R$$

In other words, $I_{xy} = \{T_s : s \in I\!R, \|T_s\| \leq 1\}$.

As in [34], pre- or post-multiplying by diag$(sgn(x_1), sgn(x_2))$, diag$(sgn(y_1), sgn(y_2))$ and permutation matrices, if necessary — each of which is an isometry — we may assume $x_1 \geq x_2 \geq 0$, $y_1 \geq y_2 \geq 0$.

Now, $T_s(f_p(x,r)) = f_q(y, rs)$, and it follows from the above discussion that for any $r \neq 0$, $\|T_s(f_p(x,r))\|^q = F_q(y, rs)$ is strictly increasing in $s \geq 0$ and strictly decreasing in $s \leq 0$, and $F_q(y, rs)$ is unbounded in s. Now, if $r \neq 0$,

$$F_q(y,0) = 1 = [F_p(x,0)]^{q/p} < [F_p(x,r)]^{q/p}$$

so, there exists unique $s_+(x,y,r) > 0$ and unique $s_-(x,y,r) < 0$ such that

$$F_q(y, rs_\pm) = [F_p(x,r)]^{q/p} \tag{4}$$

And the quantity on the LHS becomes smaller or larger than the one in the RHS according as $|s|$ gets smaller or larger. Evidently, such $s_\pm$ also exist for $(x^o)^{p-1}$, which we denote by $f_p(x,\infty)$. In fact, in this case, $|s_\pm(x,y,\infty)| = \|(x^o)^{p-1}\|/\|(y^o)^{q-1}\|$. Notice that for fixed x, y, $s_\pm$ is a continuous function of $r \neq 0$ and elementary examples show that $\lim_{r\to 0} s_\pm(x,y,r)$ may not even exist. Let

$$\begin{aligned} s_+^*(x,y) &= \inf\{s_+(x,y,r) : r \neq 0\} & s_+^{**}(x,y) &= \liminf_{r\to 0} s_+(x,y,r) \\ s_-^*(x,y) &= \sup\{s_-(x,y,r) : r \neq 0\} & s_-^{**}(x,y) &= \limsup_{r\to 0} s_-(x,y,r) \end{aligned}$$

where $\liminf_{r\to 0} s_+(x,y,r) = \sup_{\varepsilon>0} \inf\{s_+(x,y,r) : |r| < \varepsilon\}$ and $\limsup_{r\to 0} s_-(x,y,r) = \inf_{\varepsilon>0} \sup\{s_-(x,y,r) : |r| < \varepsilon\}$. Clearly, $s_-^{**} \leq s_-^* \leq 0 \leq s_+^* \leq s_+^{**}$, and $T_s \in I_{xy}$ if and only if $s_-^* \leq s \leq s_+^*$, i.e., $T_{s_\pm^*}$ are end points of I_{xy} and hence are extreme. Also note that if $J_{xy} = \{s : T_s$ is contractive in a neighbourhood of $x\}$, then $s_-^{**} = \inf J_{xy}$ and $s_+^{**} = \sup J_{xy}$.

Now, either $s_\pm^*$ equals $s_\pm(x,y,r)$ for some $r \neq 0$ (including $r = \infty$), in which case $s_\pm^* \neq 0$ and $T_{s_\pm^*}$ attain their norm on two linearly independent vectors, or $s_\pm^* = s_\pm^{**}$, in which case $|s_\pm^{**}| < \infty$, $s_\pm^{**} \in J_{xy}$, in fact, $T_{s_\pm^{**}} \in I_{xy}$.

Note that T attains its norm on two linearly independent vectors if and only if T^* attains its norm on two linearly independent vectors. Moreover, any such T is exposed, and hence strongly exposed.

Fix x, y.

CASE(I) : (i) $p = 2$ and either $q = 2$ or $y_2 = 0$; (ii) $q = 2$ and either $p = 2$ or $x_2 = 0$; (iii) $p \neq 2 \neq q$ and $x_2 = 0 = y_2$.

$$\begin{aligned} T_s \text{ is a contraction} &\iff F_q(y, rs) \leq [F_p(x,r)]^{q/p} \text{ for all } r \\ &\iff 1 + |rs|^q \leq [1 + |r|^p]^{q/p} \text{ for all } r \\ &\iff |s|^q \leq \frac{[1+|r|^p]^{q/p} - 1}{|r|^q} \text{ for all } r \neq 0 \end{aligned}$$

Note that the RHS $\equiv 1$ if $p = q$ and is strictly decreasing (resp. increasing) in $|r|$ for $q > p$ (resp. $q < p$).

So, if $p = q$, $s_\pm \equiv \pm 1$, and hence, $s_\pm^* = s_\pm^{**} = \pm 1$ and $T_{s_\pm^*}$ are isometries. And, if $p \neq q$, the infimum of the RHS over $r \neq 0$ yields

$$|s_\pm^*|^q = \begin{cases} 1 & \text{if } q > p \\ 0 & \text{if } q < p \end{cases}$$

So, if $q < p$, $s_\pm^* = 0$, hence $I_{xy} = \{T_0\}$, and

$$T_0 = \begin{cases} x \otimes e_1 & \text{if } p = 2 > q \\ e_1^* \otimes y & \text{if } p > q = 2 \\ e_1^* \otimes e_1 & \text{if } p > q,\ p \neq 2 \neq q \end{cases}$$

is an extreme contraction which attains its norm at precisely one point.

And if $q > p$, $s_\pm^* = s_\pm(\infty) = \pm 1$ with $T_{\pm 1}$ attaining its norm at both x and $(x^o)^{p-1}$. It is interesting to note that if $p \neq 2 \neq q$, T_1 in this case is the identity operator.

In each of the following cases our basic aim is to find extreme contractions that do not attain their norm on two linearly independent vectors. Then necessarily $s_\pm^* = s_\pm^{**}$. So, we first assume $|s_\pm^{**}| < \infty$. If we reach a contradiction, we conclude that $s_\pm^{**} = \pm\infty$, whence $s_\pm^* \neq s_\pm^{**}$ and $T_{s_\pm^*}$ is not of the desired type. Otherwise, we check whether $s_\pm^{**} \in J_{xy}$. If it does not, we are again done. And if it does, we further check whether $T_{s_\pm^{**}}$ gives a contraction. If it does not, we are done, and if it does, we get an extreme contraction. We then check whether it attains its norm in any direction other than that of x and only if it does not, we get an extreme contraction of the desired type. This is exemplified in the analysis below.

To calculate $s_\pm^{**}$, let $\{r_n\}$ be a sequence of real numbers such that $r_n \to 0$ and $s_\pm(r_n) \to s_\pm^{**}$. Since $|s_\pm^{**}| < \infty$, $\{s_\pm(r_n)\}$ is a bounded sequence. Now, by (4),

$$F_q(y, r_n s_\pm(r_n)) = [F_p(x, r_n)]^{q/p} \tag{5}$$

CASE (II) : $q \neq 2$, $y_2 > 0$ and either $p = 2$ or $x_2 = 0$.

In this case, subtracting 1 from both side of (5), dividing by r_n^2 and taking limit as $n \to \infty$, we get by L'Hospital's rule and (2) that LHS$\longrightarrow$

$\frac{1}{2}q(q-1)s^2(y_1y_2)^{q-2}$, where $s = s_{\pm}^{**}$ and

$$\text{RHS} \longrightarrow \begin{cases} 0 & \text{if } p > 2 \\ \infty & \text{if } p < 2 \\ \dfrac{q}{2} & \text{if } p = 2 \end{cases}$$

So, if $p < 2$, we have a contradiction, whence $s_{\pm}^{**} = \pm\infty$, so that $T_{s_{\pm}^{*}}$ is not of the desired type, and if $p > 2$, $s_{\pm}^{*} = s_{\pm}^{**} = 0$, $I_{xy} = \{T_0\}$ and $T_0 = \boldsymbol{e}_1^* \otimes \boldsymbol{y}$ is extreme, and clearly of the desired type.

If $p = 2$, we have

$$s^2(q-1)(y_1y_2)^{q-2} = 1 \tag{6}$$

Now, if the s given by (6) belongs to J_{xy}, we must have

$$F_q(\boldsymbol{y}, rs) \leq [F_p(\boldsymbol{x}, r)]^{q/p} \quad \text{for all small } r \neq 0 \tag{7}$$

By (2), the Taylor expansion of the LHS above around $r = 0$ is given by

$$1 + \frac{1}{2!}q(q-1)r^2s^2(y_1y_2)^{q-2} + \frac{1}{3!}q(q-1)(q-2)r^3s^3(y_1y_2)^{q-3}[y_1^q - y_2^q]$$
$$+\frac{1}{4!}q(q-1)(q-2)(q-3)r^4s^4(y_1y_2)^{q-4}[1 - 3(y_1y_2)^q] + \cdots$$

while that of the RHS is given by

$$1 + \frac{1}{2}qr^2 + \frac{1}{8}q(q-2)r^4 + \cdots$$

Comparing the two, we see that the coefficients of 1, r and r^2 on both sides are equal, whence the inequality (7) for small r implies the corresponding inequality for the coefficient of r^3 on both sides, which, for $r > 0$ and $r < 0$, leads to the following equality :

$$\frac{1}{6}s^3q(q-1)(q-2)(y_1y_2)^{q-3}(y_1^q - y_2^q) = 0 \tag{8}$$

Combining equations (6) and (8), we have $y_1^q = y_2^q = 1/2$, $s^2 = \frac{1}{(q-1)}4^{(q-2)/q}$. But again the equality in (8) forces the inequality of the coefficient of r^4, i.e.,

$$\frac{1}{8}q(q-2) \geq \frac{1}{24}s^4q(q-1)(q-2)(q-3)(y_1y_2)^{q-4}[1 - 3(y_1y_2)^q]$$

$$\text{or} \quad 3(q-2) \geq \frac{(q-2)(q-3)}{(q-1)}$$

Now for $q < 2$, this leads to a contradiction, so that $s_{\pm}^{**} \notin J_{xy}$. And for $q > 2$, by [28, Proposition 1], T_s with these parameters is a contraction that attains its norm only in the direction of x. So, for $q > 2$, T_s with these parameters is of the desired type and for $q < 2$, $s_{\pm}^{*} \neq s_{\pm}^{**}$, so that $T_{s_{\pm}^{*}}$ is not of the desired type.

CASE (III) : $p \neq 2$, $x_2 > 0$ and either $q = 2$ or $y_2 = 0$.

This is the situation dual to case (II) and hence, if $q < 2$, $s_{\pm}^{*} = s_{\pm}^{**} = 0$, $I_{xy} = \{T_0\}$ and $T_0 = x^{p-1} \otimes e_1$ is extreme and of the desired type; if $q > 2$, $s_{\pm}^{*} \neq s_{\pm}^{**}$, so that $T_{s_{\pm}^{*}}$ is not of the desired type, and if $q = 2$, then for $p < 2$, $s_{\pm}^{*} = s_{\pm}^{**} = \pm\sqrt{(p-1)}\, 2^{(2-p)/p}$, $x_1 = x_2 = (1/2)^{1/p}$, T is extreme and of the desired type, and for $p > 2$, $T_{s_{\pm}^{*}}$ is not of the desired type.

CASE(IV) : $p \neq 2 \neq q$ and $x_2 > 0$, $y_2 > 0$.

In this case too, subtracting 1 from both side of (5), dividing by r_n^2 and taking limit as $n \to \infty$, we get by (2) and (3)

$$(q-1)(y_1y_2)^{q-2}s^2 = (p-1)(x_1x_2)^{p-2} \tag{9}$$

where $s = s_{\pm}^{**}$. Now,

$$s = \pm\left[\frac{(p-1)(x_1x_2)^{p-2}}{(q-1)(y_1y_2)^{q-2}}\right]^{1/2} = \pm\left[\frac{(p-1)(x_1x_2)^{p}}{(q-1)(y_1y_2)^{q}}\right]^{1/2} \cdot \frac{y_1y_2}{x_1x_2}$$

Put $\alpha = \left[\dfrac{(p-1)(x_1x_2)^{p}}{(q-1)(y_1y_2)^{q}}\right]^{1/2}$, then $s = \pm\alpha\dfrac{y_1y_2}{x_1x_2}$, whence

$$\begin{aligned}
T_s &= x^{p-1} \otimes y + s x^o \otimes (y^o)^{q-1} \\
&= \begin{bmatrix} x_1^{p-1}y_1 & x_2^{p-1}y_1 \\ x_1^{p-1}y_2 & x_2^{p-1}y_2 \end{bmatrix} \pm \alpha\frac{y_1y_2}{x_1x_2}\begin{bmatrix} y_2^{q-1}x_2 & -y_2^{q-1}x_1 \\ -y_1^{q-1}x_2 & y_1^{q-1}x_1 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{x_1^p y_1}{x_1} & \dfrac{x_2^p y_1}{x_2} \\ \dfrac{x_1^p y_2}{x_1} & \dfrac{x_2^p y_2}{x_2} \end{bmatrix} \pm \alpha \begin{bmatrix} \dfrac{y_1 y_2^q}{x_1} & -\dfrac{y_1 y_2^q}{x_2} \\ -\dfrac{y_2 y_1^q}{x_1} & \dfrac{y_2 y_1^q}{x_2} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{y_1}{x_1}(x_1^p \pm \alpha y_2^q) & \dfrac{y_1}{x_2}(x_2^p \mp \alpha y_2^q) \\ \dfrac{y_2}{x_1}(x_1^p \mp \alpha y_1^q) & \dfrac{y_2}{x_2}(x_2^p \pm \alpha y_1^q) \end{bmatrix}
\end{aligned} \tag{10}$$

To continue, in this case, by (2), the Taylor expansion of the LHS of (7) around $r = 0$ is given by

$$1+\frac{1}{2!}q(q-1)r^2s^2(y_1y_2)^{q-2}+\frac{1}{3!}q(q-1)(q-2)r^3s^3(y_1y_2)^{q-3}[y_1^q-y_2^q]$$
$$+\frac{1}{4!}q(q-1)(q-2)(q-3)r^4s^4(y_1y_2)^{q-4}[1-3(y_1y_2)^q]+\cdots$$

while by (3), that of the RHS is given by

$$1+\frac{1}{2!}q(p-1)r^2(x_1x_2)^{p-2}+\frac{1}{3!}q(p-1)(p-2)r^3(x_1x_2)^{p-3}[x_1^p-x_2^p]$$
$$+\frac{1}{4!}q(p-1)r^4(x_1x_2)^{p-4}[(p-2)(p-3)-3(x_1x_2)^p\cdot$$
$$\{(p-2)(p-3)+(p-q)(p-1)\}]+\cdots$$

So, if $s \in J_{xy}$, by similar arguments, we must have

$$s^3(q-1)(q-2)(y_1y_2)^{q-3}(y_1^q-y_2^q)=(p-1)(p-2)(x_1x_2)^{p-3}(x_1^p-x_2^p) \quad (11)$$

and

$$s^4(q-1)(q-2)(q-3)(y_1y_2)^{q-4}[1-3(y_1y_2)^q]\le(p-1)(x_1x_2)^{p-4}\cdot$$
$$[(p-2)(p-3)\{1-3(x_1x_2)^p\}-3(p-q)(p-1)(x_1x_2)^p] \quad (12)$$

Eliminating s from (9) and (11) and using the fact that $\boldsymbol{x}$, $\boldsymbol{y}$ are unit vectors, we get

$$\frac{(q-2)^2}{(q-1)}\left[\frac{1}{(y_1y_2)^q}-4\right]=\frac{(p-2)^2}{(p-1)}\left[\frac{1}{(x_1x_2)^p}-4\right] \quad (13)$$

Also, dividing (12) by the square of (9), we get

$$\frac{(q-2)(q-3)}{(q-1)}\left[\frac{1}{(y_1y_2)^q}-3\right]\le\frac{(p-2)(p-3)}{(p-1)}\left[\frac{1}{(x_1x_2)^p}-3\right]-3(p-q) \quad (14)$$

Notice that if $p = q$, we get from (13) that $x_i = y_i$, $i = 1, 2$, whence from (9), $s = \pm 1$, and from (11), $y_1 = y_2 = x_1 = x_2$ for $s = -1$. Also, (14) is consistent with (13) and equality holds. Now, in (10), $\alpha = 1$, and so,

$$T_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \text{ and } T_{-1}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

and hence, clearly, are isometries.

Now, let $p \neq q$. From (13) and (14), writing $\frac{1}{(x_1x_2)^p} = A$, we get

$$\frac{(p-2)^2(q-3)}{(q-2)(p-1)}(A-4) + \frac{(q-2)(q-3)}{(q-1)} \leq \frac{(p-2)(p-3)}{(p-1)}(A-3) - 3(p-q)$$

or

$$\begin{aligned} &\left[\frac{(p-2)^2(q-3)}{(q-2)(p-1)} - \frac{(p-2)(p-3)}{(p-1)}\right] \cdot A \\ \leq\ & \frac{4(p-2)^2(q-3)}{(q-2)(p-1)} - \frac{3(p-2)(p-3)}{(p-1)} - \frac{(q-2)(q-3)}{(q-1)} - 3(p-q) \\ =\ & \left[\frac{3(p-2)^2(q-3)}{(q-2)(p-1)} - \frac{3(p-2)(p-3)}{(p-1)}\right] + \left[\frac{(p-2)^2(q-3)}{(q-2)(p-1)}\right. \\ & \left. - \frac{(q-2)(q-3)}{(q-1)}\right] - 3(p-q) \end{aligned}$$

Simplifying we get

$$\frac{(q-p)(p-2)}{(p-1)(q-2)} \cdot A \leq \frac{2q(q-p)(pq-p-q)}{(p-1)(q-1)(q-2)}$$

So, if (a) $1 < q < 2 < p < \infty$, or, (b) $1 < p < q < 2$, or, (c) $2 < p < q < \infty$, we have

$$A \leq \frac{2q(pq-p-q)}{(p-2)(q-1)} \quad \text{i.e.,} \quad A - 4 \leq \frac{2(q-2)(pq-p-q+2)}{(p-2)(q-1)} \tag{15}$$

And if (d) $1 < p < 2 < q < \infty$, or, (e) $1 < q < p < 2$, or, (f) $2 < q < p < \infty$, we have

$$A \geq \frac{2q(pq-p-q)}{(p-2)(q-1)} \quad \text{i.e.,} \quad A - 4 \geq \frac{2(q-2)(pq-p-q+2)}{(p-2)(q-1)} \tag{16}$$

Now, $(x_1x_2)^p = 1/A$ and $x_1^p + x_2^p = 1$, so $0 < 1/A \leq 1/4$, i.e., $A \geq 4$ or $A - 4 \geq 0$.

But since $(pq - p - q + 2) = (p-1)(q-1) + 1$ is always positive for $1 < p, q < \infty$, if $1 < q < 2 < p < \infty$, i.e., in case (a) above, we reach a contradiction at this point, whence T is not of the desired type.

Summarising the results so far, we have the following

Theorem 4.9 *For $1 < p,\, q < \infty$, an operator $T : \ell_2^p \longrightarrow \ell_2^q$ with $\|T\| = 1$ is an extreme contraction*

(*i*) [26] *for $p = q = 2$, if and only if T is an isometry.*

(*ii*) [28] *for $p = 2 \neq q$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} \boldsymbol{x} \otimes \boldsymbol{e}_i & \text{if } q < 2 \\ \boldsymbol{x} \otimes \boldsymbol{y} + s\boldsymbol{x}^o \otimes (\boldsymbol{y}^o)^{q-1} & \text{if } q > 2 \end{cases}$$

where $\boldsymbol{x}$ is any unit vector and, in the second case, $|y_i|^q = \frac{1}{2}$ and $s = \pm\frac{1}{\sqrt{(q-1)}}\, 2^{(q-2)/q}$.

(*iii*) [28] *for $p \neq 2 = q$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} \boldsymbol{e}_i \otimes \boldsymbol{y} & \text{if } p > 2 \\ \boldsymbol{x}^{p-1} \otimes \boldsymbol{y} + s\boldsymbol{x}^o \otimes \boldsymbol{y}^o & \text{if } p < 2 \end{cases}$$

where $\boldsymbol{y}$ is any unit vector and, in the second case, $|x_i|^p = \frac{1}{2}$ and $s = \pm\sqrt{(p-1)}\, 2^{(2-p)/p}$.

(*iv*) [25] *for $p = q$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} \boldsymbol{e}_i^* \otimes \boldsymbol{y} & \text{if } p > 2,\ y_1 y_2 \neq 0 \\ \boldsymbol{x}^{p-1} \otimes \boldsymbol{e}_j & \text{if } p < 2,\ x_1 x_2 \neq 0 \end{cases}$$

(*v*) *for $1 < q < 2 < p < \infty$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *$T = \boldsymbol{x}^{p-1} \otimes \boldsymbol{y}$ with $\boldsymbol{x}$, $\boldsymbol{y}$ unit vectors and $x_1 x_2 y_1 y_2 = 0$.*

REMARK : In the discussion preceeding Theorem 4.9, the conditions (b) and (e) are dual to (c) and (f) respectively. And in the cases (b) and (c), the inequality (15) implies

$$\frac{1}{2} \leq x_1^p \leq \frac{1}{2}\left[1+\sqrt{\frac{(q-2)(pq-p-q+2)}{q(pq-p-q)}}\right] \tag{17}$$

while in cases (e) and (f), the inequality (16) implies

$$\frac{1}{2}\left[1+\sqrt{\frac{(q-2)(pq-p-q+2)}{q(pq-p-q)}}\right] \leq x_1^p < 1 \tag{18}$$

Now, in cases (b), (c), (e) and (f), we have from (11) that for $s < 0$, both sides of (11) must be 0, i.e., we must have $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$. But then in cases (e) and (f), we have a contradiction. So, in these two cases, s_-^* gives extreme contractions not of the desired type.

Also in case (d), (16) is always satisfied and (11) implies that for $s > 0$, $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$. Now, combining Propositions 2 and 3 of [28], we see easily that for $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$ for both $s > 0$ and $s < 0$, we have a contraction that attains its norm only in the direction of x, i.e., we get extreme contractions of the desired type.

Thus, we are left with the following cases unsolved :

(1) Case (b) with x_1 satisfying (17) for $s > 0$ and $x_1 = 1/2$ for $s < 0$ with y_1 given by (13).

(2) Case (e) with x_1 satisfying (18) for $s > 0$ with y_1 given by (13).

(3) Case (d) with $x_1^p > 1/2$ and $s < 0$ with y_1 given by (13).

Notice that for $\frac{1}{p}+\frac{1}{q} = 1$ $(p < 2)$, (13) gives $x_i^p = y_i^q$, $i = 1, 2$, whence in this case, T_s — as given by (10) — has a comparatively simple form.

We proceed now to find the closure of the extreme contractions in the cases described in Theorem 4.9. In case (i), the set of extreme contractions is clearly closed. Also, in the cases (ii), (iii) and (v), the set of operators of the form (b) is clearly closed. And in case (iv), the closure of the set of operators of the form (b) contains only the operators $e_i \otimes e_j$, $i, j = 1, 2$ in addition.

Let us consider the set of operators of the type (a) in cases (ii), (iv) and (v). Let $\{T_n\}$ be a sequence of operators of the type (a). Let $T_n \longrightarrow T$ in operator norm. Let $x_n = (x_{n1}, x_{n2})$ be such that $\|x_n\| = 1 = \|T_n x_n\|$. Let $T_n x_n = y_n = (y_{n1}, y_{n2})$. Then T_n is of the form

$$T_n = x_n^{p-1} \otimes y_n + s_\pm^*(x_n, y_n) x_n^o \otimes (y_n^o)^{q-1}$$

where $s_\pm^*(x_n, y_n)$ is as in our earlier discussion. For notational simplicity, write $s_\pm^*(x_n, y_n) = \pm s_n$. Passing to a subsequence, if necessary, assume $x_n \to x = (x_1, x_2)$, $y_n \to y = (y_1, y_2)$ (by compactness of the unit balls of ℓ_2^p and ℓ_2^q), and all the s_n's have the same sign, without loss of generality, positive.

Clearly, $\|T\| = 1$ and $Tx = y$, whence T is of the form

$$T = x^{p-1} \otimes y + s x^o \otimes (y^o)^{q-1}$$

Also, as $T_n \longrightarrow T$, $s_n = \|T_n - x_n^{p-1} \otimes y_n\| / \|x_n^o\| \cdot \|(y_n^o)^{q-1}\| \longrightarrow \|T - x^{p-1} \otimes y\| / \|x^o\| \cdot \|(y^o)^{q-1}\|$, i.e., $\{s_n\}$ is convergent. Clearly, $s_n \to s$.

Now, since T_n is of the type (a), there exists $z_n = x_n + r_n (x_n^o)^{p-1}$ with $r_n \neq 0 \in \mathbb{R}$ such that $\|T_n z_n\| = \|z_n\|$. Again we may assume all r_n's are of the same sign, in particular positive and $r_n \to r$, where $0 \le r \le \infty$. If $0 < r < \infty$, $z_n \to z = x + r(x^o)^{p-1}$ and $\|Tz\| = \|z\|$, i.e., T is also of the type (a). Also, if $r_n \to \infty$, let $u_n = z_n / \|z_n\|$. Then $u_n \to u = (x^o)^{p-1} / \|(x^o)^{p-1}\|$ and $\|Tu\| = 1$, so that T again is of the type (a).

Now, suppose $r_n \to 0$. Then from $\|T_n z_n\| = \|z_n\|$ we have

$$F_q(y_n, r_n s_n) - [F_p(x_n, r_n)]^{q/p} = 0 \tag{19}$$

For (ii), if $p = 2$ and $q < 2$, since T_n is of type (a), we have $y_{n1} y_{n2} \neq 0$ for all n. And if $q > 2$, we have two possibilities; either there is a subsequence for which $y_{n1} y_{n2} = 0$, or, eventually $y_{n1} y_{n2} \neq 0$. In the first case, we restrict ourselves only to that subsequence, and we have, by case(I), $s_n = 1$ for all n, whence $s = 1$. Also, $y_1 y_2 = 0$. So, $T = x \otimes e_i + x^o \otimes e_j$ $(i \neq j)$, and it is clear that T is of type (a) (see case (I)). And in the second case, we assume $y_{n1}y_{n2} \neq 0$ for all n. Then dividing (19) by r_n^2 and taking limit as $n \to \infty$, we get by L'Hospital's rule

$$(q-1)s^2(y_1y_2)^{q-2} - 1 = 0 \tag{20}$$

If $q > 2$, for $y_1y_2 = 0$, this leads to a contradiction, whence either $r_n \not\to 0$ (in which case we get an operator of type(a)), or $y_1y_2 \neq 0$. In the latter case, (6) and (20) coincides, i.e., we have $s = s_+^{**}(x,y)$. Now, our analysis in case (II) (see p 53) shows that only for $y_1^q = y_2^q = 1/2$, s_+^{**} gives a contraction (which is an extreme contraction of type (b)). And in every other case, we run into a contradiction, i.e., we must have $r_n \not\to 0$.

And if $q < 2$, for $y_1y_2 = 0$, (20) makes sense only if $s = 0$. In that case, $T = x \otimes e_i$, which, by case (I), is an extreme contraction of the type (b). And for $y_1y_2 \neq 0$, we again have $s = s_+^{**}(x,y)$ and our analysis in case (II) shows that this case always leads to a contradiction, whence $r_n \not\to 0$.

So, in both the cases, the closure of the set of operators of the type (a) contains at most operators of type (b), and therefore, the set of extreme contractions is closed.

Since case (iii) is just the dual of case (ii), the set of extreme contractions, in this case too, is closed.

In (iv), i.e., if $p = q$, by duality, it suffices to consider $p > 2$. Since T_n is of type (a), we have three possibilities; either there is a subsequence for which both $x_{n1}x_{n2} = 0$ and $y_{n1}y_{n2} = 0$, or, there is a subsequence for which $x_{n1}x_{n2} \neq 0$ and $y_{n1}y_{n2} = 0$, or, eventually both $x_{n1}x_{n2} \neq 0$ and $y_{n1}y_{n2} \neq 0$.

In the first case, we again restrict ourselves only to that subsequence, and we have, by case(I), $s_n = 1$ for all n, whence $s = 1$. Also, $x_1x_2 = 0$ and $y_1y_2 = 0$. Now again by case (I), T is of type (a).

In the second and the third case, dividing (19) by r_n^2 and taking limit — through a subsequence if necessary — as $n \to \infty$, we get in the second case

$$(x_1x_2)^{p-2} = 0$$

and in the third case

$$(y_1y_2)^{p-2}s^2 = (x_1x_2)^{p-2}$$

So, in the second case, $y_1y_2 = 0$, and we have a contradiction unless $x_1x_2 = 0$. And in that case, T is of the form $\mathrm{diag}(1, s)$ upto isometric factors of signum or permutation matrices. Now, T is a contraction for $-1 \le s \le 1$ and is extreme (in fact, an isometry) only for $s = \pm 1$.

In the third case, if $x_1x_2 = 0$, we get a contradiction unless $y_1y_2 = 0$ or $s = 0$. If $y_1y_2 \neq 0$, $s = 0 = s^*_\pm(x, y)$, whence T is extreme. And if $y_1y_2 = 0$, we get the conclusions as in the second case. If $x_1x_2 \neq 0$, $y_1y_2 = 0$ leads to a contradiction (and therefore, $r_n \not\to 0$), and if $y_1y_2 \neq 0$, $s = s^{**}_+(x, y)$ (see p 55), so that T is an isometry and hence is of type (a).

Thus, in case (iv), the closure of the operators of type (a), and so the closure of extreme contractions, can have at most operators of the form $\mathrm{diag}(1, s)$, $|s| < 1$, upto isometric factors of signum or permutation matrices as additional elements. However, we do not have precise description of the closure.

In case (v), i.e., if $1 < q < 2 < p < \infty$, since T_n is of type (a), we must have $x_{n1}x_{n2}y_{n1}y_{n2} \neq 0$ for all n. And a similar argument leads to

$$s^2(q-1)(y_1y_2)^{q-2} = (p-1)(x_1x_2)^{p-2}$$

If $x_1x_2 \neq 0$ the only situation that does not lead to any contradiction — either immediate or to the fact that T is a contraction — is both $y_1y_2 = 0$ and $s = 0$. And in that case, $s = 0 = s^*_\pm(x, y)$, so that T is extreme. And if $x_1x_2 = 0$, we must have $s = 0$, in which case, by cases (I) and (II), $s = 0 = s^*_\pm(x, y)$ and T is extreme. Thus in this case too, the set of extreme contractions is closed.

Corollary 4.9.1 *In each of the following cases of $1 < p, q < \infty$, $\ell_2^p \otimes_\pi \ell_2^q$ lacks the MIP (equivalently, the CI) :*

(*i*) *p and q are conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.*

(*ii*) *Either p or q is equal to 2.*

(*iii*) *$2 < p, q < \infty$.*

Proof : The dual of $\ell_2^p \otimes_\pi \ell_2^q$ is $\mathcal{L}(\ell_2^p, \ell_2^{q'})$, where $\frac{1}{q} + \frac{1}{q'} = 1$ and the closure of extreme contractions in each of the above cases does not contain norm 1 operators of the form $x^{p-1} \otimes y$, where x and y are unit vectors with $x_1 x_2 y_1 y_2 \neq 0$. ■

REMARK : The fact that operators of the above form do not belong to the closure of extreme contractions in any of these cases seems to suggest that this is a general phenomenon. It is possible that this is happens in higher dimensions as well. Can one give a proof of this without going through the characterisation of extreme contractions ? What seems to be required is a more tractable necessary condition for extremality, or, for belonging to the closure of extreme contractions.

Chapter 5

Exposed Points of Continuity and Strongly Exposed Points

In this chapter, we make the following change in our notations. For a Banach space X and $A \subseteq X$, by A^o, the polar of A, we mean $A^o = \{f \in X^* : f(x) \leq 1 \text{ for all } x \in A\}$ (i.e., we do not take the absolute value of $f(x)$).

If K is a closed bounded convex set in X and $f \in X^*$, we will say that $f \in X^*$ *supports* K *at* $x_o \in K$ if $f(x_o) = M(K, f)$ and we let $S(K, x_o) = \{f \in X^* : f(x_o) = M(K, f)\}$. Note that $S(K, x_o)$ is a w*-closed convex cone with vertex 0. We may sometimes abbreviate $S(K, x_o)$ by $S(x_o)$ when there is no scope of confusion.

Call $x_o \in K$ a *relatively exposed point of* K if for each $x \in K \setminus \{x_o\}$ there exists $f_x \in S(K, x_o)$ such that $f_x(x) < f_x(x_o)$. Call x_o a *vertex of* K if for each $x \in X \setminus \{x_o\}$ there exists $f_x \in S(K, x_o)$ such that $f_x(x) \neq f_x(x_o)$, or, in other words, $S(K, x_o)$ separates points of X. We can also define w*-relatively exposed points or w*-vertices similarly. Obviously, a (w*-) vertex is (w*-) relatively exposed.

5.1. The Counterexample

Let $\{\delta_n : n \geq 1\}$ denote the canonical basis of ℓ^1. Recall that ℓ^1 as a dual space has a w*-topology induced on it. Let

$$K = \overline{\text{co}}^{w^*}[\{\frac{1}{n}\delta_1 + \delta_n : n \geq 2\} \cup \{\frac{1}{n^2}\delta_1 - \frac{1}{n}\delta_n : n \geq 2\}]$$

From Milman's Theorem (see [43, p 9]) and the metrizable version of Choquet's Theorem ([1] or [43]), it follows that $x \in K$ if and only if x is of the form

$$x = \sum_{n\geq 2} \alpha_n(\frac{1}{n}\delta_1 + \delta_n) + \sum_{n\geq 2} \beta_n(\frac{1}{n^2}\delta_1 - \frac{1}{n}\delta_n)$$

with α_n, $\beta_n \geq 0$ and $\sum_{n\geq 2}(\alpha_n + \beta_n) \leq 1$ (see [7, Example 3.2.5] for the details in a similar situation). Observe that $(-1, 0, 0, \ldots) \in c_o \subseteq \ell^\infty = (\ell^1)^*$ exposes $0 = (0, 0, \ldots) \in K$. If $f = (a_n) \in \ell^\infty$ supports K at 0, then $\frac{1}{n}a_1 \pm a_n \leq 0$ for $n \geq 2$, from which it follows that $a_1 \leq 0$ and $|a_n| \leq \frac{1}{n}|a_1|$, $(n \geq 2)$ and we see therefore that $(a_n) \in c_o$. Since $f(\frac{1}{m}\delta_1 + \delta_m) = \frac{1}{m}a_1 + a_m \to 0$ as $m \to \infty$, $S(K, f, \alpha) = \{y \in K : f(y) > -\alpha\}$ contains $\frac{1}{m}\delta_1 + \delta_m$ for all sufficiently large m, whence $\operatorname{diam}[S(K, f, \alpha)] \geq \|\frac{1}{m}\delta_1 + \delta_m\| > 1$, showing that 0 is not strongly exposed.

Next, for $m \geq 2$, let $f_m = (-1, -\frac{1}{m^2}, \ldots, -\frac{1}{m^2}, -\frac{1}{m}, -\frac{1}{m}, \ldots) \in \ell^\infty$ where the mth coordinate onwards of f_m has the value $-\frac{1}{m}$ and consider the slice $S_m = \{y \in K : f_m(y) > -\frac{1}{m^3}\}$ determined by f_m. Clearly, $0 \in S_m$ and if

$$x = \sum_{n\geq 2} \alpha_n(\frac{1}{n}\delta_1 + \delta_n) + \sum_{n\geq 2} \beta_n(\frac{1}{n^2}\delta_1 - \frac{1}{n}\delta_n) \in S_m$$

we must have

$$-\frac{1}{m^3} < -\sum_{n=2}^{m-1} \alpha_n \cdot \frac{m^2+n}{m^2 n} - \sum_{n\geq m} \alpha_n \cdot \frac{m+n}{mn} - \sum_{n=2}^{m-1} \beta_n \cdot \frac{m^2-n}{m^2 n^2} + \sum_{n\geq m} \beta_n \cdot \frac{n-m}{n^2 m}$$

whence it follows that

$$\begin{aligned}
\frac{1}{m^2} &> \sum_{n=2}^{m-1} \alpha_n \cdot \frac{n+1}{n} \cdot \frac{m^2+n}{m(n+1)} + \sum_{n\geq m} \alpha_n \cdot \frac{n+1}{n} \cdot \frac{m+n}{n+1} \\
&\quad + \sum_{n=2}^{m-1} \beta_n \cdot \frac{n+1}{n^2} \cdot \frac{m^2-n}{m(n+1)} - \sum_{n\geq m} \beta_n \cdot \frac{n-m}{n^2} \\
&\geq \sum_{n=2}^{m-1} \alpha_n \cdot \frac{n+1}{n} + \sum_{n\geq m} \alpha_n \cdot \frac{n+1}{n} + \frac{m-1}{m+1} \sum_{n=2}^{m-1} \beta_n \cdot \frac{n+1}{n^2} - \frac{1}{4m}
\end{aligned}$$

on using successively the following easily verified inequalities :

(i) for all $m \geq 1$ and $n \leq m$, $\dfrac{m^2+n}{m(n+1)} \geq 1$.

(ii) for all $m \geq 1$, $\dfrac{m+n}{n+1} \geq 1$.

(iii) for all $n \leq m$, $\dfrac{m^2-n}{m(n+1)} \geq \dfrac{m-1}{m+1}$, since $\dfrac{m^2-n}{m(n+1)}$ is decreasing in n.

(iv) for all $m, n \geq 1$, $\dfrac{n-m}{n^2} \leq \dfrac{1}{4m}$.

Thus,

$$\begin{aligned}
\frac{m+1}{m-1} \cdot \frac{m+4}{4m^2} &\geq \frac{m+1}{m-1} \sum_{n\geq 2} \alpha_n \cdot \frac{n+1}{n} + \sum_{n=2}^{m-1} \beta_n \cdot \frac{n+1}{n^2} \\
&\geq \sum_{n\geq 2} \alpha_n \cdot \frac{n+1}{n} + \sum_{n=2}^{m-1} \beta_n \cdot \frac{n+1}{n^2}
\end{aligned}$$

Now,

$$\begin{aligned}
\|x\| &= \sum_{n\geq 2} \left(\frac{\alpha_n}{n} + \frac{\beta_n}{n^2} \right) + \sum_{n\geq 2} \left| \alpha_n - \frac{\beta_n}{n} \right| \leq \sum_{n\geq 2} \alpha_n \cdot \frac{n+1}{n} + \sum_{n\geq 2} \beta_n \cdot \frac{n+1}{n^2} \\
&\leq \frac{(m+1)(m+4)}{4m^2(m-1)} + \sum_{n\geq m} \beta_n \cdot \frac{n+1}{n^2} \leq \frac{(m+1)(m+4)}{4m^2(m-1)} + \frac{m+1}{m^2} \\
&= \frac{5(m+1)}{4m(m-1)} \leq \frac{15}{4m} \text{ for all } m \geq 2.
\end{aligned}$$

Hence, $\text{diam}(S_m) \leq 15/2m \to 0$ as $m \to \infty$ and we conclude that 0 is a denting point of K.

REMARKS : 1. Let F_o be a w*-cluster point of $\{\frac{1}{n}\delta_1 + \delta_n : n \geq 2\}$ in $\widetilde{K}$. Then $F_o \neq 0$ since $F_o(f_o) = 1$ where $f_o = (1, 1, \ldots, 1, \ldots) \in \ell^\infty$. As $\mathcal{S}(0) \subseteq c_o$, we must have $F_o \in \cap\{\ker(\hat{f}) : f \in \mathcal{S}(0)\}$ and hence 0 is *not* w*-relatively exposed in $\widetilde{K}$. This observation proved useful in the formulation (d) of Theorem 5.3 in the next section.

2. Since $\mathcal{S}(0) \subseteq c_o$, one in fact has $\mathcal{S}(0) = \{(a_n) \in c_o : a_1 \leq 0$ and $|a_n| \leq \frac{1}{n}|a_1|, n \geq 2\}$. It is now easy to see that $\mathcal{S}(0)$ separates points of ℓ^1, i.e., 0 is a vertex of K.

5.2. A Characterisation Theorem

We need the following lemma :

Lemma 5.1 *Let K be a compact convex set in a locally convex space E. Then*

(a) $x_o \in K$ is exposed by $f \in E^*$ if and only if $\{S(K, f, \alpha) : \alpha > 0\}$ forms a local base for the relative topology on K at x_o.

(*b*) *A relatively exposed point* $x_o \in K$ *is exposed if and only if* x_o *is a* G_δ *point.*

Proof : (a) The sufficiency is obvious and the necessity follows from the compactness of K.

(b) [We follow [1, p 119]] If x_o is exposed by $f \in E^*$, then $\{x_o\} = \cap_{n\geq 1} S(K, f, \frac{1}{n})$ is clearly G_δ.

Conversely, let x_o be relatively exposed and $\{x_o\} = \cap_{n\geq 1} W_n$, where W_n is open in K. Now, an easy compactness argument shows that for any compact subset F of K with $x_o \notin F$, there exists $f \in E^*$ such that $f \in S(x_o)$ and $f(y) < f(x_o)$ for all $y \in F$.

Since for all $n \geq 1$, W_n is open in K, by the above argument, there exists $f_n \in S(x_o)$ such that $f_n(y) < f_n(x_o)$ for all $y \in K \setminus W_n$. It is now esay to show that $f = \sum_{n\geq 1} \frac{1}{2^n} f_n$ exposes x_o. ■

Theorem 5.2 *Let* K *be a w*-compact convex set in a dual Banach space* X^* *and* $x_o^* \in K$ *be a w*-PC. The following are equivalent :*

(*a*) $x_o^* \in K$ *is w*-strongly exposed.*

(*b*) $x_o^* \in K$ *is w*-exposed.*

(*c*) $x_o^* \in K$ *is w*-relatively exposed.*

Proof : Follows from Lemma 5.1 and the fact that a w*-PC is a w*-G_δ point. ■

Theorem 5.3 *Let* K *be a closed bounded convex set in a Banach space* X *and* $x_o \in K$ *be a PC. The following are equivalent :*

(*a*) $x_o \in K$ *is strongly exposed.*

(*b*) $\hat{x}_o \in \widetilde{K}$ *is w*-strongly exposed.*

(*c*) $\hat{x}_o \in \widetilde{K}$ *is w*-exposed.*

(*d*) $\hat{x}_o \in \widetilde{K}$ *is w*-relatively exposed.*

(*e*) $X^*/\overline{\mathrm{sp}}(S(x_o)) = \overline{\pi[(K - x_o)^\circ]}$ *where* $\pi : X^* \longrightarrow X^*/\overline{\mathrm{sp}}(S(x_o))$ *is the quotient map and the closure is with respect to the norm topology.*

Proof : (*a*) $\iff$ (*b*) is immediate, while (*b*) $\iff$ (*c*) $\iff$ (*d*) follows from Theorem 5.2 and Lemma 1.4.

$(d) \iff (e)$: Using definitions, simple separation arguments and polar calculations *à la* the Bipolar Theorem, each of the statement below is easily seen to be equivalent to the next :

(1) $\hat{x}_o$ is w*-relatively exposed in $\widetilde{K}$.

(2) 0 is w*-relatively exposed in $\widetilde{K}_1$, where $K_1 = K - x_o$.

(3) $\widetilde{K}_1 \cap [\cap\{\ker(\hat{f}) : f \in S(x_o)\}] = \{0\}$.

(4) $K_1^{oo} \cap [\overline{\text{sp}}(S(x_o))]^o = \{0\}$, $\cdots (*)$
(since $\cap\{\ker(\hat{f}) : f \in S(x_o)\} = [\overline{\text{sp}}(S(x_o))]^\perp = [\overline{\text{sp}}(S(x_o))]^o$ and $\widetilde{K}_1 = K_1^{oo}$)

(5) $[K_1^o \cup \overline{\text{sp}}(S(x_o))]^o = \{0\}$.

(6) $\overline{\text{co}}[K_1^o \cup \overline{\text{sp}}(S(x_o))] = X^*$.

This last condition implies that $\overline{[\overline{\text{sp}}(S(x_o)) + K_1^o]} = X^*$ which in turn means that

$$X^*/\overline{\text{sp}}(S(x_o)) = \overline{\pi(K_1^o)}. \qquad \cdots (**)$$

Conversely, suppose that $(**)$ holds and that $\phi \in K_1^{oo} \cap [\overline{\text{sp}}(S(x_o))]^o$. We get immediately

(a) $\phi \in [\overline{\text{sp}}(S(x_o))]^o = [X^*/\overline{\text{sp}}(S(x_o))]^*$, and

(b) $M(K_1^o, \phi) \leq 1$.

But (a) and (b) together give $M(\overline{\pi(K_1^o)}, \phi) \leq 1$ and using $(**)$, we see that $\phi = 0$, thus $(*)$ holds. ■

Corollary 5.3.1 *If K is weakly compact then $x_o \in K$ is strongly exposed if and only if x_o is exposed and PC.*

Proof : Follows immediately from Theorem 5.2 or from Theorem 5.3, once we observe that if K is weakly compact, $\widetilde{K} = \widehat{K}$ and that $\hat{x}_o \in \widetilde{K}$ is w*-exposed if $x_o \in K$ is exposed. ■

Corollary 5.3.2 (a) *Let K be a closed bounded convex set in a Banach space X. $x_o \in K$ is strongly exposed if and only if x_o is contained in slices of K determined by functionals from $S(K, x_o)$ of arbitrarily small diameter.*

(b) Let K be a w-compact convex set in a dual Banach space X^*. $x_o^* \in K$ is w*-strongly exposed if and only if x_o^* is contained in w*-slices of K determined by functionals from $S(K, x_o^*) \cap X$ of arbitrarily small diameter.*

REMARKS : 1. Note that the proof of $(d) \iff (e)$ in Theorem 5.3 does not use the fact that x_o is a PC.

2. If x_o is a PC and $\hat{x}_o \in \widetilde{K}$ is a w*-vertex, or equivalently, x_o is a PC and $\mathrm{sp}(S(x_o))$ is dense in X^*, it follows from Theorem 5.3, that x_o is strongly exposed. But, if $x_o \in K$ is a strongly exposed point with a unique (upto scalar multiples) exposing functional — this happens for instance for any boundary point of the unit ball of the Euclidean space $\mathbb{R}^2$ — then $\mathrm{sp}(S(x_o))$ is 1-dimensional and thus $X^* \neq \overline{\mathrm{sp}}(S(x_o))$, i.e., this condition is, in general, not necessary. And since 0 in our example is a vertex (see Remark 2), the weaker condition that x_o is a PC and $x_o \in K$ is a vertex is no longer sufficient for x_o to be strongly exposed.

3. As pointed out earlier, $\cap_{\varepsilon>0} M_\varepsilon = \{x \in S(X) :$ the norm is Fréchet differentiable at $x\}$ and by Theorem 1.9, $D(\cap_{\varepsilon>0} M_\varepsilon) = \{$w*-strongly exposed points of $B(X^*)\}$. We note the following consequence of Corollary 5.3.2(b) :

$$\{\text{w*-strongly exposed points of } B(X^*)\} = \cap_{\varepsilon>0} D(M_\varepsilon).$$

Now, if X has the MIP, for each $\varepsilon > 0$, $D(M_\varepsilon)$ is dense in $S(X^*)$. And the necessity of Phelps' condition (a) reduces to the question whether the intersection of these dense sets is also dense. Naturally, if each of these sets was a G_δ, the answer would be yes. So one asks :

> *Does the quasi-continuity of the duality map force the image of M_ε, an open set, to be G_δ ?*

5.3. A Geometric Characterisation of Banach Spaces Containing ℓ^1

Let us say that a closed bounded convex set K in a Banach space X has Property (P) if every exposed PC in K is strongly exposed; and that a

Banach space X has Property (P) if every closed bounded convex set K in X has Property (P).

It follows from the counterexample and Corollary 5.3.1 above that

(i) If X has the Property (P) then X does not contain a copy of ℓ^1.

(ii) If X is reflexive, then X has the Property (P).

The monograph [15] is an excellent introduction to the theory of Banach spaces not containing ℓ^1. Here we show that a weaker version of the Property (P) is equivalent to X not containing ℓ^1, while among a certain class of Banach spaces, Property (P) implies reflexivity. Specifically we prove :

Theorem 5.4 *Let X be a Banach space.*

(a) X does not contain a copy of ℓ^1 if and only if every norm bounded, weakly sequentially complete *convex set in X has Property (P).*

(b) If X is weakly sequentially complete, then X is reflexive if and only if X has Property (P).

Proof : (a) Since ℓ^1 is weakly sequentially complete (see [16]), so is any closed convex subset of it, in particular, so is the set K in our example of Section 5.1. Note that if X contains ℓ^1, the set K above can be identified as a norm bounded, weakly sequentially complete convex subset of X which, by Section 5.1, lacks Property (P).

Conversely, if X does not contain a copy of ℓ^1 and $K \subseteq X$ is weakly sequentially complete then K is weakly compact. Indeed, if $\{x_n\}$ is a sequence in K, then by Rosenthal's ℓ^1 Theorem (see [15] or [12, Chapter XI]), it has a weak Cauchy subsequence which, by weak sequential completeness, is weakly convergent. And hence, by Eberlein-Smulian Theorem, K is weakly compact and, by Corollary 5.3.1, K has the Property (P).

The proof of (b) is similar. ■

REMARK : Can one relax the assumption of weak sequential completeness in any of the two statements above ?

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