Commuting squares and Principal graphs of subfactors

Uma Krishnan

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Preface

This thesis is devoted to the study of some problems related to the inclusion of a pair of (usually hyperfinite) II_1 factors R. Specifically, the following and the relation between them are studied:

(a) pairs of graphs which can occur as principal graphs for $N \subseteq M$;

(b) construction of commuting squares starting with a pair of finite graphs;

(c) computation of the higher relative commutants of subfactors constructed from specific commuting squares.

The first chapter is introductory in nature and is included for the sake of completeness and convenience of reference. It contains a rather perfunctory description of the basic construction for a pair of II_1 factors. The inclusion of finite dimensional C^*-algebras and the construction of a path-algebra on a tower of such algebras is described.

We go on to describe Ocneanu's paragroup invariant for the inclusion of a pair of II_1 factors, which includes a description of the principal graphs for $N \subseteq M$ and Ocneanu's biunitarity condition for the existence of a commuting square. An iterative procedure for constructing a pair of subfactors is described and a complete proof of Ocneanu's Compactness Theorem is given.

In the second chapter the properties of a pair of graphs which arise as principal graphs for $N \subseteq M$ are studied. Based on this, a property called weak duality, for a pair of finite, bipartite, connected graphs is defined. The technical result proved here is that a graph $\mathcal{G}$ with at most triple points, no multiple bonds and not containing two specific subgraphs can be weakly dual only to itself. Combining this with Ocneanu's triple point obstruction, it is shown that a tree, with trivial contragredient map, can occur as a principal graph only if it contains a copy of $E_6^{(1)}$.

In the third chapter, starting with two specific pairs of finite, bipartite, connected graphs, explicit constructions of commuting squares, each with these graphs as inclusions, is given. The first example is taken from the theory of hypergroups and the second occurs as a principal graph of $R \rtimes H \subseteq R \rtimes G$, where H is a particular subgroup of a specific group G.

In the last chapter, a special class of commuting squares, called vertex models, is studied. Two classes of such commuting squares are considered and the principal graphs of subfactors, constructed from these following the standard iterative procedure described in chapter 1, are computed. It is shown that one of these is related to the group dual of a suitable (closed) subgroup of $U(N)$, and the other to the Cayley graph of a (non-closed) group, modulo scalars, generated by N elements of $U(N)$.

Further, the vertex models in the case $N = 2$ are classified and the possible resulting principal graphs are identified as $A_{(2n-1)}^{(1)}, 1 \leq n \leq \infty$. A brief comment is made on some results which have been obtained in the case when $N = 3$ and the generating biunitary matrix is a permutation matrix. The most interesting result is the occurrence of infinite graphs among the possible principal graphs arising from such commuting squares.

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Contents

Chapter 1

Preliminaries

1.1 The basic construction

We start by recalling some facts about the basic construction for $N \subseteq M$, a pair of II_1 factors of finite index. Let tr denote the unique faithful, normal, normalised (so that $tr(1) = 1$) trace on M. Then, let $L^2(M, tr)$ denote the Hilbert space completion of M via the GNS construction. Let e_N denote the projection of $L^2(M, tr)$ onto $L^2(N, tr)$. Note that M acts on $L^2(M, tr)$ by left multiplication. The subalgebra of $\mathcal{L}(L^2(M, tr))$ generated by M and e_N is said to be the basic construction of $N \subseteq M$, and is denoted by M_1 and e_N is called the Jones projection for the basic construction of $N \subseteq M$.

The map $E_N : M \mapsto N$ is the restriction of e_N to M and thus satisfies $tr(E_N(x)y) = tr(xy)$ for all y in N and x in M. E_N is called the conditional expectation of M onto N.

The extension of the adjoint map in M to $L^2(M, tr)$ is denoted by J_M, and $M_1 = J_M\ N'\ J_M$. Hence, under the assumption of finite index, M_1 is also a II_1 factor.

The trace tr defines a Markov trace for $N \subseteq M$, i.e. it extends to a trace on M_1 so that $tr(xe_N) = \tau tr(x)$ for all $x \in M$ where $\tau = [M : N]^{-1}$, and its extension to M_1 is a Markov trace for $M \subseteq M_1$. It is further true that $[M_1 : M] = \tau^{-1}$.

Using this, we can iterate the basic construction and thus obtain Jones' tower of the basic construction for $N \subseteq M$:

$$N \subseteq M \subseteq M_1 \subseteq M_2 \cdots \subseteq M_n \subseteq \cdots.$$

The existence of the $*$-operation and a well defined trace in B enables us to carry out the basic construction for a pair of finite-dimensional C^*-algebras, $A \subseteq B$. The next section gives a brief description of such an inclusion and the ensuing basic construction.

1.2 Inclusions of finite dimensional algebras

We give here a brief description of the inclusions of finite dimensional C^*-algebras and the associated Bratteli diagrams. If A, B are such algebras then, being semi-simple, $A \simeq \bigoplus_{i=1}^{n} M_{l_i}(\mathbb{C})$ and $B \simeq \bigoplus_{j=1}^{m} M_{k_j}(\mathbb{C})$.

Let $\pi(X)$ denote the set of minimal central projections of X. The Bratteli diagram for $A \subseteq B$ is the graph Λ whose set of vertices is the disjoint union of two sets labelled by $\pi(A)$ and $\pi(B)$ and p_i in $\pi(A)$ is joined to q_j in $\pi(B)$ by m_{ij} bonds where m_{ij} is the multiplicity of $M_{l_i}(\mathbb{C})$ in $M_{k_j}(\mathbb{C})$ under this inclusion.

The inclusion $A \subseteq B$ can also be described by an inclusion matrix λ of size $n \times m$ where $\lambda_{ij} = m_{ij}$. Clearly the adjacency matrix of Λ is given by

$$\begin{bmatrix} 0 & \lambda \\ \lambda' & 0 \end{bmatrix}.$$

The vectors $(l_1, l_2, \cdots, l_n)$ and $(k_1, k_2, \cdots, k_m)$ are the dimension vectors of A and B respectively. Clearly, if the inclusion of A in B is unital,

$$\vec{k} = \lambda' \vec{l}. \tag{1.1}$$

(We shall always regard vectors as column vectors.)

Since there is a canonical normalized trace on $M_n(\mathbb{C})$, any trace on B is specified by a vector $(t_B^1, t_B^2, \cdots t_B^n)$, where t_B^i is the trace of a minimal projection under q_i. Then the the trace on A, obtained by the restriction of the trace on B, is given by the trace vector $\vec{t_A} = \lambda \vec{t_B}$.

If $A \subseteq B \subseteq B_1$ is an instance of the basic construction for finite dimensional algebras, then it is not a difficult fact that $\pi(A) \simeq \pi(B_1)$ and the inclusion matrix for $B \subseteq B_1$ is λ'. Thus, if $\vec{t}_{B_1}$ were to define a trace on B_1, whose restriction to B and A are given by the vectors $\vec{t_B}$ and $\vec{t_A}$ respectively, then

$$\vec{t_A} = \lambda \vec{t_B} = (\lambda\lambda')\vec{t}_{B_1}. \tag{1.2}$$

Also, $\vec{t_B}$ defines a Markov trace for $A \subseteq B$ if and only if the following equivalent conditions are satisfied :

(i) $(\lambda\lambda')\vec{t_A} = \tau^{-1}\vec{t_A}$;

(ii) $(\lambda'\lambda)\vec{t_B} = \tau^{-1}\vec{t_B}$;

in this case we necessarily have $\tau = ||\lambda||^{-2}$.

If the conditions above are satisfied, there is a trace on B_1 given by $\vec{t}_{B_1}$ whose restriction to B is given by $\vec{t}_B$ and

$$\vec{t}_{B_1} = \tau \vec{t}_A.$$

Thus, from equation 1.2 and the above equation we have

$$(\lambda\lambda')\vec{t}_{B_1} = ||\lambda||^2 \vec{t}_{B_1}.$$

So, $\vec{t}_{B_1}$ is the Perron-Frobenius eigen-vector for $(\lambda\lambda')$.

For the details and proofs see [J] and [GHJ].

1.3 The path algebra model

We give here a brief description of the path algebra associated with the inclusion of finite dimensional C^*-algebras, $A \subseteq B \subseteq C$. Let $\mathcal{G}$ and $\mathcal{H}$ be the Bratteli diagrams for the inclusions of A in B and B in C respectively. Let $\Omega_{[A,B]}$ be the space whose elements are paths in $\mathcal{G}$, from the vertices corresponding to A, indexed by $\pi(A)$, to the vertices corresponding to B, indexed by $\pi(B)$. Let $\Omega_{[A,B,C]}$ denote the space of paths from $\pi(A)$ to $\pi(C)$ through $\pi(B)$, along the graphs $\mathcal{G}$ and $\mathcal{H}$.

For α in $\Omega_{[A,C]}$ let $s(\alpha)$ denote the starting vertex of α in $\pi(A)$ and $f(\alpha)$ denote the finishing vertex of α in $\pi(C)$. Also let α_B denote the vertex in $\pi(B)$ through which α passes.

If the matrix describing the inclusion $A \subseteq B$ were λ, then from equation 1.1 we have that, in particular, $k_j = \sum_i [\lambda']_{ji}\ l_i = \sum_i [\lambda]_{ij}\ l_i$. Note that for a longer chain of inclusions this would still hold with λ replaced by the product of the inclusion matrices. So, when all the l_i's are 1, the dimension k_j of B_j is the number of distinct paths in the Bratteli diagram, ending at the vertex labelled by p_j in $\pi(B)$. When l_i's are not 1, we can introduce an inclusion $\mathbb{C} \subseteq A$, with inclusion matrix λ_{-1} of size $1 \times n$ given by $[\lambda_{-1}]_j = l_j$.

Define $\mathcal{H}_{[A,B]}$ to be a Hilbert space with orthonormal basis indexed by the paths in $\Omega[A,B]$. Then clearly, $\mathcal{L}(\mathcal{H}_{[A,B]})$ is a matrix algebra. The elements of $\mathcal{L}(\mathcal{H}_{[A,B]})$ are matrices with rows and columns indexed by paths in $\Omega_{[A,B]}$. From the discussion in the last paragraph, it is clear that, B is isomorphic to the subalgebra of $\mathcal{L}(\mathcal{H}_{[A,B]})$ given by

$$\{x \in M_{\Omega_{[A,B]}}(\mathbb{C}) : x(\alpha,\beta) \neq 0 \text{ only if } f(\alpha) = f(\beta)\}.$$

When the paths of $\Omega_{[A,B]}$ are suitably ordered, the elements of B would correspond to block diagonal matrices, the blocks being labelled by the elements of $\pi(B)$, under the above identification.

In this representation, the inclusions of the algebras are described thus: if x is in B, then for all α, β in $\Omega_{[A,C]}$, $x(\alpha,\beta) = x(\alpha_0,\beta_0)$ if and only if $\alpha = \alpha_0 \circ \xi, \beta = \beta_0 \circ \xi$ for some ξ in $\Omega_{[B,C]}$.

When we consider the path space of a long chain of algebras, $\alpha_{A]}$ is used to denote the stretch of α upto A, $\alpha_{[A}$ is used to denote the stretch after A and $\alpha_{[A,B]}$ denotes the stretch between A and B.

Clearly, for x in B', $x(\alpha,\beta)$ is non-zero only if $\alpha_{B]} = \beta_{B]}$ and $x(\alpha,\beta) = x'(\alpha_{[B},\beta_{[B})$ for some $x' \in \mathcal{L}(\mathcal{H}_{[B})$.

From this observation it follows that an element of $A' \cap C$ has the description:

$$x(\alpha,\beta) = \delta_{\alpha_{A]}\beta_{A]}}\delta_{\alpha_{[C}\beta_{[C}}X(\alpha_{[A,C]},\beta_{[A,C]})$$

where $X \in \mathcal{L}(\mathcal{H}_{[A,B,C]}) = M_{\Omega_{[A,B,C]}}(\mathbb{C})$.

Let $\mathcal{G}$ be the Bratteli diagram describing the inclusion of two finite dimensional algebras $A_0 \subseteq A_1$, then, for the tower of the basic construction:

$$A_0 \overset{\mathcal{G}}{\subseteq} A_1 \overset{\mathcal{G}'}{\subseteq} A_2 \cdots A_{2n} \overset{\mathcal{G}}{\subseteq} A_{2n+1} \overset{\mathcal{G}'}{\subseteq} A_{2n+2} \cdots ,$$

the Bratteli diagrams for the inclusions are alternately $\mathcal{G}$ and $\mathcal{G}'$ as indicated. Define

$$\mathcal{G}^{(n)} = \begin{cases} \mathcal{G}^{(0)} & \text{if } n \text{ is even} \\ \mathcal{G}^{(1)} & \text{otherwise .} \end{cases}$$

Let $\Omega_k^{(n)}$ denote the space of all oriented paths of length k in $\mathcal{G}$ which start at $\mathcal{G}^{(n)}$.

Define

$$C_k^{(n)} = \{x \in Mat_{\Omega_k^{(n)}}(\mathbb{C}) : x(\alpha, \beta) = 0 \text{ unless } (s(\alpha), f(\alpha)) = (s(\beta), f(\beta))\}$$

and

$$\tilde{A}_k = \{x \in Mat_{\Omega_k^{(0)} \times \Omega_k^{(0)}}(\mathbb{C}) : x(\alpha, \beta) = 0 \text{ unless } f(\alpha) = f(\beta)\}.$$

Then, clearly, using the path algebra model based on $\mathcal{G}$:

(a) $A_n \simeq \tilde{A}_n$.

(b) There are isomorphisms $j_{[n,n+k]}$ which map $C_k^{(n)}$ onto $A'_n \cap A_{(n+k)}$. These maps are consistent with the inclusions of $C_k^{(n)}$ into $C_{k+1}^{(n)}$. In particular $A'_n \cap A_{(n+k)} \simeq A'_{n+2} \cap A_{(n+k+2)}$.

For an element a in $C_k^{(n)}$ we use $a_{[n,n+k]}$ to denote $j_{[n,n+k]}(a)$.

For a more complete description see [S1].

1.4 Principal Graphs

For a pair of II_1 factors $N \subseteq M$, Ocneanu has defined an invariant called the paragroup invariant which is given by a tuple of the form $(\mathcal{G}, \mathcal{H}, \tau, [W])$, where $\mathcal{G}$ and $\mathcal{H}$ are finite, connected, bipartite graphs, τ is a mapping defined on the vertices of these graphs, and $[W]$ is a 'connection' defined on a square of algebras defined by $\mathcal{G}, \mathcal{H}$ and τ (cf. [O1]). This and the next section give a description of these in some detail.

The first part of Ocneanu's invariant consists of the two principal graphs $\mathcal{G}$ and $\mathcal{H}$. Ocneanu himself has followed the bimodule approach in describing the principal graphs. We discuss here the method of obtaining the principal graphs from the 'derived tower' in keeping with the spirit of this thesis.

Consider the following tower of finite dimensional (when $[M : N] < \infty$) C^*-algebras

$$N' \cap N \subseteq N' \cap M \subseteq N' \cap M_1 \subseteq \cdots,$$

obtained from Jones' tower $\{M_n\}_{n \geq -1}$ of the basic construction. This is called the derived tower of N in M. The Bratteli diagram for the above tower is constructed in the following manner: first, the Bratteli diagram for each inclusion, $N' \cap M_k \subseteq N' \cap M_{k+1}$, is drawn. Then, taking care of the way the nodes are labelled in the successive algebras, these Bratteli diagrams are stacked up to obtain the Bratteli diagram of the derived tower.

Let e_k be the Jones projection in M_{k+1} which implements the basic construction of $M_{k-1} \subseteq M_k$. It can be shown [cf. GHJ] that $tr_{M_k}|_{N' \cap M_k}$ is the Markov trace for $N' \cap M_{k-1} \subseteq N' \cap M_k$ and e_k (which is in $M'_{k-1} \cap M_{k+1}$ $(\subseteq N' \cap M_{k+1})$) implements the conditional expectation of $N' \cap M_k$ onto $N' \cap M_{k-1}$ with respect to tr_{M_k}, and hence $N' \cap M_{k+1}$ contains the basic construction of $N' \cap M_{k-1}$ in $N' \cap M_k$. From this it follows that if the graph for the inclusion of $N' \cap M_{k-1}$ in $N' \cap M_k$ is G, then the graph for the inclusion of $N' \cap M_k$ in $N' \cap M_{k+1}$ contains a 'reflection' of G.

The principal graph $\mathcal{G}$ is obtained from the Bratteli diagram for the derived tower by deleting at every stage, the reflection of the previous stage, including only the 'new' edges and nodes. It can be shown [cf. GHJ] that no information about the inclusion is lost in this process.

In some cases, for a finite k, the Bratteli diagram of $N' \cap M_k \subseteq N' \cap M_{k+1}$ consists only of the reflection of $N' \cap M_{k-1} \subseteq N' \cap M_k$, i.e., there are no 'new' edges and nodes. Since it is true that 'new' edges arise only from the new nodes of the previous stage, the principal graph stops growing. In such cases the graph is said to be of finite depth.

The dual graph $\mathcal{H}$ is obtained, in a similar way, from the other derived tower of relative commutants:

$$M' \cap M \subseteq M' \cap M_1 \subseteq M' \cap M_2 \subseteq \cdots .$$

Clearly $\mathcal{G}$ and $\mathcal{H}$ are connected bipartite graphs. The set $V(\mathcal{G})$ of vertices of $\mathcal{G}$, can be written thus: $V(\mathcal{G}) = \mathcal{G}^{(0)} \amalg \mathcal{G}^{(1)}$, where $\amalg$ denotes disjoint union. Following convention, $\mathcal{G}^{(0)}$ includes the minimal central projection p_0 corresponding to $N' \cap N$, and is called the set of even vertices of $\mathcal{G}$ and p_0 is denoted by $*_{\mathcal{G}}$. The set $\mathcal{G}^{(1)}$ is called the set of odd vertices. These sets can be described thus:

$$\mathcal{G}^{(0)} = \{p \in \coprod_{k=-1}^{\infty} \pi(N' \cap M_{2k+1}) : p \neq J_{N' \cap M_{2k}} q J_{N' \cap M_{2k}} \text{ for any } q \in \pi(N' \cap M_{2k-1}), k \geq 1\}$$

and

$$\mathcal{G}^{(1)} = \{p \in \coprod_{k=0}^{\infty} \pi(N' \cap M_{2k}) : p \neq J_{N' \cap M_{2k-1}} q J_{N' \cap M_{2k-1}} \text{ for any } q \in \pi(N' \cap M_{2k-2}), k \geq 1\}$$

The vertices of $\mathcal{H}$ are also described similarly:

$$V(\mathcal{H}) = \mathcal{H}^{(0)} \amalg \mathcal{H}^{(1)}$$

$$\mathcal{H}^{(0)} = \{p \in \coprod_{k=0}^{\infty} \pi(M' \cap M_{2k}) : p \neq J_{M' \cap M_{2k-1}} q J_{M' \cap M_{2k-1}} \text{ for any } q \in \pi(M' \cap M_{2k-2}), k \geq 1\}$$

and

$$\mathcal{H}^{(1)} = \{p \in \coprod_{k=0}^{\infty} \pi(M' \cap M_{2k+1}) : p \neq J_{M' \cap M_{2k}} q J_{M' \cap M_{2k}} \text{ for any } q \in \pi(N' \cap M_{2k-1}), k \geq 1\}$$

We use the canonical anti-unitary J to describe the contragredient map. Consider the following instance of the basic construction

$$N \subseteq M_k \subseteq M_{2k+1}.$$

Then $J_{M_k} N' J_{M_k} = M_{2k+1}$ and $J_{M_k} M_{2k+1} J_{M_k} = N'$. Hence $J_{M_k}(N' \cap M_{2k+1}) J_{M_k} = N' \cap M_{2k+1}$. Thus the contragredient map τ maps $\mathcal{G}_{[k]}^{(0)}$ into itself, i.e., it acts as a permutation on $\mathcal{G}_{[k]}^{(0)}$, where $\mathcal{G}_{[k]}$ denotes the vertices in $\mathcal{G}$ at distance k from $*$.

Similarly for the even vertices of $\mathcal{H}$, consider the following instance of the basic construction :

$$M \subseteq M_k \subseteq M_{2k}.$$

Since $J_{M_k}(M' \cap M_{2k}) J_{M_k} = M' \cap M_{2k}$, τ maps $\mathcal{H}_{[k]}^{(0)}$ into itself.

For the odd vertices consider

$$N \subseteq M \subseteq M_k \subseteq M_{2k} \subseteq M_{2k+1}.$$

Now, $J_{M_k}(N' \cap M_{2k}) J_{M_k} = M' \cap M_{2k+1}$ and hence $\mathcal{G}_{[k]}^{(1)}$ gets mapped to $\mathcal{H}_{[k]}^{(1)}$.

1.5 Commuting Squares

Let A, B, C and D be finite von-Neumann algebras with a finite faithful normal trace tr on D. Then, the following:

$$\begin{array}{ccc} C & \subseteq & D \\ \cup\!| & & \cup\!| \\ A & \subseteq & B \end{array} \qquad (\dagger)$$

is said to be a commuting square if and only if $E_C|_B = E_A|_B$, the conditional expectations being defined on D with respect to the trace tr.

We describe here the 'biunitarity condition' due to Ocneanu, which is equivalent to the commuting square condition for ($\dagger$). For a proof of the equivalence see [SCH]. Let $\mathcal{G}, \mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ be the Bratteli diagrams for the inclusions in ($\dagger$) as indicated below:

$$\begin{array}{ccc} C & \overset{\mathcal{L}}{\subseteq} & D \\ {}_{\mathcal{K}}\cup\!| & & \cup\!|_{\mathcal{H}} \\ A & \underset{\mathcal{G}}{\subseteq} & B \end{array}$$

The graphs are bipartite and are assumed to be connected. Again, as described in §1.2, the inclusions can also be described by inclusion matrices of the appropriate sizes. Let G, H, K and L be the matrices describing $\mathcal{G}, \mathcal{H}, \mathcal{K}$, and $\mathcal{L}$ respectively. The commuting square ($\dagger$) can also be written as below:

$$\begin{array}{ccc} C & \overset{L}{\subseteq} & D \\ {}_{K}\cup\!| & & \cup\!|_{H} \\ A & \underset{G}{\subseteq} & B \end{array} \qquad (\ddagger)$$

The consistency of the inclusions implies that we necessarily have $GH = KL$. Then we define a set Υ of unitary matrices of the following form

$$U = \oplus u^{ij}, (i,j) \in \pi(A) \times \pi(D)$$

where each u^{ij} is a unitary matrix of size $(GH)_{ij}$ whose rows are indexed by paths

$$\{\alpha \in \Omega_{[A,B,D]}, \alpha = a \circ b : s(a) = i, f(b) = j, f(a) = s(b) \in \pi(B)\}$$

and columns by

$$\{\alpha \in \Omega_{[A,C,D]}, \alpha = c \circ d : s(c) = i, f(d) = j, f(c) = s(d) \in \pi(C)\}$$

For U in Υ define $\tilde{V}(U)$ to be a matrix of the following form :

$$\tilde{V}(U) = \oplus \tilde{v}^{kl}, (k,l) \in \pi(B) \times \pi(C)$$

where each $\tilde{v}^{kl}$ is of size $(G'K)_{kl} \times (HL')_{kl}$ and is given by the following prescription:

$$\tilde{v}^{kl}(a \circ b, c \circ d) = \sqrt{\frac{\alpha(i)\delta(j)}{\beta(k)\gamma(l)}}\ \overline{u^{ij}(\tilde{a} \circ c, b \circ \tilde{d})}$$

where $\tilde{x}$ denotes the 'reflection' of an edge x of a graph, $s(a) = s(c) = k \in \pi(B), f(b) = f(d) = l \in \pi(C)$, $f(a) = s(b) = i \in \pi(A), f(c) = s(d) = j \in \pi(D)$, and α, β, γ and δ are the vectors defining the restriction of the trace tr to the algebras A, B, C and D respectively.

Ocneanu's biunitarity condition: The diagram (‡) above is a commuting square if and only if:
(a) $GH = KL$;
(b) $G'K \leq HL'$ (meaning entry-wise inequality); and
(c) there exists a U in Υ such that each summand $\tilde{v}^{kl}$ of $\tilde{V}(U)$ is an isometric matrix.

Again, the same diagram is said to be a symmetric commuting square if $G'K = HL'$, and so each $\tilde{v}^{kl}$ in (c) (and hence also $\tilde{V}(U)$) is a unitary matrix.

Further, for a symmetric commuting square, it can be shown (cf. [SCH]) that $||G|| = ||L||, ||H|| = ||K||$; δ is the Perron-Frobenius eigenvector for $H'H$ as well as $L'L$ and gives the Markov trace for the inclusions $B \subseteq D$ as well as $C \subseteq D$; β and γ are the Perron-Frobenius eigenvectors for $G'G$ and $K'K$ respectively and give the Markov trace for $A \subseteq B$ and $A \subseteq C$ respectively; and α is the Perron-Frobenius eigenvector for GG'.

In the path algebra representation, it can be seen that U maps the Hilbert space $\mathcal{H}_{[A,B,D]}$ onto $\mathcal{H}_{[A,C,D]}$ and defines a spatial isomorphism between the representations of D on the two algebras $\mathcal{L}(\mathcal{H}_{[A,B,D]})$ and $\mathcal{L}(\mathcal{H}_{[A,C,D]})$.

1.6 The paragroup invariant for $N \subseteq M$

Associated with a pair of II_1 factors, $N \subseteq M$, of finite depth, is the following commuting square:

$$\begin{array}{ccc} N' \cap M_{2k} & \overset{G'}{\subseteq} & N' \cap M_{2k+1} \\ {}_{H}\cup| & & \cup|_{G'} \\ M' \cap M_{2k} & \underset{H}{\subseteq} & M' \cap M_{2k+1} \end{array}$$

Note that for $2k$ greater than the depth, the Bratteli diagram for the inclusion of $N' \cap M_{2k-1} \subseteq N' \cap M_{2k}$ is (the principal graph) $\mathcal{G}$ and that for $M' \cap M_{2k} \subseteq M' \cap M_{2k+1}$ is (the dual graph) $\mathcal{H}$. Also,

$$N' \cap M_{2k-1} \subseteq N' \cap M_{2k} \subseteq N' \cap M_{2k+1}$$

and

$$M' \cap M_{2k-1} \subseteq M' \cap M_{2k} \subseteq M' \cap M_{2k+1} (= J_{M_k}(N' \cap M_{2k})J_{M_k})$$

are instances of the basic construction, and hence,by the definition of τ - see the last three paragraphs of §1.4 - the inclusions are as indicated.

Clearly, the above is a commuting square. (For $x \in M' \cap M_{2k+1}$, $E_{N' \cap M_{2k}}(x) = E_{M_{2k}}(x)$, which clearly commutes with M.) Hence it has a 'biunitary' matrix defined for it. This matrix defines the 'connection' W, the last ingredient of Ocneanu's paragroup invariant for $N \subseteq M$.

Now we describe a method for obtaining a pair of subfactors starting with an admissible tuple $(\mathcal{G}, \mathcal{H}, \tau, W)$. For convenience of exposition, however, we only consider the case of trivial contragredient maps: i.e.,weassume that $\mathcal{G}^{(1)} = \mathcal{H}^{(1)}$ and that at the even levels $\tau = id$.

Let A_0, A_1, B_0 and B_1 be finite dimensional algebras with inclusions given by:

$$\begin{array}{ccc} B_0 & \overset{\mathcal{H}'}{\subseteq} & B_1 \\ {\scriptstyle \mathcal{G}}\cup\vert & & \cup\vert{\scriptstyle \mathcal{H}'} \\ A_0 & \underset{\mathcal{G}}{\subseteq} & A_1 \end{array} \qquad (\star)$$

so that W defines a biunitary matrix for the above square.

Now let B_2 be the algebra obtained by the basic construction of B_0 in B_1. Suppose e_1 is the Jones projection bringing this about. Let A_2 be the algebra generated by A_1 and e_1. Then by the biunitarity of W and the involutive nature of the operation $U \mapsto \tilde{V}(U)$ defined in §1.5, the following is also a commuting square :

$$\begin{array}{ccc} B_1 & \subseteq & B_2 \\ \cup\vert & & \cup\vert \\ A_1 & \subseteq & A_2 \end{array} .$$

By iterating the above process we obtain the following towers of algebras:

$$\begin{array}{ccccccccccc}
B_0 & \overset{\mathcal{H}'}{\subseteq} & B_1 & \overset{\mathcal{H}}{\subseteq} & B_2 & \ldots & \overset{\mathcal{H}}{\subseteq} & B_{2n} & \overset{\mathcal{H}'}{\subseteq} & \ldots \\
{}_{\mathcal{G}}\cup\!| & & {}_{\mathcal{H}'}\cup\!| & & {}_{\mathcal{G}}\cup\!| & & & {}_{\mathcal{G}}\cup\!| & & \\
A_0 & \underset{\mathcal{G}}{\subseteq} & A_1 & \underset{\mathcal{G}'}{\subseteq} & A_2 & \ldots & \underset{\mathcal{G}'}{\subseteq} & A_{2n} & \underset{\mathcal{G}}{\subseteq} & \ldots
\end{array}$$

All the squares in the above diagram are commuting squares. Let $R_0 = (\bigcup_n A_n)''$ and $R_1 = (\bigcup_n B_n)''$.

If we follow the above iterative procedure for the canonical commuting square associated with a pair of II_1 factors $N \subseteq M$ viz.,

$$\begin{array}{ccc}
M' \cap M_{2k-1} & \overset{H'}{\subseteq} & M' \cap M_{2k} \\
{}_{{}^tG^r}\cup\!| & & \cup\!|_{{}^tH'^r} \\
M_1' \cap M_{2k-1} & \underset{G}{\subseteq} & M' \cap M_{2k}
\end{array}$$

if the given inclusion has finite depth, and if k is sufficiently big, then it is a fact (cf. [P]) that the pair $R_0 \subseteq R_1$ obtained is isomorphic to $N \subseteq M$. On the other hand, starting with a suitable tuple, $(\mathcal{G}, \mathcal{H}, r, [W])$ or equivalently a commuting square, following the iterative construction above, we obtain a pair of subfactors $R_0 \subseteq R_1$. Now, if we could compute the higher relative commutants of R_0 in R_1, i.e., $\{R_0' \cap R_k\}_k$, we can construct the principal graphs $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$ for the inclusion $R_0 \subseteq R_1$. When $(\tilde{\mathcal{G}}, \tilde{\mathcal{H}}) \simeq (\mathcal{G}, \mathcal{H})$ the connection W is said to be 'flat'. For instance this always occurs when $[M : N] < 4$. In general, the graphs are not isomorphic, but $\tilde{\mathcal{G}}$ is the 'flat' part of $\mathcal{G}$. We give below a result due to Ocneanu, which gives, in principle, a method of computing the higher relative commutants of R_0 in R_1. For the sake of completeness, we also include a proof of this fact.

THEOREM 1.6.1 **(Ocneanu Compactness)**

Let $R_0 \subseteq R_1$ be the pair of subfactors obtained by the iterative procedure applied to the symmetric commuting square $(\star)$ as described above.

Then $R_0' \cap R_1 = A_1' \cap B_0$.

Proof: To show that $A_1' \cap B_0 \subseteq R_0' \cap R_1$, we first show that $A_1' \cap B_0 = A_m' \cap B_n$ for any (m,n) such that $m > n$.

Now, $A_m =< A_{n+1}, e_{n+1}, e_{n+2}, \cdots e_{m-1} >$.

So, $A_m' = A_{n+1}' \cap \{e_{n+1}\}' \cap \{e_{n+2}\}' \cdots \{e_{m-1}\}'$.

Also, $B_n \subseteq \{e_{n+1}\}'$,

$B_n \subseteq B_{n+1} \subseteq \{e_{n+2}\}'$ and so on.

Therefore,

$$
\begin{aligned}
B_n \cap A_m' &= B_n \cap A_{n+1}' \\
&= B_n \cap A_n' \cap \{e_n\}' \\
&= B_{n-1} \cap A_n' \text{ (since } B_n \cap \{e_n\}' = B_{n-1} \text{)} \\
&= B_{n-2} \cap A_{n-1}' \text{ (by the same argument)} \\
&\vdots \\
&= B_0 \cap A_1'
\end{aligned}
$$

Since the above equality holds for all (m,n) such that $m > n$ it follows that $A_1' \cap B_0 \subseteq R_0' \cap R_1$.

The following lemma is needed to prove the reverse inclusion.

LEMMA 1.6.2 *Let H be a finite dimensional Hilbert space. Let E, F be subspaces of H. Then there exists a $k > 0$ such that*

$$d(x, E \cap F) \leq k(d(x, E) + d(x, F))$$

for all x in H.

Proof: Let $p_1(x) = d(x, E \cap F)$ and $p_2(x) = d(x, E) + d(x, F)$. Then p_1 and p_2 define seminorms on H, both having kernel $E \cap F$. They define two norms $\tilde{p_1}$ and $\tilde{p_2}$ on $H/(E \cap F)$ which is also finite dimensional.Thus $\tilde{p_1}$ and $\tilde{p_2}$ are equivalent and so there exists $k > 0$ such that $\tilde{p_1} \leq k\tilde{p_2}$ for all x in $H/(E \cap F)$. Using the fact that p_1 and p_2 are seminorms on H, it follows that $p_1(x) \leq k\ p_2(x)$. □

Continuing with the proof of the theorem, consider the following finite dimensional algebras:
$H_{2n} = A'_{2n} \cap B_{2n+1}$,
$E_{2n} = A'_{2n} \cap B_{2n}$,
$F_{2n+1} = A'_{2n+1} \cap B_{2n+1}$.

Note that the two path spaces $\Omega_{[A_0,A_1,B_1]}$ and $\Omega_{[A_{2n},A_{2n+1},B_{2n+1}]}$ are isomorphic since the inclusion matrices are the same. So H_0 is spatially isomorphic to H_{2n}.

Let this isomorphism be denoted by L_n. Clearly, from the nature of this isomorphism, $L_n(F_1) = F_{2n+1}$ and $L_n(E_0) = E_{2n}$.

Assertion: There exists a $c > 0$ such that

$$c^{-1}||x|| \leq ||L_n(x)|| \leq c||x||$$

for all x in H_0, for all n, where, of course, $||x||^2 = tr(x^*x)$.

We will first prove the theorem assuming the truth of the assertion.

Let G_0 be any closed subspace of H_0 and let G_n be $L_n(G_0)$. Using the assertion above, we have :

$$c^{-1}d(x_0, G_0) \leq d(x_n, G_n) \leq c\ d(x_0, G_0).$$

for any x_0 in H_0, where $x_n = L_n(x_0)$. Now, let $G_0 = E_0 \cap F_1$ so that $G_n = E_{2n} \cap F_{2n+1}$. Then we have

$$\begin{aligned} d(x_n, E_{2n} \cap F_{2n+1}) &\leq c\ \ d(x_0, E_0 \cap F_1) \\ &\leq c\ \ k_0(d(x_0, E_0) + d(x_0, F_1)) \text{ by Lemma 1.6.2} \\ &\leq c^2\ k_0(d(x_n, E_{2n}) + d(x_n, F_{2n+1})) \end{aligned}$$

for all n. So there exists a $K > 0$ such that

$$d(x_n, E_{2n} \cap F_{2n+1}) \leq K(d(x_n, E_{2n}) + d(x_n, F_{2n+1}))$$

for all x_n in H_n, for all n.

For x in $A' \cap B$, define $x_n = E_{B_n}(x)$. Then $x_n \to x$ as $n \to \infty$ in L^2 norm.

Since $x \in A'$, it follows that $x \in A'_n$ for all n and therefore $x\ a_n = a_n x$ for all a_n in A_n.

Taking conditional expectation onto B_n we have $E_{B_n}(x)a_n = a_n E_{B_n}(x)$, i.e., $x_n \in A'_n$ for all n. So, $d(x_n, B_n \cap A'_n) = 0$ for all n.

Now, $E_{2n} \cap F_{2n+1} = A'_{2n} \cap B_{2n} \cap A'_{2n+1} \cap B_{2n+1} = B_{2n} \cap A'_{2n+1}$.

For x_{2n} , $d(x_{2n}, E_{2n}) = 0$ and $d(x_{2n}, F_{2n+1}) \leq ||x_{2n} - x_{2n+1}||_2$.

So, $d(x_{2n}, E_{2n} \cap F_{2n+1}) \leq K||x_{2n} - x_{2n+1}||_2 \to 0$ as $n \to \infty$.

But $B_{2n} \cap A'_{2n+1} = B_0 \cap A'_1$ for all n. Therefore $x \in A'_1 \cap B_0$.

Proof of the assertion: First observe that, using the path algebra model $H_0 = A_0' \cap B_1$ is a subalgebra of $\mathcal{L}(\mathcal{H}_{[A_0,A_1,B_1]})$, which is a direct sum of matrix algebras

$$H_0 = \bigoplus_{(i,j)\in\pi(A_0)\times\pi(B_1)} M_{n_{ij}}(\mathbb{C})$$

where n_{ij} is the number of paths from $i \in \pi(A_0)$ to $j \in \pi(B_1)$ along the Bratteli diagram for the inclusions $A_0 \subseteq A_1 \subseteq B_1$.

Thus suppose the minimal central projections of H_0 are $\{p_{ij}^{(0)} : i \in \pi(A_0), j \in \pi(B_1)\}$. Clearly, then, $L_n(p_{ij})^{(0)}) = p_{ij}^{(n)}$ where $\{p_{ij}^{(n)} : i \in \pi(A_{2n}) \simeq \pi(A_0), j \in \pi(B_{2n+1}) \simeq \pi(B_1)\}$ are the minimal central projections of H_{2n}.

Note that it is enough to prove the assertion for p_{ij}^0 in $\pi(H_0)$ i.e., the set of minimal central projections in H_0. This is because $Tr_n : x \mapsto tr(L_n(x_0))$ is a trace on $H_0 p_{ij}$, which is a factor, so that there exists a positive constant c_n such that, for all x in $H_0 p_{ij}$, we have

$$\frac{Tr_n(x)}{tr(x)} = c_n = \frac{tr(L_n(p_{ij}))}{tr(p_{ij})}.$$

We wish to show that there exists a positive constant c such that

$$c^{-1} \leq \frac{tr(p_{ij}^{(n)})}{tr(p_{ij}^{(0)})} \leq c, \forall\, i, j.$$

Note that it is enough to show that $\lim_{n\to\infty} tr(p_{ij}^{(n)})$ exists and is greater than 0 for each i, j.

$$\begin{aligned} tr(p_{ij}^{(n)}) &= \sum_k (GG')^n_{ki}(GH)_{ij} t_j^{B_{2n+1}} \\ &= \sum_k (GG')^n_{ki}(GH)_{ij}(t_j^{B_1}/\lambda^n) \quad (\text{where } \lambda = \|H'H\| = \|GG'\|) \end{aligned}$$

$$= \sum_k ((GG')^n_{ki}/\lambda^n)(GH)_{ij} t_j^{B_1}$$
$$= \sum_k ((GG')^n_{ki}/\lambda^n) tr(p_{ij}^{(0)}).$$

So,

$$\frac{tr(p_{ij}^{(n)})}{tr(p_{ij}^{(0)})} = \sum_k \frac{(GG')^n_{ki}}{\lambda^n}.$$

Let $\{v_1, v_2, \cdots, v_m\}$ be an orthonormal basis for $\mathbb{C}^m$, and let $\lambda = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ be such that $GG'v_j = \lambda_j v_j$ for all j. It follows that if $w \in \mathbb{C}^m$ and n are arbitrary, then

$$(GG')^n w = \sum_j \lambda_j^n < w, v_j > v_j$$

and hence

$$\lim_{n \to \infty} \frac{(GG')^n w}{\lambda^n} = < w, v_1 > v_1.$$

Put $w = (1, 1, \cdots, 1)'$ and deduce from the the last equation that

$$\begin{aligned} \lim_{n \to \infty} \frac{trp_{ij}^{(n)}}{trp_{ij}^{(0)}} &= \lim_{n \to \infty} \sum_k \frac{(GG')^n_{ik}}{\lambda^n} \\ &= \lim_{n \to \infty} \frac{((GG')^n w)_i}{\lambda^n} \\ &= < w, v_1 > (v_1)_i \\ &= (v_1)_i^2 \end{aligned}$$

which is positive since v_1 is the Perron-Frobenius eigenvector of the irreducible matrix GG'. □

Chapter 2

Weak duality

2.1 Introduction

This chapter is devoted to a study of the properties of a pair $(\mathcal{G}, \mathcal{H})$ of pointed finite bipartite graphs which arise as a part of Ocneanu's paragroup invariant of a finite index subfactor. We single out a property possessed by a pair of principal graphs of a subfactor for which the contragredient maps are trivial, which we term 'weak duality'. The main result here is that if a pair of graphs $\mathcal{G}$ and $\mathcal{H}$ are weakly dual, then $\mathcal{G}$ is necessarily isomorphic to $\mathcal{H}$ if $\mathcal{G}$ satisfies some conditions - at most triple points, no double bonds, and the absence of two specific kinds of subgraphs. When suitably combined with Ocneanu's triple point obstruction, it leads fairly easily to a proof of the result that a tree, with trivial contragredient map, can occur as a principal graph, only if it contains a copy of $E_6^{(1)}$.

2.2 Weak duality.

In this chapter, we will be dealing with *pointed bipartite graphs.* By a graph, we mean what is sometimes called an undirected multigraph - by which, of course, is meant a pair $(V(\mathcal{G}), E(\mathcal{G}))$ where, as usual, the symbol $V(\mathcal{G})$ denotes the set of 'vertices' of the graph, and the symbol $E(\mathcal{G})$ denotes the set of 'edges' of the graph (with the understanding that there may be several edges joining the same pair of vertices). The graph $\mathcal{G}$ is said to be bipartite if the vertex set $V(\mathcal{G})$ is partitioned into two sets - which we shall, for convenience, call the sets of even and odd vertices respectively - in such a way that every edge in $\mathcal{G}$ joins an odd vertex to an even vertex. By a pointed bipartite graph, we will mean a bipartite graph $\mathcal{G}$, together with a distinguished even vertex - usually denoted by $*_{\mathcal{G}}$ - with the property that the Perron-Frobenius eigenvector of the adjacency matrix of $\mathcal{G}$ assumes the smallest value at $*_{\mathcal{G}}$. (To be sure, we can talk of 'the' Perron-Frobenius eigenvector only if the graph is connected - but we shall only be dealing with such graphs.)

If two pointed bipartite graphs $\mathcal{G}, \mathcal{H}$ arise as the two principal graphs corresponding to a finite-index subfactor, then the sets $\mathcal{G}^0, \mathcal{H}^0$ of even vertices of the two graphs are naturally equipped with the contragredient map. We shall be concerned with subfactors for which both these involutions are trivial; for brevity, we shall simply say that the subfactor has trivial contragredient maps when this happens. For such a subfactor - i.e., one with trivial contragredient maps -it follows from the discussion in §1.4 that the graphs $\mathcal{G}$ and $\mathcal{H}$ are 'weakly dual' in the sense of the next definition.

DEFINITION 2.2.1 Two pointed finite connected bipartite graphs $(\mathcal{G}, *_{\mathcal{G}})$ and $(\mathcal{H}, *_{\mathcal{H}})$ are said to be 'weakly dual' if the following conditions are satisfied:

(1) $\mathcal{G}^1 = \mathcal{H}^1$.

(2) $G^t(*_\mathcal{G}) = H^t(*_\mathcal{H})$ (i.e. the neighbours of $*$ in $\mathcal{G}$ and $\mathcal{H}$ are the same).

(3) $G^tG(\xi^1, \eta^1) = H^tH(\xi^1, \eta^1)$ for all $\xi^1, \eta^1 \in \mathcal{G}^1$, (i.e. the number of paths, of length 2, between ξ^1 and η^1 is the same in $\mathcal{G}$ and $\mathcal{H}$) .

(We would like to acknowledge our gratitude to Uffe Haagerup for pointing out that if the graph $\mathcal{G}$ 'looks like an A_n up to a certain distance from $*_\mathcal{G}$', so also must $\mathcal{H}$, a remark which led us to think along the above lines - *cf* Remark 2.4.2)

Remark Note that when $\mathcal{G}$ and $\mathcal{H}$ are a pair of principal graphs, the identification of the odd vertices in (1) is via the contragredient map τ. When the contragredient map on the even vertices is nontrivial the graphs do not satisfy condition (3), but they satisfy $G^t\tau_{\mathcal{G}^0}G = H^t\tau_{\mathcal{H}^0}H$.

2.3 Some illustrative examples

We turn now to pairs of graphs which are not isomorphic but which are weakly dual. The simplest known example comes from the principal graphs for the inclusion $N \subset M$, when M is the crossed-product of N with a non-abelian group of outer automorphisms of N. In this example, as is well-known, the graph $\mathcal{H}$ has multiple bonds while $\mathcal{G}$ does not. The following is an example of graphs without multiple bonds.

EXAMPLE 2.3.1 We are grateful to Bhaskar Bagchi for this combinatorial example which, besides being pretty, illustrates the phenomenon of two pointed, bipartite, connected graphs $\mathcal{G}$ and $\mathcal{H}$ which are isomorphic but not weakly dual:

Both the graphs $\mathcal{G}, \mathcal{H}$ have $15 = \binom{6}{2}$ odd vertices indexed by the 15 edges of the complete graph K_6. Each graph has 10 even vertices indexed by certain subgraphs of K_6 (isomorphic to $C_3 \amalg C_3$ or C_6, where C_k denotes a k-cycle - thus C_6 is a hexagon, etc.) In both graphs , an odd vertex is adjacent to an even vertex precisely when the relevant edge belongs to the relevant subgraph.

The even vertices of $\mathcal{H}$ correspond to all the $10 = (\binom{6}{3})/2$ subgraphs of K_6 isomorphic to $C_3 \amalg C_3$.

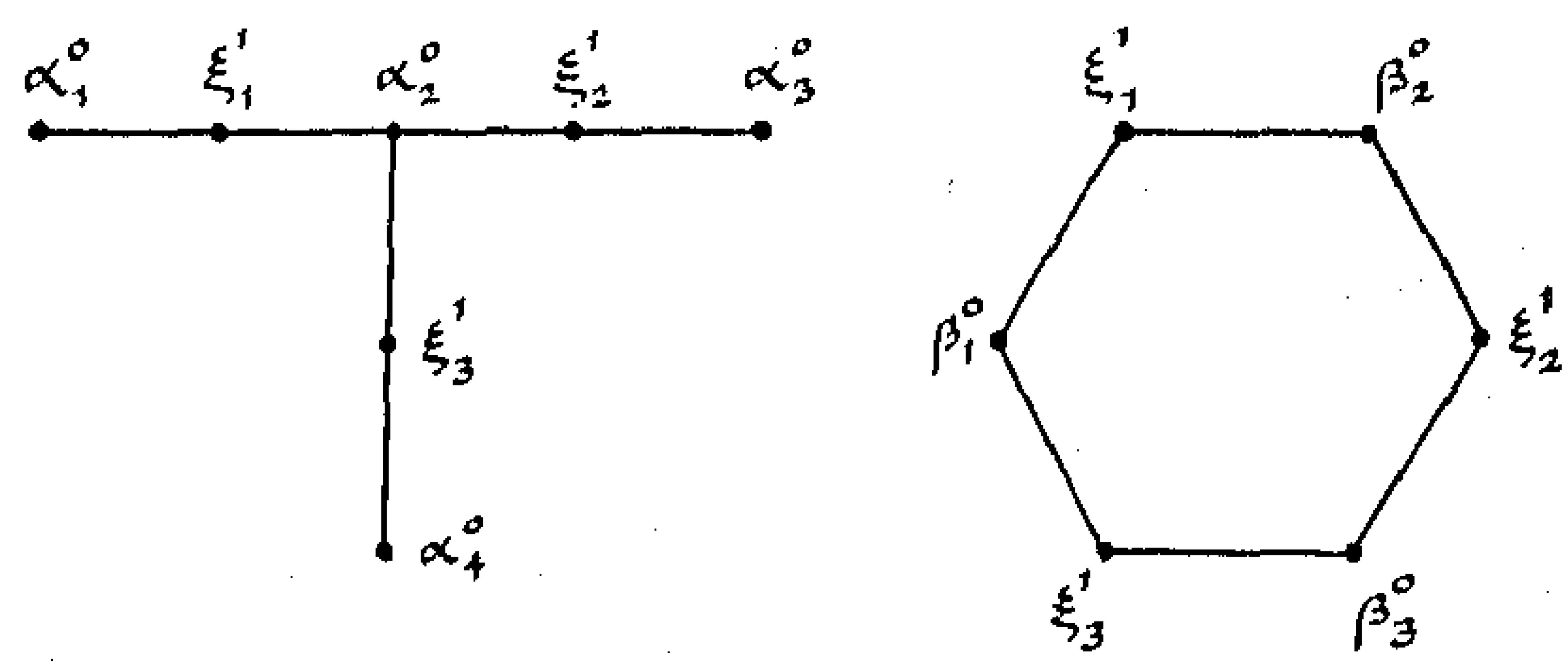
The graph $\mathcal{G}$ also has ten even vertices ; these correspond to six subgraphs isomorphic to $C_3 \amalg C_3$ and four subgraphs isomorphic to C_6. They are : $\{(124)\amalg(356), (125)\amalg(346), (136)\amalg(245), (145)\amalg(236), (134)\amalg(256), (146)\amalg (235)\}$ and $\{(123456), (126453), (156423), (153426)\}$ - where we have used the obvious notation $(v_1 \ldots v_k)$ to denote the k-cycle that successively passes through the vertices $v_1, \ldots, v_k$.

Both graphs share the following properties: (i) each odd vertex has degree 4 and each even vertex has degree 6 (and hence the value of the Perron-Frobenius eigenvector at a vertex depends only on the parity of the vertex); (ii) given any two distinct odd vertices, the number of paths of length two which join them is 1 or 2 according as the corresponding edges (in K_6) share a common vertex or not. These facts ensure that the graphs are weakly dual as asserted, provided the distinguished vertices of both $\mathcal{G}$ and $\mathcal{H}$ are taken to be the same graph isomorphic to $C_3 \amalg C_3$.

Another such, but smaller, example is given below.

EXAMPLE 2.3.2 We begin by discussing a pair of graphs which are 'almost' weakly dual, but just fail to be so; nevertheless they have near relatives which do furnish an example of a pair of graphs which are weakly dual but not isomorphic. The non-example is discussed here mainly because of the key role these two graphs play in Proposition 2.4.1.

(a) Consider the following pair of graphs, with even and odd vertices labelled as indicated.

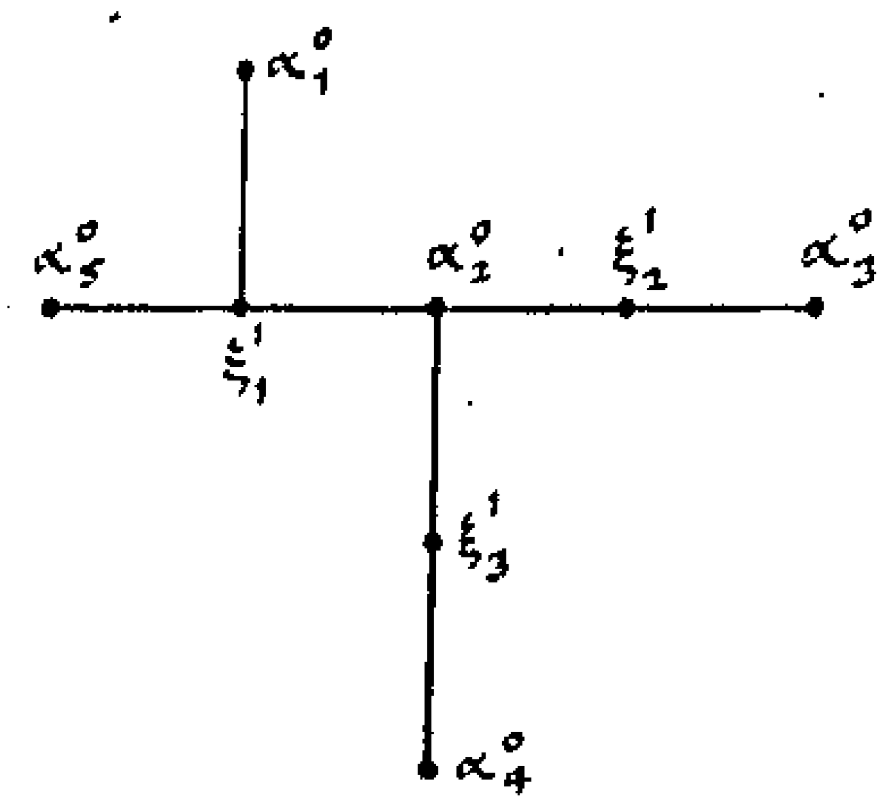


$\mathcal{G} = T$ $\qquad\qquad$ $\mathcal{H} = C_6$

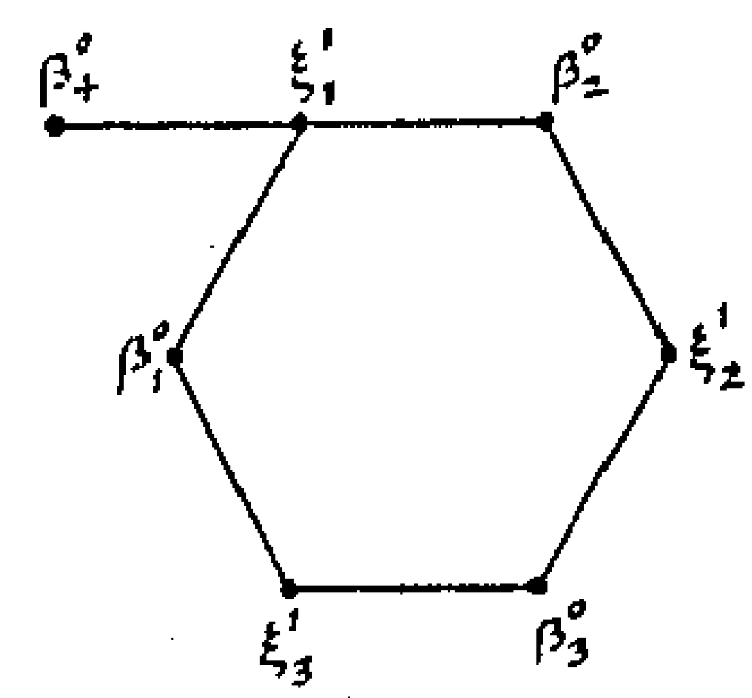
(Here and in the sequel, we write T to denote the 'T-graph' each of whose arms is two edges long. This graph is denoted by $E_6^{(1)}$ in [GHJ], but we use the notation T because it is more suggestive.)

It is easily verified that the condition $G^tG = H^tH$ is satisfied. Since every even vertex in $\mathcal{H}$ has degree 2 while none in $\mathcal{G}$ does, clearly these two graphs cannot be weakly dual (by condition (2)).

(b) Consider these graphs with labelling as indicated.



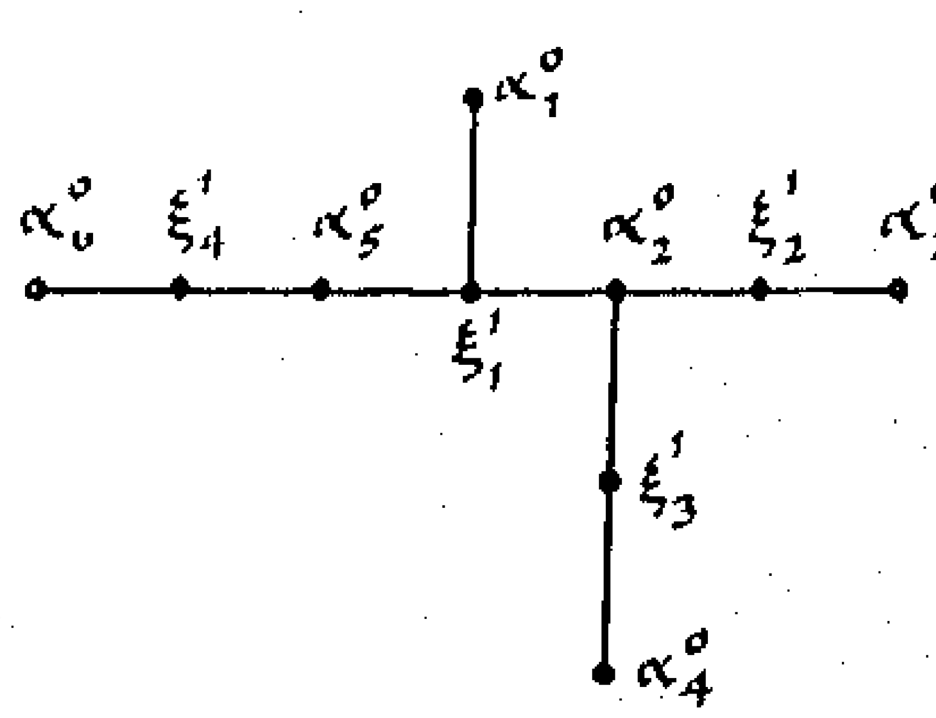
$\mathcal{G} = A_2 \# T$



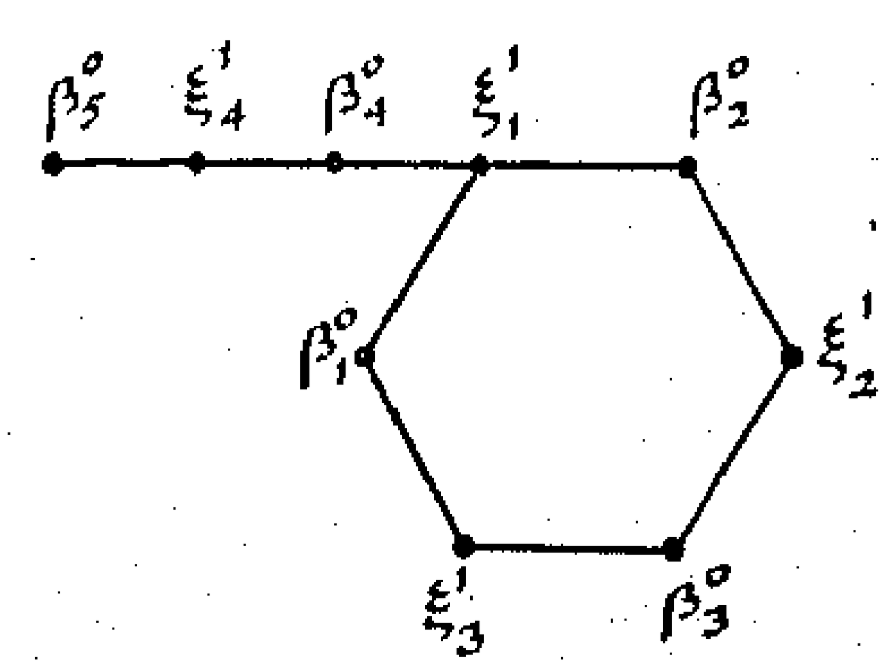
$\mathcal{H} = A_2 \# C_6$

(Here and elsewhere, we use the symbol # to denote 'connected sum', whereby we mean that a pair of vertices, one from each of the graphs in question, has been identified; to be sure, there are several ways of forming such a connected sum.) Again, the condition $G^tG = H^tH$ is satisfied. While the vertices α_5^0 and β_4^0 have the same degree, what fails now is that the minimum value of the Perron-Frobenius eigenvector of $A(\mathcal{G})$ occurs not at α_5^0 but at the vertices α_3^0 and α_4^0.

(c) Finally, the desired example comes from the following graphs, with labelling as indicated.



$\mathcal{G} = A_4 \# T$



$\mathcal{H} = A_4 \# C_6$

Here, it is the case that $*_{\mathcal{G}} = \alpha_6^0$ and $*_{\mathcal{H}} = \beta_5^0$.

(d) It goes without saying that by extending the A-part of the graphs more and more, the graphs $\mathcal{G}$ and $\mathcal{H}$ generate a whole sequence of pairs of non-isomorphic graphs - namely $A_{2n} \# T$ and $A_{2n} \# C_6$ - which are weakly dual.

2.4 Weakly self-dual graphs

We are now ready to prove the following proposition which gives some criteria on a bipartite graph $\mathcal{G}$ which ensure that the only graph, up to isomorphism, which is weakly dual to $\mathcal{G}$ is $\mathcal{G}$ itself. (Observe that in view of Example 2.3.2(c), (d), the conditions (3) and (4) in the proposition are almost necessary.)

PROPOSITION 2.4.1 *Suppose $\mathcal{G}$ is a finite connected bipartite graph satisfying the following conditions:*

(1) *no vertex of $\mathcal{G}$ has degree greater than 3;*

(2) *$\mathcal{G}$ does not have double bonds;*

(3) *$\mathcal{G}$ has no 6-cycles; and*

(4) *$\mathcal{G}$ has no subgraph isomorphic to T such that each of the vertices of degree 1 in T is an even vertex in $\mathcal{G}$ whose degree in $\mathcal{G}$ is still 1.*

Then the identification $\mathcal{G}^1 = \mathcal{H}^1$ extends to a graph isomorphism of $\mathcal{G}$ onto $\mathcal{H}$.

Before proceeding to the proof proper, we set up some notation. We shall use the notation $(\xi_0 - \xi_1 - \ldots - \xi_n) \in \mathcal{G}$ to signify that $\xi_0, \xi_1, \ldots \xi_n$ are vertices of the graph $\mathcal{G}$ such that ξ_{i-1} is adjacent to ξ_i in $\mathcal{G}$ for $1 \leq i \leq n$.

The set of neighbours of α in $\mathcal{G}$ will be denoted by $\mathcal{N}^{\mathcal{G}}_\alpha$. In the following, since we shall be dealing with a pair of weakly dual graphs $\mathcal{G}$ and $\mathcal{H}$, which have the same odd vertices, we shall, without fear of confusion, write simply $\mathcal{N}_\alpha$ when α is an even vertex of either $\mathcal{G}$ or $\mathcal{H}$.

We shall also employ the following notation: for vertices ξ, η in $\mathcal{G}$:

(i) the symbol $\mathcal{G}(\xi, \eta)$ will denote the set of common neighbours in $\mathcal{G}$ of ξ and η, i.e. $\mathcal{G}(\xi, \eta) = \mathcal{N}^{\mathcal{G}}_\xi \cap \mathcal{N}^{\mathcal{G}}_\eta$; (note that, in the absence of double bonds, $|\mathcal{G}(\xi^1, \eta^1)| = G^t G(\xi^1, \eta^1)$, whence $|\mathcal{G}(\xi^1, \xi^1)|$ is the degree of ξ^1 in $\mathcal{G}$;)

(ii) the symbol $I^{\mathcal{G}}(\xi)$ will denote the set of degree one neighbours of ξ in $\mathcal{G}$; i.e., $I^{\mathcal{G}}(\xi) = \{\beta \in \mathcal{N}^{\mathcal{G}}_\xi : \deg_{\mathcal{G}}(\beta) = 1\}$;

(iii) the symbol Λ will denote the set of triple points (i.e., vertices of degree 3) in $\mathcal{G}^0$; suppose $\Lambda = \{\lambda^0_1, \lambda^0_2, \ldots, \lambda^0_l\}, l \geq 0$.

Proof: It is not hard to see that the above conditions (1), (2) and (4) of the proposition imply the conditions $(1'), (2')$ and $(4')$ below. (To be precise, conditions (1) and (2) are together equivalent to conditions $(1')$ and $(2')$; while condition (4) is equivalent to $(4')$.) What we shall prove is that conditions $(1')$, $(2'), (3)$ and $(4')$ suffice to ensure the validity of the conclusion of the Proposition. (We have, however, chosen to state the proposition as we have, since we feel that this formulation is more 'visual' and easier to verify.)

(1′) $(G^tG)(\xi^1, \xi^1) \leq 3$, for all ξ^1 in $\mathcal{G}^1$;

(2′) for all $\beta^0 \in \mathcal{G}^0$ $\deg(\beta^0) \leq 3$; and

(4′) for all $\lambda^0 \in \Lambda$ there exists $\xi^1_{\lambda^0} \in \mathcal{N}_{\lambda^0}$ such that $\mathcal{I}^{\mathcal{G}}(\xi^1_{\lambda^0}) = \phi$.

In the proof we would have occasion to use the following condition (3′) which can be seen to be implied by (3).

(3′) If $\Omega = \{\omega^1_1, \omega^1_2, \omega^1_3\}$ is a subset of $\mathcal{G}^1$ such that between any two vertices in Ω there is a path of length 2 in $\mathcal{G}$, i.e., $(G^tG)(\omega^1_i, \omega^1_j) \neq 0$ for $i \neq j$, then, Ω is the set of neighbours of some triple point in $\mathcal{G}$, i.e., $\Omega = \mathcal{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

We break the proof, which is somewhat involved, into the following steps.

Step 1 $\mathcal{H}$ has no double bonds.
Reason : $H^tH(\xi^1, \xi^1) = G^tG(\xi^1, \xi^1) \leq 3$ for all $\xi^1 \in \mathcal{G}^1$.

Step 2 Each vertex in $\mathcal{H}$ has degree at most 3.
Reason : For the same reason as in Step 1, this is clear for the odd vertices. Suppose, now, that there is an even vertex δ^0 in $\mathcal{H}^0$ such that $\deg(\delta^0) \geq 4$.

Case(1) There is an even vertex δ^0 in $\mathcal{H}^0$ such that $\deg(\delta^0) > 4$. Then δ^0 has at least five neighbours, ξ^1_1, ξ^1_2, ξ^1_3, ξ^1_4, and ξ^1_5. The path $(\xi^1_i - \delta^0 - \xi^1_j)$ in $\mathcal{H}$ ensures that $G^tG(\xi^1_i, \xi^1_j) = H^tH(\xi^1_i, \xi^1_j) \neq 0$ for all i and j. By (3′), for any choice of distinct i, j and k, $\{\xi^1_i, \xi^1_j, \xi^1_k\} = \mathcal{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$, i.e. each ξ^1_i is adjacent to $(\binom{4}{2} =)$ 6 distinct triple points in $\mathcal{G}$, which contradicts (1).

Case(2) Suppose there is an even vertex δ^0 in $\mathcal{H}^0$ such that $\deg(\delta^0) = 4$. Then δ^0 has four neighbours ξ^1_1, ξ^1_2, ξ^1_3, and ξ^1_4. By the same reasoning as above, for distinct i, j and k, $\{\xi_i^1,\ \xi_j^1,\ \xi_k^1\} = \mathcal{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$. So there are 4 triple points $\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0$ in $\mathcal{G}$ such that $\mathcal{N}_{\lambda_i^0} = \{\xi_j^1 : j \neq i\}$. Let $\mathcal{G}'$ be the induced subgraph of $\mathcal{G}$ on the vertices $\{\xi_j^1 : 1 \leq j \leq 4\} \cup \{\lambda_i^0 : 1 \leq i \leq 4\}$. Since each λ_i^0 and ξ_j^0 has degree 3 in $\mathcal{G}'$, the conditions (1) and (2) imply that $\mathcal{G}'$ is a connected component of $\mathcal{G}$, and hence $\mathcal{G}' = \mathcal{G}$ by the assumed connectedness of $\mathcal{G}$.

Since $\mathcal{G}$ and $\mathcal{H}$ are weakly dual, we have the following:

(i) $\mathcal{H}^1 = \{\xi_1^1,\ \xi_2^1,\ \xi_3^1,\ \xi_4^1\}$.

(ii) $H^tH(\xi_i^1,\ \xi_j^1\ ) = 2$ for $1 \leq i \neq j \leq 4$.

(iii) $H^tH(\xi_i^1, \xi_i^1) = 3$ for $1 \leq i \leq 4$.

We proceed to deduce that there must exist another even vertex $\delta_1^0 \neq \delta^0$ of degree 4 in $\mathcal{H}$ such that $\mathcal{N}_{\delta_1^0} = \mathcal{N}_{\delta^0} = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

By (ii) there are unique even vertices κ_{ij}^0, distinct from δ^0, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathcal{H}$. Then for any ξ_i^1 and $j \neq i$, we have $(\xi_i^1 - \delta^0)$, and $(\xi_i^1 - \kappa_{ij}^0)$, are in $\mathcal{H}$. But $\deg(\xi_i^1) \leq 3$. Therefore for each i, $\kappa_{ij}^0 = \kappa_{ik}^0$ for some $j \neq k$. Now $(\xi_j^1 - \kappa_{ij}^0 = \kappa_{ik}^0 - \xi_k^1)$ is in $\mathcal{H}$. But κ_{jk}^0 is the unique vertex other than δ^0 such that $(\xi_j^1 - \kappa_{jk}^0 - \xi_k^1)$ is in $\mathcal{H}$. So $\kappa_{ij}^0 = \kappa_{ik}^0 = \kappa_{jk}^0$, which is then a vertex of degree at least three. Hence each ξ_i^1 is connected to a $\kappa_i^0 \neq \delta^0$ such that $\deg(\kappa_i^0) \geq 3$. We now show that all the κ_i^0 are the same.

Now, for $1 \leq i, j \leq 4$, we see that,

$$\begin{aligned} |\mathcal{N}_{\kappa_i^0} \cap \mathcal{N}_{\kappa_j^0}| &= |\mathcal{N}_{\kappa_i^0}| + |\mathcal{N}_{\kappa_j^0}| - |\mathcal{N}_{\kappa_i^0} \cup \mathcal{N}_{\kappa_j^0}| \\ &\geq 3 + 3 - |\mathcal{H}^1| = 2. \end{aligned}$$

Let $1 \leq i \neq j \leq 4$. Then there exist $1 \leq k \neq l \leq 4$ such that $\xi^1_k, \xi^1_l \in \mathcal{N}_{\kappa^0_i} \cap \mathcal{N}_{\kappa^0_j}$. Then since $\mathcal{H}(\xi^1_k, \xi^1_l) \supset \{\delta^0, \kappa^0_i, \kappa^0_j\}$ and $\delta^0 \neq \kappa^0_i, \kappa^0_j$, the property (ii), stated above, implies that $\kappa^0_i = \kappa^0_j$.

So there does indeed exist $\delta^0_1 \neq \delta^0 \in \mathcal{H}^0$, such that $\deg(\delta^0_1)=4$, and $\mathcal{N}_{\delta^0_1} = \{\xi^1_1, \xi^1_2, \xi^1_3, \xi^1_4\}$.

By (iii) there must exist even vertices β^0_1, β^0_2, β^0_3, β^0_4 in $\mathcal{H}^0$, such that $(\xi^1_i - \beta^0_i)$ are in $\mathcal{H}$ and $\deg \beta^0_i = 1$.

Thus the graphs $\mathcal{G}$ and $\mathcal{H}$ are fully determined. Observe that all the even vertices of $\mathcal{G}$ $(\lambda^0_i, 1 \leq i \leq 4)$ have degree 3, while the even vertices of $\mathcal{H}$ have degree either 4 ($\deg(\delta^0)=\deg(\delta^0_1) = 4$) or 1 ($\deg(\beta^0_i) = 1$ for all i). So there can be no choice of $*_{\mathcal{G}}$ and $*_{\mathcal{H}}$ such that $G^t(*_{\mathcal{G}}) = \mathcal{H}^t(*_{\mathcal{H}})$.

This completes the proof of Step 2.

Let $\mathcal{M} = \{\mu^0_1, \mu^0_2, \ldots, \mu^0_m\}, m \geq 0$, be the set of triple points in $\mathcal{H}^0$. Consider the following partition of the sets of even vertices of $\mathcal{G}$ and $\mathcal{H}$ respectively, obtained by considering the degrees of the even vertices:-

$$\begin{aligned} \mathcal{G}^0 &= \bigsqcup_{\xi^1 \in \mathcal{G}^1} \mathcal{I}^{\mathcal{G}}(\xi^1) \bigsqcup_{\substack{\xi^1,\, \eta^1 \in \mathcal{G}^1 \\ \xi^1 \neq \eta^1}} ((\mathcal{G}(\xi^1, \eta^1) \setminus \Lambda) \bigsqcup \Lambda. \\ \mathcal{H}^0 &= \bigsqcup_{\xi^1 \in \mathcal{H}^1} \mathcal{I}^{\mathcal{H}}(\xi^1) \bigsqcup_{\substack{\xi^1,\, \eta^1 \in \mathcal{H}^1 \\ \xi^1 \neq \eta^1}} ((\mathcal{H}(\xi^1, \eta^1) \setminus \mathcal{M}) \bigsqcup \mathcal{M}. \end{aligned}$$

To establish that $\mathcal{G}$ is isomorphic to $\mathcal{H}$, it is enough to set up bijections between the corresponding components of the above partition for $\mathcal{G}^0$ and $\mathcal{H}^0$, which preserve neighbours - i.e., if $f : \mathcal{H}^0 \to \mathcal{G}^0$ is the resulting 'grand bijection', then $\mathcal{N}_{\alpha^0} = \mathcal{N}_{f(\alpha^0)}$ for all α^0 in $\mathcal{H}^0$.

Step 3 In order that there exist a bijection between $\mathcal{G}^0$ and $\mathcal{H}^0$ as in the preceding sentence, it is necessary and sufficient that the following conditions (A) - equivalently (A′) - and (B) are satisfied:

(A) There is a bijection $f : \mathcal{M} \mapsto \Lambda$ so that $\mathcal{N}_\mu = \mathcal{N}_{f(\mu)}$ for all μ in $\mathcal{M}$.

(A′) For any three element subset $\mathcal{N}$ of $\mathcal{G}^1$, $|\{ \lambda_i^0 \in \Lambda : \mathcal{N}_{\lambda_i^0} = \mathcal{N} \}| = |\{ \mu_i^0 \in \mathcal{M} : \mathcal{N}_{\mu_i^0} = \mathcal{N} \}|$

(B) $|\mathcal{I}^{\mathcal{G}}(\xi^1)| = |\mathcal{I}^{\mathcal{H}}(\xi^1)|$ for all ξ^1 in $\mathcal{G}^1$.

Reason: The necessity of the conditions (A) and (B) is easy to see, as is the equivalence of the conditions (A) and (A′).

As for sufficiency, suppose the conditions (A′) and (B) are met. Note that $|\mathcal{G}(\xi^1,\eta^1)| = G^tG(\xi^1, \eta^1) = H^tH(\xi^1,\eta^1) = |\mathcal{H}(\xi^1,\eta^1)|$; on the other hand, the condition (A′) implies that, for all ξ^1, η^1 in $\mathcal{G}^1$, we have the equality

$$|\{\lambda_i^0 \in \Lambda : (\xi^1 - \lambda_i^0 - \eta^1) \in \mathcal{G}\}| = |\{\mu_i^0 \in \mathcal{M} : (\xi^1 - \mu_i^0 - \eta^1) \in \mathcal{H}\}|.$$

Therefore, for all ξ^1, η^1 in $\mathcal{G}^1$, we have

$$|\mathcal{G}(\xi^1, \eta^1) \setminus \Lambda| = |\mathcal{H}(\xi^1, \eta^1) \setminus \mathcal{M}|,$$

which establishes a bijection between the vertices of degree two connecting ξ^1 and η^1 in $\mathcal{G}$ and $\mathcal{H}$. This completes the proof of Step 3.

Hence, in order to complete the proof of the proposition, we only need to verify the validity of (A′) and (B). The proof of (A′) will be achieved in Steps 4 and 5, while Step 6 will prove (B).

Step 4 For $\mathcal{N} = \{\xi_1^1, \xi_2^1, \xi_3^1\} \subseteq \mathcal{G}^1$, $|\{\lambda^0 \in \Lambda : \mathcal{N}_{\lambda^0} = \mathcal{N}\}| \geq |\{\mu^0 \in \mathcal{M} : \mathcal{N}_{\mu^0} = \mathcal{N}\}|$.

Reason: We consider three cases according to the number of triple points $\mu^0 \in \mathcal{M}$ such that $\mathcal{N} = \mathcal{N}_{\mu^0}$, (which cannot exceed 3 since the odd vertices can have degree at most 3, in either graph).

Case(i) Let $\mathcal{N} = \mathcal{N}_{\mu^0}$ for some $\mu^0 \in \mathcal{M}$. Then $G^tG(\xi_i^1, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1)$
$\neq 0$ for all i, j. By (3′) $\mathcal{N}_{\mu^0} = \mathcal{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

Case(ii) Suppose there exist $\mu_1^0, \mu_2^0 \in \mathcal{M}$ such that $\mu_1^0 \neq \mu_2^0$ and $\mathcal{N}_{\mu_1^0} = \mathcal{N}_{\mu_2^0} = \mathcal{N}$. By (i) above, there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are at least two triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, $G^tG(\xi_i^1, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1) \geq 2$. Therefore, there exist $\kappa_{ij}^0 \neq \lambda_1^0$ in $\mathcal{G}^0$, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathcal{G}$ for all distinct i and j.

If all the κ_{ij}^0's were distinct, $(\xi_1^1 - \kappa_{12}^0 - \xi_2^1 - \kappa_{23}^0 - \xi_3^1 - \kappa_{31}^0 - \xi_1^1)$ would form a 6-cycle in $\mathcal{G}$. Therefore for some $j \neq k$, $\kappa_{ij}^0 = \kappa_{ik}^0$, which then is a triple point, λ_2^0, in $\mathcal{G}^0$, such that $\mathcal{N}_{\lambda_2^0} = \mathcal{N}$.

Case(iii) Suppose there exist three distinct points $\mu_1^0, \mu_2^0, \mu_3^0 \in \mathcal{M}$ such that $\mathcal{N}_{\mu_i} = \mathcal{N}$ for all i. By (i) above there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are three triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, we have $H^tH(\xi_1^1, \xi_2^1) = 3$. So there exist distinct κ_1^0, κ_2^0, distinct from λ^0, such that $(\xi_1^1 - \kappa_1^0 - \xi_2^1)$, $(\xi_1^1 - \kappa_2^0 - \xi_2^1)$ are in $\mathcal{G}$. Now $G^tG(\xi_1^1, \xi_3^1) = 3$ and $\mathrm{Deg}(\xi_1^1) \leq 3$. Therefore $(\xi_1^1 - \kappa_1^0 - \xi_3^1)$ and $(\xi_1^1 - \kappa_2^0 - \xi_3^1)$ are in $\mathcal{G}$. So we have $\{\lambda_1^0, \lambda_2^0 = \kappa_1^0, \lambda_3^0 = \kappa_2^0\} \in \Lambda$ such that $\mathcal{N}_{\lambda_i^0} = \mathcal{N}$.

Step 5 End of proof of (A′).

For all $\xi^1 \in \mathcal{G}^1$

$$|\mathcal{G}(\xi^1, \xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{G}(\xi^1, \eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| + |\mathcal{I}^{\mathcal{G}}(\xi^1)| \qquad (1)$$

and

$$|\mathcal{H}(\xi^1,\xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{H}(\xi^1,\eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 — \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|. \qquad (2)$$

(Reason: While $|\mathcal{G}(\xi^1,\ \eta^1)|$ counts the number of even vertices β^0 such that $(\xi^1 — \beta^0 — \eta^1)$ is in $\mathcal{G}$, the first summation on the right side counts such vertices of degree two precisely once and vertices of degree three twice.)

If ξ^1 is such that $|\mathcal{I}^{\mathcal{G}}(\xi^1)|$ =0, then

$$\begin{aligned}|\mathcal{G}(\xi^1,\xi^1)| &= \sum_{\eta^1 \neq \xi^1} |\mathcal{G}(\xi^1,\eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 — \xi^1) \in \mathcal{G}\}| \text{ and} \\ |\mathcal{H}(\xi^1,\xi^1)| &= \sum_{\eta^1 \neq \xi^1} |\mathcal{H}(\xi^1,\eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 — \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|.\end{aligned}$$

Now $|\mathcal{G}(\xi^1,\xi^1)| = |\mathcal{H}(\xi^1,\xi^1)|$, and $|\mathcal{G}(\xi^1,\eta^1)| = |\mathcal{H}(\xi^1,\eta^1)|$.

Therefore,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 — \xi^1) \in \mathcal{G}\}| = |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 — \xi^1) \in \mathcal{H})\}| - |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

And hence,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 — \xi^1) \in \mathcal{G}\}| \leq |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 — \xi^1) \in \mathcal{H}\}|.$$

So, we have,

$$\begin{aligned}0 &\geq |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 — \xi^1) \in \mathcal{G}\}| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 — \xi^1) \in \mathcal{H}\}| \\ &= |\{\lambda_i^0 \in \Lambda : \xi^1 \in \mathcal{N}_{\lambda_i^0}\}| - |\{\mu_i^0 \in \mathcal{M} : \xi^1 \in \mathcal{N}_{\mu_i^0}\} \\ &= \sum_{\mathcal{N} \ni \xi^1} (|\{\lambda_i^0 \in \Lambda : \mathcal{N}_{\lambda_i^0} = \mathcal{N}\}| - |\{\mu_i^0 \in \mathcal{M} : \mathcal{N}_{\mu_i^0} = \mathcal{N}\}|) \\ &\geq 0 \text{ (since each term in the sum is positive by Step 4 above).}\end{aligned}$$

Hence each term in the sum is zero; i.e.,

$$|\{\lambda_i^0 \in \Lambda : \mathcal{N}_{\lambda_i^0} = \mathcal{N}\}| = |\{\mu_i^0 \in \mathcal{M} : \mathcal{N}_{\mu_i^0} = \mathcal{N}\}$$

for all $\mathcal{N} \subseteq \mathcal{G}^1$, containing an element ξ^1 such that $|\mathcal{I}^{\mathcal{G}}(\xi^1)| = 0$.

But, by (4′), every $\mathcal{N}_\lambda$ has an element ξ^1_λ with $|\mathcal{I}^{\mathcal{G}}(\xi^1_\lambda)| = 0$. If $\mathcal{N} \neq \mathcal{N}_\lambda$ for any λ, there is no triple point in $\mathcal{G}$ whose set of neighbours equals $\mathcal{N}$ and so, by Step 4, there is no such triple point in $\mathcal{H}$ either. So we have (A′).

Step 6 *Proof of* (B)

By (A′) we know that for any ξ^1 in $\mathcal{G}^1$

$$\begin{aligned} |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| &= \sum_{\mathcal{N} \ni \xi^1} |\{\lambda_i^0 \in \Lambda : \mathcal{N}_{\lambda_i^0} = \mathcal{N}\}| \\ &= \sum_{\mathcal{N} \ni \xi^1} |\{\mu_i^0 \in \mathcal{M} : \mathcal{N}_{\mu_i^0} = \mathcal{N}\}| \\ &= |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}|. \end{aligned}$$

Therefore by comparing (1) and (2) we have

$$|\mathcal{I}^{\mathcal{G}}(\xi^1)| = |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

The proof of the proposition is finally complete. □

REMARK 2.4.2 (1) Each of the graphs A_n, D_n and E_n satisfies the four hypotheses of the last proposition. (For $n > 8$, we write $E_n = A_{n-8} \# E_8$, where the $*$'s of the two graphs are identified.)

(2) For a bipartite graph $\mathcal{G}$, there is a natural induced metric on $V(\mathcal{G})$. For each integer $n \geq 0$, write $\mathcal{G}_{n]}$ for the induced subgraph of $\mathcal{G}$ on the set of vertices at distance at most n from $*_{\mathcal{G}}$. (Thus, $\mathcal{G}_{0]} = \{*\}, \mathcal{G}_{1]} = \mathcal{N}^{\mathcal{G}}_* \cup \{*\}$, etc..) It is clear that if $\mathcal{G}$ and $\mathcal{H}$ are weakly dual, so are $\mathcal{G}_{2n]}$ and $\mathcal{H}_{2n]}$. In particular, we recapture Haagerup's observation: if $\mathcal{G}$ and $\mathcal{H}$ are the principal

graphs of a finite-index subfactor and if $\mathcal{G}_{n]} = A_{n+1}$, then $\mathcal{H}_{n]} = A_{n+1}$ for even n.(To be sure, it must be verified that the contragredient map is trivial on the even vertices; but for this it is enough to note that for all n the set $\mathcal{G}^0_{2n+2]} - \mathcal{G}^0_{2n]}$ is invariant under the involution on the even vertices.) The above statement is also valid for odd n; this follows from the case of even n and the connectedness of the principal graphs.

(3) It is tempting to call the subgraph conditions - *c.f.* (3) and (4) of Proposition 2.4.1 - a 'double of a star-triangle' relation; more precisely, is there more than just a superficial similarity between the two notions?

2.5 The non-occurrence of some graphs as principal graphs

We now recall the following observation made by Ocneanu (see [K] for the statement and [OK] for a proof).

OBSERVATION 2.5.1 Suppose a graph $\mathcal{G}$ satisfies the following conditions:

(1) $\mathcal{G}$ does not contain a subgraph isomorphic to C_4;

(2) $\mathcal{G}$ contains a triple point; and

(3) $||G|| > 2$.

Then it is not possible to construct a commuting square of the following form:

$$\begin{array}{ccc} C & \overset{\mathcal{G}''}{\subseteq} & D \\ {}_{\mathcal{G}}\cup\vert & & \cup\vert_{\mathcal{G}'} \\ A & \underset{\mathcal{G}}{\subseteq} & B \end{array}$$

In particular, there does not exist a finite depth subfactor $N \subset M$ with trivial contragredient maps, both of whose principal graphs are $\mathcal{G}$.

The above observation, in conjunction with the preceding proposition, has the following interesting consequence.

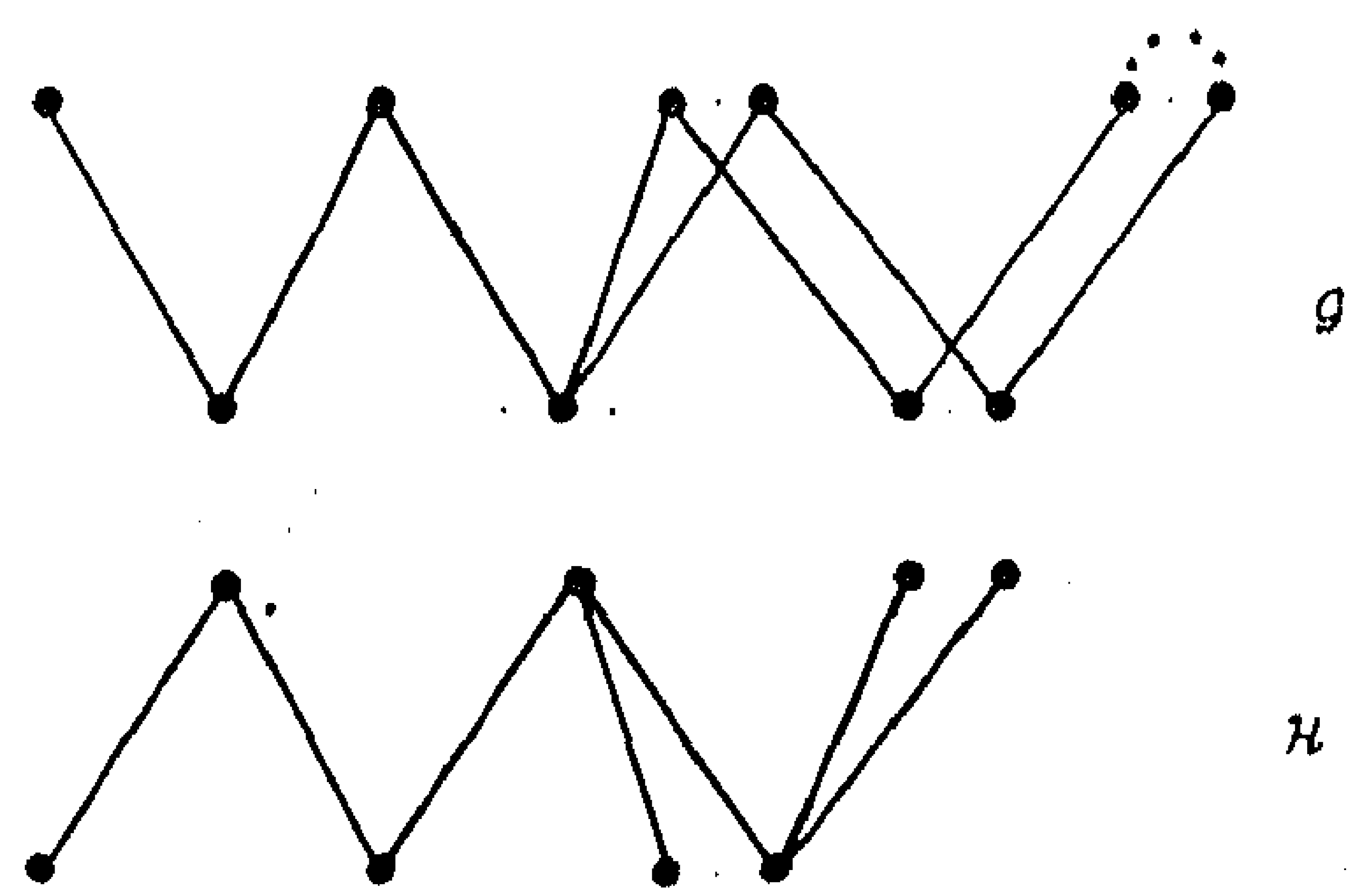
THEOREM 2.5.2 *Let $N \subset M$ be a pair of II_1 factors such that $[M : N] > 4$. Assume that the subfactor N has trivial contragredient maps. If one of the associated principal graphs $\mathcal{G}$ is a finite tree, each of whose vertices has degree at most three, then $\mathcal{G}$ contains a subgraph isomorphic to T such that the vertices of degree one in T are even vertices of degree one in $\mathcal{G}$.*

Proof. Suppose now that a graph $\mathcal{G}$ arises as in the statement of the theorem. The hypothesis ensures that $\mathcal{G}$ satisfies conditions (1), (2) and (3) of Proposition 2.4.1, as well as conditions (1) and (3) of Observation 2.5.1. Since $||G|| > 2$, the graph $\mathcal{G}$ is not A_n for any n. Since $\mathcal{G}$ is assumed to have at most triple points, it follows that $\mathcal{G}$ also satisfies condition (2) of Observation 2.5.1.

On the other hand, it must be obvious that a finite graph cannot arise as a principal graph of a subfactor with trivial contragredient maps, if it satisfies conditions (1) - (4) of Proposition 2.4.1 as well as conditions (1) - (3) of Observation 2.5.1.

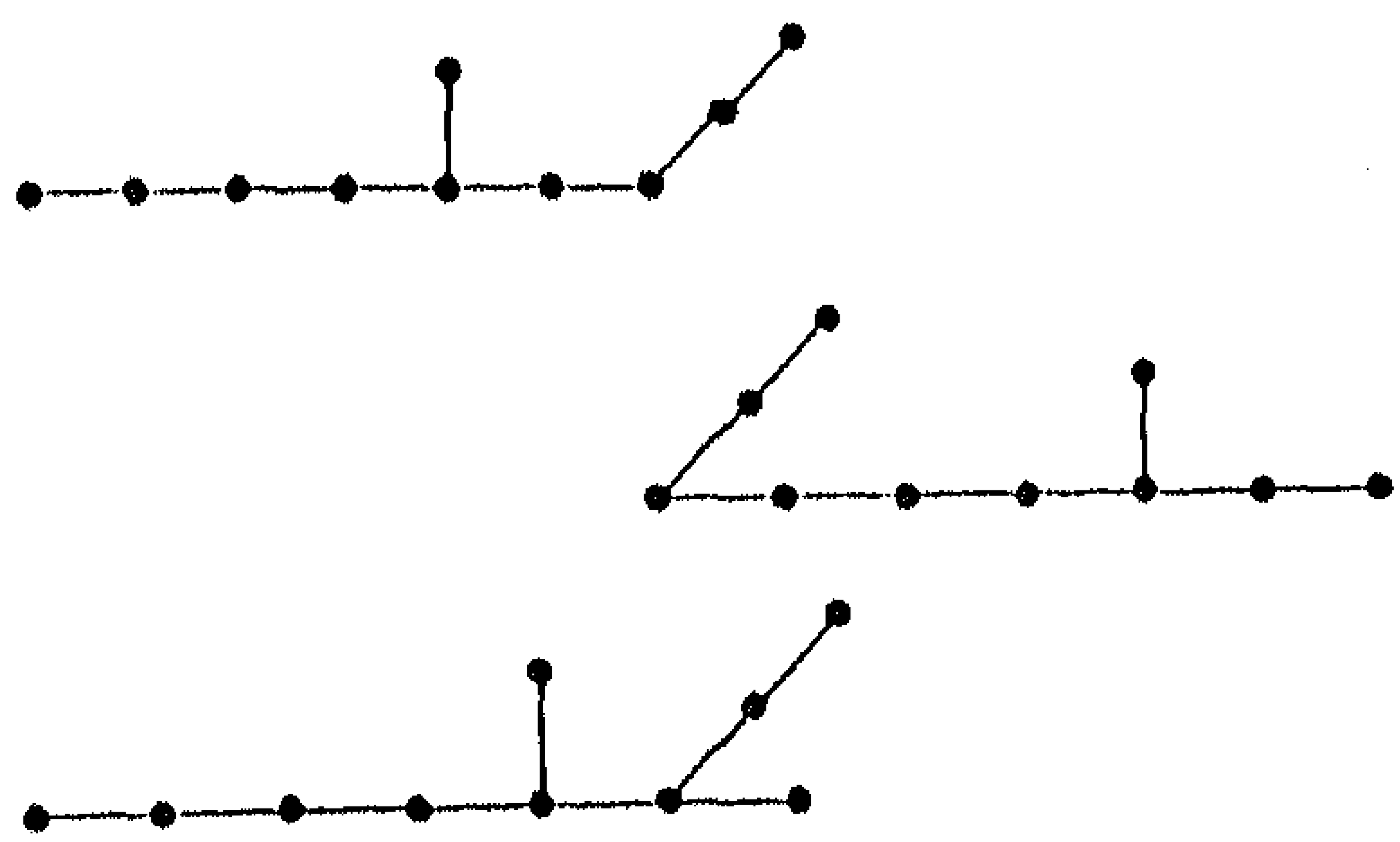
Hence it must be the case that $\mathcal{G}$ violates condition (4) of Proposition 2.4.1, and the proof is complete. □

REMARK 2.5.3 The hypothesis about trivial contragredient maps is essential. It has been shown by Haagerup (cf. [HS]) that there exists a subfactor whose principal graphs are as shown below, where the non-trivial contragredient mapping in the graph $\mathcal{G}$ is indicated by the dotted line :

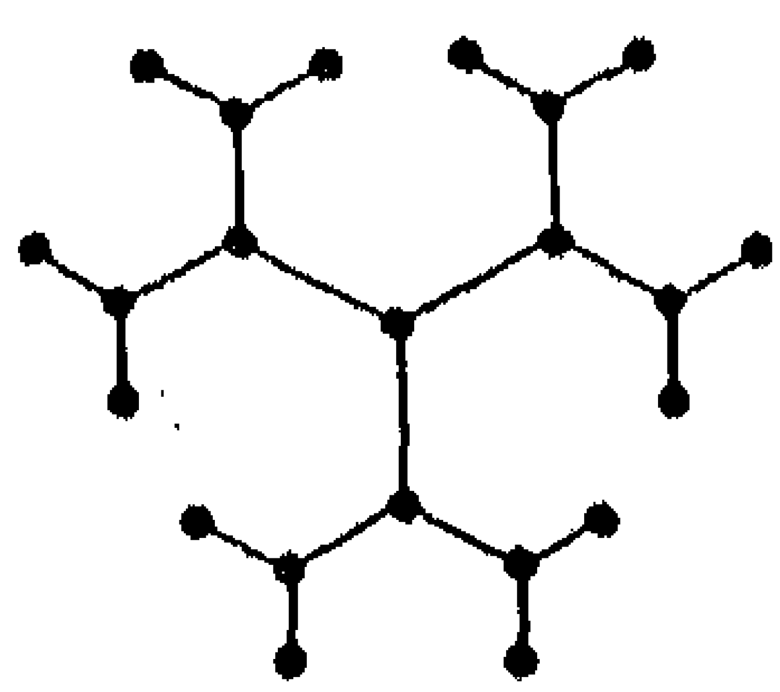


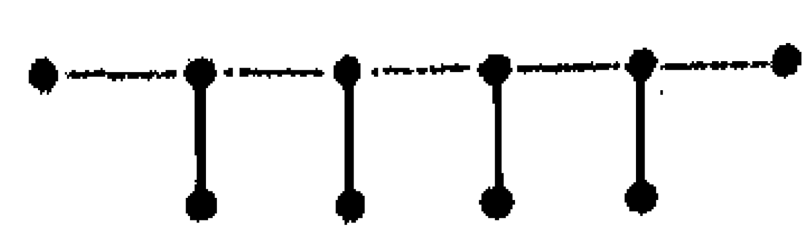
The above theorem shows that no *T-graph* - i.e., a graph with a unique vertex of degree 3, with all other vertices of degree at most two - with norm greater than 2 can arise as a principal graph of a subfactor with trivial contragredient maps.

We conclude this chapter by describing some more graphs which cannot arise as a principal graph of a subfactor with trivial contragredient maps, as these are trees which satisfy condition (4) of Proposition 2.4.1 (and which have norm greater than 2 and have no vertex of degree greater than 3):
(i) any version of a connected sum of $A_n, n \geq 2$ and E_8 in which a vertex of degree one from A_n has been identified with one of the vertices of degree one in E_8 or one of their degree two neighbours;



(ii) the Cayley tree and many other subgraphs of the Bethe lattice.





Chapter 3

Construction of some commuting squares

3.1 Introduction

In this chapter, we give the explicit construction of a couple of commuting squares. The first example is motivated by the theory of hypergroups and the second turns out to be the principal graph for a pair of subfactors of the type, $R \rtimes H \subseteq R \rtimes G$, where H is a subgroup of a finite group G.

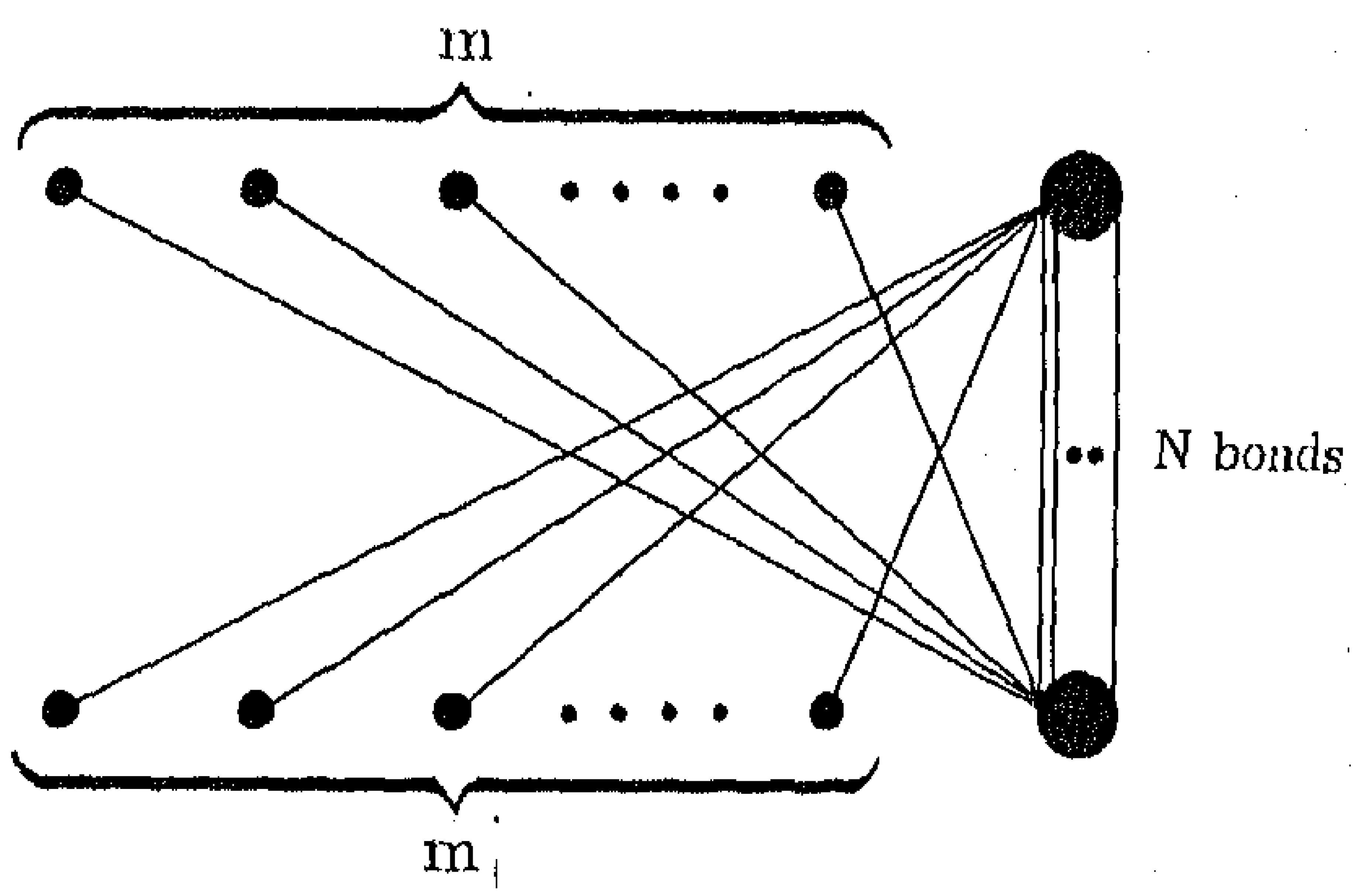
3.2 A commuting square using Hadamard matrices

Consider a hypergroup consisting of a group of m elements, $\{\alpha_1, \alpha_2 \cdots \alpha_m\}$ together with a single element, δ. The hypergroup operation is defined by matrices corresponding to each element. For details see [SV]. In this example

$\delta^2 = \sum_i^m \alpha_i + N\delta$. The matrix corresponding to δ is :

$$L = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & N \end{bmatrix}.$$

If L were the inclusion matrix of two finite dimensional algebras the corresponding graph (Bratteli diagram) would be:



Both the sets of vertices (even and odd) are labelled $\{1, 2, \cdots, m, N\}$ and the edges between N and N are labelled by the set $\{a_1, \cdots a_N\}$. The Perron-Frobenius eigenvalue of the symmetric matrix L, denoted by λ , is equal to $(N + \sqrt{N^2 + 4m})/2$. The Perron-Frobenius eigenvector has entries 1 corresponding to the vertices $1, \cdots, m$ and λ corresponding to the vertices N.

We wish to construct a commuting square of the form:

$$\begin{array}{ccc} C & \overset{L'}{\subseteq} & D \\ {}_L\cup\!\mid & & \cup\!\mid_{L'} \\ A & \underset{L}{\subseteq} & B \end{array}$$

By the biunitarity condition described in the first chapter, to do this it is enough to construct unitary matrices U and V satisfying the following criteria:

(a)

$$U = \bigoplus_{i,j=1}^{m} u_{ij} \quad \oplus \quad \bigoplus_{i=1}^{m} u_{iN} \quad \oplus \quad \bigoplus_{i=1}^{m} u_{Ni} \quad \oplus \quad u_{NN}$$

and

$$V = \bigoplus_{i,j=1}^{m} v_{ij} \quad \oplus \quad \bigoplus_{i=1}^{m} v_{iN} \quad \oplus \quad \bigoplus_{i=1}^{m} v_{Ni} \quad \oplus \quad v_{NN}$$

where u_{ij}, v_{ij} are 1×1 matrices, u_{iN}, u_{Ni}, v_{Ni} and v_{iN} are $N \times N$ matrices, and u_{NN} and v_{NN} are $(N^2 + m) \times (N^2 + m)$ matrices;

(b) the direct summands of V are defined as follows:

(i) $v_{ij} = \lambda\overline{(u_{NN})_{ji}}$ for $i, j = 1 \cdots m$;

(ii) $(v_{iN})_{ab} = \sqrt{\lambda}\ \ \overline{(u_{NN})_{ba,i}}$ for $a, b \in \{a_1, \cdots a_N\}, i = 1, \cdots m$;

(iii) $(v_{Ni})_{ab} = \sqrt{\lambda}\ \ \overline{(u_{NN})_{i,ba}}$ for $a, b \in \{a_1, \cdots a_N\}, i = 1, \cdots m$;

(iv) $(v_{NN})_{ij} = \frac{1}{\lambda}\overline{u_{ji}}$ for $i, j = 1, \cdots m$;

(v) $(v_{NN})_{i,ab} = \frac{1}{\sqrt{\lambda}}\overline{(u_{Ni})_{ba}}$ for $a, b \in \{a_1, \cdots a_N\}, i = 1, \cdots m$;

(vi) $(v_{NN})_{ab,i} = \frac{1}{\sqrt{\lambda}}\overline{(u_{iN})_{ba}}$ for $a, b \in \{a_1, \cdots a_N\}, i = 1, \cdots m$;

(vii) $(v_{NN})_{ab,cd} = \overline{(u_{NN})_{db,ca}}$.

One way of ensuring the unitarity of V is to impose the condition $V = \overline{U}$. Then the problem reduces to finding a unitary U of the above form satisfying the following criteria:

(i) u_{ij} for $i, j = 1, \cdots, m$, are 1×1 matrices;

(ii) u_{iN} and u_{Ni} are $N \times N$ matrices;

(iii) u_{NN} is an $(N^2 + m) \times (N^2 + m)$ matrix such that

(a) $(u_{NN})_{ij} = \frac{1}{\lambda} u_{ji}$ for $i, j \in \{1, 2, \cdots m\}$

(b) $(u_{NN})_{i,ab} = \frac{1}{\sqrt{\lambda}} (u_{Ni})_{ba}$ for $i \in \{1, 2, \cdots, m\}, a, b \in \{a_1, a_2, \cdots a_N\}$

(c) $(u_{NN})_{ab,i} = \frac{1}{\sqrt{\lambda}} (u_{iN})_{ba}$ for $i \in \{1, 2, \cdots, m\}, a, b \in \{a_1, a_2, \cdots a_N\}$.

(d) $(u_{NN})_{ab,cd} = (u_{NN})_{db,ca}$ for $a, b, c, d \in \{a_1, a_2, \cdots a_N\}$.

In other words, we wish to construct an $(N^2 + m) \times (N^2 + m)$ unitary matrix u_{NN} of the following form

$$\begin{bmatrix} A & S \\ R & K \end{bmatrix}$$

where,

(a) A is an $m \times m$ matrix with $|a_{ij}| = 1/\lambda$;

(b) each row of $\sqrt{\lambda} S$ is an $N \times N$ unitary matrix written out row after row.

(c) each column of $\sqrt{\lambda} R$ is an $N \times N$ unitary matrix written out column after column; and

(d) K is an $N^2 \times N^2$ matrix, which when labelled suitably to form $N \times N$ blocks, i.e., $K(a \circ c, d \circ b) = K_{ab}(c, d)$, satisfies the condition $K_{ab} = K_{ba}$.

Here we would like to state a couple of results used in the construction of u. (For the proof see [S2].)

LEMMA 3.2.1 Let $A \in M_m(\mathbb{R}), B \in M_{m\times N}(\mathbb{R})$. Then the following conditions are equivalent :

(i) there exists a self-adjoint, orthogonal matrix

$$P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

(ii) $A = A^*$ and $A^2 + BB^* = I_m$.

□

LEMMA 3.2.2 Let A, B be matrices satisfying the conditions of Lemma 3.2.1. Then there exists an orthogonal matrix $Q \in M_{m+N^2}(\mathbb{R})$ of the form:

$$Q = \begin{bmatrix} A & B_1 & B_2 & \cdots & B_N \\ B_1^* & K_{11} & K_{12} & \cdots & K_{1N} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ B_N^* & K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}$$

where

(i) B_j is the $m \times N$ matrix with p-th row given by $B_{pj}e_j^{(N)}$, where $e_j^{(N)}$ denotes the j-th standard basis vector of $\mathbb{R}^N$; and

(ii) $K_{pq} \in M_N(\mathbb{R}), 1 \leq p, q \leq N$, and these matrices satisfy

$$K_{pq} = K_{qp}.$$

□

So if we could find

(i) an $m \times m$ self adjoint matrix A such that $|a_{ij}| = 1/\lambda$ for all $i,j = 1,2,\cdots m$;

(ii) an $m \times N$ matrix B such that $|B_{ij}| = 1/\sqrt{\lambda}$ for all $i = 1,2,\cdots m, j = 1,2,\cdots N$;

(iii) so that A and B satisfy $A^2 + BB^* = I_m$,

then A and B would satisfy the hypotheses of Lemma 3.2.1 and then from Lemma 3.2.2 we would have the matrix Q which would satisfy all the requirements (a) - (e) of the matrix u_{NN}.

If $u_{NN} = Q$, then the matrix U is completely specified in terms of Q and in turn, in terms of A and B, using the conditions (i) - (iii) stated before Lemma 3.2.1.

So we would be through if we could find A and B satisfying the conditions (i) - (iii) above.

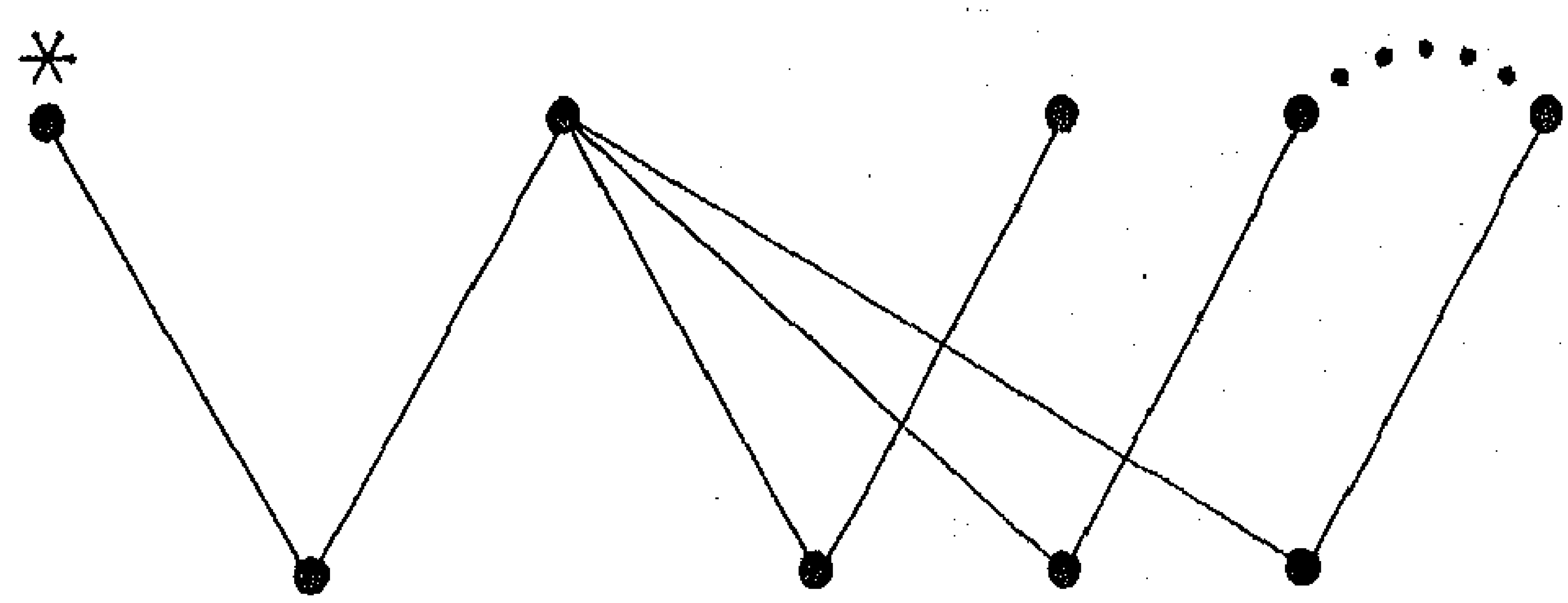
Note that if we could choose A to be a $(\sqrt{m}/\lambda)\tilde{A}$ where $\tilde{A}$ is a unitary matrix, and B to be $(\sqrt{N/\lambda})(\tilde{B})$ where the rows of $\tilde{B}$ are orthonormal then condition (iii) follows from the (eigenvalue) equation $m + N\lambda = \lambda^2$.

Recall that a matrix is called an Hadamard matrix if it is a real orthogonal matrix all of whose entries have equal modulus. (Thus, for instance, it is known that if $k > 2$, a necessary condition for the existence of a $k \times k$ Hadamard matrix is that k is divisible by 4. It is conjectured that the above condition is also sufficient for the existence of an Hadamard matrix of order k; this conjecture, although as yet unresolved, has been verified for fairly large values of k.)

Hence, if N (resp., m) is an Hadamard (resp., symmetric Hadamard) integer - i.e., a number which admits a real Hadamard (resp., real symmetric Hadamard) matrix of that size - and if $N > m$, then we could choose $\tilde{A}$ to be a real symmetric $m \times m$ Hadamard matrix and $\tilde{B}$ to be the first m rows of an $N \times N$ real Hadamard matrix. Then, by the defining property of Hadamard matrices, all the entries are of equal modulus and all the conditions (i) - (iii) are met.

3.3 A biunitary matrix on a 4-star

For the second example - which is an example with non-trivial contragredient map - we start with the following graph:



The dotted lines indicate the action of the contragredient map. The adjacency matrix for $\mathcal{G}$ is given by

$$\begin{bmatrix} 0 & G \\ G' & 0 \end{bmatrix}$$

where G is the $\mathcal{G}^{(0)} \times \mathcal{G}^{(1)}$ matrix :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Perron-Frobenius eigenvalue for $G'G$ denoted by λ is equal to 5. The entries of the Perron-Frobenius eigenvector, corresponding to the odd vertices are all equal to $\sqrt{5}$. Corresponding to the even vertices all the entries are 1, except for the vertex labelled 2, for which the entry is 4.

The contragredient map, restricted to the even vertices, is given by :

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We wish to construct a symmetric commuting square of the type :

$$\begin{array}{ccc} C & \overset{G'}{\subseteq} & D \\ {}_{\tau G}\cup\vert & & \cup\vert_{G'\tau} \\ A & \underset{G}{\subseteq} & B \end{array}$$

Now,

$$G'(\tau G) = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \quad G(G'\tau) = (\tau G)G' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

So, in order to construct a commuting square as above, we need to construct a unitary matrix U which satisfies the following criteria:

(i) U is of the following form:

$$U = \bigoplus_{(i,j)\in I} u_{ij} \bigoplus u_{22}$$

where $I = \{(0,0),(0,2),(2,0),(2,4),(2,6),(2,8),(4,2),(4,6),(6,2),(6,4),(8,2),(8,8)\}$ and u_{ij} is a 1×1 matrix for each $(i,j) \in I$, and $u_{22} = [a_{kl}]_{k,l\in\{1,3,5,7\}}$ is a 4×4 matrix.

(ii) V, defined as follows, is a unitary matrix :

$$V = \bigoplus_{(k,l)\in J_1 \cup J_2} v_{kl}$$

where $J_1 = \{(1,3),(1,5),(1,7),(3,1),(3,3),(3,7),(5,1),(5,5),(5,7),(7,1),(7,3),(7,5)\}$ and $J_2 = \{(1,1),(3,5),(5,3),(7,7)\}$, and

(a) if $(k,l) \in J_1$, then v_{kl} is the 1×1 matrix

$$v_{kl} = 4/\sqrt{5}\ \ a_{lk};$$

(b) for $(k,l) \in J_2$, we have

$$v_{11} = 1/\sqrt{5}\begin{bmatrix} u_{00} & 2u_{20} \\ 2u_{02} & 4a_{11} \end{bmatrix}, v_{35} = 1/\sqrt{5}\begin{bmatrix} u_{64} & 2u_{24} \\ 2u_{62} & 4a_{53} \end{bmatrix},$$

$$v_{53} = 1/\sqrt{5}\begin{bmatrix} u_{46} & 2u_{26} \\ 2u_{42} & 4a_{35} \end{bmatrix}, v_{77} = 1/\sqrt{5}\begin{bmatrix} u_{88} & 2u_{28} \\ 2u_{82} & 4a_{77} \end{bmatrix}.$$

We require both U and V to be unitary which means that all the summands must be unitary. From this it is not hard to see that the moduli of the entries in u_{22} are as in the following matrix :

$$1/4\begin{bmatrix} 1 & \sqrt{5} & \sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} & 1 & \sqrt{5} \\ \sqrt{5} & 1 & \sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} & \sqrt{5} & 1 \end{bmatrix}.$$

If we choose u_{22} to be :

$$u_{22} = 1/4\begin{bmatrix} 1 & \sqrt{5} & \sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} & -1 & -\sqrt{5} \\ \sqrt{5} & -1 & -\sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} & \sqrt{5} & -1 \end{bmatrix},$$

u_{00} to be $[-1]$ and all the other u_{ij}'s to be $[1]$, it is not hard to see that all the summands of V are unitary.

It is known (cf. [KY] and [RS]) that this graph actually occurs as a principal graph for a pair of subfactors of the type, $R \rtimes H \subseteq R \rtimes G$ where H is a subgroup of G. To be precise, G is the group of affine transformations of the field Z_5, and H is the non-normal subgroup of linear tranformations (i.e., those that fix 0); thus G is the semi-direct product of the normal subgroup K of translations, by H.

Chapter 4

Vertex models

4.1 Introduction

This chapter is devoted to the study of subfactors built out of a commuting square corresponding to a 'vertex model', i.e., a commuting square of the form

$$\begin{array}{ccc} M_N(\mathbb{C}) \otimes 1 & \subseteq & M_N(\mathbb{C}) \otimes M_N(\mathbb{C}) \\ \cup\!| & & \cup\!| \\ \mathbb{C} & \subseteq & (Ad\, u)(M_N(\mathbb{C}) \otimes 1) \end{array}$$

where u is a (bi)unitary in $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$.

First, we use the Compactness Theorem due to Ocneanu -see Theorem1.6.1 to obtain a formula for computing the higher relative commutants of a pair of subfactors $R_0 \subseteq R_1$, built out of a commuting square of the form:

$$\begin{array}{ccc} B & \overset{G'}{\subseteq} & D \\ {}_G\cup\!| & & \cup\!|_{G'} \\ A & \underset{G}{\subseteq} & uBu^* \end{array}$$

Then in the third section we apply the results of the second section to two specific families of commuting squares, to compute the principal graph of the resulting subfactors.

The fourth section is devoted to 'classifying' such commuting squares, at least when $N = 2$; the conclusion is that the resulting subfactor always has non-trivial relative commutant and (as a result of the analysis of the second section) the principal graphs of subfactors of index 4 that are so constructed, are precisely the extended diagrams $A^{(1)}_{(2n-1)}, n = 1, 2, \cdots, \infty$.

Finally, we state, without proof, some results that have been obtained in the case when u is a biunitary permutation matrix.

Acknowledgement : We would like to thank Vaughan Jones for suggesting that we should look at these vertex models. Also he pointed out that the formula for the higher relative commutants (and hence the subsequent computations) can be simplified by considering the tower of the basic construction instead of the tunnel as we did in [KSV]. We would also like to thank Vishwambhar Pati who helped us clarify our thinking along the lines which led to the classification result in the fourth section.

4.2 Computing higher relative commutants

In this section, we start with a commuting square of the form

$$\begin{array}{ccc} B & \overset{G'}{\subseteq} & D \\ {}_{G}\cup\mid & & \cup\mid_{G'} \\ A & \underset{G}{\subseteq} & uBu^* \end{array} \qquad (*)$$

where u is a unitary element of $A' \cap D$, construct a subfactor R_0 of the hyperfinite factor $R(= R_1)$ in the usual manner - cf. §1.4 - and describe a method for computing the higher relative commutants $R_0' \cap R_n$, where $\{R_n\}$ denotes the tower obtained by iterating the basic construction to the initial inclusion $R_0 \subset R$.

We shall use (and build upon) the notation of path-algebras introduced in Chapter 1 (in sections 1.3 and 1.5). Thus, we fix a bipartite graph $\mathcal{G}$, with the set of even (resp., odd) vertices denoted by $\mathcal{G}^{(0)}$ (resp., $\mathcal{G}^{(1)}$), and with 'incidence matrix' given by $G \in Mat_{\mathcal{G}^{(0)} \times \mathcal{G}^{(1)}}(\mathbb{Z}_+)$. For convenience, we define $\mathcal{G}^{(n)}$ (resp., $G^{(n)}$) to be $\mathcal{G}^{(0)}$ (resp., G) or $\mathcal{G}^{(1)}$ (resp., $G^{(1)}$) according as n is even or odd. As before, we write $\Omega_k^{(n)}$ for the space of paths in $\mathcal{G}$ which start in $\mathcal{G}^{(n)}$ and we write $C_k^{(n)}$ for the path-algebra given by

$$C_k^{(n)} = \{x \in Mat_{\Omega_k^{(n)}}(\mathbb{C}) \ : \ x(\alpha,\beta) = 0 \text{ unless } (s(\alpha), f(\alpha)) = (s(\beta), f(\beta))\}.$$

Thus the biunitarity condition says that the square of algebras

$$\begin{array}{ccc} C & \overset{G^{(n+1)}}{\subseteq} & D \\ {\scriptstyle G^{(n)}}\cup\!\mid & & \cup\!\mid{\scriptstyle G^{(n+1)}} \\ A & \underset{G^{(n)}}{\subseteq} & B \end{array} \qquad (**)$$

is a commuting square (with respect to the Markov trace) if and only if there exists a unitary element $u \in C_2^{(n)}$ such that $\tilde{V}(u) (\in C_2^{(n+1)})$ is also unitary, where $\tilde{V}(u)$ is defined as in §1.5. (Recall that

$$(\tilde{V}(u))(a \circ b, c \circ d) = \sqrt{\frac{\mu(f(a))\mu(s(d))}{\mu(s(a))\mu(f(d))}} \ \overline{u(\tilde{a} \circ c, b \circ \tilde{d})}$$

where μ denotes the Perron-Frobenius eigenvector of the adjacency matrix of the bipartite graph $\mathcal{G}$, and of course, the symbol $\tilde{a}$ denotes the reflection of the path a.)

For convenience of notation, we define

$$BU(C_2^{(n)}) = \{u \in C_2^{(n)} : \textit{both } u \textit{ and } \tilde{V}(u) \textit{ are unitary}\}. \qquad (4.1)$$

Thus, the square (**) is a commuting square precisely when the set $BU(C_2^{(n)})$ (of 'biunitary matrices') is non-empty.

It must be clear that $u \in BU(C_2^{(n)})$ if and only if $\tilde{V}(u) \in BU(C_2^{(n+1)})$, and so we shall think of $\tilde{V}$ as a map from $BU(C_2^{(n)})$ to $BU(C_2^{(n+1)})$ for all n. (Thus, for instance, if we identify $BU(C_2^{(n)})$ with $BU(C_2^{(n+2)})$ in the natural manner, then $\tilde{V}^2 = \tilde{V} \circ \tilde{V}$ is the identity map on $BU(C_2^{(n)})$.)

For our purposes, a slight variation of $\tilde{V}$, which we shall denote by V, will be more useful. Thus for $u \in C_2^{(n)}$, define $V(u) \in C_2^{(n+1)}$ by

$$(V(u))(a \circ b, c \circ d) = \sqrt{\frac{\mu(f(a))\mu(s(d))}{\mu(s(a))\mu(f(d))}}\ \overline{u(d \circ \tilde{b}, \tilde{c} \circ a)}.$$

(It is a fact that $u \in BU(C_2^{(n)})$ if and only if $V(u) \in BU(C_2^{(n+1)})$.) More generally, we would have occasion to use the following result from [S3] several times. (It must be remarked here that although there is a standing additional assumption in [S3] - called 'rotational symmetry' there - that assumption is not needed for the validity of the conclusions stated here.)

THEOREM 4.2.1 *Let $A_0 \overset{G}{\subset} A_1$ be an inclusion of finite-dimensional C^*-algebras, and let*

$$A_0 \subseteq A_1 \subseteq A_2 \cdots A_n \subseteq A_{n+1} \cdots$$

be the resulting tower of the basic construction.

(a) Let $u \in BU(C_2^{(0)})$ be arbitrary; define

$$u_n = \left\{ \begin{array}{ll} u_{[n-1,n+1]} & \text{if } n \text{ is odd} \\ (V(u))_{[n-1,n+1]} & \text{if } n \text{ is even} \end{array} \right\}$$

and define

$$w_n = u_1 u_2 \cdots u_n, \quad n = 1, 2, \cdots.$$

Then

(i)

$$\begin{array}{ccc} A_n & \subseteq & A_{n+1} \\ \cup\!\mid & & \cup\!\mid \\ (Ad\ w_{n-1})A_{n-1} & \subseteq & (Ad\ w_n)A_n \end{array}$$

is a commuting square, for all n , *where* $Ad\ w$ *denotes the map* $x \mapsto wxw^*$;

(ii)

$$(Ad\ w_{n-2})A_{n-2} \subseteq (Ad\ w_{n-1})A_{n-1} \subseteq (Ad\ w_n)A_n$$

is an instance of the basic construction, for each n ; *and*

(iii) if $R_0 \subseteq R_1$ *is the subfactor-factor pair constructed in the usual fashion, starting from the initial commuting square*

$$\begin{array}{ccc} A_1 & \subseteq & A_2 \\ \cup\!\mid & & \cup\!\mid \\ A_0 & \subseteq & (Ad\ u)A_1 \end{array} ,$$

then the equation

$$\alpha(x) = \lim_{n\to\infty} (Ad\ w_n)(x)$$

defines an endomorphism α *of* R_1 *such that* $\alpha(R_1) = R_0$.

(b) If n *is any positive integer and if* $u \in BU(C_2^{(n)})$ *is arbitrary, then*

$$(Ad\ (u_{[n-2,n]}(V(u))_{[n-1,n+1]})\ )(e_{n-1}) = e_n, \qquad (4.2)$$

where e_n *denotes, here and throughout this section, the Jones projection in* A_{n+1} *which implements the conditional expectation of* A_n *onto* A_{n-1}.

We shall assume throughout this section that we are given a commuting square

$$\begin{array}{ccc} A_1 & \overset{G'}{\subseteq} & A_2 \\ {}_G\cup\!\mid & & \cup\!\mid_{G'} \\ A_0 & \underset{G}{\subseteq} & uA_1u^* \end{array}$$

where u is a unitary element in $A_0' \cap A_2$, and that $R_0 \subseteq R_1$ is the factor-subfactor pair obtained in the usual manner. We shall assume that

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq R_{n+1} \cdots$$

is the resulting tower of the basic construction with f_n being the Jones' projection implementing the conditional expectation of R_n onto R_{n-1}.

Define a grid $\{A_{n,k} : n, k \geq 0\}$ of finite dimensional C^*-algebras as follows:

(i) $A_{1,k} = A_{k+1}$ for all k;

(ii)$A_{0,k} = (Ad\ w_k)A_k$ for all k;

(iii)$A_{n+1,k} = < A_{n,k}, f_n >$ for all k and $n \geq 1$.

We begin with some observations about the grid $\{A_{n,k}\}$.

(i) Each of the following is a commuting square:

$$\begin{array}{ccc} A_{n+1,k} & \subseteq & A_{n+1,k+1} \\ \cup\!\mid & & \cup\!\mid \\ A_{n,k} & \subseteq & A_{n,k+1} \end{array}$$

(ii) For all $n \geq 1, k \geq 0$ $A_{n-1,k} \subseteq A_{n,k} \subseteq A_{n+1,k}$, is an instance of the basic construction, with Jones' projection f_n.

(iii) $R_n = (\cup_k A_{n,k})''$.

The proof of these facts can be found in [GHJ].

PROPOSITION 4.2.2 *Let R_n and $A_{n,k}$ be as described above; then*

$$R_0' \cap R_n = A_{n,0} \cap A_{0,1}'$$

Before proceeding to a proof of the proposition we gather together some results about biunitary matrices:

LEMMA 4.2.3 *For $u \in BU(C_2^{(n)})$, define $\tilde{u} \in BU(C_2^{(n)})$ by $\tilde{u} = (V*)^2(u)$, where we write $*$ for the mapping $u \mapsto u^*$. Then*

(i) $\tilde{\tilde{u}} = u, i.e. (V*)^4 = id$, and hence $\tilde{u} = (*V)^2(u)$.

(ii) $V(\tilde{u}) = \widetilde{V(u)}$.

Proof: (i) Note that $V(V(u)) = u$ and $\tilde{u}(a \circ b, c \circ d) = u(\check{d} \circ \check{c}, \check{b} \circ \check{a})$; hence $\tilde{\tilde{u}} = u$, or $(V*)^4 = id$. It follows that $(*V)^2 = (V*)^{-2} = (V*)^2$.

(ii)

$$\begin{aligned} V(\tilde{u}) &= V(V*)^2(u) \\ &= (*V*)(u) \\ \text{and } \widetilde{V(u)} &= (*V)^2 V(u) \\ &= (*V*)(u). \end{aligned}$$

□

(Observe, incidentally, that $\tilde{V}(u) = \widetilde{V(u)}$.)

LEMMA 4.2.4 *For $u \in BU(C_2^{(n)})$, define*

$$\theta_1^{(n)} = \begin{cases} u & \text{if } n \text{ is even} \\ \tilde{u} & \text{otherwise} \end{cases}$$

and $\theta_{k+1}^{(n)} = V(\theta_k^{(n)})$ *for* $k \geq 1$. *Then* $\widetilde{\theta_k^{(n)}} = \theta_k^{(n+1)}$.

Proof: The proof is by induction on k. The statement is true for $k = 1$ by definition and Lemma 4.2.3 (ii). For the inductive step,

$$\begin{aligned}\widetilde{\theta_{k+1}^{(n)}} &= \widetilde{V(\theta_k^{(n)})} \\ &= V(\tilde{\theta}_k^{(n)}) \text{ by Lemma 4.2.3 (ii)} \\ &= V(\theta_k^{(n+1)}) \text{ by induction hypothesis} \\ &= \theta_{k+1}^{(n+1)}.\end{aligned}$$

□

LEMMA 4.2.5 *If* $u \in BU(C_2^{(n)})$ *and* $\theta_k^{(n)}$ *is as in Lemma 4.2.4, define* $u_k^{(n)} = j_{[k-1,k+1]}(\theta_k^{(n)})$ *for* $k \geq 1$. *Then, for all* $k \geq 1$,

(i) $u_k^{(n)} \in A'_{k-1} \cap A_{k+1}$.

(ii) $(Ad\ (u_{k+1}^{(n+1)} u_k^{(n)})\ )(e_{k+1}) = e_k$.

(iii) $(Ad\ (u_k^{(n)} u_{k+1}^{(n)}))(e_k) \ = \ e_{k+1}$.

Proof: The first statement follows from the definition of the map $j_{[k-1,k+1]}$. As for the second, write $u_0 = (\theta_k^{(n)})^*$. Then, $u_0 = (*V)(\theta_{k+1}^{(n)})$ since $V(V(u)) = u$, and hence,

$$\begin{aligned}V(u_0) &= (V * V)(\theta_{k+1}^{(n)}) \\ &= *(\widetilde{\theta_{k+1}^{(n)}}) \\ &= *(\theta_{k+1}^{(n+1)})\end{aligned}$$

and so, $j_{[k-1,k+1]}(u_0) = u_k^{(n)^*}$ and $j_{[k,k+2]}(V(u_0)) = u_{k+1}^{(n+1)^*}$. It follows now from equation 4.2 that $(Ad\ (u_k^{(n)^*} u_{k+1}^{(n+1)^*})\ )(e_k) = e_{k+1}$, or equivalently, that $(Ad\ (u_{k+1}^{(n+1)} u_k^{(n)})\ )(e_{k+1}) = e_k$.

For the third assertion, put $u_1 = \theta_k^{(n)}$, note that $v_1 = V(u_1) = \theta_{k+1}^{(n)}$, and proceed as in the proof of (ii). □

DEFINITION 4.2.6 *Let w_k, u_k be constructed from u as in Theorem 4.2.1(a), and define $w^*_{[k,l]} = u^*_k u^*_{k-1} \cdots u^*_l$ for $k \geq l$. Also, define $\lambda_{(k,l)} = u_k^{(0)*} u_{k+1}^{(1)*} \cdots u_l^{(l-k)*}$ for $l \geq k$.*

LEMMA 4.2.7 *For $n \geq 0, k \geq 1$ and $l \geq m$ the following statements are valid:*

(i) $w^*_k \in U(A_{k+1})$ *and hence* $(Ad\ w^*_l)A_k = A_k$ *for all* $l < k$.

(ii) $u_{(k+1)} \in A'_k$ *and hence*

(a) $(Ad\ w_k)A_k = (Ad\ w_m)A_k$ *for all* $m > k$, *and*

(b) $(Ad\ w^*_m)A_k = (Ad\ w^*_{[m,k]})A_k$ *for all* $m > k$.

(iii) $(Ad\ \lambda_{(l,m)})(A_m) = (Ad\ \lambda_{(l,n)})(A_m)$ *for* $m \leq n$.

(iv) $(Ad\ \lambda_{(l,m)})(A_m) = (Ad\ \lambda_{(l,m)}\lambda_{(k,n)})(A_m)$ *for* $l \leq m$ *and* $k \leq n < m$.

Proof:

(i) The first statement is clear from the definition of w^*_k.
(ii) This statement follows from the fact that $u_{k+i}^{(n)}$'s commute with A_k, for $i \geq 1$.
(iii) Note that the unitaries $u_{m+k}^{(p)}$ commute with A_m for $k > 0$ and that proves (iii).
(iv) This statement follows from the fact that $u_{m-k}^{(p)}$ are in A_m for $k > 0$. □

In order to prove Proposition 4.2.2 we first fix a positive integer m and consider a finite grid of algebras $\{B_{n,k}, n, k = 0, \cdots m\}$ as follows:

(i) $B_{0,k} = A_k$ for $0 \leq k \leq m$.

(ii) $B_{1,k} = (Ad\, w^*_{[m,k+1]})(A_{k+1})$ for $0 \leq k \leq m$.

(iii) $B_{n,k} = (Ad\ (\lambda_{(m,m+n-1)}\lambda_{(m-1,m+n-2)} \cdots \lambda_{(k+1,k+n)}))(A_{n+k})$ for all $n > 1$ and $0 \leq k \leq m$.

LEMMA 4.2.8 *With the* $B_{n,k}$ *as above, we have :*

(i)

$$B_{m,k-1} \subseteq B_{m,k} \subseteq B_{m,k+1}$$

is an instance of the basic construction, and the Jones projection in $B_{m,k+1}$ *which implements the conditional expectation of* $B_{m,k}$ *onto* $B_{m,k-1}$ *is nothing but* e_k *, and this is valid independent of* m*; and*

(ii) if $0 \leq r < n \leq m$, $0 \leq l \leq m$ *are arbitrary, then*

$$B_{r,l} = (Ad\ (\lambda_{(m,m+n-1)}\lambda_{(m-1,m+n-2)} \cdots \lambda_{(l+1,l+n)}))(A_{l+r}).$$

Proof:

(i) First observe that

$$B_{m,k-1} \subseteq B_{m,k} \subseteq B_{m,k+1}$$

can be written as follows, using Lemma 4.2.7(iv):

$$(Ad\ \lambda_{(m,2m-1)}\lambda_{(m-1,2m-2)} \cdots \lambda_{(k,m+k-1)})\{A_{m+k-1} \subseteq A_{m+k} \subseteq A_{m+k+1}\}.$$

This is clearly an instance of a basic construction, with implementing Jones' projection being given by:

$$
\begin{aligned}
&(Ad\ (\lambda_{(m,2m-1)}\lambda_{(m-1,2m-2)}\cdots\lambda_{(k,m+k-1)}))e_{m+k} \\
&= (Ad\ ((u_m^{(0)*}u_{m+1}^{(1)*}\cdots u_{2m-1}^{(m-1)*})(u_{m-1}^{(0)*}u_m^{(1)*}\cdots u_{2m-2}^{(m-1)*})\cdots \\
&\qquad \cdots (u_k^{(0)*}\cdots u_{m+k-2}^{(m-2)*}u_{m+k-1}^{(m-1)*})))e_{m+k} \\
&= (Ad\ ((u_m^{(0)*}u_{m-1}^{(0)*}\cdots u_k^{(0)*})(u_{m+1}^{(1)*}u_m^{(1)*}\cdots u_{k+1}^{(1)*})\cdots \\
&\qquad \cdots (u_{2m-1}^{(m-1)*}\cdots u_{m+k}^{(m-1)*}u_{m+k-1}^{(m-1)*})))e_{m+k} \\
&= (Ad\ ((u_m^{(0)*}u_{m-1}^{(0)*}\cdots u_k^{(0)*})(u_{m+1}^{(1)*}u_m^{(1)*}\cdots u_{k+1}^{(1)*})\cdots \\
&\qquad \cdots (u_{2m-2}^{(m-2)*}\cdots u_{m+k-1}^{(m-2)*}u_{m+k-2}^{(m-2)*})))e_{m+k-1} \\
&\ \vdots \\
&= (Ad\ (u_m^{(0)*}\cdots u_{k+1}^{(0)*}u_k^{(0)*}))e_{k+1} \\
&= e_k.
\end{aligned}
$$

using Theorem 4.2.1(b), and the fact that e_l commutes with $u_q^{(p)}$ whenever $q > l+1$.

(ii) This follows from the fact that

$$
\begin{aligned}
&\lambda_{(m,m+n-1)}\lambda_{(m-1,m+n-2)}\cdots\lambda_{(l+1,l+n)} \\
&= (u_m^{(0)*}u_{m+1}^{(1)*}\cdots u_{m+n-1}^{(n-1)*})(u_{m-1}^{(0)*}u_m^{(1)*}\cdots u_{m+n-2}^{(n-1)*})\cdots(u_{l+1}^{(0)*}u_{l+2}^{(1)*}\cdots u_{l+n}^{(n-1)}) \\
&= (u_m^{(0)*}u_{m-1}^{(0)*}\cdots u_{l+1}^{(0)*})(u_{m+1}^{(1)*}u_m^{(1)*}\cdots u_{l+2}^{(1)*})\cdots(u_{m+n-1}^{(n-1)*}u_{m+n-2}^{(n-1)*}\cdots u_{l+n}^{(n-1)*})
\end{aligned}
$$

together with the fact that $u_q^{(p)} \in A_l'$ whenever $q > l$. □

Proof of Proposition 4.2.2:

Deduce first, using Lemma 4.2.8(ii) and Lemma 4.2.7, that for $0 \le k < m$, the grid

$$\begin{array}{ccccccc} A_{k+1} & \subseteq & A_{k+2} & & \cdots & \subseteq & A_{m+k+1} \\ \cup| & & \cup| & & & & \cup| \\ A_k & \subseteq & Ad\ \lambda_{[k+1,k+1]} A_{k+1} & & \cdots & \subseteq & Ad\ \lambda_{[k+1,k+m]} A_{k+m} \end{array}$$

is mapped onto the grid

$$\begin{array}{ccccccccc} B_{0,k+1} & \subseteq & B_{1,k+1} & \subseteq & B_{2,k+1} & \cdots \subseteq & B_{m-1,k+1} & \subseteq & B_{m,k+1} \\ \cup| & & \cup| & & \cup| & & \cup| & & \\ B_{0,k} & \subseteq & B_{1,k} & \subseteq & B_{2,k} & \cdots \subseteq & B_{m-1,k} & \subseteq & B_{m,k} \end{array}$$

by the map $Ad\,(\lambda_{(m,2m-1)}\lambda_{(m-1,2m-2)}\cdots\lambda_{(k+2,m+(k+1))})$. For $u = u^t_{k+1}$, using Lemma 4.2.5, the above grid is clearly of the type considered in Theorem 4.2.1. Since this is true for all $k \ge 1$ and for all $m \ge 2$, from Theorem 4.2.1 we have:

(i)

$$B_{n-1,k} \subseteq B_{n,k} \subseteq B_{n+1,k}$$

is an instance of the basic construction for all $1 \le n \le m$ and $0 \le k < m$; and

(ii) the Jones' projection implementing the above basic construction is independent of k.

Notice now that the grids $\{B_{n,k}\}_{0\leq n,k\leq m}$ and $\{A_{n,k}\}_{0\leq n,k\leq m}$ satisfy the following properties :

(a) any string of three algebras along any column of either grid is an instance of the basic construction, and the implementing Jones projection does not depend upon which column one started with; (for the $B_{n,k}$ -grid, this has just been observed - see (ii) above; while the $A_{n,k}$-grid has this property by construction;) and

(b) by definition of $B_{n,k}, k = 0, 1$, the grid $\{A_{n,k} : 0 \leq n \leq m, 0 \leq k \leq 1\}$ is mapped isomorphically onto the grid $\{B_{n,k} : 0 \leq n \leq m, 0 \leq k \leq 1\}$ by the map $Ad\ w_m^*$.

It follows that the grids $\{B_{n,k}\}_{0\leq n,k\leq m}$ and $\{A_{n,k}\}_{0\leq n,k\leq m}$ are isomorphic and, in particular,the following algebras satisfy the hypotheses of the Ocneanu Compactness theorem:

$$\begin{array}{ccccccccc}
A_{n,0} & \subseteq & A_{n,1} & \subseteq & A_{n,k} & \cdots \subseteq & A_{n,k+1} & \subseteq & \cdots \\
\cup\mid & & \cup\mid & & \cup\mid & & \cup\mid & & \\
A_{0,0} & \subseteq & A_{0,1} & \subseteq & A_{0,k} & \cdots \subseteq & A_{0,k+1} & \subseteq & \cdots
\end{array}$$

hence we have

$$R_0' \cap R_n = A_{0,1}' \cap A_{n,0}.$$

□

We again use Theorem 4.2.1 to compute $A'_{0,1} \cap A_{n,0}$. First, note that the commuting square

$$\begin{array}{ccc} A_{1,0} & \subseteq & A_{1,1} \\ \cup\!\!| & & \cup\!\!| \\ A_{0,0} & \subseteq & A_{0,1} \end{array}$$

is by definition

$$\begin{array}{ccc} A_1 & \subseteq & A_2 \\ \cup\!\!| & & \cup\!\!| \\ A_0 & \subseteq & Ad\,(u)A_1 \end{array}$$

which can be written as follows:

$$Ad\,(u)\left\{\begin{array}{ccc} Ad\,(u^*)A_1 & \subseteq & A_2 \\ \cup\!\!| & & \cup\!\!| \\ A_0 & \subseteq & A_1 \end{array}\right\}.$$

Therefore,

$$\begin{array}{ccccccccc} A_{0,1} & \subseteq & A_{1,1} & \subseteq & A_{2,1} & \cdots\subseteq & A_{n,1} & \subseteq & \cdots \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \\ A_{0,0} & \subseteq & A_{1,0} & \subseteq & A_{2,0} & \cdots\subseteq & A_{n,0} & \subseteq & \cdots \end{array}$$

is isomorphic to

$$\begin{array}{ccccccccc} A_1 & \subseteq & A_2 & \subseteq & A_3 & \cdots\subseteq & A_n & \subseteq & \cdots \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \\ A_0 & \subseteq & Ad\,(\tilde{w}_1)A_1 & \subseteq & Ad\,(\tilde{w}_2)A_2 & \cdots\subseteq & Ad\,(\tilde{w}_n)A_n & \subseteq & \cdots \end{array}$$

where $\tilde{w}_n$ is constructed out of u^* in exactly the same manner as w_n was constructed out of u (in Theorem 4.2.1(a)).

Therefore $R'_0 \cap R_n \simeq A'_{0,1} \cap A_{n,0} \simeq A'_1 \cap (Ad\,\tilde{w}_n)(A_n)$, where $\tilde{w}_n$ is as in the last sentence. (This is the description of the higher relative commutants that we shall use in the next section.)

4.3 Computation of principal graphs for the vertex model

We start here with the simple bipartite graph $\mathcal{G}$ consisting of two vertices with N bonds joining them, and consider biunitary matrices u associated with this graph. Thus, we assume that

$$\begin{array}{ccc} A_1 & \subseteq & A_2 \\ \cup\!| & & \cup\!| \\ A_0 & \subseteq & (Ad\ u)A_1 \end{array},$$

is a commuting square where $A_k \simeq M_{N^k}(\mathbb{C}), k = 0, 1, 2,$. We let $R_0 \subseteq R_1$ be the subfactor constructed in the usual fashion, as in Theorem 4.1.1, starting from this initial commuting square. Such a construction of subfactors of the hyperfinite II_1 factor is referred to as a 'vertex model' due to considerations from statistical mechanics.

In this section, we start with some special classes of biunitary matrices which give rise, as above, to vertex models, and use the machinery of the last section to compute the relative commutants $R_0' \cap R_n, n = 0, 1, 2, \cdots$, (where $R_0 \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots$ is the tower obtained from the basic construction applied to the initial inclusion $R_0 \subseteq R_1$) as well as the associated principal graph.

Thus, throughout this section, we fix an integer $N \geq 2$, and use the following notation: $A_n = M_N(\mathbb{C}) \otimes M_N(\mathbb{C}) \otimes \cdots \otimes M_N(\mathbb{C})$ (n-terms); and thus A_n is viewed as a subalgebra of A_{n+1} via $x \to x \otimes 1$ and $R = R_1$ is the von Neumann algebra completion of $\cup A_n$ with respect to the unique trace on $\cup A_n$.

When convenient, we think of A_n as $\mathcal{L}(\otimes^n \mathbb{C}^N)$, and describe elements of A_n by their matrix with respect to the basis

$\{\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_n} : 1 \leq i_1, \cdots, i_n \leq N\}$, where $\{\epsilon_1, \cdots, \epsilon_N\}$ is the standard basis of $\mathbb{C}^N$. Thus, for instance, there is a natural unitary representation $S_n \mapsto A_n$ given by $\sigma \mapsto F_\sigma$ where

$$F_\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\sigma^{-1}(1)} \otimes \cdots \otimes \xi_{\sigma^{-1}(n)}$$

for all $\xi_1, \cdots, \xi_n \in \mathbb{C}^N$. The inclusions $S_n \subseteq S_{n+1}$ and $A_n \subseteq A_{n+1}$ are consistent with respect to the above definitions and so we have a unitary representation $S_\infty \mapsto R$. It follows, for instance, that if $\sigma \in S_n$, then

$$(Ad\ F_\sigma)(x_1 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}\ \forall\, x_1 \cdots x_n \in M_N(\mathbb{C}). \quad (4.3)$$

It is fairly easy to see that if $\sigma \in S_\infty$ and $t \in A_n$, then $(Ad\ F_\sigma)t$ depends only on the n-tuple $(\sigma(1), \cdots, \sigma(n))$, and so we shall use the notation

$$t_{\sigma(1)\cdots\sigma(n)} = (Ad\ F_\sigma)t \quad \forall\ t \in A_n,\ \sigma \in S_\infty. \quad (4.4)$$

(Thus, for instance, if $t = x \otimes y$ with $x, y \in M_N(\mathbb{C})$, then $t_{41} = y \otimes 1 \otimes 1 \otimes x$; also, if $t \in A_n$, then $t = t_{12\cdots n}$.)

For most of this section, the symbols $u_{(1)}, \cdots, u_{(N)}$ wil denote a fixed (but entirely arbitrary) collection of N unitary elements of $M_N(\mathbb{C})$ and the symbol t will denote the element of A_2 defined by

$$t(\xi \otimes \epsilon_j) = u_{(j)}\xi \otimes \epsilon_j, \quad \forall\, \xi \in \mathbb{C}^N, 1 \leq j \leq N. \quad (4.5)$$

For instance, if $N = 2$, $u_{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $u_{(2)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the matrix of t with respect to the ordered orthonormal basis $\{\epsilon_1 \otimes \epsilon_1, \epsilon_2 \otimes \epsilon_1, \epsilon_1 \otimes \epsilon_2, \epsilon_2 \otimes \epsilon_2\}$ of $\otimes^2 \mathbb{C}^N$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

(It is also true, for general n and arbitrary choices of the $u_{(i)}$'s, that the matrix of t with respect to the ordered orthonormal basis $\{\epsilon_1 \otimes \epsilon_1, \cdots, \epsilon_N \otimes \epsilon_1; \epsilon_1 \otimes \epsilon_2, \cdots, \epsilon_N \otimes \epsilon_2; \cdots; \epsilon_1 \otimes \epsilon_N, \cdots, \epsilon_N \otimes \epsilon_N\}$ is the direct sum of the matrices $u_{(i)}$.)

This section is devoted to a proof of the following theorems. (In both results the symbol t has the meaning given by equation 4.5.)

THEOREM 4.3.1 *Let t be as before; assume $u_{(1)}$ is the identity matrix. Let $u = t_{12}F_{(12)} \in A_2$. Then u is biunitary, and if $R_0 \subseteq R_1 = R$ is the subfactor as constructed in §4.1, then the principal graph $\mathcal{G}$ corresponding to the tower $\{R_0' \cap R_n : n \geq 0\}$ of relative commutants has the following description :*

Let G be the closed subgroup of $U(N)$ generated by $u_{(2)}, \cdots u_{(N)}$. Consider the matrix Λ whose rows and columns are indexed by the set $\hat{G}$ of distinct (inequivalent) irreducible representations of G, defined by setting Λ_{ij} equal to the multiplicity with which the representation j features in the tensor-product of i and π, where π denotes the standard representation of G on $\mathbb{C}^N$. Let $\tilde{\mathcal{G}}$ denote the bipartite graph, with even vertices indexed by $\hat{G} \times \{0\}$ and odd vertices indexed by $\hat{G} \times \{1\}$, with adjacency matrix Λ. Let $\mathcal{G}$ denote the connected component of $\tilde{\mathcal{G}}$ containing $(*, 0)$, where $*$ denotes the trivial representation of G. Then $\mathcal{G}$ is the desired principal graph, with $(*, 0)$ as the distinguished vertex.

THEOREM 4.3.2 *Let* $u = F_{(12)}t_{12} \in A_2$. *Then* u *is biunitary and the principal graph* $\mathcal{G}$ *corresponding to the tower* $\{R_0' \cap R_n : n \geq 0\}$ *of relative commutants has the following description:*

Let $\tilde{G}$ *be the (non-closed) subgroup of* $U(N)$ *generated by* $u_{(1)}, \cdots, u_{(N)}$, *and let* $G = \tilde{G}/\tilde{G} \cap \{\omega.1_{\tilde{G}} : \omega \in \mathbb{C}, |\omega| = 1\}$. *Consider the matrix* Λ *with rows and columns indexed by* G, *where* $\Lambda_{[s_1][s_2]}$ *is defined to be equal to the number of* $j, 1 \leq j \leq N$ *such that* $[s_2] = [s_1][u'_{(j)}]$. *(Here we write* $[s]$ *for the element of* G *corresponding to the element* s *of* $\tilde{G}$.*) Let* $\tilde{\mathcal{G}}$ *be the bipartite graph, with even vertices indexed by* $G \times \{0\}$ *and odd vertices indexed by* $G \times \{1\}$, *with adjacency matrix given by* Λ. *Let* $\mathcal{G}$ *be the connected component of* $(*, 0)$, *where* $*$ *is the identity element of* G. *Then* $\mathcal{G}$ *is the desired principal graph, with* $(*, 0)$ *as the distinguished vertex.*

Before proceeding to the proof of the theorems, let us pause to interpret the notion of biunitarity in the context of the examples of this section.

To be precise, we have $A_0 = \mathbb{C} \subseteq A_1 = M_N(\mathbb{C}) \otimes 1 \subseteq A_2 = M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$; a unitary element $u \in A_2$ is a 'biunitary' matrix (as per the discussion of §4.1) precisely when the element $v = V(u) \in A_2$ defined by $v_{kl}^{ij} = \overline{u_{ki}^{lj}}$ is also unitary. (Here, we write $a_{kl}^{ij} = < a(e_k \otimes e_l), e_i \otimes e_j >$ for $a \in A_2$.) For such a biunitary u, the matrices $\tilde{u}, \tilde{v} \in A_2$ are also unitary, where $\tilde{u}_{kl}^{ij} = u_{ji}^{lk}$ and $\tilde{v}_{kl}^{ij} = \overline{u_{jl}^{ik}}$. Note that if $F = F_{(12)} \in A_2$, then $F_{kl}^{ij} = \delta_l^i \delta_k^j$, and in particular, it follows that $\tilde{u} = Fu'F$ (and $\tilde{v} = Fv'F$).

Now we proceed to prove the theorems stated above.

Proof of Theorem 4.3.1:

If $u = tF$ as in the statement of the theorem, observe that

$$\begin{aligned}
(u^*)^{ij}_{kl} &= \; < Ft^*(\epsilon_k \otimes \epsilon_l), \epsilon_i \otimes \epsilon_j > \\
&= \; < t^*(\epsilon_k \otimes \epsilon_l), \epsilon_j \otimes \epsilon_i > \\
&= \; < u^*_{(l)}\epsilon_k \otimes \epsilon_l, \epsilon_j \otimes \epsilon_i > \\
&= \; \delta^i_l \, (u^*_{(l)})^j_k
\end{aligned}$$

and hence

$$V(u^*)^{ij}_{kl} = \overline{u^*}^{lj}_{ki} = \delta^l_i \, \overline{(u^*_{(i)})}^j_k = \delta^l_i \, (u'_{(i)})^j_k = \widetilde{V(u)}'^{ij}_{kl} \tag{4.6}$$

from which it is seen that $V(u^*) = Ft'$; hence $V(u^*)$ is unitary and u is a biunitary matrix.

We henceforth denote $V(u)$ by v for convenience.

(In the following, we shall have to deal with expressions such as $x_1 x_2^* x_3 x_4^* \cdots x_n^{(*)}$, where $x_n^{(*)}$ is x_n or x_n^* according as n is odd or even. Similarly, we will use such expressions as $x_1 \bar{x}_2 x_3 \bar{x}_4 \cdots x_n^{(-)}$.)

Continuing with the notation of the last section (see the last paragraph of §4.1), we see that

$$\begin{aligned}
\tilde{w}_n &= u^*_{12} \tilde{v}^*_{23} u^*_{34} \cdots (n \text{ terms}) \\
&= F_{12} t^*_{12} F_{23} t'_{23} \cdots F_{n,n+1} {t^*_{n,n+1}}^{(-)} \\
&= F_{12} F_{23} \cdots F_{n,n+1} t^*_{1,n+1} t'_{2,n+1} \cdots {t^*_{n,n+1}}^{(-)};
\end{aligned}$$

i.e.,

$$\tilde{w}_n = F_{(12\cdots(n+1))} t_{12\cdots(n+1)},$$

$$\begin{aligned}
t_{12\cdots(n+1)} &= t^*_{1,n+1} t'_{2,k+1} \cdots {t^*_{n,n+1}}^{(-)} \\
F_{(12\cdots(n+1))} &= F_{12} F_{23} \cdots F_{n,n+1}.
\end{aligned}$$

It is seen from equation 4.5 that

$$t_{12\cdots(n+1)}(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n \otimes \epsilon_j) = u_{(j)}^* \xi_1 \otimes u'_{(j)} \xi_2 \otimes \cdots \otimes u^{(\prime)}_{(}j)^{(-)} \xi_n \otimes \epsilon_j \quad (4.7)$$

for all $\xi_1, \cdots, \xi_n \in \mathbb{C}^N, 1 \leq j \leq N$, and $n = 1, 2, \cdots$.

It follows from the analysis of the last section that if $C_n = R'_0 \cap R_n$, then

$$\begin{aligned} C_n &= A'_{0,1} \cap A_{n,0} \\ &= A'_1 \cap ((Ad\ \tilde{w}_n)(A_n)) \\ &= A'_1 \cap A_{n+1} \cap ((Ad\ (F_{(12\cdots(n+1))} t_{12\cdots(n+1)}))(A_n)) \\ &= (Ad\ F_{(12\cdots(n+1))})(((Ad F^*_{(12\cdots(n+1))})(A'_1 \cap A_{n+1})) \cap ((Ad\ t_{12\cdots(n+1)})A_n)) \\ &= (Ad\ F_{(12\cdots(n+1))})(A_n \cap (Ad t_{12\cdots(n+1)})(A_n)). \end{aligned}$$

Hence $Ad\ F_{(12\cdots(n+1))}$ maps $\tilde{C}_n$ isomorphically to C_n where

$$\begin{aligned} \tilde{C}_n &= A_n \cap (Ad\ t_{12\cdots(n+1)})(A_n) \\ &= \{z \in A_n : \exists x \in A_n \ni (Ad(t_{12\cdots(n+1)})^*)(z \otimes 1) = x \otimes 1\} \end{aligned}$$

It follows easily from equation 4.7 that for each $n = 1, 2 \cdots$, the unitary operator $t_{12\cdots(n+1)}$ (and hence also its adjoint operator $t^*_{12\cdots(n+1)}$) on $\otimes^{n+1}\mathbb{C}^N$ leaves each subspace of the form $(\otimes^n \mathbb{C}^N) \otimes \epsilon_j, 1 \leq j \leq N$, invariant, so that there exists an operator $\tau^{(j)}_{n+1}$ on $\otimes^n \mathbb{C}^N$ such that

$$t^*_{12\cdots(n+1)}(\xi \otimes \epsilon_j) = \tau^{(j)}_{n+1} \xi \otimes \epsilon_j \quad (4.8)$$

for all $\xi \in \otimes^n \mathbb{C}^N$ and $1 \leq j \leq N$. In fact, it is seen from equation 4.7 that

$$\tau^{(j)}_{n+1} = u_j \otimes \bar{u}_j \cdots \otimes u_j^{(-)}. \quad (4.9)$$

So it follows that if $z \in A_n$, then

$$(t^*_{12\cdots(n+1)}(z \otimes 1) t_{12\cdots(n+1)})(\xi \otimes \epsilon_j) = \tau^{(j)}_{n+1} z \tau^{(j)*}_{n+1} \xi \otimes \epsilon_j \quad (4.10)$$

for all $\xi \in \otimes^n \mathbb{C}^N$ and $1 \leq j \leq N$; hence, $(Ad\ t^*_{12\cdots(n+1)})(z \otimes 1) = x \otimes 1$ for some $x \in A_n$ if and only if $\tau^{(1)}_{n+1} z \tau^{(1)*}_{n+1} = \tau^{(2)}_{n+1} z \tau^{(2)*}_{n+1} = \cdots = \tau^{(N)}_{n+1} z \tau^{(N)*}_{n+1}$.

Thus we find that $z \in \tilde{C}_n \Leftrightarrow z \in A_n$ and z commutes with $\tau^{(1)*}_{n+1}\tau^{(j)}_{n+1}$ for $1 \leq j \leq N$.

It follows from equation 4.9 that

$$\tau^{(1)*}_{n+1}\tau^{(j)}_{n+1} = (u_{(j)} \otimes \overline{u_{(j)}} \otimes u_{(j)} \otimes \overline{u_{(j)}} \otimes \cdots \otimes u^{(-)}_{(j)})$$

We deduce finally that if $z \in A_n$, then

$$z \in \tilde{C}_n \Leftrightarrow z \in \{\overbrace{u_{(j)} \otimes \overline{u_{(j)}} \otimes u_{(j)} \otimes \overline{u_{(j)}} \otimes \cdots \otimes u^{(-)}_{(j)}}^{n} : 2 \leq j \leq N\}';$$

in other words, if G is the closed subgroup of $U(N)$ generated by $u_{(2)}, \cdots u_{(N)}$, then

$$\tilde{C}_n = (\overbrace{\pi \otimes \bar{\pi} \otimes \pi \otimes \bar{\pi} \otimes \cdots \otimes \pi^{(-)}}^{n})(G)' \tag{4.11}$$

where π denotes the standard representation of G on $\mathbb{C}^N$.

Note now that $z \mapsto z \otimes 1$ defines an embedding $\tilde{C}_n \hookrightarrow \tilde{C}_{n+1}$ and clearly, the following is a commutative diagram:

$$\begin{array}{ccc} \tilde{C}_{n+1} & \overset{Ad\ F_{12\cdots(n+2)}}{\rightarrow} & C_{n+1} \\ \cup\!| & & \cup\!| \\ \tilde{C}_n & \overset{Ad\ F_{12\cdots(n+1)}}{\rightarrow} & C_n \end{array}$$

Hence the tower $\{C_n : n \geq 0\}$ is isomorphic to the tower $\{\tilde{C}_n : n \geq 0\}$, and it is fairly easy now to deduce, from the description of $\tilde{C}_n$ as $(\pi \otimes \bar{\pi} \otimes \cdots \otimes \pi^{(-)})(G)'$, that the principal graph is described as stated in the theorem.

Proof of Theorem 4.3.2:

If $s = (Ad\ F_{(12)})(t)$, it must be clear that $s(\epsilon_i \otimes \xi) = \epsilon_i \otimes u_{(i)}\xi$ for all $\xi \in \mathbb{C}^N, 1 \leq i \leq N$, and that $u = Ft = sF$, so that

$$\begin{aligned} u^{ij}_{kl} &= < sF(\epsilon_k \otimes \epsilon_l), \epsilon_i \otimes \epsilon_j > \\ &= < \epsilon_l \otimes u_{(l)}\epsilon_k, \epsilon_i \otimes \epsilon_j > \\ &= \delta^i_l (u_{(l)})^j_k \end{aligned}$$

and hence

$$u^{*ij}_{kl} = \delta^j_k (u^*_{(k)})^i_l. \tag{4.12}$$

Hence $(V(u^*))^{ij}_{kl} = \bar{u}^{*lj}_{ki} = \delta^k_j (u_{(j)})^i_l$, so that $V(u^*) = Fs$. It follows that u is a biunitary matrix.

Arguing exactly as in the proof of Theorem 4.3.1, we see that

$$\tilde{w}_n = F_{(12\cdots n+1)} s_{12\cdots n+1}, \tag{4.13}$$

where

$$s_{12\cdots n+1} = s^*_{1,n+1} s_{2,n+1} s^*_{3,n+1} \cdots s^{(*)}_{n,n+1}. \tag{4.14}$$

From the fact that $s(\epsilon_i \otimes \xi) = \epsilon_i \otimes u_{(i)}\xi$ for all $\xi \in \mathbb{C}^N, 1 \leq i \leq N$, we have

$$s_{12\cdots n+1}(\epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \cdots \otimes \epsilon_{i_n} \otimes \xi) = \epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \cdots \otimes \epsilon_{i_n} \otimes u^*_{i_1} u_{i_2} \cdots u^{(*)}_{i_n} \xi. \tag{4.15}$$

As before, we see that if $C_n = R_0' \cap R_n$, then $C_n = (Ad\ F_{(12\cdots(n+1))})(\tilde{C}_n)$, where

$$\begin{aligned} \tilde{C}_n &= A_n \cap (Ad\ s_{12\cdots n+1})(A_n) \\ &= \{z \in A_n : \exists x \in A_n \ni (Ad\ s^*_{12\cdots(n+1)})(z \otimes 1) = x \otimes 1\}. \end{aligned}$$

For typographical convenience, let us use the following notation : for any positive integer n and any integers $1 \leq i_1, \cdots, i_n \leq N$, we write $\mathbf{i} = (i_1, \cdots, i_n)$, $e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_n}$, $u_{\mathbf{i}} = u^{*}_{(i_1)} u_{(i_2)} \cdots u^{(*)}_{(i_n)}$.

With the above notation, we see that if $z \in A_n$ and $1 \leq i, j \leq N$, an easy computation shows that for all multi-indices $\mathbf{i}, \mathbf{j}$ of size n,

$$< ((Ad\ s^{*}_{12\cdots n+1})(z \otimes 1))(e_{\mathbf{i}} \otimes e_i, e_{\mathbf{j}} \otimes e_j >=< ze_{\mathbf{i}}, e_{\mathbf{j}} >< u_{\mathbf{i}} e_i, u_{\mathbf{j}} e_j > . \quad (4.16)$$

It follows from the above equation that $z \in \tilde{C}_n$ if and only if $z \in A_n$ and $< ze_{\mathbf{i}}, e_{\mathbf{j}} >= 0$ unless $u_{\mathbf{i}}$ and $u_{\mathbf{j}}$ are scalar multiples of one another.

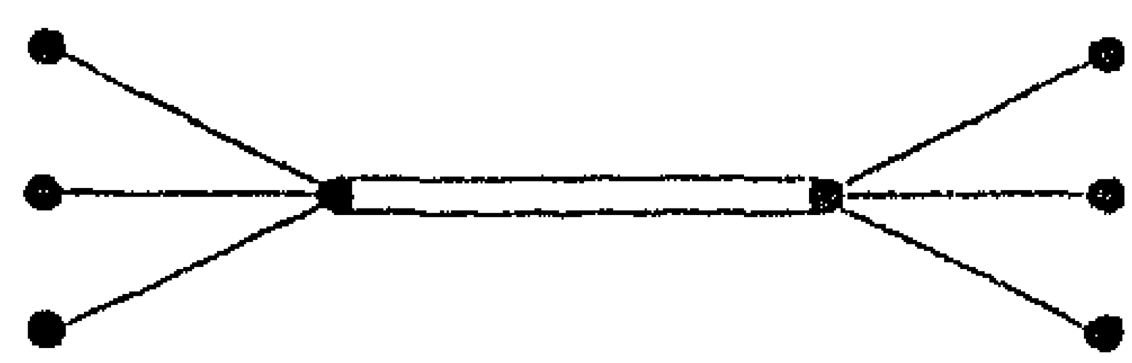
Notice, from the description of $\tilde{C}_n$, that $z \in \tilde{C}_n \Rightarrow z \otimes 1 \in \tilde{C}_{n+1}$; i.e. $\tilde{C}_n \hookrightarrow \tilde{C}_{n+1}$ via the embedding $z \mapsto z \otimes 1$. A moment's reflection should convince the reader that the Bratteli diagram of the tower $\{\tilde{C}_n\}_{n \geq 0}$ is described by the Cayley graph of the group, modulo scalars, that is generated by $u^{*}_{(1)}, \cdots, u^{*}_{(N)}$ - as explained in the statement of Theorem 4.3.2.

To complete the proof of the theorem, we only need to observe that the Bratteli diagrams for the towers $\{C_n\}_{n \geq 0}$ and $\{\tilde{C}_n\}_{n \geq 0}$ are the same, which is clear from the nature of the isomorphism $Ad\ F_{(12\cdots(n+1))}$, which maps $\tilde{C}_n$ to C_n. □

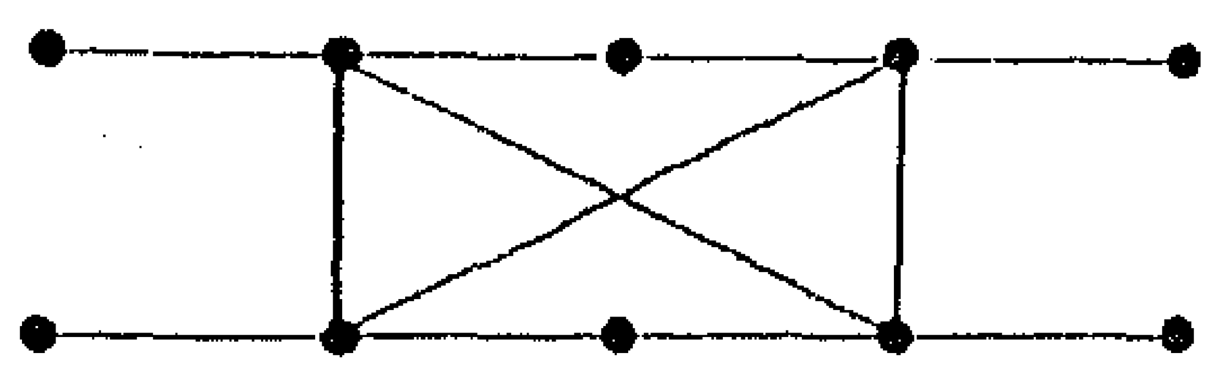
We conclude this section with a few examples of principal graphs of subfactors constructed as in Theorem 4.3.1.

Consider the finite subgroups of $SO(3)$ corresponding to the symmetries of the platonic solids - viz. A_4 (tetrahedron), S_4 (cube/octahedron) and A_5 (icosahedron/dodecahedron). Each of these groups is generated by two elements - to be specific, by $\{(12)(34), (123)\}$, $\{(12), (1234)\}$ and $\{(12)(34), (12345)\}$ respectively. If we let $u_{(2)}$ and $u_{(3)}$ denote the images of the two generators under the standard representation on $\mathbb{C}^3$, we find that the three subfactors of $\otimes^\infty M_3(\mathbb{C})$, obtained as in Theorem 4.3.1, have principal graphs which are given as follows:

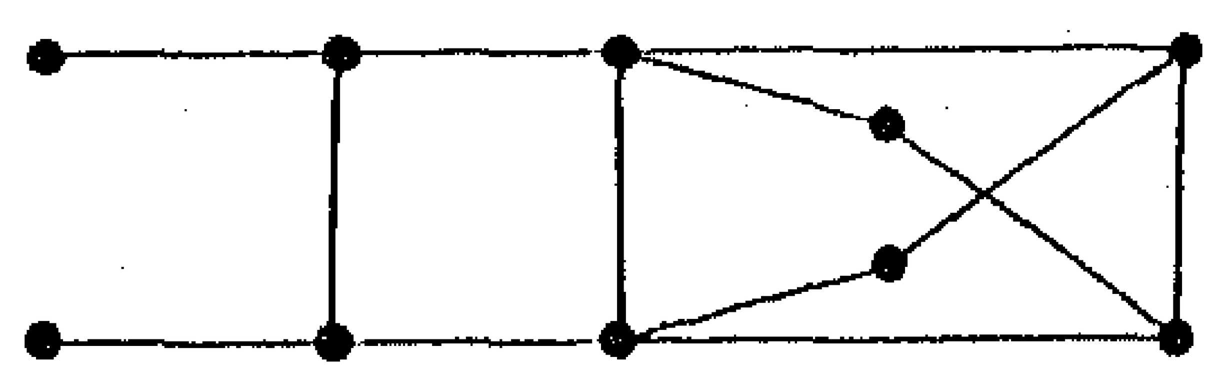
A_4:



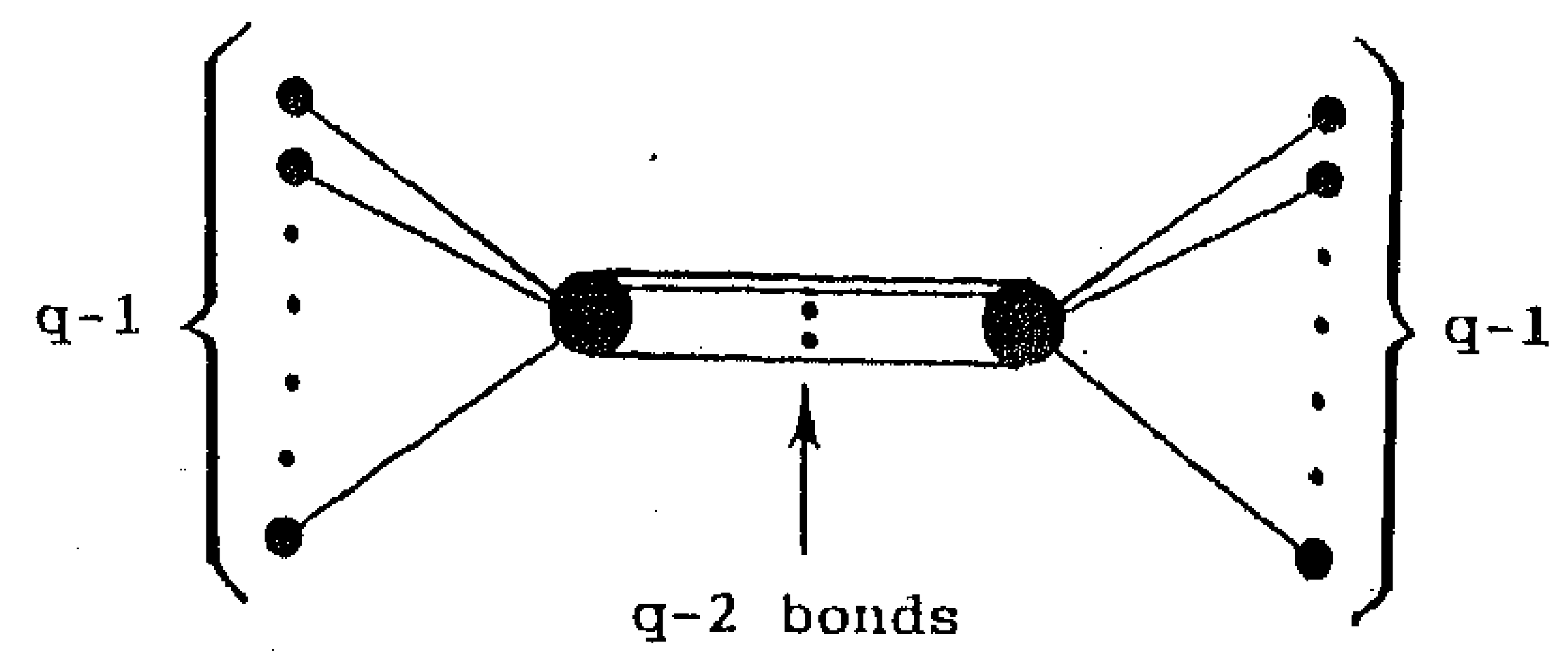
S_4:



A_5:



There is another class of examples that generalise the A_4 example. Let F be the finite field of order $q = p^n, p$ a prime, with $q \geq 3$; the additive group $(F, +)$ is isomorphic to $\underbrace{Z_p \times \cdots Z_p}_{n}$, while the multiplicative group $F^\times$, being cyclic, is isomorphic to Z_{q-1}. Thus Z_{q-1} acts on Z_p^n, via scalar multiplication, in such a way that there are only two orbits, $\{0\}$ and $F \backslash \{0\}$. It follows that if $\tilde{G}$ is the semi-direct product of Z_p^n and Z_{q-1}, then $\tilde{G}$ is generated by two elements - namely, any non-zero element α of Z_p^n and a generator σ of Z_{q-1}. It follows from the general Mackey theory that $\tilde{G}$ has $(q-1)$ distinct 1-dimensional representations and one $(q-1)$-dimensional irreducible representation π. If we now set $G = \pi(\tilde{G})$, then $\tilde{G}$ is a subgroup of $U(q-1)$ which is generated by $u_{(2)} = \pi(\alpha), u_{(3)} = \pi(\sigma), u_{(4)} = \cdots = u_{(q-1)} = 1$. If we now define the biunitary u using these $u_{(j)}$'s as in Theorem 4.3.1, it is not hard to see that the principal graph of the subfactor so obtained is as follows:



4.4 An equivalence relation on biunitary matrices

In this section we restrict ourselves to vertex models as in §4.2. We introduce an equivalence relation on biunitary matrices which preserves the resulting factor, subfactor pair (R_0, R_1) up to isomorphism. In particular, we show that all 4×4 biunitaries can be reduced by means of this equivalence to a 'canonical form' using which all possible principal graphs obtainable from the vertex model can be characterised.

We use the convention of writing elements of $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ as elements of $M_{N^2}(\mathbb{C})$. The ordering of basis vectors in $\mathbb{C}^N \otimes \mathbb{C}^N$ is such that in block form $a \otimes b = ((b_{ij}\, a))$.

Consider a unitary $N^2 \times N^2$ matrix u written in block form as $u = ((x_{ij}))$

where x_{ij} is an $N \times N$ matrix for $i, j = 1, 2, \cdots N$. Then the following conditions are equivalent:

(i) u is a biunitary matrix (i.e. $V(u)$ as defined in §4.1 is also unitary).

(ii) The x_{ij}'s form an orthonormal basis (with respect to the Hilbert-Schmidt inner product) for $M_{N^2}(\mathbb{C})$.

(iii) $\sum_{i,j} x_{ij}\ y\ x_{ij}^* = \mathrm{Tr}(y) 1_{N\times N}$ for any $y \in M_N(\mathbb{C})$.

Given a biunitary $N^2 \times N^2$ matrix u, consider the matrix $u_0 = (a \otimes b)u(c \otimes d)$, where $a, b, c, d \in U(N)$; then u_0 is a biunitary matrix since it can be easily verified that

$$V(u_0) = (d^* \otimes \bar{b})V(u)(\bar{c} \otimes a^*).$$

We therefore define the following equivalence relation in the set of biunitary matrices :

$$u_1 \sim u_2 \text{ if and only if } u_1 = (a \otimes b)u_2(c \otimes d) \text{ for some } a, b, c, d \text{ in } U(N). \tag{4.17}$$

Recall from §4.2 that if (R_0, R_1) are obtained from the commuting square

$$\begin{array}{ccc} A_1 & \overset{G'}{\subset} & A_2 \\ {}_G\cup & & \cup_{G'} \\ A_0 & \overset{G}{\subset} & uA_1u^* \end{array},$$

then there exists an endomorphism α_u of R_1 such that $R_0 = \alpha_u(R_0)$. This endomorphism is defined by

$$\alpha_u(x) = \lim_{n\to\infty}(Ad\ w_n)(x) \ \ \forall x \in R_1.$$

What makes the equivalence relation 4.17 useful is the following:

Assertion : Equivalent biunitary matrices yield conjugate subfactors.

Proof : Let $w_n(u)$ denote the w_n associated to the biunitary matrix u as in Theorem 4.2.1; then an easy computation shows that

$$w_n((a\otimes b)u(c\otimes d)) = \begin{cases} a_1(b_2\bar{b}_3b_4\bar{b}_5\cdots\bar{b}_{n+1})w_n(u)(c_1\bar{c}_2c_3\bar{c}_4\cdots\bar{c}_n)a^*_{n+1} & (n \text{ even}) \\ a_1(b_2\bar{b}_3b_4\bar{b}_5\cdots b_{n+1})w_n(u)(c_1\bar{c}_2c_3\bar{c}_4\cdots c_n)d_{n+1} & (n \text{ odd}) \end{cases} \tag{4.18}$$

(where the subscripts have the interpretation described in §4.3.)

Consequently,

$$\alpha_{(a\otimes b)u(c\otimes d)} = \theta_1 \circ \alpha_u \circ \theta_2$$

where θ_1, θ_2 are the automorphisms of R_1 defined by

$$\begin{aligned} \theta_1(x) &= \lim_{n\to\infty}(Ad\ (a_1(b_2\bar{b}_3b_4\bar{b}_5\cdots\bar{b}_{2n+1}))\ )\ (x) \\ \theta_2(x) &= \lim_{n\to\infty}(Ad\ ((c_1\bar{c}_2c_3\bar{c}_4\cdots\bar{c}_n)a^*_{2n+1})\ )\ (x) \end{aligned}$$

for all $x \in R_1$. In particular θ_1 is an automorphism of R_1 which maps the subfactor $\alpha_u(R_1)$ onto the subfactor $\alpha_{(a\otimes b)u(c\otimes d)}(R_1)$.

As a consequence we only have to look at a restricted collection of (inequivalent) biunitary matrices. In the two dimensional situation ($N = 2$ in the notation of §4.2), it turns out that inequivalent biunitaries are parametrized by the circle.

PROPOSITION 4.4.1 *Any 4×4 biunitary is equivalent to one of the form*

$$u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta \end{bmatrix} \tag{4.19}$$

with $|\eta| = 1$.

Proof : Following the notation introduced earlier, we write

$$u = ((x_{ij}))_{i,j=1,2}$$

(i) We may, without loss of generality, assume that x_{11} has rank one.

Reason : If x_{11} is non-singular, premultiply u by a matrix of the form $1 \otimes \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $(-\alpha/\beta)$ equal to an eigenvalue of $x_{11}^{-1}x_{12}$. This ensures that the new x_{11} is singular, hence rank 1 (since the biunitarity condition forces x_{11} to have Hilbert-Schmidt *norm* 1.)

(ii) We may, without loss of generality, assume that $x_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Reason : Pre- and post-multiply by $a \otimes 1$ and $b \otimes 1$ with suitable a and b.

(iii) Using biunitarity and by pre- and post-multiplying by allowed diagonal matrices it is easy to see that u can be reduced to the form tF or Ft (see §4.2) with $u_{(1)} = 1$ and $u_{(2)} \in SU(2)$.

Consider the case $u = tF$. Let $\lambda, \bar{\lambda}$ be the eigenvalues of $u_{(2)}$ and let a be a unitary matrix that diagonalizes $u_{(2)}$. Let b be the matrix $diag(1, \bar{\lambda})$. Pre-multiplying u by $(a \otimes b)$ and post-multiplying by $(a^* \otimes 1)$ reduces u to the form displayed in equation 4.19.

A similar analysis applies to the case $u = Ft$.

In view of the above proposition, the principal graphs of all subfactors of index 4 constructed from the above 'vertex model' prescription are described by Theorem 4.3.1. In particular, since any singly generated closed subgroup of $U(2)$ is abelian, no such subfactor is irreducible.

It follows from [P1] that the only possible principal graphs for reducible subfactors of index 4 of the hyperfinite factor are $A^{(1)}_{2n-1}$ and $A^{(1)}_{\infty}$. These graphs are obtainable from the above vertex model corresponding to a biunitary u in the 'canonical form' of Proposition 4.4.1, when η is of order n or ∞.

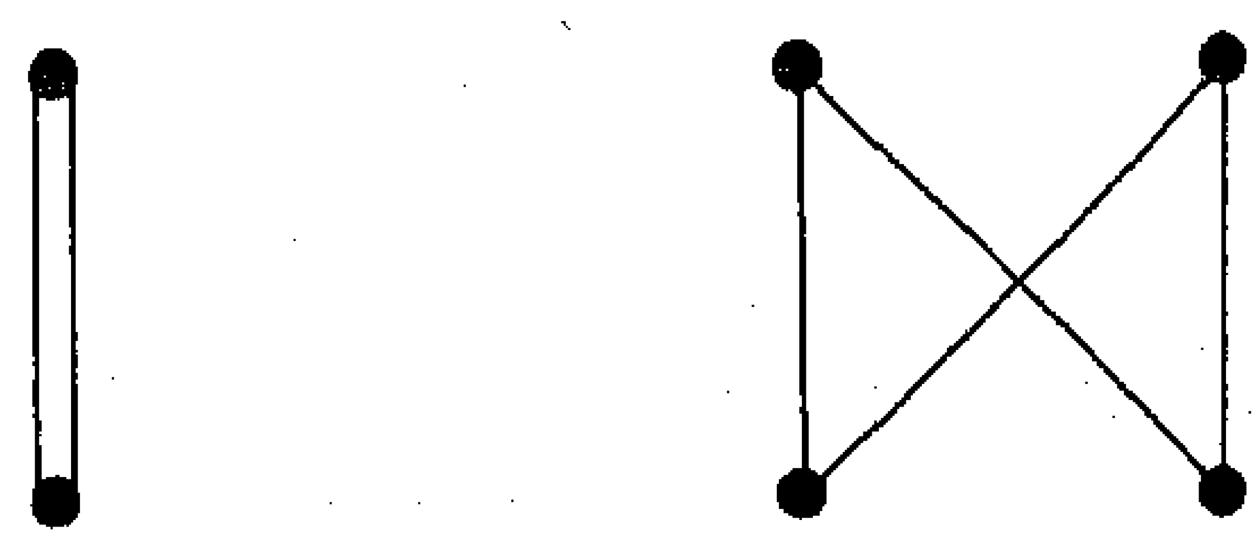
4.5 Biunitary permutation matrices

Lastly, we would like to comment on a class of commuting squares which have been examined in some detail in [KS]. These are vertex models given by biunitary matrices, which also happen to be permutation matrices, i.e., commuting squares of the form:

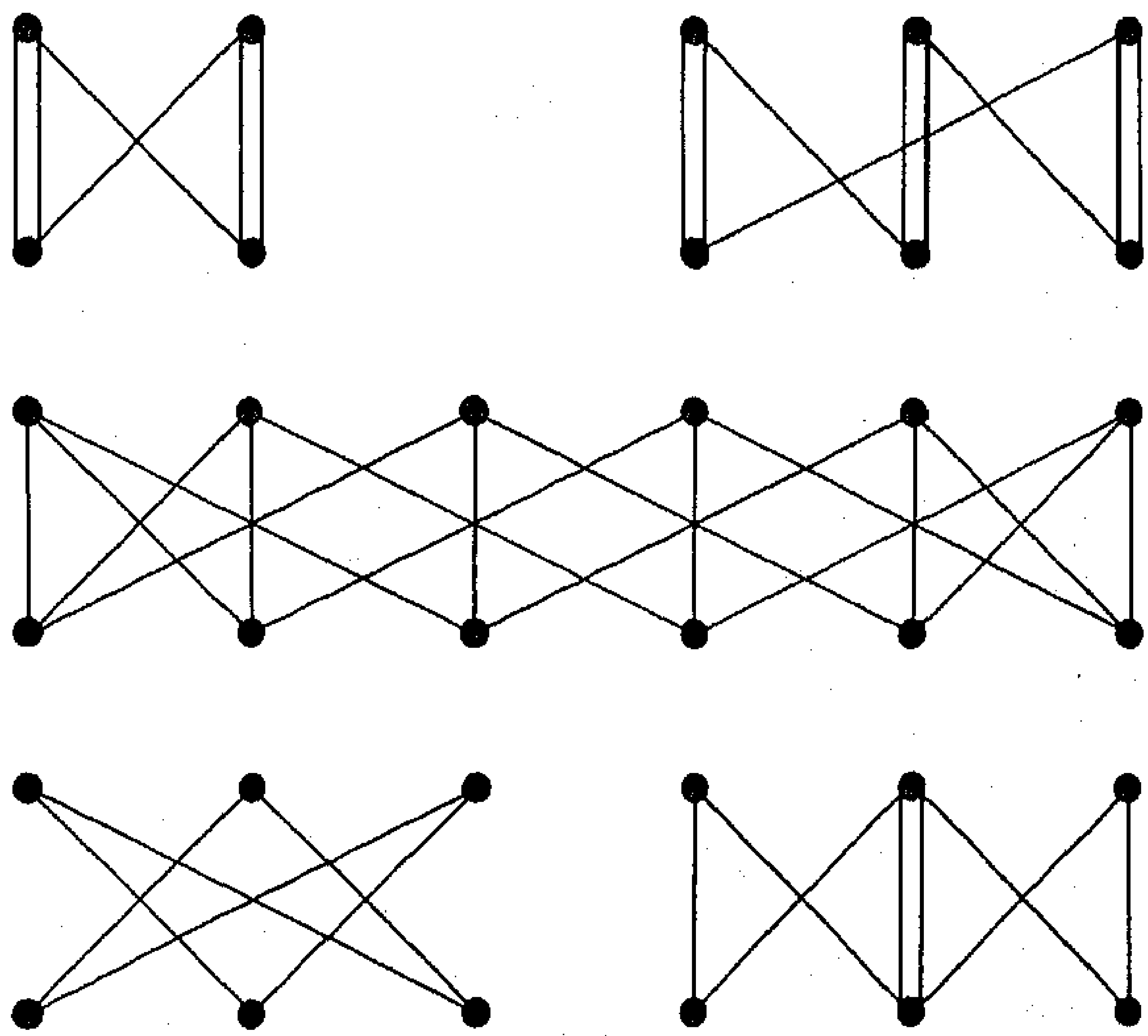
$$\begin{array}{ccc} M_N(\mathbb{C}) \otimes 1 & \subseteq & M_N(\mathbb{C}) \otimes M_N(\mathbb{C}) \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathbb{C} & \subseteq & u(M_N(\mathbb{C}) \otimes 1)u^* \end{array}$$

where u is a permutation matrix in $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$.

Of special interest are the cases when $N \leq 3$. For $N = 2$, all the possible biunitary permutation matrices are of the kind discussed in §4.2 and hence the principal graphs can be computed using Theorems 4.3.1 and 4.3.2, and these are:

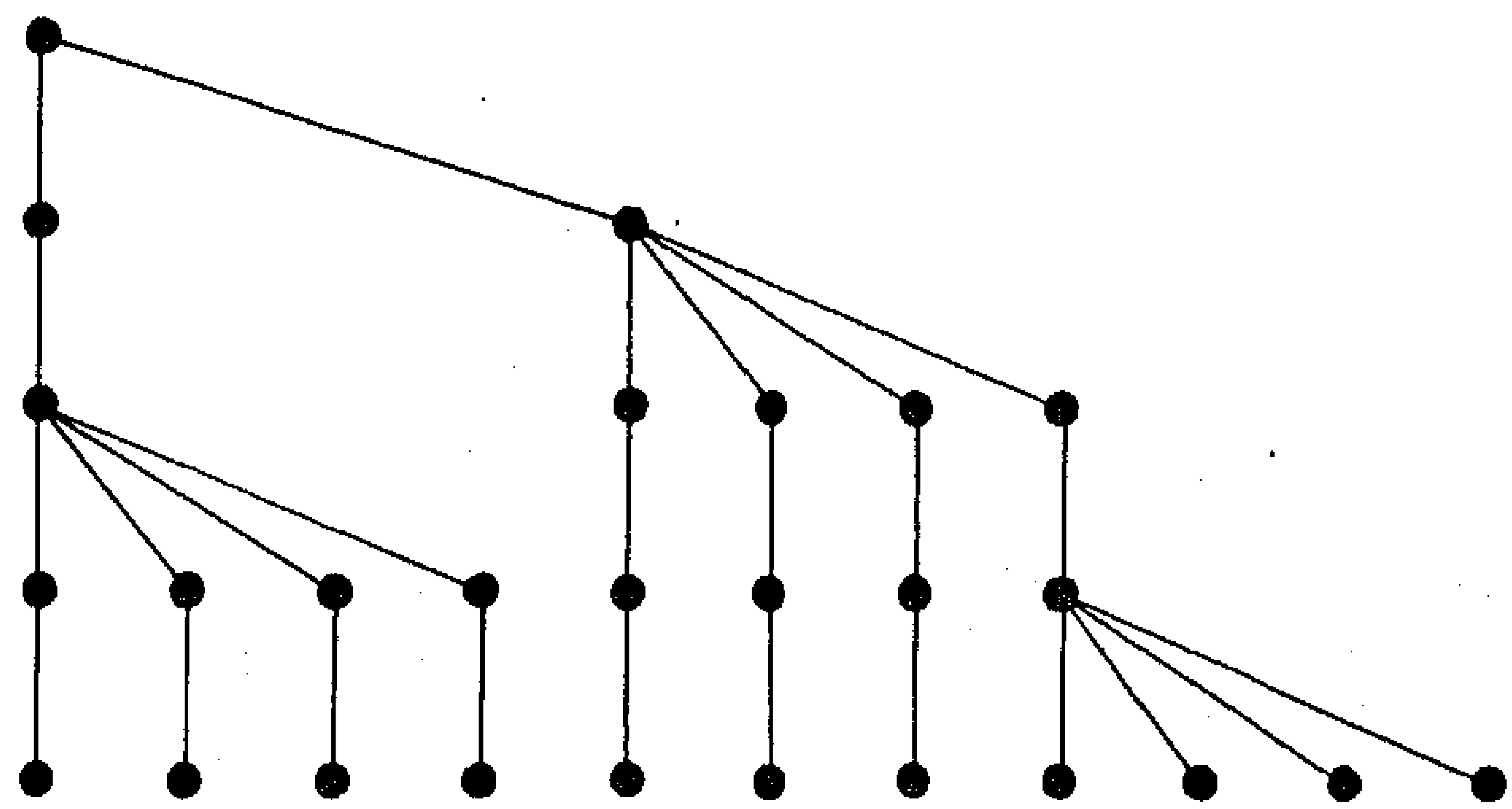


For $N = 3$, it is found that there are 18 eqivalence classes of biunitary permutation matrices. Of these, 15 cases yield finite principal graphs, and they are of the kind described in Theorems 4.3.1 and 4.3.2, i.e.,the Cayley graph of a group or a group dual corresponding to a finite subset of the group. Given below is a sample of the principal graphs arising from these cases:

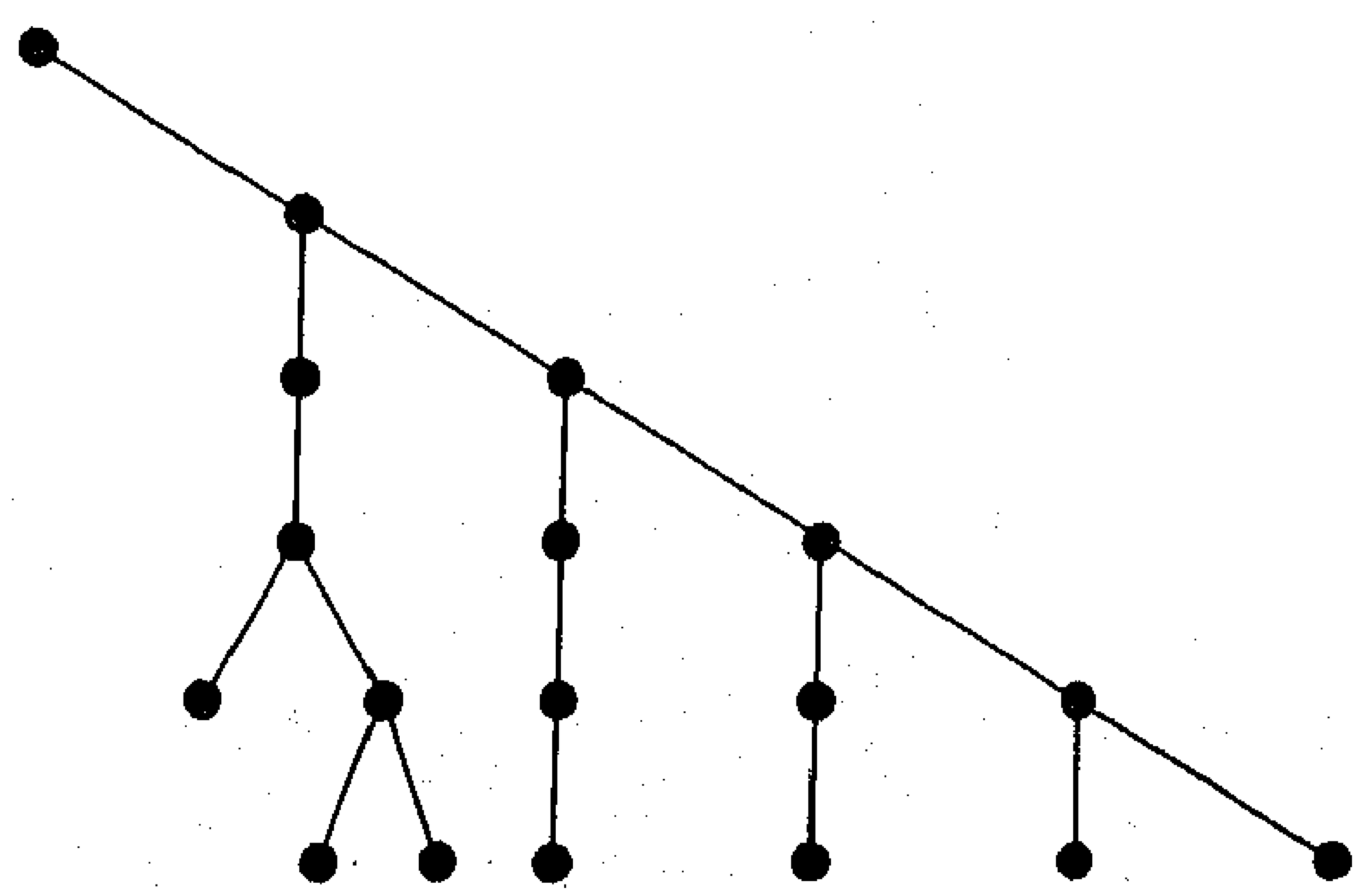


Two of the remaining cases, surprisingly, give rise to principal graphs of infinite depth.

Without going through the computations, for the first such case, we just show what the subgraph of the principal graph induced by the set of vertices at distance at most 4 from $*$ looks like. (This amounts to computing the Bratteli diagrams for the tower $\{N' \cap M_k\}_{-1 \leq k \leq 3}$.)



For the second case yielding an infinite graph, we give here the subgraph of the principal graph induced by the set of vertices at distance at most 5 from $*$. For further details see [KS].



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