A STUDY OF MAPPINGS AND MANIFOLDS OF FUETER, HYPERCOMPLEX AND ALMOST COMPLEX TYPES

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CHAPTER I

INTRODUCTION AND OPEN PROBLEMS

This thesis is concerned chiefly with certain families of smooth mappings from domains in $\mathbb{R}^n$ to $\mathbb{R}^n$ which arise as natural generalisations of complex analytic mappings. These ''Fueter'' families have important ''hypercomplex'' subfamilies defined via convergent Laurent series in one quaternionic or octonionic variable (with central coefficients). The various families and subfamilies are closed under composition and inversion (when defined) of mappings - consequently they form pseudogroups on which one can model manifolds. The geometry of manifolds which carry such ''Fueter'' or ''hypercomplex'' structure turns out to be quite rich. In particular, they have canonical foliations and the leaves of foliations carry natural complex manifold structures. Further, such manifolds come equipped with a fibre-bundle projection to spheres of appropriate dimension. These properties allow us to investigate the topological and analytic structure of Fueter and hypercomplex manifolds using the methods of algebraic topology and function theory. For example, we prove in Chapter IV, using fairly subtle topology, that the only compact simply connected manifolds which can possibly allow quaternionic (respectively octonionic) structures are S^4 (respectively S^8) (these are the quaternionic and octonionic projective spaces) and $S^2 \times S^2$ (respectively $S^2 \times S^6$).

In order to study these manifolds equipped with Fueter and/or hypercomplex coordinate systems it is of course imperative to first understand the geometric and analytic nature of the Fueter and hypercomplex mappings themselves. This study is carried out in Chapters II and III. Indeed, we find that the hypercomplex mappings satisfy certain generalised Cauchy-Riemann relations. We are able to supplement the Cauchy-Riemann relations with some extra algebro-differential identities satisfied by our mappings. Thus we can characterise Fueter and hypercomplex mappings by a fixed system of partial differential equations.

We determine conditions for the K-quasiconformality of our mapping in the sense of Ahlfors [1].

The problem of whether a given C^{∞} manifold can be assigned hypercomplex/Fueter structure is of course intimately related to whether the structure group of the tangent bundle of the manifold can be reduced to the group of Jacobians of hypercomplex/Fueter diffeomorphisms. We therefore study the Lie groups of Jacobian matrices and their corresponding Lie algebras. We find that the invertible Jacobian matrices (which we explicitly determine) form a family of Lie subgroups of $GL(n, \mathbb{R})$ - all members of the family being mutually isomorphic subgroups. The whole family turns out to be parametrised by certain real projective spaces. For higher type-p Fueter mappings, the Lie groups are again all isomorphic of dimension $(2p^2 + p)$ and the whole family is parametrised by the quotient of $S^{n-2} \times \dots \times S^{n-2}$ (p factors) modulo a certain involution. The results of Chapter IV might therefore be approachable by pure differential geometric methods using the conclusions of the previous Chapters.

Imaeda and Imaeda ([17],[18]), have also pursued analytic functions of hypercomplex variables, extending work of Fueter et al.. We describe the connection between our functions and Imaedas' functions. See Datta [9] .

We would like to mention here that our geometrical interpretation of the Fueter transform gives a precise meaning to a series of the form $\Sigma a_n V^n = f(V)$ where V is a vector variable from <u>any</u> $\mathbb{R}^n$ (irrespective of whether this $\mathbb{R}^n$ carries an algebra structure or not). When $n = 2, 4, 8$ our interpretation exactly coincides with the usual interpretation as a series in a complex, quaternionic or octonionic variable.

In Chapter IV we start to implement the programme of characterising hypercomplex and Fueter manifolds topologically and analytically by utilising our understanding of the character of these classes of mappings that we have achieved from the previous Chapters.

We define and study pseudogroups of Fueter and hypercomplex diffeomorphisms. A 4-dimensional (or 8-dimensional) manifold modelled on these 'Fueter pseudogroups' turns out to be a quaternionic (respectively octonionic) manifold.

We characterise compact Fueter manifolds as being products of compact Riemann surfaces with appropriate dimensional spheres. It then transpires that a connected compact quaternionic ($\mathbb{H}$) (respectively octonionic ($\mathbb{O}$)) manifold X, minus a finite number of circles (its 'real set'), is the orientation double covering of the product $Y \times \mathbb{P}^2$, (respectively $Y \times \mathbb{P}^6$), where Y is a connected surface equipped with a canonical conformal structure and $\mathbb{P}^n$ is n-dimensional real projective space. A corollary is that the only simply connected compact manifolds which can allow $\mathbb{H}$ (respectively $\mathbb{O}$) structure are S^4 and $S^2 \times S^2$ (respectively S^8 and $S^2 \times S^6$), See Nag, Hillman and Datta [24].

Marchiafava [21] and Salamon [25] have studied very closely-related classes of manifolds by differential geometric methods. They discovered characterisation theorems similar to ours. We explain the connection between their structures and ours.

In Chapter II we have discussed in detail a geometric characterisation of Fueter mappings and the Fueter transforms. This precipitates a rather surprising application of our theory in Chapter V. Indeed, we can characterise the location of the zeroes of quaternionic and octonionic analytic functions defined by convergent power or Laurent series with central coefficients.

We prove that the zero set of any quaternionic (or octonionic) analytic function f with central (i.e., real) coefficients is the disjoint union of codimension two spheres in $\mathbb{R}^4$ (respectively in $\mathbb{R}^8$)

and certain purely real points. In particular, for polynomials with real coefficients, the complete root-set is geometrically characterisable from the lay-out of the roots in the complex plane. The root-set becomes the union of a finite number of codimension two Euclidean spheres together with a finite number of real points. We also find the preimages $f^{-1}(A)$ for any quaternion (or octonion) A.

We demonstrate that this surprising phenomenon of complete spheres being part of the solution is very markedly a special 'real' phenomenon. For example, the quaternionic or octonionic N^{th} roots of any non-real quaternion (respectively octonion) turn out to be precisely N distinct points.

An amusing topological application of these results is to exhibit natural self maps of the Euclidean unit spheres of dimension 3 and 7 (viz. the quaternionic and octonionic unit spheres) which are of topological degree N (N any integer) such that every fibre has precisely $|N|$ distinct points, while all the exceptional fibres actually contain codimension one subspheres. The number of exceptional fibres is one for $N = 2$ and two otherwise. Using the Fueter transform we are also able to study a natural generalisation of these self-mappings on spheres of arbitrary dimension. See Datta and Nag [11] .

The penultimate Chapter of this thesis is devoted to placing our work in proper context with respect to the past and present literature in the theory of hypercomplex mappings and manifolds. From the basic references Yano [28] , Ishihara and Yano [20] , Ishihara [19] we learnt about closely related types of geometric structure on manifolds considered by other authors. We compare these various differential geometric and analytic structures on smooth manifolds.

We have proved that a n-dimensional Fueter manifold M_n has 2 and (n-2)-dimensional transverse foliations with a natural complex structure on the 2-dimensional leaves. Also there is a class of canonical biholomorphisms between the 2-dimensional leaves and the 2-dimensional foliation is obtained from a submersion $g : M_n \longrightarrow S^{n-2}$. Similar results are true for higher type Fueter structure.

We have also established the relation between Fueter structure and Yano's f-structure. Indeed, a Fueter manifold is a smooth manifold carrying a canonical integrable f-structure.

The relations between Fueter and hypercomplex structures and Ishihara's quaternion structure have been discussed. See Datta [10] .

In Chapter VII we prove the interesting theorem that $S^{2p} \times S^{2q}$ never allows almost complex structure except for a small finite number of cases. We know precisely which cases allow. This question became important for us in setting up some counter-examples to compare and contrast the various types of hypercomplex structures we met in Chapter VI. The proof of this theorem uses characteristic class techniques suggested by Prof. M.S. Narasimhan and Prof. M.S. Raghunathan.

I am extremely grateful for their guidance. The calculation using Chern classes were carried out in special cases first by S. Subramanian and was later generalised by us. See Datta and Subramanian [12]. This chapter is rather independent of the rest of the thesis and contains a single theorem which is quite interesting in its own right. The result has applications in Chapter VI – see example 6.3.6.

We wish to conclude this introduction by indicating just a few open-ends dangling from the work of this thesis. It is clear that suitable quaternionic series in two (or more) variables can be also used to form pseudogroups. Unfortunately the higher type Fueter transform of holomorphic function of two (or more) complex variables (see Chapter II) do not seem to be directly related to the corresponding quaternionic series obtained by replacing the complex variables by quaternionic variables. (This was the case for one quaternionic and octonionic variable.) So how should one study the properties of mappings and manifolds of more than one quaternionic variable ? (It is to be noted that since any two octonions generate an associative subalgebra, we were in no trouble when dealing with series in one octonionic variable with central coefficients. The difficulties arising from non-associativity when dealing with series in two or more octonionic variables would be formidable. We do not dare to even ask the questions .)

Another important area in which to extend our research would be to deal with variables from arbitrary Clifford algebra and consider power series etc. formed with these variables. Ahlfors [2] has made a start in studying these functions with a view to applications for higher dimensional Möbius transformation groups, (and consequently to higher dimensional hyperbolic manifolds).

A third important area of research would be to investigate whether complex or almost complex structure exists on twisted sphere bundles over spheres using techniques similar to the ones utilised in Chapter VII. Prof. M.S. Narasimhan raised this query because the Penrose twistor theory shows that a certain S^2 bundle over S^4 does carry complex structure – the complex structure of $\mathbb{C}P^3$.

It may be worthwhile to pursue these and related questions in the future.

CHAPTER II

FUETER AND HYPERCOMPLEX MAPPINGS

2.1 The Fueter transform

In this section we shall define Fueter transforms of complex analytic mappings of one and several complex variables. We shall also introduce the Fueter mappings and the Laurent series of hypercomplex variables and explain the relation between them. We then prove some essential properties of Fueter mappings.

2.1.1

$\mathbb{R}^n$ will denote the usual Euclidean n-dimensional space with coordinates $(x_0, x_1, \ldots, x_{n-1})$ and standard unit vectors $e_0, e_1, \ldots, e_{n-1}$. We identify $(x_0, x_1, \ldots, x_{n-1})$ with $x_0 e_0 + \ldots + x_{n-1} e_{n-1}$ and with $x_0 + x_1 e_1 + \ldots + x_{n-1} e_{n-1}$.

Set $'\mathbb{R}^n$ to be $\mathbb{R}^n$ minus the x_0 - axis, i.e.,

$$'\mathbb{R}^n = \mathbb{R}^n - \left\{ (x_0, 0, \ldots, 0) : x_0 \in \mathbb{R} \right\} \tag{1}$$

S^n will denote the unit sphere in $\mathbb{R}^{n+1}$.

For $n = 4$ and 8 we will often think of $\mathbb{R}^n$ as the space of quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$) respectively. We recall here the laws of the algebras $\mathbb{H}$ and $\mathbb{O}$.

$$\text{In } \mathbb{H} : e_r^2 = e_1 \cdot e_2 \cdot e_3 = -1, \; r = 1,2,3 \tag{2}$$

where $e_p \cdot e_q \cdot e_r = -1$ means

$$e_p e_q = -e_q e_p = e_r,\ e_q e_r = -e_r e_q = e_p,\ e_r e_p = -e_p e_r = e_q \tag{3}$$

In $\mathbb{O}$ (using same interpretation as above) :

$$e_r^2 = e_1 e_2 e_4 = e_2 e_3 e_5 = e_3 e_4 e_6 = e_4 e_5 e_7$$
$$= e_5 e_6 e_1 = e_6 e_7 e_2 = e_7 e_1 e_3 = -1 . \tag{4}$$

$\mathbb{O}$ is a non associative algebra. But it is well-known that any two octonions generate an associative subalgebra. This will be quite crucial for us because all our arguments will consequently hold uniformly for any of the algebras we deal with.

<u>2.1.2</u>

Let D be a region in U. (U is the standard upper half-plane in $\mathbb{C}$.) Let $\varphi : D \longrightarrow \mathbb{C}$ be a complex analytic (holomorphic) mapping with real and imaginary part decomposition $\varphi \equiv \xi + i\eta$.

The n-dimensional ($n \geq 2$) <u>Fueter transform</u> $F_n(\varphi) : F_n(D) \longrightarrow \mathbb{R}^n$ of φ is defined as,

$$F_n(\varphi)\left(x_0 + \sum_{j=1}^{n-1} x_j e_j\right) = \xi(x_0, y) + \sum_{j=1}^{n-1} \frac{x_j e_j}{y} \eta(x_0, y) , \tag{5}$$

where,

$$F_n(D) = \left\{ (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n : (x_0, y) \in D \right\} \subseteq \mathbb{R}^n \tag{6}$$

and, $y = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$; (the positive squre root is used always).

If moreover φ has real boundary values where the real axis abuts D then a direct application of the Schwarz reflection principle guarantees that $F_n(\varphi)$ can be defined real analytically on the revolved domain $F_n(D)$ together with the corresponding portion of the x_0-axis. This is completely clear.

REMARK : In this connection note that, we also can define Fueter transform of a holomorphic mapping whose domain of definition D is a subset of the lower half-plane L by the rule :

$$F_n(\Psi) = F_n(\widetilde{\Psi}) \tag{7}$$

where $\widetilde{\Psi} : j_2(D) \longrightarrow \mathbb{C}$ given by

$$\widetilde{\Psi}(z) = \overline{\Psi(\bar{z})}$$

$$(\text{i.e. } \widetilde{\Psi} = j_2 \circ \widetilde{\Psi} \circ j_2).$$

(Here j_2 is the complex conjugation, i.e., $j_2(z) = \bar{z}$.)

Then for a symmetric holomorphic mapping φ on a domain D which is itself symmetric about the x_0-axis, (i.e. φ commutes with j_2 and $j_2(D) = D$), one checks that $F_n(\varphi|_{D\cap U}) = F_n(\varphi|_{D\cap L})$.

A mapping of a domain in $\mathbb{R}^n$ to $\mathbb{R}^n$ will be called a <u>Fueter mapping</u>, if it is (a restriction on an open subset of) some Fueter transform of a holomorphic mapping as defined above.

2.1.3

We now define the higher type Fueter transforms of analytic mappings of several complex variables. (For the sake of simplicity of notation we restrict ourselves to the case of two complex variables; no new ideas are necessary for more complex variables.)

Let, $\varphi \equiv (\varphi_1, \varphi_2) : D(\subseteq U \times U) \longrightarrow \mathbb{C}^2$

be an analytic mapping. The type-2 Fueter transform $F_n^{(2)}(\varphi) : F_n^{(2)}(D) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ of φ is defined as,

$$F_n^{(2)}(\varphi)\left(x_{0,1} + \sum_{j=1}^{n-1} e_j x_{j,1},\ x_{0,2} + \sum_{j=1}^{n-1} e_j x_{j,2}\right)$$

$$= \left(\xi_1 + \frac{\eta_1}{y_1} \sum_{j=1}^{n-1} e_j x_{j,1},\ \xi_2 + \frac{\eta_2}{y_2} \sum_{j=1}^{n-1} e_j x_{j,2}\right) \tag{8}$$

where, $$F_n^{(2)}(D) = \left\{\left(x_{0,1} + \sum_{j=1}^{n-1} e_j x_{j,1},\ x_{0,2} + \sum_{j=1}^{n-1} e_j x_{j,2}\right) : (x_{0,1} + iy_1,\ x_{0,2} + iy_2) \in D\right\} \subseteq {}'\mathbb{R}^n \times {}'\mathbb{R}^n \tag{9}$$

$$\varphi_k = \xi_k + i\,\eta_k,\ y_k = (x_{1,k}^2 + \dots + x_{n-1,k}^2)^{1/2},\ k = 1,2.$$

REMARK : The Fueter transform F_n defined in 2.1.2 would be $F_n^{(1)}$ in the present notation. Whenever we write F_n it will be understood that one complex variable is being used.

2.1.4 PROPOSITION:

(a) $F_n(a\varphi) = a F_n(\varphi)$, for $a \in \mathbb{R}$.

(b) $F_n(\varphi_1 + \varphi_2) = F_n(\varphi_1) + F_n(\varphi_2)$.

(c) $F_n(\varphi_1 \circ \varphi_2) = F_n(\varphi_1) \circ F_n(\varphi_2)$, (whenever $\varphi_1 \circ \varphi_2$ is defined) .

(d) $F_n(j_2 \circ \varphi) = j_n \circ F_n(\varphi)$, (where j_n is the conjugation in $\mathbb{R}^n$

i.e., $j_n(x_0, x_1, \ldots, x_{n-1}) = (x_0, -x_1, \ldots, -x_{n-1})$) .

(e) $F_n(\varphi^{-1}) = F_n(\varphi)^{-1}$,

(whenever φ^{-1} is well defined with domain in U) .

(f) $F_n(\varphi_1 \cdot \varphi_2) = F_n(\varphi_1) \cdot F_n(\varphi_2)$, for $n = 4$ and 8.

Here, we are identifying $\mathbb{R}^4$ with $\mathbb{H}$ and $\mathbb{R}^8$ with $\mathbb{O}$.

Proof : (a), (b), (c) and (d) follow from definition.

(e) follows from (c).

For (f), let $\varphi_k = \xi_k + i\,\eta_k$, $k = 1, 2$.

Then,

$$\varphi_1 \cdot \varphi_2 = (\xi_1 \xi_2 - \eta_1 \eta_2) + i\,(\xi_1 \eta_2 + \xi_2 \eta_1)$$

Now,

$$(F_4(\varphi_1) \cdot F_4(\varphi_2))(x_0, x_1, x_2, x_3) = \left[\xi_1(x_0, y) + \frac{e_1 x_1 + e_2 x_2 + e_3 x_3}{y}\,\eta_1(x_0, y)\right] \cdot$$

$$\left[\xi_2(x_0, y) + \frac{e_1 x_1 + e_2 x_2 + e_3 x_3}{y}\,\eta_2(x_0, y)\right]$$

(where $y = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.)

$$= (\xi_1\xi_2 - \eta_1\eta_2)(x_0, y) + \frac{e_1x_1 + e_2x_2 + e_3x_3}{y}(\xi_1\eta_2 + \xi_2\eta_1)(x_0, y)$$

$$= F_4(\varphi_1 \cdot \varphi_2)(x_0, x_1, x_2, x_3) .$$

The result follows . Similarly for $F_8(\varphi_1 \cdot \varphi_2)$. ////

2.1.5

In [23] Nag studied convergent Laurent series on hypercomplex (i.e. quaternionic and octonionic) variables with real (i.e. central) coefficients around real centres. Namely,

$$\sum_{n=0}^{\infty} a_n (V - c)^n + \sum_{m=1}^{\infty} b_m (V - c)^{-m} \tag{10}$$

(a_n , b_m , c are reals .)

We observe (for example by using Proposition 2.1.4) that these series (and hence in particular power series) of hypercomplex variables with central coefficients are the Fueter transforms of the analytic mappings which have the ''same'' Laurent expansions in one complex variable.

Namely, let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n (z - c)^n + \sum_{m=1}^{\infty} b_m (z - c)^{-m} , \tag{11}$$

(a_n , b_m , c are reals, the annulas of convergence being $r < |z-c| < R$),

then,

$$F_4(\varphi)(V) = \sum_{n=0}^{\infty} a_n (V - c)^n + \sum_{m=1}^{\infty} b_m (V - c)^{-m}$$

where, $V = x_0 + e_1x_1 + e_2x_2 + e_3x_3$ is a quaternionic variable. Similarly, $F_8(\varphi)$ will be represented by the same Laurent series (10) where V is an octonionic variable. We note that the domain of convergence for (10) is exactly the ring domain $r < ||V-c|| < R$ in Euclidean space $\mathbb{R}^4$ and $\mathbb{R}^8$ respectively. Indeed, these ring domains are F_n $(r < |z-c| < R)$, as is to be expected. The convergence in these domains follows because Hadamard's radius of convergence formula holds for power series with variable in any Banach algebra.

REMARK : It is worth noting here that in the light of the above fundamental correspondence between Fueter and hypercomplex mappings, our general Fueter transform in any dimension n can be thought of as giving a precise meaning to a series of the form $\Sigma\, a_n V^n$ where V, is a vector variable from any $\mathbb{R}^n$ and a_n's real. (This is irrespective of whether $\mathbb{R}^n$ carries an algebra structure or not.) In case $n = 2,4,8$ our interpretation is seen to exactly coincide with the usual and natural interpretation of $\Sigma\, a_n V^n$ as a series in a complex, quaternionic or octonionic variable.

2.2 Geometrical Ideas about the Fueter transforms

In this section we will show that the Fueter transform of an analytic mapping φ is obtainable by a certain rigid geometrical rotation of φ around the x_0-axis.

2.2.1

We can identify $'\mathbb{R}^n$ with $\mathcal{U} \times S^{n-2}$ using the mapping.

$$x_0 + e_1x_1 + \dots + e_{n-1}x_{n-1} \longmapsto (x_0 + iy, (\frac{x_1}{y}, \dots, \frac{x_{n-1}}{y})) \qquad (12)$$

where $y = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$.

One can think of $\mathcal{U}_k \equiv \mathcal{U} \times \{k\}$, for any $k \in S^{n-2}$, as the rotated position of the standard half-plane $\mathcal{U} \equiv \mathcal{U} \times \{(1, 0, \dots, 0)\}$ in $\mathbb{R}^n$. The axis of rotation is of course the x_0-axis. Let us also set $\mathbb{C}_k = \mathcal{U}_k \cup \{x_0\text{-axis}\} \cup \mathcal{U}_{-k}$ to be the k-rotated position of $\mathbb{C} \equiv \mathbb{C} \times \{(1,0,\dots,0)\}$. Note that $\mathbb{C}_k = \mathbb{C}_{-k}$.

To be quite specific we explain the identification of $\mathcal{U}$ with U_k. The relevant ''k-roation'' mapping is

$$(x_0, y, 0,\dots,0) \longmapsto (x_0, k_1y,\dots,k_{n-1}y) \qquad (13)$$

for $k = (k_1,\dots,k_{n-1}) \in S^{n-2}$. Notice that this mapping is nothing but the restriction to (x_0, y) half-plane of any orientation preserving Euclidean isometry (i.e., SO(n) matrix) of the form

$$M(k) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & k_1 & & & \\ \dots\dots & & & N & \\ \dots\dots & & & & \\ 0 & k_{n-1} & & & \end{bmatrix} \qquad (14)$$

where N is arbitrary as long as $M(k) \in SO(n)$.

Notice that

$$M(k) \circ \varphi = F_n(\varphi) \circ M(k)$$

on the (x_0, y) plane.

The Fueter mapping $F_n(\varphi)$ on $F_n(D)$ (where D is the domain of φ within U) is then the 'function of revolution' obtained by 'revolving' the function φ and its domain D around the x_0-axis. We state this as the : Revolution Principle : A Fueter mapping $F_n(\varphi)$ preserves each half-plane $\mathcal{U}_k$ in the sense that $F_n(\varphi)$ maps $\mathcal{U}_k$ into $\mathbb{C}_k$, and $F_n(\varphi)$ restricted to $\mathcal{U}_k \cap F_n(D)$, for any $k \in S^{n-2}$, is identifiable with the original mapping φ on D when $\mathcal{U}_k$ is identified with $\mathcal{U}$ by the explained k-rotation.

REMARK : Note that since $D \subseteq \mathcal{U}$, each $\mathbb{C}_k$ intersects $F_n(D)$ in two components, each component being a rotated version of D. Here we identify $\mathbb{C}_k$ with $\mathbb{C}$ under either of the k or $-k$ rotations. If $\mathbb{C}_k \cap F_n(D) = D_k^+ \cup D_k^-$, then $F_n(\varphi)$ restricted to D_k^+ or D_k^- will coincide with φ on D, provided one uses the same rotation on the domain and range.

Alternatively, we can start by extending φ by symmetry to a holomorphic function $\hat{\varphi}$ on the (two-component) region $D \cup (j_2(D))$ via $\hat{\varphi}(z) = \varphi(z)$ on D and $\hat{\varphi}(z) = \overline{\varphi(\bar{z})}$ on $j_2(D)$.

(As before j_2 is the conjugation in C, i.e., $j_2(z) = \bar{z}$.) Then $F_n(\hat{\varphi})$ restricted to $\mathbb{C}_k \cap F_n(D)$ is precisely this extension of φ where $\mathbb{C}_k$ is identified with $\mathbb{C}$ under either of the two possible rotations, provide we make the same rotation to identify the range $\mathbb{C}_k$ with $\mathbb{C}$. In this connection Remark (after definition 2.1.2) may please be noted.

More explicitly we have the following

<u>2.2.2 THEOREM</u> : Let D be an open subset of the upper half-plane. The mapping $f \equiv (f_0,\ldots,f_{n-1})$ defined on some $F_n(D)$ (or defined on an open subset $\widetilde{D}$ of some $F_n(D)$ is a Fueter mapping if and only if the following conditions are satisfied.

(a) $f_1(x_0,\ldots,x_{n-1}) : f_2(x_0,\ldots,x_{n-1}) : \ldots : f_{n-1}(x_0,\ldots,x_{n-1})$

$$\equiv x_1 : x_2 : \ldots : x_{n-1},$$

for all $(x_0,\ldots,x_{n-1}) \in F_n(D)$.

(b) $f_0(x_0,\ldots,x_{n-1}) = f_0(x_0, (x_1^2 + \ldots + x_{n-1}^2)^{1/2}, 0, \ldots, 0)$.

(c) $(f_1^2 + \ldots + f_{n-1}^2)(x_0,\ldots,x_{n-1})$

$$= (f_1^2 + \ldots + f_{n-1}^2)(x_0, (x_1^2 + \ldots + x_{n-1}^2)^{1/2}, 0,\ldots,0)$$

(d) $\varphi\ (\equiv f\,|_D) : D \longrightarrow \mathbb{C}$ (that is,

$\varphi(x_0+iy) = f_0(x_0,y,0,\ldots,0) + i\, f_1(x_0, y, 0,\ldots,0)$).

is complex analytic on D.

Proof : The definition of Fueter transform shows that for a Fueter mapping (a), (b), (c) and (d) hold.

Conversely, if f satisfies (a), (b), (c) and (d) then $f = F_n(\varphi)$ where φ is the restriction of f on $F_n(D) \cap \mathcal{U}$, i.e., $\varphi(x_0, y) = f(x_0, y, 0, \ldots, 0)$. ////

2.3 Analytic Characterisation of Fueter and hypercomplex Mappings

2.3.1 THEOREM : Let $f \equiv (f_0, \ldots, f_{n-1}) = F_n(\varphi) : F_n(D) \longrightarrow \mathbb{R}^n$ be a Fueter mapping. Then f satisfies the following relations :

(a) $\partial_0 f_j = -\partial_j f_0 \quad (j > 0), \quad (\partial_p \equiv \dfrac{\partial}{\partial x_p} \quad p = 0, \ldots, n-1)$.

(b) $\partial_k f_j = \partial_j f_k, (j,k > 0)$.

(c) $\langle \nabla f_0, \nabla f_j \rangle = 0, \ (j > 0)$.

(d) Supertrace of Jac (f) $(= \partial_0 f_0 - \sum_{j=1}^{n-1} \partial_j f_j) = (2-n) \dfrac{f_p}{x_p}$

whenever $x_p \neq 0, p = 1, \cdots, n-1$.

$= \pm(2-n) (f_1^2 + \ldots + f_{n-1}^2)^{1/2} / (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$ at $(x_0, \ldots, x_{n-1})$

(In case $n = 2$, (d) becomes $\partial_0 f_0 - \partial_1 f_1 = 0$ which is the second Cauchy - Riemann relation.)

(e) $(\partial_0 f_0) x_k = \sum_{j=1}^{n-1} (\partial_k f_j) x_j, \quad k > 0$

(In case $n = 2$, (e) also reduces to the Second Canchy - Riemann relation.)

(f) $\partial_0 f_0$ is a function of x_0 and $(x_1^2 + \dots + x_{n-1}^2)^{1/2} = y$ only, equivalently,

(f') $$\frac{\partial_1(\partial_0 f_0)}{x_1} = \frac{\partial_2(\partial_0 f_0)}{x_2} = \dots = \frac{\partial_{n-1}(\partial_0 f_0)}{x_{n-1}}$$

for $x_1, \dots, x_{n-1} \neq 0$)

(g) $$\frac{\partial_k f_j}{x_k x_j} = \frac{\partial_p f_q}{x_p x_q}$$ for all $p, q, j, k > 0$, $x_j \cdot x_k \cdot x_p \cdot x_q \neq 0$, and $k \neq j$, $p \neq q$.

(h) $\frac{\partial_k f_j}{x_k x_j}$ is a function of x_0 and $(x_1^2 + \dots + x_{n-1}^2)^{1/2} = y$ only,

for $k \neq j$ and $j, k > 0$.

Equivalently,

(h') $$\partial_1\left(\frac{\partial_k f_j}{x_k x_j}\right) \cdot \frac{1}{x_1} = \partial_2\left(\frac{\partial_k f_j}{x_k x_j}\right) \cdot \frac{1}{x_2} = \dots = \partial_{n-1}\left(\frac{\partial_k f_j}{x_k x_j}\right) \cdot \frac{1}{x_{n-1}}$$

for $x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \neq 0$.

(i) $$f_p(x_0, \dots, x_{n-1}) = \left(\partial_0 f_0 - (x_1^2 + \dots + x_{n-1}^2)\frac{\partial_k f_j}{x_k x_j}\right) x_p$$

for $k, j > 0$, $k \neq j$, $x_k \cdot x_j \neq 0$, $p > 0$.

(j) $$y\,\partial_0^3 f_0 - y^3 \partial_0^2 \left(\frac{\partial_k f_j}{x_k x_j}\right) + 2y\,\frac{\partial_k(\partial_0 f_0)}{x_k} + \frac{y^2}{x_p}\partial_p\left(\frac{\partial_q(\partial_0 f_0)}{x_q} \cdot y\right)$$
$$- 6y\,\frac{\partial_k f_j}{x_k x_j} - \frac{6y^3}{x_r}\partial_r\left(\frac{\partial_k f_j}{x_k x_j}\right) - \frac{y^4}{x_s}\partial_s\left(\frac{y}{x_m}\partial_m\left(\frac{\partial_k f_j}{x_k x_j}\right)\right) = 0$$

for $k \neq j$, $k, j, r, p, q, s, m > 0$, $x_j \cdot x_k \cdot x_p \cdot x_q \cdot x_r \cdot x_s \cdot x_m \neq 0$, where $y = (x_1^2 + \dots + x_{n-1})^{1/2}$

((j) is unambiguous because of (f') and (h')).

(k) $y^2 \frac{\partial_p(\partial_0 f_0)}{x_p} - 3y^2 \frac{\partial_k f_j}{x_k x_j} - \frac{y^4}{x_q}\partial_q\left(\frac{\partial_k x_j}{x_k x_j}\right) = 0$

$x_j \cdot x_k \cdot x_p \cdot x_q \neq 0$, $k \neq j$ and $j, k, p, q > 0$.

REMARK : Notice that (c) says that the f_0 = constant hypersurfaces intersect orthogonally the f_j = constant hypersurfaces for all $j \geq 1$. When interpreted as quaternionic or octonionic analytic functions, this principle is a clear generalisation of the fact that the real part and imaginary part level curves of any holomorphic function form an orthogonal net. The various other relations above are also interesting symmetrics satisfied by the Jacobian matrices of any Fueter (or hypercomplex) mapping.

Proof : Here $f = F_n(\varphi)$, $\varphi = \xi + i\eta$ is holomorphic.

(a) $$(\partial_0 f_j)(x_1,\ldots,x_{n-1}) = \frac{x_j}{y}\partial_0 \eta\,(x_0, y),\; y = (x_1^2 + \ldots + x_{n-1}^2)^{1/2},$$

$$= -\frac{x_j}{y}\frac{\partial}{\partial y}\xi\,(x_0, y)$$

$$= -\frac{x_j}{y}\frac{\partial}{\partial y} f_0\,(x_0,\ldots,x_{n-1})$$

$$= -(\partial_j f_0)(x_0,\ldots,x_{n-1})$$

(b) For $k \neq j$.

$$\partial_k f_j\,(x_0,\ldots,x_{n-1}) = \partial_k\left(\frac{x_j}{y}\eta\,(x_0, y)\right)$$

$$= -\frac{x_j x_k}{y^3}\eta\,(x_0, y) + \frac{x_j}{y}\partial_k \eta\,(x_0, y)$$

$$= -\frac{x_j x_k}{y^3}\eta\,(x_0, y) + \frac{x_j x_k}{y^2}\eta_y\,(x_0, y) \tag{15}$$

$$= \partial_j f_k\,(x_0,\dots,x_{n-1})$$

(c) $\langle \nabla f_0,\ \nabla f_j \rangle = \langle (\partial_0 f_0,\dots,\partial_{n-1} f_0),\ (\partial_0 f_j,\dots,\partial_{n-1} f_j)\rangle$

$$= \langle (\partial_0 f_0, -\partial_0 f_1,\dots,-\partial_{n-1} f_0),\ (\partial_0 f_j,\dots,\partial_{n-1} f_j)\rangle$$

$$= \Big\langle \Big(\eta_y, \frac{-x_1}{y}\eta_{x_0},\dots,\frac{-x_{n-1}}{y}\eta_{x_0}\Big), \Big(\frac{x_j}{y}\eta_{x_0}, \frac{x_1 x_j}{y^2}\eta_y - \frac{x_1 x_j}{y^3}\eta,\dots,\frac{x_j x_j}{y^2}\eta_y - \frac{x_j^2}{y^3}\eta + \frac{\eta}{y},\dots,\frac{x_{n-1}x_j}{y^2}\eta_y - \frac{x_{n-1}x_j}{y^3}\eta\Big)\Big\rangle$$

$$= \frac{x_j}{y}\eta_{x_0}\eta_y + \frac{x_j}{y^4}(x_1^2+\dots+x_{n-1}^2)\eta\,\eta_{x_0} - \frac{x_j}{y^3}(x_1^2+\dots+x_{n-1}^2)\eta_{x_0}\eta_y - \frac{x_j}{y^2}\eta\,\eta_{x_0}$$

$$= \frac{x_j}{y}\eta_{x_0}\eta_y + \frac{x_j}{y^2}\eta\,\eta_{x_0} - \frac{x_j}{y}\eta_{x_0}\eta_y - \frac{x_j}{y^2}\eta\,\eta_{x_0}$$

$$= 0\,.$$

(d) $\partial_j f_j = \frac{\eta}{y} - \frac{x_j^2}{y^3}\eta + \frac{x_j^2}{y^2}\eta_y$ and therefore at $(x_0,\dots,x_{n-1})$

$$\partial_0 f_0 - \sum_{j=1}^{n-1}\partial_j f_j = \xi_{x_0} - (n-1)\frac{\eta}{y} + \frac{x_1^2+\dots+x_{n-1}^2}{y^3}\eta - \frac{x_1^2+\dots+x_{n-1}^2}{y^2}\eta_y$$

$$= \eta_y + (1-n)\frac{\eta}{y} + \frac{\eta}{y} - \eta_y$$

$$= (2-n)\frac{\eta}{y} = (2-n)\frac{f_j}{x_j}$$

whenever $x_j \neq 0,\ j = 1,\dots,n-1$.

(e) $\partial_k f_j = \dfrac{x_j x_k}{y^2}\eta_y - \dfrac{x_j x_k}{y^3}\eta$, for $j \neq k$, $j,k \geq 0$

$$\partial_k f_k = \frac{x_k^2}{y^2}\eta_y - \frac{x_k^2}{y^3}\eta + \frac{\eta}{y}$$

Therefore,

$$\sum_{j=1}^{n-1} (\partial_k f_j)x_j = \sum_{\substack{j=1\\ j\neq k}}^{n-1} (\partial_k f_j)x_j + (\partial_k f_k)x_k$$

$$= \left[\sum_{j\neq k} \left(\frac{x_j^2 x_k}{y^2}\eta_y - \frac{x_j^2 x_k}{y^3}\eta\right)\right] + \left[\frac{x_k^3}{y^2}\eta_y + \frac{y^2 - x_k^2}{y^3} x_k \eta\right]$$

$$= \left[\frac{x_k \eta_y}{y^2} \sum_{j=1}^{n-1} x_j^2\right] - \left[\frac{x_k}{y^3}\eta \sum_{j=1}^{n-1} x_j^2\right] + \frac{x_k}{y}\eta$$

$$= x_k \eta_y - \frac{x_k}{y}\eta + \frac{x_k}{y}\eta$$

$$= x_k \eta_y = x_k \xi_0 = x_k \partial_0 f_0$$

(f) follows from the fact that

$$(\partial_0 f_0)(x_0,\ldots,x_{n-1}) = \xi_{x_0}(x_0, y), \quad \text{where} \quad y = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$$

(g) and (h) follow from equation (15).

For (i), we have from equation (15)

$$\frac{\partial_k f_j}{x_k x_j} = \frac{\eta_y}{y^2} - \frac{\eta}{y^3} \quad , \text{ for } j \neq k \ .$$

Therefore,

$$x_p \left(\partial_0 f_0 - (x_1^2 + \dots + x_{n-1}^2) \frac{\partial_k f_j}{x_k x_j} \right) = x_p \left(\xi_{x_0} - \left(\eta_y - \frac{\eta}{y}\right)\right)$$

$$= x_p \frac{\eta}{y} = f_p$$

(j) From (i)

$$\eta = y\,(\partial_0 f_0) - y^3 \left(\frac{\partial_k f_j}{x_k x_j}\right) \qquad j \neq k,\ \ j,k > 0$$

$$\partial_0^2 \eta = y\,(\partial_0^3 f_0) - y^3 \partial_0^2 \left(\frac{\partial_k f_j}{x_k x_j}\right),$$

And,

$$\frac{\partial^2}{\partial y^2}\eta = 2\frac{\partial}{\partial y}(\partial_0 f_0) + y\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(\partial_0 f_0)\right) - 6y\frac{\partial_k f_j}{x_k x_j}$$

$$- 6y^2 \frac{\partial}{\partial y}\left(\frac{\partial_k f_j}{x_k x_j}\right) - y^3 \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial_k f_j}{x_k x_j}\right)\right)$$

$$= \frac{2y}{x_k}\partial_k(\partial_0 f_0) + \frac{y^2}{x_p}\partial_p\left(\frac{y}{x_q}\partial_q(\partial_0 f_0)\right) - 6y\frac{\partial_k f_j}{x_k x_j}$$

$$- \frac{6y^3}{x_r}\partial_r\left(\frac{\partial_k f_j}{x_k x_j}\right) - \frac{y^4}{x_s}\partial_s\left(\frac{y}{x_m}\partial_m\left(\frac{\partial_k f_j}{x_k x_j}\right)\right)$$

Hence (j) follows by using Laplace equation $\Delta \eta = 0$.

(k) From (i) $\eta = y(\partial_0 f_0) - y^3 (\frac{\partial_k f_j}{x_k x_j})$ $k \neq j,\ j, k > 0$.

By the Cauchy-Riemann relations for φ :

$$\frac{\partial_0}{\partial x_0} f_0 = \frac{\partial}{\partial y} (y \partial_0 f_0 - y^3 \frac{\partial_k f_j}{x_k x_j})$$

$$= \partial_0 f_0 + \frac{y^2}{x_p} \partial_p (\partial_0 f_0) - 3y^2 \frac{\partial_k f_j}{x_k x_j} + \frac{y^4}{x_q} \partial_q (\frac{\partial_k f_j}{x_k x_j})$$

$p, q, k, j > 0$, $j \neq k$. ////

Now, we have the following theorem ,

<u>2.3.2 THEOREM:</u>

$f \equiv (f_0, \ldots, f_{n-1}) : D (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is a Fueter mapping

if and only if it satisfies formulas (a), (f), (g), (h), (i), (j) and (k)

<u>Proof</u> : Let $y = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$.

Put $\eta (x_0, y) = y (\partial_0 f_0 - y^2 \frac{\partial_k f_j}{x_k x_j})$

(it is unambiguous because of (f), (g) and (h)) then (by (i))

$$f_k = \frac{x_k}{y} \eta , \qquad k > 0$$

The equation (a) says

$$\partial_k f_0 = -\partial_0 f_k = -\frac{x_k}{y} \eta_{x_0} , \quad \text{for } k > 0 .$$

Therefore, $\frac{\partial_1 f_0}{x_1} = \dots = \frac{\partial_{n-1} f_0}{x_{n-1}}$

equivalently f_0 depends on x_0 and y only (which means that on x_0 and y constant loci f_0 takes constant values).

One may therefore unambiguously define

$$\xi (x_0, y) = f_0 (x_0 + e_1 x_1 + \dots + e_{n-1} x_{n-1}).$$

Then by equation (a) $\xi_y = -\eta_{x_0}$

and by equation (k) $\xi_{x_0} = \partial_0 f_0 = \eta_y$.

The equation (j) says η is harmonic in the relevant domain of the (x_0, y) plane.

Now one verifies that $f = F_n (\xi + i \eta)$ on the relevant domain . ////

2.4 Connection between Fueter and Imaeda's regular mappings

K. Imaeda and M. Imaeda [17] had defined some generalisations of analytic functions of an octonionic variable which are similar to those for quaternionic variables defined and discussed by R. Fueter [15] . Imaeda and Imaeda [18] also generalised these concepts over variables from more general algebras. The Fueter maps and the regular maps defined by Imaeda and Imaeda actually form disjoint classes (except for the constant maps, being members of both). However, there is some connection between these two classes which we are going to discuss in this section.

2.4.1 DEFINITION (due to K. Imaeda and M. Imaeda) of regular functions:

A mapping $f \equiv (f_0, \ldots, f_{n-1}) = e_0 f_0 + \ldots + e_{n-1} f_{n-1} ; W (\subseteq \mathbb{R}^n)$ $W \to \mathbb{R}^n$ is called left regular (respectively right regular) if $Df = 0$ (respectively $fD = 0$) where $D = \sum_{j=0}^{n-1} e_j \partial_j$ ($\partial_j = \frac{\partial}{\partial x_j}$) and $e_j e_k + e_k e_j = 2\delta_{j,k}$.

2.4.2 PROPOSITION : A mapping f is both Fueter and regular (in the sense of Imaeda) if and only if it is a real constant.

Proof : Let $f : W (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be both left regular and Fueter in W i.e., $Df(x_0, \ldots, x_{n-1}) = 0$ for $(x_0, \ldots, x_{n-1}) \in W$ and there exists holomorphic mapping $\varphi = \xi + i\eta$ on the relevant domain, such that $f = F_n(\varphi)|_W$. Equivalently,

$$(\partial_0 + \sum_{j=1}^{n-1} e_j \partial_j)(\xi(x_0, y) + \sum_{j=1}^{n-1} \frac{e_j x_j}{y} \eta(x_0, y)) = 0$$

(where $y = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$)

$$\text{or } \Big(\xi_{x_0} + \frac{e_1 x_1}{y}\eta_{x_0} + \ldots + \frac{e_{n-1} x_{n-1}}{y}\eta_{x_0}\Big)$$
$$+ e_1\Big(\xi_y \frac{x_1}{y} + e_1\Big(\frac{y - x_1^2/y}{y^2}\eta + \frac{x_1^2}{y}\eta_y\Big) + e_2 \frac{x_2 x_1}{y^2}\eta_y + \ldots + e_{n-1}\frac{x_{n-1} x_1}{y^2}\eta_y\Big)$$
$$+ e_2\Big(\xi_y \frac{x_2}{y} + e_1 \frac{x_1 x_2}{y^2}\eta_y + e_2\Big(\frac{y - x_2^2/y}{y^2}\eta + \frac{x_2^2}{y^2}\eta_y\Big) + \ldots + e_{n-1}\frac{x_{n-1} x_2}{y^2}\eta_y\Big)$$
$$+ \ldots + e_{n-1}\Big(\xi_y \frac{x_{n-1}}{y} + e_1 \frac{x_1 x_{n-1}}{y^2}\eta_y + \ldots + e_{n-1}\Big(\frac{y - x_{n-1}^2/y}{y^2}\eta + \frac{x_{n-1}^2}{y^2}\eta_y\Big)\Big)$$
$$= 0$$

or $\xi_{x_0} - (\frac{y - x_1^2/y}{y^2}\eta + \frac{x_1^2}{y^2}\eta) - \dots - (\frac{y - x_{n-1}^2/y}{y^2}\eta + \frac{x_{n-1}^2}{y^2}\eta_y) = 0$

(Here we are using Cauchy-Riemann relations: $\xi_{x_0} = \eta_y$ and $\xi_y = \eta_{x_0}$.)

$$\text{or} \quad \xi_{x_0} - \frac{(n-1)\, y^2 - (x_1^2 + \dots + x_{n-1}^2)}{y^3}\eta - \eta_y = 0 .$$

And therefore again by Cauchy-Riemann relations between ξ and η

$$-(n-2)\frac{\eta(x_0, y)}{y} = 0$$

Therefore for $n > 2$ we have $\eta = 0$, which implies φ and therefore f are constant maps with constant real values. Similarly for right regularity. ////

For even dimensions one can generate regular mappings in the sense of Imaeda from Fueter mappings. For interest's sake we state this connection below (see Imaeda [17])

2.4.3 PROPOSITION

If $f : W (\subseteq \mathbb{R}^{2s}) \longrightarrow \mathbb{R}^{2s}$ is a Fueter mapping then $\Box^{s-1} f$ is both left and right regular. (where $\Box = D\bar{D} = \sum_{j=0}^{2s-1} \partial_j^2$).

REMARK : Note that for the trivial 2-dimensional case Fueter maps and Imaeda's regular maps all coincide with usual holomorphic mappings.

CHAPTER III

DIFFERENTIAL ANALYSIS OF FUETER AND HYPERCOMPLEX MAPPINGS

3.1 Pseudogroups of Fueter and hypercomplex mappings

We will introduce two important classes of real analytic diffeomorphisms on $\mathbb{R}^n$ – the Fueter pseudogroups and hypercomplex pseudogroups. Naturally, our chief interest will be in analysing smooth manifolds modelled on these pseudogroups.

3.1.1

(a) Let $\mathcal{A}$ denote the pseudogroup of all diffeomorphisms which are Fueter mappings obtained by taking Fueter transform of holomorphic mappings with domains in U and ranges are also in U. (Proposition 2.1.4 ((c) and (d)) guarantees that these classes form pseudogroups.)

(b) Obviously the above classes have higher type generalisations. Namely,

$$\mathcal{A}^p = \left\{ \text{Fueter diffeomorphisms which are } F_n^{(p)} \text{ of analytic mappings of } p \text{ complex variables with domains and ranges both in } U^p \right\}$$

(a') An n-dimensional manifold M_n with atlas of coordinates such that all the transition functions are in $\mathcal{A}$ will be called a n-dimensional Fueter manifold. Of course, M_n is a smooth (in fact real-analytic) manifold, (since all Fueter mappings are clearly real-analytic).

(b') Any pn-dimensional manifold M_{pn} with atlas of coordinates such that all the transition functions are in $\mathcal{A}^p$ will be called a pn-dimensional Fueter type - p manifold. Again any such manifold is clearly again real-analytic.

3.1.2

The family of diffeomorphisms which are restrictions of convergent Laurent series ((10) as introduced in 2.1.5) with central coefficients will also form pseudogroups (in dimension 4 and dimension 8). Manifolds (4 and 8 dimensional) modelled on such pseudogroups (i.e. the coordinate transition functions are from these pseudogroups) will be called 1-dimensional central quaternionic (respectively octonionic) manifolds. Briefly we will call them 'hypercomplex manifolds' (with $\mathbb{H}$ and $\mathbb{O}$ structure respectively). Of course, such manifolds are real analytic.

3.2 Jacobians of Fueter and hypercomplex mappings

We start with a little algebra describing those matrices in $GL(n, \mathbb{R})$ which will turn out to be the Jacobian matrices of invertible Fueter mappings.

Our interest will be in studying the Lie groups and Lie algebras formed by the Jacobians of Fueter mappings arises from a differential geometric attempt at determining which smooth manifolds will allow Fueter (or hypercomplex) structure. Naturally, the idea is to try to see whether the structure group of the tangent bundle can be reduced to these Lie groups which we determine. This is of course just a first step in attacking the question of existence of 'almost Fueter' or 'almost hypercomplex' structure.

<u>3.2.1 DEFINITION</u> : For any $k = (k_1, \ldots, k_{n-1}) \in S^{n-2}$, we consider a subgroup of $GL(n, \mathbb{R})$:

$$J_n(k) = \left\{ \lambda(a,b,c,k) = \begin{bmatrix} a & -bk_1 & -bk_2 & \cdots & -bk_{n-1} \\ bk_1 & a-(1-k_1^2)c & ck_1k_2 & \cdots & ck_1k_{n-1} \\ bk_2 & ck_1k_2 & a-(1-k_2^2)c & \cdots & ck_2k_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ bk_{n-1} & ck_1k_{n-1} & ck_2k_{n-1} & \cdots & a-(1-k_{n-1}^2)c \end{bmatrix} \right.$$

$$\left. a,b,c \text{ reals, } (a,b) \neq (0,0) \text{ and } a \neq c \right\} \subseteq GL(n, \mathbb{R}) \qquad (1)$$

<u>3.2.2 PROPOSITION</u> :

(a) $\lambda(a,b,c,k) = M(k)\, \lambda(a,b,c,1)\, M(k)^t$

where the 'base-point' $(1,0,\ldots,0)$ in S^{n-2} is identified with 1 and

$$M(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & k_1 & & & \\ 0 & k_2 & & N & \\ \vdots & \vdots & & & \\ 0 & k_{n-1} & & & \end{bmatrix} \qquad (2)$$

N being any $(n-1) \times (n-2)$ matrix such that $M(k)$ is an orthogonal matrix i.e., $M(k)^t = M(k)^{-1}$.

(b) $\det(\lambda(a,b,c,k)) = (a^2 + b^2)(a - c)^{n-2}$

Proof of (a) :

$M(k)\,\lambda(a,b,c,1)\,M(k)^t$

$$= M(k)\left((a-c)I_n + \begin{bmatrix} c & -b & 0 & \cdots & 0 \\ b & c & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right) M(k)^t \qquad (3)$$

$$= (a-c)I_n + M(k) \begin{bmatrix} c & -b & 0 & \cdots & 0 \\ b & c & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M(k)^t$$

$$= (a-c)I_n + \begin{bmatrix} c & -bk_1 & -bk_2 & \cdots & -bk_{n-1} \\ bk_1 & ck_1^2 & ck_1k_2 & \cdots & ck_1k_{n-1} \\ bk_2 & ck_2k_1 & ck_2^2 & \cdots & ck_2k_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ bk_{n-1} & ck_{n-1}k_1 & ck_{n-1}k_2 & & ck_{n-1}^2 \end{bmatrix} \qquad (4)$$

$$= \lambda(a,b,c,k)$$

Proof of (b) : From (a) $\det(\lambda(a,b,c,k)) = \det(\lambda(a,b,c,1))$

$$= (a^2 + b^2)(a - c)^{n-2} .$$

(Also by direct calculation it can be proved that

$\det(\lambda(a,b,c,k)) = (a^2 + b^2)(a - c)^{n-2})$. ////

3.2.3 THEOREM

(a) $J_n(k)$ are 3-dimensional commutative Lie subgroups of $GL(n, \mathbb{R})$,

(b) Any two such subgroups are isomorphic to each other.

(c) $J_n(k) = J_n(-k)$.

(d) For $(k_1^{(1)},\dots, k_n^{(1)}) = k^{(1)} \neq \pm k^{(2)} = \pm (k_1^{(2)},\dots,k_n^{(2)})$,

$J_n(k^{(1)}) \cap J_n(k^{(2)}) = \{aI_n : a \neq 0, I_n$ is the $n \times n$ identity matrix$\}$.

Proof of (a)

Commutativity follows from the fact that

$$\lambda(a_1,b_1,c_1,k)\cdot\lambda(a_2,b_2,c_2,k) = \lambda(a_2,b_2,c_2,k)\cdot\lambda(a_1,b_1,c_1,k)$$

$$= \lambda(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2, a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2, k) \qquad (5)$$

and $a_1 \neq c_1$, $a_2 \neq c_2$ implies that $a_1a_2 - b_1b_2 \neq a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2$.

Also $$\lambda(a,b,c,k)^{-1} = \lambda\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}, \frac{-(ac+b^2)}{(a^2+b^2)(a-c)}, k\right) \qquad (6)$$

and $a \neq c$ implies $\dfrac{a}{a^2+b^2} \neq \dfrac{-(ac+b^2)}{(a^2+b^2)(a-c)}$.

That $J_n(k)$ is a 3-dimensional Lie subgroup is obvious any way but also proved in 3.3.

Proof of (b) : $\lambda(a,b,c,k^{(1)}) \longmapsto \lambda(a,b,c,k^{(2)})$ gives the desired isomorphism.

Proof of (c) : Follows from the definition of $J_n(k)$.

Proof of (d) : Observe that

$$aI_n = \lambda(a,0,0,k^{(1)}) = \lambda(a,0,0,k^{(2)}) \text{ belongs to } J_n(k^{(1)}) \cap J_n(k^{(2)})$$

Conversely, let $\lambda(a_1,b_1,c_1,k^{(1)}) = \lambda(a_2,b_2,c_2,k^{(2)}) \in J_n(k^{(1)}) \cap J_n(k^{(2)})$.

If $b_1 \neq 0$ ($\therefore b_2 \neq 0$) comparing the terms of $\lambda(a_1,b_1,c_1,k^{(1)})$ and $\lambda(a_2,b_2,c_2,k^{(2)})$ we get, either both $k_i^{(1)}$ and $k_i^{(2)}$ are zero or both non zero and for non zero $k_i^{(1)}$ and $k_i^{(2)}$

$$\frac{(k_i^{(1)})^2}{(k_i^{(2)})^2} = \frac{\sum_{k_i^{(1)} \neq 0} (k_i^{(1)})^2}{\sum_{k_i^{(2)} \neq 0} (k_i^{(2)})^2} = 1$$

which implies $k_i^{(1)} = \pm k_i^{(2)}$ and therefore $b_1 = \pm b_2$, consequently $k^{(1)} = \pm k^{(2)}$

Similarly, if we assume $c_1 \neq 0$ ($\therefore c_2 \neq 0$) then $k_i^{(1)} = 0$

if and only if $c_1 k_1^{(1)} k_i^{(1)} = \dots = c_1 k_{n-1}^{(1)} k_i^{(1)} = 0$

if and only if $c_2 k_1^{(2)} k_i^{(2)} = \dots = c_2 k_{n-1}^{(2)} k_i^{(2)} = 0$

if and only if $k_i^{(2)} = 0$ and hence by the same argument $k^{(1)} = \pm k^{(2)}$.

Thus for $k^{(1)} \neq \pm k^{(2)}$ we have $b_1 = 0 = c_1$ (and therefore $b_2 = 0 = c_2$). ////

REMARK : Note that in virtue of (c) and (d) of the theorem 3.2.3, the distinct subgroups are parametrised by $\mathbb{P}^{n-2}(\mathbb{R})$ (real projective space).

3.2.4 THEOREM : For any $\lambda(a,b,c,k) \in J_n(k) = J_n(-k)$, $(k = (k_1,\ldots,k_{n-1}) \in S^{n-2}$ and $n \geq 2)$, and for any point p in $\{$the 2-plane generated by the vectors $(1,0,0,\ldots,0)$ and $(0, k_1,\ldots, k_{n-1})\} - \{$real line$\}$ (note, if $p = (x_0, x_1,\ldots, x_{n-1})$ then $x_1 : k_1 \equiv x_2 : k_2 \equiv \cdots \equiv x_{n-1} : k_{n-1}$), there exists a Fueter mapping $f \equiv (f_0,\ldots,f_{n-1})$ (i.e., there exists holomorphic mapping φ with $f = F_n(\varphi)$) such that

$$d_p f = [\mathrm{Jac}(f)]_p = \lambda(a,b,c,k)$$

Proof : $\lambda(a,b,c,k) \in J_n(k)$

Let $p = (p_0, p_1k_1,\ldots, p_1k_{n-1})$

we may assume $p_1 > 0$

(Since if $p_1 < 0$ we can replace p_1 by $-p_1$, k by $-k$ and b by $-b$).

Consider the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \lambda^*(a,b)$

There exists a complex analytic function $\varphi = \xi + i\eta$ defined in a neighbourhood D of (p_0, p_1) in $\mathcal{U}$ to $\mathbb{C}$ with $\eta(p_0, p_1) = (a-c)p_1$ such that

$$[\mathrm{Jac}(\varphi)]_{(p_0,p_1)} = \lambda^*(a,b)$$

Consider the Fueter function $f = F_n(\varphi)$

i.e. $f(x_0,\dots,x_{n-1}) = \xi\,(x_0, y) + \frac{e_1x_1+\dots+e_{n-1}x_{n-1}}{y}\,\eta(x_0, y)$

where $y = (x_1^2+\dots+x_{n-1}^2)^{1/2}$.

Then,

$$\left[\mathrm{Jac}(f)\right]_p = \begin{bmatrix} \partial_0 f_0 & \cdots & \partial_{n-1}f_0 \\ \partial_0 f_1 & \cdots & \partial_{n-1}f_1 \\ \cdots & \cdots & \cdots \\ \partial_0 f_{n-1} & \cdots & \partial_{n-1}f_{n-1} \end{bmatrix}_p =$$

$$\begin{bmatrix} \eta_y & -\eta_{x_0}\frac{x_1}{y} & -\eta_{x_0}\frac{x_2}{y} & \cdots & -\eta_{x_0}\frac{x_{n-1}}{y} \\ \eta_{x_0}\frac{x_1}{y} & \frac{x_1^2}{y^2}(\eta_y-\frac{\eta}{y})+\frac{\eta}{y} & \frac{x_1x_2}{y^2}(\eta_y-\frac{\eta}{y}) & \cdots & \frac{x_1x_{n-1}}{y^2}(\eta_y-\frac{\eta}{y}) \\ \eta_{x_0}\frac{x_2}{y} & \frac{x_1x_2}{y}(\eta_y-\frac{\eta}{y}) & \frac{x_2^2}{y^2}(\eta_y-\frac{\eta}{y})+\frac{\eta}{y} & \cdots & \frac{x_2x_{n-1}}{y^2}(\eta_y-\frac{\eta}{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \eta_{x_0}\frac{x_{n-1}}{y} & \frac{x_1x_{n-1}}{y^2}(\eta_y-\frac{\eta}{y}) & \frac{x_2x_{n-1}}{y^2}(\eta_y-\frac{\eta}{y}) & \cdots & \frac{x_{n-1}^2}{y^2}(\eta_y-\frac{\eta}{y})+\frac{\eta}{y} \end{bmatrix}_p$$

$= \lambda(a,b,c,k)$ ////

3.2.5 THEOREM :

(a) $\det\left(\left[\mathrm{Jac}\,F_n(\varphi)\right]_{(x_0,x_1,\dots,x_{n-1})}\right) = \left(\frac{\eta}{y}\right)^{n-2} \det\left(\left[\mathrm{Jac}(\varphi)\right]_{(x_0,y)}\right)$

where $\varphi = \xi + i\eta$ and $y = (x_1^2 + \dots + x_{n-1}^2)^{1/2} \neq 0$.

(b) Fueter and hypercomplex (with $\mathbb{H}$ or $\mathbb{C}$ structure) manifolds are orientable.

<u>Proof of (a)</u>: $\det\left(\left[\mathrm{Jac}\, F_n(\varphi)\right]_{(x_0,x_1,\dots,x_{n-1})}\right)$

$$= \det\left(\begin{bmatrix} \eta_y & -\eta_{x_0}\frac{x_1}{y} & \cdots & \eta_{x_0}\frac{x_{n-1}}{y} \\ \eta_{x_0}\frac{x_1}{y} & \frac{x_1^2}{y^2}(\eta_y - \frac{\eta}{y}) + \frac{\eta}{y} & \cdots & \frac{x_1 x_{n-1}}{y^2}(\eta_y - \frac{\eta}{y}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{x_0}\frac{x_{n-1}}{y} & \frac{x_1 x_{n-1}}{y^2}(\eta_y - \frac{\eta}{y}) & \cdots & \frac{x_{n-1}^2}{y^2}(\eta_y - \frac{\eta}{y}) + \frac{\eta}{y} \end{bmatrix}_{(x_0,y)}\right)$$

$= \det(\lambda(a,b,c,k))$

(where $y = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$, $k_i = \frac{x_i}{y}$ for $i = 1,\dots, n-1$,

$k = (k_1,\dots,k_{n-1})$, $a = \eta_y(x_0, y)$, $b = \eta_{x_0}(x_0, y)$,

$c = \eta_y(x_0, y) - \frac{\eta(x_0, y)}{y}$,)

$= (a^2 + b^2)(a - c)^{n-2} = (\frac{\eta}{y})^{n-2} \det\left(\left[\mathrm{Jac}(\varphi)\right]_{(x_0,y)}\right)$.

<u>Proof of (b)</u>: From (a) the determinants of the Jacobian matrices of the transition mappings of Fueter manifolds are positive, since η is positive.

Also, whenever the hypercomplex mapping is a diffeomorphism (equivalently $\lim_{y \to 0} \frac{\eta}{y} \neq 0$) the determinant is again positive, since in this case n is even. We are done. ////

3.2.6 For type-2 Fueter transformation $F_n^{(2)}$ the Jacobians are of the form

$$\lambda(a,b,c,d,e,f,g,h,p,q,k,m) =$$

$$\begin{bmatrix}
a & c & -k_1e & -m_1g & -k_2e & -m_2g & \cdots & -k_{n-1}e & -m_{n-1}g \\
b & d & -k_1f & -m_1h & -k_2f & -m_2h & \cdots & -k_{n-1}f & -m_{n-1}h \\
k_1e & k_1g & a-(1-k_1^2)p & k_1m_1c & k_1k_2p & k_1m_2c & \cdots & k_1k_{n-1}p & k_1m_{n-1}c \\
m_1f & m_1h & k_1m_1b & d-(1-m_1^2)q & m_1k_2b & m_1m_2q & \cdots & m_1k_{n-1}b & m_1m_{n-1}q \\
k_2e & k_2g & k_1k_2p & m_1k_2c & a-(1-k_2^2)p & k_2m_2c & \cdots & k_2k_{n-1}p & k_2m_{n-1}c \\
m_2f & m_2h & k_1m_2b & m_1m_2q & k_2m_2b & d-(1-m_2^2)q & \cdots & m_2k_{n-1}b & m_2m_{n-1}q \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_{n-1}e & k_{n-1}g & k_1k_{n-1}p & m_1k_{n-1}c & k_2k_{n-1}p & m_2k_{n-1}c & \cdots & a-(1-k_{n-1}^2)p & k_{n-1}m_{n-1}c \\
m_{n-1}f & m_{n-1}h & k_1m_{n-1}b & m_1m_{n-1}q & k_2m_{n-1}b & m_2m_{n-1}q & \cdots & k_{n-1}m_{n-1}b & d-(1-m_{n-1}^2)q
\end{bmatrix} \quad (7)$$

where $k, m \in S^{n-2}$, a to q being real numbers. (This is a $2n \times 2n$ matrix of course.)

Here, $\det(\lambda(a,b,c,d,e,f,g,h,p,q,k,m)) = ((ad-bc)^2 + (eh-gf)^2)(a-p)^{n-2}(d-q)^{n-2}$. (8)

<u>REMARK</u> : The above result for Jacobians of higher type Fueter mapping can also be thought of in a way analogus to the one variable case treated just before. We have,

$$\lambda(a,b,c,d,e,f,g,h,p,q,k,m)$$

$$= \begin{bmatrix} A-C & 0 & 0 & \cdots & 0 \\ 0 & A-C & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A-C \end{bmatrix} + M(k,m) \begin{bmatrix} C & -B & 0 & \cdots & 0 \\ B & C & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M(k,m)^t$$

$$= M(k,m) \begin{bmatrix} A & -B & 0 & \cdots & 0 \\ B & A & 0 & \cdots & 0 \\ 0 & 0 & A-C & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A-C \end{bmatrix} M(k,m)^t \qquad (9)$$

Notation : 0 denote the appropriate sized zero matrix

Here, $$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B = \begin{bmatrix} e & g \\ f & h \end{bmatrix}, \quad C = \begin{bmatrix} p & o \\ b & q \end{bmatrix} \qquad (10)$$

and $$M(k,m) = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & K_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & K_{n-1} & 0 & \cdots & 0 \end{bmatrix}$$

where $k = (k_1, \dots, k_{n-1})$, $m = (m_1, \dots, m_{n-1})$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$K_j = \begin{bmatrix} k_j & 0 \\ 0 & m_j \end{bmatrix}$ for $j = 1, \dots, n-1$.

In analogy with the proof of (a) of Proposition 3.2.2 it therefore seems justifiable to denote $\lambda(a,b,c,d,e,f,g,h,p,q,k,m)$ by $\lambda_2(A,B,C,K)$ where A, B, C are as above, and

0

$$K = (K_1, \dots\dots, K_{n-1}) = \begin{pmatrix} k_1 & 0 & k_2 & 0 & \dots & k_{n-1} & 0 \\ 0 & m_1 & 0 & m_2 & \dots & 0 & m_{n-1} \end{pmatrix}$$

$= ((k_1, \dots, k_{n-1}), (m_1, \dots, m_{n-1})) \in S^{n-2} \times S^{n-2}$.

Also denote $M(k,m)$ by $M_2(K)$ (in analogy with $M(k)$ seen in 3.2.2(a)).

Then

$$\lambda_2(A,B,C,K) = M_2(K)\,\lambda(A,B,C,I^*)\,M_2(K)^t \tag{11}$$

where, $I^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$.

Now (8) follows immediately.

Utilising the above convenient notation for these large matrices, it is easy to prove in analogy with theorem 3.2.3 the following :

<u>3.2.7 THEOREM</u> :

(a) $J_{2n}(K) = \{\lambda_2(A,B,C,K)$: at least one of A, B is non-singular and $A - C$ nonsingular $\}$.

(here A,B,C are as in (10).) are $(2.2^2 + 2)$-dimensional subgroups of $GL(2n, \mathbb{R})$ for each fixed K.

(b) Any two such subgroups are isomorphic to each other.

(c) $J_{2n}(K) = J_{2n}(-K)$

(d) For $K^{(1)} \neq K^{(2)}$, $J_{2n}(K^{(1)}) \cap J_{2n}(K^{(2)})$

$$= \left\{ \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A \end{bmatrix} : A \text{ any nonsingular } (2 \times 2) \text{ real matrix} \right\}$$

<u>3.2.7 COROLLARY</u> : The family of Lie subgroups $J_{2n}(K)$ as K varies is parametrised by the quotient of $S^{n-2} \times S^{n-2}$ modulo the involution α of $(S^{n-2})^2$ on itself where $\alpha(K) = -K$.

<u>REMARK</u> : For higher type-p Fueter mappings the Lie groups of invertible Jacobians are turning out to be non-commutative subgroups of $GL(pn, \mathbb{R})$. We have been showing the calculation for $p = 2$ to keep notational simplicity.

<u>3.3 The Lie algebra of the Lie groups of Jacobians</u>

<u>3.3.1</u>

Let us first deal with the usual type-1 Fueter mappings $F_n(\varphi)$. Let us set, for any fixed $k \in S^{n-2}$, $X : J_n(k) \longrightarrow \mathbb{R}^3$, to be the mapping : $\lambda(a,b,c,k) \longmapsto (a,b,c)$.

X identifies $J_n(k)$ as the open subset of $\mathbb{R}^3$ given by $\{(a,b,c) \in \mathbb{R}^3 : (a,b) \neq (0, 0), a \neq c\}$. The multiplication in $J_n(k)$ is equivalent to the mapping : $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by :

$$((a_1,b_1,c_1), (a_2,b_2,c_2)) \longmapsto (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2, a_1c_2+c_1a_2 - c_1c_2 - b_1b_2) \qquad (11)$$

This is clearly C^∞ and $\lambda(a,b,c,k) \longmapsto \lambda(a,b,c,k)^{-1}$ is equivalent to the mapping : $\underset{\sim}{\mathbb{R}}^3 \longrightarrow \underset{\sim}{\mathbb{R}}^3$ given by $(a,b,c) \longmapsto \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}, \frac{-(ac+b^2)}{(a^2+b^2)(a-c)}\right)$,

(where $\underset{\sim}{\mathbb{R}}^3 = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \neq (0,0), a \neq c\}$.) (12)

which is also clearly C^∞.

<u>REMARK</u> : This shows directly that $J_n(k)$ is a 3-dimensional Lie subgroup of $GL(n, \mathbb{R})$ for every fixed $k \in S^{n-2}$. The group structure is independent of k by equations (11) and (12) so that it is no surprise that all the $J_n(k)$ were isomorphic to each other.

<u>Notation</u> : For the calculation below we will set $(a,b,c) = (x^1, x^2, x^3)$ and $\partial_i = \partial/\partial x_i$ for convenience of notation.

We now calculate explicitly the Lie algebra $L(J_n(k))$ of the Lie group $J_n(k)$. Fix k. Let $T_e(J_n(k)) = \langle e_1,e_2,e_3 : e_i = (\partial_i)_e \rangle$ be the tangent space of $J_n(k)$ at the identity $e = \lambda(1,0,0,k)$.

Let $\mathcal{L}(J_n(k))$ be the all left invariant vector fields. Then,

$$\mathcal{L}(J_n(k)) = \langle X_1,X_2,X_3; X_i : \lambda \longmapsto L_{\lambda *}(\partial_i)_e : i = 1,2,3 \rangle$$

Here L_λ is left-translation by λ in the group. (See Brickell and Clark [5, p.219] for notation.)

The vector fields X_1, X_2, X_3 are given at the point $g \in J_n(k)$ by

$$(X_i)_g = \sum_{r=1}^{3} (\partial_i(x^r \circ L_g))_e \, (\partial_r)_g ,$$

so that if $g = \lambda(a_1, b_1, c_1, k)$ then,

$$(X_i)_g = (\partial_i(a_1x^1 - b_1x^2))_e \, (\partial_1)_g + (\partial_i(a_1x^2 + b_1x^1))_e \, (\partial_2)_g$$
$$+ (\partial_i((a_1 - c_1)x^3 + c_1x^1 - b_1x^2))_e \, (\partial_3)_g \qquad (13)$$

Therefore,

$$(X_1)_g = a_1(\partial_1)_g + b_1(\partial_2)_g + c_1(\partial_3)_g$$
$$(X_2)_g = -b_1(\partial_1)_g + a_1(\partial_2)_g - b_1(\partial_3)_g$$
$$(X_3)_g = (a_1 - c_1)(\partial_3)_g$$

which gives

$$X_1 = x^1 \partial_1 + x^2 \partial_2 + x^3 \partial_3$$
$$X_2 = -x^2 \partial_1 + x^1 \partial_2 - x^2 \partial_3 \qquad (14)$$
$$X_3 = (x^1 - x^3)\partial_3$$

Since, $J_n(k)$ is commutative, of course all Lie backets vanish.

3.3.2

We proceed to the type-2 Lie groups $J_{2n}(K) \subseteq GL(2n, \mathbb{R})$. For $(k,m) \in S^{n-2} \times S^{n-2}$, $J_{2n}(k,m) = \{\lambda(a,b,c,d,e,f,g,h,p,q,k,m)$: $a \neq p$, $d \neq q$, $(ad, eh) \neq (bc, gf)\} \subseteq GL(2n, \mathbb{R})$ is a non-commutative Lie group with multiplication as follows :

$$\lambda(a_1,b_1,c_1,d_1,e_1,f_1,g_1,h_1,p_1,q_1,k,m)\cdot\lambda(a_2,b_2,c_2,d_2,e_2,f_2,g_2,h_2,p_2,q_2,k,m)$$

$$= \lambda\,(a_1a_2+c_1b_2-e_1e_2-g_1f_2,\ b_1a_2+d_1b_2-f_1e_2-h_1f_2,\ a_1c_2+c_1d_2-e_1g_2-g_1h_2,$$
$$b_1c_2+d_1d_2-f_1g_2-h_1h_2,\ a_1e_2+c_1f_2+e_1a_2+g_1b_2,\ b_1e_2+d_1f_2+f_1a_2+h_1b_2,$$
$$a_1g_2+c_1h_2+e_1c_2+g_1d_2,\ b_1g_2+d_1h_2+f_1c_2+h_1d_2,\ c_1b_2-e_1e_2-g_1f_2+a_1p_2+$$
$$p_1a_2-p_1p_2,\ b_1c_2-f_1g_2-h_1h_2+d_1g_2+q_1d_2-q_1q_2\ ,\ k,m) \qquad (15)$$

And the Lie algebra of any of the Lie groups is

$L(J_n(k,m)) = \langle X_1,\dots,X_{10},\ \ [\,,\,]\rangle$ where

$$X_1 = x^1\,\partial_1 + x^2\,\partial_2 + x^5\,\partial_5 + x^6\,\partial_6 + x^9\,\partial_9$$
$$X_2 = x^3\,\partial_1 + x^4\,\partial_2 + x^7\,\partial_5 + x^8\,\partial_6 + x^3\,\partial_9$$
$$X_3 = x^1\,\partial_1 + x^2\,\partial_4 + x^5\,\partial_7 + x^6\,\partial_8 + x^2\,\partial_{10}$$
$$X_4 = x^3\,\partial_3 + x^4\,\partial_4 + x^7\,\partial_7 + x^8\,\partial_8 + x^{10}\,\partial_{10}$$
$$X_5 = -x^5\,\partial_1 - x^6\,\partial_2 + x^1\,\partial_5 + x^2\,\partial_6 - x^5\,\partial_9$$
$$X_6 = -x^7\,\partial_1 - x^8\,\partial_2 + x^3\,\partial_5 + x^4\,\partial_6 - x^7\,\partial_9$$
$$X_7 = -x^5\,\partial_3 - x^6\,\partial_4 + x^1\,\partial_7 + x^2\,\partial_8 - x^6\,\partial_{10}$$
$$X_8 = -x^4\,\partial_3 - x^8\,\partial_4 + x^3\,\partial_7 + x^4\,\partial_8 - x^8\,\partial_{10}$$
$$X_9 = (x^1 - x^9)\,\partial_9$$
$$X_{10} = (x^4 - x^{10})\,\partial_{10} \qquad (16)$$

and

$$\begin{aligned}
[X_1,X_3] &= [X_3,X_4] = [X_7,X_5] = [X_8,X_7] = X_3\\
[X_6,X_1] &= [X_8,X_2] = [X_4,X_6] = [X_2,X_5] = X_6\\
[X_1,X_7] &= [X_5,X_3] = [X_3,X_8] = [X_7,X_4] = X_7\\
[X_4,X_2] &= [X_6,X_8] = X_2\\
[X_2,X_7] &= [X_6,X_3] = X_8 - X_5\\
[X_6,X_7] &= X_1\\
[X_2,X_3] &= X_4 + X_{10} - X_1 - X_9\\
[X_i,X_j] &= 0 \text{ whenever } [X_i,X_j] \text{ or } [X_j,X_i] \qquad (17)
\end{aligned}$$

are not amongst the above.

The calculations for the results above are of course more cumbersome but exactly similar to the calculations shown in 3.3.1.

3.4 Quasiconformity of Fueter mappings

We have obtained Fueter mappings from holomorphic mappings. Locally invertible holomorphic mappings have the famous classic property of conformality following from the Cauchy-Riemann equations. It is natural therefore to investigate to what degree the Fueter mappings are conformal or fail to be conformal. This is all the more natural since we saw that some degree of conformality is present in Fueter and hypercomplex mappings, indeed the generalised Cauchy-Riemann relations - in particular (c) of Theorem 2.3.1 asserted that real part constant level surfaces are orthogonal to each of the 'imaginary' part constancy level surfaces for Fueter (hypercomplex) mappings.

The correct measure of conformality in high dimensions is Ahlfors' notion of quasiconformality defined below. It is to be recalled that in dimension $n \geq 3$ the only conformal maps are the Möbius transformations.

REMARK : A Möbius (conformal) transformation (in dimensions $n \geq 3$) may not be a Fueter mapping. Indeed even the translation $V \longmapsto V + b$, b not purely real, already fails to be a Fueter mapping.

3.4.1 DEFINITION : (due to Ahlfors [1]) Consider a differentiable mapping

$$f : D(\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

Define,
$$Sf = \frac{1}{2}(Df + (Df)^t) - \frac{1}{n} \mathrm{Tr}(Df) I_n \tag{18}$$

where Df is the Jacobian matrix of f and $\mathrm{Tr}(Df)$ = Trace of Df. $\|Sf\|$ = (Sum of the squares of the entries of $Sf)^{1/2}$. Then f is said to be K-quasiconformal if $(1/n^{1/2}) \|Sf\| \leq K$. A mapping is called quasiconformal if it is K-quasiconformal for some $K < \infty$.

3.4.2 PROPOSITION : Suppose $f = F_n(\varphi)$ is a Fueter mapping. f is K-quasiconformal if
$$|\eta_y - \eta/y| \leq 2K \tag{19}$$
over the domain of $\varphi = \xi + i\eta$.

Proof :

$$Sf = \begin{bmatrix} (1-\frac{2}{n})(\eta_y - \frac{\eta}{y}) & 0 & 0 & & 0 \\ 0 & (k_1^2 - \frac{2}{n})(\eta_y - \frac{\eta}{y}) & k_1k_2(\eta_y - \frac{\eta}{y}) & \cdots & k_1k_{n-1}(\eta_y - \frac{\eta}{y}) \\ 0 & k_1k_2(\eta_y - \frac{\eta}{y}) & (k_2^2 - \frac{2}{n})(\eta_y - \frac{\eta}{y}) & \cdots & k_2k_{n-1}(\eta_y - \frac{\eta}{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & k_1k_{n-1}(\eta_y - \frac{\eta}{y}) & k_2k_{n-1}(\eta_y - \frac{\eta}{y}) & \cdots & (k_{n-1}^2 - \frac{2}{n})(\eta_y - \frac{\eta}{y}) \end{bmatrix} \tag{20}$$

Therefore,

$$\| Sf \|^2 = (\eta_y - \frac{\eta}{y})^2 \Big[(1 + \frac{2}{n})^2$$

$$+\{(k_1^2 - \frac{2}{n})^2 + (k_1k_2)^2 + \cdots + (k_1k_{n-1})^2\}$$

$$+\{(k_1k_2)^2 + (k_2^2 - \frac{2}{n})^2 + \cdots + (k_2k_{n-1})^2\}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\{(k_1k_{n-1})^2 + (k_2k_{n-1})^2 + \cdots + (k_{n-1}^2 - \frac{2}{n})^2\}\Big]$$

$$= (\eta_y - \frac{\eta}{y})^2 \Big[(1 - \frac{2}{n})^2 + \{(k_1^2 - \frac{2}{n})^2 + k_1^2 (1 - k_1^2)\}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\{(k_{n-1}^2 - \frac{2}{n})^2 + k_{n-1}^2 (1 - k_{n-1}^2)\}\Big]$$

$$= (\eta_y - \frac{\eta}{y})^2 \Big[(1 - \frac{2}{n})^2 + (\frac{4}{n^2} - 4k_1^2/n + k_1^2) + \cdots$$

$$\cdots + (\frac{4}{n^2} - 4k_{n-1}^2/n + k_{n-1}^2)\Big]$$

$$= (\eta_y - \frac{\eta}{y})^2 \Big[1 - \frac{4}{n} + n \cdot \frac{4}{n^2} - \frac{4}{n}(k_1^2 + \cdots + k_{n-1}^2) + (k_1^2 + \cdots + k_{n-1}^2)\Big]$$

$$= (\eta_y - \frac{\eta}{y})^2 (1 - \frac{4}{n} + \frac{4}{n} - \frac{4}{n} + 1)$$

$$= (\eta_y - \frac{\eta}{y})^2 (2 - \frac{4}{n}) \qquad (21)$$

Hence,

$$(1/n^{1/2})\ \|Sf\| = (2n-4)^{1/2} \cdot n^{-1}\left|\eta_y - \frac{\eta}{y}\right| \leq \frac{1}{2}\left|\eta_y - \frac{\eta}{y}\right| \qquad (22)$$

The result follows. ////

3.4.3 COROLLARY : Any Fueter mapping is quasiconformal on any relatively compact subdomain.

A suitable K can be determined from the supremum of $\left|\eta_y - \eta/y\right|$ over the subdomain. ////

CHAPTER IV

COMPACT FUETER AND HYPERCOMPLEX MANIFOLDS

In this chapter we start to implement the programme of characterising hypercomplex and Fueter manifolds topologically and analytically by utilising our understanding of the character of the classes of Fueter and hypercomplex mappings that we have achieved from the previous chapters.

We define and study pseudogroups of Fueter and hypercomplex diffeomorphisms. A 4-dimensional (or 8-dimensional) manifold modelled on these 'Fueter pseudogroups' turns out to be a quaternionic (respectively octonionic) manifold.

We characterise compact Fueter manifolds as being products of compact Riemann surfaces with appropriate dimensional spheres. It then transpires that a connected compact quaternionic ($\mathbb{H}$) (respectively octonionic ($\mathbb{O}$)) manifold X, minus a finite number of circles (its 'real set'), is the orientation double covering of the product $Y \times \mathbb{P}^2$, (respectively $Y \times \mathbb{P}^6$), where Y is a connected surface equipped with a canonical conformal structure and $\mathbb{P}^n$ is n-dimensional real projective space. A corollary is that the only simply connected compact manifolds which can allow $\mathbb{H}$ (respectively $\mathbb{O}$) structure are S^4 and $S^2 \times S^2$ (respectively S^8 and $S^2 \times S^6$). See Nag, Hillman and Datta [24].

Marchiafava [21] and Salamon [23] have studied very closely-related classes of manifolds by differential geometric methods. They discovered characterisation theorems similar to ours. We explain the connection between their structures and ours.

4.1 Characterisation of Fueter Manifolds

Recall the identification $'\mathbb{R}^n = \mathbb{R}^n - \{x_0 - \text{ axis}\}$ with $\mathcal{U} \times S^{n-2}$ by the mapping;

$$(x_0 + e_1x_1 + \dots + e_{n-1}x_{n-1}) \longmapsto (x_0 + iy, (\frac{x_1}{y}, \frac{x_2}{y}, \dots, \frac{x_{n-1}}{y})) \in \mathcal{U} \times S^{n-2} \quad (1)$$

where $y = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$ (positive square root).

We will think of $\mathcal{U}_\sigma = \mathcal{U} \times \{\sigma\}$, for any $\sigma \in S^{n-2}$, as the rotated position of a standard upper half-plane $\mathcal{U} \times \{(1, 0, \dots, 0)\}$ in $'\mathbb{R}^n$. The axis of rotation is of course the x_0-axis. We can therefore identify any $\mathcal{U}_\sigma$ with $\mathcal{U}$.

From the Revolution principle (see 2.2.1) it now follows immediately that <u>for any n-dimensional Fueter manifold X there is an intrinsically defined C^∞ submersion</u>

$$g : X \to S^{n-2} \quad (2)$$

i.e. $g(x) = \sigma$ determines the half-plane $\mathcal{U}_\sigma$ in which $x \in X$ lies with respect to any Fueter coordinate chart around x. By the Revolution principle the mapping g is well-defined and <u>each fibre of g has the structure of a 1-dimensional complex manifold</u>. Note that <u>neighbouring fibres have canonical local (biholomorphic) identifications</u> determined simply by the Fueter structure of X. Indeed the local identifications are obtained by identifying U_σ with $U_{\sigma'}$ (by rotation around the x_0-axis) in the image (in ${}'\mathbb{R}^n$) of any Fueter chart. Again because of Revolution principle <u>the identifications do not depend on the Fueter chart used</u>, (charts will always be required to have connected domains).

Clearly if X is compact then the fibre of g is a compact Riemann surface R, and g is a C^∞ fibre bundle (being a proper submersion). By standard compactness arguments we can then show that there are canonical <u>global</u> biholomorphic identifications between any two fibres of g. We therefore derive :

<u>4.1.1 THEOREM</u> : Any compact n-dimensional Fueter manifold X is Fueter-category isomorphic to the product of a compact Riemann surface R with S^{n-2}. (In fact, $R \times S^{n-2}$ has a canonical Fueter structure for any Riemann surface R in an obvious way. We remark that if $R = U/G$, G an arbitrary Fuchsian group, then $R \times S^{n-2} = {}'\mathbb{R}^n/F_n(G)$ as a Fueter manifold. G can be allowed to possess elliptic elements, and every Riemann surface R then occurs as U/G).

Details of the proof of the above Theorem are omitted because they are exactly analogous to, (but much simpler than), the proofs for the more subtle hypercomplex manifold results which we explain below. In fact the proofs for the hypercomplex cases just go through verbatim except that we do not have to worry about the ''real set.''

4.2 Characterisation of hypercomplex manifolds

Recall that any Fueter or hypercomplex manifold is orientable (see 3.2.5(b)).

Thus if we choose a fixed orientation for $\mathbb{R}^4$ any $\mathbb{H}$ manifold then gets a canonical orientation.

4.2.1 DEFINITION : The set of points in a hypercomplex manifold X whose image under any hypercomplex chart is on the real (x_0)-axis is a closed 1-dimensional submanifold of X called its 'real set' $\rho_X = \rho$.

Clearly, if X is compact ρ_X is a finite union of circles smoothly embedded in X.

Now, any central $\mathbb{H}$ or $\mathbb{O}$ Laurent series will map any 2-plane containing the x_0-axis into itself ; so, using the facts for $F_4(\varphi)$, (Revolution principle), we can understand a $\mathbb{H}$ or $\mathbb{O}$ Laurent series function as a 'function of revolution' obtained by revolving a complex analytic function around the real axis. 2-planes in $\mathbb{R}^n$ through x_0 axis are parametrised by the real projective space $\mathbb{P}^{n-2}$, so on the hypercomplex manifold X we can define a natural C^∞ submersion, (in analogy with (2)),

$$g : X - \rho_X \to \mathbb{P}^{n-2} \ (n = 4 \text{ or } 8) \tag{3}$$

4.2.2 EXAMPLE : Let $X = S^4 \equiv \mathbb{R}^4 \cup \{\infty\}$. We can give this a $\mathbb{H}$-structure analogous to the complex structure of the Riemann sphere, by assigning the identity chart on $\mathbb{R}^4$ and obtaining $V \mapsto \frac{1}{V}$ as the transition function to the obvious chart covering ($\mathbb{R}^4$ - origin) $\cup \{\infty\}$.

Notice ρ is then the 'real circle' $\{x_0\text{-axis}\} \cup \{\infty\}$ and the mapping $g : X - \rho \equiv \mathbb{R}^4 \rightarrow \mathbb{P}^2$ is precisely the 2nd - component of the identification mapping (1) followed by the standard double covering $\pi : S^2 \rightarrow \mathbb{P}^2$. The fibres of g are two disjoint half-planes. (S^8 has similar octonionic description).

Thus S^4 is quaternionic projective space $\mathbb{P}^1(\mathbb{H})$.

Note that for g to be well-defined we must be mapping to $\mathbb{P}^2$, and not to S^2; because <u>if</u> (V, φ) <u>is a $\mathbb{H}$-chart on X then so is</u> $(V, -\varphi)$, and the map $\varphi : V \rightarrow \mathbb{R}^4 \equiv \mathcal{U} \times S^2$ assigns the S^2 values <u>antipodal</u> to those determined by $-\varphi : V \rightarrow \mathbb{R}^4$. Notice further that the upper half-plane element assigned by φ to any $x \in V(\subset X)$ is the <u>reflection across the y-axis</u> (in $\mathcal{U}$) of the $\mathcal{U}$-element associated to x by $-\varphi$.

<u>4.2.3 THEOREM</u>: Let X be a connected compact hypercomplex manifold with real set ρ. Then

(a) $g : X - \rho \rightarrow \mathbb{P}^{n-2}$ ($n = 4$ for $\mathbb{H}$, 8 for $\mathbb{O}$) as defined in (3) is a C^∞ fibre bundle which is not globally trivial. Let the fibre $g^{-1}(k)$ for any $k \in \mathbb{P}^{n-2}$ be donated $X(k)$ ($\subset X - \rho$).

(b) $X(k)$ is an orientable surface with at most two components. It has a canonical conformal structure induced by the hypercomplex structure of X.

(c) The closure $\overline{X(k)}$ of $X(k)$ in X is precisely $X(k) \cup \rho$ (for all $k \in \mathbb{P}^{n-2}$). $\overline{X(k)}$ is itself orientable, and if ρ is non-empty then $\overline{X(k)}$ is connected.

(d) $\overline{X(k)}$ is a compact surface with a conformal structure, (i.e. transition functions are holomorphic or conjugate-holomorphic) and there is a global conformal identification of $\overline{X(k)}$ with $\overline{X(k')}$ for all k' in a neighbourhood of k. These identifications are determined by the hypercomplex structure of X and act as the identity when restricted to ρ.

Proof : We deal only with the $\mathbb{H}$-case since now new ideas come in for $\mathbb{O}$.

First notice that g is surjective. Indeed, if ρ is empty then g being submersive and X being compact says g is onto. If ρ has a point ξ in it then any chart φ around ξ will map to a 4-dimensional open neighbourhood of $\varphi(\xi)$, ($\varphi(\xi)$ is on x_0-axis), and already every 2-plane is intersected ; so g is onto.

Note that the last argument shows that each point of ρ is a limit point of every fibre $g^{-1}(k)$, and then the first part of (c) follows easily.

Consider now any chart (V, φ), $V \subset X$; φ assigns to each point of $V-\rho$ a point in $'\mathbb{R}^4 \equiv \mathcal{U} \times S^2$. (recall (1)). Thus to any $x \in V-\rho$, φ assigns an element of $\mathcal{U}$ and an element of S^2. We denote the map φ restricted to $(V-\rho)\cap X(k)$ by $\varphi(k)$, $(k \in \mathbb{P}^2)$. We can think of $\varphi(k)$ as a chart on a small piece of $X(k)$, mapping it to U; (by cutting down the size of V we may assume $(V-\rho)\cap X(k)$ is connected – so $\varphi(k)$ maps to exactly one half-plane). If (W, ψ) is another chart around $x \in X-\rho$ then ψ assigns to x either the same S^2 – value or the opposite S^2-value to that assigned by φ. Since the hypercomplex central Laurent series are essentially Fueter mappings we see from the fundamental 'Revolution principle' that $\varphi(k)$ and $\Psi(k)$ are holomorphically related near $x \in X(k)$ if the S^2-values coincided, and are anti-holomorphically related if the S^2-values were antipodal.

In any case $X(k)$ has a conformal structure, which, by using charts at points of ρ also, clearly extends to a conformal structure on all of $\overline{X(k)}$.

Now let us explain the local conformal identification of fibres. As for Fueter manifolds, these come by using charts and rotating half-planes to fall on one another. Let (V, φ) be a 'small chart' on X with a 'small' image in $\mathbb{R}^4$, i.e. $\varphi(V)$ does not intersect any pair of opposite half-planes. In that case

$$\varphi(k')^{-1} \circ \varphi(k) \tag{4}$$

for nearby values of k and k' will be a conformal identification of a piece of $X(k)$ with a piece of $X(k')$. Notice that if we use a different chart (W, ψ) the identifications are still the same;

$$\begin{aligned}\psi(k')^{-1} \circ \psi(k) &\equiv \varphi(k')^{-1} \circ \varphi(k') \circ \psi(k')^{-1} \circ \psi(k) \circ \varphi(k)^{-1} \circ \varphi(k) \\ &\equiv \varphi(k')^{-1} \circ [\varphi(k') \circ \psi(k')^{-1}] \circ [\psi(k) \circ \varphi(k)^{-1}] \circ \varphi(k) \\ &= \varphi(k')^{-1} \circ \varphi(k) \end{aligned} \tag{5}$$

because the square-bracketed mappings cancel each other off by the revolution principle.

Thus, a point $x_1 \varepsilon X(k)$ is identified with a point $x_2 \varepsilon X(k')$ (k' near k) precisely when the U-values assigned to x_1 and x_2 are the same via any small chart containing both x_1 and x_2 in its domain. It is obvious that $\overline{X(k)}$ can now be conformally identified with $\overline{X(k')}$ (for k' in a small neighbourhood of k in $\mathbb{P}^2$) by extending these canonical identifications to ρ, the extension being the identity on ρ. (Since $\overline{X(k)}$ is compact there is no problem in using a finite number of small hypercomplex charts to cover $X(k)$, and thus get the conformal mappings globally from all of $X(k)$ to each $X(k')$, for k' sufficiently near to k).

Since we have now got a canonical way to map the fibre $X(k)$ onto $X(k')$, for all k' in a small neighbourhood of k in $\mathbb{P}^2$, it is clear that we have local triviality; and so g is a C^∞ fibre bundle.

The bundle cannot be globally trivial since a product $Y \times \mathbb{P}^2$ cannot be orientable for any surface Y whatsoever. Since $X-\rho$ is orientable we also see that the fibre $X(k)$ must be orientable since the local triviality of the bundle makes $X(k) \times$ (small 2-disc) an open subset of the orientable $X-\rho$. This says nothing, however yet, for orientability of the compact surface $\overline{X(k)} = X(k) \cup \rho$, (e.g. a Klein bottle minus a circle can be an annulus). The fibre homotopy exact sequence for g:

$$\cdots \to \pi_1 (X-\rho) \to \pi_1 (\mathbb{P}^2) \to \pi_0 (X(k)) \to \pi_0 (X-\rho) \to ,$$

shows immediately that <u>$X(k)$ has at most two components</u>.

To complete the proof of Theorem 4.2.3 we need to prove the rather subtle assertions of part (c). We abstract this situation into the following topological proposition.

<u>4.2.4 PROPOSITION</u>: Let X be a connected oriented closed smooth 4-manifold with a non-empty smooth closed 1-submanifold ρ such that there is a bundle projection $g: X-\rho \to \mathbb{P}^2$, with fibre F. Suppose that for each $k \in \mathbb{P}^2$ the closure $\overline{X(k)}$ in X of the fibre $g^{-1}(k) = X(k)$ is $X(k) \cup \rho$, and is a closed 2-submanifold in X. Then

(i) ρ is 2-sided in $\overline{X(k)}$,

(ii) $\overline{X(k)}$ is orientable,

(iii) $\overline{X(k)}$ is connected.

Note: (ii) $\Rightarrow$ (i) of course.

<u>Proof</u>: Since X and ρ are orientable, the normal bundle of ρ in X is orientable and therefore trivial; so ρ has a closed product neighbourhood N homeomorphic ($\approx$) to $\rho \times D^3$ in X. We may (either using the geometry of our hypercomplex X or by topology) choose N

so that $g|_{X-\mathrm{int}N}$ is still a bundle projection. The new fibre G is then a surface with boundary, $\mathrm{int}G \approx F$.

If $\rho_1 (\approx S^1)$ is a component of ρ, and $M_1 = \rho_1 \times S^2$ is the corresponding component of ∂N, the restriction $g|_{M_1}$ to IP^2 is again a fibre bundle. The fibre homotopy exact sequence says that *the fibre of* $g|_{M_1}$ has either 1 or 2 components - necessarily circles.

But, in fact the fibre must have 2 (circle) components because the total space of any S^1-bundle over IP^2 can never be $S^1 \times S^2 (\approx M_1)$. (The same principle holds for $S^1 \times S^n$ fibering over IP^n, $n \geq 2$. The $n = 6$ case is needed for octonionic manifolds.)

Hence the fibre of $g|_{X-\mathrm{int}N}$, say $G(k)$ above any $k \in IP^2$, has boundary $\partial G(k) = \rho \times \{-1,1\}$.

Again by the homotopy exact sequence we see that $\pi_1(M_1)$ maps trivially to $\pi_1(IP^2)$; therefore $g|_{M_1}$ factors through the double covering $\pi : S^2 \rightarrow IP^2$ via a map $\theta : M_1 \rightarrow S^2$ which is itself a bundle map with fibre S^1. Now, the only S^1-bundle over S^2 with total space homeomorphic to $S^1 \times S^2$ is the trivial bundle, so we may choose a homeomorphism $h_1 : M_1 \rightarrow S^1 \times S^2$ such that $\theta = pr_2 \circ h_1$ and so that $g|_{M_1} = \pi \circ pr_2 \circ h_1$, ($pr_2$ is projection to the second factor of course).

We make this choice of homeomorphism h_j for each component ρ_j of ρ, and clearly we can 'radially' extend the union of all the h_j

to a homeomorphism $H : N \to \rho \times D^3$, so that $H(\rho) = \rho \times \{0\}$ and $g|_{N-\rho}$ is $\pi \circ \widehat{pr}_2 \circ H$ where $\widehat{pr}_2 : \rho \times (D^3 - \{0\}) \to S^2$ projects onto the second factor and then normalizes.

We have therefore proved that $g|_{N-\rho}$ is bundle equivalent to a union of copies of the obvious bundle $S^1 \times (D^3 - \{0\}) \to \mathbb{P}^2$.

It follows that the closure in N of any fibre of $g|_{N-\rho}$ is homeomorphic to $\rho \times [-1, 1]$, and hence that for any $k \in \mathbb{P}^2$ the circles ρ are two-sided in $\overline{X(k)} = G(k) \cup_{\rho \times \{-1,1\}} \rho \times [-1,1]$.

We do not yet know that the annuli $\rho \times [-1,1]$ are attached to $G(k)$ so as to produce an orientable $\overline{X(k)}$, but this can now be derived as follows.

Note first that an orientation for a disc neighbourhood of k in $\mathbb{P}^2$ determines a transverse orientation of the normal bundle of $G(k)$ in $X - \text{int}N$, and in particular of $\partial G(k)$ in ∂N. It will suffice to check that the orientations determined in this way by $\pi \circ pr_2 : S^1 \times S^2 \to \mathbb{P}^2$ on $S^1 \times \{\sigma\}$ and $S^1 \times \{-\sigma\}$ (here $\{\sigma, -\sigma\} = \pi^{-1}(k)$) extend compatibly to $S^1 \times \delta$, where δ is the diameter in $\mathbb{R}^3$ connecting σ and $-\sigma$.

Now, an orientation on the segment δ must point inward at one end and outwards at the other end, so it determines opposing transverse orientations about σ and $-\sigma$. Conversely, since the covering involution of S^2 over $\mathbb{P}^2$ is orientation reversing, the transverse orientations we had induced at σ and $-\sigma$ from a local orientation around $k \in \mathbb{P}^2$ must again be opposite, – so they give rise to a consistent orientation of the diameter δ. (It is crucial that we are dealing with even dimensional projective spaces here !).

Thus, $G(k)$ was orientable (being within the orientable $X(k)$, and we now see that the orientation extends over the attached annuli, proving $\overline{X(k)}$ is orientable.

We prove $\overline{X(k)}$ is connected by constructing another bundle over $\mathbb{P}^2$ with closed fibre by a process analogous both to surgery and to blowing-up. Let B be the mapping cylinder of the covering $\pi : S^2 \to \mathbb{P}^2$. Then B is a 3-manifold with boundary S^2 and it fibres over $\mathbb{P}^2$ with fibre $[-1,1]$. In fact B is homeomorphic to a closed regular neighbourhood of $\mathbb{P}^2$ in $\mathbb{P}^3$, and therefore to $(\mathbb{P}^3 - \mathrm{int}D^3)$. Let $W = (X - \mathrm{int}N) \cup \rho \times B$. Then W is orientable and it fibres over $\mathbb{P}^2$ with fibre $\tilde{F} = G \underset{\rho \times \{-1,1\}}{\overset{\partial}{\cup}} \rho \times [-1,1]$. Since W is orientable $\tilde{F}$ is too (just as in the proof for orientability of $X(k)$). We cannot straight-away identify $\tilde{F}$ with $\overline{X(k)} = \overline{F}$ but from the construction we see that they have the same number of components (and indeed $\tilde{F} \approx \overline{F}$ if and only if $\overline{F}$ is orientable, which we know it is).

To see that $\tilde{F}$, and hence $\overline{X(k)}$, is connected we note that B contains a loop projecting to the non-trivial element in $\pi_1(\mathbb{P}^2)$. Therefore, since $\rho \neq \varphi$, W also has such a loop and $\pi_1(W) \to \pi_1(\mathbb{P}^2)$ is surjective. By the fibre homotopy sequence, therefore, $\tilde{F}$ is connected.

This completes Proposition 4.2.4 and Theorem 4.2.3. ////

Our prime example of manifolds with hypercomplex structure we will describe in the next :

4.2.5 PROPOSITION : Let G be torsion-free Fuchsian group operating on $\mathcal{U}$; let Y be the Riemann surface $\mathcal{U}/G$, and $\Omega = \mathcal{U} \cup L \cup \tilde{P}$ be the full region of discontinuity for G (on $\hat{\mathbb{C}}$) . (Here L is the lower half-plane and $\tilde{P}$ is the portion (possibly empty) of Ω on $\mathbb{R} \cup \{\infty\}$) . Then $X = X_G \equiv {}'\mathbb{R}^n \cup \tilde{P}/F_n(G)$ is a manifold with (central) hypercomplex structure, real set P being the ideal boundary of Y , and (with previous terminology) $X(k) = \mathcal{U}/G \cup L/G = Y \cup (-Y)$, and $\overline{X(k)} = \Omega/G$ = the Schottky double of Y .

Proof : This is clear since if $f(z) = \frac{az+b}{cz+d}$, $f \in G$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ then $F_n(f)(V) = \frac{aV+b}{cV+d}$, V being $\mathbb{H}$ or $\mathbb{D}$ variable . Clearly these give transitions in the allowed pseudogroup (expand as Laurent series around $-d/c$) . ////

X_G will be compact precisely when Ω/G is compact.

For a finer description of compact hypercomplex manifolds we need to understand the 'monodromy' of the local identifying maps between the fibres of g ; i.e. if we take a composition of a finite chain of the identifying maps between nearby fibres, say,
$X(k) \to X(k_1) \to X(k_2) \to \cdots \to X(k_n) \to X(k)$, then what will the final conformal automorphism of $X(k)$ look like ? (The identifications always extend to $\overline{X(k)}$ by the identity on P , and they always

preserve angles but not necessarily orientations) . X is assumed to be a compact connected $\mathbb{H}(\mathbb{O})$ manifold in all that follows.

4.2.6 LEMMA : The composition of any finite chain of the canonical mappings between nearby fibres of g always produces on any $X(k)$ either the identity or a certain canonical fixed-point free involution $\tau(k)$.

REMARK : $\tau(k)$ extends to $\overline{X(k)}$ by the identity on ρ , and in fact the proof following shows that near ρ $\tau(k)$ is precisely quaternion (octonion) conjugation. Thus, if ρ is non-empty then $\tau(k)$ is orientation reversing anti-conformal on the connected compact Riemann surface $\overline{X(k)}$.

Proof : Consider an equivalence relation $\sim$ on $X - \rho$ defined as follows : $x, y \in X-\rho$ are $\sim$ equivalent if there exists a finite sequence of points $x_1 = x, x_2, \dots, x_n = y$ such that each consecutive pair x_i, x_{i+1} are in the (connected) domain of some chart (V_i, φ_i) and the U-element assigned by φ_i to x_i and x_{i+1} coincide. This is clearly an equivalence relation.

Of course, if V_i is a small chart then $x_i \sim x_{i+1}$ exactly when the canonical identification of fibres maps x_i to x_{i+1}. Thus $\sim$ corresponds precisely to compositions of several local identifying mappings of the type (4).

We claim that if $x \sim z$ then there exists a <u>single</u> hypercomplex chart (V, φ) with $x, z \in V$ and $\mathcal{U}$-values coincident for x and z via φ. This will follow if we can fuse together charts and extend them 'in the S^2-factor directions'; we achieve this by applying the revolution principle.

To fix notations suppose $x \sim y$ are in (T, θ) chart and $y \sim z$ are in (W, ψ) chart, both being small charts and without loss of generality assume $W \cap T$ is connected. $y \in W \cap T$ of course. Then let $g(y) = k$, so we may assume $W \cap T \cap X(k)$ is a non-empty connected open subset of the fibre $X(k)$. Define extension of the θ-chart by

$$\theta^{ext} = \begin{cases} \theta & \text{on } T \\ F_4(\theta(k) \circ \psi(k)^{-1}) \circ \psi & \text{on } \psi^{-1}(\psi(W) \cap F_4(D)) \end{cases} \qquad (6)$$

where $\psi(k)(W \cap T \cap X(k)) = D \subset \mathcal{U}$ (upper half-plane).
Note, we have arranged $\theta(k) \circ \psi(k)^{-1}$ to be holomorphic on $D \subset \mathcal{U}$ (into $\mathcal{U}$) by replacing θ by its negative if necessary (see paragraph preceding Theorem 4.2.3). Then clearly θ^{ext} will be in the hypercomplex atlas of X and its domain contains x and z with same $\mathcal{U}$-value being assigned by θ^{ext} to both points. We can thus establish our claim by induction.

We can show that on any fibre $X(k)$ the relation $\sim$ identifies points in pairs, - and this is involutory automorphism $\tau(k)$ of $X(k)$ which we have in the statement of Lemma 4.2.6.

In fact, let $x_1 \in X(k)$ be within a small chart (V,φ) around it. Let $\varphi(x_1) = (\varsigma,\sigma) \in U \times S^2 = {}'\mathbb{R}^4$. Thus $k = \mathbb{P}^2$–class of σ. Connect σ to $-\sigma$ by a half-circle γ on S^2. We claim that there exists a chart $(W,\tilde{\varphi})$ which extends φ and $\tilde{\varphi}(W) \subset \mathbb{R}^4$ intersects all the half-planes corresponding to points of γ. If this were not true there would be a first point σ_1 on γ (starting from σ) which is not included in any such chart. But by taking limits and using compactness we see there is some point in $X-\rho$ corresponding to (ς,σ_1). We take any small chart around this point and use the previous fusing of charts argument to extend φ a little further in the S^2–direction, i.e. σ_1 cannot exist. It is not hard to see that σ_1 cannot be $-\sigma$ either.

Thus, we will have a ('big') chart (V,φ) containing x_1 and also containing a point x_2 such that $\varphi(x_2) = (\varsigma,-\sigma)$. Thus $x_1 \sim x_2$ (both on $X(k)$), and $\tau(k)$ interchanges x_1 and x_2 on $X(k)$. Because of the revolution principle a different choice of charts makes no effect on the definition of $\tau(k)$; the proof of this is similar to the equations (5) in the proof of Theorem 4.2.3.

Lemma 4.2.6 is proved. Indeed, note that our γ on S^2 represents the non-trivial element of $\pi_1(\mathbb{P}^2)$, and continuation of charts along γ has led to the involution $\tau(k)$ on $X(k)$. Continuation of charts along a γ_1 which represents the trivial element of $\pi_1(\mathbb{P}^2)$ would produce the identity identification on $X(k)$. This closely resembles a 'monodromy' map $\pi_1(\mathbb{P}^2) \to \mathrm{Aut}(X(k))$. ////

<u>4·2·7 LEMMA</u> : If ρ is non-empty then the fibres $X(k)$ must have two components, Y and $-Y$, (mirror images of each other), each with ideal boundary ρ. $\overline{X(k)}$ must be the Schottky double of Y (and hence connected), and $\tau(k)$ is the canonical reflection on the double.

<u>Proof</u> : Consider any component A of $X(k)$. $\tau(k)$ ($=\tau$ say) maps components to components, so, if $\tau(A) \cap A$ is non-empty then $\tau(A) = A$.

But A must have pieces of ρ as its boundary since we proved $\overline{X(k)} = X(k) \cup \rho$ was connected. Now, τ acts as reflection near points of ρ, (remark following Lemma 4·2·6), and this is impossible if $\tau(A) = A$. So there must be a component distinct from A and all the assertions follow. ////

<u>4·2·8 THEOREM</u> : Let X be a connected closed hypercomplex manifold with real set ρ. Then there is a natural C^{∞} mapping

$$\beta : X-\rho \to Y \times \mathbb{P}^{n-2}, \ (n=4, 8)$$

which is the orientation double covering mapping; here $Y = X(k)/\tau(k)$ is a connected surface with a conformal structure (fix any $k \in \mathbb{P}^{n-2}$).

If ρ is not empty then Y must be simply a component of the fibre $X(k)$ and the compact manifold X is isomorphic to the manifold X_G of Proposition 4·2·5 with $Y = \mathcal{U}/G$ and $X-\rho$ is isomorphic to $Y \times S^{n-2}$.

<u>ADDENDUM</u> : If ρ is empty we may separate the cases (i) $X(k)$ has 2-components, (ii) $X(k)$ is connected. In case (i) each component is a compact Riemann surface Y, $\tau(k)$ maps one component to the other and X is diffeomorphic to $Y \times S^{n-2}$. (If genus $(Y) > 1$ then X is

isomorphic to $X_G = {}' \mathbb{R}^n / \Gamma_n(G)$, as before). In case (ii) $X(k)$ is a compact Riemann surface. If $\tau(k)$ is orientation preserving then $Y = X(k)/\tau(k)$ is itself a compact Riemann surface and X is diffeomorphic to $Y \times S^{n-2}$. If $\tau(k)$ is orientation reversing then Y is a compact non-orientable surface with conformal structure, and X is the orientation double covering of $Y \times \mathbb{P}^{n-2}$.

<u>Proof</u>: The second-factor of the map β is our original fibre bundle g. Fix any $k_0 \in \mathbb{P}^2$, and define $Y = X(k_0)/\tau(k_0)$. We now take any $x \in X\text{-}\rho$, say $x \in X(k)$; identify $X(k)$ with $X(k_0)$ by any chain of the canonical local fibre-identifications. Then the image of x in $X(k_0)$ is well-defined when we go modulo the 'monodromy' $\tau(k_0)$. This defines β and shows it to be two-to-one, and therefore a covering space. Since $X\text{-}\rho$ is oriented, but $Y \times \mathbb{P}^2$ is never orienta`'e for any surface Y, we see β must be the orientation double covering. (Recall that any oriented covering space of a non-orientable manifold factors through the orientation double covering.) Y is connected because X is.

The last statement of the theorem follows by pulling back the bundle $g : X\text{-}\rho \to \mathbb{P}^2$ over S^2 and noting that the new total space $\tilde{X}\text{-}\rho$ has two components, since by Lemma 4.2.7, $X(k_0)$=fibre of g has two components; (use the exact homotopy sequence of the pullback bundle). So each component of $\tilde{X}\text{-}\rho$ must be a copy of $X\text{-}\rho$ itself, and since $\widetilde{X}-\rho$ is a double covering of $Y \times S^2$ (by pulling back β) we see that $X-\rho \cong Y \times S^2$. ////

In view of our previous results all the claims are now established without difficulty. ////

<u>4.2.9 COROLLARY</u> : The only simply-connected compact manifolds which can allow hypercomplex structure are S^4 and $S^2 \times S^2$ (for quaternionic); (S^8 and $S^2 \times S^6$ for octonionic).

<u>Proof</u> : $\pi_1(X) = 0$ implies $\pi_1(X-\rho) = 0$. Then comparing β with the double covering $Y \times S^2 \to Y \times \mathbb{P}^2$ shows $X-\rho \cong Y \times S^2$ for some surface Y. But simple connectivity says $\pi_1(Y) = 0$, and the only simply connected surfaces are $\mathcal{U}$ or S^2. Since ρ is a finite union of circles compactifying $Y \times S^2$ it must have exactly one component for $Y = \mathcal{U}$ case and no components for $Y = S^2$. The results follow. ////

<u>REMARKS</u> : We do not know whether $S^2 \times S^{n-2}$ carries $\mathbb{H}$ (or $\mathbb{O}$) structure.

Our central quaternionic manifolds are extremely akin to, (but nevertheless distinct from) the integrable almost quaternionic manifolds studied by Marchiafava [21] and Salamon [25] et. al.. Actually the derivatives of our transition mappings ((10) in 2.1.5) do not in general lie in the group $GL(1, \mathbb{H}) = \mathbb{H}^*$. Indeed, the Jacobian of one of our coordinate transitions falls in $\mathbb{H}^*$ only for $\varphi(z) = az + b$, (φ as in ((11) in 2.1.5)). See Datta and Nag [11].

Thus the hypercomplex manifolds we are dealing with need not be integrable almost quaternionic manifolds – despite the marked similarity .

It has been proved (see [21] , [25]), that amongst the class of integrable almost quaternionic manifolds the only compact simply – connected one is S^4 . The reader may compare with this our corollary above.

As general references for the work of previous authors we quote [21] , [25] , [26] .

CHAPTER V

ZERO-SETS OF HYPERCOMPLEX FUNCTIONS

In Chapter II we have discussed in detail a geometric characterisation of Fueter mappings and the Fueter transforms. This precipitates a rather surprising application of our theory in this chapter. Indeed, we can characterise the location of the zeroes of quaternionic and octonionic analytic functions defined by convergent power or Laurent series with central coefficients.

We prove that the zero set of any quaternionic (or octonionic) analytic function f with central (i.e., real) coefficients is the disjoint union of codimension two spheres in $\mathbb{R}^4$ (respectively in $\mathbb{R}^8$) and certain purely real points. In particular, for polynomial with real coefficients, the complete root-set is geometrically characterisable from the lay-out of the roots in the complex plane. The root-set becomes the union of a finite number of codimension two Euclidean spheres together with a finite number of real points. We also find the preimages $f^{-1}(A)$ for any quaternion (or octionion) A.

We demonstrate that this surprising phenomenon of complete spheres being part of the solution is very markedly a special 'real' phenomenon. For example, the quaternionic or octonionic N^{th} roots of any non-real quaternion (respectively octonion) turn out to be precisely N distinct points.

An amusing topological application of these results is to exhibit natural self maps of the Euclidean unit spheres of dimension 3 and 7 (viz. the quaternionic and octonionic unit spheres) which are of topological degree N (N any integer) such that every fibre has precisely |N| distinct points, while all the exceptional fibres actually contain codimension one subspheres. The number of exceptional fibres is one for N = 2 and two otherwise. Using the Fueter transform we are also able to study a natural generalisation of these self-mappings on spheres of arbitrary dimension. See [11].

5.1 Zero-sets of hypercomplex mappings by Fueter analysis

Let us recall (see 2.1.5) that if φ has Laurent expansion with real coefficients about real centres, that is,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n(z-c)^n + \sum_{m=1}^{\infty} b_m(z-c)^{-m}, \tag{1}$$

where a_n, b_m, c are reals, the annulus of convergence is $r < |z-c| < R$, then

$$F_4(\varphi)(V) = \sum_{n=0}^{\infty} a_n(V-c)^n + \sum_{m=1}^{\infty} b_m(V-c)^{-m} \tag{2}$$

where $V = x_0+e_1x_1+e_2x_2+e_3x_3$ is a quaternionic variable. Similarly, $F_8(\varphi)$ will be represented by the 'same' Laurent series with V an octonionic variable. The corresponding domains of convergence are the ring-domains $r < ||V-c|| < R$ in Euclidean space $\mathbb{R}^4$ and $\mathbb{R}^8$ respectively.

Also recall the Revolution principle (see 2.2.1), ''if φ is a symmetric (about real axis) function then $F_n(\varphi)$ preserves each $\mathbb{C}_\sigma$, that is, $F_n(\varphi)$ maps $\mathbb{C}_\sigma$ into itself. In fact, $F_n(\varphi)$ restricted to $\mathbb{C}_\sigma \cap F_n(D)$ (for any $\sigma \in S^{n-2}$) is identifiable with the original mapping φ on D when $\mathbb{C}$ is identified with $\mathbb{C}_\sigma$ by either of two possible rotations provided we make the same rotation on the range $\mathbb{C}_\sigma$.''

From this geometrical interpretation it is evident that, whenever φ has Laurent expansion (1) we will have :

$$\left\{V \in \mathbb{R}^n : F_n(\varphi)(V) = A\right\} \equiv F_n(\left\{z : \varphi(z) = A\right\}) \qquad (3)$$

for any real number A.

We see immediately the following :

<u>5.1.1 THEOREM.</u>: Let θ be any Laurent series with central coefficients (in $\mathbb{H}$ or $\mathbb{O}$ variable V), as in (2), convergent in $r < \|V-c\| < R$; namely

$$\theta(V) = \sum_{n=0}^{\infty} a_n(V-c)^n + \sum_{m=1}^{\infty} b_m(V-c)^{-m}$$

(a_n, b_m, c are reals). The zero-set of this function θ, namely $\left\{V ; \theta(V) = 0\right\}$, is simply the above rotation-transform F_4 or F_8 applied to the zero-set of the complex analytic function $\varphi(z)$ (defined by (1)) in $r < |z-c| < R$.

The set F_4 of a point is a 2-sphere orthogonal to each of the planes $\mathbb{C}_\sigma$ provided the point is not on the x_0-axis. On the

x_0-axis of course rotation changes nothing. In fact,

$$F_4(\{\alpha+i\beta\}) = \{V = \alpha + e_1x_1+e_2x_2+e_3x_3 : x_1^2+x_2^2+x_3^2 = \beta^2\} .$$

Similarly for F_8.

In particular, for polynomials we state what we now know separately.

5.1.2 COROLLARY : Let $\{\alpha_1 \pm i\beta_1,\dots,\alpha_m \pm i\beta_m,\gamma_1,\dots,\gamma_k : \alpha_j,\beta_j,\gamma_p$ reals, $\beta_j > 0 ; j = 1,\dots,m, p = 1,\dots,k\}$ be the set of complex roots of the polynomial equation

$$a_N V^N + \dots + a_1 V + a_0 = 0,\ a_j \in \mathbb{R} ;\ j = 0,\dots,N . \tag{4}$$

Then the quaternionic (octonionic) roots of (4) form the set

$$\bigcup_{j=1}^{m} S_{\alpha_j,\beta_j} \cup \{\gamma_1,\dots,\gamma_k\}$$

where $S_{\alpha_j,\beta_j} = F_n(\{\alpha_j \pm i\beta_j\})$, $n = 4$ (or 8) .

REMARK : In the above situation it appears reasonable to think of the sphere S_{α_j,β_j} as occurring with multiplicity $2m_j$, where m_j is the multiplicity of the root $(\alpha_j + i\beta_j)$ of the complex polynomial (4). The total multiplicity over all components of the quaternionic or octonionic solution set then adds up to the degree N.

When we wish to solve $\theta(V) = A$, $A \notin \mathbb{R}$, we can still apply our rotation process. Since $A \notin \mathbb{R}$, all the roots must lie in precisely the same $\mathbb{C}_\sigma$ which contains A itself. Therefore it only remains to rotate A into the standard position of the complex plane (namely, $A = a_0 + \sum_{j=1}^{n-1} e_j a_j \longmapsto a_0 + ia$, where $a = (\sum_{j=1}^{n-1} a_j^2)^{1/2} > 0$, $n=4$ (or 8) and consequently the roots of $\theta(V) = A$ are nothing but the roots of $\varphi(z) = a_0 + ia$ rotated back into the $\mathbb{C}_\sigma$ position. Note, σ here is $(a_1/a, \ldots, a_{n-1}/a) \in S^{n-2}$. We state therefore the following theorem.

<u>5.1.3 THEOREM</u> : The root-set of $\theta(V) = A$, $A \in \mathbb{R}$, is

$$S = \left\{ \alpha + \beta \sum_{n=1}^{n-1} e_j \frac{a_j}{a} : \alpha + i\beta \text{ is a root of } \varphi(z) = a_0 + ia \text{ in } \mathbb{C} \right\}$$

where $A = a_0 + \sum_{j=1}^{n-1} e_j a_j$, $a = (\sum_{j=1}^{n-1} a_j^2)^{1/2} > 0$. The multiplicity of $\alpha + \beta \sum_{j=1}^{n-1} e_j a_j / a$ is the same as that of $\alpha + i\beta$ as a root of $\varphi(z) = a_0 + ia$.

(Here $n = 4$ or 8 according as V is a quaternionic or octonionic variable.)

<u>REMARK</u> : The set identity (3) for any quaternion $A = a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3$, is

$$\left\{ V \in \mathbb{R}^n : F_n(\varphi)(V) \in F_n(\{A\}) \right\} \equiv F_n(\{z : \varphi(z) = a_0 + ia\}) \quad (5)$$

where, $a = (a_1^2 + a_2^2 + a_3^2)^{1/2} > 0$.

<u>5.2 Zero sets of quaternionic and octonionic polynomials with central coefficients by algebraic method</u>.

The Corollary 5.1.2 can be proved by straightforward algebra. We deal only with the quaternionic case since no new ideas come in for octonions.

Firstly note that any quaternionic polynomial with real coefficients can be factored into the product of quadratic and linear polynomials with real coefficients. Namely,

$$V^N + a_{N-1}V^{N-1} + \dots + a_1V + a_0 = (V^2 + b_1V + c_1)\cdots$$
$$\cdots(V^2 + b_mV + c_m)(V + d_{2m+1}) \cdots (V + d_N) \qquad (6)$$

with $b_j^2 - 4c_j < 0$; $j = 1,\dots,m$. Consider therefore such a quadratic polynomial

$$V^2 + bV + c = 0 \qquad (7)$$

where $b, c \in \mathbb{R}$, $b^2 - 4c < 0$ (and therefore $c > 0$).
If $V = x_0 + e_1x_1 + e_2x_2 + e_3x_3$ is a root of (7) then one notes that $\|V\|^2 = c$ and $V^2 = 2x_0 V - \|V\|^2 = 2x_0V - c$. Consequently, $2x_0 + b = 0$. This implies that V lies on the sphere

$$\left\{V = x_0 + e_1x_1 + e_2x_2 + e_3x_3 : x_0 = -b/2, x_1^2 + x_2^2 + x_3^2 = (4c - b^2)/4\right\}.$$

The Corollary 5.1.2 now follows since there are no zero divisors in $\mathbb{H}$ or $\mathbb{O}$.

It is to be noted that when there exist infinitely many roots of a polynomial like (6) then the polynomial actually allows infinitely many distinct factorisations.

5.3 A recursive representation of powers of a hypercomplex variable, with applications.

As any non-zero quaternion is the product of a non-negative real number and a quaternion of norm 1, we will consider only quaternions or octonions with norm 1. In fact,

$$\{V : V^N = A\} \equiv \{\|A\|^{1/N} \times W : W^N = A/\|A\|\}, (\|A\|^{1/N} > 0).$$

Now $\|V\| = 1$ is equivalent to

$$V^2 = 2x_0V-1, \text{ where } V = x_0+e_1x_1+e_2x_2+e_3x_3 \quad (8)$$

(Again V could be an octonion without any extra trouble.) This gives, inductively,

$$V^k = P_k(x_0)\,V - P_{k-1}(x_0), \quad (9)$$

where P_k is a real polynomial of degree $(k-1)$ in the single (real) variable x_0. The P_k satisfy the recursive relations

$$P_{k+2} - 2x_0 \cdot P_{k+1} + P_k = 0 \quad (10)$$

Note that $P_1 = 1$ and $P_2 = 2x_0$ from equation (8). Equations (9) and (10) provide a convenient representation of powers of a $\mathbb{H}$ or $\mathbb{O}$ variable.

As an application note the analysis of

$$V^N = A = a_0 + e_1a_1 + e_2a_2 + e_3a_3 . \quad (11)$$

Then V is a root of (11) precisely when

$$x_0P_N(x_0) - P_{N-1}(x_0) = a_0 \quad (12)$$

and

$$P_N(x_0)x_j = a_j, \; j = 1,2,3. \quad (13)$$

If A is real then the solutions of (11) are described in Corollary 5.1.2 (and may be obtained from (12) and (13) also). If A is non-real, then note that $P_N(x_0) \neq 0$, because otherwise $V^N = -P_{N-1}(x_0)$ would be real. In this case (12) has exactly N

solutions, all real. (Indeed, the real parts of the complex roots of $z^N = a_0 + i(a_1^2 + a_2^2 + a_3^2)^{1/2}$ are precisely the roots of (12). Therefore, by (13), $V^N = A$ has exactly N distinct solutions in $\mathbb{H}$. This confirms the conclusion of Theorem 5.1.3.

NOTE. The roots of $1/V^N = A$ also behave similarly, since

$$\frac{1}{V^N} = \frac{\bar{V}^N}{\|V\|^{2N}} \tag{14}$$

REMARK : The polynomials $P_n(x)$ above are universal for all the algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. They are essentially related to Tchebysheff's polynomials T_n, defined by $T_n(\cos\theta) = \cos n\theta$. In fact, both systems are solutions of the same difference equation (10), with the respective initial conditions:

$$\begin{cases} P_1(x) = 1 \\ P_2(x) = 2x \end{cases} \qquad \begin{cases} T_0(x) = 1 \\ T_1(x) = x \, . \end{cases} \tag{15}$$

They are related by the following formulae :

$$\begin{aligned} 2T_n &= P_{n+1} - P_{n-1} \\ P_{n+1} &= 2(T_n + T_{n-2} + \cdots) \, . \end{aligned} \tag{16}$$

In the case of $\mathbb{C}$ these formulae can be thought of as consequences of de Moivre's formula.

5.4 Applications to topology

Consider the natural maps $f_N(V) = V^N$, $(N = \pm 1, \pm 2, \ldots,)$, as self-maps of the unit spheres S^3 or S^7 – V being a quaternionic (respectively octonionic) variable of unit norm. From Theorem 5.1.3 and by Section 5.3 we know that the preimage via f_N of every non-real point is precisely N distinct points. The preimages of the two real points ± 1 are described in Corollary 5.1.2. Let us denote by $\lambda(N, \pm 1)$ the number of codimension one subspheres which are contained in $f_N^{-1}(\pm 1)$. Then, Corollary 5.1.2 implies :

$$\lambda(N,1) = \begin{cases} \frac{|N|-1}{2} \text{ if } N \text{ is odd,} \\ \frac{|N|}{2} - 1 \text{ if } N \text{ is even,} \end{cases}$$

$$\lambda(N,-1) = \begin{cases} \frac{|N|-1}{2} \text{ if } N \text{ is odd,} \\ \frac{|N|}{2} - 1 \text{ if } N \text{ is even.} \end{cases} \qquad (17)$$

It is convenient to note that we need not restrict to spheres of dimensions 3 or 7 only because we have the Fueter transform of $\varphi_N(z \longmapsto z^N)$ at our disposal in any dimension. Thus $f_N = F_{d+1}(\varphi_N)$ is again a real-analytic self-map of S^d. The fibres of these general f_N mappings are also describable just as above because the 'Revolution Principle' applies.

It is natural to ask for the topological degree of the mappings f_N and whether their restriction above $S^d - \{\pm(1,0,\cdots,0)\}$ is a $|N|$-sheeted covering space or not. The answers are interesting and provided below.

First of all we notice from Corollary 5.1.2 that the $\lambda(N,1) + \lambda(N,-1)$ $(=|N|-1)$ codimension one subspheres in S^d separate $S^d - \{\pm(1,0,\cdots,0)\}$ into $|N|$ cylinders (that is $S^{d-1} \times (0,1)$) each of which maps homeomorphically onto $S^d - \{\pm(1,0,\cdots,0)\}$. The two ideal boundary components in each cylinder are getting collapsed to the points $(1,0,\cdots 0)$ and $(-1,0,\cdots,0)$. Thus:

5.4.1 PROPOSITION: The restriction of f_N to $S^d - \{f_N^{-1}(\pm(1,0,\cdots,0))\}$ is a trivial $|N|$-fold covering of $S^d - \{\pm(1,0,\cdots,0)\}$.

As for the degree, the answers are given in:

5.4.2 PROPOSITION: If d is odd, $f_N : S^d \to S^d$ has degree $\deg(f_N) = N$. If d is even,

$$\deg(f_N) = \begin{cases} +1 & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

REMARK: Algebraic topologists have already been interested in special cases of the above. See Dold [13, p.65] for the complex and quaternionic case.

It is convenient to prove a lemma first.

5.4.3 LEMMA: For $N \geq 1$, f_N is homotopic to $I * (-I)*I*(-I)*\cdots*((-1)^{N-1}I)$. Here I denotes the identity mapping on S^d, $-I$ denotes the antipodal mapping, and $*$ denotes the usual operation by which mappings are composed in the homotopy group $\pi_d(S^d)$. (We will give a formal definition of $*$ in the proof.)

<u>Proof</u> : Let us parametrise S^d by polar coordinates $(\theta_0,\cdots,\theta_{d-1})$:

$$\begin{aligned} x_0 &= \cos\theta_0 , \\ x_1 &= \sin\theta_0 \cos\theta_1 , \\ x_2 &= \sin\theta_0 \sin\theta_1 \cos\theta_2 , \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ x_{d-1} &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \cdots \cos\theta_{d-1} , \\ x_d &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1} , \end{aligned} \tag{18}$$

$\theta_1,\cdots,\theta_{d-1} \in [0,\pi)$, $\theta_0 \in [-\pi,\pi)$. In that case one realises (say by the 'Revolution principle') that $F_{d+1}(\varphi_N)(\theta_0,\cdots,\theta_{d-1}) = (N\theta_0,\theta_1,\cdots,\theta_{d-1})$.

Now define $*$ as follows : we will say two self-maps f, g : $S^d \to S^d$ are $*$-composable if

$$g(0, \theta_1,\cdots,\theta_{d-1}) = f(\pm\pi,\theta_1,\cdots,\theta_{d-1}) \text{ for all } (\theta_1,\cdots,\theta_{d-1}) \in [0,\pi)^{d-1}$$

Then define

$$(f*g)(\theta_0,\cdots,\theta_{d-1}) = \begin{cases} f(2\theta_0,\theta_1,\cdots,\theta_{d-1}) & \text{if } \theta_0 \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \\ g(2\theta_0-\pi,\theta_1,\cdots,\theta_{d-1}) & \text{if } \theta_0 \in \left[-\pi,-\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2},\pi\right] . \end{cases} \tag{19}$$

This is clearly the usual * product in the definition of composition in $\pi_d(S^d)$. It is now trivial to check that

$$I*(-I)*I*\cdots*((-1)^{N-1}I)(\theta_0,\ldots,\theta_{d-1}) = (N\theta_0,\theta_1,\ldots,\theta_{d-1})$$

for all $(\theta_0,\ldots,\theta_{d-1}) \in [-\pi,\pi) \times [0,\pi)^{d-1}$.

Proof of 5.4.2 Recall the following standard facts about the degree :

$$\deg(f \circ g) = \deg(f)\deg(g) \tag{20}$$

$$\deg(f * g) = \deg(f)+\deg(g) . \tag{21}$$

Using these relations we immediately obtain the claimed values of $\deg(f_N)$ for N positive. For N negative, notice that

$$V^N = V^{-|N|} = \overline{V}^{|N|} = j_{d+1} \circ V^{|N|} ,$$

where j_{d+1} is the 'conjugation map,' in $\mathbb{R}^{d+1}$ that is ,

$$j_{d+1}(x_0,x_1,\ldots,x_d) = (x_0,-x_1,\ldots,-x_d) .$$

But $\deg(j_{d+1}) = (-1)^d$ (see for example Vick [27]), consequently the proposition is proved completely.

CHAPTER VI

FUETER STRUCTURE, HYPERCOMPLEX STRUCTURE, YANO'S F-STRUCTURE, ISHIHARA'S QUATERNION STRUCTURE AND FOLIATIONS ON SMOOTH MANIFOLDS

This chapter compares various differential geometric and analytic structures on smooth manifolds which have previously studied in the literature. All the structures are related to quaternionic and octonionic structures.

In particular we treat the relations between (a) Fueter structure (Datta [9] , Fueter [14]), (b) hypercomplex structure (Nag, Hillman and Datta [24]), (c) Yano's f-structure (Yano [28] , Ishihara and Yano[20]) and quaternion structure (Ishihara [19]).

In Section 6.1,we have proved that a n-dimensional Fueter manifold M_n has 2 and n-2 dimensional transverse foliations with a natural complex structure on the two dimensional leaves. Moreover, the 2-dimensional foliation is obtained from a submersion $g : M_n \longrightarrow S^{n-2}$. Similar results are true for higher type Fueter structures.

In Section 6.2, we have established the relation between Fueter structure and Yano's f-structure. A Fueter manifold is a smooth manifold with integrable f-structure.

In Section 6.3, the relations between Fueter and hypercomplex and Ishihara's quaternion structure have been discussed. In this section we we utilise the theorem on non-existence of almost complex structures on products of even-dimensional spheres.

6.1 Foliations on Fueter manifolds

In the final section of this chapter we will see from a different point of view that Fueter manifolds are foliated. But in this section we will show the same by finding appropriate pseudogroup on which one can model Fueter manifold and which shows that the Fueter manifolds are foliated.

Recall that Fueter manifold is a smooth manifold which is modelled on the pseudogroup of diffeomorphisms which are Fueter mappings.

Let M_n be a Fueter manifold. Let $x^* \in M_n$ and (W,φ) be a coordinate chart with $\varphi(x^*) = (x_0^*, x_1^* k_1^*, \dots, x_1^* k_{n-1}^*)$, where $(k_1^*, \dots, k_{n-1}^*) \in S^{n-2}$ and $x_1^* > 0$. (See Chapter II for the definitions.)

If $(k_1^*, \dots, k_{n-1}^*) \neq (0, \dots, 0, 1)$, choose

$$\widetilde{W} = \{ x \in W : (k_1, \dots, k_{n-1}) \neq (0, \dots, 0, 1), \text{ where } \varphi(x) = (x_0, x_1 k_1, \dots, x_1 k_{n-1}) \} \quad (1)$$

and define $\widetilde{\varphi} : \widetilde{W} \to \mathbb{R}^n$ as

$$\widetilde{\varphi}(x) = (x_0, x_1, \frac{k_1}{1-k_{n-1}}, \cdots, \frac{k_{n-2}}{1-k_{n-1}}) \tag{2}$$

(The last coordinates correspond to sterographic projection from $(0,\cdots,0,1)$ of S^{n-2} on $\mathbb{R}^{n-2}$.)

Suppose $(k_1^*,\cdots,k_{n-1}^*)$ happens to be the ''north pole'' $(0,\cdots,0,1)$

Consequently $\varphi(x^*) = (x_0^*,0,\cdots,0,x_1^*)$, with $x_1^* > 0$, choose

$$\widetilde{W} = \{ x \in W : (k_1,\cdots,k_{n-1}) \neq (0,\cdots,0,-1) \text{ where } \varphi(x) = (x_0,x_1k_1,\cdots,x_1k_{n-1}) \} \tag{3}$$

and define $\widetilde{\varphi} : \widetilde{W} \longrightarrow \mathbb{R}^n$ given by

$$\widetilde{\varphi}(x) = (x_0,x_1,k_1/(1+k_{n-1}),\cdots,k_{n-2}/(1+k_{n-1})). \tag{4}$$

(Sterographic projection from ''south pole'' now!). Then,

$$\{(\widetilde{W},\widetilde{\varphi}) : (W,\varphi) \text{ are Fueter charts}\} \tag{5}$$

form an atlas, and with respect to this atlas M_n is a Foliated manifold modelled on the pseudogroup of local diffeomorphisms.

$G_n^1 = \{ f \equiv (f^1,\cdots,f^n) : D(\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^n : f^1$ and f^2 depend only on first two variables and $f^1 + if^2$ is complex analytic (in the obvious sense) and either $(f^3,\cdots,f^n)(x_1,\cdots,x_n) = (x_3,\cdots,x_n)$ or $(f^3,\cdots,f^n)(x_1,\cdots,x_n)$

$$= (\frac{x_3}{x_3^2+\cdots+x_n^2}, \cdots, \frac{x_n}{x_3^2+\cdots+x_n^2}), \text{ when } (x_1,x_2,0,\cdots,0) \notin D \} . \tag{6}$$

$$G_n^1 = \Big\{ f : D(\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}^n \ ; \ f(x_1,\dots,x_n) = (f^1(x_1,x_2),\ f^2(x_1,x_2),\ x_3,\dots,x_n)$$
$$\text{and } f^1 + if^2 \text{ is complex analytic in } x_1 + ix_2 \Big\}$$
$$\cup \Big\{ f : D(\subseteq \mathbb{R}^n - \{(x_1,x_2,0,\dots,0) : x_1,x_2 \in \mathbb{R}\}) \longrightarrow \mathbb{R}^n :$$
$$f(x_1,\dots x_n) = \Big(f^1(x_1,x_2), f^2(x_1,x_2), \frac{x_3}{x_3^2+\dots+x_n^2}, \dots, \frac{x_n}{x_3^2+\dots+x_n^2}\Big)$$
$$\text{and } f^1+if^2 \text{ is complex analytic in } x_1 + ix_2 \Big\} \qquad (7)$$
$$= P_n^1 \cup Q_n^1 \quad \text{(say)}$$

Thus we have (from the definition of G_n^1).

<u>6.1.1 THEOREM</u> : Each Fueter manifold M_n has 2 and $n-2$ dimensional transverse foliations with natural complex structure on 2-dimensional leaves induced by the Fueter structure. ////

Conversely, let M_n be modelled on the pseudogroup $G_n^1 = P_n^1 \cup Q_n^1$ and Φ be the complete atlas corresponding to G_n^1. Define a relation $\sim$ on Φ as : $(W,\varphi)\sim(V,\psi)$ if there exist $(U_1,\varphi_1),\dots,(U_m,\varphi_m) \in \Phi$ with $(U_1,\varphi_1) = (W,\varphi)$, $(U_m,\varphi_m) = (V,\psi)$, $U_i \cap U_{i+1}$ non empty and $\varphi_{i+1} \circ \varphi_i^{-1} \in P_n^1$ for $i = 1,\dots,m-1$. Then, $\sim$ is an equivalence relation which define a partition of Φ into two classes (as M_n is path connected) Φ_1 and Φ_2 (say).

Let $(W,\varphi) \in \Phi$. If $(W,\varphi) \in \Phi_1$, then define $\underset{\sim}{\varphi} : W \longrightarrow \mathbb{R}^n$ as

$$\underset{\sim}{\varphi}(x) = \Big(x_0, \frac{2x_1x_2}{x_2^2+\dots+x_{n-1}^2+1}, \dots, \frac{2x_1x_{n-1}}{x_2^2+\dots+x_{n-1}^2+1}, \frac{x_1(x_2^2+\dots+x_{n-1}^2-1)}{x_2^2+\dots+x_{n-1}^2+1}\Big) \qquad (8)$$

where, $\varphi(x) = (x_0, x_1, \cdots, x_{n-1})$. (The formulas come from the inverse sterographic projection.)

On the other hand, if $(W,\varphi) \in \Phi_2$, then define $\underset{\sim}{\varphi} : W \longrightarrow \mathbb{R}^n$ as

$$\underset{\sim}{\varphi}(x) = \left(x_0, \frac{2x_1x_2}{x_2^2+\cdots+x_{n-1}^2+1}, \cdots, \frac{2x_1x_{n-1}}{x_2^2+\cdots+x_{n-1}^2+1}, \frac{x_1(1-x_2^2-\cdots-x_{n-1}^2)}{x_2^2+\cdots+x_{n-1}^2+1}\right) \quad (9)$$

where $\varphi(x) = (x_0, x_1, \cdots, x_{n-1})$.

Then, $\underset{\sim}{\Phi} = \{(W, \underset{\sim}{\varphi}) : (W,\varphi) \in \Phi\}$ forms an atlas, and by direct calculation one can check that for two intersecting charts $(W, \underset{\sim}{\varphi})$ and $(V, \underset{\sim}{\psi})$, $\underset{\sim}{\psi} \circ \underset{\sim}{\varphi}^{-1}$ is a Fueter mapping.

<u>6.1.2 COROLLARY</u> : Structure groups of Fueter manifolds are reducible to $U(1) \times O(n-2)$.

<u>Proof</u> : This follows from the fact that Fueter manifolds can be modelled on G_n^1 and the Jacobians of maps in G_n^1 are matrices in $U(1) \times O(n-2)$ clearly.

<u>REMARK</u> : In fact, the foliation discussed above is obtained from a submersion $g : M_n \longrightarrow S^{n-2}$. (See Section 4.1.) More explicitly let M_n be a Fueter manifold with Fueter atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$. Then for $x \in M_n$, if $\varphi_\alpha(x) = (x_0, x_1k_1, \cdots, x_1k_{n-1})$ with $x_1 > 0$, where $x \in U_\alpha$, then for any other $\beta \in I$ with $x \in U_\beta$, we have $\varphi_\beta(x)$ is of one form $(\xi, \eta k_1, \cdots, \eta k_{n-1})$. So, the mapping $g : M_n \longrightarrow S^{n-2}$, given by :

$$g(x) = (k_1, \cdots, k_{n-1}), \quad (10)$$

is well defined. Clearly this g is a submersion.

Similar types of results are true for Fueter manifolds of higher type (See Section 3.1 for definition). Let M_{pn} be a higher type-p Fueter manifold. In this case there exists a submersion

$$g^p : M_{pn} \longrightarrow (S^{n-2})^p \qquad (11)$$

and M_{pn} has two transverse foliations of respective leaf dimensions $2p$ and $(n-2)p$. As before the 2p-dimensional leaves carry naturally the structure of p-dimensional complex manifolds.

6.2 Relations between Fueter structure and Yano's f-structure

K. Yano [28] had defined f-structure on manifolds which are some generalisation of complex structure. In this section we will show how f-structure comes in a very natural way on a Fueter manifold. And we also show that the f-structure is integrable (in the sense of Ishihara and Yano [20]).

Let us first recall some definitions and propositions about f-structure (Yano and Kon [29, p.379]).

6.2.1 DEFINITION : A structure on an n-dimensional manifold M_n given by a non-null tensor field f of type (1,1) satisfying

$$f^3 + f = 0 . \qquad (12)$$

is called an f-structure.

6.2.2 PROPOSITION (Yano and Kon [29, p.379]) : The rank of f is a constant, say r. (i.e. for each $x \in M_n$ the rank of $f_x : T_x(M_n) \longrightarrow T_x(M_n)$ is r.)

6·2·3 PROPOSITION (Yano and Kon [29, p·386]) : A necessary and sufficient condition for an n-dimensional manifold M_n admit an f-structure f of rank r is that r is even (say r = 2m) and the group of the tangent bundle of M_n be reducable to $U(m) \times O(n-2m)$ ·

6·2·4 OBSERVATION : From Corollary 6·1·2 and Proposition 6·2·3 it follows that each *Fueter manifold* has rank 2 f-structure·

In this section we will show this explicitly·

6·2·5 DEFINITION : An f-structure f is said to be integrable if there exists a coordinate system in which f has the constant components

$$f = \begin{bmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (13)$$

$r = 2m$ having rank of f·

To show the existence of f-structure on Fueter manifold we will first show that there is a natural way to define rank 2 f-structure on $'\mathbb{R}^n = \mathbb{R}^n - \{x_0 \text{ axis}\}$, which can be identified with $U \times S^{n-2}$ in a fixed fashion· Indeed, we map :

$$(x_0+e_1x_1+\cdots+e_{n-1}x_{n-1}) \longmapsto (x_0+iy,(\frac{x_1}{y},\ldots,\frac{x_{n-1}}{y})) \in U \times S^{n-2} \qquad (14)$$

where $y = (x_1^2+\cdots+x_{n-1}^2)^{1/2}$ · (See Section 2·2·)

Let us also identify $T('\mathbb{R}^n)$ by $'\mathbb{R}^n \times \mathbb{R}^n$ ·

For $k = (k_1,\ldots,k_{n-1}) \in S^{n-2}$, recall $\lambda(a,b,c,k)$

$$= \begin{bmatrix} a & -bk_1 & -bk_2 & \cdots & -bk_{n-1} \\ bk_1 & a-(1+k_1^2)c & ck_1k_2 & \cdots & ck_1k_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ bk_{n-1} & ck_1k_{n-1} & ck_2k_{n-1} & \cdots & a-(1+k_{n-1}^2)c \end{bmatrix} \qquad (15)$$

These were studied in Chapter III, since they are the Jacobian matrices of Fueter mappings. See Section 3.2 of Chapter III.

Consider the map J on $T(\mathbb{R}^n)$ given by :

$$J((z,k),v) = ((z,k),\lambda(0,1,0,k)v) \qquad (16)$$

Then $J^3 = -J$ (i.e., $J^3+J = 0$)

Define a f-structure $f_{\mathbb{R}^n}$ on $\mathbb{R}^n$ by

$$(f_{\mathbb{R}^n}(X))((z,k)) = J(X(z,k)) \qquad (17)$$

Then $f^3_{\mathbb{R}^n} + f_{\mathbb{R}^n} = 0$ and clearly it is of rank 2.

Now, let M_n be a Fueter manifold. Take $x \in M_n$ and let (U,φ) be a Fueter chart around x and $\varphi(x) = (z,k)$. Define,

$$f_{M_n}(x) = (d_x\varphi)^{-1} \circ f_{\mathbb{R}^n}(z,k) \circ d_x\varphi \qquad (18)$$

f_{M_n} is well defined, since $f_{\mathbb{R}^n}$ depends only on k and $f_{\mathbb{R}^n}$ commutes with the Jacobians of Fueter maps, since they are of the form $\lambda(a,b,c,k)$ (See Section 3.2). This f_{M_n} is a f-structure on

M_n and the matrices of f_{M_n} with respect to Fueter charts are of the form $\lambda(0,1,0,k)$ at the points of $(g)^{-1}(k)$.

We intend to show that this f-structure on M_n is integrable.

Therefore, consider the atlas of coordinate charts of $'\mathbb{R}^n$ consisting of two charts (W^*,φ^*) and (V^*,ψ^*) where

$$W^* = {}'\mathbb{R}^n - \{(x_0,0,\cdots,0,x_{n-1}) : x_{n-1} > 0\}$$

$$V^* = {}'\mathbb{R}^n - \{(x_0,0,\cdots,0,x_{n-1}) : x_{n-1} < 0\} \qquad (19)$$

$\varphi^* : W^* \longrightarrow \mathbb{R}^n$ given by :

$$\varphi^*(x_0,\cdots,x_{n-1}) = (x_0,y,\frac{x_1}{y-x_{n-1}},\cdots,\frac{x_{n-2}}{y-x_{n-1}})$$

and $\psi^* : V^* \longrightarrow \mathbb{R}^n$ given by :

$$\psi^*(x_0,\cdots,x_{n-1}) = (x_0,y,\frac{x_1}{y+x_{n-1}},\cdots,\frac{x_{n-2}}{y+x_{n-1}}) \qquad (20)$$

where $y = (x_1^2+\cdots+x_{n-1}^2)^{1/2}$.

Then, with respect to these new coordinate charts, $f_{'\mathbb{R}^n}$ is of the form $\lambda(0,1,0,1)$, and hence with respect to the charts $(\tilde{W},\tilde{\varphi})$ defined in (1), (2), (3) and (4) where (W,φ) are Fueter charts, f_{M_n} is of the form $\lambda(0,1,0,1)$. That is,

$$f_{M_n} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (21)$$

Consequently, f_{M_n} is clearly integrable. Thus we have

6.2.6 THEOREM : Every Fueter manifold has a natural integrable rank-2 f-structure of Yano.

6.3 Relations between Fueter, hypercomplex and Ishihara's quaternion structures on 4n-dimensional smooth manifolds

Ishihara's ([19]) quaternion structure is the following :

6.3.1 DEFINITION : Let M_n be an n-dimensional manifold with a rank-3 real vector bundle V consisting of tensors of type (1,1) over M_n satisfying the following conditions :

In any coordinate neighbourhood W of M_n, there exists a local basis F, G, H of V such that

$$F^2 = G^2 = H^2 = -I, \text{ (I denote the identity tensor)}$$

$$GH = F = -HG, \; HF = G = -FH, \; FG = H = -GF \; .$$

Then the bundle V is called an almost quaternion structure in M_n, and (M_n, V) an almost quaternion manifold.

6.3.2 OBSERVATION (Ishihara [19]) : An almost quaternion manifold is necessarily of dim. $n = 4m$.

6.3.3 DEFINITION : The almost quaternion structure V in M_n is called integrable if there exists a coordinate system in which F, G, H have constant components of the form

$$F = \begin{bmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & -I_m \\ 0 & 0 & -I_m & 0 \\ 0 & I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{bmatrix} \qquad (22)$$

where $n = 4m$, I_m is the identity $m \times m$ matrix.

6.3.4 OBSERVATION : Every almost quaternion manifold is almost complex and integrable almost quaternion manifold is a complex manifold.

Proof : Each of the tensor fields F, G, H defines an almost complex structure. In case the almost quaternion structure is integrable (by Definition 6.3.3) each F, G, H is locally constant – consequently the almost complex structures F, G, H are integrable.

The following examples will demonstrate that there is no necessary logical relationship between Fueter structure and Ishihara's quaternion structure, and also between hypercomplex (See Section 3.1 of Chapter III) and Ishihara's quaternion structure.

6.3.5 EXAMPLE : Let, $M = S^1 \times S^1 \times S^1 \times S^1$. From Theorem 4.1.1 (Chapter III) it is easily seen that M does not admit any Fueter structure. But M has an integrable quaternion structure. To show this, consider the almost quaternion structure on $\mathbb{R}^4$ given (globally) by :

$$F = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Now, M can be modelled on the pseudogroup

$$\left\{ f : D\ (\subseteq \mathbb{R}^4) \longrightarrow \mathbb{R}^4 : f(x_0,x_1,x_2,x_3) = (x_0+m_0, x_1+m_1, x_2+m_2, x_3+m_3)\ , \text{ where } m_p \in \mathbb{Z}\ ,\ p = 0,1,2,3 \right\} \tag{24}$$

Then, of course, F, G, H commute with the derivatives of the maps in this pseudogroup. So, we can transfer the almost quaternion structure of $\mathbb{R}^4$ to M by means of coordinate charts corresponding to the above pseudogroup. Thus, if (W,φ) is a chart around $x \in M$, then

$$\tilde{F}_x = (d_x\varphi)^{-1} \circ F_{\varphi(x)} \circ (d_x\varphi), \quad \tilde{G}_x = (d_x\varphi)^{-1} \circ G_{\varphi(x)} \circ (d_x\varphi),$$

$$\tilde{H}_x = (d_x\varphi)^{-1} \circ H_{\varphi(x)} \circ (d_x\varphi) \tag{25}$$

define $\tilde{F}, \tilde{G}$ and $\tilde{H}$ on M. And as F, G, H (and therefore $\tilde{F}, \tilde{G}$ and $\tilde{H}$ locally) are of the form (23), this quaternion structure is integrable.

6.3.6 EXAMPLE : Let $M_{4n} = S^2 \times S^{4n-2}$, ($n \geq 3$ be any positive integer).

From Theorem 4.1.1 (Chapter IV), it follows that M_{4n} admits Fueter structure (of type-1 and dimension $4n$). But M_{4n} does not admit any Ishihara's almost quaternion structure. This follows from observation 6.3.4 and the fact that $S^2 \times S^{2m}$, for $m \geq 4$, never allows any almost complex structure. See Chapter VII for a proof of this last fact.

6.3.7 EXAMPLE : Let $M_n = S^n \equiv \mathbb{R}^n \cup \{\infty\}$, $n = 4$ or 8. We can give on this a hypercomplex (respectively quaternionic or octonionic) structure analogous to the complex structure of the Riemann sphere, by assigning the identity chart on $\mathbb{R}^n (n = 4 \text{ or } 8)$ and obtaining $V \longmapsto \frac{1}{V}$ as the transition function to the obvious chart covering $(\mathbb{R}^n - \text{origin}) \cup \{\infty\}$. Indeed M_n is quaternionic or octonionic projective space.

But this does not admit any almost quaternion structure. This follows from the Observation 6.3.4, since S^4 and S^8 do not permit any almost complex structure (Borel and Serre [3]).

CHAPTER VII

NON-EXISTENCE OF ALMOST COMPLEX STRUCTURES ON PRODUCTS OF EVEN-DIMENSIONAL SPHERES

In this chapter we prove the theorem that $S^{2p} \times S^{2q}$ allows almost complex structure if and only if $(p,q) = (1,1), (2,1), (3,1), (3,3)$ (harmlessly assuming $p \geqslant q$). This question became important for us in setting up some counter-examples to compare and contrast the various types of hyper-complex structures we met in Chapter VI. The proof of the theorem uses characteristic class techniques - the chief tools being Bott's periodicity and integrality theorems. See Datta and Subramanian [12]. This chapter is rather independent of the rest of the thesis and contains a single theorem which is quite interesting in its own right. The result has applications in Chapter VI - see example 6.3.6.

7.1 As is very well-known, Borel and Serre [3] had proved in 1953 that S^2 and S^6 are the only even dimensional spheres allowing almost complex structures. In passing it may be remarked that Calabi and Eckmann [7] had shown that complex structures do exist on $S^{2p+1} \times S^{2q+1}$ for all $p,q \geq 0$. Here we prove the following

7.1.1 THEOREM: The only products of even dimensional spheres that allow almost complex structures are $S^2 \times S^2$, $S^6 \times S^2$, $S^6 \times S^6$, $S^4 \times S^6$, $S^4 \times S^2$, $S^2 \times S^4$, $S^2 \times S^6$

REMARK: Of course $S^2 \times S^2$, $S^6 \times S^2$, and $S^6 \times S^6$ allow almost complex structures by Borel and Serre [3]. It is easy to see that $S^4 \times S^2$ is diffeomorphically embeddable in $\mathbb{R}^7$. In fact, for any $n, m > 0$, $S^n \times S^m$ is embedded in $\mathbb{R}^{n+m+1}$ in the following way :

$$S^n \times S^m \xrightarrow{\alpha \times I} (\mathbb{R}^n \times \mathbb{R}^+) \times S^m \xrightarrow{\beta} \mathbb{R}_n^{n+m+1} \xrightarrow{i} \mathbb{R}^{n+m+1}$$

where, $\alpha(x_0,\dots,x_n) = (x_0,\dots,x_{n-1},x_n+2)$,

i is the inclusion mapping,

I is the identity mapping.

Here $\mathbb{R}_n^{n+m+1} = \mathbb{R}^{n+m+1} - \{(x_0,\dots,x_{n-1},0,\dots,0) : x_j \in \mathbb{R}\} \equiv \mathbb{R}^{n+m+1} - \mathbb{R}^n$ and we define

$$\beta((x_0,\dots,x_{n-1},y),(k_1,\dots,k_{m+1})) = (x_0,\dots,x_{n-1},yk_1,\dots,yk_{m+1}).$$

Thus $\beta^{-1}(x_0,\dots,x_{n-1},x_n,\dots,x_{n+m}) = ((x_0,\dots,x_{n-1},y),(\frac{x_n}{y},\dots,\frac{x_{n+m}}{y}))$

where, $y = (x_n^2+\dots+x_{n+m}^2)^{1/2} > 0$. The mapping β is a diffeomorphism.

Therefore by Calabi's result [6], which says that any 6-dimensional orientable manifold immersed in $\mathbb{R}^7$ allows almost complex structure, we see that $S^4 \times S^2$ and $S^2 \times S^4$ do allow almost complex structure.

<u>REMARK</u> : Notice that our diffeomorphism β between $\mathbb{R}_n^{n+m+1}$ and $(\mathbb{R}^n \times \mathbb{R}^+) \times S^m$ is a natural generalisation of the ideas we used critically in Chapter II, see 2.1.1.

<u>7.1.2 DEFINITION</u> : The Chern character $ch(W)$ of a complex vector bundle W of rank n over a base B is defined to be the formal sum

$$n + \sum_{k=1}^{\infty} \frac{s_k(c_1(W),\dots,c_n(W))}{k!} \in H^*(B,Q)$$

Here s_k is the polynomial $s_k(c_1,\ldots,c_n) = \sum_{i=1}^{n} t_i^k$, where c_j, the j-th Chern class of W, is the j-th elementary symmetric function in the variables $t_1,\ldots,t_n$. See Milnor and Stasheff [22, p. 195] for more details.

<u>Proof of the theorem</u>

Let, $T(S^{2p} \times S^{2q})$ be the tangent bundle of $S^{2p} \times S^{2q}$. Suppose there is an almost complex structure on $S^{2p} \times S^{2q}$. Then we have

$$T(S^{2p} \times S^{2q}) \otimes C = V \oplus \bar{V} \qquad (2)$$

where V is a complex vector bundle and $\bar{V}$ is its conjugate vector bundle; V is isomorphic to $T(S^{2p} \times S^{2q})$ as a real vector bundle.

First of all <u>we claim that the Chern character of</u> V <u>is integral</u> i.e.

$$ch(V) \in H^*(S^{2p} \times S^{2q}; \mathbb{Z}).$$

<u>Proof of the Claim</u>

Recall :

(a) The integrality theorem of Bott, which says that for any complex vector bundle W on S^{2k}

$$ch(W) \in H^*(S^{2k}; \mathbb{Z}). \qquad (3)$$

(See Husemoller [16, p. 280].)

(b) The Bott periodicity theorem, which says that

$$K(S^{2p} \times S^{2q}) = K(S^{2p}) \otimes K(S^{2q}) \qquad (4)$$

(See Husemoller [16, p.137].)

By (b) V can be written as a direct sum of tensor products of bundles on S^{2p} and on S^{2q}. Consequently, the multiplicativity of Chern character implies, utilising (a), that our claim of integrality is true.

Now, $H^{2k}(S^{2p} \times S^{2q}) = 0$ for $k \neq 0, p, q, p+q$ and $H^{2k}(S^{2p} \times S^{2q}, \mathbb{Z}) = \mathbb{Z}$ for $k = 0, p, q, p+q$ except when $p = q = k$. (5)

Therefore by our previous claim $s_k/k!$ is an integer except when $p = q = k$ and if $p = q = k$ then $(s_k/k!)^2$ is an integer.

Also $c_k = 0 = s_k$ for $k \neq 0, p, q, p+q$. (6)

Our strategy is to obtain a contradiction to integrality.

From Newton's relation (Milnor and Stasheff [22, p.196])

$$s_n - c_1 s_{n-1} + c_2 s_{n-2} + \dots + (-1)^{n-1} c_{n-1} s_1 + (-1)^n nc_n = 0 \tag{7}$$

Assume $p > q \geq 1$

By using (6) we have

$$s_q + (-1)^q qc_q = 0 \tag{8}$$

For, $p - q \neq q$ i.e., $p \neq 2q$

$$s_p + (-1)^q c_q s_{p-q} + (-1)^{p-q} c_{p-q} s_q + (-1)^p p c_p = 0$$

or $\quad s_p + (-1)^p p c_p = 0 \qquad$ (by (6)) $\qquad$ (9a)

For, $p - q = q$ i.e., $p = 2q$

$$s_{2q} + (-1)^q c_q s_q + 2q c_{2q} = 0. \tag{9b}$$

$$s_{p+q} + (-1)^q c_q s_p + (-1)^p c_p s_q + (-1)^{p+q} (p+q) c_{p+q} = 0 \tag{10}$$

Recall also, $c_k\,(V \oplus \bar{V}) = \sum_{m+n=k} c_m(V)\; c_n(\bar{V})$

and $c_j(\bar{V}) = (-1)^j\, c_j(V)$

Therefore,

$$c_{p+q}(V \oplus \bar{V}) = c_{p+q}(V) + c_p(V)\; c_q(\bar{V}) + c_q(V)\; c_p(\bar{V}) + c_{p+q}(\bar{V})$$

$$= (1+(-1)^{p+q})(c_{p+q}(V)+(-1)^q\, c_p(V)\; c_q(V)) \qquad (11)$$

For $p \neq 2q$ i.e., $p - q \neq q$

$$c_p(V \oplus \bar{V}) = c_p(V) + c_{p-q}(V)\; c_q(\bar{V}) + c_q(V)\; c_{p-q}(\bar{V}) + c_p(\bar{V})$$

$$= c_p(V) + c_p(\bar{V}) = (1 + (-1)^p)\; c_p \qquad (12a)$$

For $p = 2q$

$$c_p\,(V \oplus \bar{V}) = c_{2q}(V \oplus \bar{V}) = c_{2q}(V) + c_q(V)\; c_q(\bar{V}) + c_{2q}(\bar{V})$$

$$= 2c_{2q} + (-1)^q\, c_q^2 \qquad (12b)$$

$$c_q\,(V \oplus \bar{V}) = c_q(V) + c_q(\bar{V}) = (1 + (-1)^q)c_q \qquad (13)$$

Since the tangent bundle of a sphere, and hence of $S^{2p} \times S^{2q}$ is stably trivial, all the Chern classes of $V \oplus \bar{V}$ are zero. So,

$$(1+(-1)^{p+q})\;(c_{p+q} + (-1)^q\, c_p\, c_q) = 0 \qquad (14)$$

$$(1+(-1)^p)\; c_p = 0 \;\text{ if }\; p \neq 2q \qquad (15a)$$

$$2c_p + (-1)^q\, c_q^2 = 0 \;\text{ if }\; p \neq 2q \qquad (15b)$$

$$(1+(-1)^q)\; c_q = 0 \qquad (16)$$

Let us first consider the <u>case when</u> $p \neq 2q$ and $p > q$.

From (8), (9a) and (10) we have,

$$s_{p+q} = (-1)^{p+q}\,(p+q)\,(c_p c_q - c_{p+q}) \qquad (10a)$$

Subcase - I: p and q both are odd

From (14) we have

$$c_{p+q} = c_p \times c_q = \frac{s_p}{p} \times \frac{s_q}{q} \quad \text{(by (8) and (9a))}$$

Therefore,

$$\frac{c_{p+q}}{(p-1)!\,(q-1)!} = \frac{s_p}{p!} \times \frac{s_q}{q!} \in \mathbb{Z} \quad \text{(by (6))}$$

i.e. $\frac{\chi(S^{2p} \times S^{2q})}{(p-1)!(q-1)!} \in \mathbb{Z}$

Here we have used the fact that top Chern class = Euler characteristic

i.e., $c_{p+q}(V) = \chi(V) = \chi(T(S^{2p} \times S^{2q})) = 4.$

Therefore, $\frac{4}{(p-1)!(q-1)!} \in \mathbb{Z}$

This is a contradiction for $p \geq 5$.

Subcase - II: p is odd and q is even

We have from (16)

$$c_q = 0$$

which implies $s_q = 0$. (by (8))

Therefore, $s_{p+q} = (p+q)\, c_{p+q}$ (by (10a))

or $\frac{c_{p+q}}{(p+q-1)!} = \frac{s_{p+q}}{(p+q)!} \in \mathbb{Z}$

i.e. $\frac{4}{(p+q-1)!} = \frac{\chi(V)}{(p+q-1)!} \in \mathbb{Z}$

This is a contradiction for q even and $p > q$.

Subcase – III : p is even and q is odd.

From (15a) we have

$$c_p = 0$$

Therefore, $s_{p+q} = (p+q)\, c_{p+q}$

which implies $\frac{4}{(p+q-1)!} = \frac{c_{p+q}}{(p+q)!} \in \mathbb{Z}$

This is a contradiction for $p \geq 4$

Subcase – IV: p and q both are even

From (14) we have

$$c_{p+q} = - c_p\, c_q$$

$$= - \frac{s_p}{p} \times \frac{s_q}{q}$$

or $$\frac{c_{p+q}}{(p-1)!(q-1)!} = - \frac{s_p}{p!} \times \frac{s_q}{q!} \in \mathbb{Z}$$

i.e. $$\frac{4}{(p-1)!(q-1)!} \in \mathbb{Z}$$

This is a contradiction for p, q positive even integers with $p > q$.

Thus the theorem is proved for all positive integers p, q with $p > q \geq 1$ and $p \neq 2q$. ($(p, q) \neq (3, 1)$.)

Now, we are considering the case when $p = 2q$. We have from (8), (9b) and (10)

$$s_q + (-1)^q\, q\, s_q = 0 \qquad (8)$$

$$s_{2q} + (-1)^q\, c_q\, s_q + 2q\, c_{2q} = 0 \qquad (9b)$$

$$s_{3q} + (-1)^q\, c_q\, s_{2q} + c_{2q}\, s_q + (-1)^q\, 3q\, c_{3q} = 0 \qquad (10b)$$

which give

$$c_q = (-1)^{q-1}\, \frac{s_q}{q}$$

$$c_{2q} = -\frac{s_{2q}}{2q} + \frac{s_q^2}{2q^2}$$

$$(-1)^q 3 c_{3q} = -\frac{s_{3q}}{q} + \frac{3}{2} \times \frac{s_q s_{2q}}{q^2} - \frac{s_q^3}{2q^3}$$

or

$$\frac{(-1)^q\, 6\mathcal{X}(S^{4q} \times S^{2q})}{((q-1)!)^3} = -2\, \frac{s_{3q}}{(3q)!} \times \frac{(3q)!}{q!(q-1)!(q-1)!}$$

$$+\; 3\, \frac{s_q}{q!} \times \frac{s_{2q}}{(2q)!} \times \frac{(2q)!}{q!(q-1)!} - \left(\frac{s_q}{q!}\right)^3 \qquad (17)$$

Each term in the right hand side is an integer.

So, $\dfrac{24}{((q-1)!)^3} \in \mathbb{Z}$

This is a contradiction for $q \geq 4$.

For $q = 3$ and therefore $p = 6$ we have,

$$2c_6 - c_3^2 = 0 \qquad \text{(by (15b))}$$

and $s_3 - 3c_3 = 0$ or $c_3 = \frac{s_3}{3}$ (by (8))

From (9b) we have,

$$s_6 - c_3 s_3 + 6\,c_6 = 0$$

or $$s_6 - 3c_3^2 + 3c_3^2 = 0$$

or $$s_6 = 0$$

Therefore from (10b)

$$s_9 + \frac{1}{2}\,s_3^3 - 9 \times 4 = 0$$

or $$\left(\frac{s_9}{9!}\right) \cdot \frac{9!}{12} + \left(\frac{s_3}{3!}\right)^3 - 3 = 0$$

or $$\left(\frac{s_3}{3!}\right)^3 = 3\left(1 - 2 \times (7!)\,\frac{s_9}{9!}\right)$$

which implies 3 is a factor of $\left(\frac{s_3}{3!}\right)^3$ and hence 3 is a factor of $\frac{s_3}{3!}$

which implies 9 is a factor of $\left(1-2 \times (7!) \cdot \frac{s_9}{9!}\right)$. This is a contradiction since 9 is a factor of $2 \times (7!) \cdot \frac{s_9}{9!}$

Also for $q = 2$ and therefore $p = 4$, we have, (from (16))

$$c_q = 0$$

and therefore $c_{p+q} = 0$ (by (14))

This is a contradiction since $c_{p+q}(V) = \chi(V) = 4$.

Thus the theorem is proved for all p,q with $p = 2q$ $(q \geq 2)$.

Finally, we are only left with the cases when $p = q$ (> 1).

In this case we have the following relations

$$s_q + (-1)^q q c_q = 0 \qquad (8)$$

$$s_{2q} + (-1)^q c_q s_q + 2q c_{2q} = 0 \qquad (10c)$$

which give

$$s_{2q} = q(c_q^2 - 2c_{2q}) . \qquad (18)$$

And

$$c_{2q}(V \oplus \bar{V}) = c_{2q}(V) + c_q(V)c_q(\bar{V}) + c_{2q}(\bar{V})$$

or $\quad 0 = (2c_{2q} + (-1)^q c_q^2)$ (11c)

and $\quad 0 = c_q(V \oplus \bar{V}) = (1+(-1)^q)\, c_q$ (12c)

Thus if q is odd then

$$2c_{2q} = c_q^2 = (s_q/q)^2 \qquad (\text{by}(8))$$

or $\quad \dfrac{2c_{2q}}{((q-1)!)^2} = (\dfrac{s_q}{q!})^2 \in \mathbb{Z}$

or $\quad \dfrac{8}{((q-1)!)^2} \in \mathbb{Z}$

This a contradiction for $q \geq 5$.

If q is even, then by (12c)

$$c_q = 0$$

Therefore,

$$s_{2q} + 2q c_{2q} = 0 \qquad (\text{by}(10c))$$

or $\frac{c_{2q}}{(2q-1)!} = -\frac{s_{2q}}{(2q)!} \in \mathbb{Z}$

i.e. $\frac{4}{(2q-1)!} \in \mathbb{Z}$

This again the desired contradiction.

Thus we have proved the Theorem for $p = q$ $(p = q \neq 1,3)$. ////

REMARK : Notice that when using stable triviality of $T(S^{2k})$ we are tacitly assuming that the standard differentiable structure of S^{2k} is being used. We do not know how our theorem would be affected in case exotic differentiable structures (if such exist) are imposed on the spheres.

REMARK : Note that $S^4 \times S^2$ allows almost complex structures and is the trivial S^2 bundle over S^4. In fact, there is a famous twisted S^2 bundle over S^4 which also not only allows almost complex structures but actually allows the complex manifold structure of $\mathbb{C}P^3$. This arises in the Penrose twistor theory. It may be worthwhile to investigate almost complex structures and complex structures on twisted sphere bundles over spheres rather than in just trivial product bundles treated above.

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E R R A T A

1. 6th and 7th lines (..... ~ the chief tools being Bott's periodicity and integrality theorem. .....) of page 93 should read

   " ..... ~ the chief tools being Künneth isomorphism theorem and integrality theorem of Bott. ..... ".

2. The last line of page 93 should read

   " complex structures as S^2 and S^6 allow. ....." .

3. The last three lines of page 95 should read

   " (b) The Künneth isomorphism, which says that
   $H^*(S^{2p}) \otimes H^*(S^{2q}) \cong H^*(S^{2p} \times S^{2q})$ (4)
   (See Milnor and Stasheff [22, p.268] .)" .

4. The first three lines of page 96 should read

   " The multiplicativity of Chern character implies, utilising (a) and (b), that our claim of integrality is true " .

5. One extra Remark of the end of Chapter VII

   <u>REMARK</u> : From the last part ($S^{2p} \times S^{2p}$ allows almost complex structures implies $p = 1$ or 3) it follows that " S^n allows almost complex structures implies $n = 2$ or 6 " .

6. One more example at the end of Chapter VI.

   <u>6.3.B EXAMPLE</u> : Let $M = S^3 \times S^5$. Then TM is trivial (since, TS^3 is trivial and TS^5 is stably trivial) and therefore M allows Ishihara's quaternion structure. But from Corollary 4.2.9 (Chapter IV) M does not allow Hypercomplex structure.