EXACT ORDER MATRICES AND THE LINEAR COMPLEMENTARITY PROBLEM

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TO MY
PARENTS

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PREFACE

A real n by n matrix M is called an $N(P)-$ matrix of exact order k, if the principal minors of M of order upto $(n-k)$, are negative (positive) and $(n-k+1)$ to n are positive (negative). In this dissertation, we study the properties of these matrix classes using the linear complementarity problem $lcp(q, M)$, for each $q \in R^n$. Emphasis is placed on N and P-matrices of exact orders 0,1 and 2.

Chapter 1 provides the necessary background on linear complementarity and its connection with game theory. Lemke's algorithm is introduced and a brief survey, on some already known classes of matrices in relation to the $lcp(q, M)$ is brought out.

A complete characterization of the class of exact order 0 based on the number of solutions to the $lcp(q, M)$ for each $q \in R^n$, is presented in chapter 2. Also, a sign reversal property for N-matrices is proved. Counterexample to a well-known characterization on P-matrices is given in the end, while a proof of the same result is provided for the size of the matrix, $n \leq 3$.

Chapter 3 deals with matrices of exact order 1. Here, results on the number of solutions to the $lcp(q, M)$ for each $q \in R^n$ is presented for both the categories of exact order 1. In the end, a generalization of exact order one is given, and a characterization of these matrices in terms of the $lcp(q, M)$ is brought out.

Chapter 4 is on matrices of exact order 2 or more; we at first define three different categories that evolve in these matrices and study their Q-nature. A complete characterization of the class of exact order 2 and a partial one of the general exact order k are presented. We also look into the following question: When $v(M) < 0$ and M is of exact order k, can we say that $-M \in Q$? We present a few modifications of the already known algorithms that would process the $lcp(q, M)$ when M is a matrix of exact order 2. Also, the difficulties that crop up as we go up the hierarchy in these classes, are cited in the end.

C^1-differentiable maps with the Jacobians being matrices exact order k, are studied in chapter 5. Gale-Nikaido result is extended for C^1-maps with Jacobians being exact order k of the first category. Finally, a result on the global univalence of C^1-maps when the Jacobian is a matrix of exact order 2 is proved.

NOTATION

1. R^n: Euclidean n-space.

2. R^n_+: The nonnegative orthant of R^n.

3. $|J|$: The cardinality of the set J.

4. $R^{n \times n}$: The space of all real n by n matrices. I will denote the identity matrix of appropriate order.

5. A vector is regarded as a column and superscript t is used to denote transposition. 'e' denotes the vector of all 1s.

6. We say that a vector $x \in R^n$ is unisigned, if either $x_i \geq 0$ for all $1 \leq i \leq n$, or $x_i \leq 0$ for all $1 \leq i \leq n$.

7. Let $M \in R^{n \times n}$. For subsets $J, K \subseteq \{1, 2, \ldots, n\}$, we denote by M_{JK} and M^{JK}, the submatrices of M and M^{-1} respectively, with rows and columns corresponding to the index sets J and K. For $J = \{1, \ldots, n\}, M_{JK}$ is written for simplicity as $M_{.K}$.

8. The matrix M_{JJ} for$J \subseteq \{1, \ldots, n\}$ denotes the principal submatrix of M. When $|J| = k$, M_{JJ} is called the principal submatrix of order k. Then, the determinant of M_{JJ}denoted by $\det(M_{JJ})$, is called a principal minor of order k.

9. The $(i, j)th$ entry of M and M^{-1} are denoted by m_{ij} and m^{ij} respectively. For any $J \subseteq \{1, 2, \ldots n\}, \bar{J}$ denotes the set $\{1, 2, \ldots, n\} \backslash J$.

10. $v(M)$: The minimax value of the two-person zero-sum game, with M as the payoff matrix.

11. $J \Delta K$: $(J \cup \overline{K}) \cap (\overline{J} \cup K)$, the symmetric difference between J and K.

12. For a $q \in R^n$, the number of solutions $lcp(q, M)$ has, will be denoted by $m(q)$.

Contents

Chapter 1

PRELIMINARIES

1.1 Exact order matrices

Matrix theory is one of the fundamental tools in mathematical sciences. Since the subject began, the theory of matrices has come to have ramifications in many areas of pure and applied mathematics. In the later years, its growth has been directed towards introducing and studying various classes that arose naturally in numerous applications. One such renowned and classical class, is the class of positive definite matrices. This class is well-studied, and has a long history in matrix theory.

Gale and Nikaido [18], generalised positive definiteness to positivity of principal minors, which they named as P-matrices (symmetry in positive definiteness being omitted). Inada [21], introduced N-matrices whose principal minors are all negative, and presented results on these two classes (P and N), to answer some questions that were raised by economists. Olech, Parthasarathy and Ravindran ([46] and [47]) in sequel defined almost P (almost N) matrices, whose proper principal submatrices are P (N) with the determinants alone being negative (positive). Putting these classes together, one can in a nutshell, define a matrix of exact order k, as follows:

Definition: A matrix $M \in R^{n \times n}$ is called an *N-matrix (P-matrix) of exact* *order* k, $1 \leq k \leq n$, if every principal submatrix of order $(n-k)$ is an N-matrix, (P-matrix), and if every principal minor of order r, $n-k < r \leq n$, is positive (negative). M is called a *matrix of exact order* k, if it is either a P-matrix or an N-matrix of exact order k.

An $N(P)-$matrix, is an $N(P)-$matrix of exact order 0, and an almost N (almost P) matrix is an $N(P)-$matrix of exact order 1. Thus, the classes of exact order matrices unify some of the already known classes of matrices. Our aim in this dissertation, is to characterize some of these classes of matrices.

The motivation for studying these classes, besides their matrix theoretic importance, arises from a mathematical programming problem, known as the linear complementarity problem. The next section takes us to this.

1.2 The problem and its importance

In this section, we introduce the linear complementarity problem and see some of its applications.

The problems that crop up in an industry, where there is an objective to be met under a set of constraints, like time factor, labour force, etc., can all be formulated as mathematical programming problems. The advantage of such a formulation is that, some of the mathematical programming problems have efficient techniques in finding out a solution and in turn providing an answer to the industry. Many such mathematical programming problems can be formulated as what is known in the literature as the linear complementarity problem.

For a given n-vector q and $M \in R^{n \times n}$, the linear complementarity problem, denoted by $lcp(q, M)$, is that of finding nonnegative vectors $w \in R^n$, $z \in R^n$ such that

$$\begin{aligned} w - Mz &= q \\ w^t z &= 0, \end{aligned} \tag{1.1}$$

A pair of vectors (w, z) that satisfies (1.1), is said to be a solution for the $lcp(q, M)$.

This problem arises in mathematical economics. In mathematical programming, the linear programming problem and the convex quadratic programming problem can be transformed into linear complementarity problems [42]. In certain problems in engineering, like plastic analysis of structures, plastic fluctual behaviour of reinforced beams and free-boundary problems of journal bearings, the linear complementarity problem finds a wide applicability. For more details on this refer to Duval [13], Ingleton [22], Maier [32], Kaneko [23] and [24], Pang et al [49], Samelson et al [62] and Cottle and Dantzig [8]. The problem of finding Nash equilibrium points of bimatrix games was first posed as a linear complementarity problem by Lemke [30]; and as a result, he proposed an algorithm, now well-known as Lemke's algorithm, for solving $lcp(q, M)$ for any $q \in R^n$, for some classes of matrices. We will see this algorithm later in this chapter, in detail.

A considerable amount of literature in linear complementarity problem deals with the following questions:

(a) When does the $lcp(q, M)$ have a solution ?

(b) Given that the $lcp(q, M)$ has a solution for a $q \in R^n$, can the number of solutions it has, be determined ?

Several classes of matrices have special importance regarding these questions. We call a matrix $M \in R^{n \times n}$, a Q-matrix if the $lcp(q, M)$ has a solution for every $q \in R^n$. A sufficient condition for $M \in Q$ was given by Karamardian [26]; later Murty [43] and Saigal [58] generalized these conditions. Unfortunately, an efficient method for determining membership in the class of Q has not yet been discovered.

There have been classes of matrices defined in the linear complementarity theory, based on the signs of their principal minors. They have been studied in detail, and some of their properties identified are found useful in other areas of mathematical programming. See Ingleton [22], Murty [42] and Cottle [6]. One of our aims in this dissertation, is to study the nature of solutions for the $lcp(q, M)$ for each $q \in R^n$, where the matrix M under consideration is an exact order matrix. Besides their connection with linear complementarity, the niceties of the matrix structure, the inverse sign pattern and the game theoretic importance of these classes of matrices are also brought out. In the end, global univalence results of C^1-differentiable functions, with Jacobian matrix being an exact order matrix are discussed.

1.3 Definitions and well-known facts in complementarity

In this section, we introduce the required terminologies for the $lcp(q, M)$. Unless otherwise stated, M always stands for a square matrix of order n.

We introduce the notation here, that is followed in this thesis. For $\phi \neq J, K \subseteq \{1, \ldots, n\}$, M_{JK} is a submatrix obtained from $M = (m_{ij}) \in R^{n\times n}$ by retaining the rows indexed by J and columns indexed by K. Similarly, when the matrix M is nonsingular, by M^{JK} we denote the submatrix of M^{-1} whose row and column indices are given by J and K respectively; m^{ij} will denote the (i, j)th entry of M^{-1}. $\overline{J}$ denotes the index set $\{1, \ldots, n\}\backslash J$. When $J = \{1, \ldots, n\}$, we write M_{JK} as $M_{.K}$ and M^{JK} as $M^{\cdot K}$; the jth column of M and M^{-1} are denoted by $M_{.j}$ and $M^{\cdot j}$ respectively. M is often written in the partitioned form as

$$M = \begin{bmatrix} M_{JJ} & M_{J\overline{J}} \\ M_{\overline{J}J} & M_{\overline{J}\overline{J}} \end{bmatrix}, \tag{1.2}$$

for some $J \subseteq \{1, \ldots, n\}$. When M_{JJ} is nonsingular, the schur complement of M with respect to M_{JJ} is denoted by (M/M_{JJ}) and is given by

$$(M/M_{JJ}) = M_{\overline{J}\overline{J}} - M_{\overline{J}J}M_{JJ}^{-1}M_{J\overline{J}}.$$

I denotes the identity matrix of appropriate order and $e \in R^n$ stands for the vector whose entries are all 1s. A vector $q \in R^n$, for $\phi \neq J \subset \{1, \ldots, n\}$ is written in the partitioned form as $q = (q_J, q_{\overline{J}})^t$ (after a suitable rearrangement of rows and columns if necessary).

As in linear programming, one deals with feasible and optimal bases, in the theory of complementarity we talk about complementary bases.

Complementary matrix: Consider $[I \vdots -M]$. For $j \in \{1, \ldots, n\}$, a pair of column vectors $\{I_{.j}, -M_{.j}\}$ of $[I \vdots -M]$ is called a complementary pair. A matrix $B \in R^{n \times k}$ with $B_{.j}$ the j-th column of B, being either $I_{.j}$ or $-M_{.j}$ for $1 \leq j \leq k \leq n$, is called a *complementary matrix* of $[I \vdots -M]$.

Let B be a complementary matrix of order n by n. Let

$$J = \{j : -M_{.j} \text{ is a column of } B\}. \tag{1.3}$$

We then write B as (if necessary after a principal rearrangement),

$$B = \begin{bmatrix} -M_{JJ} & 0 \\ -M_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix}. \tag{1.4}$$

Hence, we sometimes denote such a complementary matrix as $B(J)$. Here, if $J = \phi$, then $B(J) = I$.

Complementary basis: Let B be a matrix of order $n \times n$ whose columns are columns of $[I \vdots -M]$. B is said to be a *basis*, if its columns are linearly independent. B is called a *complementary basis*, if its columns are complementary also.

A solution (w, z) of the *lcp*(q, M) is said to be a *complementary basic feasible solution*, if the set of columns $I_{.j}$ for j such that $w_j > 0$ and the set of columns $-M_{.k}$ for k such that $z_k > 0$ form a linearly independent set. We note that this need not in general contain n columns.

For a $q \in R^n$, the number of solutions the *lcp*(q, M) has will be denoted by $m(q)$.

Nondegenerate q: A solution (w, z) for the *lcp*(q, M) is said to be *nondegenerate*, if it has exactly n coordinates positive. A vector $q \in R^n$ is said to be *nondegenerate* with respect to M, if each solution (w, z) for the *lcp*(q, M) is nondegenerate.

Complementary cones: The nonnegative cone generated by a complementary matrix B of $[I \vdots -M]$ denoted by pos(B), is defined as

$$\text{pos}(B) = \{Bx : \ x \in R^n, \ x \geq 0\}.$$

When B is a square matrix of order n, pos(B) is called a complementary cone of $[I \vdots -M]$. The complementary cone pos(B) is called a *full cone*, if $\det(B) \neq 0$. We call pos(B), a *degenerate cone*, if B is singular and a *strongly degenerate cone*, if there exists a $0 \neq x \geq 0, \ x \in R^n$ such that $Bx = 0$.

Hence, if the $lcp(q, M)$ has a solution for a $q \in R^n$, there exists a complementary cone pos(B) of $[I \vdots -M]$ such that $q \in$ pos(B). We use the notation $q \in \text{pos}(B)^c$ to mean that $q \notin pos(B)$. There are atmost 2^n complementary cones in $[I \vdots -M]$. We denote the union of complementary cones of $[I \vdots -M]$ by $D(M)$. It can be seen that

$$D(M) = \{q : q \in R^n, \ lcp(q, M) \text{ has a solution}\}.$$

So, a matrix M is a Q-matrix if and only if $D(M) = R^n$.

Principal rearrangement: By $\bar{M}$, a principal rearrangement of $M \in R^{n \times n}$, we mean that there exists a permutation matrix $P \in R^{n \times n}$ (which is a matrix of 0's and 1's with every column and every row having exactly one nonzero entry), such that $M = PMP^t$.

We have the following easy consequence.

Theorem 1.1 *If M is a Q-matrix, then PMP^t is Q for all permutation matrices P.*

This can be observed easily, as only the indices get rearranged in w and z in a similar way as in PMP^t.

Principal pivot transform: The concept of principal pivot transforms is due to Tucker [68]. Let $B(J)$ be a nonsingular complementary matrix. Note that we can write the $n \times 2n$ matrix $[I \vdots -M]$ as $[B(J) \vdots \bar{B}(J)]$ where $\bar{B}(J)$ is the matrix of columns of $[I \vdots -M]$ not in $B(J)$. We can transform the original problem $lcp(q, M)$ to an equivalent problem $(\bar{q}, \mu_J(M))$ where $\mu_J(M) =$

$B(J)^{-1}\bar{B}(J)$ and $\overline{q} = B(J)^{-1}q$. The matrix $\mu_J(M)$ is then called a principal pivot transform of M with respect to the complementary matrix $B(J)$. Given the partitioned form of M, with respect to the index set J, the principal pivot transformed matrix (PPT in short) $\mu_J(M)$, with respect to the nonsingular $B(J)$ is written as follows:

$$\mu_J(M) = \begin{bmatrix} M_{JJ}^{-1} & -M_{JJ}^{-1}M_{J\overline{J}} \\ M_{\overline{J}J}M_{JJ}^{-1} & M_{\overline{J}\overline{J}} - M_{\overline{J}J}M_{JJ}^{-1}M_{J\overline{J}} \end{bmatrix}.$$

More often we write, for the simplicity of notation, a PPT of M by $\overline{M}$.

We have the following lemma which relates the principal minors of M to those of its PPT, $\overline{M}$.

Lemma 1.1 *Let $B(J)$ be a nonsingular complementary matrix with the index set as defined in (1.2). Let $\overline{M}$ be the principal pivot transform of M with respect to $B(J)$. Then for any $K \subseteq \{1, \ldots, n\}$,*

$$det\overline{M}_{KK} = detM_{K\Delta J}/detM_{JJ} \tag{1.5}$$

where $K\Delta J$ is the symmetric difference between K and J.

Proof: For a proof of this, we refer to Cottle [5].

Proper and reflecting facets: Consider a submatrix C of order n by $(n-1)$, of $[I \vdots -M]$ which is a complementary matrix. We call pos(C) an $(n-1)$-face of $[I \vdots -M]$ if rank $(C) = n-1$. Let $F =$ pos(C) be an $(n-1)$-face of $[I \vdots -M]$. A complementary cone pos(B) is said to be *incident on* F if the columns of C are also columns of B. Thus, for any $(n-1)$-face F, there are exactly two complementary cones incident on it. If we assume that the matrix M is nondegenerate (i.e. none of the principal minors is zero), then it is valid to say that any n by $(n-1)$ complementary matrix of $[I \vdots -M]$ is an $(n-1)$-face of $[I \vdots -M]$. Further the subspace generated by such an $(n-1)$-face will be a hyperplane in R^n. We say that two complementary cones incident on an $(n-1)$-face F are *properly situated* if they lie on opposite sides of the hyperplane generated by F. F is called a *proper face*, if the two complementary cones incident on F are properly situated. If the two cones incident on it are not properly situated, then F is called a *reflecting face*. The above notions have been introduced by Saigal [59]. He proves a lemma which we require in the subsequent chapters.

Lemma 1.2 *Let $M \in R^{n\times n}$. Let F be an $(n-1)$-face of $[I \vdots -M]$, with the two complementary cones incident on it being $pos(B)$ and $pos(B')$. Let $det(B) \neq 0$. F is proper if and only if*

$$det(B')/det(B) \leq 0. \tag{1.6}$$

These concepts are dealt with in detail by Saigal and Stone [61]. Saigal [59] observed the relationship between the complementary cones and the faces of $[I \vdots -M]$ and those of $[I : -\overline{M}]$, where $\overline{M}$ is a PPT of M, which is stated as follows:

Theorem 1.2 *Let $\overline{M}$ be a PPT of $M \in R^{n\times n}$. Then there exists a one to one correspondence between the cones and faces of $[I \vdots -M]$ and those of $[I \vdots -\overline{M}]$. Hence, if $M \in Q$, then $\overline{M} \in Q$.*

1.4 Two-person Zero-Sum Games

Our results in $lcp(q, M)$ depend on a well-known minimax theorem due to Von Neumann for two-person zero-sum games, which we describe here. For more details on this subject we refer to Parthasarathy and Raghavan [52].

A two-person zero-sum matrix game can be described as follows:

Player 1 chooses an integer i ($i = 1, 2, \ldots, m$) and player 2 selects an integer j ($j = 1, 2, \ldots, n$) simultaneously. Then player 1 pays player 2 an amount m_{ij} (which may be positive, zero or negative). $M = ((m_{ij}))$ is called the pay-off matrix of the game. Since player 2's gain is player 1's loss, the game is said to be zero-sum.

A strategy for player 1 is a probability vector $(p_1, p_2, \ldots, p_m)$. The idea is that he will choose integer i with probability p_i. Analogously, strategy q for player 2 is defined. Von Neumann's fundamental minimax theorem is as follows:

Theorem 1.3 *Consider a two-person zero-sum game with a pay-off matrix M. There exist strategies $(p_1, p_2, \ldots, p_m)$, $(q_1, q_2, \ldots, q_n)$ and a real number v such that*

$$\sum_i p_i m_{ij} \leq v \quad \textit{for all} \quad j = 1, 2, \ldots, n$$

$$\sum_j q_j m_{ij} \geq v \quad \textit{for all} \quad i = 1, 2, \ldots, m.$$

This v is called the *minimax value* associated with the matrix M or simply value of the game and the strategies are called *optimal strategies* for the two players. In the game described above, player 1 is the minimizer (that is he wants to give player 2 as little as possible) and player 2 is the maximizer. We write $v(M)$ to denote the value of the game corresponding to M.

Sometimes we may change the roles of the two players - in other words player 1 will be the maximizer and player 2 will be the minimizer. The following elementary lemma is quite useful. Here, we consider the column chooser to be the maximizer.

Lemma 1.3 *Let M be a nonsingular matrix with $v(M) > 0$. Then $v(\overline{M})$ is also positive, where $\overline{M}$ is a principal pivot transform of M.*

Proof: As $v(M) > 0$ there exists a probability vector $x > 0$ such that $Mx > 0$. Hence we have,

$$\begin{bmatrix} I_{JJ} & 0 & -M_{JJ} & -M_{J\overline{J}} \\ 0 & I_{\overline{J}\overline{J}} & -M_{\overline{J}J} & -M_{\overline{J}\overline{J}} \end{bmatrix} \begin{bmatrix} y_J \\ y_{\overline{J}} \\ x_J \\ x_{\overline{J}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $y = Mx$. Multiplying the first J rows by $-M_{JJ}^{-1}$ we have

$$\begin{bmatrix} -M_{JJ}^{-1} & 0 & I_{JJ} & M_{JJ}^{-1}M_{J\overline{J}} \\ 0 & I_{\overline{J}\overline{J}} & -M_{\overline{J}J} & -M_{\overline{J}\overline{J}} \end{bmatrix} \begin{bmatrix} y_J \\ y_{\overline{J}} \\ x_J \\ x_{\overline{J}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Adding the last $\overline{J}$ with the first J rows multiplied by $M_{\overline{J}J}$ and rearranging the columns, we get

$$\begin{bmatrix} I_{JJ} & 0 & -M_{JJ}^{-1} & M_{JJ}^{-1}M_{J\overline{J}} \\ 0 & I_{\overline{J}\overline{J}} & -M_{\overline{J}J}M_{JJ}^{-1} & M_{\overline{J}\overline{J}} - M_{\overline{J}J}M_{JJ}^{-1}M_{J\overline{J}} \end{bmatrix} \begin{bmatrix} x_J \\ y_{\overline{J}} \\ y_J \\ x_{\overline{J}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we have for the principal pivot transform $\overline{M}$ a vector $z^t = (y_J, x_{\overline{J}})^t$ such that $z > 0$, $\overline{M}z > 0$, which implies $v(\overline{M}) > 0$. ∎

A strategy x is pure if it is of the form $(0, 0, 1, \ldots, 0)$; otherwise it is called a mixed strategy. A mixed strategy x is said to be *completely mixed*, if $x > 0$. If the only optimal strategies in a game are completely mixed, we shall call the game *completely mixed.* We present next, two famous theorems due to Kaplansky [25], on completely mixed games.

Theorem 1.4 *Let M denote the payoff matrix of order m by n, of a two person zero-sum game.*

1. *If player 1 has a completely mixed optimal strategy $p = (p_1, p_2, \ldots, p_m)$, then any optimal strategy $q = (q_1, q_2 \ldots, q_n)$ for player 2 satisfies $\sum_j m_{ij} q_j = v, \ \forall\, i = 1, \ldots, m$.*

2. *If $m = n$, and the game is not completely mixed, then both the players have optimal strategies that are not completely mixed.*

3. *A game with value zero is completely mixed if and only if*

   (a) its matrix is square, i.e., $m = n$ and has rank $(n-1)$ and

   (b) all the cofactors M_{ij} are different from zero and have the same sign (cofactor M_{ij} denotes the determinant of the principal submatrix of M got by deleting the ith row and the jth column).

4. *The value v of a completely mixed game is given by*

$$v = \frac{|M|}{\sum \sum M_{ij}}$$

   where $|M|$ is the determinant of M.

Theorem 1.5 *Let M be a matrix of order m by n. Let $V = ((v_{ij}))$ denote the matrix of order m by n, where each v_{ij} is the minimax value of the game played with the principal submatrix of M got by deleting the ith row and the jth column, as the payoff matrix. Then the game M is not completely mixed if and only if the game V has a pure saddle point, i.e., $\exists$ a pair i_0, j_0 such that*

$$v_{i_0 j} \leq v_{i_0 j_0} \leq v_{i j_0}, \ \forall\, i, j \tag{1.7}$$

and $v_{i_0 j_0}$ is the value of the game M.

For proofs of these results, see Kaplansky [25] and also Parthasarathy and Raghavan [52].

Completely mixed games with value zero were considered by Eagambaram and Mohan [15] in connection with the linear complementarity problem. They prove under these conditions over the matrix M that $D(M)$ is convex. They also propose a variant of the Lemke's algorithm that processes the $lcp(q, M)$ whenever the game is completely mixed with $v(M) = 0$.

A result on the Q-nature of a special class of completely mixed games is proved in Chapter 4.

1.5 Lemke's Algorithm for the $lcp(q, M)$

This algorithm was originally proposed by Lemke and Howson [31] for finding an equilibrium point of a bimatrix game. Later, Lemke [30] modified it for solving a class of linear complementarity problems.

This algorithm is very much similar to the simplex method for the linear programming problem in the sense that it makes use of the minimum ratio technique in the choice of its leaving variables.

If q is nonnegative, then it is clear that $lcp(q, M)$ has the trivial solution $(w = q; z = 0)$. If $q \not\geq 0$, an artificial variable is introduced. Then we have the following system:

$$\begin{aligned} w &= Mz + q + ez_0 \\ w^t z &= 0,\ w \geq 0,\ z \geq 0, \end{aligned} \tag{1.8}$$

where z_0 is an artificial variable which takes a large positive initial value so that $w \geq 0$. This is known as the primary ray. Let

$$z_0 = \max_i \{-q_i : 1 \leq i \leq n\}. \tag{1.9}$$

The triplet (w, z, z_0) gives rise to the initial solution to the system (1.7). The algorithm is based on pivot steps. It aims at making the artificial variable leave the basis, thereby obtaining a solution to the $lcp(q, M)$.

Before we describe the algorithm, we require to know about the almost complementary solutions and adjacent almost complementary solutions to the system (1.7).

Definition: Consider the system (1.7). A feasible solution (w, z, z_0) to this system is called an *almost complementary solution* if

(i) (w, z, z_0) is a basic feasible solution to (1.7).

(ii) Neither w_s nor z_s is basic, for some $s \in \{1, \ldots, n\}$.

(iii) z_0 is basic, and exactly one variable from the complementary pair (w_j, z_j) is basic for $j = 1, \ldots, n, \;\; j \neq s$.

An *adjacent almost complementary basic feasible solution* of an almost complementary basic feasible solution with w_s and z_s nonbasic, is got by introducing either w_s or z_s into the basis in the place of a basic variable $y \neq z_0$.

Let s be the index of the row at which the z_0 value in (1.8) is attained. At any iteration, the variable that is entering the basis is denoted by (x_s) for the sake of simplicity.

Now we describe the algorithm.

Step 1: Let d_s be the updated column in the current tableau under the variable x_s. If $d_s \leq 0$, go to step 4. Otherwise, determine the index r by the following minimum ratio test, where $\overline{q}$ is the updated right hand side column denoting the values of the basic variables.

$$\frac{\overline{q}_r}{d_{rs}} = \min_i \{\frac{\overline{q}_i}{d_{is}} : d_{is} > 0\}.$$

If the basic variable at row r is z_0, go to step 3. Otherwise, go to step 2.

Step 2: The basic variable at row r is either w_l or z_l, for some $l \neq s$. The variable x_s enters the basis and the tableau is updated by pivoting at row r and the d_s column. If the variable that just left the basis is w_l, then let $x_s = z_l$, and if the variable that just left the basis is z_l, then let $x_s = w_l$. Go to step 1.

Step 3: Here, x_s enters the basis, and z_0 leaves the basis. Pivot at the d_s column and the row z_0, producing a complementary basic feasible solution. We stop.

Step 4: The algorithm ends in a secondary ray. A ray $\{(w, z, z_0)+\lambda d : \lambda \geq 0\}$, where $z \neq 0$ and $d \neq e$ denotes the extreme direction, is found such that every point in this ray satisfies the system (1.7).

When the algorithm ends in step 4, the problem $lcp(q, M)$ might still have a solution. But the Lemke's algorithm is unable to determine it.

An algorithm is said to *process the* $lcp(q, M)$ if and only if whenever the $lcp(q, M)$ has a solution, it should be able to find it, or else conclude that the $lcp(q, M)$ has no solution. Lemke in his paper [30], proved that his algorithm can process the $lcp(q, M)$ for the class of copositive plus matrices (for definition, see section 6). There have been attempts in defining various classes of matrices for which the Lemke's algorithm processes the $lcp(q, M)$. We would see them in detail, in the next section.

1.6 Classes of matrices

Several classes of matrices have been defined and studied in connection with the linear complementarity problem. In this section, we list some of the classes.

Nondegenerate matrix: A matrix M is said to be nondegenerate if its principal minors are nonzero. Murty [43] proves a nice characterization of nondegenerate matrices which we state below:

Theorem 1.6 *Let $M \in R^{n \times n}$. M is a nondegenerate matrix if and only if the $lcp(q, M)$ has finitely many solutions, for every $q \in R^n$.*

A vector q nondegenerate with respect to M is said to have an odd (even) parity, if $m(q)$ is odd (even). Murty [43] also proved the following:

Theorem 1.7 *Let $M \in R^{n \times n}$ be a nondegenerate matrix. Then every vector q nondegenerate with respect to M has the same parity.*

As a consequence, we note that if M is nondegenerate and $lcp(q, M)$ has an odd parity for some $q \in R^n$ nondegenerate with respect to M, then M is a Q-matrix. This was observed by Saigal [58] also.

We have already defined that $M \in R^{n \times n}$ is called a $Q-$matrix if and only if $D(M) = R^n$. When $D(M)$ is convex, M is called a Q_0-*matrix.* Kelly and Watson [27] had brought an altogether different perspective of studying the problems of Q and Q_0-matrices. We refer to Watson [71] for a spherical geometric approach to these problems. In what follows, we define matrices based on the sign of the quadratic form $x^t M x$.

Copositive matrix: A matrix M is said to be *copositive* if, whenever $x \geq 0 \Rightarrow x^t M x \geq 0$. This class has a very long history in matrix theory. See for instance, Motzgin [40] and Cottle et al [9]. A matrix M is said to be *copositive plus*, if M is copositive and

$$x^t M x = 0, \; x \geq 0, \; x \in R^n \;\Rightarrow\; (M + M^t)x = 0.$$

Clearly, a positive definite matrix is copositive plus. Lemke's algorithm processes the $lcp(q, M)$ when M is copositive plus. This was proved by Lemke [30]. Further classes, based on the behaviour of the quadratic forms are defined and studied by Valiaho [69].

The following classes of matrices are based on the structure of M and the signs of its principal minors.

Z-matrices: A matrix M is said to be a Z-matrix, if $m_{ij} \leq 0, \quad \forall i, j \in \{1, \ldots, n\}, i \neq j$. There are various characterizations of these matrices given by Fiedler and Ptak [17].

Define for a $q \in R^n$ and a matrix M,

$$X(q, M) = \{z \geq 0 \;:\; q + Mz \geq 0\}.$$

A point x is called a *least element* of a set X if $x \in X$ and $y \in X \Rightarrow x \leq y$. Following characterization of Z-matrices is due to Tamir [66]:

Theorem 1.8 *$M \in Z$ if and only if for any q such that $X(q, M)$ is nonempty, $X(q, M)$ has a least element which solves the $lcp(q, M)$.*

The theory of least elements is very much connected to the study of q and M for which $lcp(q, M)$ is equivalent to a linear program. See Mangasarian [33], Cottle and Pang [10] and Murthy [42]. Saigal [60] proved that Lemke's algorithm processes the $lcp(q, M)$ when $M \in Z$. Mohan [35] proved that through linear programming formulation of the $lcp(q, M)$, the simplex method processes the $lcp(q, M)$ in atmost n steps when M is a Z-matrix. Chandrasekaran [4] developed an efficient algorithm for the $lcp(q, M)$ when the matrix under consideration is a Z-matrix. Ramamurthy [56] proposed an efficient algorithm to determine whether a given Z-matrix has all its principal minors nonnegative.

Signature matrix: We call a diagonal matrix S of order n, a signature matrix if $s_{ii} = \pm 1$, for all $1 \leq i \leq n$. S is sometimes written as S_J where J denotes the set of all indices i with $s_{ii} = +1$.

The principal minors of M and SMS keep the same sign for any signature matrix S.

We present below, the classes of matrices that are defined based on the number of solutions to the $lcp(q, M)$ for some fixed $q \in R^n$. As this literature is vast, we present here only those classes of matrices that we frequent in this dissertation.

Regular matrix: M is said to be regular if

$$\begin{aligned} z \geq 0, \quad & t \geq 0, \\ M_{i.}z + t = 0 \quad & \text{if } i \text{ is such that } z_i > 0 \\ M_{i.}z + t \geq 0 \quad & \text{if } i \text{ is such that } z_i = 0. \end{aligned} \tag{1.10}$$

Consequently, M is regular if and only if the $lcp(q, M)$ has a unique solution $\forall\, q \geq 0$. A matrix M for which $lcp(0, M)$ has a unique solution is denoted by R_0. These classes were defined by Karamardian [26]. He also proved that if $M \in R_0$ and $lcp(q, M)$ has a unique nondegenerate solution for some $q \in R^n$, then M is a Q-matrix. This result was earlier observed by Ingleton [22]. His arguments showed that Lemke's algorithm would process the $lcp(q, M)$ under these conditions. In this dissertation, we often make use of the following result of R_0, due to Murty [43] and Saigal [58]:

Theorem 1.9 *If $M \in R^{n \times n}$ is such that $lcp(0, M)$ has a unique solution and for a $q \in R^n$, $lcp(q, M)$ has an odd number of nondegenerate solutions, then M is a Q-matrix.*

E_0 matrix: A matrix M is called an E_0-matrix (also known as a semimonotone matrix), if $\forall x \in R^n$, $x \geq 0$, there exists an index i such that $x_i > 0$ and $(Mx)_i \geq 0$. Eaves [16] calls this class of matrices as L_1-matrices. The following result gives the connection between E_0 and the $lcp(q, M)$:

Theorem 1.10 *A matrix M is E_0 if and only if the $lcp(q, M)$ has a unique solution whenever $q > 0$.*

This is due to Eaves [16]. Periera [54] proved that if M is symmetric, then M is semimonotone if and only if it is copositive.

Strictly semimonotone matrix: A matrix M is said to be strictly semimonotone, if for every n vector $0 \neq x \geq 0$, $\exists$ an index i such that, $x_i(Mx)_i \geq 0$. M is strictly semimonotone if and only if for every principal submatrix $\overline{M}$ of M, the system

$$\overline{M}x < 0, \ \ x \geq 0$$

is inconsistent. Another equivalent way of defining this class is as follows: M is said to be strictly semimonotone if the $lcp(q, M)$ has a unique solution whenever $q \geq 0$. For results on this class, we refer to Eaves [16], Cottle and Dantzig [8], Karamardian [26] and Lemke [30]. Lemke denotes this class by E. Cottle defined and studied the class of completely Q-matrices in [7]. He also proved that they are precisely the class of strictly semimonotone matrices. Van der Heyden [70] studied this class in the name of $V-$matrices and developed an algorithm for the processing of the $lcp(q, M)$ when $M \in V$.

Fully semimonotone matrix: A matrix M is said to be fully semimonotone if M and all its principal pivot transforms are semimonotone. A matrix M is fully semimonotone if and only if $lcp(q, M)$ has a unique solution for each q in the interior of every full cone of $[I \vdots - M]$. This result has been observed by Cottle and Stone [11].

Garcia's class: M is said to satisfy Garcia's condition, if, whenever (w, z) is a solution of the $lcp(q, M)$ with $z \neq 0$, there exists a $v \in R^n$, $v \geq 0$, such that

$$u = -M^t v \geq 0 \text{ and } z \geq v, \; w \geq u. \tag{1.11}$$

For more details, we refer to Garcia [19].

Doverspike's class: We say that M satisfies the Doverspike's condition, if all the strongly degenerate complementary cones of $[I \vdots - M]$ lie on the boundary of $[I \vdots - M]$. We denote the class of matrices satisfying this condition by ${E_0}'$.

Doverspike [12] proved that when $M \in {E_0}'$, and if $lcp(q, M)$ has a unique nondegenerate solution for a $q > 0$, then M is Q_0.

Todd's class: We say that a matrix M is in Todd's class $L'(d)$ for some vector $d > 0$, if the following conditions are satisfied:

1. $M \in {E_0}'$

2. (w, z) is a solution of $lcp(d, M)$, $z \neq 0$ implies for any J,

$$\{i : z_i > 0\} \subseteq J \subseteq \{i : w_i = 0\},$$

$\det(M_{JJ}) > 0$.

We refer to Todd [67], for results on this class.

For more details about the classes of matrices that are defined and studied in relation to the linear complementarity problem, we refer to Murty [42].

1.7 Global Univalence results

In this section, we review some univalence results that are known in the literature of C^1-differentiable mappings of functions from R^n to R^n. Global univalence theory has gained immense importance because of its applications to several fields. Mathematical economists study univalence in connection with the uniqueness of a competitive equilibrium. The study of global univalence using Jacobian matrices has been of interest for a long time. We present in chapter 5, results on global univalence of C^1-differentiable functions, whose Jacobian matrices are exact order matrices.

To start with, we define what we mean by a C^1-differentiable map.

Definition: Let Ω be a rectangular region in R^n. A real-valued mapping $F : \Omega \rightarrow R^n$ is said to be differentiable at t_0 if $\exists$ a linear transformation L (depending on t_0) such that,

$$\lim_{h \rightarrow 0} \frac{1}{||h||}[(F(t_0 + h) - F(t_0)) - L(h)] = 0.$$

Here, we write $F = (f_1, f_2, \ldots, f_n)$ where each f_i is a real-valued function from Ω to R. Let their partial derivatives be $\delta f_i/\delta x_j, \forall i, j \in \{1, \ldots, n\}$. A mapping F is differentiable at t_0, if and only if each of its components f_i is differentiable at t_0, for $i = 1, \ldots, n$. If F is differentiable at t_0, then denote the matrix of partial derivatives $((\delta f_i/\delta x_j))$ by $J(t_0)$. This is called *the Jacobian matrix* of F at t_0. F is said to be differentiable, if it is differentiable at each point of Ω. It is said to be C^1-differentiable, if it is differentiable and all its partial derivatives are continuous. We next define an univalent map.

Definition: A mapping $F : \Omega \rightarrow R^n$ is said to be *globally univalent* if, whenever $x \neq y,\ x, y \in \Omega \Rightarrow F(x) \neq F(y)$.

In one-dimension, the problem of univalnece is easy; non-vanishing of the derivative throughout Ω ensures global univalence. But in the n-dimensional Euclidean space, it only ensures local univalence at a point. This is given by the local univalence theorem. We state it below without proof.

Theorem 1.11 *(Local Univalence Theorem): Let $F : \Omega \rightarrow R^n$ be a mapping, where Ω is a rectangular region. We have the following:*

(i) If F is differentiable at $t_0 \in \Omega$, and $detJ(t_0) \neq 0$, then there is a neighbourhood U of t_0, such that $F(y) = F(t_0) \Rightarrow y = t_0$.

(ii) If F is C^1-differentiable in a neighbourhood of an interior point t_0 of Ω, and $\det J(t_0) \neq 0$, then there is a neighbourhood U of t_0 where F is univalent, i.e., $F(y) = F(z),\ y, z \in \Omega \Rightarrow\ y = z$.

Even if the Jacobian J is non-vanishing throughout the domain of definition, the mapping F need not be univalent. Gale and Nikaido [18] gave an example of such a function. Take $F = (f, g)$ with

$$f = e^{2x} - y^2 + 3,\ \ g = 4e^{2x}y - y^3. \tag{1.12}$$

Here, $\delta f/\delta x > 0$ and $\det J_F > 0$ on R^2. However, F is not univalent, since $F(0, \pm 2) = F(0, 0)$.

Most of the results in univalence are initiated by a conjecture due to Samuelson, which is as follows: F is globally univalent if all the leading principal minors, $\det[\ (\delta F_i/\delta x_j)(p),\ 1 \leq i, j\ \leq r\ ]$ of the Jacobian $J_F(p)$ are nonzero, for all $p \in R^n$ and all $1 \leq r \leq n$.

But this turned out to be wrong and the Gale-Nikaido's example (1.11) serves as a counterexample to this.

Hence, it becomes necessary to look for more conditions over the principal minors for the existence of univalence. The following is the question that remains open:

Question : Let $F : \Omega \to R^n$ be a C^1-differentiable mapping, where Ω is a rectangular region. If the Jacobian matrix $J(x)$ associated with F has all its principal minors nonzero, is it true that F is univalent ?

This has been answered affrmatively, for $n \leq 3$, by Ravindran [57]. Beyond R^3, the answer is not known. There have been attempts at answering the above problem under certain further restrictions over the Jacobian matrix. The following is known as the Fundamental theorem of global univalence due to Gale, Nikaido and Inada. For a proof of this, we refer to Parthasarathy [51].

Theorem 1.12 *Let $F : \Omega \to R^n$ be a differentiable mapping where Ω is a rectangular region in R^n. Then F is globally univalent on Ω if either one of the following conditions holds good:*

(a) $J(x)$ is a P-matrix $\forall\, x \in \Omega$.

(b) $J(x)$ is an N-matrix and the partial derivatives are continuous $\forall\, x \in \Omega$.

Univalence results have been proved for C^1-differentiable functions with Jacobians, being matrices of exact order one by Olech et al in [46] and [47]. We make a mention of these results in chapter 5 and prove in detail, a result on C^1-functions with Jacobians, being exact order two matrices.

Chapter 2

THE CLASSES OF N AND P-MATRICES

In this chapter, some characterization theorems are proved for the class of N-matrices. We start with P-matrices, which have a long history in matrix theory. Most of the results of this chapter are from [38]. The last section results are based on [39].

2.1 Some known results on P-matrices

The class of P-matrices arises in a number of applications. There are many well-known equivalent characterizations of these matrices. In Berman and Plemmons [3], one can see a list of fifty equivalent characterizations of the class $P \cap Z$, a subclass of P-matrices.

The following theorem states a few of the equivalent characterizations of P-matrices.

Theorem 2.1 *The following conditions on a matrix M are equivalent:*

1. *M is a P-matrix.*

2. *For every nonzero vector x there exists an index i such that $x_i(Mx)_i > 0$.*

3. *For every nonzero vector x there exists a positive diagonal matrix D such that $x^t MDx > 0$.*

4. *The real eigenvalues of all the principal submatrices of M are positive.*

5. *For every signature matrix S there exists a positive vector x such that $SMSx > 0$.*

6. *For every vector $q \in R^n$, the lcp(q, M) has a unique solution.*

7. *Every principal pivot transform of M has its diagonal entries positive.*

The equivalence of the first two conditions and the fourth is due to Feidler and Ptak [17]; Condition 3 and 5 are due to Gale and Nikaido [18]. Condition 6 is due to Samelson, Wesler and Thrall [62]; the last one is by Tucker [68]. We also refer to Murty [42] for proofs of these results.

2.2 Sign pattern of N-matrices

As mentioned earlier, N-matrices were introduced by Inada [21] with regard to studying univalence. But though P-matrices attracted a lot of attention from mathematicians, N-matrices, introduced around the same period in the literature, did not receive an equal amount of importance. It was Saigal [59], in the seventies, who at first classified these matrices into two different categories and studied them in the context of the linear complementarity problem.

Let $M \in R^{n \times n}$ be an N-matrix, i.e., M has all its principal minors negative. M is said to be an N-matrix of *the first category*, if it has a positive entry; otherwise, it is said to be of *the second category.*

For the first category N-matrices, Inada [21] proved the following:

Lemma 2.1 *Let M be an N-matrix of the first category. Then the system,*

$$Mx \leq 0, \;\; x \geq 0$$

has only the trivial solution $x = 0$.

In other words, this lemma asserts that for an N-matrix of the first category, the minimax value of the game is positive.

In what follows, we study the structure of the entries of an N-matrix. When M is an N-matrix, then no entry of M can be zero. The following lemma is due to Ravindran [57].

Lemma 2.2 *Let M be a square matrix of order n, whose principal minors of order 3 or less negative. Then there is a signature matrix S such that*

$$SMS < 0. \tag{2.1}$$

This lemma determines the sign pattern of entries of an N-matrix. To obtain an explicit form, we define the following:

Definition: Let $x, y \in R^n$ have nonzero coordinates; we say that x and y have the same sign pattern if $x_i y_i > 0$, for all $i = 1, \ldots, n$. If x and y have the same sign pattern, they are said to be *sign equivalent.* We have the following lemma on the sign equivalence of matrices whose principal minors of order upto 3 are negative.

Lemma 2.3 *Let M be a square matrix of order n, whose principal minors of order 3 or less are negative. Then the sign equivalence is an equivalence relation on the set of columns of M, which partitions the columns of M into atmost two equivalent classes.*

Proof: Let $M_{.i}$ and $M_{.k}$ be two columns of M. Suppose $m_{ii} m_{ik} > 0$. Then clearly, $m_{ik} < 0$, and considering the 2 by 2 principal submatrix,

$$\begin{bmatrix} m_{ii} & m_{ik} \\ m_{ki} & m_{kk} \end{bmatrix}$$

we see that $m_{ki} < 0$. Thus $m_{ki} m_{kk} > 0$. We now claim that $m_{ji} m_{jk} > 0$, $\forall j$. Suppose for some $r \neq i$ or k, $m_{ri} m_{rk} < 0$. Consider the 3 by 3 principal submatrix,

$$\begin{bmatrix} m_{ii} & m_{ir} & m_{ik} \\ m_{ri} & m_{rr} & m_{rk} \\ m_{ki} & m_{kr} & m_{kk} \end{bmatrix}$$

The sign pattern of this matrix is either

$$\begin{bmatrix} - & + & - \\ + & - & - \\ - & - & - \end{bmatrix}$$

or

$$\begin{bmatrix} - & - & - \\ - & - & + \\ - & + & - \end{bmatrix}$$

depending upon whether $m_{ri} > 0$ or $m_{ri} < 0$.

But these are not the sign patterns of an N-matrix of order 3. See Parthasarathy and Ravindran [53]. Hence if $m_{ii}m_{ik} > 0$, then $m_{ji}m_{jk} > 0\ \forall j$. Similarly, we can show that if $m_{ii}m_{ik} < 0$, then $m_{ji}m_{jk} < 0$, for any $i, j, k \in \{1, \ldots, n\}$.

Now consider the index set

$$J = \{l : m_{11}m_{1l} > 0, 1 \leq l \leq n\}.$$

J is nonempty and it follows that all the columns of M whose indices are in J are sign equivalent; J and $\overline{J}$ induce the desired partition of the columns of M. If $J = \{1, \ldots, n\}$, then $M < 0$. ∎

Remark 2.1: Let us define, for the signature matrix S such that $SMS < 0$ in (2.1), the index set J, $\phi \neq J \subseteq \{1, 2, \ldots, n\}$ as

$$J = \{i : s_{ii} = +1\}. \tag{2.2}$$

Then the partition induced by J in M can be written as (if necessary, after a principal rearrangement of its rows and columns)

$$M = \begin{bmatrix} M_{JJ} & M_{J\bar{J}} \\ M_{\bar{J}J} & M_{\bar{J}\bar{J}} \end{bmatrix} \tag{2.3}$$

with $M_{JJ} < 0, M_{\bar{J}\bar{J}} < 0$ and $M_{J\bar{J}}, M_{\bar{J}J} > 0$. Thus if $n \geq k + 3$ and M is an N-matrix of exact order k, then M has the sign pattern as given in (2.3).

2.3 Some known results on N-matrices

In the theory of linear complementarity, it has been of special interest to characterize a class of matrices based on the number of solutions the $lcp(q, M)$ has for each $q \in R^n$. We have mentioned such a result, for the class of P-matrices due to Samelson, Wesler and Thrall [62] in Theorem 2.1.

The linear complementarity problem $lcp(q, M)$ with M as an N-matrix has earlier been studied by Saigal [59] and Kojima and Saigal [28]. They prove in [28], the following theorem.

Theorem 2.2 *If M is an N-matrix of the second category, then $lcp(q, M)$ has exactly two solutions for any $q > 0$, and no solution for any $q \in R^n$, $q \not\leq 0$. If M is an N-matrix of the first category, then for each $q \not\geq 0$, $lcp(q, M)$ has a unique solution and for a $q \in R^n_+$, $lcp(q, M)$ has exactly three solutions.*

However, until recently there has been no published proof of the converse, viz., a characterization of N-matrices using the number of solutions to the $lcp(q, M)$. Recently, Parthasarathy and Ravindran [53] , proved the following for the second category N-matrices.

Theorem 2.3 *Let $M < 0$. The following statements are equivalent :*

(i) The $lcp(q, M)$ has exactly two solutions for every $q > 0$.

(ii) $v(SMS) > 0$, $\forall$ signature matrices S, where $S \neq \pm I$.

(iii) For any vector $x \in R^n$, $x_i(Mx)_i \leq 0 \Rightarrow$ either $x \geq 0$, or $x \leq 0$.

Another characterization of N-matrices of the second category was given by Maybee [34].

A main result in this section is the converse of the Kojima-Saigal result for N-matrices of the first category. This was posed as an open problem in [53].

The following theorem is about the number of solutions $lcp(q, M)$ has for each $q \in R^n_+$ when M is an N-matrix of the first category. This result has been observed by Kojima and Saigal [28]. But here, we give a different proof and correct some errors in their paper [28].

Theorem 2.4 *Let $M \in R^{n \times n}$ be an N-matrix of the first category. Then for each $q > 0$, $lcp(q, M)$ has exactly 3 solutions. If $q \geq 0$ with $q_i = 0$ for some $i \in \{1, \ldots, n\}$, then $lcp(q, M)$ has exactly 2 solutions.*

Proof: Let J be as defined in (2.2); let the matrix M be partitioned as in (2.3).

Consider a $q > 0$, and let $q = (q_J, q_{\overline{J}})^t$. As M_{JJ} is an N-matrix of the second category, from Theorem 2.3, we see that (q_J, M_{JJ}) has exactly two solutions. With the solution (w_J^*, z_J^*) of (q_J, M_{JJ}) where $z_J^* \neq 0$, define $w \in R^n, z \in R^n$ by

$$w_J = w_J^*; \qquad z_J = z_J^*;$$
$$w_{\overline{J}} = q_{\overline{J}} + M_{\overline{J}J} z_J^*, \qquad z_{\overline{J}} = 0.$$

It is easy to see that (w, z) solves $lcp(q, M)$ and $z_J \neq 0$. Since (w_J^*, z_J^*) uniquely determines w_J, there is exactly one solution with $z_J \neq 0$, $z_{\overline{J}} = 0$.

By a similar argument, we can show that there is exactly one solution (u, v) for which $v_J = 0, v_{\overline{J}} \neq 0$. In addition, we have the trivial solution to the $lcp(q, M)$, viz., $w = q, z = 0$; hence we have three solutions for the $lcp(q, M)$.

To show that $lcp(q, M)$ has exactly three solutions, we show that there is no other solution (x, y) to the $lcp(q, M)$ in which y_J and $y_{\overline{J}}$ both have nonzero coordinates. Suppose on the contrary, there is a solution (x, y) to the $lcp(q, M)$ with $y_J \neq 0$ and $y_{\overline{J}} \neq 0$. Let

$$L = \{s : y_s > 0, 1 \leq s \leq n\}.$$

By our hypothesis, $L \cap J \neq \phi$ and $L \cap \overline{J} \neq \phi$. Therefore, the principal submatrix M_{LL} is an N-matrix of the first category; further we have

$$-q_L = M_{LL} y_L, \ y_L > 0$$

which contradicts Lemma 2.1. Hence $lcp(q, M)$ has exactly three solutions for $q > 0$.

If $q \geq 0, q \neq 0$ with at least one coordinate of it being zero, then the above arguments show that there are exactly two solutions to the $lcp(q, M)$. This completes the proof of the theorem. ∎

Remark 2.2: Let us define two classes of complementary cones of M. Let

$$\begin{aligned} \mathcal{C}_1 &= \{pos(B) : B \text{ is a compl. matrix of } [I \vdots -M] \text{ with} B_{.k} = I_{.k}\ \forall k \in \bar{J}\} \quad (2.4) \\ \mathcal{C}_2 &= \{pos(B) : B \text{ is a compl. matrix of } [I \vdots -M] \text{ with } B_{.k} = I_{.k}\ \forall k \in J\}. \end{aligned}$$

Geometrically the above theorem shows that the complementary cones in $\mathcal{C}_1$, other than $pos(I)$, intersected with R^n_+, make a partition of the positive orthant (if there is only one complementary cone in $\mathcal{C}_1$, it covers the whole of $pos(I)$). So are the cones in $\mathcal{C}_2$.

Remark 2.3: The above theorem corrects a wrong assertition in the statement of Theorem 3.3 of Kojima and Saigal [28], which claims that the number of solutions to the $lcp(q, M)$ when $q > 0$ is not nondegenerate with respect to M, is two. This and some other corrections have been noted by Stone ([63] and [64]), also.

Remark 2.4: Theorem 3.4 stated in Kojima and Saigal [28] on the number of solutions to the $lcp(q, M)$ when q is contained on a face of $pos(I)$ is also wrong. It asserts that the number of solutions of $lcp(q, M)$ is exactly two, when $q \geq 0$, with $q_i = 0$, for at least one index i. The following example shows that this need not be so.

Example:

$$M = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

is an N-matrix of the first category. Here, for $q = (0, 0, 1)$, $lcp(q, M)$ has a unique solution.

Remark 2.5: A refinement of the above theorem, stating clearly the number of solutions for the $lcp(q, M)$ when q lies on a face of pos(I) has been proved by Gowda [20], using degree theory.

Kojima and Saigal [28] present a lemma on the types of solutions for the $lcp(q, M)$, when M is an N-matrix. This is stated below.

Lemma 2.4 *Let M be an N-matrix. If $lcp(q, M)$ has a solution $(\overline{w}, \overline{z})$, with $\overline{w}_i = 0$ for some i, then every other solution (w, z) of the $lcp(q, M)$ has $w_i > 0$.*

2.4 Characterization theorems for N-matrices

We prove some theorems characterizing N-matrices of the first category. The first theorem is a converse of the Kojima-Saigal result [28] on the number of solutions to the $lcp(q, M)$ when M is an N-matrix of the first category. We start with a lemma.

Lemma 2.5 *Suppose $X \in R^{n \times n}$ is a matrix of nonzero principal minors. Let the two complementary cones incident on any $(n-1)$ face of $[I \vdots -X]$, which is not a face of $pos(-X)$ be properly situated. Then all the proper principal minors of X are positive.*

Proof: The proof is by induction on the order of the principal minors of X. We at first, show that all the principal minors of order 1 of X are positive. To show that x_{jj}, the (j, j)th entry of X is positive, $1 \leq j \leq n$, consider

$$pos(B^1) = pos(I_{.1}, \ldots, I_{.j-1}, -X_{.j}, I_{.j+1}, \ldots, I_{.n})$$

and let

$$pos(B) = pos(I).$$

Since these two cones are properly situated on the $(n-1)$ face

$$F = pos(I_{.1}, \ldots, I_{.j-1}, I_{.j+1}, \ldots, I_{.n})$$

using Lemma 1.2, it follows that

$$\det(B^1)/\det(B) < 0$$

which $\Rightarrow \det(B^1) = -x_{jj} < 0$, or $x_{jj} > 0$, $\forall\, 1 \leq j \leq n$.

Let us assume that all the principal minors of order upto r, $(r < (n-2))$ of X are positive; consider a principal submatrix of order $(r+1)$. Let it be X_{JJ}, where $|J| = r+1$. Let $s \in J$ and $L = J \backslash \{s\}$; consider the two cones

$$pos(B^1) = pos\{-X_{.j}, \forall j \in L, I_{.j}, \ \forall j \notin L\}$$

$$pos(B) = pos\{-X_{.j}, \forall j \in J, I_{.j}, \ \forall j \notin J\}$$

and the face

$$F = pos\{-X_{.j}, \forall j \in L, I_{.j}, \forall j \notin L \text{ and } j \neq s\}.$$

Since $pos(B)$ and $pos(B^1)$ are properly situated on F, using (1.5),

$$\frac{\det(B^1)}{\det(B)} < 0.$$

We have $\det(B) = (-1)^r \det X_{LL}$ and by induction, $\det X_{LL} > 0$. Therefore, $\det(B) < 0$ if r is odd and $\det(B) > 0$ if r is even; it follows that, $\det(B^1) > 0$ when r is odd and $\det(B^1) < 0$ when r is even. Since $\det(B^1) = (-1)^{r+1}\det X_{JJ}$, it is clear that $\det(X_{JJ}) > 0$ in either case. The proof is complete. ∎

The following theorem characterizes N-matrices of the first category based on the number of solutions to the $lcp(q, M)$ for each $q \in R^n$.

Theorem 2.5 *Let $M \in R^{n \times n}$ be such that $M_{.j} \not\leq 0, \forall j, 1 \leq j \leq n$. Suppose $lcp(q, M)$ has a unique solution whenever $q \not\geq 0$ and a finite number of solutions whenever $q \geq 0$, with $lcp(q, M)$ having more than one solution for at least one $q > 0$. Then, M is an N-matrix of the first category.*

Proof: Since $lcp(q, M)$ has a finite number of solutions for any $q \in R^n$, from Theorem 1.6, it follows that none of the principal minors of M is zero.

We will show that, if F is an $(n-1)$ face of a complementary cone of $[I \vdots -M]$ which is not a face of pos(I), then the two complementary cones incident on it are properly situated. Suppose not. Let F be an $(n-1)$ face in $[I \vdots -M]$ generated by k columns of I and $(n-k-1)$ columns of $-M$, $1 \leq k \leq (n-2)$, such that the two complementary cones pos(B) and pos(B^1) incident on it, lie on the same side of F. If $F \subseteq \text{pos}(I)$, then for some $r \in \{1, \ldots, n\}$, $-M_{.r}$ which is in the set of columns generating F, is in $\text{pos}(I)$, contrary to the hypothesis. Hence $F \not\subseteq \text{pos}(I)$.

Suppose the complementary pair of vectors left out in generating F are $-M_{.s}$ and $I_{.s}$. Since $\text{pos}(B)$ and $\text{pos}(B^1)$ lie on the same side of F and $F \not\subseteq \text{pos}(I)$, we can find a $q' \in F$, $q' \notin \text{pos}(I)$ and an $\epsilon > 0$ such that

$$q(\text{say}) \; = q' + \epsilon(-M_{.s}) \in \text{pos}(B) \cap \text{pos}(B^1).$$

But $q \not\geq 0$ and for this vector q, $lcp(q, M)$ has at least two solutions, which is a contradiction. Hence our assertion follows.

Let $X = -M^{-1}$. By Theorem 1.2, it follows that if F is an $(n-1)$ face of $[I \vdots -X]$ other than $\text{pos}(-X)$, then the two complementary cones incident on it are properly situated. We note from Lemma 2.5, that all the proper principal minors of X are positive.

Now if $\det(X) > 0$, then X and hence $X^{-1} = M$ is a P-matrix, which contradicts Theorem 2.1 on the number of solutions to the $lcp(q, M)$. Hence $\det(X)$ and in turn $\det(M)$ is negative. From Lemma 1.2, it follows that all the proper principal minors of M are negative. This completes the proof. ■

Theorem 2.6 *Let $M \in R^{n \times n}$ be such that every column of M has a positive entry. M is an N-matrix of the first category if and only if $lcp(q, M)$ has*

(i) a unique solution for all $q \not\geq 0$,

(ii) exactly three solutions for all $q > 0$ and

(iii) atmost two solutions for any $q \in R^n_+, q \not> 0$.

Proof: This follows from Theorem 2.2 and the above theorem. ■

Next, we present a theorem characterizing an N-matrix based on the signs of the diagonal entries in each of its principal pivot transforms. This is similar to (7) of Theorem 2.1, on P-matrices.

Theorem 2.7 *Let $M \in R^{n \times n}$. M is an N-matrix if and only if the following holds:*

(i) All the diagonal entries of M are negative and

(ii) let $\phi \neq J \subseteq \{1, \ldots, n\}$. Let $\mu_J(M)$ as defined in (1.3), be the principal pivot transform of M with respect to $B(J)$. Then whenever $|J| > 1$, all the diagonal entries of $\mu_J(M)$ are positive.

Proof: (Only if): When M is an N-matrix, all the principal minors are negative (in particular, the diagonal entries). Hence we can take principal pivot transform with respect to $B(J)$ for any $\phi \neq J \subseteq \{1, \ldots, n\}$. Condition (ii) now follows easily from (1.4).

(If): By hypothesis, all the diagonal entries are negative. Consider any 2 by 2 principal submatrix M_{LL} of M. Let $L = \{i, j\}$. Consider $J = L\backslash\{i\}$. Since the diagonal element m_{jj} is negative, let $\overline{M} = \mu_J(M)$, be the principal pivot transform with respect to $B(J)$. Now let $K = \{i\}$, then using (1.4),

$$\begin{aligned} \det\overline{M}_{KK} &= \det M_{K\Delta J, K\Delta J}/\det M_{JJ} \quad (2.5) \\ &= \det M_{LL}/\det M_{JJ}. \end{aligned}$$

By hypothesis this is positive. Since $\det M_{JJ} < 0$ it follows that $\det M_{LL} < 0$. We can now complete the proof by induction on the order of the principal minors of M. ∎

2.5 Sign reversal property for N

Sign reversal property of matrices plays a key role in the theory of linear complementarity. We say that a matrix M *reverses* the sign of a vector $x \in R^n$, if $x_i(Mx)_i \leq 0$, for all $1 \leq i \leq n$. For P-matrices, condition (2) of Theorem 2.1 gives the sign reversal property.

For an N-matrix of the second category, the result on sign reversal property is mentioned in Theorem 2.3, which has been observed by Parthasarathy and Ravindran [53].

The next theorem gives a characterization of N-matrices of the first category in terms of the sign reversal property.

Theorem 2.8 *Let $M \in R^{n \times n}$ be partitioned as in (2.3), for some $J \subseteq \{1, 2, \ldots, n\}$. M is an N-matrix if and only if M reverses the sign of only those vectors of the form $x_J \geq 0$, $x_{\bar{J}} \leq 0$ or $x_{\bar{J}} \geq 0$, $x_J \leq 0$.*

Proof: This theorem easily follows from the sign reversal property presented in Theorem 2.3, by observing that M reverses the sign of a vector $x \in R^n$, if and only if SMS reverses the sign of Sx, where S is the signature matrix with s_{ii} being -1 for $i \in J$ and $+1$ for $i \notin J$. However, we prove this theorem based on the linear complementarity.
(Only if): Suppose M is an N-matrix of the first category, then by Lemma 2.3, M has the partitioned form as in (2.3) (after a principal rearrangement of its rows and columns if necessary), where J is as defined (2.2). It is clear from the partitioned form of M, that M reverses the sign of vectors $(x_J, x_{\bar{J}})^t$ of the form $x_J \leq 0$ and $x_{\bar{J}} \geq 0$ or of the form $x_J \geq 0$ and $x_{\bar{J}} \leq 0$. To show that M does not reverse the sign of any other vector, we proceed as follows.

Suppose M reverses the sign of x where x_J and $x_{\bar{J}}$ are nonnegative with at least one coordinate in x_J and $x_{\bar{J}}$ being positive. Consider the index set

$$L = \{i : x_i > 0, 1 \leq i \leq n\}.$$

We have

$$L \cap J \neq \phi, \; L \cap \bar{J} \neq \phi.$$

Let $(Mx)_L = q_L = M_{LL}x_L$; we note that $q_L \leq 0$. Also, it is clear that M_{LL} is an N-matrix of the first category. Thus we arrive at a contradiction to Lemma 2.1.

The only other possibility to be considered is that of M reversing the sign of a vector of mixed signs in x_J and $x_{\bar{J}}$. Let the sign of x, where x_J has both a positive and a negative coordinate, be reversed by M. Let

$$x_i^+ = \begin{cases} x_i & \text{if } x_i > 0, \\ 0 & \text{otherwise.} \end{cases} \quad x_i^- = \begin{cases} -x_i & \text{if } x_i < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now $x = x^+ - x^-$ and we see that with $u = Mx$,

$$u^+ - Mx^+ = u^- - Mx^- = \bar{q}(\text{say}).$$

Thus $lcp(\overline{q}, M)$ has two distinct solutions, (u^+, x^+) and (u^-, x^-), as $x^+ \neq 0$, $x^- \neq 0$. There are two cases:

Case(i): $\overline{q} \not\geq 0$. We have a contradiction to the result that for such a $\overline{q}$, $lcp(\overline{q}, M)$ has a unique solution.

Case(ii): $\overline{q} \geq 0$. If $x_{\overline{J}} = 0$, then we have at least three solutions to (q_J, M_{JJ}), a contradiction to Theorem 2.3. If $x_J \neq 0$, then we have a contradiction to Theorem 2.2.

Similarly, we can show that M does not reverse the sign of a vector $(x_J, x_{\overline{J}})^t$ when $x_{\overline{J}}$ has both a positive and a negative coordinate. This completes the proof of the 'only if' part.

(IF): Suppose M can be partitioned as in (2.3) and M does not reverse the sign of any nonzero vector $x = (x_J, x_{\overline{J}})^t$ except when $x_J \geq 0$ and $x_{\overline{J}} \leq 0$ or $x_J \leq 0$ and $x_{\overline{J}} \geq 0$. By taking either $x_J = 0$ or $x_{\overline{J}} = 0$ we see from Theorem 2.3 that, M_{JJ} and $M_{\overline{J}\overline{J}}$ are N-matrices of the second category. Let C_1, C_2 be the classes of complementary cones of $[I \vdots -M]$, as defined in (2.4). Then by the proof of Theorem 2.4, any $q > 0$ is exactly contained in one complementary cone from C_1 (C_2) other than pos(I). We now show that for such a $q > 0$, $lcp(q, M)$ has no solution (w, z) in which $z_i > 0$ for some $i \in J$ and $z_k > 0$ for some $k \in \overline{J}$. Suppose this is not true. Let

$$L = \{k : z_k > 0, 1 \leq k \leq n\}$$

and $L \cap J \neq \phi$, $L \cap \overline{J} \neq \phi$. We note that

$$q_L = -M_{LL} z_L.$$

Define y by taking $y_L = z_L$; $y_{\overline{L}} = 0$. Then, M reverses the sign of y, contradicting our hypothesis.

The above argument also shows that under our hypothesis about M, for any $q > 0$, $lcp(q, M)$ has exactly three solutions.

We now show that no principal subdeterminant of M (including det(M)) is zero. Suppose not. Let det$(M_{LL}) = 0$ for some set $L \subseteq \{1, \ldots, n\}$, then there is $0 \neq x \in R^{|L|}$ such that $M_{LL} x = 0$. Without loss of generality, we may assume that no coordinate of x is zero. Let $y \in R^n$ be defined by taking $y_L = x$

and $y_{\overline{L}} = 0$. Then M reverses the sign of the vector y. Note also that

$$(My)_L = M_{LL}y_L + M_{L\overline{L}}y_{\overline{L}} = 0.$$

Suppose $y_J \leq 0$ and $y_{\overline{J}} \geq 0$. From the sign of M and the fact that at least one coordinate of either y_J or $y_{\overline{J}}$ is nonzero, it follows that

$$(My)_{L\cap J} = M_{L\cap J\, L\cap J}y_{L\cap J} + M_{L\cap J\, L\cap \overline{J}}y_{L\cap \overline{J}} > 0$$

contradicting $(My)_L = 0$. Similarly the case $y_J \geq 0,\ y_{\overline{J}} \leq 0$ does not arise. Thus M reverses the sign of a vector y, not allowed by our hypothesis. This contradiction shows that no principal subdeterminant of M is zero.

In particular, it follows that M is nondegenerate. Also, the number of solutions for the $lcp(q, M)$ for any $q \in R^n$ is finite, from Theorem 1.6.

Now consider any $q \not\geq 0$. Suppose $lcp(q, M)$ has a solution. We then claim that the solution is unique. Let on the contrary, (w^1, z^1) and (w^2, z^2) be two distinct solutions to the $lcp(q, M)$. Then

$$w^1 - Mz^1 = w^2 - Mz^2 = q$$

or

$$w^1 - w^2 = M(z^1 - z^2) \qquad (2.6)$$

from which it is clear that M reverses the sign of the vector $(z^1 - z^2)$. By our hypothesis, $(z^1 - z^2)_J \leq 0$ and $(z^1 - z^2)_{\overline{J}} \geq 0$. From the sign pattern of M and the fact that $(w^1 - w^2)_J \geq 0$ and $(w^1 - w^2)_{\overline{J}} \leq 0$, it follows that $z^1_J = 0$ and $z^2_{\overline{J}} = 0$. Now it is easy to check that

$$q_{\overline{J}} = w^2_{\overline{J}} - M_{\overline{J}J}z^1_J \geq 0.$$

In a similar manner, we have $q_J \geq 0$. This however, contradicts our assumption about q. Hence the claim is proved.

Let $\overline{q} = -Me$. By the sign reversal property of M, it is easy to see that $\overline{q} \not\geq 0$. Moreover, $lcp(q, M)$ has a solution $w = 0,\ z = e$. Hence by the previous argument, the solution is unique. Since $lcp(0, M)$ has a unique solution and $lcp(\overline{q}, M)$ has a nondegenerate unique solution, we note that M is a Q-matrix using Theorem 1.9. From the previous paragraph, it follows that $lcp(q, M)$ has a unique solution whenever $\overline{q} \not\geq 0$. Thus we see that M satisfies all the hypotheses of Theorem 2.6 and hence is an N-matrix of the first category. This completes the proof of the theorem. ∎

The following theorem gives in a nutshell, the various equivalent characterizations of an N-matrix.

Theorem 2.9 *Let M have the partitioned form as in (2.3) for some $\phi \neq J \subset \{1,\ldots,n\}$. Let S_0 be the signature matrix satisfying (2.1). Then the following conditions on M are equivalent:*

1. *M is an N-matrix.*

2. *Whenever a vector $x \in R^n$ gets reversed in sign, then x is either of the form $x_J \geq 0$, $x_{\overline{J}} \leq 0$ or $x_J \leq 0$, $x_{\overline{J}} \geq 0$.*

3. *For every signature matrix $S \neq \pm S_0$, $v(SMS) > 0$.*

4. *For every vector $q \in R^n, q \not\geq 0$, the lcp(q,M) has a unique solution; lcp(q,M) has finitely many solutions $\forall\, q \in R^n$ with more than one solution for at least one $q > 0$.*

5.

   (i) All the diagonal entries of M are negative and

   (ii) let $\phi \neq J \subseteq \{1,\ldots,n\}$. Let $\mu_J(M)$ as defined in (1.3), be the principal pivot transform of M with respect to $B(J)$. Then whenever $|J| > 1$, all the diagonal entries of $\mu_J(M)$ are positive.

2.6 A counterexample to a characterization of P-matrices

In section 1 of this chapter, we had seen a characterization of the class of P-matrices due to Samelson, Wesler and Thrall. In geometric terms it can be stated as follows: M is a P-matrix if and only if the complementary cones of

$[I \vdots -M]$ partition the whole of R^n. However, to decide whether a given matrix M is a P-matrix or not, this theorem is not directly of much help. In [43], Murty presented the following result:

Theorem 2.10 *Let M be a Q-matrix and let $lcp(q, M)$ have a unique solution for each q in the finite test set*

$$\{I_{.1}, I_{.2}, \ldots, I_{.n}, -M_{.1}, -M_{.2}, \ldots, -M_{.n}\}.$$

Then M is a P-matrix.

We refer to 4.9 and 4.10 of [43] for a proof of this result. Using this result, Murty in [44] and Tamir in [65] have observed the following refinements in the characterization of P-matrices.

Theorem 2.11 *(Murty [44]): M is a P-matrix if and only if $lcp(q, M)$ has a unique solution for each q in*

$$\Gamma = \{I_{.1}, \ldots, I_{.n}, -I_{.1}, \ldots, -I_{.n}, M_{.1}, \ldots, M_{.n}, -M_{.1}, \ldots, -M_{.n}, e\}$$

where e is the vector of order n each of whose coordinates is one.

Theorem 2.12 *(Tamir [65]): M is a P-matrix if only if $lcp(q, M)$ has a unique solution for each q in*

$$\Gamma_1 = \{I_{.1}, I_{.2}, \ldots, I_{.n}, M_{.1}, M_{.2}, \ldots, M_{.n}, -M_{.1}, -M_{.2}, \ldots, -M_{.n}, e\}$$

Also Kostreva [29] had given a further refinement of this test set. We show in this section that the above theorems are in fact, not correct for $n \geq 4$. The proof 4.10 in [43], we submit, contains an error. In what follows, we present a counterexample which is a matrix of order 4. It has been brought to our notice by one of our referees of [39] that about an year ago, Professor W.Pye, Department of Mathematics, University of Southern Mississippi, has pointed out an error in Murty's proof. Later in this section, we present a proof of these theorems for $n \leq 3$.

Counterexample: Let

$$M = \begin{bmatrix} -1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & 2 & -1 & -1 \end{bmatrix} \tag{2.7}$$

We note that all the principal minors of M are negative and $M \not\leq 0$. M is an N-matrix of the first category and hence, a Q-matrix. From Theorem 2.6, it is clear that $lcp(q, M)$ has a unique solution for

$$q \in \{I_{.1} \ldots, I_{.4}, -M_{.1} \ldots, -M_{.4}\}.$$

Here, *the observation that* $lcp(I_{.j}, M)$ *has a unique solution*, $\forall\ 1 \leq j \leq 4$, follows from Lemma 2.4. It can also be proved as follows.

Let $J = \{1, 2\}$, then $\overline{J} = \{3, 4\}$. Also note that $M_{JJ} < 0, M_{\overline{J}\overline{J}} < 0$ and that these are N-matrices of the second category. $M_{J\overline{J}} > 0$, and $M_{\overline{J}J} > 0$. Suppose $lcp(I_{.1}, M)$ has a solution (w, z) with $z \neq 0$. Let $K = \{i\ :\ z_i > 0\}$. Suppose that $K \cap J \neq \phi$, and $K \cap \overline{J} \neq \phi$. Hence, M_{KK} is an N-matrix of the first category. The equations

$$\begin{aligned} w - Mz &= I_{.1} \\ w^t z &= 0 \end{aligned} \tag{2.8}$$

imply that

$$-M_{KK} z_K \geq 0,$$

i.e., the system

$$M_{KK} z_K \geq 0,\ z_K \geq 0$$

has a solution, which contradicts Lemma 2.1. We thus have $K \subseteq J$ or $K \subseteq \overline{J}$. Consider the case $K \subseteq J$. Since $M_{JJ} < 0$ and $z_J \neq 0$, we have $-M_{JJ} z_J > 0$. As $z_{\overline{J}} = 0$ we have a contradiction to the assumption that (w, z) solves $lcp(I_{.1}, M)$. Similarly, we can show that there is no solution (w, z) to $lcp(I_{.j}, M)$ with $z_J \neq 0$ and $z_{\overline{J}} \neq 0$. It follows that the solution to $lcp(I_{.j}, M)$ is unique. Thus $lcp(I_{.j}, M)$ *has a unique solution for* $1 \leq j \leq 4$. Thus, M given in (2.8) satisfies the hypothesis of Theorem 2.10 and is not a P-matrix.

Let M be the matrix given in (2.8). Then M^{-1} satisfies the hypotheses of Theorem 2.11 and 2.12 and is not a P-matrix.

The reason for the failure of these theorems is that the part of the proof which shows that under the hypothesis of 4.10 of [43] every principal submatrix of order $(n-2)$ is also a Q-matrix, i.e., the concluding part of the 4.13 [43] is incorrect. The matrix in (2.8) is a counterexample. Murty's finite set characterization of P-matrices has inspired many researchers to look forward for a similar characterization for other classes of matrices so far, without much of a success. We see that the cardinality of the sets Γ and Γ_1 are linear in n. Our observations here leave the question of finding such a nice set (i.e., a set whose cardinality is bounded above by a polynomial function in n) open even for P-matrices.

In order to prove the above mentioned results for $n \leq 3$, we first state some lemmas.

Lemma 2.6 *Let M be a real square matrix of order n. If for every $q \in \{-M_{.1}, \ldots, -M_{.n}\}$, $lcp(q, M)$ has a unique solution, then $lcp(0, M)$ has a unique solution.*

Proof: The unique solution for each $lcp(-M_{.j}, M); 1 \leq j \leq n$ is given by $(w = 0,\ z = e_j)$. Suppose $lcp(0, M)$ has a nontrivial solution $(\overline{w}, \overline{z})$ with $\overline{z}_k \neq 0$ for some kth coordinate. Then we can easily verify that $(\overline{w}, e_k + \overline{z})$ solves the problem $lcp(-M_{.k}, M)$, which is a contradiction. ∎

Now we prove a result in R^2.

Theorem 2.13 *Let M be a square matrix of order 2 with $M_{.1} < 0$ and $M_{.2} > 0$. Then the following are equivalent:*

(i) M is a Q-matrix.

(ii) $lcp(I_{.2}, M)$ has exactly two solutions.

(iii) $det(M) > 0$.

Proof: (i)$\Rightarrow$ (ii): It is easy to see from the given conditions and (i) that, if $q = (x, y)^t$ with $x < 0$ and $y > 0$ then $q \in \text{pos}\{-M_{.1}, -M_{.2}\}$. Since $\text{pos}\{-M_{.1}, -M_{.2}\}$ is also closed, it follows that $I_{.2} \in \text{pos}\{-M_{.1}, -M_{.2}\}$. It is also clear that $I_{.2} \in \text{pos}(I)$ and $\text{pos}\{-M_{.1}, I_{.2}\}$ but not contained in $\text{pos}\{I_{.1}, -M_{.2}\}$. Hence (ii) follows.

$(ii) \Rightarrow (iii)$: Since $lcp(I_{.2}, M)$ has two solutions, from the given conditions, it is easy to check that $I_{.2} \in \text{pos}(-M)$. Thus there exists a vector $x \geq 0$ such that

$$-Mx = e_2.$$

Let $x = (x_1, x_2)^t$. Note that $x_2(-\text{det}M/m_{11}) = 1$, which implies that $\text{det}M > 0$.

$(iii) \Rightarrow (i)$: Since none of the principal minors of M are zero, $lcp(0, M)$ has a unique solution. Also it is easy to check that $lcp(-I_{.1}, M)$ has the unique solution $x = (x_1, x_2)^t$ where

$$x_1 = m_{22}/\text{det}M$$

$$x_2 = -m_{21}/\text{det}M$$

and the solution is nondegenerate. Thus M is a Q-matrix. ∎

Lemma 2.7 *Let M be a Q-matrix of order 2. Then $lcp(0, M)$ has the unique solution* $(w = 0, z = 0)$.

Proof: This can be verified very easily, by drawing the complementary cones in R^2. Also one can see exercise 3.103 in [42, p.248]. ∎

Lemma 2.8 *Let M be a square matrix of order 2 with $m_{11} < 0$. If M is a Q-matrix, then its entries must be nonzero and it must have one of the following sign patterns of its entries,*

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix}$$

or

$$\begin{bmatrix} - & + \\ + & - \end{bmatrix}.$$

Proof: This is easy to verify. ■

Now, we prove the desired theorem.

Theorem 2.14 *Let M be a square matrix of order n, with $n \leq 3$. M is a P-matrix if and only if for all*

$$q \in \Gamma_2 = \{I_{.1}, I_{.2}, \ldots, I_{.n}, -M_{.1}, -M_{.2}, \ldots, -M_{.n}, q^1\}$$

$lcp(q, M)$ has a unique solution, where q^1 is in the interior of some complementary cone of $[I \vdots -M]$.

Proof: Suppose $lcp(q, M)$ has a unique solution for all $q \in \Gamma_2$, then by Lemma 2.6, $lcp(0, M)$ has a unique solution. Since $lcp(q^1, M)$ also has a unique non-degenerate solution, it follows from Theorem 1.9, that M is a Q-matrix. Now, we proceed as in 4.11 of Murty [43] to show that all the principal submatrices of order $(n-1)$ of M are Q-matrices.

Suppose A, a principal submatrix of M of order $(n-1)$ is obtained by striking off the first row and the first column of M, and A is not a Q-matrix. Then $\exists\, q \in R^{(n-1)}$ such that the problem

$$\begin{aligned} w - Az &= q \\ w^t z &= 0,\ w, z \geq 0 \end{aligned} \tag{2.9}$$

does not have a solution. Let $\overline{q} \in R^n$ be defined as $\overline{q} = (\alpha, q^t)^t$ for α, a scalar. If $lcp(q, M)$ has a solution (w, z) when $q = \overline{q}$, then $z_1 > 0$, for otherwise $(w_2, w_3; z_2, z_3)$ will be a solution for (2.10). Hence every point on the line

$$\{\overline{q} : \overline{q} = (\alpha, q^t)^t, \alpha \text{ a scalar}\}$$

corresponds only to solutions in which $z_1 > 0$. Since there are only a finite number of complementary cones and each one is convex, there must exist an α_0 such that the half line

$$\{q : q = (\alpha_0, q^t)^t + \theta I_{.1},\ \theta \geq 0\}$$

lies fully in a complementary cone. This complementary cone should have the column $-M_{.1}$ as one of its generators. This implies that $\text{pos}(I_{.1})$ also lies in the same complementary cone, i.e., $lcp(I_{.1}, M)$ has another solution, which is a contradiction. By a similar argument, we conclude that all principal submatrices order $(n-1)$ are Q-matrices.

If $n = 2$, this shows that the diagonal entries are positive. Now the conclusion of the theorem follows from (7) of Theorem 2.1.

For $n = 3$, we have all the order 2 principal submatrices being Q-matrices; we have to show that all the diagonal entries are positive. We proceed as follows, by considering different cases. At first, our claim is that the diagonal entries are nonzero. Suppose on the contrary that $m_{11} = 0$. Since all the submatrices of order 2 are Q-matrices, by Lemma 2.7 we conclude that $m_{21} < 0, m_{31} < 0$. But this implies that $lcp(-M_{.1}.M)$ has at least two solutions contradicting our hypothesis. Thus m_{11} and consequently all m_{ij} are nonzero.

Suppose $m_{11} < 0$. We have the following two cases.

Case 1: $m_{21} \leq 0, m_{31} \leq 0$. Then $lcp(-M_{.1}.M)$ has at least two solutions, a contradiction.

Case 2: $m_{21} > 0, m_{31} \geq 0$ or $m_{21} \geq 0, m_{31} > 0$. In either of these we have, $I_{.1} \in \text{pos}\{-M_{.1}, I_{.2}, I_{.3}\}$ contradicting the uniqueness of solution to the $lcp(I_{.1}, M)$.

Case 3: $m_{21} < 0, m_{31} > 0$ or $m_{21} > 0, m_{31} < 0$. If $m_{21} < 0$ and $m_{31} > 0$, then by Lemma 2.7, the sign pattern of the entries of M is given by

$$\begin{bmatrix} - & + & + \\ - & + & - \\ + & + & - \end{bmatrix}.$$

Let N be the principal submatrix of order 2 of M obtained by deleting its last row and the last column. By Theorem 2.13, for $q = (0,1)^t$, (q, N) has exactly two solutions. From the proof of Theorem 2.13, we see that $(0, \bar{z})$ is a solution to (q, N), where $\bar{z} > 0$. Choose a scalar $x_3 > 0$, so that

$$-m_{31}\bar{z}_1 - m_{32}\bar{z}_2 + x_3 = 0.$$

This is possible by the signs of m_{31} and m_{32}. Now note that $(0, z)$ solves $lcp(I_{.2}, M)$ where $z = (\bar{z}^t, x_3)^t$, which contradicts the uniqueness hypothesis. Similarly, we can arrive at a contradiction for the case $m_{21} > 0, m_{31} < 0$ also.

Thus $m_{11} > 0$. By a similar argument, we can show that $m_{22} > 0$ and $m_{33} > 0$. The conclusion now follows from (7) of Theorem 2.1. ∎

Chapter 3

MATRICES OF EXACT ORDER ONE

Like N-matrices, we would at first categorize the matrices of exact order one into two different categories. (As mentioned earlier, these matrices are known as almost N and almost P-matrices, in the literature). Most of the results in this chapter and the later ones are based on the papers [36] and [37].

Definition: Let $M \in R^{n \times n}$ be an N-matrix of exact order one, for $n \geq 4$. M is said to be of the *first category* if both M and M^{-1} contain at least one positive entry each; otherwise, it is said to be of the *second category*.

From the definition, it is clear that if $M \in R^{n \times n}$ is an N-matrix of exact order one, then so is M^{-1}.

Definition: Let $M \in R^{n \times n}$ be a P-matrix of exact order one. M is said to be of the *first category*, if M^{-1} has a positive entry; otherwise it is said to be of the *second category*.

3.1 P-matrices of exact order one

Characterizing P-matrices of exact order one becomes easier, due to the fact that a P-matrix of exact order one is the inverse of an N-matrix.

We present below a complete list of all characterizations known so far about P-matrices of exact order one of the second category.

Theorem 3.1 *Let M be a nonsingular matrix with $v(M) < 0$. Then the following statements are equivalent:*

1. *M is a P-matrix of exact order one.*
2. *Whenever a nonzero vector $x \in R^n$ gets reversed in sign, then either $x > 0$, or $x < 0$.*
3. *For every signature matrix $S \neq \pm I$, $v(SMS) > 0$.*
4. *For $q \in R^n$, $q \in intD(M)$, $lcp(q, M)$ has exactly two solutions.*
5. *Let $A = SMS$, where S is a signature matrix, $S \neq \pm I$. For every vector $q \in R^n, q \notin pos(-A)$, $lcp(q, A)$ has a unique solution; $lcp(q, A)$ has finitely many solutions $\forall\ q \in R^n$, and has more than one solution for at least one $q \in pos(-A)$.*

Proof: The equivalence of first and third is due to Parthasarathy and Ravindran [53]; they also prove condition 4. From the earlier chapter, we get the last condition. Also, Gowda [20] proves the last condtion for a matrix of exact order one using degree theory. Hence, only the equivalence of conditions 1 and 2 remains to be established.

Let M be a P-matrix of exact order one and as noticed earlier, M^{-1} is an N-matrix; since $v(M) < 0$, M^{-1} is an N-matrix of the second category.

Suppose M reverses the sign of a nonzero vector $x \in R^n$. Then M^{-1} reverses the sign of y, $y = Mx$. Hence from Theorem 2.3, y is unisigned, i.e., either $y \geq 0$ or $y \leq 0$. As M is nonsingular, $y \neq 0$. When $y \geq 0$, $x = M^{-1}y < 0$. Similarly $y \leq 0$ implies $x > 0$. Hence condition 2 follows.

Suppose the condition 2 holds for a nonsingular matrix M with $v(M) < 0$. Let $N \in R^{(n-1)\times(n-1)}$ be a principal submatrix of M; N does not reverse the sign of any nonzero vector. Otherwise, if $y \in R^{(n-1)}$, the vector $y \neq 0$ gets reversed in sign by N, then the vector $(y^t, 0)^t \in R^n$ (after a suitable rearrangement of its entries) gets reversed in sign by M contradicting our assumption. Hence all the proper principal submatrices of M are P-matrices. We conclude, from the value of M being negative that M is a P-matrix of exact order one. ∎

3.2 N-matrices of exact order one

Throughout the rest of this chapter, we consider matrices of order greater than three.

Several equivalent characterizations of these classes of matrices were given by Olech, Parthasarathy and Ravindran [46]. We state without proof, the following theorem from [46].

Theorem 3.2 *Let $M < 0$ be a nonsingular matrix. Then the following statements are equivalent:*

1. *M is an N-matrix of exact order one.*

2. *$v(SMS) > 0$ for all signature matrices S with the exception of two, viz., $S = \pm I$, and $S = \pm S_0(\neq I)$ where the signature matrix S_0 is such that $S_0 M^{-1} S_0 < 0$.*

3. *Whenever a vector $x \in R^n$ gets reversed in sign, then either x is unisigned or $S_0 x$ is unisigned (this S_0 is as given in condition 2).*

4. *$SMS \in Q$ for all signature matrices S, except for $S = \pm I$ and $S = \pm S_0$ (S_0 as given in condition 2).*

Olech et al [46] also proved a result on the number of solutions $lcp(q, M)$ has for some vectors q, $q \in R^n$:

Theorem 3.3 *Let M be an N-matrix of exact order one and $q > 0$ be an n vector. Then*

(i) if M is of the first category, then $lcp(q, M)$ has exactly three solutions;

(ii) if $M^{-1} < 0$, then $lcp(q, M)$ has exactly four solutions.

The main task of this section is to provide a complete characterization of N-matrices of exact order one in terms of the number of solutions the $lcp(q, M)$ has for each $q \in R^n$. This was posed as an open problem in [46].

To start with, we prove a lemma which plays a crucial role in these results.

Lemma 3.1 *Let M be an N-matrix of exact order one. If $lcp(q, M)$ for $q \in R^n$ has two solutions (w^1, z^1) and (w^2, z^2) with $w_i^1 = w_i^2 = 0$, for some $i = 1, 2, \ldots, n$, then $q \in pos(-M)$.*

Proof: Without loss of generality, assume that $i = 1$, i.e., $w_1^1 = w_1^2 = 0$. We consider two cases :

case(i): $M^{-1} < 0$. Since $lcp(q, M)$ has a solution (w^1, z^1), considering the system

$$z - M^{-1}w = -M^{-1}q \tag{3.1}$$

$$z \geq 0,\ w \geq 0 \text{ and } z^t w = 0.$$

We see that the $lcp(-M^{-1}q, M^{-1})$, has a solution (z^1, w^1). As $M^{-1} < 0$, we get $-M^{-1}q \geq 0$ and hence $q \in pos(-M)$.

case (ii): M^{-1} has a positive entry. We notice that (z^1, w^1) and (z^2, w^2) are two solutions to (3.1) with $w_1^1 = w_1^2 = 0$. Let $\overline{A}$ be the principal submatrix of M^{-1} got by omitting its first row and first column. Extracting the system

$$\overline{z} - \overline{A}\overline{w} = \overline{q} \tag{3.2}$$

obtained by dropping the first entry of $(-M^{-1}q)$ in (3.1), we note that all principal minors of $\overline{A}$ are negative and the reduced $lcp(\overline{q}, \overline{A})$ has two solutions.

Let them be denoted as $(\overline{w}^1, \overline{z}^1)$ and $(\overline{w}^2, \overline{z}^2)$. As $\overline{A}$ is an N-matrix, this implies that $\overline{q} \geq 0$, i.e., $(-M^{-1}q)_i \geq 0$, for $i = 2, \ldots, n$.

If $(-M^{-1}q)_1 \geq 0$, then $q \in pos(-M)$ and Lemma 3.1 follows. On the contrary let $(-M^{-1}q)_1 < 0$. From (3.1) we note that

$$z_1^1 - \sum_{j=1}^{n} m^{1j} w_j^1 = z_1^1 - \sum_{j=2}^{n} m^{1j} w_j^1 = (-M^{-1}q)_1 < 0 \tag{3.3}$$

Similarly,

$$z_1^2 - \sum_{j=2}^{n} m^{1j} w_j^2 = (-M^{-1}q)_1 < 0$$

It follows from (3.3) that there exist indices j and k, $1 \leq j, k \leq n$ such that,

$$w_j^1 > 0, \qquad m^{1j} > 0 \tag{3.4}$$

and

$$w_k^2 > 0, \qquad m^{1k} > 0$$

If in (3.4) $j = k$, then by complementarity, $z_j^1 = z_j^2 = 0$, and this violates Lemma 2.4 when applied to $(\overline{q}, \overline{A})$. Hence $j \neq k$ and the following must hold;

$$w_j^1 > 0, z_j^1 = 0; w_j^2 = 0, z_j^2 > 0$$

$$w_k^1 > 0, z_k^1 = 0; w_k^2 = 0, z_k^2 > 0 \tag{3.5}$$

Note that $\overline{A}$ reverses the sign of $(\overline{w}^1 - \overline{w}^2)$ which is not unisigned, in view of (3.5). Hence by Theorem 2.3, $\overline{A}$ is not an N-matrix of the second category, and $\overline{A}$ should have at least one positive entry. By Lemma 2.2, there is a $\phi \neq J \subset \{l, \ldots, n\}$ such that $(\overline{w}^1 - \overline{w}^2)_J \leq 0, (\overline{w}^1 - \overline{w}^2)_{\overline{J}} \geq 0$. Further $j \in J$ and $k \in \overline{J}$, by (3.5). Hence we have

$$m^{ij} m^{ik} < 0 \text{ for all } 1 \leq i \leq n.$$

The partitioned form of M^{-1} is as given in (2.3),

$$M^{-1} = \begin{bmatrix} M^{LL} & M^{L\overline{L}} \\ M^{\overline{L}L} & M^{\overline{L}\overline{L}} \end{bmatrix}$$

$$\begin{aligned} \text{where either } L &= J \cup \{1\}, \ \overline{L} = \overline{J} \text{ or} \\ L &= J; \ \overline{L} = \overline{J} \cup \{1\}. \end{aligned}$$

In either case using Lemma 2.3, we get $m^{1j}m^{1k} < 0$, a contradiction to (3.4). Hence $q \in pos(-M)$ and the proof is complete. ∎

We state below a lemma, proved in [46], regarding the minimax value of exact order one.

Lemma 3.2 *Let $M \in R^{n \times n}$, $n \geq 4$, be a matrix of exact order one. Suppose M has a positive entry. Then exactly one of the following holds:*

(i) There exists a positive vector u such that $Mu > 0$.

(ii) For all $u \geq 0$, $Mu \leq 0$; that is $M^{-1} < 0$.

Proof: For N-matrix of exact order one, it is already known [46]; if M is a P-matrix of exact order one, the proof follows from the fact that M^{-1} is an N-matrix. ∎

The following theorems present the number of solutions $lcp(q, M)$ has, for each $q \in R^n$, when M is an N-matrix of exact order one.

Theorem 3.4 *Let $M < 0$ be an N-matrix of exact order one of the second category. Then*

(i) $m(q) = 4$, for $q \in int[pos(-M)]$.

(ii) $m(q) = 2$, for $q > 0, q \notin pos(-M)$.

Before we prove this theorem, we remark that a more precise version of this theorem has been obtained by Gowda [20], using results on degree theory.

Proof: (of Theorem 3.4): Since $M < 0$, we have $D(M) = R^n_+$ and it is sufficient to show that if $m(q) > 2$, then $q \in pos(-M)$, and $m(q) = 4$. Observe that $lcp(q, M)$ has at least two solutions for $q > 0$, due to Theorem 1.7 on parity of solutions. Suppose for some $q > 0$, there exist two solutions for $lcp(q, M)$, other than the trivial solution, $w = q$, $z = 0$; we have two solutions (w^1, z^1) and (w^2, z^2) such that $z^1 \neq z^2 \neq 0$. M reverses of the sign of the nonzero vector $(z^1 - z^2)$. Then, by Theorem 3.2, either $(z^1 - z^2)$ is unisigned or there is a signature matrix $S_0 \neq \pm I$, such that $S_0(z^1 - z^2)$ is unisigned.

Suppose $(z^1 - z^2)$ is unisigned. Without loss of generality, we can assume that $(z^1 - z^2) \geq 0$. Since $z^2 \neq 0$, there exists an index i, $1 \leq i \leq n$, such that $z_i^2 > 0$ and $z_i^1 > 0$. Hence $w_i^1 = w_i^2 = 0$, and applying Lemma 3.1, we have $q \in pos(-M)$. If $S_0(z^1 - z^2)$ is unisigned, we have $S_0 M^{-1} S_0 < 0$ and let the partition of M^{-1} induced by S_0 be

$$M^{-1} = \begin{bmatrix} M^{JJ} & M^{J\bar{J}} \\ M^{\bar{J}J} & M^{\bar{J}\bar{J}} \end{bmatrix}$$

Further, we have $(z^1 - z^2)_J \geq 0$ and $(z^1 - z^2)_{\bar{J}} \leq 0$. If there is an index i such that $z_i^1 > 0$ and $z_i^2 > 0$, $1 \leq i \leq n$, we are done as in the previous paragraph; Otherwise, we have

$$z_J^1 \geq 0; z_J^2 = 0$$
$$z_{\bar{J}}^1 = 0; z_{\bar{J}}^2 \geq 0.$$

In this case, we notice that,

$$-M^{\bar{J}\bar{J}} z_{\bar{J}}^2 = (-M^{-1}q)_{\bar{J}} \geq 0$$
$$-M^{JJ} z_J^1 = (-M^{-1}q)_J \geq 0$$

and hence $-M^{-1}q \geq 0$ or in other words, $q \in pos(-M)$. Hence if $lcp(q, M)$ for $q > 0$ has more than two solutions, then $q \in pos(-M)$.

To complete the proof, we note from thereom 3.3, that $lcp(q, M^{-1})$ has exactly 4 solutions if $q > 0$. As there is a one to one correspondence between the complementary cones of $[I : -M]$ and that of $[I : -M^{-1}]$, it follows that $m(q) = 4$, if $q \in int[pos(-M)]$. This completes the proof. ∎

Remark 3.1: From Theorem 3.4, it follows that, if $M \not< 0$ is an N-matrix of exact order one of the second category, then $m(q) = 4$ for $q > 0$, and $m(q) = 2$, for $q \in int[D(M) \cap pos(I)^C]$.

Remark 3.2: For $M \not< 0$, an N-matrix of exact order one of the second category, one can easily check that $lcp(q, M)$ has exactly 1 or 2 solutions, for $q \geq 0$, with $q_i = 0$, for at least one $i, 1 \leq i \leq n$, depending on the sign pattern of M. The proof is similar to the one given in Theorem 2.4.

Theorem 3.5 *$M \in R^{n\times n}$ be an N-matrix of exact order one of the first category. Then*

(i) $m(q) = 3$, if $q \in int[pos(-M)]$ or $int[pos(I)]$;

(ii) $m(q) = 1$, if $q \in [pos(I) \cup pos(-M)]^C$.

Proof: It is clear from Theorem 3.3, that $m(q) = 3$, if $q > 0$. As M^{-1} is also an N-matrix of exact order one of the first category, it follows that if $q \in int[pos(-M)]$, then $m(q) = 3$.

Since we know that M is a Q-matrix using Theorem 3.3, we need to prove that if $m(q) > 1$, then either $q \in pos(-M)$ or $q \in pos(I)$.

Suppose that $lcp(q, M)$ has two distinct solutions (w^1, z^1) and (w^2, z^2) with $z^1 \neq z^2$. Then M reverses the sign of the vector $x = (z^1 - z^2)$. This implies that there exist signature matrices S_0, S_1, such that either S_0x is unisigned or S_1x is unisigned, where $S_0, S_1 \neq \pm I$ and $S_0MS_0 < 0, S_1M^{-1}S_1 < 0$.

Consider $S_0MS_0 < 0$. The signature matrix S_0 induces a partition as in (2.3) for some $\phi \neq J \subset \{1, 2, \ldots, n\}$ and we have $(z^1 - z^2)_J \geq 0, (z^1 - z^2)_{\bar{J}} \leq 0$. If there is an index i such that $z_i^1 \geq z_i^2 > 0$ or $z_i^2 \geq z_i^1 > 0$, for $1 \leq i \leq n$, then as in the proof of Theorem 3.4, it follows that $q \in pos(-M)$; otherwise, we have

$$z_J^1 \geq 0; z_J^2 = 0$$

$$z_{\bar{J}}^1 = 0; z_{\bar{J}}^2 \geq 0.$$

From the sign pattern of the partitioned form in (2.3) it follows that $q_J \geq 0$ and $q_{\bar{J}} \geq 0$, and hence $q \in pos(I)$.

We can proceed similarly, in the case of the signature matrix S_1, with $S_1M^{-1}S_1 < 0$, using M^{-1}.

This completes the proof of Theorem 3.5. ∎

The following theorems establish the converse of Theorem 3.4 and Theorem 3.5, respectively.

Theorem 3.6 *Let $M < 0$ be a square matrix of order $n \geq 4$. Suppose $m(q)$ satisfies the following conditions:*

(i) $m(q) < \infty$ *for all* $q \in R^n$;

(ii) $m(q) > 2$ *if* $q \in int[pos(-M)]$;

(iii) $m(q) \leq 2$, *if* $q \notin pos(-M)$.

Then M is an N-matrix of exact order one of the second category.

Proof: Since $m(q) < \infty$, for all $q \in R^n$, none of the principal minors of M is zero. We shall show that if F is an $(n-1)$-face of$[I \vdots -M]$, which is not a face of pos (I) or pos $(-M)$ then F is proper. Let F be an $(n-1)$-face generated by k columns of I and $(n-k)$ columns of $-M$, $1 < k < (n-1)$, such that the two complementary cones pos(B) and pos(B^1) incident on it are not properly situated.

Without loss of generality, let us assume that

$$F = pos(I_1, \ldots, I_{.k}, -M_{k+2}, \ldots, -M_{.n})$$

with $\{I_{.k+1}, -M_{k+1}\}$ as the left out complementary pair. Since $M < 0$, it is easy to see that the vector

$$q = \sum_{r=1}^{k} I_{.r} + \delta \sum_{s=k+2}^{n} (-M_{.s}) \in F$$

is not contained in $pos(-M)$ for δ sufficiently small and $q > 0$. As F is not proper, there exists an $\varepsilon > 0$, such that

$$q + \varepsilon(I_{.k+1}) \notin pos(-M) \text{ and}$$

$$q + \varepsilon(I_{.k+1}) \in pos(B) \cup pos(B^1)$$

We then find the $(q + \varepsilon(I_{.k+1}), M)$ has at least 3 solutions, contrary to our hypothesis. Hence our claim that F, which is not a face of $pos(I)$ or $pos(-M)$, is proper follows.

Now by Lemma 2.5, it follows that all the proper principal minors of M have the same sign. Since $M < 0$, all the proper principal minors are negative. If $\det(M) < 0, M$ will be an N-matrix. This however contradicts Theorem 2.3.

Therefore $\det(M) > 0$, and M is an N-matrix of exact order one. ∎

Theorem 3.7 *Suppose for $M \in R^{n \times n}$, with $n \geq 4$, we have the following:*

(i) $m(q) < \infty$ for all $q \in R^n$;

(ii) $m(q) = 2$ or 0, if $q \notin R^n_+$;

(iii) $m(q) > 2$ for all $q > 0$.

Also suppose that $pos(I) \subset pos(-M)$. Then, M is an N-matrix of exact order one of the second category.

Proof: Notice that as $pos(I) \subset pos(-M)$, $pos(-M^{-1}) \subset pos(I)$; therefore, $M^{-1} < 0$ and M^{-1} satisfies all the conditions of Theorem 3.6. Hence, the conclusion follows. ∎

Theorem 3.8 *Let $M \in R^{n \times n}$, $n \geq 4$, be such that M as well as M^{-1} have the partitioned form as in* (2.3) *and suppose that M satisfies the following:*

(i) $m(q) < \infty$ for all $q \in R^n$;

(ii) $m(q) = 1$, if $q \notin [pos(I) \cup pos(-M)]$;

(iii) $m(q) > 1$ for $q \in int[pos(I)]$ or $q \in int[pos(-M)]$.

Then, M is an N-matrix of exact order one of the first category.

Proof: Since $m(q) < \infty$, for all $q \in R^n$, it follows that none of the principal minors of M is zero. We now claim that if F is an $(n-1)$ face of $[I \vdots - M]$, which is not a face of $pos(I)$ or $pos(-M)$, then F is proper. Suppose this is not true. Then there exists a k, $1 < k < (n-1)$, and an $(n-1)$ face F, generated by k columns of I and $(n-k)$ columns of $-M$, such that the two complementary cones that are incident on F lie on the same side of it. Without loss of generality, let us assume that

$$F = pos(I_{.1}, \ldots, I_{.k} - M_{.k+2}, \ldots, -M_{.n}).$$

We now construct a $q \in R^n$, contained in the relative interior of F which is neither in $pos(I)$ nor in $pos(-M)$. First note that, for any j, $1 \leq j \leq n$,

$I_{.j} \notin pos(-M)$, for otherwise $-M^{\cdot j} \in pos(I)$, contradicting our hypothesis about M^{-1}. Also by hypothesis, $M_{\cdot k+2}$ contains a positive entry, i.e. $\exists$ an index r, such that $m_{rk+2} > 0$. Consider for $\lambda > 0$, $s \neq r$, $\tilde{q} = -M_{\cdot k+2} + \lambda I_{\cdot s} \not\geq 0$, where $1 \leq s \leq k$. Hence $\tilde{q} \notin pos(I)$.

Also, as $I_{\cdot s} \notin pos(-M)$, there exists a $\lambda_0 > 0$ such that

$$-M_{\cdot k+2} + \lambda_0 I_{\cdot s} \notin pos(-M)$$

Let $q = -M_{\cdot k+2} + \lambda_0 I_{\cdot s} + \delta \sum_{j=1}^{k} I_{\cdot j} + \delta \sum_{j=k+3}^{n}(-M_{\cdot j}) \in F$. For sufficiently small δ, q is neither contained in $pos(I)$ nor in $pos(-M)$. Now, we note that there exists an $\epsilon > 0$, such that $lcp(q + \epsilon(-M_{\cdot k}),\ M)$ has at least two solutions and $q + \epsilon(-M_{\cdot k}) \notin pos(I)$ or $pos(-M)$, which contradicts our hypothesis.

The remaining part of the proof follows in the same lines as that of Theorem 3.6. ∎

3.3 Generalizations of exact orders 0 and 1

The class of matrices with nonnegative principal minors are called P_0-matrices. It had been of interest in complementarity, to find constructive characterizations of $P_0 \cap Q_0$. We refer to [2] and [41]. Among the class of P_0-matrices, Q-matrices are precisely the R_0-matrices. This was observed by Aganagic and Cottle [1].

A similar attempt has been made to characterize the class of $N_0 \cap Q$-matrices, i.e.,Q-matrices whose principal minors are all nonpositive. Eagambaram and Mohan [14] and Pye [55] have shown that the Aganagic-Cottle result holds good for nonsingular N_0-matrices with a positive entry.

In this section, we investigate the class of N_0-matrices, which are got as limit points of N-matrices and obtain a result regarding their Q-property. Our result generalizes the results of Pye [55] and Eagambaram and Mohan [14]. Unlike P_0-matrices, there are N_0-matrices which are not limit points of N-matrices. Hence, the question of characterizing the class of $N_0 \cap Q$ completely, still remains open.

The class of $\bar{N}$ -matrices are defined as follows:

Definition: A matrix $M \in R^{n\times n}$ is said to be $\bar{N}$, if and only if $\exists$ a sequence $\{M^{(k)}\}$, $M^{(k)} \in R^{n\times n}$ of N-matrices, such that $m_{ij}^k \to m_{ij}$, $\forall i,j \in \{1,\ldots,n\}$.

For P_0-matrices, by perturbing the diagonal entries alone one can get a sequence of P-matrices (whose principal minors are all positive) that converges to P_0. It is not so with N_0-matrices; one of the reasons is that an N-matrix needs to have all its entries nonzero. Hence the matrix

$$M = \begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \tag{3.6}$$

though it is N_0, cannot be got as a limit point of N-matrices by perturbing the diagonal; but $M \in \bar{N}$, as the sequence

$$M^{(k)} = \begin{bmatrix} -1 & -1 & 2 \\ -\frac{2}{k} & -\frac{1}{k} & 2 \\ 1 & 1 & -1 \end{bmatrix} \tag{3.7}$$

converges to M as $k \to \infty$.

We note that $\bar{N}$ includes the following matrices: .

(i) N_0 with nonzero diagonal entires;

(ii) Symmetric N_0 and

(iii) Nonsingular N_0.

The following example shows that $\bar{N}$ does not include all the N_0-matrices:

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \tag{3.8}$$

In order to characterize the class of $\bar{N}$ -matrices, we at first study the following class of almost $\mathcal{P}$ -matrices:

Definition: Let $M \in R^{n\times n}$. M is called an almost $\mathcal{P}$ (almost $\mathcal{P}_0$)-matrix, if $\exists$ a $\phi \neq K \subseteq \{1,\ldots,n\}$ such that $\forall\, J \subseteq \{1,\ldots,n\}$, $J \neq K$, $\det(M_{JJ}) > 0 (\geq 0)$, and $\det(M_{KK}) < 0$. Throughout our discussion, we fix the index set to be K with respect to which the class almost $\mathcal{P}$ is defined. The above definition of almost $\mathcal{P}$ and almost $\mathcal{P}_0$ generalizes the class of P of exact order one.

The reason for such a generalization of the class of P of exact order one stems from the fact that if M is an almost $\mathcal{P}$ -matrix, then $\mu_K(M)$, its PPT with respect to the fixed index set K, is an N-matrix. Hence some of the results of section 1 can be reformualted in terms of almost $\mathcal{P}$. But such a generalization of N of exact order one will result in altogether a new class and hence, is not done here.

As a consequence, we have $v(M) \neq 0$ whenever M is an almost $\mathcal{P}$ -matrix. Also, there exists a signature matrix S_J for some $J \subseteq \{1,\ldots,n\}$ such that $v(S_J M S_J) < 0$. when M is almost $\mathcal{P}$.

Following is a lemma on the signs of minimax values of M and M^t when M is an almost $\mathcal{P}$ -matrix.

Lemma 3.3 *Let $M \in R^{n\times n}$ be an almost $\mathcal{P}$ -matrix for $\phi \neq K \subset \{1,\ldots,n\}$. If $v(M) < 0$ then $v(M^t) > 0$.*

Proof: The proof follows from the relation between the PPT of M and the PPT of M^t, viz.,

$$\mu_K(M^t) \;=\; S_K(\mu_K(M))S_K \tag{3.9}$$

and the fact that $\mu_K(M)$ is an N-matrix of the second category. ∎

Next we prove a sign reversal property for almost $\mathcal{P}$ -matrices.

Theorem 3.9 *Let $M \in R^{n\times n}$ be an almost $\mathcal{P}$ -matrix with $v(M) < 0$. Then whenever*

$$\begin{aligned} x_i(Mx)_i \leq 0 \Longrightarrow \quad & \textit{either} \quad x = 0, \\ & \textit{or} \quad x_K > 0, x_{\overline{K}} \leq 0, \\ & \textit{or} \quad x_K < 0, x_{\overline{K}} \geq 0, \end{aligned} \tag{3.10}$$

Proof: Let $y = Mx$ and $y_i x_i \leq 0,\ \forall\, i \in \{1, \dots, n\}$. Writing $y = Mx$ in the partitioned form,

$$\begin{bmatrix} I_{KK} & 0 & -M_{KK} & -M_{K\overline{K}} \\ 0 & I_{\overline{K}\overline{K}} & -M_{\overline{K}K} & -M_{\overline{K}\overline{K}} \end{bmatrix} \begin{bmatrix} y_K \\ y_{\overline{K}} \\ x_K \\ x_{\overline{K}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.11}$$

Multiplying the first K rows by $-M_{KK}^{-1}$ we have

$$\begin{bmatrix} -M_{KK}^{-1} & 0 & I_{KK} & M_{KK}^{-1}M_{K\overline{K}} \\ 0 & I_{\overline{K}\overline{K}} & -M_{\overline{K}K} & -M_{\overline{K}\overline{K}} \end{bmatrix} \begin{bmatrix} y_K \\ y_{\overline{K}} \\ x_K \\ x_{\overline{K}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.12}$$

Adding the last $\overline{K}$ with the first K rows multiplied by $M_{\overline{K}K}$ and rearranging the columns, we get

$$\begin{bmatrix} I_{KK} & 0 & -M_{KK}^{-1} & M_{KK}^{-1}M_{K\overline{K}} \\ 0 & I_{\overline{K}\overline{K}} & -M_{\overline{K}K}M_{KK}^{-1} & M_{\overline{K}\overline{K}} - M_{\overline{K}K}M_{KK}^{-1}M_{K\overline{K}} \end{bmatrix} \begin{bmatrix} x_K \\ y_{\overline{K}} \\ y_K \\ x_{\overline{K}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.13}$$

Thus we have, for the principal pivot transform $\mu_K(M)$ a vector $z^t = (y_K, x_{\overline{K}})^t$ such that $z_i(\mu_K(M)\,z)_i \leq 0$. As $v(\mu_K(M)\,) < 0$, $\mu_K(M)$ is an N-matrix of the second category. From Theorem 2.3, we see that z is an unisigned vector, i.e., either $z \leq 0$ or $z \geq 0$. When $z \geq 0,\ z \neq 0,\ \mu_K(M)\,z < 0$ which implies $x_K < 0$ and $x_{\overline{K}} \geq 0$. Similarly, when $z \leq 0,\ z \neq 0,\ \mu_K(M)\,z > 0$ which implies $x_K > 0$ and $x_{\overline{K}} \leq 0$. That M reverses the sign of no other vector follows from the sign reversal nature of $\mu_K(M)$. This completes the proof. ∎

For $\bar{N}$ matrices, we have the following result:

Theorem 3.10 *Let $M \in R^{n \times n}$ be an $\bar{N}$ -matrix, with $v(M) > 0$. Then the following are equivalent:*

(i) $M \in R_0$

(ii) $M \in Q$.

Remark 3.3: Before we proceed to prove Theorem 3.10, we remark that this includes the already known results in the class of $N_0 \cap Q$. Pye [55] and Eagambaram and Mohan [14] prove Theorem 3.10 for the class of nonsingular N_0-matrices with a positive entry. Since nonsingular N_0-matrices have inverse almost-P_0 [55], one can easily verify that they belong to $\bar{N}$. The class of $\bar{N}$ matrices includes more than nonsingular N_0-matrices, is clear from the example in (3.8).

Now we present the proof of Theorem 3.10.

Proof: At first we note that M has no zero column, for otherwise M is neither Q nor R_0. See [42].

(i) $\Rightarrow$ (ii): This has already been observed by Eagambaram and Mohan [14]; we give a different proof here.

Let $M \in R_0 \cap \bar{N}$, with $v(M) > 0$. Since $M \in \bar{N}$, $\exists\ \phi \neq J \subseteq \{1,\ldots,n\}$,

$$M = \begin{bmatrix} M_{JJ} & M_{J\bar{J}} \\ M_{\bar{J}J} & M_{\bar{J}\bar{J}} \end{bmatrix} \tag{3.14}$$

where $M_{JJ} \leq 0,\ M_{\bar{J}\bar{J}} \leq 0,\ M_{\bar{J}J} \geq 0,\ M_{J\bar{J}} \geq 0$, using (2.3).

Consider the principal submatrix M_{JJ} of M. Every column of M_{JJ} should have a negative entry, for otherwise $(0, M)$ will have a nontrivial solution, violating the condition that $M \in R_0$. Hence $\exists\ x \in R^{|J|}$, $x > 0$, such that $x^t M_{JJ} < 0$. Thus, the sufficient condition of Theorem 4.3 of [58] is met, and for any $q_J > 0$, q_J nondegenerate with respect to M_{JJ}, (q_J, M_{JJ}) has exactly two solutions. We choose and fix a $q_J > 0$. By symmetry of the structure, $(q_{\bar{J}},\ M_{\bar{J}\bar{J}})$ also has exactly two solutions, for any $q_{\bar{J}} > 0$, $q_{\bar{J}}$ nondegenerate with respect to $M_{\bar{J}\bar{J}}$. Thus for $q \in R^n$, $q > 0$, $lcp(q, M)$ has at least three solutions. As M is a limit point of a sequence of N-matrices with $v(M) > 0$, we conclude from Theorem 2.6, that for $q > 0$ nondegenerate with respect to M, $lcp(q, M)$ has no other solution. Using Theorem 1.9, we have $M \in Q$.

(ii) $\Rightarrow$ (i): Let $M \in \tilde{N} \cap Q$. From [42, Exercise 3.103], it is clear that $\exists$ a $\phi \neq K \subseteq \{1, \ldots, n\}$, such that $\det(M_{KK}) < 0$. Consider the PPT matrix $\mu_K(M)$ of M.

Let $M \notin R_0$. Then $\mu_K(M) \notin R_0$ and there exists a nontrivial solution (w, z) for $(0, \mu_K(M))$, which can be written in the matrix form, for some $\phi \neq L \subseteq \{1, \ldots, n\}$ as

$$\begin{bmatrix} -\mu_K(M)_{LL} & 0 \\ -\mu_K(M)_{\bar{L}L} & I_{\bar{L}\bar{L}} \end{bmatrix} \begin{bmatrix} z_L \\ w_{\bar{L}} \end{bmatrix} = \begin{bmatrix} q_L \\ q_{\bar{L}} \end{bmatrix} \tag{3.15}$$

where $z_L > 0$. Now, we claim that for any $q \in R^n$, with $(q_L < 0,\ q_{\bar{L}} > 0)$, $lcp(q, M)$ has no solution.

On the contrary, suppose (u, v) is a solution of $(q, \mu_K(M))$. Let $y = v - \lambda z$, for some $\lambda > 0$. As in the proof of Theorem 1 [1], one can verify that

$$y_i(\mu_K(M)\, y)_i \ < \ 0, \ \text{ for } y_i \neq 0,\ i = 1, \ldots, n, \tag{3.16}$$

for sufficiently small $\lambda > 0$, with $y_L > 0$ and $y_{\bar{L}} \leq 0$.

Let $T = \{i : y_i \neq 0\}$ and N be the principal submatrix of $\mu_K(M)$ with respect to the index set T. Let the cardinality of the index set be s. From (3.16), clearly $K \subseteq T$. By the choice of λ, we have $L \subseteq T$. Let us denote the sets $T \backslash K$ and $T \backslash L$ by $\hat{K}$ and $\hat{L}$ respectively. Then for $q \in R^s$, $(q_L < 0, q_{\hat{L}} > 0)$, (q, N) has a solution, viz., (u_T, v_T).

$M \in \tilde{N}$ implies $\exists$ a sequence $\{M^{(r)}\}$, $M^{(r)} \in R^{n \times n}$, $M^{(r)}$ an N-matrix, such that $M^{(r)}$ converges to M. As $\det(M_{KK}) < 0$, we have the sequence $\{\mu_K(M^{(r)})\}$, where $\mu_K(M^{(r)})$ the PPT of $M^{(r)}$, converging to $\mu_K(M)$. If $N^{(r)}$ denotes the principal submatrix of $\mu_K(M^{(r)})$ with respect to the index set T, then $\{N^{(r)}\}$ converges to N. Let $B^r = N^{(r)}$, for the simplicity of notation. Therefore, for r large enough, we have

$$y_i(B^r y)_i \ < \ 0,\ \forall\, i \in T. \tag{3.17}$$

B^r has the principal subdeterminant $\det(B^r_{KK})$ negative, and B^r is an almost $\mathcal{P}$-matrix. For the signature matrix $S_L \in R^{s \times s}$, $v(S_L B^r S_L) < 0$. Rewriting (3.17),

$$(S_L y)_i (S_L B^r S_L S_L y)_i \ < \ 0,\ \forall\, i \in T. \tag{3.18}$$

From Theorem 3.9, it follows that

$$\begin{aligned} \text{either } (S_L y)_K > 0, \quad & (S_L y)_{\hat{K}} < 0, \\ \text{or} (S_L y)_K < 0, \quad & (S_L y)_{\hat{K}} > 0, \end{aligned} \tag{3.19}$$

Let $(S_L y)_K > 0$, and $(S_L y)_{\hat{K}} < 0$. Then (3.19) implies that

$$\begin{aligned} y &= (y_{L \cap K} > 0, y_{L \cap \hat{K}} < 0, y_{\hat{L} \cap K} < 0, y_{\hat{L} \cap \hat{K}} > 0) \\ &= (y_{(L \cap K) \cup (\hat{L} \cap \hat{K})} > 0, \ y_{(\hat{L} \cap K) \cup (L \cap \hat{K})} > 0) \end{aligned} \tag{3.20}$$

But from (3.16), we have $y_L > 0$ and $y_{\hat{L}} < 0$, and hence

$$\begin{aligned} L &= (L \cap K) \cup (\hat{L} \cap \hat{K}) \\ \hat{L} &= (\hat{L} \cap K) \cup (L \cap \hat{K}) \end{aligned} \tag{3.21}$$

or in other words,

$$L \subseteq K \text{ and } \hat{L} \subseteq K$$

As $L \cup \hat{L} = T$ and $K \subseteq T$, we get $T = K$. This implies that $\det(N) < 0$ and N is an almost P_0-matrix as defined by Pye [55]. But the fact that (q, N) has a solution for $q \in R^s$ with $(q_L < 0, q_{\hat{L}} > 0)$ yields a contradiction to Theorem 5 of [55]. Hence the theorem follows. ∎

Finally, we state without proof a similar theorem for the class of almost $\mathcal{P}_0$-matrices.

Theorem 3.11 *Let $M \in R^{n \times n}$ be an almost $\mathcal{P}_0$-matrix with $det(M_{KK}) < 0$ for $\phi \neq K \subseteq \{1, \ldots, n\}$, for which there exists a sequence of almost $\mathcal{P}$-matrices that converges to it. Let $v(M) > 0$. Then the following are equivalent:*

(i) $M \in R_0$

(ii) $M \in Q$.

Proof: This follows from Theorem 3.10. ∎

Chapter 4

MATRICES OF EXACT ORDER TWO

In this chapter, we study the properties of matrices of exact order 2. Characterizing these matrices, regarding their Q-nature, forms the main result of this chapter.

We start with some examples.

Example 4.1: Consider the matrix

$$M = \begin{bmatrix} -.9 & -2 & -2 & 2 & -2 \\ -1 & -.9 & -3 & 3 & -1 \\ -1 & -3 & -.9 & 3 & -1 \\ 1 & 3 & 3 & -.9 & 1 \\ -2 & -2 & -2 & .2 & -.9 \end{bmatrix}$$

One can directly verify that, every principal minor of order 1,2 or 3 of M is negative and principal minors of order 4 and the determinant of M are positive. Hence M is an N-matrix of exact order 2.

Example 4.2:

$$B = \begin{bmatrix} 1 & 2 & 0 & 1.453378 & 0 \\ 1 & 4.989 & 1 & 1.368317 & 1 \\ 0 & 2 & 1.2 & 1.168878 & 0 \\ 1 & 2.6 & 1 & 2.842317 & 1.41 \\ 0 & 2 & 0 & 1.168878 & 1.2 \end{bmatrix}$$

As before, by looking at the principal minors of B, one can see that B is a P-matrix of exact order 2.

4.1 The three categories of exact order

The following notation is followed in this chapter only. A square matrix $M \in R^{n \times n}$, is assumed to be of order at least 5. $B_i \in R^{(n-1)\times(n-1)}$, $1 \leq i \leq n$, will denote the principal submatrix of M, got by deleting the ith row and the ith column of M. By $v_{ij}, 1 \leq i, j \leq n$, we mean the value of the game whose pay-off matrix, is the submatrix of M obtained by deleting the ith row and the jth column of M.

We notice that if M is of exact order k, then $B_i, 1 \leq i \leq n$, are matrices of exact order $(k-1)$.

Depending on the categories of the exact order one matrices in M, we classify a matrix of exact order 2, into three categories.

Definition: Let $M \in R^{n \times n}$ be an exact order 2 matrix. M is said to be of *the first category*, if $M \not< 0$ and every B_i, which is a matrix of exact order 1, $B_i \not< 0, 1 \leq i \leq n$, is of the first category; we say that M is of *the second category*, if all B_is are of the second category. M is said to be of *the third category*, if there are indices, $i, j \in \{1, \cdots, n\}$, such that B_i is of the first category, $B_j \not< 0$ is of the second category.

The matrix given in example 4.1 is an N-matrix of the exact order 2, of the third category. In example 4.2, we have a P-matrix of exact order 2 which is of the first category.

We observe the following about the principal minors of M^{-1}, if M is of exact order 2.

Lemma 4.1 *If M is an $N(P)$-matrix of exact order 2, then M^{-1} has diagonal entries positive, all proper principal minors of order greater than or equal to 2 are negative and $det(M) > 0 (< 0)$.*

In general, we observe the following relationship between the classes of P of exact order r and N of exact order r:

Lemma 4.2 *Let M be an N-matrix of exact of r and D be a principal submatrix of M^{-1} of order k by k, $n > k \geq r+1$. Then D^{-1} is a P-matrix of exact order r.*

4.2 Two-person zero-sum games with exact order matrices

We now prove some game theoretic results for these classes of matrices.

Lemma 4.3 *Let M be a matix of exact order 2. Then $v(M) \neq 0$.*

Proof: Suppose $v(M) = 0$. Then there exists a probability vector y, such that $y^t M \leq 0$. If $y > 0$, then there is a probability vecor z, such that $Mz = 0$, which contradicts the hypothesis about M. Hence, without loss of generality, assume that $y = (y_1, y_2, \ldots y_{n-1}, 0)^t$. If $B_n < 0$, (which may occur, if M is an N-matrix of exact order 2), then from the fact that $y^t M \leq 0$ and the sign pattern of M, it follows that $M < 0$, contradicting our assumption that $v(M) = 0$.

Now, it follows from Lemma 3.2 that $\bar{y} = (y_1, y_2, \ldots, y_{n-1})^t > 0$. Thus $(Mz)^t = (0, \ldots, 0, \alpha)$ for any optimal strategy z of the maximizer, where $\alpha > 0$ is a scalar.

This implies

$$z = M^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \end{bmatrix} = \alpha.M^{\cdot n}$$

where $M^{\cdot n}$ denotes the nth column of M^{-1}. By Lemma 4.1, since every 2 by 2 principal minors is negative, it follows that $M^{\cdot n} > 0$ as $z \geq 0$. Thus $v(M^{-1}) > 0$. It follows that $v(M) > 0$, contradicting our assumption. This completes the proof.

Lemma 4.3 can be restated as a theorem of alternatives as follows.

Theorem 4.1 *Let M be a matrix of exact order 2. Then exactly one of the following holds:*

(i) there exists a $y > 0$, such that $y^t M < 0$.

(ii) there exists an $x > 0$, such that $Mx > 0$.

It is well-known that $v(M)$ and $v(M^t)$ keep the same sign whenever M is a matrix of exact order 1 or 0. This result, for matrices of exact order 2, is proved next, in Theorem 4.2. We first prove two lemmas.

Lemma 4.4 *Let M be a matrix of exact order 2. If all $B_i, 1 \leq i \leq n$, are of the same category, then $v(M)$ and $v(M^t)$ have the same sign.*

Proof: If the matrix game M is completely mixed, then it is known already from Theorem 1.4 that $v(M) = v(M^t)$. Otherwise, from (1.6), there exist indices $i_0, j_0, \in \{1, \ldots, n\}$, such that

$$v_{i_0 j} \leq v(M) \leq v_{i j_0} \text{for all } 1 \leq i, j \leq n \tag{4.1}$$

Similarly, there exist $i_1, j_1, \in \{1, \ldots, n\}$, such that

$$v^t_{i_1 j} \leq v(M)^t \leq v^t_{i j_1} \text{for all } 1 \leq i, j \leq n \tag{4.2}$$

where by v^t_{ij} we mean the value of the subgame with the pay-off matrix obtained from M^t by deleting the ith row and the jth column.

Suppose $v(M) > 0$ and $v(M^t) < 0$. From (4.1), (4.2) and Lemma 3.2, we conclude that B_{j_0} is of the first category and B_{i_1} is of the second category, contrary to our hypothesis. This concludes the proof. ∎

More precisely, for the first category matrices, we have the following:

Corollary 4.1: Let M be a matrix of exact order 2. If it is of the first category, then $v(M)$ and $v(M^t)$ are positive.

Now, we prove our desired theorem.

Theorem 4.2 *Let M be a matrix of exact order 2. Then $v(M)$ and $v(M^t)$ have the same sign.*

Proof: If M is either of the first category, or of the second category, the theorem follows from Lemma 4.4.

So, let M be an exact order 2 matrix of the third category. Then there exists an $i \in \{1, \dots, n\}$ such that, $B_i \not< 0$ is of the second category. Assume, without loss of generality, that $i = 1$. Let us also assume that $v(M) < 0$.

By Lemma 4.1, M^{-1} has no zero entry, with the diagonal entries being positive. Suppose the first row of M^{-1}, i.e., $M^{1\cdot} > 0$. Then $M^{\cdot 1} > 0$, and hence $v(M^{-1}) > 0$.

This contradicts our assumption that $v(M) < 0$ by Lemma 1.3. Hence $M^{\cdot 1}$ contains a negative entry. Thus, there is a $k \in \{2, \dots, n\}$ such that $m^{1k} < 0$.

Define the vector $(w_2, \dots, w_n)^t \in R^{n-1}$ by taking

$$w_i = \begin{cases} 0, & \text{if } i \neq k \\ -1 & \text{otherwise, } i = 2, \dots, n. \end{cases}$$

Let $y = {B_1}^{-1} w$; As $B_1^{-1} < 0$, $y > 0$.

By taking $u = \begin{bmatrix} 0 \\ y \end{bmatrix}$ as an n-vector, we have

$$Mu = \begin{bmatrix} w^* \text{ (say)} \\ w \end{bmatrix} \text{ and } M^{-1} \begin{bmatrix} w^* \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Now w^* can be determined from the equation

$$m^{11} w^* + m^{1k}(-1) = 0$$

Then $w^* < 0$. We have $Mu = \begin{bmatrix} 0 \\ y \end{bmatrix} \leq 0$, or $u^t M^t \leq 0$.

This, with Lemma 4.4, implies that we have $v(M^t) < 0$. Similarly, one can prove the theorem with $v(M) > 0$.

This completes the proof of the theorem. ■

4.3 Completely mixed game and Q-matrices

The following result which we present for a class of completely mixed games with value greater than zero, will be made use of in the sequel, in characterizing the class of exact order matrices.

Theorem 4.3 *Let $M \in R^{n \times n}$. Let $v(B_i) < 0$, for $1 \leq i \leq n$. Then the following are equivlent:*

(i) $v(M) > 0$

(ii) $M \in Q$

(iii) M is nonsingular and $M^{-1} > 0$.

Proof: (i) $\Rightarrow$ (ii): We shall show that, for a $q \in R^n$, $lcp(q, M)$ has a unique solution, viz., $(w = 0; z > 0)$, which by definition, is nondegenerate.

Let $v = v(M) > 0$, and $v_i = v(B_i)$, for $1 \leq i \leq n$.

Consider the vector $q = -ve$, where e is the n-vector of 1's. Since $v > 0$, and $v_i < 0$, for all $1 \leq i \leq n$, it follows that the game is completely mixed; hence, there exists a $z > 0, z \in R^n$ such that $Mz = -ve$, or $(0, z)$ solves $lcp(q, M)$.

Suppose (w^1, z^1) is another solution to $lcp(q, M)$. We then have the following equations:

$$Mz + q = 0 \tag{4.3}$$

$$Mz' + q = w' \tag{4.4}$$

If $w' = 0$, then $z = z'$, since the game is completely mixed. So assume that $w' \neq 0$; Suppose the first coordinate $w'_1 > 0$. Then $z'_1 = 0$. We note that

$$B_1 \bar{z}_1 + \bar{q} = \bar{w}^1 \tag{4.5}$$

Where $\bar{x}$ denotes an $(n-1)$-vector obtained from $x \in R^n$ by omitting its first coordinate.

Equation (4.5) implies that $v(B_1) = v_1 > 0$, contradicting our hypothesis. Thus $(0, z)$ is the only solution to $lcp(q, M)$. Similarly, we can show that $(0, 0)$ is the only solution to the $lcp(0, M)$.

It now follows from Theorem 1.9, that M is a Q-matrix.

(ii) $\Rightarrow$ (iii): Take $q = -e_i$, for $i \in \{1, \ldots, n\}$, where e_i is the ith column vector of I. Since $M \in Q$, $lcp(q, M)$ should have a solution.

We can observe, as in the previous case, that $lcp(q, M)$ has a unique solution, $(0, z)$, with $z > 0$. In otherwords, $-e_i \in pos(-M)$, for $1 \leq i \leq n$. This implies, $pos(-I) \subset pos(-M)$, and hence $M^{-1} > 0$.

That (iii) implies (i) follows from Lemma 1.2. ∎

Remark 4.1: As a consequence of the above theorem, if $v(M) > 0$ for the second category matrices of exact order 2, then we have $M \in Q$. But it is clear that with $B_i \not< 0$, $1 \leq i \leq n$, this will not happen at least for the size of the matrix being odd, as it would contradict Theorem 1.7 on parity of solutions to $lcp(q, M)$.

In fact, the value of the matrix game is never positive for the second category matrices. This is proved in the next theorem.

Theorem 4.4 *Let $M \in R^{n \times n}$ be a matrix of exact order 2. If M is of the second category, then $v(M) < 0$.*

Proof: Let us assume that M has at least one positive entry, as otherwise there is nothing to prove.

Let $v(M) > 0$. Suppose $B_i \not< 0$ for all $1 \leq i \leq n$. There are two cases possible based on the size of the matrix.

Case (i): 'n' is odd. From Remark 4.1, it is clear that $v(M) < 0$, if M is of the second category.

Case (ii): 'n' is even. Let $v(M) > 0$. We now claim that M is of the second category if and only if $lcp(I_{.j}, M)$ has at least n solutions for each $j = 1, \ldots, n$.

As $v(M) > 0$, and M is of the second category, each subproblem $lcp(I_{.j}, B_i)$ for any fixed j, $1 \leq j \leq n$, has a nontrivial solution for all $i \neq j, i = 1, \ldots, n$. Since each of them can be extended to the $lcp(q, M)$, with $q = I_{.j}$, $lcp(I_{.j}, M)$ has at least n solutions. The solutions are distinct from the fact that in each subproblem $lcp(I_{.j}, B_i)$, we have $I_{.j} \in \text{int}pos(-B_i)$ for $i \neq j$.

Conversely, suppose $lcp(I_{.1}, M)$ has at least n solutions. Let a nontrivial solution (w, z) for the $lcp(I_{.1}, M)$ be written in the matrix form, for $\phi \neq J \subset \{1, \ldots, n\}$, as

$$\begin{bmatrix} -M_{JJ} & 0 \\ -M_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix} \begin{bmatrix} z_J \\ w_{\bar{J}} \end{bmatrix} = \begin{bmatrix} I_{.1} \\ 0 \end{bmatrix}$$

i.e. the system,

$$-M_{JJ} z_J \geq 0, z_J > 0$$

has a solution. This means that the matrix M_{JJ} cannot be an $N(P)$-matrix; hence, M_{JJ} has to be a matrix of exact order one, i.e., $M_{JJ} = B_i$ for some $2 \leq i \leq n$. Using Lemma 3.2, we conclude that $v(B_i) < 0$. In a similar manner, we note that $v(B_i) < 0$ for all $i = 1, \ldots, n$ and hence, M is of the second category. Thus our claim is established. That is, M being of the second category is equivalent to assuming that the $lcp(I_{.j}, M)$ has at least n solutions for each $j = 1, \ldots, n$.

For a fixed j, say $j = 1$, from the earlier paragraph we see that, $lcp(I_{.1}, M)$ has solutions in the complementary cones,

$$pos(I),\ pos(-M_{.1}, \ldots, -M_{.i-1}, I_{.i}, -M_{.i+1}, \ldots, -M_{.n}),\ i = 2, \ldots, n.$$

Let $N = M^{-1}$. Under our hypothesis, $N > 0$. Due to the 1-1 correspondence that exists between the cones of $[I : -M]$ and that of $[I : -N]$ from Theorem 1.2, we note that the $lcp(-N_{.1}, N)$ has at least n solutions viz.,

$$pos(-N),\ pos(I_{.1}, \ldots, I_{.i-1}, -N_{.i}, I_{.i+1}, \ldots, I_{.n}),\ i = 2, \ldots, n.$$

Thus one can list down all the n solutions of $(I_{.j}, N), j = 1, \ldots, n$ explicitly.

Let $\tilde{N}$ be the principal submatrix of N, leaving the last row and the last column of N. We can see that $\tilde{N} > 0$ and by Lemma 4.2, $\tilde{N}^{-1}$ is a matrix of exact order 2, of size $(n-1)$ by $(n-1)$. But $(-\tilde{N}_{.j}, \tilde{N})$ has at least $(n-1)$ solutions for each $j = 1, \ldots, n-1$, with $v(\tilde{N}) > 0$ implies that $\tilde{N}^{-1}$ is a matrix of exact order of the second category. This gives rise to a contradiction to case (i), as the order of $\tilde{N}^{-1}$ is even. Hence $v(M) < 0$.

If there exists an i, $1 \leq i \leq n$, such that $B_i < 0$, say B_n then the roles of case (i) and case (ii) get interchanged and one can prove in a similar way that $v(M) < 0$. This completes the proof of Theorem 4.4. ∎

4.4 Results on first category exact order two matrices

We now have a result on the Q-nature of the exact order 2 matrices of the first category.

Theorem 4.5 *Let M be a matrix of exact order 2 of the first category. Then $M \in Q$.*

Proof: Since M is nondegenerate, $(0, M)$ has a unique solution. We treat P and N-matrices of exact order 2 separately, below.

Suppose M is a P-matrix of exact order 2, of the first category. Consider a $q \in R^n, q > 0$. We claim that $lcp(q, M)$ has a unique solution $w = q, z = 0$.

Let there exist a solution (w^1, z^1) for $lcp(q, M)$ with $z^1 \neq 0$. The solution can be written in the matrix form, for some $\phi \neq J \subseteq \{1, \ldots, n\}$ as

$$\begin{bmatrix} -M_{JJ} & 0 \\ -M_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix} \begin{bmatrix} z_J^1 \\ w_{\bar{J}}^1 \end{bmatrix} = \begin{bmatrix} q_J \\ q_{\bar{J}} \end{bmatrix} \tag{4.6}$$

i.e. the system,

$$-M_{JJ} z_J > 0, z_J > 0 \tag{4.7}$$

has a solution, which implies that $v(M_{JJ}^t) \leq 0$. We know that $v(M_{JJ}) = v(M_{JJ}^t) > 0, J \subseteq \{1, \ldots, n\}$, for a P-matrix of exact order 2 of the first category, using Corollary 4.1. Hence (4.7) is impossible and $lcp(q, M)$ has a unique solution for $q > 0$. By Theorem 1.9, we have $M \in Q$.

Suppose now, M is an N-matrix of exact order 2, of the first category. Two cases arise.

Case(i): There is an $i_0, 1 \leq i_0 \leq n$, such that $B_{i_0} < 0$. We may assume, without loss of generality, that $i_0 = 1$; the sign pattern of M can be written as

$$M = \begin{bmatrix} - & + & + & \dots & + \\ + & & & & \\ + & & & B_1 & \\ \vdots & & & & \\ + & & & & \end{bmatrix} = \begin{bmatrix} m_{11} & d^t \\ c & B_1 \end{bmatrix} \text{(say)}$$

with $B_1 < 0$. Consider a $q > 0$, whose partitioned form is $q = (q_1, \tilde{q})^t$, where $\tilde{q} \in R^{n-1}$. Choose $\tilde{q} \in R_+^{n-1}$, such that $\tilde{q} \in int[pos(-B_1)]$. Since B_1 is an N-matrix of exact order 1, of the second category, and $\tilde{q} \in int[pos(-B_1)]$, by Theorem 3.3, it follows that the $lcp(\tilde{q}, B_1)$ has exactly four solutions. Also, if $(\overline{w}, \overline{z})$ solves $lcp(\tilde{q}, B_1)$, then the pair $(w, z), w \in R^n, z \in R^n$, defined by

$$w_1 = q_1 + d^t\overline{z}; z_1 = 0$$

$$w_j = \overline{w}_{j-1}; z_j = \overline{z}_{j-1}, 2 \leq j \leq n,$$

solves $lcp(q, M)$. Thus we obtain 4 solutions to the $lcp(q, M)$. We construct another solution as follows:

$$\begin{aligned} \text{Take } z_1^1 &= q_1/(-m_{11}); \ w_1^1 = 0 \\ z_i^1 &= 0; \ w_i^1 = \tilde{q}_{i-1} + z_1^1.c_i, \ \forall 2 \leq i \leq n. \end{aligned}$$

Then (w^1, z^1) solves $lcp(q, M)$ and (w^1, z^1) is different from the four solutions constructed before. Thus we have 5 solutions to $lcp(q, M)$ and q nondegenerate with respect to M, by our construction. Now, for this $q \in R_+^n$, we proceed to prove that $lcp(q, M)$ has no other solution.

Suppose (u, v) is a solution to $lcp(q, M)$ distinct from the 5 listed above. Let

$$L = \{i : v_i > 0\}$$

Then since (u, v) is different from the aforesaid 5 solutions, it follows that, the index $1 \in L$ and $L \cap \{2, \dots, n\} \neq \phi$. Now the equation

$$u - Mv = q,$$

leads us to

$$M_{LL}v_L < 0$$

where either M_{LL} is an N-matrix of exact order 1 of the first category or $L = \{1, \ldots, n\}$. But this gives rise to a contradiction to the property of an N-matrix of exact order 1 or 0, or when $L = \{1, 2, \ldots, n\}$, to our assumption on the $v(M)$. This shows that there are exactly 5 solutions to $lcp(q, M)$, and by our choice of $\tilde{q} \in int[pos(-B_1)], q$ is nondegenrate with respect to M. This along with $lcp(0, M)$ having a unique solution implies (using Theorem 1.9) that M is a Q-matrix.

Case (ii) $\nexists\ i_0$, such that $B_{i_0} < 0$. M can be written in the partitioned form, for some $\phi \neq J \subset \{1, \ldots, n\}$ as

$$M = \begin{bmatrix} M_{JJ} & M_{J\overline{J}} \\ M_{\overline{J}J} & M_{\overline{J}\overline{J}} \end{bmatrix}$$

where $M_{JJ} < 0, M_{\overline{J}\overline{J}} < 0$ and $M_{J\overline{J}}, M_{\overline{J}J} > 0$ with $1 < |J| < n-1$. We proceed as before, finding a vector $q > 0$, q nondegenerate with respect to M and $lcp(q, M)$ has exactly three solutions. By Theorem 1.9, the result follows. ∎

4.5 Results on second category exact order 2 matrices

It is well-known that if M is an N-matrix of exact order 1 or 0, or a P-matrix of exact order 1, of the second category, then $-M \in Q$. In view of this, one may consider the following question:

Question: Let M be a matrix of exact order 2 of the second category. Is $-M$, a Q-matrix?

Our Theorem 4.7, that follows will provide an affirmative answer to this question. To do this, we need the following results.

Lemma 4.5 *Let M be a matrix of exact order 2. Let B_1 and B_2 be matrices* *of exact order 1 of the second category, with $B_i \not< 0$, $i = 1, 2$. Suppose that $v(M) < 0$. Then m^{12}, the $(1,2)$th entry of M^{-1} is negative.*

Proof: By hypothesis, $B_1^{-1} < 0$. Let $y = (w_2, 0, \ldots, 0)^t \in R^{n-1}$, where $w_2 < 0$. Then $\exists$ a $\tilde{z} \in R^{n-1}$, $\tilde{z} > 0$ such that

$$y = B_1 \tilde{z}$$

Let $z = (0, \tilde{z}^t)^t \in R^n$. We note that

$$Mz = \begin{bmatrix} w_1 \\ w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ for some } w_1.$$

In other words,

$$z = M^{-1} w. \tag{4.8}$$

From Lemma 4.1, the diagonal entries of M^{-1} are positive, and its 2 by 2 principal minors, negative. Hence $m^{12} \neq 0$. From (4.8) we have the first equation

$$m^{11} w_1 + m^{12} w_2 = 0, \tag{4.9}$$

Now suppose $m^{12} > 0$. It then follows from (4.9), that $w_1 > 0$. As B_2 is also of the second category, $B_2 \not< 0$, we have $B_2^{-1} < 0$.

Let $x = (-w_1, 0, \ldots, 0)^t \in R^{n-1}$, where w_1 is determined from (4.9). Then there exists a vector $\tilde{v} \in R^{n-1}$, $\tilde{v} > 0$ such that

$$B_2 \tilde{v} = x \tag{4.10}$$

Let $v \in R^n$ be defined by

$$v_i = \begin{cases} \tilde{v}_i, & \text{for } i \in \{1\} \cup \{3, 4, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note from (4.10) that

$$Mv = \begin{bmatrix} -w_1 \\ w_2^* \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } w_2^*.$$

This implies

$$M(v+z) = \begin{bmatrix} 0 \\ w_2^{**} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{where } w_2^{**} = w_2^* + w_2.$$

Let $u = z + v$. As $u > 0$, and $v(M) < 0$, we see that $w_2^{**} < 0$. On the other hand, the first coordinate of u, i.e.,

$$u_1 = (M^{-1}Mu)_1 = m^{12}w_2^{**} < 0$$

a contradiction. Hence $m^{12} < 0$, and this concludes the proof. ∎

The next theorem characterizes the class of inverse Z-matrices (matrices whose inverses belong to Z), within exact order two matrices.

Theorem 4.6 *Let M be a matrix of exact order 2, $M^{-1} \in Z$ if and only if M is of the second category with each $B_i \not< 0$.*

Proof: Let M be of the second category exact order 2 with each $B_i \not< 0$, $\forall\, 1 \leq i \leq n$. Then by Theorem 4.4, we have $v(M) < 0$ and the assumptions of Lemma 4.5 are satisfied; hence, it follows that $M^{-1} \in Z$.

Conversely, let M^{-1} be a Z-matrix. By Lemma 4.1, the principal minors of order r, $3 \leq r < n$, are negative. Hence for a $0 \neq z \geq 0, z \in R^n$, we have $z^t M^{-1} \leq 0$. Hence $v(M^{-1}) \leq 0$. From Lemma 1.3, it follows that $v(M) \leq 0$. Using Lemma 4.3, we conclude that $v(M) < 0$. We shall show that $B_1 \not< 0$ and B_1 is of the second category. This will complete the proof.

Clearly, $B_1 < 0$ is not possible as M^{-1} is a Z-matrix. Suppose B_1 is a matrix of exact order 1, of the first category. Then there exists a vector $\overline{y} \in R^{n-1}$, $\overline{y} > 0$ such that

$$B_1\overline{y} > 0 \tag{4.11}$$

See Lemma 3.2.

Let $y \in R^n$ be defined as $y = \begin{bmatrix} 0 \\ \overline{y} \end{bmatrix}$.

$$\text{Note that } M_{1.}y \leq 0, \text{for otherwise, } My > 0 \tag{4.12}$$

and hence $v(M) > 0$, contradictory to our earlier conclusion. We have

$$y_1 = (M^{-1}My)_1 = 0. \tag{4.13}$$

However, we also note that, $(M^{-1}My)_1 = \sum_{j=1}^{n} m^{1j}(My)_j < 0$ from (4.11), (4.12) and the fact that $m^{1j} \leq 0$, for $j \neq 1$, which contradicts (4.13). This contradiction implies that B_1 is of the second category. This concludes the proof of the theorem. ■

We now present a theorem that answers our earlier question. For notational convenience in the next theorem, we denote the exact order matrix by A and the negative of its inverse, by M. B_i, $1 \leq i \leq n$ will stand for the principal submatrix of A, leaving the ith row and the ith column.

Theorem 4.7 *Let $A \in R^{n\times n}$, be a matrix of exact order 2 of the second category. Then $-A$ is a Q-matrix.*

Proof: If $A < 0$, then $-A$ is trivially a Q-matrix. We refer to Murty [43]. Hence, we assume that A has at least one positive entry.

Let $M = -A^{-1}$. We prove that, for some principal pivot transform $\bar{M}$ of M, there exists a $q > 0$, such that $(q, \bar{M})$ satisfies Todd's conditions and hence $M \in Q$.

Note that $lcp(0, M)$ has a unique solution as no principal minor of M is zero.

Let us take $q = -I_{.1}$. Suppose $lcp(q, M)$ has a solution (w, z).

Then the solution $(\overline{w}, \overline{z})$ can be written as

$$\begin{bmatrix} -M_{LL} & 0 \\ -M_{\overline{L}L} & I_{\overline{L}} \end{bmatrix} \begin{bmatrix} \overline{z}_{L} \\ \overline{w}_{\overline{L}} \end{bmatrix} = \begin{bmatrix} q_L \\ q_{\overline{L}} \end{bmatrix} \tag{4.14}$$

for some $\phi \neq L \subseteq \{1, \ldots, n\}$. As M is nondegenerate, it is clear that $1 \in L$. Let $|L| = k$. We claim that $k = (n-2)$.

Suppose $1 \leq k \leq (n-2)$. Then

$$-M_{LL} z_L = A_{LL} z_L \leq 0, \ z_L > 0$$

implies that A_{LL} is an N-matrix of the second category. But, $A_{LL} z_L = q_L$ with $z_L > 0$ is impossible, since $A_{LL} < 0$. Hence $k > (n-2)$. Again, $k = n$ is not possible, for otherwise,

$$-M^{-1} q = A^{-1} q > 0$$

implies that $a^{11} < 0$, contradicting Lemma 4.1 that A^{-1} has diagonal entries positive.

Thus, if ever $lcp(q, M)$ has solutions, for $q = -I_{.j}$ for any $1 \leq j \leq n$, it has solutions only in the complementary cones, containing exactly $(n-1)$ columns of $-M$ as generators. Now, we proceed to show that $lcp(q, M)$ has a solution and in turn, all its solutions are nondegenerate, for a $q = -I_{.j}$, for some $1 \leq j \leq n$.

Without loss of generality, let B_n the principal submatrix of M got by deleting the last row and the last column of M, be such that $B_n \not> 0$ (this is feasible, as $A \not< 0$); and M be partitioned as

$$\begin{bmatrix} B_n & c \\ d & f \end{bmatrix}$$

where f is a scalar and d and c are the last row and column vectors of M leaving the diagonal entry, respectively.

M^{-1} in the partitioned form is given by

$$\begin{bmatrix} B_n^{-1} + B_n^{-1} c (M/B_n)^{-1} d B_n^{-1} & -c B_n^{-1} (M/B_n)^{-1} \\ -d B_n^{-1} (M/B_n)^{-1} & (M/B_n)^{-1} \end{bmatrix} \tag{4.15}$$

Let pos(C) be the complementary cone of $[I \vdots -M]$ with the first $(n-1)$ columns of $-M$. Then from the hypothesis that $-B_n^{-1} < 0$ and the fact that dB_n^{-1} must have a negative entry from the partitioned form of M^{-1}, it follows that $C^{-1}q > 0$ for some $q = -I_{.j}$, $1 \leq j \leq (n-1)$, and hence $q \in pos(C)$. Thus $lcp(q, M)$ has at least one solution for $q = -I_{.j}$ for some $1 \leq j \leq n$. Also we note that, dB_i^{-1} has no zero entry for any $1 \leq i \leq n$, from Lemma 4.2; this implies that every solution of the $lcp(q, M)$ is nondegenerate for $q = -I_{.j}$, $1 \leq j \leq n$.

Let us fix $q = -I_{.1}$. We do a principal pivot transform of M, with respect to any one of these complementary cones in which $lcp(q, M)$ has a solution. If $\bar{M} = B^{-1}\bar{B}$ is the PPT matrix, where $q \in pos(B)$, then by taking $\overline{q} = B^{-1}q$, we see that $\bar{M}$ with respect to $\overline{q}$ (along with $lcp(0, \bar{M})$ having a unique solution) satisfies the Todd's conditions and hence Lemke's algorithm, when applied to the $lcp(\overline{q}, \bar{M})$ will never end in a secondary ray. Therefore $\bar{M}$, and hence M is a Q-matrix. This concludes the proof of Theorem 4.7. ∎

Remark 4.2: In the above theorem, we have only made use of the fact that every 2 by 2 principal submatrix of M is an N-matrix of the first category. Hence we have, if $M \in R^{n \times n}$, $n \geq 2$ is such that every 2 by 2 principal submatrix of M is an N-matrix of the first category, then $M \in Q$.

Remark 4.3: A comparison matrix of M is defined as A, where

$$a_{ij} = \begin{cases} |m_{ij}| & \text{if } i = j \\ -|m_{ij}| & \text{if } i \neq j. \end{cases}$$

Let M be an N-matrix of exact order k. Then above theorem and Remark 4.2 imply that for its comparison matrix A, $-A \in Q$.

4.6 A characterization of the third category

In this section, we turn our attention to matrices of exact order 2 of the third category. Before we proceed to give a characterization theorem on their Q-

property, we present a few lemmas, for the class of matrices of exact order 2.

Lemma 4.6 *Let M be a matrix of exact order 2, with $v(M) > 0$. Suppose $B_1 \not< 0$ is a matrix of exact order 1 of the second category. then $M^{\cdot 1} > 0$, and $M^{1\cdot} > 0$.*

Proof: Clearly from Lemma 4.1, $m^{ij} \neq 0$, for all $1 \leq i, j \leq n$ and the diagonal entries of M^{-1} are positive.

Suppose $m^{12} < 0$. Taking $w = (w_2, 0, \ldots, 0) \in R^{n-1}$ for some $w_2 < 0$, there exists a $\overline{y} > 0, \overline{y} \in R^{n-1}$, such that

$$B_1\overline{y} = \begin{bmatrix} w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let $y \in R^n$ be defined as $y^t = (0; \overline{y}^t)$. Then, $M_{1.}y > 0$, for otherwise $v(M) \leq 0$. Now,

$$(M^{-1}My)_1 = 0 \text{ or}$$

$$m^{11}(M_{1.}y) + m^{12}w_2 = 0$$

However, we note that, since $m^{12} < 0, \ w_2 < 0$,

$$m^{11}(M_{1.}y) + m^{12}w_2 > 0$$

which is a contradiction. Hence $m^{12} > 0$. This completes the proof. ∎

Lemma 4.7 *Let M be a matrix of exact order 2, with $v(M) < 0$. If B_1 is of the first category, then $M^{1\cdot}$ (as well as $M^{\cdot 1}$) has*

$$m^{1j} < 0, m^{1k} > 0, for\ some\ j, k \in \{2, \ldots n\}.$$

Proof: This follows from the proof of Lemma 4.3. ∎

The next lemma characterizes, first category matrices through the sign pattern of M^{-1}.

Lemma 4.8 *Let M be a matrix of exact order 2, with $v(M) > 0$. If every row (or column) of M^{-1} has a negative entry, then $M \in Q$.*

Proof: Suppose $m^{12} < 0$. We will prove that either B_1 and B_2 are of the first category, or $B_i < 0$, for some $i \in \{1, 2\}$. Suppose $B_2 \not< 0$ is of the second category. Consider $(w_2, 0, \ldots, 0) \in R^{n-1}$, with $w_2 < 0$. Then there exists a $\overline{y} > 0$, $\overline{y} \in R^{n-1}$,

$$B_1\overline{y} = \begin{bmatrix} w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By taking $y = (0, \overline{y}^t)^t \in R^n$, as before we have

$$(M^{-1}My)_1 = 0$$

implies

$$m^{11}(M_{1.}y) + m^{12}w_2 = 0$$

which implies that $M_{1.}y < 0$ and hence $My \leq 0$, for $y > 0$.

This contradicts our assumption that $v(M) > 0$. Hence all B_is are of the first category, except possibly, for one i, $B_i < 0$. Thus M is a matrix of exact order 2, of the first category and by Theorem 4.5, $M \in Q$. ∎

The next theorem gives a characterization of exact order 2 matrices of the third category.

Theorem 4.8 *Let M be a matrix of exact order 2 of the third category, with $v(M) > 0$. Define*

$$L = \{i : B_i \not< 0, B_i \textit{ is of the second category}, 1 \leq i \leq n\}.$$

M is a Q-matrix iff the cardinality of L is even.

Proof: **If part:** Suppose the cardinality of L is even. As in the proof of Theorem 4.5, let us construct a $q \in R^n_+$, q nondegenerate with respect to M, such that $lcp(q, M)$ has an odd number of distinct solutions.

Now, with each $B_i \not< 0$, B_i a second category matrix , $i \in \{1, \ldots, n\}$, we produce a solution to the $lcp(q, M)$ which is different from the ones stated. Let $B_1 \not< 0$, be a second category matrix. Let us partition M, as

$$M = \begin{bmatrix} m_{11} & d^t \\ c & B_1 \end{bmatrix}$$

where $d, c \in R^{n-1}$; let $\overline{q} = (q_2, \ldots, q_n) \in R^{n-1}$ be the $(n-1)$-vector got from q by deleting its first coordinate. As $B_1^{-1} < 0$,

$$\overline{q} \in pos(-B_1)$$

i.e., $\exists$ an $x > 0, x \in R^{n-1}$, such that $-B_1 x = \overline{q}$; if $d^t x \leq 0$, then $M \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} d^t x \\ -\overline{q} \end{bmatrix} \leq 0$, and we have a contradiction to $v(M) > 0$. Hence $d^t x > 0$.

Let $w \in R^n, z \in R^n$ be defined as

$$w_1 = q_1 + d^t x; \qquad z_1 = 0$$
$$w_i = 0; \qquad z_i = x_i \text{ for } 2 \leq i \leq n.$$

Thus, (w, z) is a solution to the $lcp(q, M)$ and it is distinct from the earlier solutions, mentioned.

As $|L|$ is even, altogether $lcp(q, M)$ has exactly an odd number of solutions, all of them being nondegerate. Using Theorem 1.9, it follows that M is a Q-matrix.

Only if part: Let $M \in Q$. Assuming that $|L|$ is odd, we will arrive at a contradiction.

We prove for $|L| = 1$, and the proof is similar for $|L|$ being odd, $|L| > 1$. Let us assume without loss of generality, that $B_1 \not< 0$, is the only matrix of the second category in M.

Since $v(M) > 0$, using Lemma 4.6, M^{-1} can be written as

$$M^{-1} = \begin{bmatrix} + & + & \ldots & + \\ + & & & \\ \vdots & & D & \\ + & & & \end{bmatrix}$$

where $D \in R^{(n-1)\times(n-1)}$. As before, we can produce a $q \in R^n_+$, q nondegenerate with respect to M, $lcp(q, M)$ has an even number of solutions; hence $lcp(q, M)$ has an even parity for all q nondegenerate with respect to M. Since $M^{\cdot 1} > 0$, for $q = I_{.1}$, (q, M^{-1}) has a unique solution. This, alongwith $M \in Q$ implies that $D \in Q$. We refer 4.9 of Murty [43].

If D has a row (and its corresponding column) of positive entries, we repeat this argument, until we get a principal submatrix $F = M^{JJ}$, for $\phi \neq J \subset \{1, \ldots, n\}$, $|J| = k, 2 \leq k \leq n$, such that $F \in Q$ and every row of F has a negative entry.

Since $F \in Q$, $v(F) > 0$, and F^{-1} is a P-matrtix of exact order 2 of the first category, using Lemma 4.8. For $\overline{q} \in R^k_+$, $lcp(\overline{q}, F^{-1})$ has a unique solution from the proof of Theorem 4.5, which is given by $(\overline{w} = \overline{q}, .\overline{z} = 0)$.

Now, define a vector $q \in R^n$ as

$$\begin{aligned} q_i &= (-F\overline{q})_i, \quad \text{for } i \in J. \\ &= 0, \quad \text{otherwise} \end{aligned}$$

The $lcp(q, M^{-1})$ has a solution, which is given as follows:

$$\begin{aligned} w_i = 0; &\qquad z_i = \overline{q}_i \text{ for } i \in J. \\ w_i = \sum m^{ij} z_j; &\qquad z_i = 0, \text{ for } i \notin J. \end{aligned}$$

As $m^{ij} > 0$, for $i \notin J, j \in J, (w, z)$ is a solution to the $lcp(q, M^{-1})$. We claim that $lcp(q, M^{-1})$ has no other solution. Suppose $lcp(q, M^{-1})$ has a solution (w^1, z^1) which can be written as

$$\begin{bmatrix} -M^{LL} & 0 \\ -M^{\overline{L}L} & I^{\overline{L}\overline{L}} \end{bmatrix} \begin{bmatrix} {z^1}_L \\ {w^1}_L \end{bmatrix} = \begin{bmatrix} q_L \\ q_{\overline{L}} \end{bmatrix}$$

where $\phi \neq L \leq \{1, \ldots, n\}$, with $L \cap J \neq \phi$, and $L \cap \overline{J} \neq \phi$.

But q_L has a zero entry, corresponding to which M^{LL} has a row of positive entries; thus there does not exist a

$$z^1_L \geq 0 \text{ such that } -M_{LL} z^1_L = q_L.$$

Hence, (q, M^{-1}) has a unique solution, which is nondegenerate. But this contradicts Theorem 1.7 that $lcp(q, M)$ has an even parity for all q nondegenerate with respect to M. Thus $|L| \neq 1$. This completes the proof. ∎

Thus a complete characterization of exact order matrices regarding their Q-nature, can be stated as follows:

Theorem 4.9 *Let M be a matrix of exact order 2 with $v(M) > 0$. Then $M \in Q$ if and only if the cardinality of the set L is even, where*

$$L = \{i : B_i \not< 0, B_i \text{ is of the second category}, 1 \leq i \leq n\}.$$

4.7 Some more examples of exact order

Here, we present few examples, to illustrate some of the results proved in this chapter.

Example 4.3: Let

$$M = \begin{bmatrix} -5.3846 & 1.5385 & 1.5385 & 1.5385 & -20 \\ 1.5385 & -.1538 & -.6538 & -.6538 & .1 \\ 1.5385 & -.6538 & -.1538 & -.6538 & 1 \\ 1.5385 & -.6538 & -.6538 & -.1538 & 7 \\ -30 & 2 & 2 & 4 & -1 \end{bmatrix}$$

In this example of an N-matrix of exact order 2, B_1 is of the first category, while B_i, for $2 \leq i \leq 5$, is of the second category; we find that

$$M^{-1} = \begin{bmatrix} .0025 & -.1439 & -.1439 & .0418 & -.0464 \\ -.5732 & .3918 & -1.6032 & -1.4688 & -.0348 \\ -.5732 & -1.6082 & .3918 & -1.4688 & -.0348 \\ .5685 & -.4941 & -.4941 & 1.7564 & -.0626 \\ -.0951 & -.0928 & -.0928 & -.1021 & .0023 \end{bmatrix}$$

Note that $v(M) < 0$; Now since $B_i \not\leq 0$, are of the second category, for $2 \leq i \leq 5$, we find that $m^{ij} < 0$, for $i \neq j$, $2 \leq i, j \leq 5$, as anticipated by Lemma 4.5. Also, as asserted by Lemma 4.7, the first row and first column of M^{-1}, each contains a positive entry.

Now, take $D \in R^{4\times 4}$ to be the principal submatrix of M^{-1}, leaving the first row and first column. Clearly, D is a Z-matrix, and it can be verified that D^{-1}

is a P-matrix of exact order 2, of the second category, with $v(D) < 0$, as stated in Theorem 4.6.

Example 4.4: Consider the matrix

$$M^{-1} = \begin{bmatrix} 1+\varepsilon & 2 & 3 & 4 & 5 \\ 6 & 7+\varepsilon & 8 & 9.1 & 10 \\ 110 & 120 & 130+\varepsilon & 140 & 150 \\ 16 & 17.1 & 18 & 19+\varepsilon & 20 \\ 21 & 22 & 23 & 24 & 25+\varepsilon \end{bmatrix}$$

where ε is taken to be 0.0795766.

Now $M = (M^{-1})^{-1}$, can be verified to be an N-matrix of exact order 2. In M, B_1 and B_5 are of the second category, while B_2, B_3, B_4 are of the first category. As asserted by Theorem 4.8, we note that $M \in Q$, since $M^{-1} > 0$.

This example also shows that, the converse of Lemma 4.6 is not true. We notice that though $M^{\cdot 1} > 0$ and $M^{1\cdot} > 0, B_1$ is of the first category.

4.8 Exact order k, $k \geq 3$

As we go up the hierarchy in the classes of exact order matrices, the results we derive here, require more calclulations. In a similar manner as done in Section 2, one can classify the exact oder k matrices $k \geq 3$, into three different categories, based on the exact order one principal submatrices present in them:

Definition: A matrix M of exact order k, is of *the first category*, if $M \not< 0$ and every principal submatrix of order $(n-k+1)$, which is a matrix of exact order 1, is of the first category; we say that it is of *the second category*, if all order $(n-k+1)$ matrices are of the second category. M is said to be of *the third category*, if there are at least two principal submatrices of M of order $(n-k+1)$, such that one of them is of the first category and the other, of the second category.

For proving results on general exact order k in a similar manner as in the earlier sections, the size of the matrix under consideration needs to be greater than or equal to $(k+3)$. We can prove the following for the first category matrices of order $n \geq 3$:

Theorem 4.10 *Let $M \in R^{n \times n}$, be a matrix of exact order k of the first category. Then M is a Q-matrix.*

Proof: At first, we observe, when M is of the first category, that $v(M) \neq 0$. If, for $k=3$, $v(M)=0$, then the game has to be completely mixed and there exists a probability vector $x>0$ such that $Mx=0$. But this is impossible for M is nonsingular. Similarly one can prove that for any order k of the first category, the value is nonzero. We will in fact prove, inductively over k, that $v(M)>0$. We know for $k=2$ from Corollary 4.1, that $v(M)>0$. If, for $k=3$, $v(M)$ is negative, then the game is completely mixed and $M^{-1}<0$. But using the determinantal expression given in (1.7), we see that the inverse of an exact order k matrix ($k \geq 2$) has all the diagonal entries positive. By repeated use of this argument, we see that $v(M)>0$. As in the proof of Theorem 4.5, there exists a nondegenerate q with respect to M, $q>0$ for which $lcp(q,M)$ has an odd number of solutions (in particular, when M is a P-matrix of exact order k, we observe that for $q \in R^n$, $q>0$, the $lcp(q,M)$ has a unique solution). This along with the fact that $lcp(0,M)$ has a unique solution implies that M is a Q-matrix. ∎

A subclass of second category exact order matrices will not belong to the class Q. This is observed in the next theorem.

Theorem 4.11 *Let $M \in R^{n \times n}$ be a matrix of exact order k of the second category with each principal submatrix, of exact order one, having at least one positive entry. Then $v(M)<0$.*

Proof: We give the proof for $k=3$ and for $k>3$, the theorem can be proved in a similar way. Let M be written in a partitioned form, as

$$M = \begin{bmatrix} A & b & c \\ d & m_{(n-1)(n-1)} & m_{(n-1)n} \\ f & m_{n(n-1)} & m_{nn} \end{bmatrix}$$

where A is a matrix of exact order one. Doing a principal pivot transform with respect to A the resulting matrix $\overline{M}$ is given by

$$\overline{M} = \begin{bmatrix} A^{-1} & -A^{-1}b & -A^{-1}c \\ A^{-1}d & & \\ A^{-1}f & & (M/A) \end{bmatrix}$$

From Theorem 4.6, we have $-A^{-1}b < 0$ and $-A^{-1}c < 0$. Since $A^{-1} < 0$, $v(\overline{M}) < 0$ and hence the theorem follows. ∎

As the size of the matrix to be considered becomes larger, when we go up the hierarchy, studying the classes of exact order k matrices beyond $k = 2$ becomes difficult. In fact, the problems we looked at in this thesis, remain open for the general exact order k matrices.

4.9 Algorithms that *process* the exact order

In this section, we consider the question of finding an algorithm to compute a solution to the $lcp(q, M)$ when M is a matrix of exact order $0, 1$, or 2. Algorithms for some of the subclasses are already known. In this section we sum up the known results and present some results new to the literature.

It is known from Chapter 1, that Lemke's algorithm [30] will find a solution to the $lcp(q, M)$ for any $q \in R^n$ when M is a P-matrix. It is also known that a solution to the $lcp(q, M)$ can be obtained from a solution to $lcp(-M^{-1}q, M^{-1})$ computed by Lemke's algorithm when M is an N-matrix of the first category. We refer to Saigal [59]. This result also takes care of the case when M is a P-matrix of exact order 1 of the first category, since such a matrix is just the inverse of an N-matrix of the first category.

For N-matrices of exact order 1 of the first category, we have the following result.

Theorem 4.12 *Let M be a matrix of exact order 1 of the first category. Let the rth coordinate of $M_{\cdot 1}$ be positive and let $d = \lambda_1(-M_{\cdot 1}) + \sum_{j \neq 1, s} I_{\cdot j} + \mu I_{\cdot s}$ where*

$1 \leq s \leq n$, $s \neq r$, λ_1 *is a fixed number such that* $\lambda_1(-m_{r1}) + 1 < 0$ *and* μ *is sufficiently large. Then Lemke's algorithm initiated with the vector* d *in the complementary cone* $pos(B)$ *where* $B = \{-M_{.1}, I_{.j}, 2 \leq j \leq n\}$ *computes a solution to the* $lcp(q, M)$ *for any* $q \in R^n$.

Proof: Notice that the rth coordinate of d, d_r is negative by the choice of λ_1. Thus $d \notin pos(I)$. Also, since $I_{.s} \notin pos(-M)$, $s \neq r$, it follows that there is a $\mu_0 > 0$ such that for $\mu > \mu_0$, $d \notin pos(-M)$: Thus for μ sufficiently large by Theorem 3.4, (d, M) has a unique solution and the theorem follows. ∎

Remark 4.4: Using standard methodology and the above theorem, we can develop a computational scheme for computing a solution to the $lcp(q, M)$ whenever M is an N-matrix of exact order 1 of the first category..

The following result can be easily seen for P-matrices of exact order k of the first category.

Theorem 4.13 *Let* M *be a* P*-matrix of exact order* k *of the first category. Then for any* $q \in R^n$, *a solution to the* $lcp(q, M)$ *can be computed by using Lemke's algorithm initiating it with any positive vector* d.

Proof: This follows from Theorem 4.10. ∎

We now restrict our attention, to matrices of exact order 2.

Theorem 4.14 *Suppose* M *is a matrix of exact order 2 of the second category with* $B_i \not< 0$, *for* $1 \leq i \leq n$. *Then a solution to the* $lcp(q, M)$, *if one exists, can be computed by obtaining a solution to the* $lcp(-M^{-1}q, M^{-1})$.

Proof: As $v(M) < 0$ from Theorem 4.6, it follows that $M^{-1} \in Z$. There are a number of methods to solve the $lcp(-M^{-1}q, M^{-1})$ which will produce a solution if it exists, or show that there is no solution. See Chandrasekaran [4], Mohan [35] and Ramamurthy [56]. ∎

Theorem 4.15 *Let* M *be an* N*-matrix of exact order 2 of the second category. Then, Lemke's algorithm processes the* $lcp(q, -M)$.

Proof: This follows from the proof of Theorem 4.7. ∎

When M is of exact order 2 of the second category, we proved in Theorem 4.7 that $-M$ is a Q-matrix. For this class of matrices, we can notice that for each $q \in R^n$, q nondegenerate with respect to M, the $lcp(q, -M)$ has more than one solution; hence, Theorem 4.15 asserts the fact that Saigal's result and Todd's condition for proving the Q-nature of a matrix are improvements over the earlier known results.

Chapter 5

GLOBAL UNIVALENCE OF MAPS WITH EXACT ORDER JACOBIANS

In this chapter, we present as an application, a univalence result for C^1-differentiable functions when the Jacobians are exact order 2 matrices. This result is well-known for exact order 0 (Jacobian) matrices, due to Gale-Nikaido [18] and Inada [21], and for exact order 1 (Jacobian) matrices, due to Olech et al., [46] and [47]. Such results are quite useful in mathematical economics; see for instance Inada [21].

5.1 Gale-Nikaido result for the first category exact order k

We prove the following, for a C^1-differentiable map with the Jacobian matrix $J(x)$ being an exact order k matrix for every x in the domain:

Theorem 5.1 *Let $F : \Omega \subset R^n \to R^n$, be a C^1-differentiable map where Ω is a rectangular region. Suppose the Jacobian $J(x)$ of F, $J = ((f_{ij}))$, is a matrix of* *exact order k of the first category for every $x \in \Omega$. Then, for any a and x in Ω, the inequalities $F(x) \leq F(a)$ and $x \geq a$ have only one solution $x = a$.*

Proof: Though this result follows immediately from Theorem 1 of [47], we present here a proof by induction over n. Clearly, this theorem is true for $n = 3$ from [51] and [57].

Let

$$X = \{x : x \in \Omega \text{ and } F(x) \leq F(a),\ x \geq a\}.$$

By definition of X, $a \in X$. The proof is complete if we show that X contains only a. So we at first we observe that a is an isolated point of X. As F is differentiable, we have

$$\lim_{x \to a} \frac{1}{||x-a||} ||F(x) - F(a) - J(a)(x-a)|| = 0.$$

Since $J(a)$ is a matrix of exact order k of the first category, there is a positive number $\delta > 0$, such that for any $x \geq a$, some coordinate of $J(a)\frac{x-a}{||x-a||} \geq \delta > 0$. Consequently, in any neighbourhood of a, some component of $F(x) - F(a)$ is positive for $x \geq a$ in Ω. This shows that a is an isolated point of X. Suppose $b \in X$ with $b \neq a$. Clearly $b \geq a$. Define $Y \subset X$ as follows.

$$Y = \{x : a \leq x \leq b \text{ and } F(x) \leq F(a)\}.$$

It can be seen that Y is compact and since a is an isolated point, $Y - \{a\}$ is compact. Let $\overline{x}$ be a smallest element of $Y - \{a\}$ in the sense that no other element y of $Y - \{a\}$ satisfies $y \leq \overline{x}$. As $\overline{x} \in Y - \{a\}$, only two possibilities can occur, either $\overline{x} > a$ or $\overline{x} \leq a$.

Case (i): $\overline{x} > a$. Because $v(J(x)) = v(J(x)^t) > 0$, this ensures the existence of a vector u, satisfying $u < 0,\ J(\overline{x})u < 0$. Define $x(t) = \overline{x} + tu$. As $u < 0,\ \overline{x} > a$ and Ω is a rectangular region, for sufficiently small t, $x(t) \in \Omega$. Moreover, by differentiability,

$$\frac{F(x(t)) - F(\overline{x})}{t||u||} - J(\overline{x})\frac{u}{||u||}$$

can be made as small as possible by letting t approach 0. Since$J(\overline{x})u < 0$, it follows for small positive t, $F(x(t)) < F(\overline{x}) \leq F(a)$ and consequently $x(t) \in$

$Y - \{a\}$. But $x(t) < \overline{x}$ and this contradicts the minimality of $\overline{x}$. Thus case (i) cannot arise.

Case (ii): $\overline{x} \leq a$. Now we apply induction. We will assume without loss of generality, that $\overline{x}_1 = a_1$ where $\overline{x}_1$ and a_1 are the first coordinates of $\overline{x}$ and a respectively. We now define a new differentiable mapping $G : \hat{\Omega} \to R^{n-1}$ where

$$\hat{\Omega} = \{(x_2, x_3, \ldots, x_n) : (a_1, x_2, \ldots, x_n) \in \Omega\}$$

and

$$g_i(x_2 \ldots, x_n) = f_i(x_1, x_2, \ldots, x_n),\ i = 2, \ldots, n.$$

Jacobian matrix of G is plainly a principal submatrix of $J(x)$ and hence it is a matrix of exact order $(k-1)$. Further,

$$\begin{aligned} g_i(\overline{x}_2 \ldots, \overline{x}_n) &\leq g_i(a_2 \ldots, a_n) \\ \overline{x}_i &\geq a_i, \text{for } i = 2, 3 \ldots, n. \end{aligned} \tag{5.1}$$

In order to complete the induction, we need to prove that $J_G(\overline{x})$ is also of the first category. Define $\hat{J}(x_2, \ldots, x_n)$ = principal minor of $J(x)$ got by omitting the first column and the first row where $x = (a_1, x_2, \ldots, x_n)$. Clearly, $\hat{J}$ is a matrix of exact order $(k-1)$. If $J(x)$ is a P-matrix of the exact order k of the first category, then so is $\hat{J}$ and we are done. When $J(x)$ is an N-matrix of the first category, it is possible that $\hat{J}$ is nonpositive. Due to the fact that all the partial derivatives are assumed to be continuous throughout Ω, if $f_{1j}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) > 0$, for $j = 2, \ldots, n$, then $f_{1j} > 0$ for every $x \in \Omega$. But since $\overline{x}_1 = a_1$, $\overline{x}_i \geq a_i$, $i = 2, \ldots, n$, with strict inequality for at least one $i \geq 2$, we have $f_1(x) > f_1(a)$ contradicting $F(x) \leq F(a)$. Therefore $f_{1j_0}(\overline{x}_1, \ldots, \overline{x}_n) \leq 0$ for some j_0. Define a vector $v = e_{j_0}$. This vector would be a nontrivial solution to $J(\overline{x})v \leq 0$ which contradicts the fact that J is a matrix of exact order k of the first category.

Therefore, by induction hypothesis, in (5.1) we must have $\overline{x}_i = a_i$ for $i = 2, \ldots, n$. Hence we have $\overline{x} = a$, and this terminates the proof of the theorem. ∎

5.2 Univalence theorem for exact order 2

We require KKM Theorem (Knaster, Kuratowski and Mazurkiewicz), to prove our main result. We state a new version of this here, without proof. For more details on this we refer to [47].

Let S be a closed simplex in R^n with vertices $s_1, s_2, \ldots, s_n$; that is

$$S = \{x : x = \mu_1 s_1 + \mu_2 s_2 + \ldots \mu_n s_n,\ \mu_1 + \mu_2 + \ldots + \mu_n = 1,\ \mu_i \geq 0,\ \forall i\}$$

and denote by F_i the face of S opposite to s_i; that is F_i is the simplex with vertices $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$. We quote the following result, from [47]:

Theorem 5.2 *If $A_1, \ldots, A_n$ are n closed sets such that*

$$\begin{aligned} S &= A_1 \cup A_2 \cup \ldots \cup A_n, \\ (\cap_{j\in I} A_j) \cap (\cap_{j \in \overline{I}} F_j) &= \phi \ \forall I \subset \{1, \ldots, n\}, \end{aligned}$$

where $\overline{I} = \{1, \ldots, n\} \backslash I$, then $A_1 \cap \ldots \cap A_n$ is not empty.

Using the above stated theorem, Olech, Parthasarathy and Ravindran ([46] and [47]) proved that a C^1-differentiable map from R^n to R^n with Jacobian $J(x)$ being a matrix of exact order one for each x is univalent.

We are now ready to state a univalence theorem, which is proved in similar lines.

Theorem 5.3 *Let $F : R^n \to R^n$ be a C^1-differentiable function. Write $F = (f_1, f_2, \ldots, f_n)$ where each $f_i : R^n \to R^1$. Suppose J_F, the Jacobian of F, is an N-matrix of exact order 2 for every $x \in R^n$. Then F is globally one to one and consequently is a homeomorphism of R^n into R^n.*

Remark 5.1: Theorem 5.3 remains true when J_F is a P-matrix of exact order 2 also, for every $x \in R^n$.

Remark 5.2: For $n = 3$, Ravindran [57] has proved Theorem 5.3 and for $n = 4$ a similar proof can be given. So we shall give a proof of Theorem 5.3 when $n \geq 5$.

Proof(of Theorem 5.3): Suppose $F(b) = F(a)$, $b \neq a$. We will assume without loss of generality, that $b > a$ because if $b_i = a_i$ for some i, using induction, we can conclude that $b_i = a_i$ for all i. So we let $b > a, F(b) = F(a)$. Now the following four cases are possible.

(i) $v(J) > 0$ and $v(J_i) > 0$ for every i, for every x

(ii) $v(J) < 0$ and $v(J_i) < 0$ for every i, for every x

(iii) $v(J) > 0$ and $v(J_i) < 0$ for $i = 1, \ldots, k$, and $v(J_i) > 0$ for $i = k+1, \ldots, n$, for every x

(iv) $v(J) < 0$ and $v(J_i) < 0$ for $i = 1, \ldots, k$, and $v(J_i) > 0$ for $i = k+1, \ldots, n$, for every x

where $v(J)$ = value of the matrix game J, $v(J_i)$ = value of the submatrix obtained from J by omitting the i^{th} row and the i^{th} column.

If case (i) occurs, the proof follows from Theorem 5.1 and Gale-Nikaido [18].

If case (ii) occurs we proceed as follows: Since $v(J) < 0$, there is a negative u such that the solution $x(t, u, a)$ of Wazewski equation [72],

$$x' = F'(x)^{-1}u, \ x(0, u, a) = a$$

is positive for small t; that is it enters the interior of the cube C

$$C = \{x : a \leq x \leq b\}.$$

This solution can be extended to the maximal interval of existence and it has to leave the cube C. In other words, there exists a point $x(s, u, a) \in \delta C$ = the boundary of C with $x_k(s, u, a) = b_k$ and

$$F(x(s, u, a) - F(a)) < 0.$$

This is possible as C is compact and $v(J) < 0$ throughout C. As the above inequality is strict coordinatewise, we can without loss of generality, assume that $x_k = b_k$ and $x_i < b_i$. From our assumption $F(b) = F(a)$, we have $f_i(x) < f_i(b)$ for every $i = 1, \ldots, n$.

Let $G = (f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n)$. We have $G(x) < G(b)$ where $x_k = b_k$ and $a_i \leq x_i < b_i$. As $v(J_k) < 0$, two subcases arise, viz.,

(α) $J_k < 0$, or

(β) J_k is a second category exact order 1 matrix with a positive entry.

If (α) occurs, $G(x) < G(b)$ is impossible. If (β) occurs, we proceed as follows. Without loss of generality assume that $k = n$.

Let $c, d \in R^{n-1}$ be such that $c_i = x_i$ and $d_i = b_i$ for all $i = 1, \ldots, n-1$. We consider the following Wazewski's equation,

$$y' = G'(y)^{-1}z, \quad y(0) = d, v \in S \tag{5.2}$$

where S is the space of probability vectors of dimension $(n-2)$. We denote by $y(t, z)$ the solution of (5.2). It exists and has the property that

$$G(y(t, z)) = G(d) + tz. \tag{5.3}$$

In fact, the derivative of the left-hand side of (5.3) is a constant and equal to $G(d) + z$. Since G is a local diffeomorphism (as F is assumed to be so), the above equation defines $y(t, z)$ uniquely. Clearly, it follows that $y(t, z)$ is continuous in z. Since $G'(x)^{-1} < 0$, $y(t, z)$ is decreasing in t for each fixed z. Therefore, there exists a $t(z)$ such that $c < y(t, z) < d$, $0 < t < t(z)$ and $\exists$ an i such that $x_i(t(z), z) = c_i$. Define

$$A_i = \{z : c_i = y_i(t(z), z)\}.$$

Since A_i is closed, we have $\cup A_i = S$. We shall now verify the conditions of Theorem 5.2; we need only to check that $\cap_{i \in I} A_i \cap \cap_{i \in \bar{I}} F_i = \phi$ for every proper subset I of $\{1, \ldots, n-1\}$. In order to do this, let us fix $I \subset \{1, \ldots, n-1\}$. Suppose $\cap_{i \in I} A_i \cap \cap_{i \in \bar{I}} F_i \neq \phi$ and let z be an element of it. Then

$$y(t(z), z) = \Sigma_{i \in \bar{I}} y_i(t(z), z) e_i + \Sigma_{i \in I} c_i e_i$$

and $z = \Sigma_{i \in I} z_i e_i$. From equation (5.3) and above we have

$$\Sigma_{i \in \bar{I}} G_i(y(t(z), z)) e_i = 0. \tag{5.4}$$

Consider the map $H(y) = \Sigma_{i \in \bar{I}} G_i(\Sigma_{i \in \bar{I}} y_i e_i + \Sigma_{i \in \bar{I}} c_i e_i) e_i$, from a proper subspace of R^{n-1} to itself. The Jacobian matrix $J_H(y)$ is an N-matrix since it is a proper principal submatrix of $J_G(y)$ and the latter is an exact order one matrix. Thus from Inada's result, H is univalent and $H(q) = H(c)$ only if $y_i = c_i$ for each $i \in \bar{I}$. Thus (5.4) implies that $y(t(z), z) = c$. Because of (5.4) this is possible only if $t(z) = 0$ or $c = d$. Since $c < d$, $\cap_{i \in I} A_i \cap \cap_{i \in \bar{I}} F_i = \phi$ for every proper subset I of $\{1, \ldots, n-1\}$. Thus $\cap_{i=1}^{n-1} A_i \neq \phi$ by Theorem 5.2. Hence there exists a $z \in A_i$, $\forall i$, which implies that $y(t(z), z) = c$ and by (5.4) it follows that $t(z) = 0$ and $c = d$. This contradicts the assumption that $c_i = x_i < b_i = d_i$ for all $i = 1 \ldots, n-1$. This concludes the argument for case (ii).

Suppose case (iii) holds. That is, $v(J) > 0$, $v(J_i) < 0$ for $i = 1, \ldots, k$, and $v(J_i) > 0$ for every $i > k$. From Lemma 4.6 it follows that $J^{-1}(= F'(x)^{-1})$ will have the first k columns and the first k rows filled up with positive entries, for every x, as F is C^1-differentiable function.

Write $G = -F$ and we have

$$G(b) = G(a),\ b > a, \text{as } F(b) = F(a).$$

We will also assume $G(b) = G(a) = 0, b > a$ as it entails no loss of generality. As before, consider Wazewski's equation for every $v \in \mathbf{S_k}$, where $\mathbf{S_k} = \{v = (v_1, \ldots, v_k, 0, \ldots, 0),\ v_i \geq 0 \text{ and } \sum^k{}_1 v_i = 1\}$,

$$x' = G'(x)^{-1} v,\ x(0) = b,\ v \in \mathbf{S_k}.$$

Then the solution $x(t, v)$ exists and it has the property that

$$G(x(t, v)) = G(b) + tv = tv. \tag{5.5}$$

Since G is a local diffeomorphism, it follows that the solution $x(t, v)$ is unique and $x(t, v)$ is continuous in v. Observe that $x(t, v)$ is decreasing in t for each fixed $v \in \mathbf{S_k}$. Thus it follows that there is a $t(v)$ such that $a < x(t, v) < b$ for $0 < t < t(v)$ and there is an i with $x_i(t, v) = a_i$. Such a $t(v)$ is uniquely defined and continuous. Define

$$A_i = \{v : x_i(t(v), v) = a_i\} \text{ for } i = 1, \ldots, n. \tag{5.6}$$

Then each A_i is closed and $\cup_i A_i = \mathbf{S_k}$. If $A_{i_0} \neq \phi$, for some $i_0 \geq k+1$, we have from (5.6)

$$f_i(x(t(v), v) \leq 0 = f_i(a),\ \text{for every } i \neq i_0$$

where

$$x_{i_0}(t(v), v) = a_{i_0} \text{ and } x_i \geq a_i \text{ for } i \neq i_0.$$

Since $v(J_{i_0}) > 0$, from Gale-Nikaido [18], we have $x_i = a_i$ for every i and consequently $t(v) = 0$ or $a = b$, leading to a contradiction. So assume that $A_i = \phi$ for $i = k+1, \ldots, n$. In other words, $\cup_{i=1}^{k} = \mathrm{S_k}$. Now one can proceed as in [p 122, 46] and arrive at a contradiction.

If (iv) holds, then a proof can be given along the same lines as indicated in the other cases to arrive at a contradiction and we omit the details. This terminates the proof of Theorem 5.3. ∎

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