The Fourier Transforms Of Very Rapidly Decreasing Functions On Certain Lie Groups

M. Sundari

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Chapter 0

Introduction

0.1 Introduction

Recall that for a function $f \in L^1(\mathbf{R}^n)$, its Fourier transform $\hat{f}$ is defined by :

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{i\langle \xi, x\rangle} dx \qquad (0.1.1)$$

where $\langle .,. \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and dx the Lebesgue measure on $\mathbf{R}^n$. A celebrated theorem of L. Schwartz asserts that a function f on $\mathbf{R}$ is 'rapidly decreasing' (or in the 'Schwartz class') if and only if its Fourier transform is 'rapidly decreasing'. In sharp contrast to Schwartz's theorem, is a result due to Hardy ([18]) which says that f and $\hat{f}$ cannot both be "very rapidly decreasing". More precisely, if $| f(x) | \leq Ae^{-\alpha x^2}$ and $| \hat{f}(\xi) | \leq Be^{-\beta\xi^2}$ for some positive constants

α, β and $\alpha\beta > \frac{1}{4}$, then $f \equiv 0$. Hardy's theorem can also be viewed as a sort of uncertainty principle. Roughly speaking, various uncertainty principles, including the celebrated Heisenberg uncertainty principle, say that a non-trivial function and its Fourier transform cannot be simultaneously 'concentrated'. Depending on the definition of 'concentration', we get a host of uncertainty principles (-see [2], [3], [4], [7], [9], [24], [25], [28], [29], [30], [33], [38], [40] etc). Clearly, Hardy's theorem belongs to this class of results where 'concentration' is measured in terms of rate of decay of f and $\hat{f}$ at infinity. Some of the uncertainty principles seem to be valid even in very abstract situations. For instance, in [5], M. F. E. De Jeu has shown that the uncertainty principle due to Donoho and Stark ([7]) is valid whenever one has an integral operator for which a "Plancherel theorem" holds. For an account of uncertainty principles and their connections with physics etc see [8] or [34].

Since the theorem of Schwartz is of fundamental importance in harmonic analysis, there is a whole body of literature (-see for instance [35], p.151 and [43]-) devoted to generalizing this result to other Lie groups. However, as far as we are aware, until very recently no systematic attempt was made to generalize Hardy's theorem in the context of harmonic analysis on Lie groups. In this thesis, we shall give generalizations of Hardy's theorem to the Heisenberg group, the n-dimensional Euclidean motion group and a sub class of noncompact semi-simple Lie groups.

Let G be a locally compact, unimodular group satisfying the second axiom of countability. Moreover, assume that G is postiliminaire. (For the precise definitions, the reader may refer to [6], pp.303 and [23], pp.196.) Let dm_G denote the Haar measure on G. Let $\widehat{G}$ be its unitary dual, i.e. the set of equivalence classes of continuous, irreducible, unitary representations of G. Given $f \in L^1(G)$, we define the group Fourier transform $\hat{f}$ of f by :

$$\hat{f}(\pi) = \pi(f) = \int_G f(x)\pi(x)dm_G(x), \qquad \pi \in \widehat{G}. \tag{0.1.2}$$

(For $\pi \in \widehat{G}$, let $\mathcal{H}_\pi$ be the underlying Hilbert space on which G acts. The above integral is to be interpreted suitably as an element of $\mathcal{B}(\mathcal{H}_\pi)$, the collection of bounded linear operators on $\mathcal{H}_\pi$.) Then, by the abstract Plancherel theorem, there exists a measure structure and a unique positive measure μ on $\widehat{G}$ such that for $f \in L^1(G) \cap L^2(G)$,

$$\int_G | f(x) |^2 \, dm_G(x) = \int_{\widehat{G}} tr(\pi(f)^*\pi(f))d\mu(\pi). \tag{0.1.3}$$

Implicit in the above is the fact that for $f \in L^1(G) \cap L^2(G)$, $\pi(f)$ is of Hilbert-Schmidt class for μ-almost all π, $\pi \in \widehat{G}$ (-see [44] for details). The main goals of basic harmonic analysis on locally compact groups are the following :

(a) To describe $\widehat{G}$ as explicitly as possible;

(b) to give an explicit formula for the Plancherel measure μ and

(c) to investigate the relationship between the behaviour of the function f and that of the group Fourier transform $\hat{f}$.

In the case of many locally compact groups (abelian or non abelian) we have an explicit description of the unitary dual and the Plancherel measure μ. So we can ask ourselves whether an analogue of Hardy's theorem holds in this set-up :

Suppose $f \in L^1(G)$ is such that both f and $\hat{f}$ are "very rapidly decreasing". Then is $f = 0$ a.e.?

We have been able to answer this question for a wide class of Lie groups. In Chapter 1, we take up the case of the Heisenberg group, in Chapter 2, the motion groups, and in Chapter 3, noncompact semi-simple Lie groups and symmetric spaces of the noncompact type. Ofcourse, in each case, the meaning of "very rapid decay" has to be made precise.

Before ending this section, we give below the precise statement of Hardy's theorem :

Theorem 0.1.1 (*Hardy [18]*) *Suppose f is a measurable function on $\boldsymbol{R}$ such that*

$$| f(x) | \leq Ce^{-\alpha x^2}, | \hat{f}(\xi) | \leq Ce^{-\beta\xi^2}, \; x, \; \xi \in \boldsymbol{R} \qquad (0.1.4)$$

where α, β are positive constants. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$ there are infinitely many linearly independent functions satisfying (0.1.4) *and if $\alpha\beta = \frac{1}{4}$ then $f(x) = Ce^{-\alpha x^2}$.*

A proof of this theorem is also found in [8], pp.156-158. In [3], Cowling and

Price proved an "$L^p - L^q$" version of Hardy's theorem. The theorem of Beurling in [25] is similar in spirit to Hardy's theorem, although far more general, and indeed Hardy's theorem as well as the result of Cowling and Price can be deduced from it as special cases.

0.2 Hardy's theorem for $\boldsymbol{R}^n$

Hardy's theorem continues to be valid for $\boldsymbol{R}^n$, $n > 1$, and although this fact is probably well known to many experts, we have been unable to find a reference in the literature for the higher dimensional case. In this section, we present a proof of Hardy's theorem for $\boldsymbol{R}^n$, $n > 1$ as an application of the Radon transform. The proof proceeds by reducing the n-dimensional case to the one dimensional case, using the Radon transform. We continue to denote the standard inner product on $\boldsymbol{R}^n$ by $\langle \cdot, \cdot \rangle$ and dx is the Lebesgue measure on $\boldsymbol{R}^n$, $n \geq 1$. Denote the norm on $\boldsymbol{R}^n$ by $\| \cdot \|$. The following is an analogue of Hardy's theorem for $\boldsymbol{R}^n$, $n > 1$ ([32], section 2) :

Theorem 0.2.1 *Let f be a measurable function on $\boldsymbol{R}^n$ and α, β two positive constants. Further assume that*

$$| f(x) | \leq C\, e^{-\alpha \|x\|^2}, | \hat{f}(\xi) | \leq C\, e^{-\beta \|\xi\|^2},\ x,\ \xi \in \boldsymbol{R}^n \qquad (0.2.1)$$

If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$, there are infinitely many linearly independent solutions for (0.2.1) and if $\alpha\beta = \frac{1}{4}$, f is a constant multiple of $e^{-\alpha\|x\|^2}$.

Proof : In view of Theorem 0.1.1, assume that $n \geq 2$. Recall that the Radon transform Rg of an integrable function g on $\boldsymbol{R}^n$ is a function of two variables (ω, s) where $\omega \in S^{n-1}$ and $s \in \boldsymbol{R}$ and is given by

$$Rg(\omega, s) = \int_{\langle x,\omega\rangle = s} g(x)dx, \tag{0.2.2}$$

where dx is the Euclidean measure on the hyperplane $\langle x, \omega\rangle = s$. Actually, for each fixed ω, the above makes sense for almost all $s \in \boldsymbol{R}$ which may depend on ω. However for functions with sufficiently rapid decay at infinity it makes sense for all s. For details about the Radon transform we refer the reader to [11] and [21].

From equation (0.1.1), the Fourier transform of a function f on $\boldsymbol{R}^n$ is given by,

$$\hat{f}(\xi) = \int_{\boldsymbol{R}^n} f(x)e^{i\langle x,\xi\rangle}dx. \tag{0.2.3}$$

Here dx denotes the Lebesgue measure on $\boldsymbol{R}^n$. Then it can be easily seen that

$$\hat{f}(s\omega) = \widehat{Rf}(\omega, s) \tag{0.2.4}$$

where $s \in \boldsymbol{R}$, $\omega \in S^{n-1}$ and $\widehat{Rf}$ stands for the Fourier transform of Rf in the s-variable alone. From the definition of the Radon transform Rf and relation (0.2.4), the conditions on f and $\hat{f}$ translate into conditions on Rf and $\widehat{Rf}$. For each fixed ω, we therefore get

$$| Rf(\omega, r) | \leq C\, e^{-\alpha r^2},\ r \in \boldsymbol{R} \qquad (\ 0.2.5\ )$$

$$| \widehat{Rf}(\omega, s) | \leq C\, e^{-\beta s^2},\ s \in \boldsymbol{R} \qquad (\ 0.2.6\ )$$

By appealing to Hardy's theorem for $\boldsymbol{R}$ we conclude that, for $\alpha\beta > \frac{1}{4}$, $Rf(\omega, .) = 0$ for almost all ω. Since $f \mapsto Rf$ is one-to-one we conclude that $f = 0$ a.e.

When $\alpha\beta = \frac{1}{4}$, $\widehat{Rf}(\omega, s) = \hat{f}(s\omega) = A(\omega)e^{-\alpha s^2}$, where A is a measurable function on the unit sphere S^{n-1}. Because $f \in L^1(\boldsymbol{R}^n)$, $\hat{f}$ is continuous at zero and by taking $s \to 0$ we obtain $A(\omega) = \hat{f}(0)$. Hence $\hat{f}(\xi) = \hat{f}(0)e^{-\beta\|\xi\|^2}$ so that $f(x) = C\, e^{-\alpha\|x\|^2}$ for some constant C.

For each multi-index $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, let Φ_μ denote the corresponding Hermite function on $\boldsymbol{R}^n$. (For the definition and properties of Hermite functions, we refer the reader to [41]). Finally, if $\alpha\beta < \frac{1}{4}$, suitably scaled Hermite functions Φ_μ satisfy the conditions (0.2.1).

Chapter 1

The Heisenberg group, H^n

In this chapter, we first describe the unitary dual of the Heisenberg group H^n, and then prove an analogue of Hardy's theorem for functions on H^n. We also show that though the exact analogue fails for the reduced Heisenberg group, a slightly modified version continues to hold.

1.1 The Heisenberg group

For the material covered in this section and the next, the reader may refer to [12] and [41]. We follow closely the notation of the latter.

Recall that as a set the Heisenberg group H^n is just $\mathbb{C}^n \times \mathbb{R}$. Multiplication on H^n is given by

$$(z,t)(w,s) = (z + w,\ t + s + \frac{1}{2}\ Im\ \ z \cdot \bar{w}) \qquad (\ 1.1.1\ )$$

where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$ and $z \cdot \bar{w}$ denotes the usual inner product on $\mathbb{C}^n$. With this multiplication, H^n is a two step, simply connected nilpotent Lie group with Haar measure $dzdt$. In order to define the group Fourier transform we need to recall some facts about the representations of the Heisenberg group. For each λ in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, there is an irreducible unitary representation π_λ of H^n realised on $L^2(\mathbb{R}^n)$ and is given by

$$(\pi_\lambda(z,t)\phi)(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)} \phi(\xi + y), \tag{1.1.2}$$

where $z = x + iy$, $\phi \in L^2(\mathbb{R}^n)$ and '$\cdot$' denotes the usual inner product on $\mathbb{R}^n$. A theorem of Stone - von Neumann says that all the infinite dimensional irreducible unitary representations of H^n upto unitary equivalance are given by $\pi_\lambda, \lambda \in \mathbb{R}^*$. The Plancherel measure μ is supported on $\mathbb{R}^*$ and is given by $d\mu(\lambda) = |\lambda|^n d\lambda$. (There is another family of one-dimensional representations of H^n which does not play a role in the Plancherel theorem.)

Given a function f in $L^1(H^n)$, its group Fourier transform $\hat{f}$ is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dzdt \tag{1.1.3}$$

The above integral is to be interpreted suitably and for each $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$. As shown in [12] and [41], $\hat{f}(\lambda)$ is an integral operator with kernel K_f^λ given by

$$K_f^\lambda(\xi,\eta) = \mathcal{F}_{13}f\,(\frac{1}{2}(\lambda(\xi+\eta)),\ \xi-\eta,\ \lambda), \qquad (1.1.4)$$

where we have written $f(z,t) = f(x,y,t)$, and $\mathcal{F}_{13}f$ stands for the Euclidean Fourier transform of f in the first and the third set of variables. For f in $L^1(H^n) \cap L^2(H^n)$ a simple calculation shows that

$$\|\hat{f}(\lambda)\|_{HS}^2 = C\ \mid \lambda \mid^{-n} \int_{\mathbb{C}^n} \mid \mathcal{F}_3 f(z,\lambda) \mid^2 dz \qquad (1.1.5)$$

(for a suitable constant C), where $\|.\|_{HS}$ is the Hilbert - Schmidt norm. (For more details about integral operators the reader may refer to [17].) From this and the Euclidean Plancherel theorem, the Plancherel theorem for the Heisenberg group follows :

$$\|f\|_2^2 = \ C_n \int_{\mathbf{R}^*} \|\hat{f}(\lambda)\|_{HS}^2 d\mu(\lambda), \qquad (1.1.6)$$

where $d\mu(\lambda)\ = \mid \lambda \mid^n d\lambda$ and C_n is a constant depending only on the dimension.

We now state and prove the following analogue of Hardy's theorem for H^n ([32], section 2).

Theorem 1.1.1 *Suppose f is a measurable function on H^n satisfying the estimates*

$$\mid f(z,t) \mid \leq \ g(z)e^{-\alpha t^2},\ \ z \in \mathbb{C}^n,\ \ t \in \mathbf{R}, \qquad (1.1.7)$$

$$\|\hat{f}(\lambda)\|_{HS} \leq Ce^{-\beta|\lambda|^2},\ \ \lambda \in \mathbf{R}^*, \qquad (1.1.8)$$

where $g \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$ *and* α, β *are positive constants. Then, if* $\alpha\beta > \frac{1}{4}$, $f = 0$ *a.e.; if* $\alpha\beta < \frac{1}{4}$ *there are infinitely many linearly independent functions satisfying the above estimates.*

Proof : For a function f on H^n, define $\tilde{f}$ to be the function $\tilde{f}(z,t) = \overline{f(z,-t)}$ and let $f \star_3 \tilde{f}$ stand for the convolution of f and $\tilde{f}$ in the t - variable. Then, a simple calculation shows that

$$\int_{H^n} (f \star_3 \tilde{f})(z,t) e^{i\lambda t} dz dt = \int_{\mathbb{C}^n} \mathcal{F}_3 (f \star_3 \tilde{f})(z,\lambda) dz = \int_{\mathbb{C}^n} | \mathcal{F}_3 f(z,\lambda) |^2 \, dz, \quad (1.1.9)$$

which, in view of (1.1.5), equals $C \mid \lambda \mid^n \|\hat{f}(\lambda)\|_{HS}^2$ for some constant C. Define a function h on $\boldsymbol{R}$ by

$$h(t) = \int_{\mathbb{C}^n} (f \star_3 \tilde{f})(z,t) dz. \quad (1.1.10)$$

Then one has from (1.1.5) and (1.1.9)

$$\mathcal{F}h(\lambda) = C \mid \lambda \mid^n \|\hat{f}(\lambda)\|_{HS}^2, \quad (1.1.11)$$

where '$\mathcal{F}h$' denotes the Euclidean Fourier transform of h. Now the conditions (1.1.7) and (1.1.8) on f and $\hat{f}$ translate into the following conditions on h and its Euclidean Fourier transform $\mathcal{F}h$:

$$| h(t) | \leq C e^{-\frac{\alpha}{2} t^2}, | \mathcal{F}h(\lambda) | \leq C \mid \lambda \mid^n e^{-2\beta|\lambda|^2}, \; t \in \boldsymbol{R}, \; \lambda \in \boldsymbol{R}^*. \quad (1.1.12)$$

Now it is easy to see that, for any $\epsilon > 0$, $\mid \lambda \mid^n e^{-2\beta|\lambda|^2} \leq C' e^{-2(\beta-\epsilon)|\lambda|^2}$, $\lambda \in \boldsymbol{R}^*$, for some constant C' depending on ϵ. So we can choose a β' such that $\alpha\beta' > \frac{1}{4}$ or $< \frac{1}{4}$

according as $\alpha\beta > \frac{1}{4}$ or $< \frac{1}{4}$ and the following estimate is satisfied by $\mathcal{F}h$:

$$| \mathcal{F}h(\lambda) | \leq C' e^{-2\beta'|\lambda|^2}, \quad \lambda \in \mathbb{R}^*. \tag{1.1.13}$$

Thus, if $\alpha\beta > \frac{1}{4}$, then $\alpha\beta' > \frac{1}{4}$, and hence Hardy's theorem for $\mathbb{R}$ together with (1.1.12), (1.1.13) implies that $h = 0$ a.e. This means $\|\hat{f}(\lambda)\|_{HS}^2 = 0$ for all $\lambda \in \mathbb{R}^*$ and consequently $f = 0$ a.e. by the Plancherel theorem for H^n. If $\alpha\beta < \frac{1}{4}$, then any function of the form $g(z)h_k(t)$, where h_k is a suitably scaled Hermite function on $\mathbb{R}$, satisfies the hypothesis of the theorem.

By abuse of notation, we denote the norm on $\mathbb{C}^n$ also by $\| \cdot \|$.

The following is the exact analogue of Hardy's theorem for H^n, which follows immediately from the above theorem.

Corollary 1.1.1 *Suppose f is a measurable L^1 - function on H^n and*

$$| f(z,t) | \leq C \, e^{-\alpha(\|z\|^2+|t|^2)}, \; z \in \mathbb{C}^n, \; t \in \mathbb{R} \tag{1.1.14}$$

$$\|\hat{f}(\lambda)\|_{HS} \leq C \, e^{-\beta|\lambda|^2}, \; \lambda \in \mathbb{R}^* \tag{1.1.15}$$

for some positive constants α and β. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e. If $\alpha\beta < \frac{1}{4}$, then there are infinitely many such linearly independent functions.

1.2 The reduced Heisenberg group

In this section, we prove an analogue of Hardy's theorem for the reduced Heisenberg group. By and large we continue to use the notation introduced in section 1.1, but with slight modifications.

As a set, the reduced Heisenberg group H^n_{red} is just $\mathbb{C}^n \times S^1$. The multiplication law on H^n_{red} is defined as in (1.1.1) except for the understanding that t is a real number modulo 2π. The reduced Heisenberg group H^n_{red} is also a two step nilpotent Lie group with Haar measure $dzdt$, where dt denotes the normalized Lebesgue measure on S^1. For each $m \in Z^* = Z\setminus\{0\}$, there is an irreducible unitary representation π_m of H^n_{red}, realised on $L^2(\mathbb{R}^n)$, and defined exactly as in (1.1.2). As in the case of H^n, we get (upto unitary equivalence) that all the infinite dimensional irreducible unitary representations of H^n_{red} are given by $\pi_m, m \in Z^*$. Apart from this there is a class of one dimensional representations, $\pi_{a,b}, a, b \in \mathbb{R}^n$ given by

$$\pi_{a,b}(z,t) = e^{i(ax+by)} \quad for \quad (z,t) \in H^n_{red}. \qquad (1.2.1)$$

Thus, the dual $\widehat{H^n_{red}}$ can be thought of as the disjoint union of Z^* and $\mathbb{R}^{2n}$. The Plancherel measure is the counting measure on Z^* with a weight function $C \mid m \mid^n$ (for a suitable constant C) and the Lebesgue measure on $\mathbb{R}^{2n}$.(This is in sharp contrast to the case of the Heisenberg group).

Though it is probably well known to many experts, as we have been unable to find a reference in the literature to the Plancherel theorem on H^n_{red}, we present a sketch of the proof below.

Given f in $L^1(H^n_{red})$, we can write

$$f(z,t) = \sum_{k=-\infty}^{\infty} \Psi_k(z) e^{ikt} \qquad (1.2.2)$$

as a Fourier series in the central variable t (Here f can be thought of as the L^1 - limit of the Cesàro means of the right hand side of (1.2.2)). Hence, as in the case of H^n, if we compute the group Fourier transform $\hat{f}(\pi_m)$, also denoted by $\hat{f}(m)$, $m \in Z^*$ we see that it is an integral operator with kernel K^m_f given by

$$K^m_f(\xi,\eta) = \mathcal{F}_1\Psi_{-m}(\frac{1}{2}(m(\xi+\eta)),\ \xi-\eta) \qquad (1.2.3)$$

where $\mathcal{F}_1\Psi_{-m}$ stands for the Fourier transform of Ψ_{-m} in the first set of variables. Therefore, for $f \in L^1(H^n_{red}) \cap L^2(H^n_{red})$, a simple calculation shows that

$$\|\hat{f}(m)\|^2_{HS} = |\, m \,|^{-n} \, \|\mathcal{F}_1\Psi_{-m}\|^2_{L^2(\mathbb{C}^n)},\ m \in Z^*. \qquad (1.2.4)$$

On the other hand, we have

$$\begin{aligned} \hat{f}(\pi_{a,b}) &= \int_{\mathbb{C}^n \times S^1} f(z,t)\pi_{a,b}(z,t)\,dz dt \\ &= \int_{R^{2n} \times S^1} f(x,y,t)\ e^{i(ax+by)} dx dy dt \\ &= \mathcal{F}_{12}\Psi_o(a,b), \qquad (1.2.5) \end{aligned}$$

where $\mathcal{F}_{12}\Psi_o$ denotes the Euclidean Fourier transform of Ψ_o in the first and second set of variables. Now using the Plancherel theorems on R^{2n} and S^1 we get the

Plancherel theorem for H^n_{red} :

$$\|f\|_2^2 = C_n \sum_{m \in Z^*} | m |^n \|\hat{f}(m)\|_{HS}^2 + C'_n \int_{R^{2n}} | \hat{f}(\pi_{a,b}) |^2 \, da db \qquad (1.2.6)$$

where C_n and C'_n are constants depending only on n.

We now show, by an example, that the exact analogue of Hardy's theorem on H^n_{red} is not valid. Since t varies over a compact set in this case, one might be tempted to consider the following analogue of Hardy's theorem :

Suppose f is a measurable L^1 - function on H^n_{red} and f satisfies the following estimates :

$$| f(z,t) | \leq Ce^{-\alpha\|z\|^2}, \; z \in C^n, \; t \in S^1,$$

$$\|\hat{f}(m)\|_{HS} \leq Ce^{-\beta|m|^2}, \; m \in Z^*, | \hat{f}(\pi_{a,b}) | \leq Ce^{-\beta(a^2+b^2)}, \; a, \; b \in R^n \qquad (1.2.7)$$

for positive constants α, β. Then if $\alpha\beta$ is sufficiently large, is $f = 0$ a.e. ?

However, the following demonstrates that this is not the case.

Observe that as f satisfies (1.2.7), f belongs to $L^1(H^n_{red}) \cap L^2(H^n_{red})$ and the series in (1.2.2) converges to f in L^2- sense. Now take $f(z,t) = e^{-\alpha\|z\|^2} e^{ik_o t}$, for some $k_o \in Z^*$. Then for $m \in Z^*$, and $\phi \in L^2(R^n)$, it is easy to see that

$$\begin{aligned} (\hat{f}(m)\phi)(\xi) &= \int_{C^n} \int_{S^1} f(z,t)(\pi_m(z,t)\phi)(\xi) dz dt \\ &= \int_{R^{2n}} \int_{S^1} e^{-\alpha(x^2+y^2)} e^{ik_o t} e^{imt} e^{im(x\cdot\xi + \frac{1}{2}x\cdot y)} \phi(\xi + y) \; dx dy dt \end{aligned}$$

$$= \begin{cases} \int_{R^{2n}} e^{-\alpha(x^2+y^2)} e^{-ik_o(x\cdot\xi+\frac{1}{2}x\cdot y)} \phi(\xi+y)\, dxdy, & m = -k_o, \\ 0 & m \neq -k_o. \end{cases}$$

Therefore $\|\hat{f}(m)\|_{HS}^2 = 0$ if $m \neq -k_o$. Further,

$$\begin{aligned} \hat{f}(\pi_{a,b}) &= \int_{R^{2n}} \int_{S^1} e^{-\alpha(x^2+y^2)} e^{ik_o t} e^{i(ax+by)} dxdydt \\ &= 0, \end{aligned}$$

as $k_o \in Z^*$. Hence for a suitable constant C, we can see that f is a non-trivial function satisfying the estimates (1.2.7), with $\alpha\beta$ as large as we please.

However a slightly modified version of Hardy's theorem *still* holds(-see Remark 7, section 2 of [32]) :

Theorem 1.2.1 *Fix $t_o \in S^1$ and let $d(\cdot,\cdot)$ denote the standard metric on S^1. Suppose f is a measurable L^1-function on H^n_{red} satisfying*

$$|\, f(z,t)\,| \leq Ce^{-\|z\|^2} e^{-\frac{1}{d(t,t_o)^\alpha}}, \;\; z \in \mathbb{C}^n, \;\; t \in S^1, \qquad (1.2.8)$$

$$\|\hat{f}(m)\|_{HS} \leq C\, e^{-\beta|m|}, \;\; m \in Z^*, \qquad (1.2.9)$$

for some positive constants α and β. Then $f = 0$ a.e.

Proof : For each z, $f(z,\cdot)$ has the expansion as in (1.2.2). From (1.2.4) and the Euclidean Plancherel theorem, we have $\|\mathcal{F}_1\Psi_{-m}\|_{L^2(\mathbb{C}^n)} = \|\Psi_{-m}\|_{L^2(\mathbb{C}^n)}$. By a similar argument as in the proof of Theorem 1.1.1, we can choose a positive constant

β' slightly smaller than β such that

$$\|\Psi_{-m}\|_{L^2(\mathbb{C}^n)} \leq Ce^{-\beta'|m|}. \qquad (1.2.10)$$

Take an orthonormal basis $\{\phi_l\}_{l \in \mathbb{N}}$ of $L^2(\mathbb{C}^n)$ such that each ϕ_l is a Schwartz class function (-eg. the Hermite functions). Then consider the function $F_l(t)$ given by

$$F_l(t) = \int_{\mathbb{C}^n} \phi_l(z)\ f(z,t)\ dz. \qquad (1.2.11)$$

The Fourier coefficients of $F_l(t)$ are just

$$a_k^{(l)} = \int_{\mathbb{C}^n} \phi_l(z)\ \Psi_k(z)\ dz, \qquad k \in Z. \qquad (1.2.12)$$

By Cauchy-Schwarz inequality it follows that $|\ a_k^{(l)}\ | \leq Ce^{-\beta'|k|}$. One can show that, because of this very rapid decay of $a_k^{(l)}$, $F_l(t)$ is a real-analytic function. (In fact, considering S^1 as a subset of $\mathbb{C}$, $F_l(t)$ will be the restriction to S^1 of a complex analytic function in an annulus containing S^1.) Using (1.2.8) it can be proved that F_l and all derivatives of F_l at $t = t_o$ are zero and hence, by real-analyticity, $F_l = 0$. This shows that for each fixed t, $\int_{\mathbb{C}^n} \phi_l(z)\ f(z,t)\ dz = 0$ and hence since $\{\phi_l\}_{l \in N}$ is an orthonomal basis of $L^2(\mathbb{C}^n)$, $f(z,t) = 0$ for a.e. z for each fixed t. Thus f is the zero function.

Remark 1.2.1 (i) Actually an examination of the proof shows that we can replace the condition (1.2.8) by

$$|\ f(z,t)\ | \leq \alpha(z)\gamma(t), \quad z \in \mathbb{C}^n, \quad t \in S^1$$

where α is any function with reasonably rapid decay at infinity and γ is any function that vanishes to infinite order at some point $t_o \in S^1$.

(ii) Since S^1 is compact the point t_o can be "viewed" as the point at infinity and therefore condition (1.2.8) can be thought of as the analogue of the decay of the function at infinity.

Chapter 2

The Euclidean motion group, $M(n)$

In this chapter, we prove an analogue of Hardy's theorem for the n-dimensional Euclidean motion group, $M(n)$, $n \geq 2$. An analogue of Hardy's theorem for the special case of $M(2)$, the Euclidean motion group of the plane, has been proved in [32]. While the proof in [32] for $M(2)$ proceeds by reducing the theorem to the Euclidean case, the proof of the general case presented here is more direct and involves some simple estimates of the $SO(n)$-finite matrix coefficients of irreducible representations of $M(n)$.

In the next section, we give a complete description of the unitary dual of $M(n)$, and in section 2.2 we state and prove the main theorem of this chapter. We denote the standard inner product on $\mathbb{R}^n$ as well as on $\mathbb{C}^k$ by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\| \cdot \|$.

2.1 Description of the unitary dual of $M(n)$

The group $G = M(n)$ is a semi-direct product of $\mathbb{R}^n$ with the special orthogonal group, $K = SO(n)$. A typical element of G is denoted by (a, k) where $a \in \mathbb{R}^n$ and $k \in K$. If da denotes the Lebesgue measure on $\mathbb{R}^n$ and dk the normalized Haar measure on K, then the Haar measure on G is given by $da\, dk$. The natural action of K on $\mathbb{R}^n$ is denoted by $k \cdot \nu$, $k \in K$, $\nu \in \mathbb{R}^n$. (Since the 'natural' action is left multiplication by the matrix k, $\mathbb{R}^n$ should really be thought of as the space of column vectors.) For any unexplained terminology and notation in this section, the reader may refer to [16].

We now describe $\widehat{G}$, the unitary dual of G.

Let $\nu \in \mathbb{R}^n$ and $\nu \neq 0$. Let U_ν denote the stabilizer of ν in K under the natural action of K on $\mathbb{R}^n$. Then U_ν is conjugate to the subgroup $\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(n-1)\}$. We identify this subgroup with $SO(n-1)$. Fix an irreducible unitary representation λ of U_ν acting on the Hilbert space H_λ(i.e. $\lambda \in \widehat{U_\nu} \cong \widehat{SO(n-1)}$). Since U_ν is compact, we can identify H_λ with $\mathbb{C}^{d_\lambda}$ where d_λ is the dimension of H_λ. Let

$$H(K, \lambda) = \{ \psi : K \to \mathbb{C}^{d_\lambda} : \psi \text{ measurable }, \quad \psi(uk) = \lambda(u)(\psi(k)), u \in U_\nu, \\ k \in K \text{ and } \int_K \|\psi(k)\|^2 dk < \infty\},$$

where $\|\cdot\|$ denotes the norm on $\mathbb{C}^{d_\lambda}$. It is easy to see that $H(K, \lambda)$ is a Hilbert space with respect to the inner product defined by

$$(\psi_1, \psi_2) = d_\lambda \int_K < \psi_1(k), \psi_2(k) > dk$$

where $< ., . >$ denotes the usual inner product on $\mathbb{C}^{d_\lambda}$, and ψ_1, $\psi_2 \in H(K, \lambda)$. Define $T_{\nu,\lambda}$ on $H(K, \lambda)$ by

$$(T_{\nu,\lambda}(a,k)\psi)(k_o) = e^{i<k_o^{-1}\cdot\nu,a>} \psi(k_o k), \quad \psi \in H(K, \lambda) \qquad (2.1.1)$$

for $a \in \mathbb{R}^n$, $k, k_o \in K$. We also use $< ., . >$ to denote the inner product on $\mathbb{R}^n$. One can easily verify that $T_{\nu,\lambda}$ is a unitary representation of G on $H(K, \lambda)$. Further, it can be shown that (-see [13], [16])

(a) $T_{\nu,\lambda}$ is irreducible for all $\nu \in \mathbb{R}^n$, $\nu \neq 0$, $\lambda \in \widehat{SO(n-1)} (\cong \widehat{U_\nu})$.

(b) Every infinite dimensional irreducible unitary representation of G is equivalent to some $T_{\nu,\lambda}$, ν and λ as above.

(c) For two non-zero vectors ν, $\nu_1 \in \mathbb{R}^n$, $\lambda \in \widehat{U_\nu}$, $\lambda_1 \in \widehat{U_{\nu_1}}$, $T_{\nu,\lambda}$ is equivalent to T_{ν_1,λ_1} if and only if ν, ν_1 belong to the same K-orbit (i.e. ν, ν_1 have the same Euclidean norm) and the representations λ, λ_1 are equivalent under the obvious identification of U_ν with U_{ν_1}.

If $\|\nu\| = \|\nu_1\| = r$, $r \in \boldsymbol{R}^+$, then by abuse of notation, we denote the n-tuple $(0, 0, \cdots, 0, r)^t$ also by r and write U_r for U_ν and the representative of the equivalence class of $T_{\nu,\lambda}$ as $T_{r,\lambda}$. Here $\|\cdot\|$ denotes the Euclidean norm on $\boldsymbol{R}^n$ and t denotes the transpose. Apart from these infinite dimensional representations, the finite dimensional unitary representations of K also yield finite dimensional unitary representations of G, but these do not enter into the Plancherel formula(-see [16] for details).

The Plancherel measure μ is supported on the subset of $\widehat{G}$ given by $\{T_{r,\lambda}\}_{\lambda \in \widehat{SO(n-1)}, r \in \boldsymbol{R}^+}$, and on each "piece" $\{T_{r,\lambda}\}_{r \in \boldsymbol{R}^+}$, for a fixed $\lambda \in \widehat{SO(n-1)}$, it is given by $C_n r^{n-1} dr$, where C_n is a constant depending only on n.

Before we end this section, we state the following lemma, from complex analysis, that plays a crucial role in the proof of our main theorem :

Lemma 2.1.1 *Suppose h is an entire function on $\boldsymbol{C}$ such that $h(z) = O(e^{a|z|^2})$, $z \in \boldsymbol{C}$ and $h(t) = O(e^{-at^2})$ for $t \in \boldsymbol{R}$ where 'a' is a positive constant. Then $h(z) = Const.e^{-az^2}$, $z \in \boldsymbol{C}$.*

This lemma follows from the following result in [42], pp.175 (-see the first half of the proof of Lemma 3.1.1) :

Let h be an entire function on $\boldsymbol{C}$ such that $h(z) = O(e^{a|z|})$ for $z \in \boldsymbol{C}$ and $h(t) = O(e^{-at})$ for $t \in \boldsymbol{R}^+$, where 'a' is a positive constant. Then $h(z) = Const.e^{-az}$, $z \in \boldsymbol{C}$.

2.2 Analogue of Hardy's theorem for $M(n)$

Given a function f in $L^1(G)$ and $\pi \in \widehat{G}$, the group Fourier transform $\hat{f}$ of f at π is the operator

$$\hat{f}(\pi) = \pi(f) = \int_{\mathbb{R}^n} \int_K f(a,k)\pi(a,k)dkda, \qquad (2.2.1)$$

as discussed in earlier chapters. Then, by the Plancherel theorem, we know that for $f \in L^1(G) \cap L^2(G)$, $\hat{f}$ is a Hilbert-Schimdt operator for almost all π (with respect to the Plancherel measure), and we denote its Hilbert-Schimdt norm by $\|\hat{f}(\pi)\|_{HS}$. We now state and prove an analogue of Hardy's theorem for the motion group G ([37]) :

Theorem 2.2.1 *Suppose f is a measurable function on G satisfying the following estimates :*

$$| f(a,k) | \leq Ce^{-\alpha\|a\|^2}, \qquad (a,k) \in G \qquad (2.2.2)$$

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \leq C_\lambda e^{-\beta r^2}, \qquad r \in \mathbb{R}^+ \qquad (2.2.3)$$

for some positive constants C_λ, α, β and C, where C_λ depends only on λ. If $\alpha\beta > \frac{1}{4}$ then $f = 0$ a.e.

Remark 2.2.1 Since functions on $\mathbb{R}^n$ can be thought of as functions on G invariant under right action by K, Hardy's theorem for $\mathbb{R}^n$ shows that $\frac{1}{4}$ is the best possible constant.

Proof : Observe that by identifying $-r$ with the n-tuple $(0,\cdots,0,-r)^t$ for $r \in \mathbb{R}^+$ we can define $T_{-r,\lambda}$. Now, $T_{-r,\lambda}$ and $T_{r,\lambda}$ are equivalent as representations of G. Hence $\|\hat{f}(T_{-r,\lambda})\|_{HS} = \|\hat{f}(T_{r,\lambda})\|_{HS}$ and we thus have

$$\|\hat{f}(T_{r,\lambda})\|_{HS} \leq C_\lambda e^{-\beta r^2}, \qquad r \in \mathbb{R}. \tag{2.2.4}$$

For $r \in \mathbb{R}$ and $\lambda \in \widehat{SO(n-1)} (\cong \widehat{U_r})$, let $S = \{ e_i^\lambda \ : \ i \in \mathbb{N} \}$ be a basis of $H(K,\lambda)$ consisting of K-finite vectors. (For fixed λ, notice that the representation $T_{r,\lambda}$ restricted to K is just the right regular action of K on $H(K,\lambda)$.) Note that if ϕ is a K-finite vector then $\phi \in C^\infty(K, \mathbb{C}^{d_\lambda})$, the space of smooth functions defined on K taking values in $\mathbb{C}^{d_\lambda}$. It is enough to show that if $\alpha\beta > \frac{1}{4}$ then for all $i,j \in \mathbb{N}$, $(\hat{f}(T_{r,\lambda})e_i^\lambda, e_j^\lambda) \equiv 0$ as a function of r and λ. Fix $i_o, j_o \in \mathbb{N}$ and consider for $r \in \mathbb{R}$,

$$(\hat{f}(T_{r,\lambda})e_{i_o}^\lambda, e_{j_o}^\lambda) = \int_K \int_{\mathbb{R}^n} f(a,k)(T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda) da dk. \tag{2.2.5}$$

Let $\Phi_{r,\lambda}^{i_o,j_o}(a,k) = (T_{r,\lambda}(a,k)e_{i_o}^\lambda, e_{j_o}^\lambda)$ for $r \in \mathbb{R}$, $\lambda \in \widehat{SO(n-1)}$, $i_o, j_o \in \mathbb{N}$, and $(a,k) \in G$. Then by definition of $T_{r,\lambda}$, we have

$$\begin{aligned}
\Phi_{r,\lambda}^{i_o,j_o}(a,k) &= d_\lambda \int_K < (T_{r,\lambda}(a,k)e_{i_o}^\lambda)(k_o), e_{j_o}^\lambda(k_o) > dk_o \\
&= d_\lambda \int_K e^{i<k_o^{-1}\cdot r, a>} < e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o) > dk_o \\
&= d_\lambda \int_K e^{i<r, k_o \cdot a>} < e_{i_o}^\lambda(k_o k), e_{j_o}^\lambda(k_o) > dk_o
\end{aligned} \tag{2.2.6}$$

Here the real number r is identified with $(0,\cdots,0,r)^t$ and $< .,. >$ denotes both inner product on $\mathbb{R}^n$ as well as $\mathbb{C}^{d_\lambda}$. Notice that the integral on the right hand side makes sense even when $r \in \mathbb{C}$ where we identify $r \in \mathbb{C}$ with $(0, \cdots, 0, r)^t$ in $\mathbb{C}^n$ and $< .,. >$ now denotes inner product on $\mathbb{C}^n$ also. Hence $\Phi^{i_o,j_o}_{r,\lambda}(a,k)$, for a fixed (a,k), as a function of r extends to the whole complex plane. We will also call these functions $\Phi^{i_o,j_o}_{z,\lambda}(a,k)$, $z \in \mathbb{C}$ and from (2.2.6), one can easily see that for fixed (a,k), $z \mapsto \Phi^{i_o,j_o}_{z,\lambda}(a,k)$ is an entire function on $\mathbb{C}$. (*Note that if $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^n$, then for a fixed $a \in \mathbb{R}^n$, $z \mapsto \langle z, a \rangle$ is an entire function on $\mathbb{C}^n$. In the final integral in (2.2.6), z is always in the first position and the second position is a vector in $\mathbb{R}^n$. Hence $\Phi^{i_o,j_o}_{z,\lambda}(a,k)$ as a function of z is entire.*) Moreover, for $z \in \mathbb{C}$,

$$\begin{aligned} | \Phi^{i_o,j_o}_{z,\lambda}(a,k) | &\leq d_\lambda \int_K | e^{i<z,k_o \cdot a>} || e^\lambda_{i_o}(k_o k) || e^\lambda_{j_o}(k_o) | dk_o \\ &\leq C \int_K e^{-<(Im\ z)e_n, k_o \cdot a>} dk_o \end{aligned} \qquad (2.2.7)$$

where $e_n = (0, \cdots, 0, 1)^t$ in $\mathbb{R}^n$, $(a,k) \in G$, and C is a constant which depends on λ, i_o, j_o. (Notice that $e^\lambda_{i_o}$ and $e^\lambda_{j_o}$ are smooth functions on K and hence bounded.) Since f satisfies (2.2.4) and $\beta > \frac{1}{4\alpha}$, we have

$$| (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) | \leq C_\lambda e^{-\beta r^2} \leq C_\lambda e^{-\frac{r^2}{4\alpha}}, \quad r \in \mathbb{R}. \qquad (2.2.8)$$

By definition of $\Phi^{i_o,j_o}_{r,\lambda}(a,k)$ we have from (2.2.5),

$$(\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o}) = \int_K \int_{\mathbb{R}^n} f(a,k) \Phi^{i_o,j_o}_{r,\lambda}(a,k) da dk. \qquad (2.2.9)$$

Since f satisfies (2.2.2) and from (2.2.7), $| \Phi^{i_o,j_o}_{z,\lambda}(a,k) | \leq Ce^{|z||a|}$, we conclude that the function $r \mapsto (\hat{f}(T_{r,\lambda})e^\lambda_{i_o}, e^\lambda_{j_o})$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that $z \mapsto (\hat{f}(T_{z,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})$ is an entire function. Further, a simple calculation using (2.2.2) and (2.2.7) shows that

$$\begin{aligned}
|\,(\hat{f}(T_{z,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})\,| &\leq \int_K \int_{R^n} |\, f(a,k)\,||\, \Phi^{i_o,j_o}_{z,\lambda}(a,k)\,|\, dadk \\
&\leq C \int_K \int_{R^n} e^{-\alpha\|a\|^2} \left(\int_K e^{-<(Im\ z)e_n, k_o\cdot a>} dk_o\right) dadk \\
&= C \int_K \int_{R^n} e^{-\alpha\|a\|^2} e^{-<(Im\ z)e_n, k_o\cdot a>} dadk_o \\
&= C \int_{R^n} e^{-\alpha\|a\|^2} e^{-<(Im\ z)e_n, a>} da \\
&= Ce^{\frac{\|(Im\ z)e_n\|^2}{4\alpha}} \int_{R^n} e^{-<\sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}}, \sqrt{\alpha}a + \frac{(Im\ z)e_n}{2\sqrt{\alpha}}>} da \\
&\leq C' e^{\frac{|Im\ z|^2}{4\alpha}} \\
&\leq C' e^{\frac{|z|^2}{4\alpha}} \qquad (2.2.10)
\end{aligned}$$

for $z \in \mathbb{C}$ and some constants C, C'.

It is clear from (2.2.8) and (2.2.10) that the function $z \mapsto (\hat{f}(T_{z,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})$ satisfies the hypothesis of Lemma 2.1.1. Hence, it follows that $(\hat{f}(T_{r,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})$ $= \ Const.e^{-\frac{r^2}{4\alpha}}$. So $|\ (\hat{f}(T_{r,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})\ | = |\ Const.e^{-\frac{r^2}{4\alpha}}\ | \leq\ C_\lambda e^{-\beta r^2}$ from (2.2.4) and since $\beta - \frac{1}{4\alpha} > 0$, it follows that $(\hat{f}(T_{r,\lambda})e^{\lambda}_{i_o}, e^{\lambda}_{j_o})\ \equiv 0$ as a function of r. Since i_o, j_o and λ were arbitrary, $\hat{f}(T_{r,\lambda})\ \equiv\ 0$ for all $r \in R^+$ and $\lambda \in \widehat{SO(n-1)}$. Hence, by the injectivity of the group Fourier transform, we get that $f\ =\ 0$ a.e. This completes the proof of the theorem.

Remark 2.2.2 Actually an examination of the proof shows that we have proved the following stronger result :

Let δ_1, $\delta_2 \in \widehat{K}$ and χ_{δ_1} and χ_{δ_2} the corresponding characters. Then $T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})$ is a finite rank operator(with rank bounded by a constant depending only on δ_1, δ_2, λ) which is zero on the orthogonal complement of a subspace whose dimension is again bounded by a constant depending only on δ_1, δ_2, λ. If $| f(a,k) | \leq Ce^{-\alpha\|a\|^2}$ and $\|T_{r,\lambda}(\chi_{\delta_1})T_{r,\lambda}(f)T_{r,\lambda}(\chi_{\delta_2})\|_{HS} \leq C_{\lambda,\delta_1,\delta_2}e^{-\beta r^2}$, where C, α, β are positive constants and $C_{\lambda,\delta_1,\delta_2}$ is a positive constant depending only on δ_1, δ_2, λ, then $f \equiv 0$ if $\alpha\beta > \frac{1}{4}$.

Chapter 3

Semi-simple Lie Groups

We establish an analogue of Hardy's theorem for a sub class of non-compact semi-simple Lie groups and all symmetric spaces of the non-compact type in this chapter.

In the next section we set up the required notation and describe the support of the Plancherel measure for those semi-simple Lie groups for which we present an analogue of Hardy's theorem. In section 3.2, we give the proof of the main theorem, and as a consequence we give an analogue of Hardy's theorem for all symmetric spaces of non-compact type in section 3.3. In section 3.4, we prove an analogue of Hardy's theorem for $SL(2, \mathbb{R})$. Further, we show, by considering the particular case of $SL(2, \mathbb{C})$, and with the normalizations used in this thesis, that $\frac{1}{4}$ is the best possible constant in Hardy's theorem.

3.1 Notation and Preliminaries

Let G be a connected, non-compact, semi-simple Lie group with finite centre and K a fixed maximal compact subgroup of G. Let $\mathcal{G}$, $\mathcal{K}$ denote the Lie algebras of G and K respectively. Suppose $\mathcal{G} = \mathcal{K} \bigoplus \mathcal{P}$ is a Cartan decomposition of $\mathcal{G}$ and B is the Cartan-Killing form of $\mathcal{G}$. It is known that B restricted to $\mathcal{P}$ is positive definite. Therefore B defines an inner product on the real vector space $\mathcal{P}$. Let $P = \exp\, \mathcal{P}$. Then G is diffeomorphic to $K \times P$ under the map $(k, u) \mapsto ku$ for $k \in K$ and $u \in P$. Therefore each $g \in G$ can be uniquely written as $g = g_K g_P$ with $g_K \in K$ and $g_P \in P$. Since P and $\mathcal{P}$ are diffeomorphic under the exponential map, $g_P = \exp\, X$ for a unique $X \in \mathcal{P}$. Define $\|g\|_G = B(X, X)^{\frac{1}{2}}$.

Fix a maximal abelian subspace $\mathcal{A}$ of $\mathcal{P}$. Let the dimension of $\mathcal{A}$ be l.('l' is called the real rank of $\mathcal{G}$.) The restriction $B|_{\mathcal{A}\times\mathcal{A}}$ gives an inner product on $\mathcal{A}$ and we can identify $\mathcal{A}$ with $\mathbb{R}^l$ under this inner product. Let Δ denote the set of roots for the adjoint action of $\mathcal{A}$ on $\mathcal{G}$. Fix a Weyl-chamber $\mathcal{A}^+$ of $\mathcal{A}$ and let Δ^+ be the corresponding set of positive roots(-see [20] for details). Let $A = \exp \mathcal{A}$ and $A^+ = \exp\, \mathcal{A}^+$. If $\overline{A^+}$ denotes the closure of A^+ in G then it is known that $G = K\overline{A^+}K$, the polar decomposition of G i.e. each $x \in G$ can be written as $x = k_1 a k_2$, for k_1, $k_2 \in K$ and $a \in \overline{A^+}$. If $\mathcal{G}_\alpha$ denotes the root space corresponding to $\alpha \in \Delta$, then we can choose a Haar measure dx on G such that relative to the polar decomposition it is given by $dx = J(a)\, dk_1\, da\, dk_2$ where

$J(a) = \prod_{\alpha\in\Delta^+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{n(\alpha)}$, $n(\alpha) = \dim \mathcal{G}_\alpha$ and 'log' is the inverse of the map 'exp' on $\mathcal{A}$ i.e. $\int_G f(x)dx = \int_K \int_{\overline{A^+}} \int_K f(k_1 a k_2)\ J(a)\ dk_1 da dk_2$, where da is the Haar measure on A. Let $G = KAN$ be the corresponding Iwasawa decomposition of G(-see [20] for details). The Iwasawa decomposition gives rise to the projection mappings $\kappa : G \to K$, $\mathbf{a} : G \to A$, and $\mathbf{n} : G \to N$. Then we have

$$x = \kappa(x) \exp H(x) \mathbf{n}(x),$$

where $\kappa(x) \in K$, $H(x) \in \mathcal{A}$, $H(x) = \log \mathbf{a}(x)$, $\mathbf{n}(x) \in N$.

If M denotes the centralizer of A in K then $P = MAN$ is the minimal parabolic subgroup of G. Fix $\xi \in \widehat{M}$ and let $\boldsymbol{H}_\xi$ be the finite dimensional Hilbert space on which ξ acts, $d(\xi) = \dim \boldsymbol{H}_\xi$. For $\lambda \in \mathcal{A}^*$(the real dual of $\mathcal{A}$), define a representation (ξ, λ) of P by :

$$(\xi, \lambda)(man) = \xi(m)\ \exp((i\lambda + \rho)(\log a)),$$

where $\log : A \to \mathcal{A}$ is the inverse of the map $\exp : \mathcal{A} \to A$ and $\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} n(\alpha)\alpha$, $m \in M$, $a \in A$, $n \in N$. From this representation we get, by induction, a representation $\pi_{\xi,\lambda}$ of G acting on the Hilbert space

$$\boldsymbol{H}_\xi^o = \{\ g : K \to \boldsymbol{H}_\xi \text{ measurable } :\ g(km) = \xi(m^{-1})g(k), k \in K, m \in M \text{ and } \int_K \|g(k)\|^2 dk < \infty\ \}$$

where $\|\cdot\|$ denotes the norm on $\boldsymbol{H}_\xi$. The induced representation $\pi_{\xi,\lambda}$ acts unitarily on $\boldsymbol{H}_\xi^o$ by the formula

$$(\pi_{\xi,\lambda}(x)g)(k) = e^{-(i\lambda+\rho)(H(x^{-1}k))}g(\kappa(x^{-1}k)) \qquad (3.1.1)$$

for $x \in G$, $k \in K$, $g \in \boldsymbol{H}_\xi^o$. Note that the action of K on $\boldsymbol{H}_\xi^o$ is just the left regular action.

Given $\xi \in \widehat{M}$, it is known that one can find a dense open subset O_ξ of $\mathcal{A}^*$ such that $\pi_{\xi,\lambda}$ is irreducible for all $\lambda \in O_\xi$(-see [26], pp.174 for details). Let W be the Weyl group of the pair $(\mathcal{G}, \mathcal{A})$. Then there is a natural action of W on $\widehat{M} \times \mathcal{A}^*$ and the only identifications among the irreducible representations in these series of representations are the identifications given by the Weyl group action(-see [26], pp.174 for details).

For the remaining part of this section we assume that G has only one conjugacy class of Cartan subgroups. Given f in $L^1(G)$, we can define the group Fourier transform on $\widehat{M} \times \mathcal{A}^*$ by

$$\hat{f}(\xi, \lambda) = \hat{f}(\pi_{\xi,\lambda}) = \pi_{\xi,\lambda}(f) = \int_G f(x)\ \pi_{\xi,\lambda}(x)\ dx \qquad (3.1.2)$$

for $(\xi, \lambda) \in \widehat{M} \times \mathcal{A}^*$, where the integral is to be interpreted in a suitable way. If $f \in L^1(G) \cap L^2(G)$, we have the Plancherel theorem for such G :
There exists an explicitly computable measure μ on $\widehat{M} \times \mathcal{A}^*$ such that

$$\int_G |\ f(x)\ |^2\ dx = \int_{\widehat{M} \times \mathcal{A}^*} tr(\pi_{\xi,\lambda}(f)\pi_{\xi,\lambda}(f)^*)\ d\mu(\xi, \lambda) \qquad (3.1.3)$$

For fixed $\xi \in \widehat{M}$, this measure is of at most polynomial growth on $\mathcal{A}^*$(-see [26], pp.511 and [19] for details). Let $\mathcal{A}^*_{\mathbb{C}} = \mathcal{A}^* \otimes \mathbb{C}$. Since B is positive definite on $\mathcal{A}$, it defines an inner product on $\mathcal{A}$. Hence there is a natural inner product on $\mathcal{A}^*$, and the corresponding norm on $\mathcal{A}^*$ will be denoted by $\|\cdot\|$. This real inner product can be extended in a unique fashion as an inner product on the complex vector space $\mathcal{A}^*_{\mathbb{C}}$ and the corresponding norm on $\mathcal{A}^*_{\mathbb{C}}$ will also be denoted by $\|\cdot\|$. By abuse of notation, the norm induced by B on $\mathcal{A}$ will also be denoted by $\|\cdot\|$.

If 1 is the trivial representation in $\widehat{M}$, then we denote $\pi_{1,\lambda}$ by π_λ. The set of representations $\{\pi_\lambda\}_{\lambda \in \mathcal{A}^*}$ are called the *class* $-$ **1** principal series representations of G, and they are realized on the Hilbert space $L^2(K/M)$. Let Φ_λ be the "elementary spherical function"(-see [14] for details) corresponding to $\lambda \in \mathcal{A}^*_{\mathbb{C}}$. Then for $\lambda \in \mathcal{A}^*$,

$$\Phi_\lambda(x) = < \pi_\lambda(x)1, 1 >, \qquad x \in G \tag{3.1.4}$$

where 1 is the constant function 1 on K/M. Also one has ;

$$\begin{aligned} \Phi_\lambda(x) &= \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))} dk \\ &= \int_K e^{(i\lambda-\rho)(H(xk))} dk \end{aligned} \tag{3.1.5}$$

for $\lambda \in \mathcal{A}^*_{\mathbb{C}} (\cong \mathbb{C}^l)$. Moreover for $\lambda \in \mathcal{A}^* (\cong \mathbb{R}^l)$ and any $a \in \overline{A^+}$, we have the following estimate :

$$| \Phi_{i\lambda}(a) | \leq e^{\lambda^+(\log a)} \tag{3.1.6}$$

where λ^+ is the element in the fundamental Weyl chamber corresponding to λ(-see [15] for details).

Finally, we end this section with a lemma from complex analysis that is crucial for the proof of our main theorem in this chapter. We shall also denote the standard Euclidean norms on $\mathbb{R}^n$ and $\mathbb{C}^n$ by $\|\cdot\|$.

Lemma 3.1.1 *Let $n \geq 1$. Let h be an entire function on $\mathbb{C}^n$ such that*

$$|\, h(z) \,| \leq \; Ce^{a\|z\|^2}, \qquad z \in \mathbb{C}^n, \tag{3.1.7}$$

$$|\, h(t) \,| \leq \; Ce^{-a\|t\|^2}, \qquad t \in \mathbb{R}^n, \tag{3.1.8}$$

for some positive constants a and C. Then $h(z) \;=\; Const.e^{-a(z_1^2+\cdots+z_n^2)}$, $z \;=\; (z_1, \cdots, z_n) \in \mathbb{C}^n$.

Proof : To prove this, we will need the following result ([42], pp.175) that has already been stated in Chapter 2. For the sake of convenience, we shall recall the statement of it here :

$$\left.\begin{array}{l} \text{Let } h \text{ be an entire function on } \mathbb{C} \text{ such that } h(z) \;=\; O(e^{a|z|}) \text{ for } z \in \mathbb{C} \\ \text{and } h(t) \;=\; O(e^{-at}) \text{ for } t \in \mathbb{R}^+, \text{ where `}a\text{' is a positive constant.} \\ \text{Then } h(z) \;=\; Const.e^{-az}, \;\; z \in \mathbb{C}. \end{array}\right\} \qquad (*)$$

We shall prove Lemma 3.1.1 in two steps. First, we prove the lemma for the case $n = 1$, and then proceed to prove it in general.

Let h be an entire function on $\mathbb{C}$ satisfying the following estimates :

$$|\, h(z) \,| \leq \; Ce^{a|z|^2}, \;\; z \in \mathbb{C}, \tag{3.1.9}$$

$$| h(t) | \leq Ce^{-at^2}, \quad t \in \mathbb{R}, \tag{3.1.10}$$

for some positive constants a and C. If h is even, then by applying $(*)$ to $\phi(z) = h(\sqrt{z})$, the result will follow immediately.(Note that since h is even and entire $\phi(z) = h(\sqrt{z})$ is an entire function and will satisfy the assumptions of $(*)$.)

Suppose h is an odd, entire function and h satisfies (3.1.9) and (3.1.10). Then the function $\phi(z) = h(z)/z$ is an even, entire function on $\mathbb{C}$ satisfying the estimates (3.1.9) and (3.1.10). Therefore, by the even case, we have, $\phi(z) = h(z)/z = C'e^{-az^2}$, $z \in \mathbb{C}$, for some constant C'. In particular, $h(t) = C'te^{-at^2}$, $t \in \mathbb{R}$. Then by (3.1.10) it will follow that :

$$| C'te^{-at^2} | \leq Ce^{-at^2}, \quad t \in \mathbb{R},$$

which is impossible, unless $C' = 0$. Hence $h \equiv 0$.

If h is an entire function on $\mathbb{C}$ satisfying the estimates (3.1.9) and (3.1.10), then write $h(z) = (h(z) + h(-z))/2 + (h(z) - h(-z))/2 = h_{even}(z) + h_{odd}(z)$, as the sum of even and odd entire functions. Since h satisfies (3.1.9) and (3.1.10), it is easy to see, in view of the expressions for h_{even} and h_{odd}, that they also satisfy (3.1.9) and (3.1.10) respectively. Applying the even and odd cases to h_{even} and h_{odd} respectively, we conclude that $h(z) = Const.e^{-az^2}$, $z \in \mathbb{C}$. This proves the lemma in the case when $n = 1$.

Now consider the case $n > 1$. For fixed $(u_1, \cdots, u_{n-1})$ in $\mathbb{R}^{n-1}$, let $g(z) = h(u_1, \cdots, u_{n-1}, z)$, $z \in \mathbb{C}$. Clearly, g is an entire function on $\mathbb{C}$ in the variable z.

Since h satisfies (3.1.7) and (3.1.8), for fixed $(u_1, \cdots, u_{n-1}) \in \boldsymbol{R}^{n-1}$, we have :

$$| g(z) | = | h(u_1, \cdots, u_{n-1}, z) | \leq Ce^{a(|u_1|^2+\cdots+|u_{n-1}|^2)}e^{a|z|^2}, \ z \in \mathbb{C},$$

$$| g(t) | = | h(u_1, \cdots, u_{n-1}, t) | \leq Ce^{-a(|u_1|^2+\cdots+|u_{n-1}|^2)}e^{-at^2}, \ t \in \boldsymbol{R}.$$

Applying the one dimensional case to g we can conclude that

$$g(z) = C_n(u_1, \cdots, u_{n-1})e^{-az^2}, \quad z \in \mathbb{C}, \quad (u_1, \cdots, u_{n-1}) \in \boldsymbol{R}^{n-1},$$

where C_n depends only on $u_1, \cdots, u_{n-1}$. Setting $z = 0$, we have $C_n(z_1, \cdots, z_{n-1}) = g(0) = h(z_1, \cdots, z_{n-1}, 0)$ for $(z_1, \cdots, z_{n-1}) \in \boldsymbol{R}^{n-1}$. Thus

$$h(z_1, \cdots, z_{n-1}, z_n) = h(z_1, \cdots, z_{n-1}, 0)e^{-az_n^2} \qquad (3.1.11)$$

for all $(z_1, \cdots, z_{n-1}, z_n) \in \boldsymbol{R}^n$. However, both sides are entire functions on $\mathbb{C}^n$ and hence (3.1.11) must actually hold for all $(z_1, \cdots, z_{n-1}, z_n) \in \mathbb{C}^n$. Here we are using the fact that two entire functions on $\mathbb{C}^n$ which agree on $\boldsymbol{R}^n$ have to actually agree on $\mathbb{C}^n$. Now from (3.1.7) and (3.1.8) it follows that

$$h(z_1, \cdots, z_{n-1}, 0) = O(e^{a(|z_1|^2+\cdots+|z_{n-1}|^2)}), \quad (z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1},$$

and

$$h(t_1, \cdots, t_{n-1}, 0) = O(e^{-a(|t_1|^2+\cdots+|t_{n-1}|^2)}), \quad (t_1, \cdots, t_{n-1}) \in \boldsymbol{R}^{n-1},$$

and applying exactly the same argument as before we will have

$$h(z_1, \cdots, z_{n-1}, 0) = h(z_1, \cdots, z_{n-2}, 0, 0)e^{-az_{n-1}^2},$$

and so

$$h(z_1, \cdots, z_n) = h(z_1, \cdots, z_{n-2}, 0, 0)e^{-a(z_{n-1}^2 + z_n^2)}.$$

Repeating the above, we finally have

$$h(z_1, \cdots, z_n) = h(0, 0, \cdots, 0)e^{-a(z_1^2 + \cdots + z_n^2)}, \quad (z_1, \cdots, z_n) \in \mathbb{C}^n,$$

and the proof of the lemma is complete.

In the next section, we state and prove an analogue of Hardy's theorem for a class of semi-simple Lie groups.

3.2 Semi-simple Lie groups with one conjugacy class of Cartan subgroups

We retain the notation introduced in section 3.1. However we assume that G has only one conjugacy class of Cartan subgroups. Thus, throughout this section, G will denote a connected non-compact semi-simple Lie group with finite centre and having only one conjugacy class of Cartan subgroups. For such groups, as described in section 3.1, the Plancherel measure is entirely supported on the various principal series representations associated with the minimal parabolic subgroup. Examples of such groups include $SL(n, \mathbb{C})$ (-in fact all complex semi-simple Lie groups-) and $SO_o(n, 1)$ with n odd. We now state and prove an analogue of Hardy's theorem for such groups ([31]).

Theorem 3.2.1 *Suppose f is a measurable function on G satisfying the following estimates :*

$$| f(x) | \leq Ce^{-\alpha\|x\|_G^2}, \quad x \in G \qquad (3.2.1)$$

$$\|\hat{f}(\xi,\lambda)\|_{HS} = \|\pi_{\xi,\lambda}(f)\|_{HS} \leq C_\xi e^{-\beta\|\lambda\|^2}, \quad (\xi,\lambda) \in \widehat{M} \times \mathcal{A}^* \qquad (3.2.2)$$

where C, C_ξ, α and β are positive constants and C_ξ depends on ξ. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

(**Note :** (i) The very rapid decay of f implies $f \in L^1(G)$. Hence $\pi_{\xi,\lambda}(f)$ is defined for all $\xi \in \widehat{M}$, $\lambda \in \mathcal{A}^*$.

(ii) If $x = k_1ak_2$, $a = \exp H$, then $\|x\|_G = \|H\|$, where $\|\cdot\|$ is the norm on $\mathcal{A}$ as described in section 3.1.)

Proof : For $\xi \in \widehat{M}$ let $\{ e_j^\xi : j \in \boldsymbol{N} \}$ be a basis of $\boldsymbol{H}_\xi^o$ consisting of K-finite vectors. (As observed earlier the action of K on $\boldsymbol{H}_\xi^o$ is just left regular action.) Let $\langle \cdot, \cdot \rangle_\xi$ denote the inner product on $\boldsymbol{H}_\xi^o$. We shall show that if $\alpha\beta > \frac{1}{4}$, $\langle \pi_{\xi,\lambda}(f)e_m^\xi, e_n^\xi \rangle_\xi = 0$, for all $\lambda \in \mathcal{A}^*$, $m, n \in \boldsymbol{N}$. Fix $m_o, n_o \in \boldsymbol{N}$. We have by (3.1.2) :

$$\langle \pi_{\xi,\lambda}(f)e_{m_o}^\xi, e_{n_o}^\xi \rangle_\xi = \int_G f(x) \langle \pi_{\xi,\lambda}(x)e_{m_o}^\xi, e_{n_o}^\xi \rangle_\xi dx \qquad (3.2.3)$$

Let $\Phi^{m_o,n_o}_{\xi,\lambda}(x) = \langle \pi_{\xi,\lambda}(x)e^{\xi}_{m_o}, e^{\xi}_{n_o}\rangle_{\xi}$ for $x \in G$. Then it can be shown from the definition of $\pi_{\xi,\lambda}(x)$ acting on H^o_{ξ} that :

$$\Phi^{m_o,n_o}_{\xi,\lambda}(x) = \int_K e^{-(i\lambda+\rho)(H(x^{-1}k))}\langle e^{\xi}_{m_o}(\kappa(x^{-1}k)), e^{\xi}_{n_o}(k)\rangle dk \qquad (3.2.4)$$

where $\langle\cdot,\cdot\rangle$ inside the integral is the inner product on H_{ξ}. Thus

$$\langle \pi_{\xi,\lambda}(f)e^{\xi}_{m_o}, e^{\xi}_{n_o}\rangle_{\xi} = \int_G f(x)\ \Phi^{m_o,n_o}_{\xi,\lambda}(x)dx \qquad (3.2.5)$$

The basis vectors $e^{\xi}_{m_o}$, $e^{\xi}_{n_o}$ being K-finite, actually belong to $C^{\infty}(K, H_{\xi})$ and hence are bounded as functions into H_{ξ}. Therefore it follows easily that for each $x \in G$, the integral defining $\Phi^{m_o,n_o}_{\xi,\lambda}$ makes sense even for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ and in fact, for each fixed x, the function $\lambda \mapsto \Phi^{m_o,n_o}_{\xi,\lambda}(x)$ extends as an entire function of $\lambda \in \mathcal{A}^*_{\mathbb{C}}(\cong \mathbb{C}^l)$. Writing $\lambda = \lambda_R + i\lambda_I$, one has the following easy estimate from the above integral :

$$|\ \Phi^{m_o,n_o}_{\xi,\lambda}(x)\ | \leq \ Const. \int_K e^{(\lambda_I-\rho)(H(x^{-1}k))}dk \qquad (3.2.6)$$

where the constant depends only on m_o, n_o and ξ. The integral on the right is just the elementary spherical function $\Phi_{i\lambda_I}$ and hence we have the following easy estimate

$$|\ \Phi^{m_o,n_o}_{\xi,\lambda}(x)\ | \leq \ Const.\ \Phi_{i\lambda_I}(x) \qquad (3.2.7)$$

Using the K-biinvariance of $\Phi_{i\lambda_I}$, one therefore finally has, if x is written as $x = k_1 a k_2$, k_1, $k_2 \in K$, $a \in \overline{A^+}$,

$$|\ \Phi^{m_o,n_o}_{\xi,\lambda}(x)\ | \leq \ Const.\ e^{\lambda_I^+(\log a)} \qquad (3.2.8)$$

where $\lambda_I^\dagger$ is the element in the fundamental Weyl chamber corresponding to λ_I(-see [15] for details). Now define

$$G(\lambda) = \int_G \Phi_{\xi,\lambda}^{m_o,n_o}(x)\, f(x) dx, \quad \lambda \in \mathcal{A}^*. \tag{3.2.9}$$

Then $G(\lambda) = \langle \pi_{\xi,\lambda}(f) e_{m_o}^{\xi}, e_{n_o}^{\xi} \rangle_\xi$ for $\lambda \in \mathcal{A}^*$. Also observe that as f decays very rapidly (3.2.1), the analyticity of $\lambda \mapsto \Phi_{\xi,\lambda}^{m_o,n_o}(x)$ on $\mathcal{A}_{\mathbb{C}}^*(\cong \mathbb{C}^l)$ for each fixed $x \in G$, the estimate (3.2.8) together with (3.2.5) will imply that the integral defining the function $G(\lambda)$ makes sense for $\lambda \in \mathcal{A}_{\mathbb{C}}^*$ and in fact defines an entire function. Moreover, for $\lambda = \lambda_R + i\lambda_I \in \mathcal{A}_{\mathbb{C}}^*$,

$$| G(\lambda) | \leq \int_G | f(x) | \, | \Phi_{\xi,\lambda}^{m_o,n_o}(x) | \; dx. \tag{3.2.10}$$

Now using polar coordinates, (3.2.8) and the fact that if $x = k_1 a k_2$, $\|x\|_G = \|a\|_G$, the integral on the right hand side is majorized by $Const. \int_{A^+} e^{-\alpha\|a\|_G^2} e^{\lambda_I^\dagger(\log a)} \mid J(a) \mid da$, where da denotes the Haar measure on A. If $H \in \mathcal{A}$ is the unique element such that $\exp H = a$ and dH denotes the Lebesgue measure on $\mathcal{A}$, then it can be easily seen that there exists a constant C such that $\mid J(a) \mid \leq Const.\ e^{C\|H\|}$ and the integral on the right hand side is majorized by $Const.\ \int_{\mathcal{A}} e^{-\alpha\|H\|^2} e^{\lambda_I^\dagger(H)} e^{C\|H\|} dH$, where now $\|\cdot\|$ is the norm on $\mathcal{A}$ induced by the Cartan-Killing form.

Now let H_{λ_I} be the unique element of $\mathcal{A}$ such that $\lambda_I^\dagger(H) = \langle H, H_{\lambda_I} \rangle$ for all H, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{A}$ induced by the Cartan-Killing form.

Then there exists $0 < \alpha' < \alpha$ such that we continue to have $\alpha'\beta > \frac{1}{4}$ and $e^{-\alpha\|H\|^2 + C\|H\|} \leq Const.e^{-\alpha'\|H\|^2}$. So :

$$\begin{aligned}
\int_{\mathcal{A}} e^{-\alpha\|H\|^2} e^{\lambda_I^{\dagger}(H)} e^{C\|H\|} dH &= \int_{\mathcal{A}} e^{-\alpha\|H\|^2 + (H, H_{\lambda_I}) + C\|H\|} dH \\
&\leq Const. \int_{\mathcal{A}} e^{-\alpha'\|H\|^2 + (H, H_{\lambda_I})} dH \\
&= Const.e^{\frac{1}{4\alpha'}\|H_{\lambda_I}\|^2} \int_{\mathcal{A}} e^{-\alpha'(H - \frac{1}{2\alpha'}H_{\lambda_I}, H - \frac{1}{2\alpha'}H_{\lambda_I})} dH \\
&= Const.e^{\frac{1}{4\alpha'}\|H_{\lambda_I}\|^2} \int_{\mathcal{A}} e^{-\alpha'\|H\|^2} dH
\end{aligned}$$

(by translation invariance of Lebesgue measure). But by the choice of inner product on $\mathcal{A}^*$, $\|H_{\lambda_I}\| = \|\lambda_I^{\dagger}\|$. Further the action of the Weyl group preserves the norm on $\mathcal{A}^*$ and hence $\|\lambda_I^{\dagger}\| = \|\lambda_I\| \leq \|\lambda\|$. So finally we get the estimate

$$| G(\lambda) | \leq Ce^{\frac{1}{4\alpha'}\|\lambda\|^2}, \quad \lambda \in \mathcal{A}^*_{\mathbb{C}} \cong \mathbb{C}^l$$

for some constant C. But for $\lambda \in \mathcal{A}^*$ by (3.2.2),

$$| G(\lambda) | \leq C_{\xi} e^{-\beta\|\lambda\|^2}.$$

Since $\alpha'\beta > \frac{1}{4}$, $-\beta < -\frac{1}{4\alpha'}$ and so we have $| G(\lambda) | \leq Ce^{\frac{1}{4\alpha'}\|\lambda\|^2}$ for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ and $| G(\lambda) | \leq C_{\xi} e^{-\frac{1}{4\alpha'}\|\lambda\|^2}$ for $\lambda \in \mathcal{A}^*$. So by Lemma 3.1.1, we have $G(\lambda) = Const.e^{-\frac{1}{4\alpha'}\|\lambda\|^2}$, $\lambda \in \mathcal{A}^*$. Therefore we have for $\lambda \in \mathcal{A}^*$,

$$| Const.e^{-\frac{1}{4\alpha'}\|\lambda\|^2} | = | G(\lambda) | \leq C_{\xi} e^{-\beta\|\lambda\|^2}.$$

But $\beta - \frac{1}{4\alpha'} > 0$ and hence we would have

$$| Const.e^{(\beta - \frac{1}{4\alpha'})\|\lambda\|^2} | \leq C_{\xi}, \quad \lambda \in \mathcal{A}^*$$

and this is impossible unless the constant on the left hand side is zero i.e. $G(\lambda) \equiv 0$ i.e. for arbitrary $\xi \in \widehat{M}$, $m_o, n_o \in N$, $\langle \pi_{\xi,\lambda}(f)e^{\xi}_{m_o}, e^{\xi}_{n_o}\rangle_{\xi} \equiv 0$ as a function of λ. Hence it follows that $\pi_{\xi,\lambda}(f) \equiv 0$ on $\widehat{M} \times \mathcal{A}^*$ and since the Plancherel measure is supported on $\widehat{M} \times \mathcal{A}^*$, it follows that $f = 0$ a.e.

3.3 Arbitrary semi-simple Lie groups

We continue to retain the notation introduced in section 3.1. In this section, G will denote an arbitrary connected noncompact semi-simple Lie group with finite centre i.e. we drop the assumption that G has only one conjugacy class of Cartan subgroups. Instead, we impose some restrictions on the kind of functions being considered; we will consider only right K-invariant functions. For the harmonic analysis of such functions, only the class-1 principal series representations are relevant. Let $\{\pi_\lambda\}_{\lambda \in \mathcal{A}^*}$ denote the class- 1 principal series representations of G(i.e. $\pi_\lambda = \pi_{1,\lambda}$ where 1 is the trivial representation of M). These can all be realized on $L^2(K/M)$. Let v_o be the constant function 1 on K/M i.e. v_o is the essentially unique K-fixed vector in $L^2(K/M)$ for the representation π_λ. Then one knows that if v is any other K-finite vector in $L^2(K/M)$ which is <u>not</u> a multiple of v_o, then $\pi_\lambda(f)v = 0$. Thus $\pi_\lambda(f)$ is completely determined by $\pi_\lambda(f)v_o$ and moreover $\|\pi_\lambda(f)\|_{HS} = \|\pi_\lambda(f)v_o\|$, where $\|\cdot\|$ denotes the usual norm in $L^2(K/M)$. Thus the group theoretic Fourier transform can be thought of as a function on $\mathcal{A}^*$ alone, taking values in the Hilbert space $L^2(K/M)$. Keeping these considerations in mind, an examination of the proof of Theorem 3.2.1 immediately yields the following result ([31]) :

Theorem 3.3.1 *Suppose f is a measurable right K-invariant function on G (i.e. $f(xk) = f(x)$, $x \in G$, $k \in K$), satisfying the following estimates for some positive constants C, α and β :*

$$| f(x) | \leq Ce^{-\alpha\|x\|_G^2}, \qquad x \in G,$$

$$\|\pi_\lambda(f)v_o\| \leq Ce^{-\beta\|\lambda\|^2}, \qquad \lambda \in \mathcal{A}^*.$$

If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

(One can view the above as a theorem about functions on G/K, which is a symmetric space of the noncompact type; the group theoretic Fourier transform can be reinterpreted as the Fourier transform on the symmetric space, as introduced by Helgason(-see [22]). A brief discussion from this point of view can be found in [28]. A sketch of the proof of a special case of Theorem 3.2.1 can also be found in [28].)

3.4 Further remarks

3.4.1 $SL(2, \mathbb{R})$

Thus in section 3.2, we have established an analogue of Hardy's theorem for a class of semi-simple Lie groups which include all complex groups and real rank - 1 groups without Discrete Series representations. However we would like to conjecture that a result of a similar nature is valid for all noncompact semisimple Lie groups. For instance, in the case when $G = SL(2, \boldsymbol{R})$, we shall prove the exact analogue of Theorem 3.2.1. For this we need to recall some facts from the representation theory of $SL(2, \boldsymbol{R})$. The reader can find the details of the material covered in this section in [10], [27] and [35].

In this section, we shall continue to use the notation introduced in section 3.1 except that the norm defined on $\mathcal{A}^*$ in section 3.1 is denoted by $\| \cdot \|_{\mathcal{A}^*}$.

For the time being, let $G = SL(2, \boldsymbol{R})$. Then $K = SO(2)(\simeq S^1)$, $\mathcal{A} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \boldsymbol{R} \right\}$ and $M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Therefore $A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \boldsymbol{R} \right\}$. The polar decomposition of an element $g \in G$ can be written as $g = k_{\theta_1} a_t k_{\theta_2}$ where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. In this case, there are only two irreducible representations of M. Corresponding to the two irreducible representations of M, one gets two sets of principal series representations denoted by $\pi_{1,\lambda}$, $\pi_{-1,\lambda}$ of G, defined exactly as in section 3.1. Each set of principal series representations is parametrized by $\mathcal{A}^* \simeq \boldsymbol{R}$. The series $\pi_{1,\lambda}$ is irreducible for all $\lambda \in \boldsymbol{R}$ and $\pi_{-1,\lambda}$ is irreducible for all $\lambda \in \boldsymbol{R} \backslash \{0\}$. There is another set of irreducible, unitary representations of G called the Discrete series. For each $n \in Z$, $| n | \geq 2$, denote by D_n the corresponding discrete series representation of G. For details about the K-module structure and the spaces on which these representations are realized, see [27], [35] etc. (Apart from these, there is another collection of irreducible, unitary representations of G, called the complementary series, which do not play a role in the Plancherel measure.) The Plancherel measure μ is supported on the set of principal and discrete series representations (-see [27] for details).

For $n \in Z$, define χ_n on $SO(2)(\simeq S^1)$ by $\chi_n(k_\theta) = e^{in\theta}$. Let π be an irreducible unitary representation of G on the Hilbert space $\boldsymbol{H}_\pi$. If $0 \neq v \in \boldsymbol{H}_\pi$ is such that $\pi(k_\theta)v = \chi_l(k_\theta)v$, $l \in Z$, $k_\theta \in K$, then we say "v transforms according to χ_l". If such a non-zero v exists then it is unique upto scalar multiplication (-see [10]) and we say that "χ_l occurs in π". If m and n are fixed integers such that χ_m and χ_n occur in π, let v_m, v_n be the essentially unique unit vectors transforming according to χ_m and χ_n respectively. Denote by $\Phi_\pi^{m,n}$ the function defined by $\Phi_\pi^{m,n}(g) = \langle \pi(g)v_n, v_m\rangle$ - this is the "elementary spherical function of type (m,n) corresponding to π". Here $\langle\cdot,\cdot\rangle$ denotes the inner product on $\boldsymbol{H}_\pi$. If D_l is a Discrete series representation of G and χ_m, χ_n occur in D_l, then explicit formulae are available for $\Phi_{D_l}^{m,n}$(-see [10] and [35]). Denote $\Phi_{D_l}^{m,n}$ by $\Phi_l^{m,n}$. It turns out that for $a_t \in A$, $\Phi_l^{m,n}(a_t)$ is a rational function of e^{-t} such that as $t \to \infty$, $\Phi_l^{m,n}(a_t) \to 0$. (Note that $e^{-t} \to 0$ as $t \to \infty$!)

Now we are in a position to state and prove the exact analogue of Theorem 3.2.1 for $G = SL(2, \boldsymbol{R})$ ([31]):

Theorem 3.4.1 *Suppose f is a measurable function on G satisfying the following estimates :*

$$| f(x) | \leq Ce^{-\alpha\|x\|_G^2}, \quad x \in G, \qquad (3.4.1)$$

$$\|\pi_{\xi,\lambda}(f)\|_{HS} \leq Ce^{-\beta\|\lambda\|_{\mathcal{A}^*}^2}, \; \xi \in \widehat{M}, \; \lambda \in \mathcal{A}^*, \qquad (3.4.2)$$

where C, α, β are positive constants. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

Proof : Any $f \in L^1(G)$ can be written (in the sense of distributions) as $f \simeq \sum_{m,n \in Z} \chi_m * f * \chi_n = \sum_{m,n \in Z} f_{mn}$. Note that each f_{mn} has the property that

$$f_{mn}(k_{\theta_1} g k_{\theta_2}) = \chi_m(k_{\theta_1}) f(g) \chi_n(k_{\theta_2}), \qquad (3.4.3)$$

k_{θ_1}, $k_{\theta_2} \in K$. One can easily show that if f satisfies estimates (3.4.1) and (3.4.2), then so does each f_{mn}. Clearly, if each $f_{mn} = 0$, then so is f. Hence, without loss of generality, assume that f satisfies (3.4.3) for some fixed m, $n \in Z$. Then exactly as in the proof of Theorem 3.2.1, it will follow that $\pi_{1,\lambda}(f) \equiv 0$ and $\pi_{-1,\lambda}(f) \equiv 0$ as functions of λ. Thus by the inversion formula(-see [27] and [35]) for functions that satisfy (3.4.3), it follows that f is a linear combination of elementary spherical functions of type (m, n) of finitely many Discrete series representations.(Note that for fixed m, $n \in Z$, there are only finitely many Discrete series representations in which χ_m and χ_n occur.) Since each $\Phi_l^{m,n}$ evaluated at $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in G$,

$t > 0$, is a rational function of e^{-t}, it follows that $f(a_t)$ is a rational function of e^{-t}. Also as noted, when $t \to \infty$ i.e. $e^{-t} \to 0$, each $\Phi_l^{m,n} \to 0$. So $f(a_t) \to 0$ as $e^{-t} \to 0$.

Suppose $f(a_t) \not\equiv 0$ as a function of t. We will arrive at a contradiction. Since $f(a_t)$ is a rational function of e^{-t}, $f(a_t) = (e^{-t})^l g(e^{-t})$ for some positive integer l, where $g(e^{-t})$ is also a rational function e^{-t} and coverges to a finite non zero limit γ as $t \to \infty$ (i.e. $e^{-t} \to 0$). On the other hand

$$| f(a_t) | \leq Ce^{-\alpha' t^2},$$

where α' is a positive constant depending on α and the way the norm is defined on $\mathcal{A}$. Hence we would have

$$| (e^{-t})^l g(e^{-t}) | \leq Ce^{-\alpha' t^2}$$

as $t \to \infty$. But since $g(e^{-t}) \to \gamma$, and γ is non zero, this clearly leads to a contradiction. This completes the proof of the theorem.

3.4.2 The sharpness of the constant $\frac{1}{4}$

For the group $G = SL(2,\mathbb{C})$, using the normalizations in this chapter, we will show that $\frac{1}{4}$ is the best possible constant ([31]). First, we recall a couple of facts. If f is an L^1-function invariant under the right action of K, then $\pi_{\xi,\lambda}(f) = 0$ unless ξ is the trivial representation of M. Thus, for such functions, it is enough to consider $\{\pi_{1,\lambda}\}_{\lambda \in \mathcal{A}^*}$. As before, denote $\pi_{1,\lambda}$ by π_λ. Now let v_o be the essentially unique

K-fixed vector in $L^2(K/_M)$ for the representation π_λ.(v_o is the constant function 1 on $K/_M$.) Then, as observed in section 3.3, for such f, $\|\pi_\lambda(f)\|_{HS} = \|\pi_\lambda(f)v_o\|$, where $\|\cdot\|$ denotes the usual norm in $L^2(K/_M)$. Further, if f is also left K-invariant i.e. f is K-biinvariant, then $\pi_\lambda(f)v_o = \langle\pi_\lambda(f)v_o, v_o\rangle v_o$ and hence $\|\pi_\lambda(f)\|_{HS} = |\langle\pi_\lambda(f)v_o, v_o\rangle|$. Now $\langle\pi_\lambda(f)v_o, v_o\rangle = \int_G \langle\pi_\lambda(x)v_o, v_o\rangle f(x)dx$, where dx is the Haar measure on G. So, as before, if we denote the function $x \mapsto \langle\pi_\lambda(x)v_o, v_o\rangle$ by $\Phi_\lambda(x)$, we need to consider only the integral $\int_G f(x)\Phi_\lambda(x)dx$. The collection $\{\Phi_\lambda\}_{\lambda \in \mathcal{A}^*}$ form a subset of the set of 'elementary spherical functions' and we are actually looking at the 'spherical Fourier transform' of f. So let $g(\lambda) = \int_G f(x)\Phi_\lambda(x)dx$. Since f is K-biinvariant, f is completely determined by its restriction to A. Thus to prove our assertion that $\frac{1}{4}$ is the best possible constant, it is enough to produce a function f which is

(a) K-biinvariant

(b) for every $\epsilon > 0$, $| f(a) | \leq C_\epsilon e^{-(\frac{1}{16}-\epsilon)\|a\|_G^2}$, $a \in A$, and

(c) $| g(\lambda) | = e^{-4\|\lambda\|^2_{\mathcal{A}^*}}$, where $\|\cdot\|_{\mathcal{A}^*}$ is the norm on $\mathcal{A}^*$ induced by the Killing form.

Each $\lambda \in \boldsymbol{R}$ can be identified with an element in $\mathcal{A}^*$ via the identification

$$\mathcal{A} \ni \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mapsto \lambda x.$$

With this identification the elementary spherical functions are given by

$$\Phi_\lambda(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}) = \frac{2\sin\lambda t}{\lambda\sinh 2t}$$

(-see [39], Vol. 2, pp. 313-314). Also $\|\lambda\|_{\mathcal{A}^*} = \frac{|\lambda|}{4}$ and $\| \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \|_G = 4 \mid t \mid$.

Let $g(\lambda) = e^{-\frac{\lambda^2}{4}}$, $\lambda \in \boldsymbol{R} \simeq \mathcal{A}^*$. In view of the "rapid decay" of g, by appealing to the Trombi - Varadarajan theorem ([43]), there exists a unique K-biinvariant f such that the spherical Fourier transform of f is precisely g.

Since in this case (i.e. $G = SL(2, \mathbb{C})$), the Plancherel formula and the inversion formula can be explicitly written down(-see [39], Vol. 2, pp.313-314), we have

$$\begin{aligned} f(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}) &= Const. \int_{\boldsymbol{R}} g(\lambda) \frac{2\sin\lambda t}{\lambda\sinh 2t} \mid \lambda \mid^2 d\lambda \\ &= \frac{Const.}{\sinh 2t} \int_{\boldsymbol{R}} \lambda e^{-\frac{\lambda^2}{4}} \sin\lambda t d\lambda \end{aligned}$$

which is equal to $Const. \frac{te^{-t^2}}{\sinh 2t}$, using routine Euclidean Fourier transform calculations. Clearly, $\mid f(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}) \mid = Const. \frac{t}{\sinh 2t} e^{-\frac{\|a_t\|_G^2}{16}} \leq C_\epsilon e^{-(\frac{1}{16}-\epsilon)\|a_t\|_G^2}$, for each $\epsilon > 0$ and $\mid g(\lambda) \mid = e^{-4\|\lambda\|^2_{\mathcal{A}^*}}$ where $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Thus the fact that $\frac{1}{4}$ is the best possible constant has been established.

Concluding Remarks

In this thesis, we have mainly focussed on one particular uncertainty principle - namely Hardy's theorem for Fourier transform pairs. For various uncertainty principles and their generalizations see [1], [5], [7], [36] etc. The investigations in [28], [33] and [36] also show that uncertainty principles are not just associated with Fourier transform pairs, but actually with more general eigenfunction expansions. However, in this thesis, we have restricted ourselves to the group Fourier transform.

Finally, we conclude with a couple of open problems :

(a) Formulate and prove an analogue of Hardy's theorem for all simply connected nilpotent Lie groups.

(It is possible to show that an analogue of the theorem is valid for a sub class of simply connected two-step nilpotent Lie groups, called H-type groups [38].)

(b) Formulate and prove an analogue of Hardy's theorem for all connected non-compact semi-simple Lie groups with finite centre.

(The results of Chapter 3 strongly suggest that such a result should be true.)

Bibliography

[1] Beckner W, *Pitt's inequality and the uncertainty principle*, Proc. Amer. Math. Soc., 123(1995), 1897-1905.

[2] Benedicks M, *On Fourier transforms of functions supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl. 106(1985), 180-183.

[3] Cowling M, Price J, *Generalisations of Heisenberg's inequality*, Harmonic analysis, Proceedings, Cortona, Italy(1982), Mauceri G, Ricci F and Weiss G, Eds., Lecture notes No.992, Springer-Verlag, 1983.

[4] Cowling M, Price J and Sitaram A, *A qualitative uncertainty principle for semi-simple Lie groups*, J. Australian Math. Soc. (Series A) 45(1988), 127-132.

[5] De Jeu M F E, *An uncertainty principle for integral operators*, J. Func. Anal. 122(1994), 247-253.

[6] Diximier J, *C^* - algebras*, North-Holland Mathematical Library, 1982.

[7] Donoho D L and Stark P B, *Uncertainty principle and signal recovery*, SIAM J. Appl. Math. 49(1989), 906-931.

[8] Dym H and McKean H P, *Fourier series and integrals*, Academic Press, New York, 1972.

[9] Echteroff S, Kaniuth E and Kumar A, *A qualitative uncertainty principle for certain locally compact groups*, Forum Math. 3(1991), 355-369.

[10] Fell J M G, *The representation theory of the real* 2×2 *unimodular group*, Unpublished lecture notes.

[11] Folland G B, *Introduction to partial differential equations*, Mathematical notes No. 17, Princeton Univ. Press, Princeton.

[12] Folland G B, *Harmonic analysis in phase space*, Annals of Mathematics Studies, No. 122, Princeton Univ. Press, Princeton, 1989.

[13] Folland G B, *Abstract harmonic analysis*, Unpublished lecture notes.

[14] Gangolli R, *Spherical functions on semisimple Lie groups in "Symmetric Spaces"*,(Boothby W and Weiss G, Eds.), 41-92, Marcel - Dekker, New York, 1972.

[15] Gangolli R and Varadarajan V S, *Harmonic analysis of spherical functions on real reductive groups*, Springer-Verlag, 1988.

[16] Gross K I and Kunze R A, *Fourier decompositions of certain representations in "Symmetric Spaces"*,(Boothby W and Weiss G, Eds.), Marcel-Dekker, New York, 1972, 119-139.

[17] Halmos P R and Sunder V S, *Bounded integral operators on L^2 spaces*, Springer-Verlag, 1978.

[18] Hardy G H, *A theorem concerning Fourier transforms*, Journal London Math. Soc. 8(1933), 227-231.

[19] Harish Chandra, *The Plancherel formula for complex semi-simple Lie groups*, Trans. Amer. Math. Soc. 76(1954), 485-528.

[20] Helgason S, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.

[21] Helgason S,. *The Radon transform*, Progress in Mathematics, No. 5, Birkhäuser, 1980.

[22] Helgason S, *Groups and geometric analysis* , Academic Press, 1984.

[23] Hewitt and Ross, *Abstract harmonic analysis*, Vol. 1, Springer-Verlag, 1979.

[24] Hogan J A, *A qualitative uncertainty principle for unimodular groups of type I*, Trans. AMS, 340, No. 2, 1993, 587-594.

[25] Hörmander L, *A uniqueness theorem of Beurling for Fourier transform pairs*, Arkiv för Matematik, 29(1991), No. 2, 237-240.

[26] Knapp A W, *Representation theory of semi-simple Lie groups, an overview based on examples*, Princeton University Press, Princeton, 1986.

[27] Lang S, $SL_2(\mathbb{R})$, Springer-Verlag, 1985.

[28] Pati V, Sitaram A, Sundari M and Thangavelu S, *An uncertainty principle for eigenfunction expansions*, to appear in J. Fourier Anal. and Appl.

[29] Price J and Sitaram A, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups*, J. Func. Anal. 79(1988), 166-182.

[30] Price J and Sitaram A, *Local uncertainty inequalities for locally compact groups*, Trans. AMS, 308(1988), 105-114.

[31] Sitaram A and Sundari M, *An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups*, to appear in Pacific J. Math.

[32] Sitaram A, Sundari M and Thangavelu S, *Uncertainty principles on certain Lie groups*, Proc. Indian Acad. Sci.(Math. Sci.), 105(1995), No. 2, 135-151.

[33] Strichartz R S, *Uncertainty principles in harmonic analysis*, J.Func.Anal., 84(1989), 97-114.

[34] Strichartz R S, *A guide to distribution theory and Fourier transforms*, Studies in advanced mathematics, CRC Press, 1994.

[35] Sugiura M, *Unitary representations and harmonic analysis, an introduction*, Kodansha scientific books,Tokyo, 1975.

[36] Sun L, *An uncertainty principle on hyperbolic space*, Proc. Amer. Math. Soc. 121(1994), 471-479.

[37] Sundari M, *Hardy's theorem for the n-dimensional Euclidean motion group*, to appear in Proc. Amer. Math. Soc.

[38] Sundari M, *An uncertainty principle for H-type groups*, In preparation.

[39] Terras A, *Harmonic analysis on symmetric spaces and applications*, Vol.2, Springer-Verlag, 1988.

[40] Thangavelu S, *Some uncertainty inequalities*, Proc. Indian Acad. Sci., 100, No. 2(1990), 137-145.

[41] Thangavelu S, *Lectures on Hermite and Laguerre expansions*, Math. Notes. No. 42, Princeton Univ. Press, Princeton, 1993.

[42] Titchmarsh E C, *Introduction to the theory of Fourier integrals*, Chelsea Publishing Company, New York, N.Y.,1986.

[43] Trombi P C and Varadarajan V S, *Spherical transforms on semi-simple Lie groups*, Ann.Math.94(1971), 246-303.

[44] Warner G, *Harmonic analysis on semi-simple Lie groups*, Vol.2, Springer-Verlag, 1972.

[45] Whittaker E T and Watson G N, *A course of modern analysis*, Cambridge Univ. Press, 1965.