ON LOCALLY MOST POWERFUL TESTS WHEN ALTERNATIVES ARE ONE SIDED

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The concept of a most powerful test with respect to a single admissable alternative to a specified hypothesis H_0 is due to Neyman and Pearson who, in an extensive series of articles, have advanced a general theory of testing ofhypotheses. A test is said to be uniformly most powerful if it is most powerful with respect to every admissable alternative. The rarity of such tests lead Neyman and Pearson to consider tests giving maximum power for alternatives in the neighbourhood of the specified hypothesis (the hypothesis being assumed measurable). Following their ideas we propose to give here tests which are locally powerful with respect to alternatives on one side only. This problem arose in connection with certain investigations that are being carried on by the authors where the altrnatives on one side alone are taken into account.

Denoting the probability density of a set of variables by $p(x, \theta)$ and the specified value of the parameter θ by θ_0 we define a region w of the sample space as locally most powerful for one-sided alternatives if

$$(i) \quad \int_w p(x, \theta_0)\ \pi dx = \alpha \text{ (level of significance)}$$

$$(ii) \quad c\int_w p'(x, \theta_0)\ \pi dx \quad \text{is maximum}$$

are satisfied where c is + 1 or − 1 according as the alternatives are $\theta > \theta_0$ or $\theta < \theta_0$. The following theorem gives the method of determining such regions.

Theorem : *If whatever may be the region w in the sample space the integral $\int_w p'(x,\theta_0)\ \pi dx$ exists, then the region w_0 within which $cp'(x,\theta_0) \geqslant \lambda p(x,\theta_0)$ and outside which $cp'(x,\theta_0) \leqslant \lambda p(x,\theta_0)$ where λ is so determined that $\int_w p(x,\theta_0) = \alpha$ is locally most powerful for alternatives $\theta > \theta_0$* and *$\theta < \theta_0$ according as c is* + 1 *or* − 1.

The proof is a direct consequence of a lemma due to Neyman and Pearson (on page 11 in *Research Memoirs* Vol 1. (1936). The boundary of the above region consists of the surfaces of constant values of $p'(x,\theta_0)/p(x,\theta_0) = \phi'(x,\theta_0))$ where ϕ is the *likelihood* defined by Fisher. We need only find the distribution of ϕ and/or tabulate values such that the probability of $\phi'(x, \theta_0)$ (*i*) exceeding them is α for alternatives on one side and (*ii*) exceeded by them is α for alternatives on the other side. On the other hand if the surfaces of constant ϕ' coincide with surfaces of constant values of a statistic T, then tabulated values of T may be used in tests of significance.

An important appliction of this result when the size of the sample is large emerges from the following theorem.

Theorem : *If (i) $x_1, x_2, \ldots x_n$ form* n *independent observations from a probability distribution, (ii) the information (as defined by Fisher) of a single observation is finite and indicated by I, then, to test the hypothesis that the value of a parameter θ, occurring in the probability distribution, is θ_0 the statistic $\phi'(x, \theta_0)/\sqrt{In}$ where ϕ is the liklihood of the sample, can for sufficiently large n be used as a normal deviate and in judging significance only one tail of the normal curve has to be considered when the alternatives are one-sided.*

The proof follows from the fact that $\phi(x, \theta_0)$ is approximately normally distributed due to central limit thereom and E [(x, θ_0)] = o and V [(x, θ)] = n I.

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