On Some Generalizations of The Linear Complementarity Problem

Samir Kumar Neogy

Indian Statistical Institute
7, S. J. S. Sansanwal Marg
New Delhi-110016
India

April 1996

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Samir Kumar Neogy
Indian Statistical Institute
New Delhi-16, India

Thesis submitted to Indian Statistical Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy

September 1995

Preface

This dissertation is devoted to the study of some generalizations of the linear complementarity problem. Given a square matrix of order n with real entries and an n dimensional vector q the linear complementarity problem (LCP) is to find an n vector z such that $Mz + q \geq 0$, $z \geq 0$ and $z^t(Mz + q) = 0$. Since several problems in optimization and engineering can be posed as LCP's, the theory of the LCP has a wide range of applications in applied science and technology. This problem also allows a number of interesting generalizations. This dissertation deals mainly with two kinds of generalizations namely, vertical linear complementarity problem (VLCP) and horizontal linear complementarity problem (HLCP). Among these generalizations most of the work pertains to the study of VLCP. In 1970, Cottle and Dantzig presented a generalization of LCP (abbreviated as VLCP in recent literature) where the matrix M is not a square matrix. This new problem is more flexible and has the potential to handle many real life problems. Lemke formulates the VLCP as an LCP and points out that there will be valuable applications in future. Cottle, Pang and Stone observe that "Rather little has been done with this model." Recently a number of applications have been reported in the literature and this has generated a lot of interest among the researchers in studying VLCP. Similarly, a lot of attention is being paid to the study of HLCP, particularly in the literature on interior point algorithms. HLCP arises in some problems of electrical engineering, structural engineering, inventory control and structural mechanics. A brief outline of the contents are presented in a chapterwise summary. Chapter 1 is introductory in nature. Here we present the required definitions and introduce the notations used in this dissertation. We also include a survey of the results from the literature which will be useful for presenting our results. In this chapter, it is also shown that Scarf's complementarity problem can be posed as a VLCP and hence as an LCP. We introduce a generalization of polymatrix game, namely *the polystochastic game* in which the criterion is the total expected discounted cost and the transition probabilities depend only on the actions of a single player and formulate the problem of finding equilibrium set of strategies and corresponding costs as a linear complementarity problem. We also examine its processability by Lemke's algorithm.

In Chapter 2, we consider several issues related to the vertical linear complementarity problem VLCP(q, A) where A is a vertical block matrix. The issues considered are listed here. We present an equivalent formulation of VLCP(q, A) as LCP(q, M) and extend some known results. The classes of matrices introduced by Garcia and Todd are generalized in the VLCP setting. The notion of generalized principal pivot transform of a vertical block matrix is introduced in the context of VLCP. A comparison of the algorithms by Cottle and Dantzig for VLCP and that of Lemke for the equivalent LCP introduced by us is provided. We consider the problem of computing the generalized Nash equilibrium point of the generalized bimatrix game introduced by Gowda and Sznajder. An alternative approach is presented for calculating the VLCP degree of a vertical block matrix. Representative submatrices have played an important role in the study of some classes of vertical linear complementarity problems. In Chapter 3, we present a slightly different approach to the study of such problems by using equivalent linear complementarity problems. It is shown that the conditions and results obtained in terms of representative submatrices for these classes of problems can be rephrased in terms of certain nontrivial submatrices of their equivalent square matrices. This approach offers a new insight and yields a new characterization result for a vertical block P-matrix. Chapter 4 deals with the algorithmic aspect of the VLCP with a vertical block Z-matrix. We prove that the pivoting algorithm of Cottle and Dantzig can process this problem by showing that this algorithm generates the same sequence of bases as does a *modified* simplex algorithm for minimizing the artificial variable z_0. We also show that a modified version of the Cottle-Dantzig algorithm can be used for determining whether a given vertical block Z-matrix is a vertical block P_0-matrix or not. In Chapter 5, we introduce the class of vertical block hidden Z-matrices and study the least element and complementarity property possessed by vertical block hidden Z-matrices. Some characterization theorems for vertical block hidden $\mathcal{K}$-matrices are also presented. We introduce a generalization of the polymatrix game (a nonzero-sum noncooperative n-person game) considered by Janovskaya and Howson in Chapter 6 and relate the problem of computing an equilibrium set of strategies for such a game to the vertical linear complementarity problem. For an even more general version of the game we prove the existence of an ε-equilibrium set of strategies. We also present a result on the stability of the equilibria based on degree theory. In Chapter 7, we consider the horizontal linear complementarity problem and provide an equivalent LCP formulation of the problem. We also prove some results using this equivalent LCP matrix.

Numbering

For internal referencing Section j in Chapter i is denoted by $i.j$ and $i.j.k$ is used to refer Item k of Section j in Chapter i. For example, the triple 2.3.5 refers to Item 5 in Section 3 of Chapter 2. All items (e.g., Lemma, Theorem, Example, Remark etc.) are identified in this fashion. For equation $(i.j.k)$ is used to refer Equation k in Section j in Chapter i. We use brackets [ ] for a bibliographical reference.

Notations at a Glance

The special notations pertaining to a particular chapter are provided in Section 2 of each chapter. The most frequently used notations are given below:

Spaces

R^n	real n-dimensional space
R	the real line
$R^{m\times k}$	the space of $m \times k$ real matrices

Vectors

x^t	the transpose of a vector x
e_m	an m-dimensional vector of all ones (m is sometimes omitted)
e^j	a vector of dimension m_j each of whose coordinates is 1 (for notational convenience we use the symbol e^j to denote e_{m_j})
$x^t y$	the standard inner product of vectors in R^n
$x \geq y$	$x_i \geq y_i,\ i = 1, \ldots, n$
$x > y$	$x_i > y_i,\ i = 1, \ldots, n$
$\min(x, y)$	the vector whose i^{th} component is $\min(x_i, y_i)$
$\max(x, y)$	the vector whose i^{th} component is $\max(x_i, y_i)$
$\|x\|$	the vector whose i^{th} component is $\|x_i\|$
x^+	$\max(0, x)$, the nonnegative part of a vector x
x^-	$\max(0, -x)$, the nonpositive part of a vector x

Sets

α	subset of $\{1, \ldots, m\}$
$\bar{\alpha}$	complement of an index set α
$\|\alpha\|$	cardinality of a finite set α
$\mathrm{Pos}(A)$	the nonnegative convex cone generated by the matrix A
	$= \{x \mid \exists\, y \geq 0,\ x = Ay\}$
J_1	see 1.2
J_r	see 1.2

Matrices

$A = ((a_{ij}))$	a matrix with real entries a_{ij}
$\det(M)$	the determinant of a square matrix M
M^{-1}	the inverse of a matrix M
A^t	the transpose of a matrix A
$A \leq B$	$a_{ij} \leq b_{ij}$ for all i and j
$A < B$	$a_{ij} < b_{ij}$ for all i and j
I	the identity matrix
$A_{\alpha\beta}$	submatrix formed by the rows and columns of A
	whose indices are in α and β, respectively
$A_{\alpha\cdot}$	submatrix formed by the rows of A
	whose indices are in α
$A_{\cdot\alpha}$	submatrix formed by the columns of A
	whose indices are in α

Miscellaneous Symbols

LCP(q, M)	the LCP with data (q, M)
VLCP(q, A)	the VLCP with data (q, A)
HLCP(q, A, B)	the HLCP with data (q, A, B)
FEA(q, M)	the feasible region of LCP(q, M)
S(q, M)	the solution set of LCP(q, M)
$K(M)$	the set of all q for which $S(q, M) \neq \emptyset$

Acknowledgements

I am indebted to Dr. S. R. Mohan, my thesis supervisor for introducing me to the field of LCP and also for his excellent guidance and encouragement to carry out this work.

My sincere thanks are due to Dr. T. Parthasarathy (a coauthor of one of my papers) for his helpful comments about the work. I express my thanks to Dr. R. Sridhar who is also a coauthor of one of my papers.

I also want to thank all the referees for their suggestions about my papers leading towards considerable improvement in the presentation of this work.

Finally, I would like to thank my colleagues in ISI for their cooperation.

S. K. Neogy

New Delhi, September 1995

Contents

Chapter 1

General Introduction and Review

1.1 Introduction

The *linear complementarity problem* can be stated as follows:

Given a square matrix M of order n with real entries and an n dimensional vector q, find n dimensional vectors w and z satisfying

$$w - Mz = q, \quad w \geq 0, \; z \geq 0 \tag{1.1.1}$$

$$w^t z = 0. \tag{1.1.2}$$

If a pair of vectors (w, z) satisfies (1.1.1), then the problem LCP(q, M) is said to have a feasible solution. A pair (w, z) of vectors satisfying (1.1.1) and (1.1.2) is called a solution to the LCP(q, M). This problem is denoted as LCP(q, M). The problem has undergone several name changes, from composite problem, to fundamental problem, to complementary pivot problem. The current name *linear complementarity problem* was proposed by Cottle [5, p. 37]. LCP is normally identified as a problem of mathematical programming and provides a unifying framework for several optimization problems. More specifically, the problems which can be posed as an LCP includes linear programming, linear fractional programming, convex quadratic programming and the bimatrix game problem. It is well studied in the literature on mathematical programming and arises in a number of applications in operations research [30], mathematical economics [72], geometry and engineering ([7],[21] and [73]). The algorithm presented by Lemke and Howson [44] to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke [43] to solve an $LCP(q, M)$ contributed significantly to the development of the linear complementarity theory. However, this algorithm does not solve every instance of the linear complementarity problem and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists. For recent books on this problem and applications see Cottle, Pang and Stone [5] and Murty [63].

1.2 Notations

We denote the n-dimensional real Euclidean space by R^n. We consider matrices and vectors with real entries. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^t denotes the row transpose of x. $I_{.j}$ denotes the vector whose j^{th} coordinate is 1 and whose other coordinates are 0's. If $x = (x_1, \ldots, x_r)^t$ and $y = (y_1, \ldots, y_r)^t$ are two vectors, we write $x < y$ if $x_i < y_i$, $\forall\ 1 \leq i \leq r$ and $x \leq y$ if $x_i \leq y_i$, $\forall\ 1 \leq i \leq r$. For any two vectors $x, y \in R^n$, we define $\min(x, y)$ as the vector whose i^{th} coordinate is $\min(x_i, y_i)$. If $x \in R^n$ and $y \in R^n$ are two vectors, the symbol $x \wedge y$ denotes the vector $u \in R^n$ whose i^{th} coordinate u_i is given by $u_i = \min(x_i, y_i)$. e^j is a column vector of 1's of order m_j. By writing $A \in R^{m \times n}$, we denote that A is a matrix of real entries with m rows and n columns. For any matrix $A \in R^{m \times n}$, a_{ij} denotes its i^{th} row and j^{th} column entry. $A_{.j}$ denotes the j^{th} column and $A_{i.}$, the i^{th} row of A. If A is a matrix of order $m \times n$, $\alpha \subseteq \{1,2,\ldots,m\}$ and $\beta \subseteq \{1,2,\ldots,n\}$ then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β respectively. $A_{\alpha\cdot}$ denotes the submatrix formed by the rows of A whose indices are in α. Similarly $A_{\cdot\alpha}$ denotes the submatrix formed by the columns of A whose indices are in α. For any set β, $|\beta|$ denotes its cardinality. For any set $\alpha \subseteq \{1,2,\ldots,m\}$, $\bar{\alpha}$ denotes its complement in $\{1,2,\ldots,m\}$. Pos(A) denotes the nonnegative cone generated by columns of A. Suppose A is a rectangular matrix of order $m \times k$ where $m \geq k$ and the rows of A are partitioned into k blocks so that the first block contains the rows 1 through m_1 and in general the r^{th} block contains the rows $\sum_{j=1}^{r-1} m_j + 1$ through $\sum_{j=1}^{r} m_j$ of A, for $r = 2,\ldots,k$. The r^{th} block of A is denoted by A^r and is a matrix of order $m_r \times k$. We then use the notation $J_1 = \{1,2,\ldots,m_1\}$ to denote the set of row indices of the first block in A and $J_r = \{\sum_{j=1}^{r-1} m_j + 1, \sum_{j=1}^{r-1} m_j + 2, \ldots, \sum_{j=1}^{r} m_j\}$ to denote the set of row indices of the r^{th} block in A for $r = 2,3,\ldots,k$. A probability vector is a vector $x \in R^n$ such that all the coordinates of x are nonnegative and $\sum_{i=1}^{n} x_i = 1$, where x_i is the i^{th} coordinate of x.

The notations pertaining to a particular chapter only is explained therein.

1.3 LCP and its Generalizations

A number of generalizations of the linear complementarity problem have been proposed to accomodate more complicated real life problems as well as to diversify the field of applications. In this dissertation two generalizations are mainly discussed namely, vertical and horizontal generalizations of the LCP.

Consider a rectangular matrix A of order $m \times k$ with $m \geq k$. Suppose A is partitioned row-wise into k blocks in the form

$$A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^k \end{bmatrix}$$

where each $A^j = ((a^j_{rs})) \in R^{m_j \times k}$ with $\sum_{j=1}^{k} m_j = m$. Then A is called a *vertical block matrix of type* $(m_1, \ldots, m_k)$. If $m_j = 1$, $\forall\, j = 1, \ldots, k$, then A is a square matrix. Thus a vertical block matrix is a natural generalization of a square matrix. The concept of a vertical block matrix was introduced by Cottle and Dantzig [4] in connection with the generalization of the linear complementarity problem introduced by them. Cottle-Dantzig's generalization [3, p. 115, footnote 3] involves a system $w - Az = q$, $w \geq 0$, $z \geq 0$ where $A \in R^{m \times k}$, $m \geq k$ and the variable $w_1, w_2, \ldots, w_m$ are partitioned into k nonempty sets $\mathcal{S}_j$, $j = 1, 2, \ldots, k$. Let $\mathcal{T}_j = \mathcal{S}_j \cup \{z_j\}$, $j = 1, 2, \ldots, k$. The problem is to find a solution $w \in R^m$ and $z \in R^k$ of the system such that exactly one member of each set $\mathcal{T}_j$ is nonbasic. This problem reduces to the standard linear complementarity problem when $k = m$ and $\mathcal{T}_j = \{w_j, z_j\}$. This generalization was formally introduced later in a paper of Cottle and Dantzig [4] in 1970. The formal statement of the problem is as follows:

Given an $m \times k$ $(m \geq k)$ vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ and a vector $q \in R^m$ where $m = \sum_{j=1}^{k} m_j$, find $w \in R^m$ and $z \in R^k$ such that

$$w - Az = q, \quad w \geq 0, \; z \geq 0 \tag{1.3.1}$$

$$z_j \prod_{i=1}^{m_j} w^j_i = 0, \; \text{for } j = 1, 2, \ldots, k. \tag{1.3.2}$$

Cottle-Dantzig's generalization was designated later by the name *vertical linear complementarity problem* [5] and this problem is denoted as VLCP(q, A).

Refering to the above generalization by Cottle and Dantzig [4], Lemke [42] observes that "It is reasonable to expect that there will be valuable applications of their results forthcoming." In recent years, a number of applications of the VLCP(q, A) have been reported in the literature. Ebiefung and Kostreva [18] introduce a *generalized Leontief* input-output linear model and formulate it as a vertical linear complementarity problem. This model can be effectively used for the problem of choosing a new technology and also for solving problems related to energy commodity demands, international trade, multinational army personnel assignment and pollution control. A slightly more general form of the VLCP(q, A) also arises in different areas of control theory through discretization of Hamilton-Jacobi-Bellman equations ([85], [86]). Another recent application of the VLCP is the formulation of the *global stability of a two-species piecewise* linear Volterra ecosystem [32]. Gowda and Sznajder [28] report an interesting extension of the bimatrix game model and the problem of computing a pair of equilibrium strategies for this extended model has a VLCP formulation. It is expected that this generalized bimatrix game will have applications in economics. This sort of applications and the potential future applications have motivated the study of the VLCP, especially the study of algorithms for the VLCP. Lemke's algorithm for the LCP(q, M) has been extended with some modifications to the VLCP(q, A) by Cottle and Dantzig in [4]. The issue of conditions for uniqueness of solutions to the VLCP has been studied by Szanc [87]. Generalizations of P_0- and Z-matrices have been studied by Ebiefung and Kostreva [17] and Sznajder and Gowda [89]. Ebiefung and Kostreva [16] characterize existence and nonexistence of solutions and present a procedure for solving VLCP(q, A). In [14], necessary and sufficient conditions for existence of solutions, a characterization of the set of Q matrices for the VLCP(q, A) and an algorithm for solving VLCP(q, A) in terms of representative submatrices are presented. Solvability by linear programs is given by Mangasarian [48]. See also [13] and [88].

A more general version of the VLCP in the setting of a finite dimensional lattice gives rise to the generalized order linear complementarity problem (GOLCP) studied by Gowda and Sznajder [25]. The generalized order linear complementarity problem (in the setting of a finite dimensional vector lattice $\mathcal{X}$) is the problem of finding a solution $x \in \mathcal{X}$ to the piecewise linear system

$$x \wedge (X^1 x + p^1) \wedge (X^2 x + p^2) \wedge \ldots \wedge (X^r x + p^r) = 0 \tag{1.3.3}$$

where X^j's are linear transformations from $\mathcal{X}$ into itself and p^j's are vectors in $\mathcal{X}$. However when specialized to R^n, the generalized order linear complementarity problem is seen to be equivalent to the VLCP. See Gowda and Sznajder [25]. The extended generalized order linear complementarity problem was considered by Goeleven [24], Gowda and Sznajder [25] and Isac and Goeleven [38]. Oh [69] has formulated a mixed lubrication problem as a generalized nonlinear complementarity problem.

Another interesting generalization of the linear complementarity problem considered in the literature namely, horizontal generalization, is motivated by certain other applications. The problem can be formally stated as follows:

Given two matrices A, $B \in R^{n\times n}$ and a vector $q \in R^n$, the horizontal linear complementarity problem is to find vectors $x \in R^n$ and $y \in R^n$ such that

$$Ax - By = q, \quad x \geq 0,\ y \geq 0 \tag{1.3.4}$$

$$x^t y = 0. \tag{1.3.5}$$

We denote this problem by HLCP(q, A, B). Clearly, this problem reduces to the standard problem LCP(q, M) when $A = I$. The HLCP was apparently introduced by Samelson, Thrall, and Wesler [81], motivated by a problem in structural engineering. The coinage of the term horizontal linear complementarity problem has been attributed to Cottle, Pang and Stone [5] by Zhang [95].

The extended horizontal linear complementarity problem can be stated as follows:

Let us consider a rectangular matrix C of order $n \times m\,(m > n)$. Suppose C is partitioned into $(k+1)$ blocks of the form

$$\begin{bmatrix} C^0 & C^1 & C^2 & \cdots & C^k \end{bmatrix}$$

where $C^j \in R^{n\times n}$, $j = 0, 1, 2, \ldots, k$ and $m = (k+1)\,n$.

Let c be a block vector which is defined as q for $k = 1$ and as $[q, d^1, d^2, \ldots, d^{k-1}]$ for $k \geq 2$, where $q \in R^n$ and $0 < d^j \in R^n$ for $j = 1, 2, \ldots, k-1$. The extended horizontal LCP(c, C) is to find vectors $x^j \in R^n$, $j = 0, 1, 2, \ldots, k$ such that

$$C^0 x^0 = q + \sum_{j=1}^{k} C^j x^j,$$

$$x^0 \wedge x^1 = 0, \;\; (d^j - x^j) \wedge x^{j+1} = 0, \;\; j = 1, 2, \ldots, k-1,$$

where for $k = 1$, only the first complementarity condition is considered. The above form of the extended HLCP has been considered by Sznajder and Gowda [89]. See also [88]. Kaneko [40] considers the extended HLCP for the case $C^0 = I$ and cites applications in mathematical programming and structural mechanics. See [41], [70] and [94] for applications in inventory theory, statistics and modelling piecewise linear electrical networks.

The study of HLCP is important due to the fact that any piecewise linear system can be formulated as a HLCP. See Eaves and Lemke [12] in this connection. Sznajder [88] has shown the equivalence of the extended HLCP with the HLCP.

In recent times much attention is being paid to the study of HLCP, particularly in the literature on interior point algorithms for linear and quadratic programming problems. Zhang [95] has proposed a convergent interior point algorithm for solving certain HLCP's. See also [31] and [96]. In [92], Tütüncü and Todd propose a polynomial time algorithm for reducing a HLCP to an LCP whenever it is possible to do so. Sznajder and Gowda [89] observe that under the assumption of column monotonicity a monotone HLCP can be rewritten as a monotone LCP.

1.4 Matrix Classes in LCP

A variety of classes of matrices are introduced in the context of the linear complementarity problem. Matrix classes play an important role for studying the theory and algorithms of LCP. Most of the matrix classes encountered in the context of LCP are commonly found in several applications. Several of these matrix classes are of interest because they characterize certain properties of the LCP and they offer certain nice features from the view point of algorithms. It is useful to review some matrix classes and their properties which will form the basis for further discussions and for subsequent generalizations.

We say that $M \in R^{n\times n}$ is a $P\,(P_0)$-*matrix* if all its principal minors are positive (nonnegative). A matrix $M \in P_0 \cap R^{n\times n}$ is said to be a *column adequate matrix* if for each $\alpha \subseteq \{1,\ldots,n\}$, $\det(M_{\alpha\alpha}) = 0$ implies that columns of $M_{\cdot\alpha}$ are linearly dependent. We say that a matrix $M = ((m_{ij}))$ of order n is a Z-*matrix* if $m_{ij} \leq 0$, $\forall i \neq j$. This class of matrices has been introduced by Fiedler and Pták [19]. We say that a matrix $M \in R^{n\times n}$ is a $\mathcal{K}\,(\mathcal{K}_0)$-*matrix* if it is in $Z \cap P\,(Z \cap P_0)$. We say that $M \in R^{n\times n}$ is *a hidden Z-matrix* if there exist square matrices X and Y of order n, $X \in Z$, $Y \in Z$ such that (i) $MX = Y$ and (ii) there exist nonnegative vectors ρ, $\sigma \in R^n$ such that $\rho^t X + \sigma^t Y > 0$. We call a square matrix M of order n an $N\,(N_0)$-*matrix* if all the principal minors of M are negative (nonpositive). A matrix $M \in N \cap R^{n\times n}$ is said to be N-*matrix of the first category* if it has at least one positive entry. $M \in N \cap R^{n\times n}$ is said to be *N-matrix of the second category* if $M < 0$. We say that a matrix $M \in R^{n\times n}$ is a Q_0-*matrix* if for any $q \in R^n$ (1.1.1) has a solution implies that LCP(q, M) has a solution. A matrix $M \in R^{n\times n}$ is said to be a Q-*matrix* if for any $q \in R^n$ LCP(q, M) has a solution. We say that $M \in R^{n\times n}$ is an R_0-*matrix* if LCP$(0, M)$ has the unique solution $w = 0, z = 0$. $M \in R^{n\times n}$ is said to be *copositive* if $x^tMx \geq 0,\ \forall\ x \geq 0$. A matrix $M \in R^{n\times n}$ is said to be *copositive-plus* if M is copositive and $x^tMx = 0,\ x \geq 0$ implies $(M + M^t)x = 0$.

The following classes of matrices have been introduced by Eaves [11].

$\mathcal{L}_1$-matrix : $M \in R^{n\times n}$ is said to be a $\mathcal{L}_1$-*matrix* if for every $0 \neq y \geq 0$, $y \in R^n$ $\exists$ an i such that $y_i > 0$ and $(My)_i \geq 0$. The class of such matrices is also denoted as E_0 and such a matrix is called *semimonotone.*

$\mathcal{L}_2$-matrix : $M \in R^{n\times n}$ is said to be a $\mathcal{L}_2$-*matrix* if for each $0 \neq \xi \geq 0$, $\xi \in R^n$ satisfying $\eta = M\xi \geq 0$ and $\eta^t\xi = 0$ $\exists$ a $0 \neq \hat{\xi} \geq 0$ satisfying $\hat{\eta} = -M^t\hat{\xi}$, $\eta \geq \hat{\eta} \geq 0,\ \xi \geq \hat{\xi} \geq 0$.

$\mathcal{L}$-matrix : $M \in R^{n\times n}$ is said to be a $\mathcal{L}$-*matrix* if it is in both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Garcia [29] introduced the classes of matrices $E(d)$ and $L(d)$ which generalizes Eaves classes $\mathcal{L}_1$ and $\mathcal{L}$.

$E(d)$: We say that a square matrix M is in the class $E(d)$ where $d \in R^n$ if $(\bar{w}, \bar{z})$, $\bar{z} \neq 0$ is a solution for the LCP(d, M) implies that there $\exists$ a $0 \neq x \geq 0$ such that $y = -M^tx \geq 0,\ \ x \leq \bar{z},\ y \leq \bar{w}$.

$E^*(d)$: We say that a square matrix M is in the class $E^*(d)$ for a $d \in R^n$ if $(\bar{w}, \bar{z})$ is a solution to the LCP(d, M) implies that $\bar{w} = d,\ \bar{z} = 0$.

Note that $E(d) = E^*(d)$ for any $d > 0$ or $d < 0$, $E(0) = \mathcal{L}_2$ of [11] and $L(d) = E(d) \cap E(0)$. Further $\mathcal{L}_1 = \cap_{d>0}E(d)$.

Todd [93] defines larger classes $\bar{E}(d)$ and $\bar{L}(d)$ by extending the classes $E(d)$ and $L(d)$ of Garcia [29] as follows:

Let (w, z) solve LCP(d, M) for some $d \in R^n$. Consider the following conditions on a given $M \in R^{n \times n}$.

(a) For all J with $\{j \mid z_j > 0\} \subseteq J \subseteq \{j \mid w_j = 0\}$ the principal submatrix of M corresponding to J has positive determinant.

(b) There is $0 \neq x \geq 0$ with $y = -M^t x \geq 0$ and $x \leq z$, $y \leq w$.

Todd defines the classes $\bar{E}(d) = \{M \mid \text{Either condition (a) or (b) is satisfied}\}$ and $\bar{L}(d) = \bar{E}(d) \cap \bar{E}(0)$. Note that $L(d) \subseteq Q_0$ [29] and $\bar{L}(d) \subseteq Q_0$ [93] if $d > 0$. We refer to $L(d)$ as **Garcia's class** and to $\bar{L}(d)$ as **Todd's class.**

1.4.1 Some Results in LCP Theory

We state a few results which will be useful for further discussions or generalizations. For proof or more details we refer the reader to the excellent book of Cottle, Pang, and Stone [5].

THEOREM 1.4.1 *((4,3) Theorem and (5,4) Theorem in [19]) Let $M \in Z$.*

(i) If there exists a vector $x \geq 0$ such that $Mx > 0$, then $M \in \mathcal{K}$ and $M^{-1} \geq 0$.

(ii) If there exists a vector $x > 0$ such that $Mx \geq 0$, then $M \in \mathcal{K}_0$.

THEOREM 1.4.2 *((3,3) Theorem in [19, p. 385] and 4.2 Theorem [64, p.75]) Let $M \in R^{n \times n}$. The following statements are equivalent:*

(i) M is a P-matrix.

(ii) M reverses the sign of no nonzero vector, i.e., $z_i(Mz)_i \leq 0 \ \ \forall\, i$ implies $z = 0$.

(iii) All real eigenvalues of M and its principal submatrices are positive.

(iv) LCP(q, M) has a unique solution for every $q \in R^n$.

THEOREM 1.4.3 *([5]) Let $M \in R^{n\times n}$. The following statements are equivalent:*

(i) M is a P_0-matrix.

(ii) For each vector $z \neq 0$ there exists an index k such that $z_k \neq 0$ and $z_k(Mz)_k \geq 0$.

(iii) All real eigenvalues of M and its principal submatrices are nonnegative.

(iv) For each $\epsilon > 0$, $M + \epsilon I$ is a P-matrix.

THEOREM 1.4.4 *([5]) Let $M \in R^{n\times n}$. The following statements are equivalent:*

(i) For all $q \in K(M)$, if z^1 and z^2 are any two solutions of LCP(q, M), then $Mz^1 = Mz^2$.

(ii) Each vector whose sign is reversed by M belongs to the nullspace of M, i.e., $z_i(Mz)_i \leq 0\ \forall\, i = 1, 2, \ldots, n$ implies $Mz = 0$.

(iii) M is a P_0-matrix and for each index set α with $\det(M_{\alpha\alpha}) = 0$, the columns of $M_{\cdot\alpha}$ are linearly dependent.

LEMMA 1.4.1 *((18) Lemma [11, p. 622]) A P_0-matrix is in $\mathcal{L}_1$.*

LEMMA 1.4.2 *((3) Lemma [11, p. 620]) Let $M \in R^{n\times n}$. The following statements are equivalent:*

(i) $M \in \mathcal{L}_1$.

(ii) LCP(q, M) has a unique complementarity solution for every $q > 0$.

THEOREM 1.4.5 *(5.2 Theorem [64, p. 84]) Let $M \geq 0$. M is a Q-matrix if and only if $m_{ii} > 0$ for each $i = 1, \ldots, n$.*

Lemke's Algorithm: For solving (1.1.1) and (1.1.2) the following algorithm based on pivot steps has been given by Lemke [43]. The initial solution to (1.1.1) and (1.1.2) is taken as

$$w = q + d\,z_0$$

$$z = 0$$

where $d \in R^n$ is any given positive vector which is called *covering vector* and z_0 is an artificial variable which takes a large enough value so that $w > 0$. This is called *primary ray.*

Step 1: Decrease z_0 so that one of the variables w_i, $1 \leq i \leq n$, say w_r is reduced to zero. We now have a basic feasible solution with z_0 in place of w_r and with exactly one pair of complementary variables (w_r, z_r) being nonbasic.

Step 2: At each iteration, the complement of the variable which has been removed in the previous iteration is to be increased. In the second iteration, for instance, z_r will be increased.

Step 3: If the variable selected at step 2 to enter the basis can be arbitrarily increased, then the procedure terminates in a *secondary ray.* If a new basic feasible solution is obtained with $z_0 = 0$, we have solved (1.1.1) and (1.1.2). If in the new basic feasible solution $z_0 > 0$, we have obtained a new basic pair of complementary variables (w_s, z_s). We repeat step 2.

Lemke's algorithm consists of the repeated applications of steps 2 and 3. If nondegeneracy is assumed, the procedure terminates either in a secondary ray or in a solution to (1.1.1) and (1.1.2). If degenerate almost complementary solutions are generated these can be resolved using the methods discussed by Eaves [11]. We say that an algorithm processes a problem if the algorithm can either compute a solution to it if one exists, or show that no solution exists. For more explanations see [5]. For $M \in L(d)$ where $d > 0$ the success of Lemke's algorithm applied to LCP(q, M) with d as the covering vector is guaranteed if it is feasible. Todd [93] proves that Lemke's algorithm with covering vector $d > 0$ processes LCP(q, M) for all matrices $M \in \tilde{L}(d)$.

1.5 Game Theory

1.5.1 Bimatrix Games

A bimatrix game is a noncooperative nonzero-sum two person game (player I and player II) in which each player has a finite number of actions (called pure strategies). Let player I have m pure strategies and player II, n pure strategies. In a game if player I chooses strategy i and player II chooses strategy j they incur the costs a_{ij} and b_{ij} respectively where $A = ((a_{ij})) \in R^{m \times n}$ and $B = ((b_{ij})) \in R^{m \times n}$ are given cost matrices.

A mixed strategy for player I is a probability vector $x \in R^m$ whose i^{th} component x_i represents the probability of choosing pure strategy i where $x_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, a mixed strategy for player II is a probability vector $y \in R^n$. If player I adopts a mixed strategy x and player II adopts a mixed strategy y then their *expected costs* are given by $x^t Ay$ and $x^t By$ respectively.

A pair of mixed strategies (x^*, y^*) with $x^* \in R^m$ and $y^* \in R^n$ is said to be a *Nash equilibrium pair* if

$$(x^*)^t Ay^* \leq x^t Ay^* \quad \text{for all mixed } strategies\ x \in R^m \text{ and}$$

$$(x^*)^t By^* \leq (x^*)^t By \quad \text{for all mixed } strategies\ y \in R^n.$$

It is easy to show that the addition of a constant to all entries of A or B leaves the set of equilibrium points invariant. Henceforth we assume that all entries of the matrices A and B are positive. We consider the following LCP:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} + \begin{bmatrix} 0 & A \\ B^t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} u \\ v \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \tag{1.5.1}$$

where e_m and e_n are m vectors and n vectors whose components are all 1's.

It is easy to see that if (x^*, y^*) is a Nash equilibrium pair then $(\bar{x}, \bar{y})$ is a solution to (1.5.1) where

$$\bar{x} = x^*/(x^*)^t B y^* \text{ and } \bar{y} = y^*/(x^*)^t A y^*. \tag{1.5.2}$$

Conversely, if $(\bar{x}, \bar{y})$ is a solution of (1.5.1) then $\bar{x} \neq 0$ and $\bar{y} \neq 0$ in (1.5.2) is ensured from the positivity of the cost matrices A and B. Therefore (x^*, y^*) is a Nash equilibrium pair where

$$x^* = \bar{x}/e_m^t \bar{x} \text{ and } y^* = \bar{y}/e_n^t \bar{y}.$$

Lemke and Howson [44] gave an efficient and constructive procedure for obtaining an equilibrium pair by solving LCP(q, M) where

$$M = \begin{bmatrix} 0 & A \\ B^t & 0 \end{bmatrix} \text{ and } q = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix}.$$

1.5.2 Polymatrix Games

A polymatrix game is an n-person nonzero-sum noncooperative game, which is a generalization of the well known bimatrix game. The credit for studying such a game for the first time has been attributed to Janovskaya [39] by Howson [37]. A description of the polymatrix game is as follows:

There are $n\ (\geq 2)$ players, player i with m_i pure strategies. When player i chooses his pure strategy s_i and player j his pure strategy s_j the partial payoff to player i is $a^{ij}(s_i, s_j)$ which does not depend on the choice of strategies by other players. If $(s_1, s_2, \ldots, s_n)$ is the vector of pure strategies chosen by players

$1, 2, \ldots, n$, the payoff to player i is given by $\sum_{j \neq i} a^{ij}(s_i, s_j)$. Let A_{ij} denote the matrix of the partial payoffs to player i resulting from the choice of pure strategies by him and player j. Note that the order of A_{ij} is $m_i \times m_j$. A mixed strategy for player i is a probability vector $x^i = (x_1^i, x_2^i, \ldots, x_{m_i}^i)^t$.

For a given set $\bar{X} = \{\bar{x}^1, \ldots, \bar{x}^n\}$ of probability vectors or mixed strategies, the expected payoff to player i is given by

$$E_i(\bar{X}) = (\bar{x}^i)^t \sum_{j \neq i} A_{ij}\, \bar{x}^j.$$

We say that a set $X^* = \{x^{1*}, x^{2*}, \ldots, x^{n*}\}$ of strategies is an equilibrium set if for all i,

$$(x^{i^*})^t \sum_{j \neq i} A_{ij}\, x^{j^*} \geq (x^i)^t \sum_{j \neq i} A_{ij}\, x^{j^*} \qquad (1.5.3)$$

for any set $X = \{x^1, x^2, \ldots, x^n\}$ of mixed strategies.

Let E^{ij} denote the matrix of 1's of order $m_i \times m_j$. It is easy to note that X^* is an equilibrium for the polymatrix game with payoff matrices A_{ij}, if and only if it is an equilibrium for the polymatrix game with payoff matrices $A_{ij} - \bar{k}_i E^{ij}$ where $\bar{k}_i$ is a constant for each i and that if the payoff matrices A_{ij}'s are replaced by $-A_{ij}$'s, X^* is an equilibrium for the new game if and only if the reverse inequality holds in (1.5.3). Given the matrices A_{ij}'s it is convenient (see Howson [37]) to replace A_{ij} by $\bar{k}_i E^{ij} - A_{ij}$ for $\bar{k}_i$'s large and consider the computation of an equilibrium for the resulting polymatrix game, which is also an equilibrium for the original game. Therefore, we shall assume without loss of generality that the A_{ij}'s are positive and that $X^* = \{x^{1*}, x^{2*}, \ldots, x^{n*}\}$ is an equilibrium set if and only if for all i,

$$(x^{i^*})^t \sum_{j \neq i} A_{ij}\, x^{j^*} \leq (x^i)^t \sum_{j \neq i} A_{ij}\, x^{j^*} \qquad (1.5.4)$$

for any set $X = \{x^1, x^2, \ldots, x^n\}$ of mixed strategies.

The problem of computing an equilibrium set of strategies for a polymatrix game has been considered by Howson [37] who formulates this problem as a linear complementarity problem and proposes a special computational scheme which shows constructively that a polymatrix game has an equilibrium point. See also Lemke [42] in this connection.

The LCP formulation of the problem of finding an equilibrium set of strategies given by Howson [37] for the polymatrix game is as follows:

Let $B = \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & 0 \end{bmatrix} \geq 0$. Then a solution to LCP$(q, X)$ where

$$q = \begin{bmatrix} 0 \\ -e \end{bmatrix} \text{ and } X = \begin{bmatrix} B & -E \\ E^t & 0 \end{bmatrix} \text{ with } E = \begin{bmatrix} e^1 & 0 & \cdots & 0 \\ 0 & e^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^n \end{bmatrix} \tag{1.5.5}$$

and $e^i \in R^{m_i}$ is a vector each of whose coordinates is 1, provides an equilibrium set of strategies. Garcia [29] provides a computational scheme that works on an augmented LCP to solve LCP(q, X). Recently, Miller and Zucker [50] prove the following:

Let $\tilde{B} = B + \Lambda$, where $\Lambda > 0$ is a matrix of all 1's. Let $\tilde{X} = \begin{bmatrix} \tilde{B} & -E \\ E^t & 0 \end{bmatrix}$. Clearly, $\tilde{X}$ is contained in the copositive-plus class and hence the LCP$(q, \tilde{X})$ can be processed by Lemke's algorithm. A solution to LCP(q, X) can be easily obtained from a solution of LCP$(q, \tilde{X})$.

In the subsection 1.5.3, we introduce a nonzero-sum noncooperative n person stochastic game called *polystochastic game* as a generalization of the polymatrix game introduced by Janovskaya [39]. The results presented here are new.

1.5.3 Polystochastic Games

Zero-sum stochastic games with two players were introduced by Shapley in [83] as a generalization of matrix games. A zero-sum stochastic game with two players is a repeated game which may be described as follows: Given a finite set S of points called *states* and a matrix $A(s)$ of order $m_s \times n_s$ associated with each $s \in S$ on the r^{th} day when the game is in state s, player I chooses one of the rows i_r, $1 \leq i_r \leq m_s$ and player II one of the column j_r, $1 \leq j_r \leq n_s$. Once the players make their choices the immediate payoff to player I from player II is determined as $a(i_r, j_r)(s)$. The game moves to a state t with probability $p(t \mid s, i_r, j_r)$ on the $(r+1)^{th}$ day and the play is repeated in state t on the $(r+1)^{th}$ day. The game is repeated over an infinite horizon of days and the total discounted payoff realized by player I is given by $\sum \beta^{r-1} a(i_r, j_r)(s_r)$ where $0 < \beta < 1$ is a discount factor, s_r is the state of the game on the r^{th} day and i_r, j_r are the indices of the row and column chosen respectively by player I and player II on the r^{th} day. Player I seeks to maximize his total expected discounted payoff and player II seeks to minimize it. Let $x^r(s)$ be a probability vector for player I on the r^{th} day for each $s \in S$ whose i^{th} coordinate $x_i^r(s)$ for $1 \leq i \leq m_s$ specifies the probability with which player I chooses row i of $A(s)$ on r^{th} day. Similarly, let the vector $y^r(s)$ for player II on the r^{th} day for each $s \in S$ be defined. Let x^r be the $(\sum_{s \in S} m_s) \times 1$ vector whose components are $x^r(s)$ and let y^r be similarly defined. A *mixed strategy* for the players I and II is a sequence $\{\pi^r\}$ where $\pi^r = \begin{bmatrix} x^r \\ y^r \end{bmatrix}$ specifies the probability vectors with which players I and II choose their actions on the r^{th} day in different states $s \in S$. A mixed strategy $\{\pi^r\}$ is said to be *stationary* if $\pi^r = \pi^1$ for all r. Shapley [83] has shown that such a game has a value vector v, whose s^{th} coordinate v_s gives the value of the game whose initial state is s and that both the players have optimal stationary strategies.

Nonzero-sum two person and n person stochastic games have been considered earlier by Fink [20] and Takahasi [90]. See also [77]. A two person nonzero-sum stochastic game is a repeated game which is a generalization of the bimatrix game. A generalized notion of Nash equilibrium point has been defined for n person noncooperative nonzero-sum stochastic games and their existence in fairly general setup is known [77], [84]. See also [76]. From now onwards, we formulate the game in terms of costs rather than payoff.

We introduce a generalization of polymatrix game namely, *polystochastic game* which is described below:

There are n players and a given finite set S with $|S| = m$ states. The i^{th} player has $m_i(s)$ possible choices while in state s, $N_i(s) = \{1, 2, \ldots, m_i(s)\}$ denoting his set of actions for $1 \leq i \leq n$. On the r^{th} day suppose the game is in state s, player i chooses an action $i_r \in N_i(s)$ and player j, $j_r \in N_j(s)$, then player i incurs a partial cost $a(i_r, j_r)(s)$ and player j, a partial cost $a(j_r, i_r)(s)$. The partial costs $a(i_r, j_r)(s)$ and $a(j_r, i_r)(s)$ incurred by player i and j do not depend on the actions chosen by the other players. The total costs incurred by player i on the r^{th} day is the sum of partial costs incurred depending on the action of player j, $j \neq i$ on that day and is therefore given by $\sum_{j \neq i} a(i_r, j_r)(s)$. Let $A_{ij}(s) = ((a(g, h)(s)))$, $g \in N_i(s)$, $h \in N_j(s)$ be the matrix of partial immediate costs incurred by player i depending on the actions of player j. This is an $m_i(s) \times m_j(s)$ matrix. Let $p(t \mid s, \tilde{a}_1, \tilde{a}_2 \ldots, \tilde{a}_n)$ be the probability that the game moves to state t on the next day, when on any day the actions chosen by the n players are $\tilde{a}_1, \tilde{a}_2 \ldots, \tilde{a}_n$.

Let $\xi^i = \begin{bmatrix} x_i(1) \\ \vdots \\ x_i(m) \end{bmatrix}$ be a $\sum_{s=1}^{m} m_i(s) \times 1$ vector with m components, the s^{th} component being a probability vector over the set $N_i(s)$, whose $\tilde{a}_i^{th}$ coordinate $x_i(\tilde{a}_i \mid s)$ gives the probability that action $\tilde{a}_i \in N_i(s)$ is chosen by player i. Thus ξ^i specifies the probabilities of the choice of actions in each state, for player i.

A *mixed strategy* for player i is a sequence $\{\xi^{ir}\}$, where ξ^{ir} specifies the probabilities of actions in different states on the r^{th} day. By the sequence $\{\pi^r\}$, where $\pi^r = \begin{bmatrix} \xi^{1r} \\ \vdots \\ \xi^{nr} \end{bmatrix}$ we denote the mixed strategies of all players, over the infinite horizon. This is called a *game plan*. Given a game plan $\{\pi^r\}$, on the r^{th} day the players use their mixed strategies ξ^{ir}. Suppose the state of the game is s on the r^{th} day. Then the probability that the game moves to state t on $(r+1)^{th}$ day under $\{\pi^r\}$ is given by

$$\rho^{(r)}(t \mid s) = \sum_{\tilde{a}_1, \ldots, \tilde{a}_n} p(t \mid s, \tilde{a}_1, \ldots, \tilde{a}_n) \prod_{i=1}^{n} x_i^r(\tilde{a}_i|s)$$

where the sum is taken over all possible $(\tilde{a}_1, \ldots, \tilde{a}_n) \in \Pi_{i=1}^n N_i(s)$. Let $Q^{(r)}(\{\pi^r\}) = ((\rho^{(r)}(t|s)))$ be an $m \times m$ matrix whose s^{th} row gives the transition probabilities to various states from state s under π^r. Let $\phi_i(\{\pi^r\})$ denote the $m \times 1$ vector of total β-discounted expected costs incurred by player i under the game plan $\{\pi^r\}$. Then it is easy to see that

$$\phi_i(\{\pi^r\}) = \sum_{r=0}^{\infty} \beta^r \prod_{k=1}^{r} Q^{(k)}(\{\pi^r\})[\psi_i^r]$$

where ψ_i^r is the $m \times 1$ vector of expected immediate costs incurred by player i on the r^{th} day whose s^{th} coordinate $\psi_i^r(s)$ is given by

$$\psi_i^r(s) = \sum_{\tilde{a}_i \in N_i(s)} x_i^r(\tilde{a}_i|s)(\sum_{k \neq i} \sum_{\tilde{a}_k \in N_k(s)} a(\tilde{a}_i, \tilde{a}_k)(s) x_k^r(\tilde{a}_k|s)).$$

Suppose $\{\xi^{ir}\}$ is a mixed strategy for player i. We say that $\{\xi^{ir}\}$ is stationary if $\xi^{ir} = \xi^{i1} = \xi^i$ for all r. We say that a game plan $\{\pi^r\}$ is stationary if each component ξ^{ir} of $\{\pi^r\}$ for $1 \leq i \leq n$ is stationary. For a stationary game plan π the expression for $\phi_i(\{\pi\})$ simplifies to $\sum_{r=0}^{\infty} \beta^r Q^r(\pi)(\psi_i)$ where $Q^r(\pi)$ is the r^{th} power of the matrix $((\rho^1(t|s)))$ and

$$\psi_i(s) = \sum_{\tilde{a}_i \in N_i(s)} x_i(\tilde{a}_i|s)(\sum_{k \neq i} \sum_{\tilde{a}_k \in N_k(s)} a(\tilde{a}_i, \tilde{a}_k)(s) x_k(\tilde{a}_k|s)).$$

A stationary strategy $\xi^i = \begin{bmatrix} x^i(1) \\ \vdots \\ x^i(m) \end{bmatrix}$ for player i is said to be a *pure stationary strategy* if for all s $\exists$ a $\tilde{a} \in N_i(s)$ such that $x_i(\tilde{a}\,|s) = 1$ and $x_i(\bar{a}|s) = 0$ if $\tilde{a} \neq \bar{a} \in N_i(s)$. A mixed strategy ξ^{ir} for player i is called *a behavior strategy* if ξ^{ir} depends on the history of the strategies of all the players and resulting states in the past upto the $(r-1)^{th}$ day.

We say that a game plan $\{\bar{\pi}^r\}$ is an equilibrium game plan if

$$\phi_i(\{(\bar{\xi}^{1r} \ldots \bar{\xi}^{ir} \ldots \bar{\xi}^{nr})^t\}) \leq \phi_i(\{(\bar{\xi}^{1r} \ldots (\xi^*)^{ir} \ldots \bar{\xi}^{nr})^t\})$$

for all $(\xi^*)^{ir}$ and for all i.

It is known that under fairly general conditions there is an equilibrium in stationary strategies. For stationary strategies the above inequality simplifies (see [20]) to the following:

We say that $\bar{\pi} = \begin{bmatrix} \bar{\pi}_i \\ \bar{\xi}^i \end{bmatrix}$ where $\bar{\pi}_i$ is obtained from $\bar{\pi}$ by omitting its i^{th} component, namely $\bar{\xi}^i$ is a *stationary equilibrium strategy* if for each $1 \leq i \leq n$

$$\phi_i\left(\begin{bmatrix} \bar{\pi}_i \\ \bar{\xi}^i \end{bmatrix}\right) \leq \phi_i\left(\left\{\begin{bmatrix} \bar{\pi}_i \\ \xi^{ir} \end{bmatrix}\right\}\right)$$

where ξ^{ir} is a behavioral strategy for player i.

The special case $n = 2$, i.e., a *bistochastic* nonzero-sum noncooperative game has earlier been considered. See [67]. In particular, the situation where the transition depend on the actions of a single player has been well studied. Parthasarathy and Raghavan [76] show that a single controller nonzero-sum noncooperative game with $n = 2$ has the *orderfield property* (i.e., an equilibrium value and the corresponding equilibrium set of strategies lie in the same ordered field as the data). Nowak and Raghavan [67] reduce the problem of finding a pair of stationary equilibrium strategies in a bistochastic noncooperative game when the criterion is either the discounted total expected cost or the average cost with the additional assumption, in this case, that under any stationary strategy, the transition probability matrix is irreducible, to the problem of computing a Nash equilibrium pair in a suitably constructed bimatrix game.

We consider the polystochastic game in which the criterion is the total expected discounted cost and the transition probabilities depend only on the actions of a single player, namely player n, mainly from the point of view of computing a stationary equilibrium set of strategies for the players and the corresponding equilibrium cost vector. We may note that the approach of Nowak and Raghavan [67] for the case $n = 2$ with the assumption of one player control transitions involve computing for each possible combination of pure stationary strategies, the corresponding total discounted expected cost vector and then constructing the associated bimatrix game and finally applying the algorithm of Lemke-Howson [44] to the associated linear complementarity problem. Although such a reduction is possible in the more general single controller polystochastic games also, it is tedious even in the case of $n = 2$. In contrast the procedure suggested below uses Lemke's algorithm on a suitable linear complementarity problem formulated with the given data only without requiring any additional computation.

Formulation

We denote the stationary strategy by $\{\pi\}$ (which is a sequence) and use $\phi_i(\pi)$ to denote the corresponding $m \times 1$ vector of total discounted expected costs to player i when the stationary strategy $\pi = (\xi^1, \xi^2, \ldots, \xi^n)$ is being used, the i^{th} component ξ^i of player i is known as stationary strategy of player i. In what follows, we shall assume that the transition probabilities depend only on the actions of player n. This means that

$$p(t|s, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = p(t|s, \tilde{a}_n).$$

As an immediate consequence of the assumption we have the following lemma.

LEMMA 1.5.1 *Let $\phi_i(\pi)$ be the vector of total β-discounted expected costs, for player $1 \leq i \leq n$, with the matrices of immediate partial costs denoted as $A_{ij}(s)$. The noncooperative game with costs as $\phi_i(\pi)$, $1 \leq i \leq n$ and the noncooperative game with costs $A_{ij}(s)$, $i \neq j$, $i \neq n$ and $\phi_n(\pi)$ (i.e., a game in which players other than n have zero immediate costs except for the first day) have the same set of equilibrium points.*

Proof. This follows in the same lines as the proof of Lemma 5.1 in [76]. ■

Let $v_1, v_2, \ldots, v_{n-1}$ be the vector of immediate expected costs for player 1 through $(n-1)$ corresponding to an equilibrium set of strategies and ϕ_n be the vector of total discounted expected costs of player n corresponding to the same equilibrium set of strategies. Suppose $((\xi^*)^1, \ldots, (\xi^*)^n)$ denote an equilibrium set of stationary strategies for the players $1, \ldots, n$.

THEOREM 1.5.1 *$\{x_i(s), 1 \leq i \leq n\}$ and $v_i(s)$, $1 \leq s \leq m$, $1 \leq i \leq n$ form a set of equilibrium strategies for players $1, \ldots, n$ and the corresponding equilibrium costs iff they satisfy the following system of inequalities and equations:*

$$w_i(s) = \sum_{j \neq i} A_{ij}(s) x_j(s) - v_i(s) e \geq 0, 1 \leq s \leq m, 1 \leq i \leq (n-1) \qquad (1.5.6)$$

$$w_n(s) = \sum_{j=1}^{n-1} A_{nj}(s) x_j(s) + \beta P(s) \phi_n - \phi_n(s) e \geq 0 \qquad (1.5.7)$$

$$\sum_{g \in N_i(s)} x_i(g|s) = 1, \; 1 \leq s \leq m, \; 1 \leq i \leq n \qquad (1.5.8)$$

$$x_i(g|s) \geq 0 \; \forall \; i, s, \text{ and } g \in N_i(s), \; 1 \leq s \leq m, 1 \leq i \leq n \qquad (1.5.9)$$

$$(w_i(s))^t x_i(s) = 0, \; 1 \leq s \leq m, \; 1 \leq i \leq n \qquad (1.5.10)$$

where ϕ_n is the vector $(\phi_n(1), \ldots, \phi_n(m))^t$ and $P(s)$ is the $m_n(s) \times m$ matrix whose $(t,g)^{th}$ entry is $p(t|s,g)$ where $g \in N_n(s)$.

Proof. This follows from Theorem 3 in [84, p. 1934]. ■

REMARK 1.5.1 Note that (1.5.6) through (1.5.10) are the system of inequalities and equations that determine an equilibrium point and a corresponding set of stationary equilibrium strategies of the following *auxiliary game:* Consider a polystochastic game in which players 1 through $(n-1)$ incur the immediate costs as specified by the matrices $A_{ij}(s)$, $j \neq i$, $1 \leq i \leq n-1$, $1 \leq j \leq n$ on the first day and zero costs on the second day onwards, whereas player n incurs costs as in the original game. Theorem 1.5.1 asserts that a set of stationary equilibrium strategies for the players in the auxiliary game is also a set of stationary equilibrium strategies in the original game and that the corresponding total discounted expected costs $\phi_i(s)$, $1 \leq i \leq (n-1)$ of players 1 through $(n-1)$ in the original game are given by $\frac{1}{1-\beta}v_i(s)$, $1 \leq i \leq (n-1)$, $1 \leq s \leq m$.

In what follows we shall use the symbol $v_n(s)$ instead of $\phi_n(s)$ in (1.5.6) through (1.5.10) for ease of notation. However it should be noted that $v_n(s) = \phi_n(s)$ whereas $\phi_i(s) = \frac{1}{1-\beta}v_i(s)$, $1 \leq i \leq (n-1)$, $1 \leq s \leq m$.

Let B, a square matrix of order $\sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ be defined as $B = \begin{bmatrix} \bar{B} \\ \tilde{B} \end{bmatrix}$ where $\bar{B}$ is a matrix of order $\sum_{i=1}^{n-1}\sum_{s=1}^{m} m_i(s) \times \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ given by

$$\begin{bmatrix}
0 & A_{12}(1) & \cdots & & A_{1n}(1) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & A_{12}(2) & \cdots & A_{1n}(2) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & \vdots & & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{12}(m) & \cdots & A_{1n}(m) \\
\vdots & \vdots & & & & & & & \vdots & & \vdots & & & \vdots & \vdots \\
A_{n-11}(1) & & \cdots & 0 & A_{n-1n}(1) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & \vdots & & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{n-11}(m) & \cdots & \cdots & A_{n-1n}(m)
\end{bmatrix}$$

and $\tilde{B}$ is the matrix of order $\sum_{s=1}^{m} m_n(s) \times \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ given by

$$\begin{bmatrix}
A_{n1}(1) & A_{n2}(1) & \cdots & A_{nn-1}(1) & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & A_{n1}(2) & 0 & \cdots & A_{nn-1}(2) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & & & & \vdots & & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{n1}(m) & A_{n2}(m) & \cdots & 0
\end{bmatrix}.$$

Let $\bar{E}$ be a matrix of order $\sum_{i=1}^{n-1}\sum_{s=1}^{m} m_i(s) \times m(n-1)$ which is of the form

$$\bar{E} = \begin{bmatrix} -e^{11} & 0 & \dots & 0 \\ 0 & -e^{12} & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & -e^{(n-1)m} \end{bmatrix}$$

where e^{is} is a vector of order $m_i(s) \times 1$ whose coordinates are all equal to 1.

Let E be a matrix of order $nm \times \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ which is of the form

$$E = \begin{bmatrix} (e^{11})^t & 0 & \dots & 0 \\ 0 & (e^{12})^t & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & (e^{nm})^t \end{bmatrix}$$

where e^{is} is as defined earlier. Let Q be the matrix of order $\sum_{s=1}^{m} m_n(s) \times m$ given by $Q = \begin{bmatrix} Q^1 \\ \vdots \\ Q^m \end{bmatrix}$ where Q^s is an $m_n(s) \times m$ matrix which is

$$Q^s = \begin{bmatrix} \beta p(1|s,1) & \dots & -1+\beta p(s|s,1) & \dots & \beta p(m|s,1) \\ \beta p(1|s,2) & \dots & -1+\beta p(s|s,2) & \dots & \beta p(m|s,2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta p(1|s,m_n(s)) & \dots & -1+\beta p(s|s,m_n(s)) & \dots & \beta p(m|s,m_n(s)) \end{bmatrix}.$$

Let M be a square matrix of order $\sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) \times nm$ which is given by

$$M = \begin{bmatrix} \bar{B} & \bar{E} & 0 \\ \tilde{B} & 0 & Q \\ E & 0 & 0 \end{bmatrix}.$$

Let the vector q be defined as $q = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -e_{mn} \end{bmatrix}$ where q^1 is of order $\sum_{i=1}^{n} \sum_{s=1}^{m} m_i(s) \times 1$ and e_{mn} is a vector of order $mn \times 1$ of 1's.

With these notations we have the following theorem.

THEOREM 1.5.2 *Consider the LCP(q, M) where q and M are defined as above. Any solution to the LCP(q, M) is a solution to the system of inequalities and equations (1.5.6) through (1.5.10) and hence yields a stationary equilibrium point of the auxiliary game.*

Proof. Note that $q + Mz \geq 0$ gives us the same set of inequalities as in (1.5.6) and (1.5.9), where z^t can be identified as

$$((x_1(1))^t, \ldots, (x_n(1))^t, \ldots, (x_n(m))^t, v_1(1), \ldots, v_1(m), \ldots, v_n(m)). \quad (1.5.11)$$

Suppose (w, z) solves LCP(q, M). Then identifying z^t as above and identifying w^t as

$$((w_1(1))^t, \ldots, (w_n(1))^t, \ldots, (w_n(m))^t, u_1(1), \ldots, u_1(m), \ldots, u_n(m)). \quad (1.5.12)$$

where $w_i(s)$ is a vector of order $m_i(s) \times 1$ defined in Theorem 1.5.1 and $u_i(s)$ are real numbers given by

$$u_i(s) = \sum_{g \in N_i(s)} x_i(g|s) - 1, \quad (1.5.13)$$

we note that $w_i(s) \geq 0$, $x_i(s) \geq 0$ and further, $\sum_{g \in N_i(s)} x_i(g|s) \geq 1$. Now note that $u_i(s)v_i(s) = 0$. This implies that if $u_i(s) > 0$ then $v_i(s) = 0$. However, since $x_i(s)$ for all i and s are nonnegative with at least one coordinate positive and as A_{ij}'s are strictly positive matrices, it follows that if $v_i(s) = 0$, then $w_i(s) > 0$. This implies that $(w_i(s))^t x_i(s) > 0$ and contradicts the hypothesis that (w, z) solves the LCP(q, M). Hence it follows that $u_i(s) = 0$ and hence $\sum_{g \in N_i(s)} x_i(g|s) = 1$.

Thus $w_i(s)$, $x_i(s)$, $v_i(s)$, $1 \leq s \leq m$, $1 \leq i \leq n$ is a solution to the system of inequalities and equations (1.5.6) through (1.5.10). Conversely, suppose we have a solution $w_i(s)$, $x_i(s)$, $v_i(s)$, $1 \leq s \leq m$, $1 \leq i \leq n$ to the system of inequalities and equations (1.5.6) through (1.5.10). Now define $u_i(s) = 0$ and take w and z as defined in (1.5.11) and (1.5.12). It is easy to see that (w, z) solves LCP(q, M).

∎

REMARK 1.5.2 Thus the problem of finding a stationary equilibrium set of strategies and corresponding costs for a discounted polystochastic game can be formulated as a problem of solving a linear complementarity problem. We shall show in the next section that a slight reformulation of this will yield an LCP that can be solved by Lemke's algorithm.

Lemke's Algorithm for Finding an Equilibrium

Lemke's algorithm when applied to the LCP(q, M) considered in Theorem 1.5.2 may terminate in a *secondary ray.* However, we shall show that a slight modification leads to an LCP$(q, \bar{M})$ where $\bar{M}$ is in the class $\mathcal{L}$ defined by Eaves [11]. For this class of matrices Lemke's algorithm with any positive covering vector will compute a solution to LCP$(q, \bar{M})$ and hence compute a Nash equilibrium point to the auxiliary game considered earlier.

Let $\bar{M}$ be a matrix obtained by replacing $B = \begin{bmatrix} \bar{B} \\ \tilde{B} \end{bmatrix}$ in M by $B + \Lambda = C$ where Λ is a square matrix of order $\sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ each of whose entries is 1. Thus

$$\bar{M} = \begin{bmatrix} \bar{C} & \bar{E} & 0 \\ \tilde{C} & 0 & Q \\ E & 0 & 0 \end{bmatrix}$$

where $\bar{C}$ and $\tilde{C}$ are the row partitions of C induced by the partitions of B as

$\begin{bmatrix} \bar{B} \\ \tilde{B} \end{bmatrix}$. We have the following lemma.

LEMMA 1.5.2 *Consider the LCP*$(q, \bar{M})$. *If* $(\bar{w}, \bar{z})$ *solves LCP*$(q, \bar{M})$ *then* (w^*, z^*) *solves LCP*(q, M) *where* $w^* = \bar{w}$, $z_r^* = \bar{z}_r$, *for* $1 \leq r \leq \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$, $z_r^* = \bar{z}_r - mn$ *for* $\sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) < r < \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) + (n-1)m$ *and* $z_r^* = \bar{z}_r - \frac{mn}{1-\beta}$ *for* $r > \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) + (n-1)m$.

Proof. This is easy to verify. ∎

THEOREM 1.5.3 *LCP*$(d, \bar{M})$ *has a unique solution when* $d > 0$, $d \in R^{m^*}$ *where* $m^* = \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) + mn$ *and when* $d = 0$.

Proof. We shall show that $\bar{M} \in \mathcal{L}_1$, the class introduced by Eaves [11] by verifying its defining condition. Suppose $0 \neq x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \geq 0$ is given where $x \in R^{m^*}$ and the partition of x is induced by the partition in $\bar{M}$. Now

$$\bar{M}x = \begin{bmatrix} \bar{C}\xi_1 + \bar{E}\xi_2 \\ \tilde{C}\xi_1 + Q\xi_3 \\ E\xi_1 \end{bmatrix}.$$

If $\xi_3 \neq 0$, then there exists a $r > \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s) + m(n-1)$ such that $\bar{x}_r > 0$ and $(\bar{M}x)_r \geq 0$ as $E\xi_1 \geq 0$. If $\xi_3 = 0$ and $\xi_2 \neq 0$, then there exists a r, $\sum_{i=1}^{n-1}\sum_{s=1}^{m} m_i(s) + m(n-1) < r < m^*$ such that $\bar{x}_r > 0$ and $(M\bar{x})_r \geq 0$ as $\tilde{C} > 0$ and $Q\xi_3 = 0$. If $\xi_3 = 0$ and $\xi_2 = 0$ then $\xi_1 \neq 0$ and since $\bar{C}\xi_1 > 0$ and $\tilde{C}\xi_1 > 0$ it follows that $\exists$ a $r \leq \sum_{i=1}^{n}\sum_{s=1}^{m} m_i(s)$ such that $x_r > 0$ and $(\bar{M}x)_r > 0$. Thus it follows that $\bar{M} \in \mathcal{L}_1$ or $M \in E(d)$ [29] for each $d > 0$.

We shall now show that LCP$(0, \bar{M})$ has a unique solution.

Suppose there is a $(\bar{w}, \bar{z})$ such that

$$\bar{w} - \bar{M}\bar{z} = 0,\ \bar{w} \geq 0,\ \bar{z} \geq 0, \text{ and } \bar{w}^t\bar{z} = 0.$$

Let $X = \begin{bmatrix} \bar{E} & 0 \\ 0 & Q \end{bmatrix}$. Also let $E^t = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ be the partition of E^t as in X, since X and E^t are of the same order. Note that $\bar{M} = \begin{bmatrix} C & X \\ E & 0 \end{bmatrix}$ and let $\bar{z} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ be the partition of $\bar{z}$ induced by this partition of $\bar{M}$. Since $\bar{w}^t\bar{z} = 0$, it follows that

$$\bar{z}^t\bar{M}\bar{z} = \eta_1^t C\eta_1 + \eta_1^t(E^t + X)\eta_2 = 0. \tag{1.5.14}$$

Noting that $C > 0$ and $E^t + X \geq 0$, we see that equation (1.5.14) implies that $\eta_1^t C\eta_1 = 0$ and $\eta_1^t(E^t+X)\eta_2 = 0$. Now $\eta_1^t C\eta_1 = 0 \Rightarrow \eta_1 = 0$. Noting that $\bar{M}\bar{z} \geq 0$, we conclude that $X\eta_2 \geq 0$. Let $\eta_2 = \begin{bmatrix} \bar{\eta}_2 \\ \tilde{\eta}_2 \end{bmatrix}$ be the partition of η_2 induced by the partition of X as $\begin{bmatrix} \bar{E} & 0 \\ 0 & Q \end{bmatrix}$. We then have $\bar{E}\bar{\eta}_2 \geq 0$ which implies that $\bar{\eta}_2 = 0$, as $\bar{E} \leq 0$. Now consider the inequality $Q\tilde{\eta}_2 \geq 0$. Note that Q is a matrix of order $\sum_{s=1}^{m} m_n(s) \times m$ matrix which has the row partition $\begin{bmatrix} Q^1 \\ \vdots \\ Q^m \end{bmatrix}$ defined earlier. Let Q_1 be the $m \times m$ matrix whose l^{th} row is the first row of Q^l for $1 \leq l \leq m$. We have $Q_1\tilde{\eta}_2 \geq 0$. Note that Q_1 is a square matrix of the form $-I + \beta R$ where R is a stochastic matrix. Thus $(I - \beta R)$ is a $Z \cap P$-matrix and the inequality $(I - \beta R)\tilde{\eta}_2 \leq 0$, $\tilde{\eta}_2 \geq 0$ has only the trivial solution $\tilde{\eta}_2 = 0$. Thus the only solution to LCP$(0, \bar{M})$ is the trivial solution $\bar{z} = 0, \bar{w} = 0$. This concludes the proof of the theorem. ■

THEOREM 1.5.4 *Lemke's algorithm processes LCP$(q, \bar{M})$ and hence computes an equilibrium point and a set of equilibrium strategies of the auxiliary game.*

Proof. This follows from Theorem 3.5 of Garcia [29]. ∎

The case $n = 2$ is discussed more often in the literature. We therefore state the following corollary.

COROLLARY 1.5.1 *Lemke's algorithm processes the linear complementarity problem associated with the problem of finding an equilibrium point and a pair of equilibrium strategies of a two person nonzero-sum stochastic game in which transition depend on the action of a single player and the criterion is the discounted total expected cost.*

REMARK 1.5.3 Note that the matrix $-Q$ is a vertical block Z-matrix. Vertical block Z matrices have been studied in Chapter 4.

REMARK 1.5.4 It is clear from the linear complementarity formulation of the problem of finding an equilibrium set of strategies and corresponding equilibrium discounted total expected cost vectors for all the players of a polystochastic game considered here, that such a game has a set of rational equilibrium strategies and corresponding rational discounted total expected cost vectors, if all the data including β are rational. Thus we have the *orderfield property* for a polystochastic game with discounted total expected cost as a criterion in which transition probabilities depend on the actions of a single player. This is not true in general even for an n person nonzero-sum noncooperative (static) game. An example is given in Nash [66] where a 3 person game with rational data is presented which has no rational equilibrium.

EXAMPLE 1.5.1 Consider the polystochastic game with $n = 3$ players and $m = 2$ states. The cost matrices $A_{ij}(s)$'s are as follows:

$$A_{12}(1) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad A_{12}(2) = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}, \quad A_{13}(1) = \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix}, \quad A_{13}(2) = \begin{bmatrix} 3 & 4 \\ 2 & 8 \end{bmatrix},$$

$$A_{21}(1) = \begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix}, \quad A_{21}(2) = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}, \quad A_{23}(1) = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}, \quad A_{23}(2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$A_{31}(1)=\begin{bmatrix}2&3\\3&2\end{bmatrix},\ A_{31}(2)=\begin{bmatrix}4&3\\3&4\end{bmatrix},\ A_{32}(1)=\begin{bmatrix}5&2\\6&4\end{bmatrix},\ A_{32}(2)=\begin{bmatrix}7&6\\6&7\end{bmatrix}.$$

Let $\beta=\frac{1}{2}$ and let the transition probabilities be given as follows:

$$p(1|1,1))=\frac{1}{2},\quad p(2|1,1))=\frac{1}{2},\quad p(1|1,2))=\frac{1}{4},\quad p(2|1,2))=\frac{3}{4},$$

$$p(1|2,1))=\frac{1}{4},\quad p(2|2,1))=\frac{3}{4},\quad p(1|2,2))=\frac{1}{3},\quad p(2|2,2))=\frac{2}{3}.$$

M and q are given by

$$\begin{bmatrix}
0&0&1&2&2&3&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&3&1&6&4&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&3&3&4&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&2&5&2&8&0&-1&0&0&0&0\\
5&3&0&0&3&4&0&0&0&0&0&0&0&0&-1&0&0&0\\
1&6&0&0&5&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&4&7&0&0&1&2&0&0&0&-1&0&0\\
0&0&0&0&0&0&3&5&0&0&2&1&0&0&0&-1&0&0\\
2&3&5&2&0&0&0&0&0&0&0&0&0&0&0&0&-1+\frac{\beta}{2}&\frac{\beta}{2}\\
3&2&6&4&0&0&0&0&0&0&0&0&0&0&0&0&-1+\frac{\beta}{4}&\frac{3\beta}{4}\\
0&0&0&0&0&0&4&3&7&6&0&0&0&0&0&0&\frac{\beta}{4}&-1+\frac{3\beta}{4}\\
0&0&0&0&0&0&3&4&6&7&0&0&0&0&0&0&\frac{\beta}{3}&-1+\frac{2\beta}{3}\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0
\end{bmatrix},\quad
\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}.$$

The solution of LCP$(q,\bar{M})$ is given by

$\bar{w}_2=3$, $\bar{w}_4=2$, $\bar{w}_5=8$, $\bar{w}_7=3$, $\bar{w}_{10}=2.408942$, $\bar{w}_{11}=0.1374046$, $\bar{z}_1=1$, $\bar{z}_3=1$, $\bar{z}_6=1$, $\bar{z}_8=1$, $\bar{z}_9=1$, $\bar{z}_{12}=1$, $\bar{z}_{13}=10$, $\bar{z}_{14}=14$, $\bar{z}_{15}=7$, $\bar{z}_{16}=12$, $\bar{z}_{17}=27.63577$ and $\bar{z}_{18}=30.90731$.

The solution for the auxiliary game is given below:

The probabilities are as follows:

$$x_1(1)=\begin{bmatrix}1\\0\end{bmatrix},\ x_2(1)=\begin{bmatrix}1\\0\end{bmatrix},\ x_3(1)=\begin{bmatrix}0\\1\end{bmatrix},\ x_1(2)=\begin{bmatrix}0\\1\end{bmatrix},\ x_2(2)=\begin{bmatrix}1\\0\end{bmatrix},\ x_3(2)=\begin{bmatrix}0\\1\end{bmatrix}.$$

The corresponding costs of the auxiliary game are $v_1(1) = 4$, $v_1(2) = 8$, $v_2(1) = 1$, $v_2(2) = 6$, $v_3(1) = 15.63577$ and $v_3(2) = 18.90731$.

The solution for the original game is obtained as follows:
The probabilities are the same as in the auxiliary game and costs for players $1 \leq i \leq n-1$ of the original game are obtained by multiplying the corresponding costs of the auxiliary game by $\frac{1}{1-\beta}$.

The costs of the original game are $\phi_1(1) = 8$, $\phi_1(2) = 16$, $\phi_2(1) = 2$, $\phi_2(2) = 12$, $\phi_3(1) = 15.63577$ and $\phi_3(2) = 18.90731$.

1.6 Some Definitions and Results in VLCP Theory

DEFINITION 1.6.1 Let A be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. A submatrix of size k of A is called a *representative submatrix* if its j^{th} row is drawn from the j^{th} block A^j of A. Clearly, a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$ has at most $\prod_{j=1}^{k} m_j$ distinct representative submatrices.

DEFINITION 1.6.2 A vertical block matrix A of type $(m_1, \ldots, m_k)$ is called a *vertical block P (P_0)-matrix,* if all its representative submatrices are P (P_0)-matrices.

DEFINITION 1.6.3 A vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ is called a *vertical block Z-matrix* if all its representative submatrices are Z-matrices.

The concepts of copositive, strictly copositive and copositive-plus vertical block matrices are similarly defined. See [4] and [14].

DEFINITION 1.6.4 We call the set of columns $\{-A_{\cdot p}\} \cup \{I_{\cdot r},\ r \in J_p\}$, the p^{th} *set of related columns* and the corresponding set $\{z_p\} \cup \{w_r,\ r \in J_p\}$ is called the p^{th} *set of related variables.*

Given a vertical block matrix A of type $(m_1, \ldots, m_k)$ consider the matrix $[I, -A]$ where I is the identity matrix of order m.

DEFINITION 1.6.5 A *proper set of column vectors* of $[I, -A]$ is a set of m column vectors forming a matrix $B = \{B_{.j},\ j = 1, \ldots, m\}$ such that
$-A_{.i}$ is a column of $B \Rightarrow \exists$ an index $l \in J_i$ such that $I_{.l}$ is not a column of B.

The matrix formed by a proper set of column vectors is called a *proper matrix.*
Note: If B is a proper matrix of $[I, -A]$ then B contains exactly m_j columns from the j^{th} set of related columns for $1 \leq j \leq k$.

DEFINITION 1.6.6 The cone generated by a proper set of column vectors is called a *proper cone.*

DEFINITION 1.6.7 We call B, a submatrix of order $m \times m$ of $(I, -A, -d)$ an *almost proper basis matrix* if

(i) $-d$ is a column of B.

(ii) $-A_{.i}$ is a column of $B \Rightarrow \exists$ an index $p \in J_i$ such that $I_{.p}$ is not a column of B.

(iii) the columns of B form a linearly independent set.

DEFINITION 1.6.8 We call B, a submatrix of order $m \times m$ of $(I, -A)$ a *proper basis matrix* if

(i) $-A_{.i}$ is a column of $B \Rightarrow \exists$ an index $p \in J_i$ such that $I_{.p}$ is not a column of B.

(ii) the columns of B form a linearly independent set.

DEFINITION 1.6.9 Let B be an almost proper or proper basis matrix. We say that B is *feasible* if $B^{-1} q \geq 0$.

DEFINITION 1.6.10 Suppose B is a feasible almost proper or proper basis matrix. We say that B is *nondegenerate* if $B^{-1}q > 0$.

DEFINITION 1.6.11 Let us consider the set

$S = \{(w, z, z_0) \mid w \in R^m, z \in R^k, z_0 \in R,\ w - Az - dz_0 = q,\ w \geq 0, z \geq 0, z_0 \geq 0\}$.

Let $y = (u^*, \bar{v}, v_0)$ be an extreme direction of S and let $(w^*, \bar{z}, z_0) \in S$. Consider the ray of points

$(w(\lambda), z(\lambda), z_0(\lambda)) := (w^*, \bar{z}, z_0) + \lambda(u^*, \bar{v}, v_0) \in S\ \ \forall \lambda \geq 0.$

We call such a ray an *almost proper ray* if

$z_j(\lambda) \prod_{i=1}^{m_j} w_i^j(\lambda) = 0, \quad \forall\ \lambda \geq 0,\ j = 1, \ldots, k.$

REMARK 1.6.1 It is easy to verify that if

$(w(\lambda), z(\lambda), z_0(\lambda)) := (w^*, \bar{z}, z_0) + \lambda\,(u^*, \bar{v}, v_0)$

is an almost proper ray, then $(u^*, \bar{v}, v_0) \geq 0,\ u^* - A\bar{v} - dv_0 = 0$ and

$\bar{v}_j \prod_{i \in J_j} u^{*j}_i = \bar{z}_j \prod_{i \in J_j} w^{*j}_i = \bar{v}_j \prod_{i \in J_j} w^{*j}_i = \bar{z}_j \prod_{i \in J_j} u^{*j}_i = 0\ \forall\, 1 \leq j \leq k.$

Many of the above notions have already been introduced in [4].

Cottle-Dantzig Algorithm [4]: The Cottle-Dantzig algorithm for the VLCP(q, A) starts with the initial solution to (1.3.1) and (1.3.2) as

$$w = q + d\,z_0$$

$$z = 0$$

where z_0 is large enough so that $w > 0$ and $d \in R^m$ is any positive vector.

Step 1: Decrease z_0 to $\bar{z}_0 = \min\{z_0 \mid q + d\,z_0 \geq 0,\ z_0 \geq 0\}$ so that one of the variables w_i, $1 \leq i \leq m$, say w_p is reduced to zero. We now have a basic feasible solution with z_0 in place of w_p. This is the initial almost proper basic feasible solution. Now let r be the unique index, $1 \leq r \leq k$, such that $p \in J_r$. We have exactly one pair of non-basic variables (z_r, w_p) which belong to the same set of related variables.

Step 2: At each iteration, there is exactly one pair of non-basic variables belonging to the same set of related variables. Of these, one has been eliminated from the set of basic variables in the previous iteration; the other is now selected to be included as a basic variable in the next iteration. For example, in the second iteration z_r is selected to be included in the set of basic variables.

Step 3: If the variable selected at step 2 to be included as a basic variable can be arbitrarily increased, then the procedure terminates in an almost proper ray, to be called a *secondary proper ray.* Otherwise, one or more of the existing basic variables are reduced to zero and one such variable is replaced by the entering variable to obtain a new almost proper or proper basic feasible solution. If the new basic feasible solution obtained has $z_0 = 0$ or z_0 is non-basic, then we have solved (1.3.1) and (1.3.2) and have a solution for the VLCP(q, A). Otherwise, we have obtained a new almost proper basic feasible solution and a new pair of nonbasic variables (x_β, y_r) belonging to the same set of related variables, say the s^{th} set, where either $(x_\beta, y_r) = (z_s, w_t)$, with $t \in J_s$ or $(x_\beta, y_r) = (w_{t_1}, w_{t_2})$, with $t_1, t_2 \in J_s$.

We repeat step 2.

The Cottle-Dantzig algorithm consists of the repeated applications of steps 2 and 3. From now onwards we refer to this algorithm as **Algorithm CD.**

THEOREM 1.6.1 *(Proposition 4.3 in [18, p. 167]) VLCP(q, A) has a complementary feasible solution if and only if there exists a representative submatrix A_G and a corresponding subvector q_G of q so that LCP(q_G, A_G) is solvable with a solution z and $Az + q \geq 0$.*

THEOREM 1.6.2 *(Theorem 3 in [4, p. 88]) Let A be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. If A is either vertical block strictly copositive or a vertical block P-matrix then VLCP(q, A) has a solution. If A is vertical block copositive-plus and Algorithm CD fails to produce a solution of VLCP(q, A) then (1.3.1) has no nonnegative solution, i.e, $\mathcal{Z} = \{z \mid q + Az \geq 0, z \geq 0\} = \emptyset$.*

Given a vertical block matrix A of type $(m_1, \ldots, m_k)$, let $u^1, \ldots, u^k$ be a collection of row vectors such that $0 \neq u^j \geq 0$ has m_j coordinates and $\sum_{j=1}^{k} m_j = m$. Then

$$U = \begin{bmatrix} u^1 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & u^k \end{bmatrix} \tag{1.6.1}$$

is of order $k \times m$. The j^{th} row of the matrix UA is the u^j-weighted sum of the rows in A^j.

THEOREM 1.6.3 *(Theorem 4 in [4, p. 88]) If A is a vertical block P-matrix of type $(m_1, \ldots, m_k)$ and U is given by (1.6.1), then UA is a P-matrix.*

THEOREM 1.6.4 *(Theorem 6 in [4, p. 89]) If A is a vertical block P-matrix of type $(m_1, \ldots, m_k)$, the system of inequalities*

$$Az > 0, \ z > 0$$

has a solution.

THEOREM 1.6.5 *(Theorem 3 in [17, p. 169]) Let A be a vertical block Z-matrix of type $(m_1, \ldots, m_k)$. Then the following are equivalent:*

(i) There exists a vector $0 \neq x \geq 0$ such that $Ax > 0$.

(ii) A is a vertical block P-matrix of type $(m_1, \ldots, m_k)$.

1.7 Scarf's Complementarity Problem

Scarf [82] introduces an interesting generalization of the linear complementarity problem involving a vertical block matrix $\bar{N}$ of type $(m_1, \ldots, m_k)$. The problem is stated as follows:

Given an $m \times k$, $m \geq k$ vertical block matrix $\bar{N}$ of type $(m_1, \ldots, m_k)$ and $\bar{q} \in R^m$ where $m = \sum_{j=1}^{k} m_j$, find $x \in R^k$ such that

$$r_j(x) = \max_{i \in J_j} (\bar{N}^j x - \bar{q}^j)_i \geq 0, \; j = 1, \ldots, k, \; x \geq 0 \tag{1.7.1}$$

$$\sum_{j=1}^{k} x_j r_j(x) = 0, \; j = 1, \ldots, k. \tag{1.7.2}$$

We will refer to this generalization as Scarf's complementarity problem and denote this problem by SCP$(\bar{q}, \bar{N})$.

Lemke [42] formulates this problem as an equivalent linear complementarity problem LCP(q^*, M^*) where

$$q^* = \begin{bmatrix} \bar{q} \\ -e_k \\ 2e_k \end{bmatrix} \text{ and } M^* = \begin{bmatrix} 0 & -\bar{N} & F \\ F^t & 0 & 0 \\ -F^t & 0 & 0 \end{bmatrix}.$$

Note that M^* is a square matrix of order $(m + 2k)$, $e_k \in R^k$ is a column vector of all 1's of order k and

$$F = \begin{bmatrix} e^1 & 0 & \ldots & 0 \\ 0 & e^2 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & e^k \end{bmatrix}$$

is an $m \times k$ matrix where e^j, $1 \leq j \leq k$ is a column vector of all 1's of order m_j. We shall show that SCP$(\bar{q}, \bar{N})$ can be formulated as a vertical linear complementarity problem.

For a given x, (1.7.1) and (1.7.2) are equivalent to the assertion that

$$w^j = \bar{q}^j - \bar{N}^j x + r_j e^j,\ x \geq 0,\ r_j \geq 0,\ w^j \geq 0,\ w^j \not> 0,\ j = 1, \ldots, k, \quad (1.7.3)$$

$$\sum_{j=1}^{k} x_j r_j = 0,\ j = 1, \ldots, k \quad (1.7.4)$$

where the scalar $r_j = r_j(x)$. We can rewrite (1.7.3) and (1.7.4) as

$$w^j = (\bar{q}^j - e^j) - \bar{N}^j x + u_j e^j,\ x \geq 0,\ u_j \geq 0,\ w^j \geq 0,\ w^j \not> 0,\ j = 1, \ldots, k \quad (1.7.5)$$

$$r_j = -1 + u_j,\ r_j \geq 0,\ j = 1, 2, \ldots, k \quad (1.7.6)$$

$$u_j \prod_{i=1}^{m_j} w_i^j = 0,\ j = 1, 2, \ldots, k \quad (1.7.7)$$

$$\sum_{j=1}^{k} x_j r_j = 0,\ j = 1, \ldots, k. \quad (1.7.8)$$

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ and $r = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}$. We note that (1.7.5) through (1.7.8) give us the VLCP(q, A) where

$A = \begin{bmatrix} F & -\bar{N} \\ I & 0 \end{bmatrix}$ is of order $(m + k) \times 2k$ and $q = \begin{bmatrix} \bar{q} - e_m \\ -e_k \end{bmatrix}$.

In the above formulation $e_m \in R^m$ and $e_k \in R^k$ are vectors of all 1's and I is the identity matrix of order k.

LEMMA 1.7.1 *SCP$(\bar{q}, \bar{N})$ has a solution if and only if VLCP(q, A) has a solution.*

Proof. Let $(\bar{r}, \bar{x})$ be a solution SCP$(\bar{q}, \bar{N})$. Let $\bar{u}_j = \bar{r}_j + 1$ and $\bar{w}^j = (\bar{q}^j - e^j) - \bar{N}^j \bar{x} + \bar{u}_j e^j$. It is easy to verify that $\left(\begin{bmatrix} \bar{w} \\ \bar{r} \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{x} \end{bmatrix} \right)$ solves VLCP(q, A).

Conversely, let $\left(\begin{bmatrix} \bar{w} \\ \bar{r} \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{x} \end{bmatrix} \right)$ solve VLCP(q, A). Note that $\bar{u}_j = \bar{r}_j + 1$, $1 \leq j \leq k$. Hence $\bar{u}_j > 0$ which implies $\bar{w}^j \not> 0$. Thus $(\bar{r}, \bar{x})$ solves SCP$(\bar{q}, \bar{N})$. ∎

1.8 Lemke's Equivalent Formulation

Given the VLCP(q, A) with a vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$, Lemke [42] observes that this can also be represented in an equivalent LCP form. To do this Lemke defines a matrix F of order $m \times k$ as in Section 1.7. The equivalent LCP considered by him is given below:

Find $w \in R^m, u \in R^m$, $v \in R^k$ and $z \in R^k$ such that

$$\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} q \\ e \end{bmatrix} + \begin{bmatrix} 0 & A \\ -F^t & 0 \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} u \\ z \end{bmatrix} \geq 0, \begin{bmatrix} w \\ v \end{bmatrix}^t \begin{bmatrix} u \\ z \end{bmatrix} = 0.$$

Note that in Lemke's equivalent formulation the matrix $\tilde{\mathcal{M}} = \begin{bmatrix} 0 & A \\ -F^t & 0 \end{bmatrix}$ is of order $(m+k) \times (m+k)$.

LEMMA 1.8.1 *([42]) VLCP(q, A) has a solution if and only if Lemke's equivalent LCP has a solution.*

Proof. Let $\left(\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} u \\ z \end{bmatrix} \right)$ solve LCP$(\tilde{q}, \tilde{\mathcal{M}})$ where $\tilde{q} = \begin{bmatrix} q \\ e \end{bmatrix}$. We partition the vector u into k subvectors $u^j \in R^{m_j}, j = 1, 2, \ldots, k$, so that $u = ((u^1)^t, \ldots, (u^k)^t)^t$. Note that $v_j = (-F^t u)_j + 1 = -\sum_{r \in J_j} u_r^j + 1$.
Now, $z_j > 0 \Rightarrow v_j = 0 \Rightarrow \sum_{r \in J_j} u_r^j = 1 \neq 0$ implies that there exists a $p(j) \in J_j$ such that $u_{p(j)}^j > 0 \Rightarrow w_{p(j)}^j = 0$ and $w^j > 0 \Rightarrow u^j = 0 \Rightarrow v_j = 1 > 0 \Rightarrow z_j = 0$. Hence $z_j \prod_{r \in J_j} w_r^j = 0, j = 1, 2, \ldots, k$. Therefore (w, z) solves VLCP(q, A).

Conversely, let (w, z) solve VLCP(q, A). Let $\alpha = \{j \mid z_j > 0\}$. For $j \in \alpha$ there exists a $p(j) \in J_j$ such that $w_{p(j)}^j = 0$. Define u^j as follows:

$$u_r^j = \begin{cases} 1 & \text{for } j \in \alpha \text{ and } r = p(j) \\ 0 & \text{for } j \in \alpha \text{ and } r \neq p(j) \\ 0 & \text{for } j \in \bar{\alpha} \text{ and for all r.} \end{cases}$$

For $j \in \alpha$, $v_j = -\sum_{r \in J_j} u_r^j + 1 = -u_{p(j)}^j + 1 = 0$ and for $j \in \bar{\alpha}$,
$v_j = -\sum_{r \in J_j} u_r^j + 1 > 0$. Hence, $w^t u = 0$ and $v^t z = 0$. Therefore
$\left(\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} u \\ z \end{bmatrix} \right)$ solves LCP$(\tilde{q}, \tilde{\mathcal{M}})$. This concludes the proof. ■

1.9 Degree Theory

We use some concepts of degree theory in Chapter 2 and Chapter 6. For the concept and the properties of the degree we refer to Lloyd [45] and Ortega and Rheinboldt [68]. For the use of this concept in linear complementarity we refer to Cottle, Pang and Stone [5], Gowda [26], Howe and Stone [36] and Morris [61].

Let Ω be a bounded open set in R^k and let $\partial\Omega$ denote its boundary and $\bar{\Omega}$ its closure. Let dist$(0, S) := \inf\{||s||, s \in S\}$ where $S \subseteq R^k$ and $0 \in R^k$ denote the distance between 0 and the set S. Let $f : \bar{\Omega} \to R^k$ be continuous with $0 \notin f(\partial\Omega)$. Then the *degree of f at 0 relative to Ω* is defined and is an integer. This degree is denoted by deg$(f, \Omega, 0)$. See [68, Definition 6.1.7]. We make use of the following properties of degree.

Property 1: (Existence property) If deg$(f, \Omega, 0) \neq 0$ then the equation $f(z) = 0$ has a solution in Ω.

Property 2: (Homotopy invariance property) Suppose that
$H : [0,1] \times \bar{\Omega} \to R^k$ is continuous and $0 \notin H(t, \partial\Omega)$ for all $t \in [0,1]$.
Then deg$(H(0,\cdot), \Omega, 0) =$ deg$(H(1,\cdot), \Omega, 0)$.
For convenience, we shall denote $H(t,\cdot)$ as $H_t(\cdot)$.

Property 3: (Nearness property) Suppose that deg$(f, \Omega, 0)$ is defined. If g is a continuous function on $\bar{\Omega}$ such that

$$\sup_{x \in \bar{\Omega}} ||g(x) - f(x)|| < \text{dist}(0, f(\partial\Omega))$$

then deg$(g, \Omega, 0)$ is defined and is equal to deg$(f, \Omega, 0)$.

Property 4: (Domain decomposition property) Suppose $\Omega = \cup_{i=1}^{m}\Omega_i$ where Ω_i's are bounded open sets such that $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$ and $\cup_{i=1}^{m}\bar{\Omega}_i = \bar{\Omega}$. Also, suppose that $0 \notin \cup_{i=1}^{m} f(\partial\Omega_i)$. Then
$\deg(f,\Omega,0) = \sum_i \deg(f,\Omega_i,0)$.

Property 5: (Excision property) Suppose that $\deg(f,\Omega,0)$ is defined and K is a compact subset of $\bar{\Omega}$ such that there is no solution of $f(x) = 0$ in K. Then $\deg(f,\Omega,0) = \deg(f,\Omega\backslash K,0)$.

Property 6: Suppose f is differentiable at z^* where $z^* \in \Omega$ is the unique point satisfying $f(z^*) = 0$. Suppose also that the Jacobian matrix $f'(z^*)$ is nonsingular. Then,

$$\deg(f,\Omega,0) = \operatorname{sgn}(\det f'(z^*))$$

where for any real number a, $\operatorname{sgn}(a) = +1$ if $a > 0$ and -1 if $a < 0$.

A particular case of Property 6 occurs when f is a piecewise linear function of the form $f(z) = z \wedge (Mz + q)$. In this case, $f(z) = 0$ is a solution of the linear complementarity problem LCP(q, M) introduced in Section 1.1. Suppose, $\bar{z}$ is the unique solution of LCP(q, M) in a bounded open set of R^k and $\bar{z} + M\bar{z} + q > 0$. It is known that ([49], Cor. 3.2) if $\mathcal{I} = \{i \mid \bar{z}_i \neq 0\}$ and $M_{\mathcal{I}\mathcal{I}}$ denotes the submatrix of M corresponding to the index set $\mathcal{I}$ then $M_{\mathcal{I}\mathcal{I}}$ is nonsingular and

$$\deg(f,\Omega,0) = \operatorname{sgn}(\det M_{\mathcal{I}\mathcal{I}}).$$

Given the LCP(q, M), define a piecewise linear map $F_M(x)$ as

$$F_M(I_{.j}) = I_{.j}$$

$$F_M(-I_{.j}) = -M_{.j} \text{ and}$$

$$F_M(x) = \sum_j x_j^+ F_M(I_{.j}) + \sum_j x_j^- F_M(-M_{.j}),\ 1 \leq j \leq n$$

for any $x \in R^n$ where $x_j^+ = \max(0, x_j)$ and $x_j^- = \max(0, -x_j)$. Then it is clear that LCP(q, M) is equivalent to finding $x \in R^n$ such that $F_M(x) = q$.

Let $M \in R^{n\times n}$ be a given nondegenerate matrix and $x \in R^n$ be a point in the interior of an orthant $\operatorname{Pos} C_I(\alpha)$, for some $\alpha \subseteq \{1, 2, \ldots, n\}$ where $\operatorname{Pos} C_I(\alpha) = \{y \in R^n \mid y_i \leq 0 \text{ for } i \in \alpha \text{ and } y_i \geq 0 \text{ for } i \notin \alpha\}$. The index of M at x denoted by $\operatorname{ind}_M(x)$ is defined as $\operatorname{sgn}(\det J_M(x))$ which is seen to be equal to $\operatorname{sgn}(\det M_{\alpha\alpha})$ where $\det M_{\emptyset\emptyset} = 1$ and $J_M(x)$ stands for the Jacobian of the map F_M at x. Thus $\operatorname{ind}_M(x) = +1$ or -1 depending on whether $\det J_M(x)$ is positive or negative. As shown in Cottle, Pang and Stone [5] the quantity $\sum_{x\in F_M^{-1}(q)} \operatorname{ind}_M(x)$ is the same for all nondegenerate $q \in R^n$ and is called the degree of F_M. We will denote the *degree of* F_M simply by $\deg F_M$. Let $q \in R^n$ be nondegenerate with respect to M. Suppose LCP(q, M) has a solution (w, z). Take $x = w - z$ and note that $F_M(x) = q$. Hence fixing this q we can rewrite the degree of F_M as

$$\deg F_M = \sum_{\{\alpha\,:\,q\in \operatorname{pos}(B_\alpha)\}} \operatorname{sgn}\ \det(M_{\alpha\alpha})$$

where $q \in R^n$ is nondegenerate with respect to M and

$$B_\alpha = \begin{bmatrix} -M_{\alpha\alpha} & 0 \\ -M_{\bar{\alpha}\alpha} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

is a complementary matrix of $[I, -M]$. The assumption of nondegeneracy of the matrix M and the vector $q \in R^n$ with respect to M made in defining the degree of the map F_M may be relaxed. See Gowda [26] for more details.

The following result based on the degree of F_M is well known in linear complementarity.

THEOREM 1.9.1 *Let $M \in R^{n\times n}$ be such that LCP$(0, M)$ has a unique solution. If $\deg F_M \neq 0$, then $M \in Q$.*

Chapter 2

The Vertical Linear Complementarity Problem

2.1 Introduction

In this chapter, we consider the vertical linear complementarity problem VLCP(q, A) where A is a vertical block matrix and present an equivalent formulation of VLCP(q, A) as LCP(q, M). This formulation is different from the equivalent formulation given by Lemke (see Section 1.8) which results in a square matrix of order $(m+k) \times (m+k)$. In contrast, the equivalent LCP presented in Section 2.3 is of order $m \times m$ and many more results on VLCP are easily obtained using this formulation.

Gowda and Sznajder [25] show that with the usual ordering on R^n, a generalized order linear complementarity problem (GOLCP) is equivalent to a VLCP. However, this formulation of the VLCP(q, A) is in the form of equations, and using this formulation they define the degree of a VLCP map.

In Section 2.2, we present the required definitions. Section 2.3 presents an equivalent formulation of VLCP(q, A) as LCP(q, M) and extends some known results. In this section, generalizations in the VLCP setting of the classes of matrices introduced by Garcia and Todd are also presented. The concept of generalized principal pivot transform of a vertical block matrix is introduced and discussed in Section 2.4. In Section 2.5, a comparison of the algorithms by Cottle and Dantzig for VLCP and that of Lemke for the equivalent LCP is made and they are shown to be one and the same. We also consider the problem of computing the generalized Nash equilibrium point of the generalized bimatrix game introduced by Gowda and Sznajder [28]. Finally, in Section 2.6, we consider the VLCP degree of a vertical block matrix.

2.2 Preliminaries

DEFINITION 2.2.1 A vertical block matrix A of type $(m_1, \ldots, m_k)$ is called a *vertical block Q-matrix* if VLCP(q, A) has a solution for each $q \in R^m$.

DEFINITION 2.2.2 A vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ is called a *vertical block Q_0-matrix* if for any $q \in R^m$, (1.3.1) has a solution implies that the VLCP(q, A) has a solution.

2.3 An Equivalent Formulation

The problem VLCP(q, A) can be formulated as LCP(q, M) as follows:
Let us consider a vertical block matrix A of type $(m_1, \ldots, m_k)$ where m_j is the size of the j^{th} block. We construct a matrix M by copying $A_{.j}$, m_j times for $j = 1, 2, \ldots, k$ (for example $A_{.1}$ is copied m_1 times, $A_{.2}$ is copied m_2 times etc.). Thus $M_{.p} = A_{.s}$ $\forall\, p \in J_s$. This construction leads to a square matrix M of order $m \times m$. We call the matrix obtained in this manner the *equivalent square matrix* of A. We note that M is singular if $m > k$.

LEMMA 2.3.1 *Given the VLCP(q, A), let M be the equivalent square matrix of A. VLCP(q, A) has a solution if and only if LCP(q, M) has a solution.*

Proof. Suppose (w, z) is a solution to VLCP(q, A). We construct a solution (u, v) to LCP(q, M) as follows:

We take $u = w$. Note that $z_j > 0$ implies $\exists\ p(j) \in J_j$ such that $w_{p(j)} = 0$. Define

$$v_r = \begin{cases} 0, & \text{if } r \neq p(j) \text{ for any } 1 \leq j \leq k \\ z_j, & \text{if } \exists \text{ a } j,\ 1 \leq j \leq k, \text{ such that } r = p(j) \end{cases}$$

Note that v_r is well defined. Now it is easy to see that (u, v) solves LCP(q, M).

Conversely, suppose (u, v) is a solution to the LCP(q, M). Define the vector $z \in R^k$ by taking

$$z_j = \sum_{i \in J_j} v_i.$$

Note that if $z_j > 0$, $\exists\ i \in J_j$ such that $v_i > 0$ and hence $u_i = 0$. Hence with $w = u$, (w, z) solves VLCP(q, A). ∎

In order to distinguish the equivalent LCP$(\tilde{q}, \tilde{M})$ introduced in Section 1.8 from the equivalent LCP(q, M) introduced in this section, we adopt the following terminology. Given VLCP(q, A), the *equivalent LCP(q, M)* refers to the problem introduced in this section and *Lemke's equivalent LCP* refers to the problem introduced in Section 1.8.

It is easy to see that a nonsingular complementary matrix of $[I, -M]$, where M is the equivalent square matrix of a vertical block matrix A of type $(m_1, \ldots, m_k)$ is a nonsingular proper matrix of $[I, -A]$. Thus nonsingular complementary matrices of $[I, -M]$ are in $1-1$ correspondence with the nonsingular proper matrices of $[I, -A]$. Thus the class of complementary cones of $[I, -M]$ with nonempty interior is the same as the class of proper cones of $[I, -A]$ with nonempty interior. Due to this fact, the following results for vertical block matrices are easy to observe.

LEMMA 2.3.2 *A is a vertical block Q-matrix if and only if the equivalent square matrix M is a Q-matrix.*

LEMMA 2.3.3 *A is a vertical block Q_0-matrix if and only if the equivalent square matrix M is a Q_0-matrix.*

In what follows, we introduce some of the classes of vertical block matrices, using the equivalent LCP.

Generalized Garcia's class: Let $A \in R^{m\times k}$ be a vertical block matrix of type $(m_1,\ldots,m_k)$. Let M be its equivalent square matrix. A is said to be in *generalized Garcia's class* if for some $d > 0$, $d \in R^m$, the equivalent square matrix $M \in L(d)$. (See Section 1.4.) We say that A is in generalized $E^*(d)$ class if $M \in E^*(d)$.

We note that the class generalized $E^*(0)$ is the same as the class of type R_0-matrices introduced by Gowda and Sznajder [25].

Generalized Todd's class: Let $A \in R^{m\times k}$ be a vertical block matrix of type $(m_1,\ldots,m_k)$. A is said to be in the *generalized Todd's class* if for some $d > 0$, $d \in R^m$, the equivalent square matrix $M \in \bar{L}(d)$. The following result now follows from linear complementarity theory.

THEOREM 2.3.1 *Let $A \in R^{m\times k}$ be a vertical block matrix of type $(m_1,\ldots,m_k)$. If $A \in L(d)$ (or $\bar{L}(d)$), then A is a Q_0-matrix.*

Using our equivalent formulation we can easily generalize Theorem 1.4.5.

THEOREM 2.3.2 *Suppose $A \geq 0$ is a vertical block matrix of type $(m_1,\ldots,m_k)$. A is vertical block Q-matrix if and only if $a_{ji} > 0 \ \forall \ j \in J_i$.*

Proof. From Lemma 2.3.2, it is clear that the equivalent square matrix M, formed out of A, has to be in the class Q, if $A \in Q$. M being nonnegative, the result follows straight from Theorem 1.4.5. ∎

Recently Murthy, et al. [65] have proved the following on nonnegative square matrices.

THEOREM 2.3.3 *Let $M \geq 0$ be an $n \times n$ matrix. M is Q_0-matrix if and only if $M_{i.} \neq 0 \Rightarrow m_{ii} > 0 \;\forall\, 1 \leq i \leq n$.*

We can state a similar result for vertical block matrices using our equivalent formulation.

THEOREM 2.3.4 *Let $A \geq 0$ be a vertical block matrix of type $(m_1, \ldots, m_k)$. A is a vertical block Q_0-matrix if and only if*

$$\forall_{1 \leq i \leq k} \forall_{j \in J_i} [A_{j.} \neq 0 \Rightarrow a_{ji} > 0]. \tag{2.3.1}$$

Proof. Consider the equivalent LCP(q, M). Note that $A_{j.} \neq 0 \Leftrightarrow M_{j.} \neq 0$ and $a_{ji} > 0,\ j \in J_i \Leftrightarrow a_{ji} = m_{jj} > 0$.

Suppose (2.3.1) holds. Then by Theorem 2.3.3, M is a Q_0-matrix. Hence it follows that A is a vertical block Q_0-matrix.

Conversely, suppose A is a vertical block Q_0-matrix. It now follows that the equivalent matrix M is a Q_0-matrix. Hence, if for some $j_0 \in J_i$, $A_{j_0.} \neq 0$ then $M_{j_0.} \neq 0$ and hence $m_{j_0 j_0} = a_{j_0 i} > 0$. ■

2.4 Generalized Principal Pivot Transforms

A useful concept in the study of the linear complementarity problem is that of a principal pivot transform (PPT). For a square matrix M suppose B is a nonsingular complementary matrix of $[I, -M]$. Let $\bar{B}$ be the matrix of columns of $[I, -M]$ not in B. We say that F is a PPT of M with respect to B if $F = -B^{-1}\bar{B}$. We note that LCP(q, M) is equivalent to LCP$(B^{-1}q, F)$. See Parsons [74]. We may introduce the notion of a generalized principal pivot transform for the VLCP(q, A) as follows:

DEFINITION 2.4.1 Suppose B is a nonsingular proper matrix. We say that F is a generalized PPT of A with respect to B if $F = -B^{-1}\bar{B}$, where $\bar{B}$ is the matrix of columns not in B. Note that $\bar{B}$ contains k columns and that F is an $m \times k$ vertical block matrix of type $(m_1, \ldots, m_k)$.

Let us now compare the generalized PPT of a vertical block matrix A of type $(m_1, \ldots, m_k)$ with that of the PPT of M, the equivalent square matrix formed from A, with respect to a nonsingular proper matrix B of $[I, -A]$ (which is also a complementary matrix of $[I, -M]$). Without loss of generality, let us assume that the first l columns of A are in B and that the rest of the columns are from the identity matrix I of order m. Let the i^{th} column of A for $1 \leq i \leq l$ occupy the j_i^{th} position in the matrix B. It is clear that $j_i \in J_i$, for each $i = 1, \ldots, l$. Then, the columns of F, which are obtained from A by the generalized principal pivot transform with respect to B are as follows:

$$F_{\cdot i} = \begin{cases} -B^{-1}I_{\cdot j_i}, & \text{if } i \in \{1, \ldots, l\} \\ B^{-1}A_{\cdot i}, & \text{if } i \in \{l+1, \ldots, k\}. \end{cases}$$

Suppose that M is the equivalent square matrix of the vertical block matrix A. As observed in the earlier section, B is also a complementary matrix of $[I, -M]$ and we can consider a principal pivot transform M^* of M, with respect to B. The columns of M^* for $i = 1, \ldots, m$ are given by

$$M^*_{\cdot i} = \begin{cases} -B^{-1}I_{\cdot i}, & \text{if } i \in \{j_1, \ldots, j_l\} \\ B^{-1}M_{\cdot i}, & \text{otherwise.} \end{cases}$$

Since for each index set J_i, the columns in M corresponding to J_i are identical by definition, we can see that M^* has several columns of negative unit vectors. Hence, if an equivalent square matrix of the generalized PPT F of A is constructed, it may not be equal to M^*. This is illustrated by the following example.

EXAMPLE 2.4.1 Consider the following vertical block matrix A of type (2,2) and its equivalent square matrix M:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $B = [I_{.1} \ \ -A_{.1} \ I_{.3} \ I_{.4}]$ be the complementary matrix with respect to which we consider the pivot transform of A and M. The resulting matrices, *i.e.*, F from A due to the generalized PPT and M^* obtained from PPT of M with respect to B are given below:

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M^* = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Clearly, if the equivalent square matrix of F is constructed, it will differ from M^* in its first column.

Is it then true that if a vertical block matrix A is in the class Q, that all its legitimate generalized PPT's will also be in Q? This does not follow straight from the already known results of linear complementarity. However, the following lemma for a square matrix M would help us answer this affirmatively.

LEMMA 2.4.1 *Let $M \in R^{m\times m}$ be a Q-matrix.*

(i) If $M_{.k} = -I_{.j}$ for some $j \neq k$, then the matrix $D \in R^{m\times m}$ defined by, $D_{.i} = M_{.i}$, $i \neq k$ and $D_{.k} = M_{.j}$ is a Q-matrix.

(ii) If $M_{.k} = M_{.j}$ for some $j \neq k$, then the matrix $D \in R^{m\times m}$ defined by, $D_{.i} = M_{.i}$, $i \neq k$ and $D_{.k} = -I_{.j}$ is a Q-matrix.

Proof. Let us consider the problem LCP(q, D), for any given $q \in R^m$. We need to prove that there exists a solution to this problem. Let (w, z) be a solution for the LCP(q, M).

If in (w, z), $z_k = 0$, then (w, z) solves LCP(q, D) also. Otherwise, we can define a pair $(\bar{w}, \bar{z})$ of vectors, where $\bar{w},\ \bar{z} \in R^m$, as follows. Let $\bar{w}_i = w_i$ and $\bar{z}_i = z_i$, for $i \neq j, k,\ i = 1, \ldots, m$. The j^{th} and k^{th} coordinates of $(\bar{w}, \bar{z})$ are defined for the parts (i) and (ii) separately, using (w, z) as below:

For part (i):

$$\bar{z}_j = 0; \qquad \bar{w}_j = w_j + z_k;$$
$$\bar{z}_k = z_j; \qquad \bar{w}_k = w_k;$$

For part (ii):

$$\bar{z}_j = z_j + z_k; \qquad \bar{w}_j = 0;$$
$$\bar{z}_k = w_j; \qquad \bar{w}_k = w_k.$$

It can be easily verified that the resulting pair $(\bar{w}, \bar{z})$ is indeed a solution for the problem LCP(q, D). Hence, D belongs to the class Q in both cases. ∎

REMARK 2.4.1 The above lemma in fact holds good for Q_0-matrices also. This can be seen from the fact that the two parts of the lemma are in some sense, complementary to one another.

We illustrate Lemma 2.4.1 by the following example.

EXAMPLE 2.4.2 Comparing the following matrices,

$$M^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

we can see that one can be derived from the other using the above lemma, and that both of them belong to the class Q. However, the matrix constructed with the two columns of M^2 interchanged, will not be in Q as it does not satisfy the conditions of the above lemma.

As a consequence of Lemma 2.4.1, we have the desired result given below, on generalized PPT's of a vertical block matrix.

THEOREM 2.4.1 *Let $A \in R^{m\times k}$ be a vertical block $Q\,(Q_0)$-matrix of type $(m_1,\ldots,m_k)$. If F is a generalized PPT of A, then $F \in$ vertical block $Q\,(Q_0)$.*

Proof. Let us consider M, the equivalent square matrix of A. By Lemma 2.3.2, $M \in Q$. Let F be a generalized PPT of A with respect to a proper complementary matrix B of $[I,-A]$. Without loss of generality, let us assume that B has the first l columns of A, $l \le k$, and they occupy respectively $j_1,\ldots,j_l^{th}$ positions in B.

Corresponding to B, let M^* be the principal pivot transform of M. From linear complementarity theory, it follows that $M^* \in Q$. Let us construct a matrix $\tilde{M}$ from M^* as follows. For $r \in J_i$ where $i > l$, let $\tilde{M}_{.r} = M^*_{.r}$. For $r \in J_i,\ i \le l$ if $M^*_{.r} = -I_{.j_i}$ then let $\tilde{M}_{.r} = M^*_{.j_i}$ and for $r \in \{j_1,\ldots,j_l\}$, let $\tilde{M}_{.r} = M^*_{.j_i}$. Now, $\tilde{M} \in Q$ using part (i) of Lemma 2.4.1. It is clear from the construction of $\tilde{M}$, that it is the equivalent square matrix of F. Now from Lemma 2.3.2, as $\tilde{M} \in Q$, F is also a Q-matrix. A similar proof goes through for the class of vertical block Q_0-matrices as well. This completes the proof. ∎

2.5 On the Cottle-Dantzig Algorithm for the VLCP

In [4], Cottle and Dantzig describe a procedure for processing a generalized linear complementarity problem, which is an extension of Lemke's algorithm for the ordinary linear complementary problem. See Chapter 1 for Lemke's algorithm and the Cottle-Dantzig algorithm (Algorithm CD). In the remainder of this section we make the ***standard nondegeneracy assumption*** that all the almost proper basic feasible solutions are nondegenerate. Under this assumption the variable that leaves the basis at step 3 of the algorithm is determined uniquely at each iteration and the procedure either terminates in a solution to the VLCP(q, A) or in a secondary proper ray.

An alternative method of solving VLCP(q, A) is to apply Lemke's algorithm to the LCP(q, M). See Lemke [43]. It is a matter of curiosity to find out the difference in the two approaches. One can make the following observation.

REMARK 2.5.1 Suppose B is an almost complementary basis matrix generated by the application of Lemke's algorithm with the artificial vector d to the equivalent problem LCP(q, M) then there is at most one index i, $i \in J_s$ such that $-M_{.i}$ is a column of B, for each $1 \leq s \leq k$. This follows from the non-singularity of B.

The following lemmas are also immediate.

LEMMA 2.5.1 *Suppose B is an almost complementary basis matrix generated by the application of Lemke's algorithm with the artificial vector d to the equivalent problem LCP(q, M) and that $-M_{.p} = -A_{.s}$ where $p \in J_s$ is the r^{th} column of B. Then the columns corresponding to the variables z_i, for $i \in J_s$ in the tableau corresponding to this basis are all equal to $I_{.r}$, where r is the column index of $-M_{.p}$ in B.*

LEMMA 2.5.2 *Suppose B is an almost complementary basis matrix generated by the application of Lemke's algorithm with the artificial vector d to the equivalent problem LCP(q, M) and suppose $-M_{.p} = -A_{.s}$ where $p \in J_s$ is the r^{th} column of B. Suppose now, the complementary pivot rule selects $-M_{.j}$ where $j \in J_s$ to be included in the next iteration. The resulting new almost complementary basis* *matrix and the new tableau have the same set of columns as the previous ones except that the set of basic variables now includes z_j in the place of z_p.*

Proof. It is obvious that when $-M_{\cdot j}$ is included in the basis, it must now replace $-M_{\cdot p}$ as $-M_{\cdot p} = -M_{\cdot j} = -A_{\cdot s}$, in view of Remark 2.5.1. Now from Lemma 2.5.2 it follows that, since the pivot column contains 0's in all the rows except the pivot row, the new tableau will be the same as the old tableau, except for the index set of basic variables, which now includes the index of z_j in place of the index of z_p. ∎

We say that Lemke's algorithm applied to the equivalent LCP(q, M) executes a trivial pivot step, when a variable z_j replaces another variable z_p, where j and p are in the same index set J_s corresponding to the s^{th} block, for some s, $1 \leq s \leq k$.

REMARK 2.5.2 Note that the generalized complementary pivot rule of Cottle and Dantzig may determine a pair of the form $(I_{\cdot j}, I_{\cdot p})$ as the unique nonbasic pair of columns where j and p are in the same index set J_s and the column to enter the basis as, say $I_{\cdot p}$. However Lemke's algorithm applied to the equivalent problem LCP(q, M) will first select $-M_{\cdot j}$ as the column to enter the basis. This will necessarily replace $-M_{\cdot p}$ by executing a trivial pivot step and in the iteration following this trivial pivot step $I_{\cdot p}$ will be included as a column of the basis matrix.

Let B_1 and B_2 be two different almost complementary basis matrices generated by Lemke's algorithm applied to the equivalent LCP(q, M). We say that B_1 and B_2 are *indistinct* if the sets of columns in B_1 and B_2 are the same; otherwise, we say that B_1 and B_2 are *distinct*.

THEOREM 2.5.1 *The sequence of almost proper basis matrices generated by the application of Algorithm CD with the artificial vector d to the VLCP(q, A) is the same as the sequence of distinct almost complementary basis matrices generated by applying Lemke's algorithm with the artificial vector d to the equivalent LCP(q, M).*

Proof. We make the standard nondegeneracy assumption. Let $q \not\geq 0$. The initial basis matrix B_0 generated by both the algorithms is the same and is given by $(B_0)_{.j} = I_{.j} \ \forall\ j \neq k$, and $(B_0)_{.k} = -d$. Algorithm CD then chooses column $-A_{.s}$, where $k \in J_s$ as the pivot column and Lemke's algorithm chooses $-M_{.k} = -A_{.s}$ as the pivot column. If at the end of this iteration either of the algorithms terminates in a ray, the other also does so, and the algorithm stops. Suppose Algorithm CD does not end in a secondary ray, but determines that w_i, where $i \in J_r$ for some r is to be eliminated from the basis. Lemke's algorithm applied to the equivalent (q, M) also determines that w_i is to leave the basis, as the minimum ratio test is the same in both cases. Therefore the columns of the basis matrix for both the algorithms are the same in the second iteration.

Let us make the induction hypothesis that the almost complementary basis matrix B_i generated by Algorithm CD at iteration i is the same as the i^{th} distinct almost complementary basis matrix generated by Lemke's algorithm, for $i \leq t$. At this stage, let Lemke's algorithm be at its l^{th} iteration. If the algorithms do not terminate at this stage we have the following two cases to be considered:

Case (i). In the t^{th} tableau of the Cottle-Dantzig algorithm, the unique pair of related nonbasic columns is $(I_{.p}, -A_{.s})$ for some $p \in J_s$ of which the generalized complementary pivot rule chooses the column $-A_{.s}$ to be included in the basis at the next iteration. This implies that the unique pair of complementary non-basic variables in the l^{th} tableau of Lemke's algorithm is $(I_{.p}, -M_{.p})$ where $-M_{.p} = -A_{.s}$. In this case the next tableau generated by Lemke's algorithm will be distinct from its l^{th} tableau and will be the same as the $(t+1)^{th}$ tableau generated by Algorithm CD.

Case (ii). In the t^{th} tableau of the Cottle-Dantzig algorithm, the unique pair of related nonbasic columns is $(I_{.p}, I_{.j})$ for some $j,\ p \in J_s,\ 1 \leq s \leq k$, of which the generalized complementary pivot rule chooses $I_{.j}$ to be included in the basis at the next iteration. The corresponding Lemke's tableau has $(I_{.p}, -M_{.p})$ as the pair of complementary nonbasic columns. From the induction hypothesis, it follows that the column eliminated from the basis in the previous iteration is $I_{.p}$. Hence at its $(l+1)^{th}$ iteration $-M_{.p}$ is included in the basis by Lemke's algorithm. Since $j, p \in J_s$, it follows that $-M_{.j}$ is a column in the basis in the previous iteration and that $-M_{.p} = -M_{.j} = -A_{.s}$. Thus the $(l+1)^{th}$ tableau is obtained by a trivial pivot step in which $-M_{.p}$ replaces $-M_{.j}$ in the basis. However, the $(l+1)^{th}$ basis matrix obtained by Lemke's algorithm is indistinct from the basis matrix at its l^{th} iteration. This trivial step is followed by the $(l+2)^{th}$ tableau in which column $I_{.j}$ is included in the basis matrix. Thus the basis matrix generated at the $(l+2)^{th}$ iteration of Lemke's algorithm which is distinct from those of l^{th} and $(l+1)^{th}$ is the same as $(t+1)^{th}$ basis matrix generated by Algorithm CD. The theorem now follows from the principle of induction. This concludes the proof. ■

REMARK 2.5.3 In view of the above, Algorithm CD for the VLCP(q, A) can be thought of as an efficient implementation of Lemke's algorithm for the equivalent LCP(q, M) in which the trivial pivots are skipped.

As a consequence of the above analysis, we have the following:

THEOREM 2.5.2 *The class of vertical block matrices A which Algorithm CD with artificial vector d can process the VLCP(q, A) is the same as the class of equivalent square matrices M obtained from A, which Lemke's algorithm can process.*

From the above theorem it follows that Algorithm CD can process the generalized Todd's class $\bar{L}(d)$ of vertical block matrices.

It is already known that Algorithm CD can process vertical block matrices in the vertical block copositive-plus class (see Cottle and Dantzig [4]).

2.5.1 Generalized Bimatrix Game

In this subsection, we shall consider the question of computing a generalized Nash equilibrium point for the generalized bimatrix game introduced by Gowda and Sznajder [28]. This generalized bimatrix game is given as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be two given finite sets of matrices, $\mathcal{A}$ containing s matrices and $\mathcal{B}$ containing r matrices, each of order $m \times n$. Player I forms his payoff matrix whose i^{th} row is chosen as the i^{th} row of some $A \in \mathcal{A}$ and then plays his choice of a mixed strategy over $\{1, 2, \ldots, m\}$. Similarly, player II (the column player) forms his payoff matrix whose j^{th} column is chosen by him as the j^{th} column of some $B \in \mathcal{B}$ and then plays his choice of mixed strategy over $\{1, 2, \ldots, n\}$. The rest of the description of the game is the same as that of a bimatrix game.

Suppose, $\mathcal{A} = \{A^p \mid p = 1, 2, \ldots, s\}$ and $\mathcal{B} = \{B^p \mid p = 1, 2, \ldots, r\}$. Consider the matrices C^j, $j = 1, 2, \ldots, m$ and D^j, $j = 1, 2, \ldots, n$ defined as follows:

$$C^j_{i\cdot} = A^i_{j\cdot}, \quad 1 \leq i \leq s$$
$$D^j_{i\cdot} = (B^i)^t_{j\cdot}, \quad 1 \leq i \leq r.$$

Without loss of generality, we may assume that each A^p, $p = 1, 2, \ldots, s$ and each B^p, $p = 1, 2, \ldots, r$ are positive matrices. Hence each C^j, $j = 1, 2, \ldots, m$ and each D^j, $j = 1, 2, \ldots, n$ are positive matrices.

$$\text{Let } X = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^m \end{bmatrix} \text{ and } Y = \begin{bmatrix} D^1 \\ D^2 \\ \vdots \\ D^n \end{bmatrix}$$

where each C^j is of order $s \times n$ and each D^j is of order $r \times m$. Note that by our assumption $X > 0$, $Y > 0$.

Consider the matrix

$$A = \begin{bmatrix} 0 & C^1 \\ 0 & C^2 \\ \vdots & \vdots \\ 0 & C^m \\ D^1 & 0 \\ D^2 & 0 \\ \vdots & \vdots \\ D^n & 0 \end{bmatrix} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$$

where the '0' blocks are of appropriate orders. This is a vertical block matrix of type $(s, \dots, s, r, \dots, r)$ where the number of blocks is $(m + n)$.

It is easy to see that a generalized Nash equilibrium point as considered by Gowda and Sznajder [28] of the game described above can be computed by obtaining a solution to the VLCP$(-e, A)$ where $-e$ is the column vector of order $(ms + nr)$, each of whose coordinate is -1.

The equivalent LCP matrix M, for the above vertical block matrix A is of the form

$$M = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$$

where $C > 0, \; D > 0$.

It is well known that Lemke's algorithm with any positive vector d as the artificial vector cannot process the equivalent LCP$(-e, M)$, since M has the form $\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ where C and D are positive and consist of sets of identical columns. However we can apply the algorithm of Lemke and Howson [44] to this problem and compute an equilibrium point. We thus have the following theorem.

THEOREM 2.5.3 *A generalized Nash equilibrium point of the generalized bimatrix game can be computed by a generalized Lemke-Howson algorithm in which* *the complementary pivot rule is replaced by the generalized complementary pivot rule given by Cottle and Dantzig.*

Proof. This follows from the fact that such an algorithm is the same as Lemke-Howson algorithm applied to the equivalent LCP$(-e, M)$, skipping the trivial pivots as discussed earlier. ■

We may also observe that it is possible to assume without loss of generality that the vertical block matrix A associated with the generalized bimatrix game has the form

$$A = \begin{bmatrix} 0 & C^1 \\ 0 & C^2 \\ \vdots & \vdots \\ 0 & C^m \\ D^1 & 0 \\ D^2 & 0 \\ \vdots & \vdots \\ D^n & 0 \end{bmatrix}$$

where each $C^j > 0$ and each $D^j < 0$.

We then have the following:

Theorem 2.5.4 *Suppose the problem of computing a generalized Nash equilibrium point of the generalized bimatrix game is formulated as finding a solution to the VLCP$(-e, A)$ where $A = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ with $X > 0$ and $Y < 0$. Then we can compute an equilibrium point by applying Algorithm CD to such a VLCP.*

Proof. The equivalent matrix M of the matrix A in this case has the form $M = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ where $C > 0$ and $D < 0$. Such a matrix has been shown to be in the class L (see Eaves [11]). Hence Lemke's algorithm initiated with any positive vector d will compute a solution of the LCP$(-e, M)$, since a solution exists. The conclusion of the theorem now follows from Theorem 2.5.2. ■

2.6 Some Results in Degree Theory

Given a vector $q \in R^m$ and an $m \times k$ vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$, consider the matrices $\bar{X}^1, \bar{X}^2, \ldots, \bar{X}^k$ each of order $r \times k$ where $r = \max(m_1, m_2, \ldots, m_k)$ is defined as follows:

Let A^i denote the i^{th} block submatrix of A of order $m_i \times k$ and q^i, the corresponding block of q. Let

$$\begin{aligned} (\bar{X}^i)_{j\cdot} &= (A^i)_{j\cdot} \text{ if } j \leq m_i \\ &= (A^i)_{1\cdot} \text{ if } m_i < j \leq r. \end{aligned}$$

Let $\bar{p} \in R^{kr}$ be defined as follows:

$$\begin{aligned} (\bar{p}^i)_j &= (q^i)_j \text{ if } j \leq m_i \\ &= (q^i)_1 \text{ if } m_i < j \leq r. \end{aligned}$$

Let $\bar{X} = \begin{bmatrix} \bar{X}^1 \\ \bar{X}^2 \\ \vdots \\ \bar{X}^k \end{bmatrix}$. Note that $\bar{X}$ is a $kr \times k$ vertical block matrix of type $(r, r, \ldots, r)$. It is easy to see that the VLCP(q, A) is equivalent to VLCP$(\bar{p}, \bar{X})$. See Gowda and Sznajder [25]. We define the matrices X^j, of order $k \times k$, $1 \leq j \leq r$, by taking

$$(X^j)_{i\cdot} = (\bar{X}^i)_{j\cdot}, \ (p^j)_i = (\bar{p}^i)_j, \ 1 \leq i \leq k.$$

Let $X = \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^r \end{bmatrix}$ be a vertical block matrix of type $(k, k, \ldots, k)$.

We can then formulate VLCP(q, A) as solving the system of equations

$$F_{(X,p)}(x) = 0 \text{ where } F_{(X,p)}(x) = x \wedge (X\,x + p)$$

where for any two vectors a, b, $a \wedge b$ denotes the component-wise minimum vector. Assuming $F_{(X,0)}(x) = 0$ has only the trivial solution, Gowda and Sznajder [25] define the GOLCP degree of X using the definition of degree as in [45].

In this chapter, we take a slightly different approach, which makes it easier to calculate the VLCP degree of a vertical block matrix A. Given the vertical block matrix A of type $(m_1, \ldots, m_k)$ consider the mapping $F_A : R^m \to R^m$ defined as follows:

Given $x \in R^m$, let x^+ and x^- be the positive and negative parts of x. Let

$$F_A(x) = x^+ - \sum_{i=1}^{k} A_{\cdot i} (\sum_{j \in J_i} x_j^-).$$

It is easy to see that given a $q \in R^m$, if there is a x such that $F_A(x) = q$, then defining $w = x^+$ and $z \in R^k$ by taking $z_i = \sum_{r \in J_i} x_r^-$, we see that (w, z) solves VLCP(q, A). Actually, it is easy to see that the VLCP map $F_A(x)$ defined above is the same as LCP map $F_M(x)$ where M is the equivalent matrix of A.

Also, it is easy to observe the following:

LEMMA 2.6.1 *Let A be a given vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. Let M be the equivalent square matrix of order m. VLCP$(0, A)$ has a unique solution if and only if the equivalent LCP$(0, M)$ has a unique solution.*

Proof. This follows from the proof of Lemma 2.3.1. ∎

DEFINITION 2.6.1 A vertical block matrix of type $(m_1, \ldots, m_k)$ is said to be a *vertical block R_0-matrix* if VLCP$(0, A)$ has the unique solution $w = 0$, $z = 0$.

LEMMA 2.6.2 *Let A be an $m \times k$ vertical block matrix of type $(m_1, \ldots, m_k)$. Suppose each of the representative submatrix of A is an R_0-matrix. Then the equivalent matrix M of A is an R_0-matrix.*

Proof. Let M be the equivalent matrix of A. Suppose (u, v), $u \in R^m$, $v \in R^m$ solves the equivalent LCP$(0, M)$. Then (w, z) with $w = u$ and $z \in R^k$ where $z_i = \sum_{r \in J_i} v_r$ solves the VLCP$(0, A)$. Let $\alpha = \{i \mid z_i > 0\}$. For each $i \in \alpha$, note that there is a $j(i) \in J_i$ such that $w_{j(i)} = 0$. Let $\beta_1 = \{j(i) \mid i \in \alpha\}$. For $i \notin \alpha$, let $j(i)$ be any arbitrary index in J_i and let $\beta_2 = \{j(i) \mid i \in \bar{\alpha}\}$. Let $\beta = \beta_1 \cup \beta_2$. Note that $A_{\beta\cdot}$ is a representative submatrix of A and that (w_β, z) solves the LCP$(0, A_{\beta\cdot})$. Since by the hypothesis of the theorem, $A_{\beta\cdot}$ is an R_0-matrix, it follows that $w_\beta = 0$, $z = 0$. Now $z = 0 \Rightarrow v = 0$ as $z_i = \sum_{r \in J_i} v_r$ and each $v_r \geq 0$. Thus the only solution to LCP$(0, M)$ is $(0, 0)$. This concludes the proof. ∎

Thus essentially A is a vertical block R_0-matrix (or $A \in$ generalized $E^*(0)$) if and only if M is an R_0-matrix. If we define the VLCP degree of A to be the degree of the piecewise linear map $F_A(x)$, then it turns out that this is also the LCP degree of the equivalent square matrix of order m.

Since the LCP degree of an R_0-matrix is well studied, (see [5], [25], [26] and [45]) and can be calculated using the index theory approach, the VLCP degree defined in this manner can be more easily calculated.

It is easy to see that the degree of a vertical block P-matrix is 1. In what follows, we shall calculate the degree of a vertical block N-matrix.

DEFINITION 2.6.2 Suppose A is a vertical block matrix of type $(m_1, \ldots, m_k)$ with $\sum m_i = m$. We say that A is a *vertical block N-matrix* if every representative submatrix is an N-matrix. We say that A is a *vertical block N-matrix of* *the first category* if A is a vertical block N-matrix and A has at least one positive entry. If A is a vertical block N-matrix all of whose entries are negative, then we say that A is a *vertical block N-matrix of the second category.*

LEMMA 2.6.3 *A is a vertical block N-matrix of the first category if and only if every representative submatrix is an N-matrix of the first category.*

Proof. Note that by definition, A is a vertical block N-matrix if and only if every representative submatrix G of A is an N-matrix.

Suppose now A has a positive entry. Let $a_{ir} > 0$, for some $i(t) = i \in J_t$. (Note that as every representative submatrix G of A is an N-matrix, for $i \in J_t$, $a_{it} < 0$ and therefore $r \neq t$ in the above.)

Look at any representative submatrix G of A, whose t^{th} row is the i^{th} row of A. Note that $g_{tr} = a_{ir} > 0$. Hence G is an N-matrix of the first category. Hence by a result in [60] G has the partitioned form

$$\begin{bmatrix} G_{\nu\nu} & G_{\nu\bar{\nu}} \\ G_{\bar{\nu}\nu} & G_{\bar{\nu}\bar{\nu}} \end{bmatrix}$$

where $G_{\nu\nu} < 0,\ G_{\bar{\nu}\bar{\nu}} < 0,\ G_{\nu\bar{\nu}} > 0,\ G_{\bar{\nu}\nu} > 0$.

Without loss of generality, we assume that $\nu = \{1, 2, \ldots, \alpha\}$, $\bar{\nu} = \{(\alpha+1), \ldots, k\}$ and that $t \in \nu$ and $r \in \bar{\nu}$. Note also that $g_{rt} = a_{i(r)t} > 0$ where we assume that $i(1), i(2), \ldots, i(k)$ are the rows of A which are the $1, 2, \ldots$ and k^{th} row of G, respectively. Now consider any g_{sp} where either $s \in \nu$ and $p \in \bar{\nu}$ or $p \in \nu$ and $s \in \bar{\nu}$. We have in our notation, $g_{sp} = a_{i(s)p} > 0$ as $s \in \nu$, $p \in \bar{\nu}$. Now consider any $j \in J_s$; $j \neq i(s)$. Consider the representative submatrix X whose s^{th} row is the j^{th} row of A and whose other rows are the same as the rows of G. In particular, $x_{ps} = g_{ps} = a_{i(p)s} > 0$, as $p \in \bar{\nu}$, $s \in \nu$. This implies that $x_{sp} = a_{jp} > 0$ as X is an N-matrix. This holds for any $j \in J_s$, for all $s \in \nu$. Hence every representative submatrix has the same sign pattern as G. The converse is obvious from the definition. This completes the proof. ∎

LEMMA 2.6.4 *Suppose A is a vertical block N-matrix of the first category, then there exists a subset $\gamma \subseteq \{1, 2, \ldots, k\}$ such that, with $\nu = \cup_{i \in \gamma} J_i$, A has the following partitioned form (after a rearrangement of columns and blocks using the same permutation)*

$$A = \begin{bmatrix} A_{\nu\gamma} & A_{\nu\bar{\gamma}} \\ A_{\bar{\nu}\gamma} & A_{\bar{\nu}\bar{\gamma}} \end{bmatrix}$$

where $A_{\nu\gamma} < 0,\ A_{\bar{\nu}\bar{\gamma}} < 0,\ A_{\nu\bar{\gamma}} > 0,\ A_{\bar{\nu}\gamma} > 0$ and γ, $\bar{\gamma}$ are nonempty sets.

Proof. This follows from Lemma 2.6.3. ∎

LEMMA 2.6.5 *Let A be a vertical block N-matrix of type $(m_1, m_2, \ldots, m_k)$. Then the equivalent matrix $M \in R_0$ and $A \in$ generalized $E^*(0)$.*

Proof. Since all the representative submatrices are N-matrices, it follows that each of the representative submatrix of A is an R_0-matrix. Thus from Lemma 2.6.2, it follows that the equivalent matrix M is an R_0-matrix. Hence $A \in$ generalized $E^*(0)$. ∎

THEOREM 2.6.1 *The degree of a vertical block N-matrix of category 1 is -1; that of a vertical block N-matrix of category 2 is 0.*

Proof. Let A be a vertical block N-matrix of category 1. There is an index set $\gamma \subseteq \{1, 2, \ldots, k\}$ such that A has the partition given by Lemma 2.6.4. Let M be the equivalent LCP-matrix. Note that M can be partitioned as

$$M = \begin{bmatrix} M_{\nu\nu} & M_{\nu\bar{\nu}} \\ M_{\bar{\nu}\nu} & M_{\bar{\nu}\bar{\nu}} \end{bmatrix}$$

where $M_{\nu\nu} < 0,\ M_{\bar{\nu}\bar{\nu}} < 0,\ M_{\nu\bar{\nu}} > 0,\ M_{\bar{\nu}\nu} > 0$.

Note also that M is an N_0-matrix. Let $q > 0$, $q \in R^m$ be nondegenerate with respect to M (i.e., (w, z) is a solution to LCP(q, M) implies that $(w + z) > 0$).

Then clearly LCP(q, M) has exactly 3 solutions; $w^1 = q,\ z^1 = 0$,

$$w^2 = \begin{bmatrix} 0 \\ w^2_{\bar\nu} \end{bmatrix};\ z^2 = \begin{bmatrix} z^2_\nu \\ 0 \end{bmatrix} \text{ where } z^2_\nu > 0 \text{ and } w^2_{\bar\nu} = q_{\bar\nu} + M_{\bar\nu\nu} z^2_\nu > 0 \text{ and}$$

$$w^3 = \begin{bmatrix} w^3_\nu \\ 0 \end{bmatrix};\ z^3 = \begin{bmatrix} 0 \\ z^3_{\bar\nu} \end{bmatrix} \text{ where } z^3_{\bar\nu} > 0 \text{ and } w^3_\nu = q_\nu + M_{\nu\bar\nu} z^3_{\bar\nu} > 0.$$

It follows from here that the LCP degree of $M = 1 - 1 - 1 = -1$. Hence the VLCP degree of A in this case is -1. Similarly, we can show that the degree of a vertical block N-matrix of category 2 is 0. ■

In the remainder of this section we shall prove that Algorithm CD can process a VLCP with a PPT of the vertical block N-matrix although it cannot process a VLCP with a vertical block N-matrix. It is well known that for the equivalent LCP(q, M) where M is the equivalent matrix of a vertical block category 1 N-matrix A of type $(m_1, \ldots, m_k)$, the facets of Pos(I) are the reflecting facets of Pos$(I, -M)$. Therefore Lemke's algorithm cannot process LCP(q, M) (see [9] and [80]) and hence Algorithm CD cannot process VLCP(q, A).

LEMMA 2.6.6 *Suppose A is a vertical block category 1 N-matrix of type $(m_1, \ldots, m_k)$. Then A is a Q-matrix.*

Proof. This follows from Theorem 2.1.1, in [45, p. 25] and the fact that the LCP degree of the equivalent M and hence VLCP degree of A is $-1, (\neq 0)$. ■

To prove our next theorem we require the following from [9].

Let M be a square matrix of order m which satisfies the following conditions:

C1: $M \in N_0$

C2: There exists an $\alpha \subseteq \{1, 2, \ldots, m\}$, $\alpha \neq \emptyset$ such that $\det(M_{\alpha\alpha}) < 0$ and $M_{\cdot\alpha}$ contains at least one positive entry.

For any nonempty $\alpha \subseteq \{1,2,\ldots,m\}$, let $C(\alpha)$ denote the complementary matrix whose columns are $-M_{.j}$, $j \in \alpha$ and $I_{.k}$, $k \in \bar{\alpha}$.

LEMMA 2.6.7 *Let A be a given vertical block N-matrix of category 1 of type $(m_1,\ldots,m_k)$ and let M be its equivalent matrix. Suppose $\alpha \subseteq \{1,2,\ldots,m\}$, $\alpha \neq \emptyset$, is such that $\alpha \cap J_i$ is either a singleton set or an empty set for each $1 \leq i \leq k$. Let M^* be the PPT of M with respect to $C(\alpha)$. Then $\det(M^*_{\beta\beta}) \geq 0$ for all $\beta \neq \alpha$ and $\det(M^*_{\alpha\alpha}) < 0$.*

Proof. This follows from a theorem in [74]. Also see Lemma 3.1 in [9]. ∎

We also require the following theorem from [9].

THEOREM 2.6.2 *(Theorem 3.1 in [9]): Let $M \in N_0$ satisfy conditions C1 and C2. Further, suppose $M \in E(0)$ and let α be as in C2. Then there exists a $d > 0$, $d \in R^m$, such that Lemke's algorithm with d can process the matrix M^* obtained by taking a PPT with respect to the complementary matrix $C(\alpha)$.*

THEOREM 2.6.3 *Let A be a vertical block N-matrix of category 1 of type $(m_1,\ldots,m_k)$ and let M be its equivalent square matrix. Then there exists a positive vector $d \in R^m$ and a PPT M^* of M such that using Lemke's algorithm with d, one can solve LCP(q^*, M^*) and hence the equivalent LCP(q, M) which solves the VLCP(q, A).*

Proof. Note that as A is a vertical block N-matrix of category 1, if for any nonempty $\alpha \subseteq \{1,2,\ldots,m\}$, $C(\alpha)$ is a nonsingular complementary matrix of $(I, -M)$, where M is the equivalent matrix of A, then conditions C1 and C2 are satisfied. Further $M \in R_0$ by Lemma 2.6.5. Thus all the conditions of Theorem 2.6.2 are satisfied and hence the conclusion of the theorem follows. ∎

Chapter 3

The Role of Representative Submatrices in Vertical Linear Complementarity Theory

3.1 Introduction

Representative submatrices have played an important role in the study of some classes of vertical linear complementarity problems. In this chapter, we present a slightly different approach to the study of such problems by using equivalent linear complementarity problems. The vertical linear complementarity problem with a vertical block P-matrix has been studied by Cottle and Dantzig [4] and Habetler and Szanc [33]. In Section 3.2, we present the required definitions. The class of vertical block P-matrices is studied in Section 3.3. Here we present alternative proofs of the properties of vertical block P-matrices obtained by Habetler and Szanc [33] in terms of their equivalent square matrices. We also present a new characterization result for a vertical block P-matrix in terms of the eigenvalues of its equivalent square matrix. However, neither the approach of representative submatrices nor the approach of the equivalent linear complementarity problem is useful in studying a VLCP with a vertical block hidden Z-matrix. This is brought out in Section 3.5. In Section 3.4, we study the class of vertical block $\mathcal{L}_1$-matrices and vertical block R_0-matrices.

3.2 Preliminaries

Let $A \in R^{m \times k}$ be a vertical block matrix of type $(m_1, \ldots, m_k)$ and let $e^1, \ldots, e^k$ be a collection of column vectors such that $e^j \in R^{m_j}$ each of whose coordinates is 1. Then,

$$\bar{U}^t = \begin{bmatrix} e^1 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & e^k \end{bmatrix}$$

is of order $m \times k$. The j^{th} row of the matrix $\bar{U}A$ is the e^j-weighted sum of the rows in A^j.

The following theorem is a special case of Theorem 1.6.3.

THEOREM 3.2.1 *If A is a vertical block P-matrix of type $(m_1, \ldots, m_k)$ then $\bar{U}A$ is a P-matrix.*

DEFINITION 3.2.1 We say that a set of indices $\alpha \subseteq \{1, 2, \ldots, m\}$ is *a representative set of indices* if $\alpha \cap J_i = \{j_i\}$ for each i, $1 \leq i \leq k$.

DEFINITION 3.2.2 Let $M \in R^{m \times m}$ be the equivalent matrix of a vertical block matrix A of type $(m_1, \ldots, m_k)$. Given an index set $\alpha \subseteq \{1, \ldots, m\}$, the principal submatrix $M_{\alpha\alpha} \in R^{|\alpha| \times |\alpha|}$ is said to be *nontrivial* if it is either a representative submatrix of A or a principal submatrix of a representative submatrix of A. Otherwise, it is called a *trivial principal submatrix.*

REMARK 3.2.1 Note that if $M_{\alpha\alpha}$ is a trivial principal submatrix then $\det(M_{\alpha\alpha}) = 0$, although, the converse is not true.

THEOREM 3.2.2 *If A be a vertical block P-matrix of type $(m_1, \ldots, m_k)$ then the equivalent matrix M is a P_0-matrix and for each index set with* $\det(M_{\alpha\alpha}) = 0$, *the columns of $M_{\cdot\alpha}$ are linearly dependent.*

Proof. Note that M is a P_0-matrix with all its nontrivial principal minors positive. Hence $\det(M_{\alpha\alpha}) = 0$ if and only if $M_{\alpha\alpha}$ is a trivial principal submatrix. The columns of a trivial principal submatrix are linearly dependent. This is clear from the construction of the equivalent matrix M. This concludes the proof. ∎

REMARK 3.2.2 The P_0-matrix M resulting from a given vertical block P-matrix is a column adequate matrix.

3.3 Vertical Block P-matrix

We provide an alternative proof of a result of Habetler and Szanc [33, Theorem 4.3]. First, we prove the following lemma.

LEMMA 3.3.1 *Suppose VLCP(q, A) has a finite number of solutions for all $q \in R^m$. Then each of the principal submatrix of any representative submatrix of A is nonsingular.*

Proof. Suppose the lemma is not true. Then there exists a $\beta = \{\beta_1, \ldots, \beta_k\}$, where $\beta_i \in J_i$ and a set $L \subseteq \{1, 2, \ldots, k\}$ such that $\det(A_{\gamma L}) = 0$ where $\gamma = \{\beta_i \mid i \in L,\ \beta_i \in \beta\}$. Now construct a $q^* \in R^m$ as follows:

$$q^* = \sum_{i \in L} (-A)_{\cdot i} + \sum_{k \in \bar{\gamma}} I_{\cdot k}$$

where $\bar{\gamma}$ is the complement of γ in $\{1, 2, \ldots, m\}$. Note that

$$q^* = \begin{bmatrix} -A_{\gamma L} & 0 \\ -A_{\bar{\gamma} L} & I_{\bar{\gamma} L} \end{bmatrix} \begin{bmatrix} z^* \\ w^* \end{bmatrix}$$

where $z^* \in R^k$ has the coordinates $(z_1^*, \ldots, z_k^*)$ given by

$$z_i^* = \begin{cases} 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L \end{cases}$$

and $w^* \in R^m$ has the coordinates $(w_1^*, \ldots, w_m^*)$ given by

$$w_j^* = \begin{cases} 1 & \text{if } j \in \bar{\gamma} \\ 0 & \text{if } j \in \gamma. \end{cases}$$

Thus (w^*, z^*) solves VLCP(q^*, A). Further, note that $\det(A_{\gamma L}) = 0$. Hence there exists a $y \in R^{|L|}$, $y \neq 0$, such that $A_{\gamma L}\, y = 0$. Let $\hat{y} \in R^k$ be defined by $\hat{y}_L = y$ and $\hat{y}_{\bar{L}} = 0$.

$$\text{Let } \bar{\lambda} = \begin{cases} \min\{\frac{1}{-\hat{y}_i} \mid \hat{y}_i < 0\} & \text{if } \hat{y}_i \ngeq 0 \\ \infty & \text{if } \hat{y}_i \geq 0. \end{cases}$$

Now note that $z(\lambda) = z^* + \lambda\hat{y}$, $w(\lambda) = w^*$ solves VLCP(q^*, A) for all $0 < \lambda < \bar{\lambda}$ and these solutions are distinct. This is a contradiction to the hypothesis. This completes the proof of the lemma. ■

THEOREM 3.3.1 *A is a vertical block P-matrix of type* $(m_1, \ldots, m_k)$ *if and only if VLCP(q, A) has a unique solution for each* $q \in R^m$.

Proof. If A is a vertical block P-matrix of type $(m_1, \ldots, m_k)$ then VLCP(q, A) has a solution by Theorem 1.6.2. By Lemma 2.3.1, the equivalent LCP(q, M) has a solution. Suppose $(\bar{w}, \bar{z})$ and $(\hat{w}, \hat{z})$, $\bar{z} \neq \hat{z}$ are any two distinct solutions for VLCP(q, A). Let $(\bar{u}, \bar{v})$ and $(\hat{u}, \hat{v})$ be the corresponding solutions to LCP(q, M). The construction of these solutions to LCP(q, M) is as follows:

We take $\bar{u} = \bar{w}$ and $\hat{u} = \hat{w}$. Note that $\bar{z}_j > 0$ implies $\exists$ a $p(j) \in J_j$ such that $\bar{w}_{p(j)} = 0$. Define

$$\bar{v}_r = \begin{cases} 0, & \text{if } r \neq p(j) \text{ for any } 1 \leq j \leq k \\ \bar{z}_j, & \text{if } \exists \text{ a } j,\ 1 \leq j \leq k \text{ such that } r = p(j). \end{cases}$$

Similarly, $\hat{v}$ is constructed. Now it is easy to see that $(\bar{u},\bar{v})$ and $(\hat{u},\hat{v})$ solve LCP(q,M). Since M is a P_0-matrix with the property that if $\alpha \subseteq \{1,2,\ldots,m\}$ and $\det(M_{\alpha\alpha})=0$ then the columns of $M_{\cdot\alpha}$ are linearly dependent and it follows by Theorem 1.4.4,

$$M\bar{v} = M\hat{v}. \tag{3.3.1}$$

Note that $\bar{z}_j = \sum_{i\in J_j} \bar{v}_i$, $\hat{z}_j = \sum_{i\in J_j} \hat{v}_i$. From (3.3.1), it follows that $A\bar{z} = A\hat{z}$. This implies that the columns of A are linearly dependent as $\bar{z} \neq \hat{z}$. This however contradicts the hypothesis that A is a vertical block P-matrix. Therefore, VLCP(q,A) has a unique solution.

Conversely, suppose VLCP(q,A) has a unique solution for each $q \in R^m$. Consider the equivalent matrix M. Suppose LCP$(\bar{q},M)$ has two solutions for some $\bar{q} \in R^m$. Let $(\bar{u},\bar{v})$, (u^*,v^*), $\bar{v} \neq v^*$ be the two solutions. Let $\bar{w} = \bar{u}$, $\bar{z}_s = \sum_{i\in J_s} \bar{v}_i$, $w^* = u^*$ and $z^*_s = \sum_{i\in J_s} v^*_i$. Consider $(\bar{w},\bar{z})$ and (w^*,z^*) where $\bar{z}$ and z^* are vectors whose coordinates are given by $\bar{z}_s, z^*_s$ respectively for $1 \le s \le k$. Note that both (w^*,z^*) and $(\bar{w},\bar{z})$ solve the VLCP$(\bar{q},A)$. By our hypothesis therefore $z^* = \bar{z}$ and hence $Az^* = A\bar{z}$. Now, it follows from here that $M\bar{v} = Mv^*$. By Theorem 1.4.4, it follows that M is a P_0-matrix such that for each index set α with $\det(M_{\alpha\alpha}) = 0$, $M_{\cdot\alpha}$ is linearly dependent. Thus all the principal minors of the representative submatrices of A are nonnegative. Now by our hypothesis about the uniqueness of solution to VLCP(q,A) and by Lemma 3.3.1, it follows that all the principal minors of representative submatrices of A are positive. This concludes the proof. ∎

THEOREM 3.3.2 *Let $A \in R^{m\times k}$ be a vertical block matrix of type $(m_1,\ldots,m_k)$. The following are equivalent:*

(i) A is a vertical block P-matrix.

(ii) The equivalent matrix M of A has $(m-k)$ zero eigenvalues and among the remaining k eigenvalues all real eigenvalues are positive. Also eigenvalues of all nontrivial principal submatrices are positive.

Proof. $(i) \Rightarrow (ii)$. Let $A \in R^{m\times k}$ be a vertical block matrix of type $(m_1, \dots, m_k)$ and $\bar{U} \in R^{k\times m}$ with $k \leq m$. Let M be the equivalent matrix of A. Then, by Theorem 1.3.20 in [35, p. 53], the equivalent matrix $M = A\bar{U}$ has the same eigenvalues as $\bar{U}A$, counting multiplicity with an additional $(m-k)$ eigenvalues equal to zero. Since by Theorem 3.2.1, $\bar{U}A$ is a P-matrix, so all real eigenvalues of $\bar{U}A$ are positive. Hence, for the equivalent matrix $M = A\bar{U}$, among the nonzero k eigenvalues, all real eigenvalues are positive and the additional $(m-k)$ eigenvalues are equal to 0. Since A is a vertical block P-matrix, M is P_0-matrix by Theorem 3.2.2. All nontrivial submatrices of M are P-matrices since they are the representative submatrices of a vertical block P-matrix. So, all real eigenvalues of nontrivial principal submatrices are positive.

$(ii) \Rightarrow (i)$. To prove the converse, note that by (ii) all real eigenvalues of nontrivial principal submatrices of M are positive, i.e., all representative submatrices of the vertical block matrix A are P-matrices. By Definition 1.6.2, it follows that A is a vertical block P-matrix. ∎

We can also study the sign nonreversal property of the equivalent matrix M of a given vertical block P-matrix A. In fact, we have the following theorem.

THEOREM 3.3.3 *Let A be an $m \times k$ vertical block matrix of type $(m_1, \dots, m_k)$. Let M be the equivalent matrix of A. Then, A is a vertical block P-matrix if and only if M does not reverse the sign of any vector $x \in R^m$ other than those belonging to the set*

$$\tilde{S} = \{y \mid y \in R^m, \ \sum_{r\in J_s} y_r = 0, \ 1 \leq s \leq k\}.$$

Proof. **(If part)** Suppose A is a vertical block P-matrix. Then clearly its equivalent matrix M is a P_0-matrix. Further, if $\alpha \subseteq \{1, 2, \ldots, m\}$ is an index set such that $\det(M_{\alpha\alpha}) = 0$ then the columns of $M_{\cdot\alpha}$ are linearly dependent, since α must include at least two indices r and s in the same set J_i, for some i, $1 \leq i \leq k$. Hence by Theorem 1.4.4, it follows that M reverses the sign of only those vectors which are in the nullspace of M. We now claim that the nullspace of M is $\tilde{S}$. Clearly if $y \in \tilde{S}$,

$$My = \sum_{s=1}^{k} A_{\cdot s}(\sum_{r \in J_s} y_r) = 0$$

and hence y is in the nullspace of M. Conversely, let y be a null vector of M. Let the vector $\bar{z} \in R^k$ be defined as follows:

$$\bar{z}_p = \sum_{r \in J_p} y_r, \; 1 \leq p \leq k.$$

By the definition of M, it follows that $A\bar{z} = My$ and hence $A\bar{z} = 0$. However, the columns of A are linearly independent as A is a vertical block P-matrix. Hence it follows that $\bar{z} = 0$ and thus $y \in \tilde{S}$.

(Only if part) Suppose M is the equivalent matrix of a given $m \times k$ vertical block matrix A of type $(m_1, \ldots, m_k)$ and suppose M does not reverse the sign of any $x \in R^k$ other than those in $\tilde{S}$. Since $\tilde{S}$ is contained in the nullspace of M, it follows from Theorem 1.4.4 that M is a P_0-matrix.

Now suppose $M_{\alpha\alpha}$ is a principal submatrix of M such that (i) $\det(M_{\alpha\alpha}) = 0$ and (ii) $M_{\alpha\alpha}$ is also a submatrix of a representative submatrix of A. Then note that

$$|\alpha \cap J_i| \leq 1, \; \forall\, i, \; 1 \leq i \leq k.$$

By Theorem 1.4.4, it follows that $M_{\cdot\alpha}$ has linearly dependent columns. Hence there exists a $x^* \in R^{|\alpha|}$, $x^* \neq 0$ such that $M_{\cdot\alpha}x^* = 0$. Define the vector $\bar{x} \in R^m$ by taking componentwise, $\bar{x}_\alpha = x^*$ and $\bar{x}_{\bar{\alpha}} = 0$. Note that $M\bar{x} = 0$. Thus M reverses the sign of $\bar{x}$. However, note that as $x^* \neq 0$ there exists a t such that $t = \alpha \cap J_r$ for some r, $1 \leq r \leq k$ and that $\bar{x}_t \neq 0$. Now it is clear that $\sum_{p \in J_r} \bar{x}_p = \bar{x}_t \neq 0$ and hence $\bar{x} \notin \tilde{S}$, which contradicts our hypothesis. It follows that if $M_{\alpha\alpha}$ is a submatrix of a representative submatrix of A, then $\det(M_{\alpha\alpha}) \neq 0$ and hence it is positive. This concludes the proof. ∎

3.4 Vertical Block $\mathcal{L}_1$- and R_0-matrices

DEFINITION 3.4.1 Let $A \in R^{m \times k}$ be a vertical block matrix of type $(m_1, \ldots, m_k)$. We define vertical block $\mathcal{L}_1$-matrix in two ways:

(i) A is said to be a vertical block $\mathcal{L}_1$-matrix if every representative submatrix of $A \in \mathcal{L}_1$.

(ii) A is a vertical block $\mathcal{L}_1$-matrix if its equivalent square matrix $M \in \mathcal{L}_1$.

We prove that both the definitions of vertical block $\mathcal{L}_1$-matrix are equivalent.

THEOREM 3.4.1 *Let A be an $m \times k$ vertical block matrix of type $(m_1, \ldots, m_k)$. Definitions (i) and (ii) are equivalent.*

Proof. $(ii) \Rightarrow (i)$. This follows from the inheritance property of $\mathcal{L}_1$-matrix [5], i.e., when $M \in \mathcal{L}_1$, every principal submatrix of $M \in \mathcal{L}_1$ and hence each representative submatrix of $A \in \mathcal{L}_1$.

$(i) \Rightarrow (ii)$. Suppose (u, v), $u \in R^m$, $v \in R^m$ solves the equivalent LCP(q, M) for some $q > 0$. Then (w, z) with $w = u$ and $z \in R^k$ where $z_i = \sum_{r \in J_i} v_r$ solves the VLCP(q, A). Let $\alpha = \{i \mid z_i > 0\}$. For each $i \in \alpha$, note that there is a $j(i) \in J_i$ such that $w_{j(i)} = 0$. Let $\beta_1 = \{j(i) \mid i \in \alpha\}$. For $i \notin \alpha$, let $j(i)$ be any arbitrary index in J_i and let $\beta_2 = \{j(i) \mid i \in \bar{\alpha}\}$. Let $\beta = \beta_1 \cup \beta_2$. Note that $A_{\beta.}$ is a representative submatrix of A and by our hypothesis $A_{\beta.}$ is an $\mathcal{L}_1$-matrix.

So LCP($q_\beta, A_{\beta\cdot}$) has a unique solution for every $q_\beta > 0$. See [11, (3) Lemma]. Let (w_β, z) solve the LCP($q_\beta, A_{\beta\cdot}$). Since $A_{\beta\cdot}$ is an $\mathcal{L}_1$-matrix, it follows that $w_\beta = q_\beta$, $z = 0$. Now, $z = 0 \Rightarrow v = 0$ as $z_i = \sum_{r \in J_i} v_r$ and each $v_r \geq 0$. Thus the only solution to LCP(q, M) is $u = q$, $v = 0$ for every $q > 0$. Therefore, M is an $\mathcal{L}_1$-matrix. This concludes the proof. ■

DEFINITION 3.4.2 Let $A \in R^{m \times k}$ be a vertical block matrix of type $(m_1, \ldots, m_k)$. A is said have the R_0 *property* if every representative submatrix of A is an R_0-matrix.

LEMMA 3.4.1 *(Lemma 2.6.2) Let A be an $m \times k$ vertical block matrix of type $(m_1, \ldots, m_k)$. Suppose A has R_0-property. Then the equivalent matrix M of A is an R_0-matrix.*

The converse of the above lemma is not true.

EXAMPLE 3.4.1 Let $A \in R^{4 \times 3}$ be a vertical block matrix of type $(1, 1, 2)$.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix} \quad \text{and the equivalent matrix} \quad M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the representative submatrix

$$G_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \notin R_0 \text{ and } G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} \notin R_0 \text{ but } M \in R_0.$$

3.5 Vertical Block Hidden Z-matrix.

DEFINITION 3.5.1 Let $A \in R^{m \times k}$ be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. A is called a *vertical block hidden Z-matrix* if there exists a Z-matrix $X = ((x_{ij})) \in R^{k \times k}$ and a vertical block Z-matrix $Y = ((y_{ij})) \in R^{m \times k}$ of the same type as A and nonnegative vectors $\rho \in R^k$, $\sigma \in R^m$ such that

(i) $AX = Y$,

(ii) $\rho^t X + \sigma^t Y > 0$.

REMARK 3.5.1 Note that if A is a vertical block Z-matrix then it is a vertical block hidden Z-matrix. This can be seen by taking $X = I$ where I is the identity matrix of order $k \times k$ and $Y = A$. We may take $\rho \in R^k$ as any positive vector and $\sigma = 0$.

Let A be a vertical block hidden Z-matrix and let X and Y be as in Definition 3.5.1. Let G be a representative submatrix of A.
Let $\alpha = \{ r \mid \text{ the } r^{th} \text{ row of } A \text{ is a row of } G\}$. Also note that since $AX = Y$ and $G = A_{\alpha\cdot}$, we have $A_{\alpha\cdot}X = Y_{\alpha\cdot}$. We refer to $Y_{\alpha\cdot}$ as the representative submatrix of Y corresponding to G.

DEFINITION 3.5.2 We say that a vertical block matrix A of type $(m_1, \ldots, m_k)$ has the hidden Z-property, if every representative submatrix of A is a hidden Z-matrix.

An obvious question is whether a vertical block hidden Z-matrix of type $(m_1, \ldots, m_k)$ has the hidden Z-property. The following example shows that this need not be so.

EXAMPLE 3.5.1 Consider the vertical block matrix A of type $(2,3,2)$ where

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 1 \\ -3 & 2 & -2 \\ -4 & 4 & -6 \\ -5 & 3 & -2 \\ -4 & 3 & 5 \\ -4 & -4 & 1 \end{bmatrix}.$$

A is a vertical block hidden Z-matrix with respect to X, Y where

$$X = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -2 \\ -1 & -2 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & -7 & 0 \\ 2 & -5 & 0 \\ -3 & 9 & -1 \\ -2 & 20 & -4 \\ -6 & 12 & -1 \\ -12 & -3 & -2 \\ -1 & -2 & 12 \end{bmatrix}.$$

We take $\rho^t = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$ and $\sigma^t = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

But the representative submatrix

$$G = \begin{bmatrix} 2 & -1 & 2 \\ -3 & 2 & -2 \\ -4 & 3 & 5 \end{bmatrix}$$

is not a hidden Z-matrix (see Cottle, Pang and Stone [5, p. 212]).

However, we can prove the following relationship.

THEOREM 3.5.1 *Suppose A is a vertical block hidden Z-matrix of type $(m_1, \ldots, m_k)$ and let X and Y be given as in the Definition 3.5.1. If X is a $\mathcal{K}$-matrix* *then A has the hidden Z-property. Conversely, suppose every representative submatrix $A_{\alpha\cdot}$ of A is hidden Z with a common X-matrix. Then A is a vertical block hidden Z-matrix of type $(m_1, m_2, \ldots, m_k)$.*

Proof. Let A be a vertical block hidden Z-matrix of type $(m_1, m_2, \ldots, m_k)$ and let X be a $\mathcal{K}$-matrix. Let $\alpha \subseteq \{1, 2, \ldots, m\}$ be such that $\alpha \cap J_i$ is a singleton set for each $1 \le i \le k$. Note that $A_{\alpha\cdot}$ is a representative submatrix of A and that

$$A_{\alpha\cdot}X = Y_{\alpha\cdot}$$

Since X is a $\mathcal{K}$-matrix there is a $\xi_1 \in R^k$, $\xi_1 \ge 0$ such that $\xi_1^t X > 0$. Choose $\xi_2 = 0$. We then have $\xi_1^t X + \xi_2^t Y_{\alpha\cdot} > 0$. Thus $A_{\alpha\cdot}$ is a hidden Z-matrix.

To prove the converse, let A be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. Suppose for every representative submatrix $A_{\alpha\cdot}$, we have

$$A_{\alpha\cdot}X = Y_{\alpha\cdot}$$

and $(\rho(\alpha))^t X + (\sigma(\alpha))^t Y_{\alpha\cdot} > 0$ where $\rho(\alpha)$, $\sigma(\alpha)$ are nonnegative vectors in R^k. It is easy to verify that $AX = Y$ where Y is obtained from $Y_{\alpha\cdot}$'s. For $1 \le i \le m$ let $\sigma_i = \sum_{\{\alpha \mid i \in \alpha\}} \sigma(\alpha)_j$ where j denotes the index of the row $A_{i\cdot}$ in $A_{\alpha\cdot}$ if $i \in \alpha$. Also let $\rho = \sum_{\alpha} \rho(\alpha)$ where α is a row representative set of indices. Now it is easy to check that $\rho^t X + \sigma^t Y > 0$. This completes the proof. ∎

We can in fact prove the following:

LEMMA 3.5.1 *Suppose A is a vertical block hidden Z-matrix of type $(m_1, m_2, \ldots, m_k)$ and let X and Y be given as in Definition 3.5.1. A has the hidden Z-property with respect to X which is common for all representative submatrices if and only if the following condition holds.*

For any index set $\alpha \subseteq \{1, 2, \ldots, m\}$ such that $\alpha \cap J_r$ is a singleton set for all $1 \le r \le k$, there is a subset α_1 of α such that

$$\begin{bmatrix} X_{\alpha_2\alpha_2} & X_{\alpha_2\bar{\alpha}_2} \\ Y_{\alpha_1\alpha_2} & Y_{\alpha_1\bar{\alpha}_2} \end{bmatrix}$$

is a $\mathcal{K}$-matrix where $\alpha_2 = \{i \mid (\alpha\backslash\alpha_1) \cap J_i \neq \phi\}$.

Proof. Suppose the stated condition holds. Let $A_{\alpha\cdot}$ be any representative square submatrix of A where α is a representative row index set. Note that $\alpha \cap J_r$ is a singleton set for each $1 \leq r \leq k$. We also note that since $AX = Y$, it follows that $A_{\alpha\cdot}X = Y_{\alpha\cdot}$ where $Y_{\alpha\cdot}$ is the submatrix of Y consisting of the rows of Y whose indices are in α.

By the given condition, there is a subset α_1 of α such that

$$B = \begin{bmatrix} X_{\alpha_2\alpha_2} & X_{\alpha_2\bar{\alpha}_2} \\ Y_{\alpha_1\alpha_2} & Y_{\alpha_1\bar{\alpha}_2} \end{bmatrix}$$

is a $\mathcal{K}$-matrix where $\alpha_2 = \{i \mid (\alpha\backslash\alpha_1) \cap J_i \neq \phi\} \subseteq \{1, 2, \ldots, k\}$ and $\bar{\alpha}_2 = \{1, 2, \ldots, k\}\backslash\,\alpha_2$. Note that $|\alpha_1| + |\alpha_2| = k$. Now there exists a positive vector v such that $v^t B > 0$. Define the vector $\rho(\alpha)$ by taking

$$\rho(\alpha)_i = \begin{cases} v_i & \text{if the } i^{th} \text{ row of } B \text{ is a row of } X \\ 0, & \text{otherwise.} \end{cases}$$

Similarly define the vector $\sigma(\alpha)$ by taking

$$\sigma(\alpha)_i = \begin{cases} v_i & \text{if the } i^{th} \text{ row of } B \text{ is a row of } Y_{\alpha\cdot} \\ 0, & \text{otherwise.} \end{cases}$$

We then note that $\rho(\alpha)$ and $\sigma(\alpha)$ are nonnegative vectors such that

$$(\rho(\alpha))^t X + (\sigma(\alpha))^t Y_{\alpha\cdot} = v^t B > 0.$$

Thus, $A_{\alpha\cdot}$ is a hidden Z-matrix. This shows that A has the hidden Z-property with respect to X, for all the representative submatrices of A.

Conversely, now suppose that every representative submatrix $A_{\alpha\cdot}$ of A is hidden Z with respect to X where $\alpha \subseteq \{1, 2, \ldots, m\}$ is any index set such that $\alpha \cap J_r$ is a singleton set for each $1 \leq r \leq k$. Now we have $A_{\alpha\cdot}X = Y_{\alpha\cdot}$ and by our hypothesis, there exist nonnegative vectors $\rho(\alpha)$ and $\sigma(\alpha)$ such that

$$(\rho(\alpha))^t X + (\sigma(\alpha))^t Y_{\alpha\cdot} > 0.$$

Hence by Theorem 3.11.17 of [5, p. 207] the required result follows. ■

Now we ask the following question:

If every representative submatrix of a vertical block matrix A is hidden Z, can we say that A is a vertical block hidden Z-matrix ?(i.e., do there exist X and Y such that conditions (i) and (ii) of Definition 3.5.1 are satisfied?)

The following example shows that this need not be true.

EXAMPLE 3.5.2 Consider the vertical block matrix A of type $(1,2,1)$ where

$$A = \begin{bmatrix} 5 & -9 & -10 \\ 8 & 2 & -2 \\ -6 & -8 & -5 \\ -9 & -1 & 0 \end{bmatrix}.$$

A has two representative submatrices G_1, G_2 where

$$G_1 = \begin{bmatrix} 5 & -9 & -10 \\ 8 & 2 & -2 \\ -9 & -1 & 0 \end{bmatrix} \text{ and } G_2 = \begin{bmatrix} 5 & -9 & -10 \\ -6 & -8 & -5 \\ -9 & -1 & 0 \end{bmatrix}.$$

G_1 is a hidden Z-matrix with respect to X_1, Y_1 where

$$X_1 = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \text{ and } Y_1 = \begin{bmatrix} 69 & -9 & -25 \\ -2 & 2 & -12 \\ -3 & -1 & 9 \end{bmatrix}.$$

We take $\rho_1^t = \begin{bmatrix} 90 & 0 & 90 \end{bmatrix}$ and $\sigma_1^t = \begin{bmatrix} 1 & 6 & 1 \end{bmatrix}$.

G_2 is a hidden Z-matrix with respect to X_2, Y_2 where

$$X_2 = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} 25 & -9 & -26 \\ -13 & -8 & -1 \\ -27 & -1 & 10 \end{bmatrix}.$$

We take $\rho_2^t = \begin{bmatrix} 10 & 20 & 25 \end{bmatrix}$ and $\sigma_2^t = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$.

Let us consider the following inequalities:

$$8\,x_{11} + 2\,x_{21} - 2\,x_{31} \leq 0$$

$$-6\,x_{11} - 8\,x_{21} - 5\,x_{31} \leq 0$$

$$x_{21},\, x_{31} \leq 0.$$

The only solution to these inequalities is $x_{11} = x_{21} = x_{31} = 0$. However if we take the 0 solution for the first column of X, then there exists no ρ, σ satisfying $\rho^t X + \sigma^t Y > 0$. Hence A is not a vertical block hidden Z-matrix.

3.6 Concluding Remark

Representative submatrices are important in the study of classes of vertical block matrices defined through the signs of determinants of the principal minors of representative submatrices, such as vertical block P, P_0 and N. Some results on vertical block N-matrices have been proved in Chapter 2. We can also use the nontrivial principal submatrices of the equivalent matrices in place of representative submatrices. However, for studying some other classes of matrices such as vertical block R_0 or hidden Z the approach of representative submatrices does not seem to be useful. The class of vertical block hidden Z-matrices is studied in Chapter 5, where it is shown that it is the largest class of vertical block matrices such that the associated VLCP has a least element property.

Chapter 4

Algorithms for the Vertical Linear Complementarity Problem with a Vertical Block $\mathcal{Z}$-Matrix

4.1 Introduction

In [15] Ebiefung and Kostreva introduce the class of vertical block Z-matrices. They present a generalization of Chandrasekaran's pivoting algorithm to solve LCP(q, M) where M *is a* Z-*matrix* (see [1]), for the VLCP(q, A) when A is a vertical block Z-matrix.

Ebiefung and Kostreva [18] point out that Cottle-Dantzig algorithm can process the VLCP(q, A) where A is a vertical block $\mathcal{K}$-matrix but are silent about whether this algorithm can process the VLCP(q, A) where A is a vertical block Z-matrix. The VLCP with a vertical block Z-matrix arises in some important practical problems (see [18], [69], [85] and [86]). This chapter is partially motivated by the question whether the algorithm given by Cottle and Dantzig (Algorithm CD) can solve the VLCP(q, A) where A is a vertical block Z-matrix. We answer this question affirmatively and thereby generalize a similar result due to Saigal [79] on the applicability of Lemke's method for LCP(q, M) when $M \in Z$. This we do, by presenting a simplex type algorithm for **Problem P**, which is:

$$\text{Minimize } z_0$$

$$\text{subject to } w - Az - dz_0 = q, \; w \geq 0, \; z \geq 0, \; z_0 \geq 0 \qquad (4.1.1)$$

where $d \in R^m$ is any positive vector and by showing that the sequence of almost proper solutions generated by our modified simplex method for the above problem is the same as the sequence of almost proper solutions generated by Algorithm CD when (1.3.1) has a solution. Actually, this idea is again a generalization of the algorithm presented in [51]. The analysis presented in this chapter leads to a greater understanding of the properties of a vertical block Z-matrix.

In Section 4.2, we present the required definitions and our algorithm. In Section 4.3, we show that when the VLCP(q, A) does not have a solution, Algorithm CD takes more iterations to detect the infeasibility than the modified simplex algorithm. In Section 4.4, we show that a modified version of Algorithm CD can be used for checking whether a given vertical block Z-matrix is a vertical block P_0-matrix or not. For the square Z-matrices such a checking algorithm has been given by Ramamurthy [78].

4.2 Preliminaries

DEFINITION 4.2.1 Let A be a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. Suppose G is a representative submatrix of A. Further, suppose that the j^{th} row of G is the r_j^{th} row of A, where $r_j \in J_j$. Given the VLCP(q, A), let $q(G)$ be the

$k \times 1$ vector whose j^{th} component is q_{r_j}. The problem LCP($q(G)$,G) is known as a *representative subproblem* associated with the VLCP(q, A).

DEFINITION 4.2.2 A vertical block matrix A of type $(m_1, m_2, \ldots, m_k)$ is called a *vertical block* $\mathcal{K}$ ($\mathcal{K}_0$)*-matrix* if it is a vertical block $Z \cap P$ ($Z \cap P_0$)-matrix.

We now look at the set

$$S = \{(w, z, z_0) \mid w \in R^m, z \in R^k, z_0 \in R,\ w - Az - dz_0 = q,\ w \geq 0, z \geq 0, z_0 \geq 0\}.$$

We say that a $(w, z, z_0) \in S$ is a *basic feasible solution* to (4.1.1) if the columns of $(I, -A, -d)$ corresponding to the positive coordinates of (w, z, z_0) are linearly independent and a *nondegenerate basic feasible solution*, if in addition, the number of positive coordinates of (w, z, z_0) is m.

Assume that all the basic feasible solutions to (4.1.1) are nondegenerate. Under this assumption the extreme points of S are in one-one correspondence with the basic feasible solutions to (4.1.1). Their correspondence is well discussed in many text books on linear programming. For instance, see Hadley ([34], p. 100). In particular, if B is a proper basis matrix which is feasible, the extreme point corresponding to it is a solution to the VLCP(q, A).

DEFINITION 4.2.3 We say that an extreme point of S is *almost proper* if the corresponding basis matrix is almost proper.

In what follows we make the standard nondegeneracy assumption that all the almost proper basic feasible solutions to (4.1.1) are nondegenerate.

Given a vertical block Z-matrix of type $(m_1, m_2, \ldots, m_k)$ and a vector $q \in R^m$ we can also apply a modified simplex method to Problem P.

The modifications we introduce are as follows:

(i) *We start with the initial basic feasible solution as* $w = q + d\bar{z}_0$ *and* $z = 0$ *where* $\bar{z}_0 = \min\{z_0 \mid q + d z_0 \geq 0,\ z_0 \geq 0\}$.

(ii) *At any iteration if of the pair of nonbasic variables belonging to the same set of related variables, there is one with a positive simplex multiplier (negative of the relative cost coefficient as in [62], p. 58), then that variable is selected for inclusion in the set of basic variables at the next iteration.*

We call the simplex method with the above modifications *Algorithm S.* While applying the above algorithm, we may encounter two types of almost proper basis matrices. We define these types below:

DEFINITION 4.2.4 We say that an almost proper basis matrix B is of *type I* if for each r, $1 \leq r \leq k$, the fact that $-A_{.r}$ is a column of B implies that for exactly one index $j \in J_r$, $I_{.j}$ is not a column of B.

DEFINITION 4.2.5 An almost proper basis matrix generated by the algorithm is said to be of *type II* if it is not of type I.

REMARK 4.2.1 Note that if an almost proper basis matrix B is of type II, then there is exactly one index s, $1 \leq s \leq k$ such that $-A_{.s}$ is a column of B and exactly two indices l and p in J_s such that $I_{.l}$ and $I_{.p}$ are not included in B.

DEFINITION 4.2.6 Let B be an almost proper basis matrix of $(I, -A, -d)$ and let π be a permutation of $\{1, 2, \ldots, m\}$, i.e., $\pi(1), \pi(2), \ldots, \pi(m)$ is a permutation of $1, 2, \ldots, m$. By permuting the rows and columns of B using the same permutation π we obtain the matrix $\bar{B}$. We call $\bar{B}$ a principal rearrangement of B.

DEFINITION 4.2.7 Let B be an almost proper basis matrix of $(I, -A, -d)$. Suppose $B_{.p} = -d$ for some $p \in J_t$. We say that B is in *standard form* if $\forall\, j \neq p$ $B_{.j}$ is either $I_{.j}$ or $-A_{.r}$ where $j \in J_r$. Further, if B is of type I then neither $I_{.p}$ nor $-A_{.t}$ are columns of B. If B is of type II then there exists an index $l \in J_t$ such that neither $I_{.p}$ nor $I_{.l}$ are columns of B.

REMARK 4.2.2 We note that by permuting the columns of B, we can obtain its standard form.

We note the following result which will be useful in Section 4.3.

LEMMA 4.2.1 *Let A be a vertical block Z-matrix. Suppose B is an almost proper basis matrix of $(I, -A, -d)$ which is in the standard form. By a principal rearrangement of the rows and columns of B, we can write B in the partitioned form as C, where*

$$C = \begin{bmatrix} I & D_1 & -\bar{d}^1 \\ 0 & D_2 & -\bar{d}^2 \\ 0 & g^t & -\bar{d}_m \end{bmatrix}.$$

Further (i) $-D_2$ is a Z-matrix and

(ii) $g^t \geq 0$ if B is of type I and at most one coordinate of g^t is negative if B is of type II.

Proof. Let p be an index such that $B_{\cdot p} = -d$. Suppose B includes exactly r columns from $-A$. Then B includes exactly $(m-r-1)$ columns from I. Let $t_1 < t_2 < \ldots < t_{m-r-1}$ be the indices such that $I_{\cdot t_j}$ is a column of B. Actually, since B is in the standard form, $B_{\cdot t_j} = I_{\cdot t_j}$. Let $L = \{l \mid -A_{\cdot l}$ is a column of $B\}$ and let $l_1 < l_2 < \ldots < l_r$ be the ordered set of indices in L.

Case 1. Suppose B is of type I. Then given any $l_i \in L$ there exists exactly one index $p_i \in J_{l_i}$ such that $I_{\cdot p_i}$ is not a column of B and $B_{\cdot p_i} = -A_{\cdot l_i}$. Now let $\pi(t_i) = i$ for $i = 1, 2, \ldots, (m-r-1)$, $\pi(p_i) = m-r-1+i$ for $i = 1, 2, \ldots, r$ and $\pi(p) = m$. Then the matrix C obtained by applying the principal rearrangement using π has the above form. Now, note that D_2 is actually a submatrix of $-A$, containing the columns $l_1, l_2, \ldots, l_r$ and the rows $p_1, p_2, \ldots, p_r$ of $-A$. Note that as $p_i \in J_{l_i}$, this is a principal submatrix of a representative submatrix of $-A$. Thus $-D_2$ is a Z-matrix. Also as g^t is obtained by a permutation of the elements of the p^{th} row of $-A$, where $p \notin J_{l_i}$, $i = 1, 2, \ldots, r$, $g^t \geq 0$. This completes the proof in this case.

Case 2. Suppose B is of type II. In this case there is exactly one index s and exactly two indices p and l in J_s such that $I_{.l}$ and $I_{.p}$ are not columns of B and $-A_{.s}$ is a column of B. We note that $s = l_i$ for some i. Since B is in the standard form there exists exactly one index $p_j \in J_{l_j}$ such that $I_{.p_j}$ is not a column of B for all j, $1 \leq j \leq r$, $j \neq i$. For $j = i$ however $l_i = s$ and we have two indices p and $l \in J_s$ such that $I_{.p}$ and $I_{.l}$ are not columns of B. We have $B_{.p_j} = -A_{.l_j}$ and either $B_{.p} = -d$ or $B_{.l} = -d$. Now we take $\pi(t_\alpha) = \alpha$, $\alpha = 1, 2, \ldots, (m - r - 1)$, $\pi(p_j) = m - r - 1 + j$, $j \neq i$. Without loss of generality suppose that $B_{.p} = -d$ then take $\pi(l) = m - r - 1 + i$ and $\pi(p) = m$. It is easy to see as in the proof of Case 1 that by permuting the rows and columns of B using π, we obtain C, which has the above form. As before we note that $-D_2$ is a principal submatrix of a representative matrix of A and therefore $-D_2 \in Z$. Note that g^t is obtained by a permutation of the p^{th} row of $-A$, where $p \notin J_{l_j}$, $1 \leq j \leq r$, $j \neq i$. But $p \in J_{l_i}$ and hence it follows that $g_j \geq 0$, $j \neq i$, and g_i may be negative. Thus at most one coordinate of g^t is negative. This concludes the proof. ∎

4.3 Validity of the Algorithms

In order to prove that both Algorithm CD and Algorithm S can process the VLCP(q, A) under the standard nondegeneracy assumption where A is a vertical block Z-matrix, we require the following lemmas.

Lemma 4.3.1 *Suppose B is an almost proper basis matrix of $(I, -A, -d)$ of type I which is in the standard form. Let C be obtained as in Lemma 4.2.1. Suppose $-D_2$ is a P-matrix. Let f^t be the last row of C^{-1}. Then $f \leq 0$.*

Proof. Since $f^t C = e_m^t$ it follows that $f_j = 0,\ j = 1, 2, \ldots, (m - r - 1)$. Also we obtain the equations

$$\bar{f}^t D_2 + f_m g^t = 0 \tag{4.3.1}$$

$$\sum_{j=m-r}^{m} f_j (-\bar{d}_j) = 1 \tag{4.3.2}$$

where $\bar{f}^t = (f_{m-r}, f_{m-r+1}, \ldots, f_{m-1})$ and $\bar{d}$ is the last column of C.

From (4.3.1) we note that

$$\begin{aligned} \bar{f}^t &= -f_m g^t (D_2)^{-1} \\ &= f_m g^t (-D_2)^{-1}. \end{aligned}$$

Since $-D_2$ is a $\mathcal{K}$-matrix and $g \geq 0$, it follows that f_m and the coordinates of $\bar{f}$ have the same sign. It now follows from (4.3.2) that $f \leq 0$. ■

LEMMA 4.3.2 *Let B be an almost proper basis matrix of $(I, -A, -d)$ of type II which is in the standard form. Let p be the index such that $B_{\cdot p} = -d$, $l_1 < l_2 < \ldots < l_r$ be the indices such that $-A_{\cdot l_j}$ is a column of B, for $j = 1, 2, \ldots, r$ and l_i be the index such that $p \in J_{l_i}$. Let C be the partitioned form obtained after a principal rearrangement of the rows and columns of B as in Lemma 4.2.1. Suppose $-D_2$ is a P-matrix. Let f^t be the last row of C^{-1}. If $f \not\leq 0$ then at least one of the two coordinates f_m and $f_{m-r-1+i}$ is positive.*

Proof. Since $f^t C = e_m^t$ we obtain the equations (4.3.1) and (4.3.2). Further $f_j = 0,\ j = 1, 2, \ldots, (m - r - 1)$.

From (4.3.1) we have

$$\begin{aligned} \bar{f}^t &= -f_m g^t (D_2)^{-1} \\ &= f_m g^t (-D_2)^{-1}. \end{aligned}$$

If $f_m > 0$ then we have nothing to prove. So let $f_m \leq 0$ and $f_{m-r-1+i} \leq 0$.

Let $\tilde{f}$ be obtained from $\bar{f}$ by omitting its i^{th} coordinate and let $\tilde{D}_2$ be the matrix obtained by omitting the i^{th} row and column of D_2. Also let $\tilde{g}$ be obtained from g by omitting its i^{th} coordinate. Then (4.3.1) leads to

$$\tilde{f}^t(-\tilde{D}_2) = f_m \tilde{g}^t + f_{m-r-1+i}(\tilde{D}_2)_{i\cdot}.$$

Note that $\tilde{g}^t$ and $(\tilde{D}_2)_{i\cdot}$ are nonnegative vectors. As $-\tilde{D}_2$ is also a $\mathcal{K}$-matrix it follows that $\tilde{f} \leq 0$ and hence $f \leq 0$.

Note also that if $f_m = 0$, then $f = 0$, which contradicts the nonsingularity of C^{-1}. This completes the proof of the lemma. ■

We now state and prove our main theorem which establishes the validity of the algorithms presented in Section 4.2.

THEOREM 4.3.1 *Let A be a vertical block Z-matrix and consider the VLCP(q, A). Suppose (1.3.1) has a solution. If we apply Algorithm S to Problem P then the sequence of almost proper basis matrices $B(t)$ generated is the same as the sequence of almost proper basis matrices generated by Algorithm CD. Further, Algorithm S and hence Algorithm CD computes a solution to the VLCP(q, A).*

Proof. Note that the initial almost proper basis matrix $B(0)$ used by both the algorithms is the same and contains $I_{\cdot j}$ for $1 \leq j \leq m$, $j \neq p$, for some p, $1 \leq p \leq m$. Further $(B(0))_{\cdot p} = -d$. Suppose $p \in J_u$. Note that $B(0)$ is of type I and let π be the permutation as in Lemma 4.2.1. Let f^t be the last row of $(C(0))^{-1}$. Let $y_{\cdot j}$ be the j^{th} column of $(I, -A, -d)$ and let $\bar{y}_{\cdot j}$ be obtained from $y_{\cdot j}$ by a permutation of its coordinates using π. Now since $f \leq 0$, if $f^t \bar{y}_{\cdot j} > 0$ for some j, then $f^t \bar{y}_{\cdot j} = \frac{a_{pu}}{d_p}$ and $y_{\cdot j} = -A_{\cdot u}$. Note also that as (1.3.1) has a feasible solution, there is such a $y_{\cdot j}$ and $-A_{\cdot u}$ is selected to be included in the basis. Thus both the algorithms select the same column $-A_{\cdot u}$, to be included in the basis at the next iteration. Further as $B(1)$ in the standard form has $(B(1))_{\cdot p} = -A_{\cdot u}$, $D_2(1)$ is a 1×1 matrix which is $(-a_{pu}) < 0$. Hence $-D_2(1)$ is a P-matrix.

We use the same notation as used in the proof of Lemma 4.2.1. In particular if $B(t)$ is the almost proper basis matrix at the t^{th} iteration which is in the standard form, $C(t)$ is obtained by using a permutation π_t as in Lemma 4.2.1.

Let us make the induction hypothesis that for all $t \leq v$ the almost proper basis matrices generated by both the algorithms are the same and that $-D_2(t)$ is a P-matrix. Let f^t be the last row of $(C(v))^{-1}$. We have two cases to consider.

Case(i). $B(v)$ is an almost proper basis matrix of type I. By Lemma 4.3.1, we have $f \leq 0$. Let $y_{.j}$ be the j^{th} column of $(I, -A, -d)$. Let $\bar{y}_{.j}$ be obtained by permuting the coordinates of $y_{.j}$ using π_v. Now let p be as in the proof of Lemma 4.2.1, i.e., $(B(v))_{.p} = -d$ and let l be such that $p \in J_l$. Now by Lemma 4.3.1, we have $f^t \leq 0$, as $-D_2(v)$ is a $\mathcal{K}$-matrix. Suppose $y_{.j} = I_{.i}$ is a column of $(I, -A, -d)$ not in $B(v)$. Then clearly $f^t \bar{y}_{.j} \leq 0$. If $y_{.j} = -A_{.s}$ for $s \neq l$ then $f^t \bar{y}_{.j} = \sum f_{m-r-1+\alpha}(-a_{\pi_v(p_\alpha)s}) \leq 0$ as $(-a_{\pi_v(p_\alpha)s}) \geq 0 \ \forall \ \alpha$. Therefore, only when $y_{.j} = -A_{.l}$, $f^t \bar{y}_{.j}$ may be positive. If $f^t \bar{y}_{.j}$ is not positive when $y_{.j} = -A_{.l}$, then Algorithm S terminates with the conclusion that (1.3.1) does not have a solution, contradicting our hypothesis. Hence $f^t \bar{y}_{.j} > 0$ for j such that $y_{.j} = -A_{.l}$ and Algorithm S selects $-A_{.l}$ to be included in $B(v+1)$. Algorithm CD also selects the same column, as w_p and z_l form the pair of nonbasic variables belonging to the same set of related variables.

Let j be the index such that $y_{.j} = -A_{.l}$ and let $x = (C(v))^{-1} \bar{y}_{.j}$. Let $x^t = (x_1, x_2, \eta)^t$ be the partition of x induced by π_v where $\eta \in R$. We have the equations

$$I x_1 + D_1(v) x_2 + (-\bar{d}^1)\eta = -\bar{y}_{Tj}$$

where $T = (t_1, t_2, \ldots, t_{m-r-1})$ is the index set as given in the proof of Lemma 4.2.1. We also have

$$D_2(v) x_2 + (-\bar{d}^2)\eta = h$$

where $h = (-a_{p_1 l}, -a_{p_2 l}, \ldots, -a_{p_r l})$.

As $\eta = f^t(\bar{y}_{\cdot j})$, we have $\eta > 0$ and we have

$$D_2(v)\, x_2 = h + \eta\, \bar{d}^2.$$

As $-D_2(v)$ is a $\mathcal{K}$-matrix and as $h + \eta\, \bar{d}^2 \geq 0$, we have $x_2 \leq 0$. This shows that $-A_{\cdot l}$ does not replace any $-A_{\cdot l_i}$ which is in $B(v)$. Therefore, $D_2(v+1)$ is given by

$$D_2(v+1) = \begin{bmatrix} D_2(v) & h \\ f & -a_{pl} \end{bmatrix}.$$

Note that $D_2(v+1) \begin{bmatrix} x_2 \\ -1 \end{bmatrix} = \eta \begin{bmatrix} \bar{d}^2 \\ \bar{d}_m \end{bmatrix}$

or $[-D_2(v+1)] \begin{bmatrix} -x_2 \\ 1 \end{bmatrix} = \eta \begin{bmatrix} \bar{d}^2 \\ \bar{d}_m \end{bmatrix} > 0.$

As $-D_2(v+1)$ is a Z-matrix, it follows from a Theorem of Fiedler and Pták ([19], p. 387) that $-D_2(v+1)$ is a P-matrix.

Note that both the algorithms use the same rule for determining the column to be replaced from $B(v)$.

If the column replaced by $-A_{\cdot l}$ is $-d$, then both the algorithms terminate with a solution to the VLCP(q, A). Otherwise they generate the same almost proper basis matrix $B(v+1)$ in the next iteration. Further $-D_2(v+1)$ is a P-matrix.

Case(ii). $B(v)$ is an almost proper basis matrix of type II. Let $B(v)_{\cdot p} = -d$. Suppose $f \leq 0$. Then $f^t \bar{y}_{\cdot j} \leq 0$ if $y_{\cdot j} = I_{\cdot t}$ for some t. Thus, none of the columns of I will be selected to enter the basis by Algorithm S.

Also, as in Case(i), for t such that $p \notin J_t$, the simplex multiplier corresponding to the nonbasic column $-A_{\cdot t}$ is nonpositive, since for $y_{\cdot j} = -A_{\cdot t}$ we have

$$f^t \bar{y}_{\cdot j} = \sum_{\alpha=1}^{r} f_{m-r-1+\alpha}\left(-a_{\pi_v(p_\alpha)t}\right)$$

where $-a_{\pi_v(p_\alpha)t} \geq 0$.

Now as in the proof of Lemma 4.2.1 let s be such that p and $l \in J_s$, and $s = l_i$. Since $-A_{\cdot l_i}$ is already a column of $B(v)$, we find that if $f \leq 0$, there is no column whose simplex multiplier is positive. It implies that z_0 cannot be further reduced and Algorithm S terminates with the conclusion that there is no solution to (1.3.1) contradicting our hypothesis. Thus $f \not\leq 0$. By Lemma 4.3.2 at least one of the two coordinates f_m and $f_{m-r-1+i}$ is positive. Note also that of the nonbasic pair $(I_{\cdot p}, I_{\cdot l_i})$ corresponding to the same set of related variables one must have been eliminated from the basis matrix $B(v-1)$. Since a column eliminated from the basis matrix cannot re-enter it immediately (see [34], p. 144), it follows that either $f_m > 0$ and $f_{m-r-1+i} \leq 0$ or $f_m \leq 0$ and $f_{m-r-1+i} > 0$. Thus both the algorithms select the same column to be included in the basis. Now since the column to leave $B(v)$ is determined by both the algorithms using the same minimum ratio rule, if this is $-d$, both the algorithms terminate with a solution to the VLCP(q, A). Otherwise they both obtain the same $B(v+1)$. Let b be either $I_{\cdot p}$ or $I_{\cdot l_i}$, whichever is selected by the algorithm to be included in $B(v+1)$, and let $\bar{b}$ be the permutation of its rows using π_v.

Let $x = (C(v))^{-1}\,\bar{b}$. Let $x^t = (x_1, x_2, \eta)^t$ be the partition of x induced by the partitioned form of $C(v)$, where $\eta \in R$.

We then have

$$C(v)\,x = \bar{b} \text{ or}$$

$$x_1 + D_1(v)\,x_2 + \eta\,(-\bar{d}^1) \geq 0 \tag{4.3.3}$$

$$D_2(v)\,x_2 + \eta\,(-\bar{d}^2) \geq 0. \tag{4.3.4}$$

Thus we have $D_2(v)\,x_2 \geq \eta\,\bar{d}^2$.

As $\eta > 0$ and $-D_2(v)$ is a $\mathcal{K}$-matrix, it follows that $x_2 \leq 0$. Thus none of the columns $-A_{\cdot l_j}$ included in $B(v)$ will be selected to leave the basis. This shows that $D_2(v+1) = D_2(v)$.

As $-D_2(v)$ is a P-matrix it follows that $-D_2(v+1)$ is also a P-matrix. It now follows, by the principle of induction that the sequence of almost proper basis matrices generated by both the algorithms is the same when (1.3.1) has a solution, and they both compute a solution to the VLCP(q, A). When (1.3.1) does not have a solution, then Algorithm S terminates with a $z_0 > 0$, thus showing that there is no solution to the VLCP(q, A). In this case Algorithm CD will terminate in a secondary proper ray. This concludes the proof. ∎

REMARK 4.3.1 From Theorem 4.3.1, it follows that when (1.3.1) has a solution both Algorithm S and Algorithm CD generate the same sequence of basis matrices. However when (1.3.1) does not have a solution, Algorithm S never requires more iterations than Algorithm CD to detect the infeasibility. There are instances when Algorithm CD may take a few more iterations than Algorithm S. The following example illustrates this. The fact that Algorithm S never requires more iterations than Algorithm CD is clear from the conditions for terminating these algorithms. Thus Algorithm S is clearly preferable to Algorithm CD for processing the VLCP(q, A) when A is a vertical block Z-matrix.

EXAMPLE 4.3.1 Consider the vertical block Z-matrix of type $(2, 3, 4)$.

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 4 & -3 & -2 \\ -1 & 3 & -1 \\ -2 & 4 & -3 \\ -1 & 6 & -1 \\ -1 & -2 & 3 \\ -3 & -1 & 5 \\ -2 & -3 & 6 \\ -1 & -1 & 4 \end{bmatrix} \quad \text{and } q = \begin{bmatrix} -4 \\ 2 \\ 3 \\ -1 \\ -3 \\ -10 \\ -5 \\ -6 \\ -8 \end{bmatrix}.$$

Algorithm CD takes 4 iterations to terminate in a secondary proper ray. Algorithm S terminates in 3 iterations with $\min z_0 > 0$, demonstrating infeasibility.

REMARK 4.3.2 When (1.3.1) has a solution, we note that Algorithm S generates at most k almost proper type I basis matrices since at every iteration with a type I basis, the algorithm introduces one of the z_j variables into the basis and a basic z_j variable never leaves the basis. However, there may be a number of almost proper type II basis matrices generated between two successive almost proper type I basis matrices.

REMARK 4.3.3 Note that when Algorithm CD or Algorithm S terminates with a solution, the corresponding proper basis matrix B can be written (after a principal rearrangement of its rows and columns) as

$$B = \begin{bmatrix} I & D_1 \\ 0 & D_2 \end{bmatrix}$$

where $-D_2$ is a $\mathcal{K}$-matrix. This is a consequence of our Theorem 4.3.1. Now it is easy to show that B is an optimal basis for the following class of linear programs as well :

$$\text{Minimize} \quad \sum_{j=1}^{k} c_j z_j$$
$$\text{subject to } w - Az = q, \;\; w \geq 0, \; z \geq 0$$

where $c = (c_1, c_2, \ldots, c_k)$ is any nonnegative vector. This shows that the solution corresponding to B is actually the least element of the set

$$X(q) = \{ (w, z) \,|\, w - Az = q, w \geq 0, z \geq 0 \}.$$

See Cottle and Veinott [6] for the definition and properties of the least element. Thus both the algorithms CD and S compute the least element of $X(q)$, whenever $X(q) \neq \emptyset$. This gives us an independent and constructive proof for the results observed by Ebiefung and Kostreva [15] and Sznajder [88].

4.4 An Application of the Cottle-Dantzig Algorithm

In this section we present a modification of Algorithm CD to determine whether a given vertical block Z-matrix A of type $(m_1, m_2, \ldots, m_k)$ is a vertical block $\mathcal{K}_0$-matrix or not. We first state a few results.

LEMMA 4.4.1 *Suppose (w^*, z^*) solves the VLCP$(0, A)$. Let $\alpha_2 = \{j \mid z_j^* > 0\}$ and $\alpha_1 = \cup_{j \in \alpha_2} J_j$. Then (i) $A_{\bar{\alpha}_1 \alpha_2} = 0$ (ii) $A_{\bar{\alpha}_1 \bar{\alpha}_2} z^*_{\bar{\alpha}_2} = 0$ (iii) $w^*_{\bar{\alpha}_1} = 0$ and (iv) $A_{\alpha_1 \alpha_2} z^*_{\alpha_2} \geq 0$.*

Proof. Since (w^*, z^*) solves the VLCP$(0, A)$ we have $w^* - A z^* = 0$. This implies that for $j \in \alpha_2$ $\exists$ $i \in J_j$ such that $w_i^* = 0$ and $\sum_{j \in \alpha_2} a_{ij} z_j^* = 0$. But for $i \in \bar{\alpha}_1$, we also have $w_i^* - \sum_{j \in \alpha_2} a_{ij} z_j^* = 0$, which implies $w_i^* = 0$ and $\sum_{j \in \alpha_2} a_{ij} z_j^* = 0$, as $a_{ij} \leq 0$ for $i \in \bar{\alpha}_1$ and $j \in \alpha_2$. It follows from here that $A_{\bar{\alpha}_1 \alpha_2} = 0$ as $z_j^* > 0$ for $j \in \alpha_2$ and $w^*_{\bar{\alpha}_1} = 0$. This proves (i) and (iii). Also, since for $i \in \alpha_1$, $w_i^* = \sum_{j \in \alpha_2} a_{ij} z_j^* \geq 0$, as $w^* \geq 0$. Hence (iv) holds. (ii) holds trivially. This completes the proof. ∎

LEMMA 4.4.2 *Suppose A is a vertical block Z-matrix of type $(m_1, m_2, \ldots, m_k)$ and let $q \in R^m$ be such that the VLCP(q, A) has no solution. Let $\Re = \{(\bar{w}, \bar{z}, \bar{z}_0) + \lambda (\hat{w}, \hat{z}, \hat{z}_0),\ \lambda \geq 0\}$ be the secondary proper ray generated by Algorithm CD. Let $\alpha_2 = \{j \mid \bar{z}_j \neq 0\}$ and $\bar{\alpha}_2 = \{j \mid \bar{z}_j = 0\}$. If $\hat{z}_0 \neq 0$ then A is not a vertical block $\mathcal{K}_0$-matrix. If $\hat{z}_0 = 0$ then (i) $A\hat{z} \geq 0$, (ii) $A_{\bar{\alpha}_1 \alpha_2} = 0$, where $\bar{\alpha}_1 = \cup_{j \in \bar{\alpha}_2} J_j$ (iii) $A_{\alpha_1 \alpha_2}$ is a vertical block $\mathcal{K}_0$-matrix where $\alpha_1 = \cup_{j \in \alpha_2} J_j$.*

Proof. It is easy to verify from Definition 1.6.11 and Remark 1.6.1 that $(\hat{w}, \hat{z}, \hat{z}_0) \geq 0$, $\hat{z}_j \prod_{i \in J_j} \hat{w}_i^j = 0$, and $\hat{w} - A\hat{z} - d\hat{z}_0 = 0$. If $\hat{z}_0 \neq 0$, then $(\hat{w}, \hat{z})$ is a nontrivial solution to the VLCP$(\hat{z}_0 d, A)$. This implies that there is a representative submatrix G of A such that $\hat{z}$ is a nontrivial solution of the LCP($\hat{z}_0\, d(G)$, G) (see Theorem 1.6.1). It now follows from Lemma 1.4.1 and Lemma 1.4.2 that G is not a P_0-matrix. Hence A is not a vertical block P_0-matrix.

If $\hat{z}_0 = 0$, it is easy to verify that $(\hat{w}, \hat{z})$ is a nontrivial solution to the VLCP$(0, A)$. Now the conclusions (i) and (ii) follow from Lemma 4.4.1. By Lemma 4.4.1, we also have $A_{\alpha_1\alpha_2}\, \hat{z}_{\alpha_2} \geq 0$ and $\hat{z}_{\alpha_2} > 0$. If G is any representative submatrix of $A_{\alpha_1\alpha_2}$ of order $|\alpha_2|$, then $G\,\hat{z}_{\alpha_2} \geq 0$, $\hat{z}_{\alpha_2} > 0$. Since G is a square Z-matrix it follows from Theorem 1.4.1 that G is a P_0-matrix. Hence $A_{\alpha_1\alpha_2}$ is a vertical block $\mathcal{K}_0$-matrix. ■

LEMMA 4.4.3 *Let A be a vertical block Z-matrix of type $(m_1, m_2, \ldots, m_k)$ and $q \in R^m$ be such that $q < 0$. Then the VLCP(q, A) has a solution if and only if A is a vertical block $\mathcal{K}$-matrix.*

Proof. Suppose $q < 0$. Let (w^*, z^*) be a solution to the VLCP(q, A). It follows that $w^* - q = Az^* > 0$. Now from Theorem 1.6.5, it follows that A is a vertical block $\mathcal{K}$-matrix. Conversely, suppose A is a vertical block $\mathcal{K}$-matrix. By Theorem 1.6.5, it follows that there is a $0 \neq x \geq 0$, $x \in R^k$ such that $Ax > 0$. Given any $q \in R^m$ choose $\mu > 0$, so that $w = q + A(\mu x) > 0$. Now it follows from Theorem 4.3.1 that VLCP(q, A) has a solution. This holds for each $q \in R^m$.
■

We now present an algorithm to verify whether a given vertical block Z-matrix A of type $(m_1, m_2, \ldots, m_k)$ is a vertical block P_0-matrix or not.

Step 0: Let A be the given vertical block matrix of type $(m_1, m_2, \ldots, m_k)$. Let $\gamma = \{1, 2, \ldots, k\}$. Choose any $q < 0$ and go to step 1.

Step 1: Let $\theta = \gamma$ and apply the following version of Algorithm CD to the VLCP$(q_{\alpha_1}, A_{\alpha_1\theta})$ where $\alpha_1 = \cup_{j\in\theta} J_j$. Execute the steps of the Algorithm

CD until either a basic z_j variable begins to decrease, or a secondary proper ray is encountered or a solution to the VLCP(q_{α_1}, $A_{\alpha_1\theta}$) is obtained. If the algorithm is terminated due to a decrease in a basic z_j variable, then we say that the algorithm is terminated prematurely.

(a) If the algorithm terminates with a solution to the VLCP(q_{α_1}, $A_{\alpha_1\theta}$), stop. A is a vertical block $\mathcal{K}_0$-matrix.

(b) If there is a premature termination, stop. A is not a vertical block $\mathcal{K}_0$-matrix.

(c) If the algorithm terminates with a secondary proper ray, then go to step 2.

Step 2: Let $\Re = \{(\bar{w}, \bar{z}, \bar{z}_0) + \lambda(\hat{w}, \hat{z}, \hat{z}_0),\ \lambda \geq 0\}$ be the secondary proper ray generated by Algorithm CD when applied to the VLCP($q_{\alpha_1}, A_{\alpha_1\theta}$). If $\hat{z}_0 \neq 0$, stop. A is not a vertical block $\mathcal{K}_0$-matrix. Otherwise go to step 3.

Step 3: Let $\alpha_2 = \{j \in \theta \,|\, \hat{z}_j \neq 0\}$ and $\bar{\alpha_2} = \{j \in \theta \,|\, \hat{z}_j = 0\}$. If $\bar{\alpha_2} = \emptyset$ stop. A is a vertical block $\mathcal{K}_0$-matrix, otherwise let $\gamma = \bar{\alpha_2}$ and return to step 1.

THEOREM 4.4.1 *The above algorithm verifies whether A is a vertical block $\mathcal{K}_0$-matrix or not in a finite number of steps.*

Proof. The matrix $A_{\alpha_1\theta}$ is a vertical block Z-matrix at any stage of the algorithm. Now if Case (a) at step 1 occurs, then $A_{\alpha_1\theta}$ is a vertical block $\mathcal{K}$-matrix by Lemma 4.4.3. However, if Case (b) occurs at step 1, $A_{\alpha_1\theta}$ is not a vertical block $\mathcal{K}_0$-matrix by our results in the previous section. If we stop with step 2 at any stage of the algorithm, then it follows from the first part of Lemma 4.4.2 that $A_{\alpha_1\theta}$ is not a vertical block $\mathcal{K}_0$-matrix. If we stop at step 3, with $\bar{\alpha_2} = \emptyset$, then for $\hat{z} > 0$, we have $A_{\alpha_1\theta}\,\hat{z} \geq 0$, which implies that $A_{\alpha_1\theta}$ is a vertical block P_0-matrix (see [17]). Hence A is a vertical block $\mathcal{K}_0$-matrix.

If $\bar{\alpha_2} \neq \emptyset$, by our Lemma 4.4.2 $A_{\bar{\alpha_3}\alpha_2} = 0$, where $\alpha_3 = \cup_{j \in \alpha_2} J_j \subseteq \alpha_1$ and $\bar{\alpha_3} = \cup_{j \in \bar{\alpha_2}} J_j \subseteq \alpha_1$ and it is easy to verify that $A_{\alpha_1 \theta}$ is a vertical block P_0-matrix if and only if $A_{\bar{\alpha_3}\bar{\alpha_2}}$ is a P_0-matrix. We also note that $|\bar{\alpha_2}| < |\theta|$. Hence the above algorithm verifies whether A is a vertical block $\mathcal{K}_0$-matrix or not in at most k repetitions of Algorithm CD. Hence our theorem follows. ∎

Chapter 5

Vertical Block Hidden Z-Matrices and the Vertical Linear Complementarity Problem

5.1 Introduction

Mangasarian [47] while studying the classes of LCP's solvable by a single linear program introduced a class of matrices, which later came to be named as the class of hidden Z-matrices in [71]. The class of hidden Z-matrices also possesses a least element property which is related to complementarity. For a study of this property see [5]. The least element theory for hidden Z-matrices was motivated by the observation of Mangasarian [47] that the LCP with a hidden Z-matrix can be solved as a single linear programming problem. For related results see also [48].

The complementarity and least element properties associated with a vertical block Z-matrix were studied recently by Ebiefung and Kostreva [15]. This chapter is motivated partly by a question which naturally arises from the work of Ebiefung and Kostreva [15] and Mangasarian ([47], [48]), which is: What is the largest class of vertical block matrices for which the associated VLCP has the least element property and hence can be solved as a single linear programming problem? This also has an implication for the class of VLCP's which has polynomial time complexity. For the class of vertical block hidden Z-matrices introduced in Chapter 3, we study the associated minimality and complementarity properties.

In Section 5.2, we present the required notations. In Section 5.3, we study the least element and complementarity property possessed by vertical block hidden Z-matrices. In Section 5.4, we present some characterization theorems for vertical block hidden $\mathcal{K}$-matrices.

5.2 Preliminaries

Let A be a vertical block matrix of order $m \times k$ and type $(m_1, \ldots, m_k)$ and $q \in R^m$ be given. The set $\text{FEA}(q, A) = \{(w, z) \mid w \in R^m,\ z \in R^k,\ (w, z) \text{ satisfies } (1.3.1)\}$ is called the *feasible region* of VLCP(q, A) and any vector in FEA(q, A) is called a *feasible vector*. Let C be a convex cone. We say that C is a *pointed* convex cone if C does not contain any linear subspace except $\{0\}$. If C is a pointed convex cone in R^n, C induces a partial ordering of vectors in R^n defined as follows: $x \preceq (C)\, y$ if $y - x \in C$. We call this partial ordering the *cone ordering* induced by C. In particular, in this chapter we consider the cone ordering induced by C where $C = \text{Pos}(X)$ for some nonsingular X. A convex cone $C = \text{Pos}(X)$ where X is nonsingular is called a *simplicial cone.*

DEFINITION 5.2.1 Let $S \subseteq R^n$ be a polyhedral set. We say that $x \in S$ is the *least element of* S *with respect to the cone ordering induced by a convex cone* C if $y - x \in C$ for any $y \in S$.

We first prove the following lemma.

LEMMA 5.2.1 *Let* A *be a vertical block hidden Z-matrix. Let* $X \in R^{k\times k}$ *be any Z-matrix and* $Y \in R^{m\times k}$ *be a vertical block Z-matrix of the same type as* A *satisfying the conditions of Definition 3.5.1. Then (i)* X *is nonsingular and (ii) there exists an index set* $\alpha \subseteq \{1, 2, \ldots, k\}$ *and a representative index set* $\beta \subseteq \{1, 2, \ldots, m\}$ *such that with* V *as the representative submatrix of* Y *corresponding to the representative submatrix* G *of* A*, where* $G = A_{\beta\cdot}$*, the matrix*

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \tag{5.2.1}$$

is in $\mathcal{K}$.

Proof. Let ρ, σ be as in Definition 3.5.1. Let $d = X^t\rho + Y^t\sigma > 0$. Then $Cx = d$ where $C = [X^t, Y^t] \in R^{k\times(m+k)}$, $x \geq 0$ has a feasible solution $x = \begin{bmatrix} \rho \\ \sigma \end{bmatrix}$.

Note that as $d > 0$ and each column of C has at most one positive entry, the system $Cy = d$, $y \geq 0$ has a nondegenerate basic feasible solution. Since $X \in Z$ and $Y \in$ vertical block Z, it is easy to see that if B is a basis in C corresponding to a basic feasible solution then B^t is of the form (5.2.1) for some index set α and for some β as defined above. Thus B (or B^t) is a $\mathcal{K}$-matrix which follows from the fact that $B \in Z$ and B is a feasible basis for a $d > 0$. This establishes (ii). To prove (i) suppose that $Xv = 0$. Then we must have $Yv = 0$. Consequently, $Wv = 0$ which implies that $v = 0$. ■

We also observe the following result.

PROPOSITION 5.2.1 *Let A be a vertical block hidden Z-matrix with X and Y as any matrices satisfying the conditions of Definition 3.5.1. Then there is at least one representative submatrix of A which is hidden Z with respect to X and the corresponding representative submatrix of Y.*

Proof. This result follows from Lemma 5.2.1. By Lemma 5.2.1, we have an index set $\alpha \subseteq \{1, 2, \ldots, k\}$ and a representative submatrix V of Y such that

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

is a $\mathcal{K}$-matrix. Let G be the corresponding representative submatrix of A. Since $W \in \mathcal{K}$, it follows that $W^t \in \mathcal{K}$ and there is a $v \in R^k$, $v \geq 0$ such that $v^t W > 0$. Let $v = (v_\alpha, v_{\bar{\alpha}})$. Take $\rho(G)^t = (v_\alpha^t, 0)$ and $\sigma(G)^t = (0, v_{\bar{\alpha}}^t)$. It is easy to verify that $GX = V$ and $\rho(G)^t X + \sigma(G)^t V = v^t W > 0$. This shows that G is a hidden Z-matrix and this completes the proof of the proposition. ∎

REMARK 5.2.1 The above proposition implies in particular that if A is a vertical block hidden Z-matrix with X and Y as any matrices satisfying the conditions of Definition 3.5.1 then there exists a matrix $U \in R^{k \times m}$ of the form

$$U = \begin{bmatrix} u^1 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & u^k \end{bmatrix}$$

where $u^r = (u_1^r, \ldots, u_{m_r}^r)$ is a semipositive (i.e., non-zero nonnegative) row vector of order $1 \times m_r$ such that UA is a square hidden Z-matrix.

REMARK 5.2.2 It is easy to see that if X is a $\mathcal{K}$-matrix then UA is a hidden Z-matrix for any matrix U of the above form. For similar results on vertical block P-matrices see [4] and for vertical block P_0 and Z-matrices see [17].

5.3 Least Element Property

In this section, we consider the least element property of vertical block hidden Z-matrices.

DEFINITION 5.3.1 $S \subset R^n$ is called a *meet semi-sublattice (under the componentwise ordering of* R^n) if for any two vectors $x, y \in S$ their meet
$z = \min(x, y) \in S$.

In what follows, let A be a vertical block hidden Z-matrix with X and Y as any matrices satisfying the conditions of Definition 3.5.1. Note that by Lemma 5.2.1, X is nonsingular. Let $S = \{v \in R^n : Xv \geq 0,\ q + Yv \geq 0\}$.

LEMMA 5.3.1 *A vector* $z \in FEA(q, A)$ *iff* $v = X^{-1}z \in S$. *Also* S *is a meet semi-sublattice.*

Proof. First note that $AX = Y$ and $w = q + Az \geq 0$ as $z \in \text{FEA}(q, A)$. Let $v = X^{-1}z$. So, $z = Xv \geq 0$. Note that $q + Az = q + AXv = q + Yv \geq 0$.
Hence $v \in S$.

Now given $v \in S$, take $z = Xv$. Note that $z \geq 0$. We have
$q + Yv = q + AXv = q + Az \geq 0$. Hence $z = Xv \in \text{FEA}(q, A)$.

Now we have to show that S is a meet semi-sublattice. Let $v^*, \bar{v} \in S$ and let $\hat{v}$ be a vector whose i^{th} coordinate is defined by $\hat{v}_i = \min(v_i^*, \bar{v}_i)$. Suppose $s \in J_i$, the set of indices of rows of A in the i^{th} block. Note that

$$\begin{aligned}
q_s + (Y\hat{v})_s &= q_s + \sum_{j=1}^{k} y_{sj}\,\hat{v}_j \\
&= q_s + y_{si}\,\hat{v}_i + \sum_{j\neq i} y_{sj}\,\hat{v}_j \\
&= q_s + y_{si}\,v_i^* + \sum_{j\neq i} y_{sj}\,\hat{v}_j, \text{ assuming (without loss of generality) } \hat{v}_i = v_i^* \\
&\geq q_s + y_{si}\,v_i^* + \sum_{j\neq i} y_{sj}\,v_j^*, \text{ since } y_{sj} \leq 0 \text{ for } j \neq i \\
&= q_s + \sum y_{sj}\,v_j^* \geq 0, \text{ since } v^* \in S.
\end{aligned}$$

Similarly, we can show that $z = X\hat{v} \geq 0$. Thus S is a meet semi-sublattice. This completes the proof of Lemma 5.3.1. ■

LEMMA 5.3.2 *S contains a least element.*

Proof. It is sufficient to verify that S is bounded below, as S is a meet semi-sublattice.

Let $v \in S$ and $\tilde{q} = \begin{bmatrix} 0 \\ q_{\bar{\alpha}} \end{bmatrix}$ where $\bar{\alpha}$ is as in Lemma 5.2.1. Let W be as in Lemma 5.2.1. Note that $W^{-1} \geq 0$ and by the definition of S, we have $Xv \geq 0$ and $q + Yv \geq 0$. Hence $\tilde{q} + Wv \geq 0$. Therefore, $v \geq -W^{-1}\tilde{q}$. This concludes the proof. ■

THEOREM 5.3.1 *Suppose that $A \in R^{m \times k}$ is a vertical block hidden Z-matrix of type $(m_1, m_2, \ldots, m_k)$. Then there exists a simplicial cone C in R^n such that $\forall\ q \in Pos(I, -A)$, $FEA(q, A)$ contains a least element $\bar{z}$ with respect to the cone ordering induced by C and $\bar{z}$ satisfies $\bar{z}_i \prod_{s=1}^{m_i} (q^i_s + (A^i \bar{z})_s) = 0\ \forall\ i = 1, 2, \ldots, k$.*

Proof. By Lemma 5.3.2, S has a least element $\bar{v}$ with respect to Pos(I). Let $\bar{z} = X\bar{v}$. Note that by Lemma 5.3.1, $\bar{z} \in$ FEA(q, A) and it follows that it is a least element of FEA(q, A) with respect to the cone ordering induced by Pos(X). Now it remains to verify that $\bar{z}_i \prod_{s=1}^{m_i} (q^i_s + (A^i \bar{z})_s) = 0$. To see this we first show that if $(X\bar{v})_i > 0$ then $\exists$ a $s \in J_i$ such that

$$q_s + (Y\bar{v})_s = 0.$$

Suppose $\forall\ s \in J_i$

$$q_s + (Y\bar{v})_s > 0.$$

Now consider a vector $v^*(\epsilon)$ whose coordinates are defined as follows:

$$v^*_j(\epsilon) = \bar{v}_j,\ j \neq i,$$

$$v^*_i(\epsilon) = \bar{v}_i - \epsilon.$$

Note that as X is a Z-matrix, for ϵ sufficiently small, $X\,v^*(\epsilon) \geq 0$. Also, it is easy to verify using the fact that Y is a vertical block Z-matrix that

$$q_s + (Y\,v^*(\epsilon))_s \geq 0, \;\; \forall \;s.$$

This however contradicts the minimality of $\bar{v}$ and completes the proof. ∎

We shall now prove the converse of Theorem 5.3.1.

THEOREM 5.3.2 *Suppose X is a $k \times k$ nonsingular matrix. Let $C = Pos(X)$. Suppose A is a given vertical block matrix. If $FEA(q, A) \neq \emptyset$ implies that $FEA(q, A)$ has a least element with respect to the ordering induced by C which is also a solution to the $VLCP(q, A)$, then A is a vertical block hidden Z-matrix.*

Proof. Let $\tilde{e}^j$ be an $m \times 1$ vector whose i^{th} coordinate $(\tilde{e}^j)_i = 1 \;\; \forall\, i \in J_j$ and 0 otherwise. Also, let e^*_j be the unit vector in R^k with $(e^*_j)_j = 1$ and $(e^*_j)_i = 0$ for $i \neq j$. Now let $q^j = \tilde{e}^j - Ae^*_j$. Clearly, $e^*_j \in$ FEA(q^j, A) and hence FEA$(q^j, A) \neq \emptyset$. Hence by our hypothesis it has a least element $\bar{z}^j$ which satisfies VLCP condition (1.3.2). Note that e^*_j does not satisfy this condition. Hence $\bar{z}^j \neq e^*_j$ and by the minimality of $\bar{z}^j$, we have $X^{-1}(\bar{z}^j) \leq X^{-1}(e^*_j)$. Let $v^j = X^{-1}(e^*_j - \bar{z}^j)$. Note that $0 \neq v^j \geq 0$. Now for $i \in \{1, 2, \ldots, k\} \setminus \{j\}$ we have $X_{i.}\,v^j = (e^*_j - \bar{z}^j)_i \leq 0$. Let $Y = AX$. Note that Y is a vertical block matrix. Now consider $Y_{s.}\,v^j$.

$$\begin{aligned} Y_{s.}\,v^j &= (Y\,v^j)_s \\ &= (A\,X\,v^j)_s \\ &= [A\,(e^*_j - \bar{z}^j)]_s \\ &= (\tilde{e}^j - q^j - A\bar{z}^j)_s \\ &= -(q^j + A\bar{z}^j)_s \text{ for } s \notin J_j. \end{aligned}$$

Therefore noting that $(q^j + A\bar{z}^j) \geq 0$, we have $Y_{s.}\,v^j \leq 0$ for $s \notin J_j$.

Let $W = (v^1, v^2, \ldots, v^k)$. Then it follows that $\tilde{X} = XW$ is a Z-matrix and $\tilde{Y} = YW$ is a vertical block Z-matrix.

We now have to show the existence of nonnegative vectors ρ and s satisfying condition (ii) of Definition 3.5.1. To do this consider the linear programming problem

$$\text{Minimize } e^t u$$

subject to

$$Xu \geq 0$$

$$Yu \geq 0$$

where e is a k-vector of 1's.

Note that u is feasible to the above problem if and only if $Xu \in \text{FEA}(0, A)$. As $0 \in \text{FEA}(0, A)$ it follows that $\text{FEA}(0, A) \neq \emptyset$ and hence it has a least element under the cone ordering induced by $\text{Pos}(X)$, which is also a solution to the $\text{VLCP}(0, A)$. Therefore, the above problem has an optimal solution. By the duality theorem, there exist nonnegative vectors ρ and σ such that $X^t\rho + Y^t\sigma = e$.

As $W \geq 0$ and no column of W is 0, we have

$$\tilde{X}^t \rho + \tilde{Y}^t \sigma = W^t (X^t \rho + Y^t \sigma) = W^t e > 0.$$

This completes the proof. ∎

REMARK 5.3.1 In view of Theorem 5.3.1, the $\text{VLCP}(q, A)$ with a vertical block hidden Z-matrix with respect to X and Y can be formulated as the linear programming problem

$$\text{Minimize } \sum_{i=1}^{k} p_i z_i$$

$$w - Az = q$$

$$w \geq 0, \quad z \geq 0$$

where $p = (p_1, p_2, \ldots, p_k)$ is any vector such that $p^t X > 0$.

REMARK 5.3.2 Thus the remarks of Cottle, Pang and Stone [5, p. 212] in the context of hidden Z-matrices also apply to the vertical block hidden Z-matrices. Thus, given an arbitrary vertical block matrix A, it is not in general easy to test whether or not it is vertical block hidden Z.

5.4 Vertical Block Hidden $\mathcal{K}$-matrices

DEFINITION 5.4.1 Let A be a vertical block hidden Z-matrix. We say that A is a *vertical block hidden $\mathcal{K}$-matrix* if every representative submatrix of A is a P-matrix.

EXAMPLE 5.4.1 Consider the vertical block matrix A of type $(3,2,2)$ where

$$A = \begin{bmatrix} 1.76 & 0.36 & 0.16 \\ 1 & 0 & 0 \\ 0.80 & -0.20 & -0.20 \\ 0.32 & 1.52 & 0.12 \\ 0.44 & 1.84 & 0.04 \\ -1.56 & -1.16 & 0.04 \\ -0.60 & -0.60 & 0.40 \end{bmatrix}.$$

It is easy to verify that A is a vertical block hidden $\mathcal{K}$-matrix with respect to X and Y where $X = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 7 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 4 & -1 \\ -1 & 5 & -2 \\ -2 & -2 & 3 \\ -1 & -2 & 4 \end{bmatrix}$.

We take $\rho^t = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ and $\sigma^t = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

The following theorem characterizes a vertical block hidden $\mathcal{K}$-matrix A assuming that it is a vertical block hidden Z-matrix.

THEOREM 5.4.1 *Let A be a vertical block hidden Z-matrix of type $(m_1, m_2, \ldots, m_k)$. Let X and Y be as in Definition 3.5.1. The following are equivalent:*

(a) A is a vertical block hidden $\mathcal{K}$-matrix.

(b) There exists an $x \in R^k$, $x > 0$ such that $Ax > 0$.

(c) There exists a vector $v > 0$ such that for any index set $\alpha \subseteq \{1,2,\ldots,k\}$, $Wv > 0$ where

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ V_{\bar{\alpha}\alpha} & V_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

as defined in Lemma 5.2.1. Further $W \in \mathcal{K}$.

(d) Every representative submatrix G of A is completely hidden $\mathcal{K}$, i.e., for every index set $\alpha \subseteq \{1,2,\ldots,k\}$, $G_{\alpha\alpha}$ is hidden $\mathcal{K}$.

Proof. (a) $\Rightarrow$ (b). Suppose A is a vertical block hidden $\mathcal{K}$-matrix. In particular, by definition A is a vertical block P-matrix. Now from Theorem 1.6.4, it follows that there is an $x \in R^k$, $x > 0$ such that $Ax > 0$.

(b) $\Rightarrow$ (c). Let $x > 0$, $x \in R^k$ be such that $Ax > 0$. Let $v = X^{-1}x$. We have $Xv > 0$, $Yv = AXv = Ax > 0$. By Lemma 5.2.1, there exists a representative submatrix V and an index set $\alpha_0 \subseteq \{1,2,\ldots,k\}$ such that

$$W_0 = \begin{bmatrix} X_{\alpha_0\alpha_0} & X_{\alpha_0\bar{\alpha}_0} \\ V_{\bar{\alpha}_0,\alpha_0} & V_{\bar{\alpha}_0\bar{\alpha}_0} \end{bmatrix}$$

is a $\mathcal{K}$-matrix. As $Xv > 0$ and $Yv > 0$ it follows that $W_0 v > 0$. This implies that $v > 0$.

Now let G be any representative submatrix of A and let H be the corresponding representative submatrix of Y. Let $\alpha \subseteq \{1,2,\ldots,k\}$ be any index set.

Consider the matrix

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ H_{\bar{\alpha}\alpha} & H_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

As $Xv > 0, \ Yv > 0$, it follows that $Wv > 0$. Since $W \in Z$ and $v > 0$, it follows that $W \in \mathcal{K}$.

(c) $\Rightarrow$ (d). This follows from Theorem 3.5.1 since by (c) X is a $\mathcal{K}$-matrix. Now we use Theorem 3.11.19 of Cottle, Pang and Stone [5, pp. 211-212] to conclude that every representative submatrix is completely hidden $\mathcal{K}$.

(d) $\Rightarrow$ (a). Note that we have $AX = Y$ with $X \in Z, \ Y \in$ vertical block Z and $\rho^t X + \sigma^t Y > 0$. Since every representative submatrix is a hidden $\mathcal{K}$-matrix, it follows that every representative submatrix is a P-matrix. Hence by definition, statement (a) follows. ■

REMARK 5.4.1 *In relation to Remark 5.3.2* if we know that A is a vertical block P-matrix and wish to test its membership in vertical block hidden $\mathcal{K}$ then it is possible to do so by solving two linear programs: one to determine if there exists a $y > 0$ such that $Ay > 0$ and the other to determine if the required X exists. Also the corresponding VLCP is solvable in polynomial time once we have determined the required X in polynomial time.

Chapter 6

Vertical Linear Complementarity in a Problem of n-Person Games

6.1 Introduction

In this chapter, we present a generalization of the polymatrix game considered in Chapter 1, which is similar to the generalization of a bimatrix game presented by Gowda and Sznajder [28] considered in Section 2.5. In the generalized polymatrix game considered here, the players not only choose their mixed strategies over their finite sets of pure strategies but also form their partial payoff matrices as follows:

Player i can form his partial payoff matrix R_{ij} with respect to player j, by choosing the r^{th} row of R_{ij} as the r^{th} row of a matrix in a given set of matrices $\mathcal{A}_{ij}$, $i \neq j$. We shall assume that the matrices in the set $\mathcal{A}_{ij}$, $i \neq j$ are all positive. In Section 6.3, we show that when the entries of the matrices in $\mathcal{A}_{ij}$'s are bounded, there is an ε-equilibrium set of strategies $X(\varepsilon)$ for the players in the sense defined by Gowda and Sznajder [28], which is a generalization of Nash's theorem. Moreover, when $\mathcal{A}_{ij}$'s are compact for all i, j, $i \neq j$, $1 \leq i \leq n$ and $1 \leq j \leq n$, then there is an equilibrium set of strategies. Our proof technique is similar to that in [28] and uses degree theory for a general complementarity system when $\mathcal{A}_{ij}$'s are not necessarily compact. In Section 6.4, we consider the question of stability of an equilibrium set of strategies for a generalized polymatrix game for given $\mathcal{A}_{ij}$, $i \neq j$. In Section 6.5, we assume that the given $\mathcal{A}_{ij}$'s are finite and formulate the problem of finding an equilibrium set as a vertical block linear complementarity problem (VLCP) and show that Howson's special scheme can solve this problem. We thus prove constructively the existence of an equilibrium set (in fact an odd number of such sets) for the finite generalized polymatrix game.

6.2 Preliminaries

Consider the generalized polymatrix game introduced in the previous section. As noted earlier we are given sets $\mathcal{A}_{ij}$ of positive matrices of order $m_i \times m_j$. In addition to choosing his mixed strategy $x^i = (x^i_1, \ldots, x^i_{m_i})^t$, being a probability vector over his pure strategies $\{s^1_i, \ldots, s^{m_i}_i\}$, player i can also choose his partial payoff matrix R_{ij}, with respect to player j, $i \neq j$, by choosing $(R_{ij})_{r\cdot}$ as $(A_{ij})_{r\cdot}$ for some $A_{ij} \in \mathcal{A}_{ij}$, for $1 \leq r \leq m_i$. R_{ij} is then called a row representative of $\mathcal{A}_{ij}$. Thus, player i chooses his mixed strategy x^i as well as his row representatives R_{ij} for $i \neq j$. We refer to this game as the generalized polymatrix game with sets $\mathcal{A}_{ij}$, $j \neq i$. We use e^j to denote a vector in R^{m_j} each of whose coordinates is 1.

DEFINITION 6.2.1 Given the nonempty sets $\mathcal{A}_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$, we say that the set $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ of probability vectors is an ε-equilibrium set of strategies for players $1, 2, \ldots, n$ if for every $\varepsilon > 0$, there exists a row representative $M_{ij}(\varepsilon)$ of $\mathcal{A}_{ij}$ such that

$$(\bar{x}^i)^t \sum_{j \neq i} M_{ij}(\varepsilon)\bar{x}^j \leq (u^i)^t \sum_{j \neq i} R_{ij}\,\bar{x}^j + \varepsilon, \ \ 1 \leq i \leq n, 1 \leq j \leq n \qquad (6.2.1)$$

for all probability vectors u^i of player i and for all row representatives R_{ij} of $\mathcal{A}_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$.

We also say that $\bar{X}$ is an equilibrium set of strategies if there exists a row representative $\bar{M}_{ij}$ of $\mathcal{A}_{ij}$ such that (6.2.1) holds with $\varepsilon = 0$ for all probability vectors u^i and for all row representatives R_{ij} for all $1 \leq i \leq n$.

Let $m = \sum_{i=1}^{n} m_i$. An equilibrium set $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ can also be thought of as a point in R^m. Given the sets $\mathcal{A}_{ij}$, $i \neq j$, $1 \leq i, j \leq n$, let $\mathcal{E}(\mathcal{A}_{ij}, i \neq j, 1 \leq i, j \leq n)$ be the collection of equilibrium sets. We simply write $\mathcal{E}(\mathcal{A}_{ij},\ i \neq j)$ to denote this collection. In what follows we shall take this point of view and consider an equilibrium set as an equilibrium point in R^m. We may also consider $\mathcal{E}(\mathcal{A}_{ij},\ i \neq j)$ as a set in R^m with each set in the collection as a point in R^m.

Let $\bar{X} \in \mathcal{E}(\mathcal{A}_{ij},\ i \neq j)$ be an equilibrium point in R^m and let $B_\epsilon(\bar{X})$ denote the open ball in R^m with centre $\bar{X}$. For two sets S, T let $d_H(S,T)$ denote the Hausdorff distance between sets. Given sets $\mathcal{A}_{ij}$, $i \neq j$ and sets $\mathcal{A}'_{ij}$, $i \neq j$, we consider the distance between the collection of sets $\{\mathcal{A}_{ij},\ i \neq j\}$ and $\{\mathcal{A}'_{ij}, i \neq j\}$ to be $\sum_{i \neq j} d_H(\mathcal{A}_{ij}, \mathcal{A}'_{ij})$.

In our proof of the main theorem in Section 6.3, we use concepts of degree theory. For this and for the properties of the degree see Section 1.9.

6.3 The Main Theorem and Its Proof

Before we state and prove the main theorem on the existence of an ε-equilibrium set of strategies for a generalized polymatrix game, we introduce the following functions and prove some lemmas.

Let us consider the functions $\phi^i_{p^i} : R^{\Sigma_{j\neq i} m_j + 1} \to R^{m_i}$, $i,j \in \{1,\ldots,n\}$ defined as follows:

$$\phi^i_{p^i}(u^1, u^2, \ldots, u^{i-1}, u^{i+1}, \ldots, u^n, v_i) = \inf_{A_{ij}\in\mathcal{A}_{ij}} [\sum_{j\neq i} A_{ij}u^j - v_i e^i] + p^i$$

where $p^i \in R^{m_i}$, $1 \leq i \leq n$ are given nonnegative vectors, $u^i \in R^{m_i}$, $v_i \in R$, $1 \leq i \leq n$ and inf refers to the componentwise infimum.

Furthermore, we define $\forall\, i \in \{1,\ldots,n\}$ the function $\psi^i : R^{m_i} \to R$ by

$$\psi^i(u^i) = \sum_j u^i_j - 1.$$

Let the function $F_{p^1,\ldots,p^n} : R^{\Sigma m_i + n} \to R^{\Sigma m_i + n}$ be defined as follows:

$$F_{p^1,p^2,\ldots,p^n}(z) = \begin{bmatrix} u^1 \wedge \phi^1_{p^1}(u^2, u^3, \ldots, u^n, v_1) \\ u^2 \wedge \phi^2_{p^2}(u^1, u^3, \ldots, u^n, v_2) \\ \vdots \\ u^n \wedge \phi^n_{p^n}(u^1, u^2, \ldots, u^{n-1}, v_n) \\ v_1 \wedge \psi^1(u^1) \\ v_2 \wedge \psi^2(u^2) \\ \vdots \\ v_n \wedge \psi^n(u^n) \end{bmatrix}, \qquad (6.3.1)$$

with $z = ((u^1)^t, \ldots, (u^n)^t, v_1, \ldots, v_n)^t$. Finally $F_{0,0,\ldots,0}(z)$ is denoted by $F(z)$.

LEMMA 6.3.1 *Suppose that the entries of the matrices in $\mathcal{A}_{ij}$ are uniformly bounded with a positive lower bound. Then there exists an equilibrium set $\bar{X}$ of probability vectors if and only if the equation $F(z) = 0$ has a solution.*

Proof. Let $z = ((\bar{x}^1)^t, (\bar{x}^2)^t, \ldots, (\bar{x}^n)^t, v_1, \ldots, v_n)^t$ be a solution to $F(z) = 0$. Then

$$\bar{x}^i \wedge \inf_{A_{ij}\in\mathcal{A}_{ij}} (\sum_{j\neq i} A_{ij}\bar{x}^j - v_i e^i) = 0, \ \forall\, i \qquad (6.3.2)$$

and

$$v_i \wedge (\sum_{j=1}^{m_i} \bar{x}_j^i - 1) = 0, \ \forall\, i. \tag{6.3.3}$$

Clearly, it follows that $\sum_{j=1}^{m_i} \bar{x}_j^i \geq 1$ and $\sum_{j=1}^{m_i} \bar{x}_j^i > 1 \Rightarrow v_i = 0$. It also follows that $\bar{x}^i$'s are nonnegative vectors and $\sum_{j \neq i} A_{ij}\, \bar{x}^j \geq v_i\, e^i$, $\forall\, A_{ij} \in \mathcal{A}_{ij}$.

Suppose $v_i = 0$ for some i. Note that each $\bar{x}^j$ is a nonzero nonnegative vector as $\bar{x}^j \geq 0$ and $\sum_{r=1}^{m_j} \bar{x}_r^j \geq 1$; since the matrices in $\mathcal{A}_{ij}$ are positively bounded below uniformly, it follows that the vector $\inf_{A_{ij} \in \mathcal{A}_{ij}} \sum_{j \neq i} A_{ij}\, \bar{x}^j > 0$ which contradicts (6.3.2). Thus it follows that $v_i > 0 \ \forall\, i$ and hence $\sum_{r=1}^{m_i} \bar{x}_r^i = 1 \ \forall\, i$. Hence, the $\bar{x}^i$'s are probability vectors.

We now construct a row representative matrix $M_{ij}(\varepsilon)$ for arbitrary $\varepsilon > 0$ as follows:

Let $1 \leq r \leq m_i$. If $\bar{x}_r^i = 0$ let $(M_{ij}(\varepsilon))_{r\cdot}$ be any $(A_{ij})_{r\cdot}$ where $A_{ij} \in \mathcal{A}_{ij}$ for $1 \leq j \leq n$, $j \neq i$.

If $\bar{x}_r^i > 0$ then note that

$$\inf_{A_{ij} \in \mathcal{A}_{ij}} \sum_{j \neq i} (A_{ij}\, \bar{x}^j)_r = v_i. \tag{6.3.4}$$

Now let $\bar{A}_{ij} \in \mathcal{A}_{ij}$ be such that

$$0 \leq \sum_{j \neq i} (\bar{A}_{ij}\, \bar{x}^j)_r - v_i \leq \varepsilon.$$

Such an $\bar{A}_{ij}$ exists since (6.3.4) holds. We then choose

$$(M_{ij}(\varepsilon))_{r\cdot} = (\bar{A}_{ij})_{r\cdot} \text{ for } 1 \leq j \leq n,\ j \neq i.$$

Such a choice satisfies the conditions of Definition 6.2.1, since

$$(\bar{x}^i)^t (\sum_{j \neq i} M_{ij}(\varepsilon) \bar{x}^j) \leq \varepsilon \sum_r \bar{x}_r^i + v_i \sum_r \bar{x}_r^i \leq v_i + \varepsilon.$$

Since for every row representative R_{ij} of $\mathcal{A}_{ij}$, $\sum_{j\neq i} R_{ij}\,\bar{x}^j \geq v_i\,e^i$, it follows that

$$(\bar{x}^i)^t(\sum_{j\neq i} M_{ij}(\varepsilon)\bar{x}^j) \leq v_i + \varepsilon \leq (u^i)^t\sum_{j\neq i} R_{ij}\,\bar{x}^j + \varepsilon\,(u^i)^t e^i$$

$$\Rightarrow (\bar{x}^i)^t(\sum_{j\neq i} M_{ij}(\varepsilon)\bar{x}^j) \leq (u^i)^t\sum_{j\neq i} R_{ij}\,\bar{x}^j + \varepsilon.$$

We shall now show the converse.

Let $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ be a set of probability vectors satisfying the conditions of Definition 6.2.1. Consider the corresponding sequence $M_{ij}(\varepsilon)$. As these matrices are uniformly bounded, we may assume without loss of generality that $\lim_{\varepsilon\downarrow 0} M_{ij}(\varepsilon)$ exists for each $i \neq j$. Let $\lim_{\varepsilon\downarrow 0} M_{ij}(\varepsilon) = M_{ij}$.

We have $(\bar{x}^i)^t\,(\sum_{j\neq i} M_{ij}\,\bar{x}^j) \leq (u^i)^t\sum_{j\neq i} R_{ij}\,\bar{x}^j$, where R_{ij} is a row representative of $\mathcal{A}_{ij}$ and u^i is any probability vector. This implies in particular that $[(\bar{x}^i)^t\,(\sum_{j\neq i} M_{ij}\,\bar{x}^j)]\,e^i \leq \sum_{j\neq i} R_{ij}\,\bar{x}^j$. Take $v_i = (\bar{x}^i)^t\,(\sum_{j\neq i} M_{ij}\,\bar{x}^j)$, $1 \leq i \leq n$ and $z = ((\bar{x}^1)^t, (\bar{x}^2)^t, \ldots, (\bar{x}^n)^t, v_1, v_2, \ldots, v_n)^t$.

Note that $\sum_{j\neq i} R_{ij}\,\bar{x}^j - v_i\,e^i \geq 0$, for all row representative R_{ij} of $\mathcal{A}_{ij}$.

Therefore, $\inf_{A_{ij}\in\mathcal{A}_{ij}} (\sum_{j\neq i} A_{ij}\bar{x}^j - v_i\,e^i) = \sum_{j\neq i} M_{ij}\,\bar{x}^j - v_i\,e^i \geq 0$.

Further, since $(\bar{x}^i)^t(\sum_{j\neq i} M_{ij}\,\bar{x}^j - v_i\,e^i) = 0$, it follows that for all $1 \leq r \leq m_i$, $\bar{x}^i_r[(\sum_{j\neq i} M_{ij}\,\bar{x}^j)_r - v_i] = 0$, as $\bar{x}^i$ and $(\sum_{j\neq i} M_{ij}\,\bar{x}^j - v_i\,e^i)$ are nonnegative vectors.

Thus $\bar{x}^i \wedge (\sum_{j\neq i} M_{ij}\,\bar{x}^j - v_i\,e^i) = 0, \ \forall\, 1 \leq i \leq n$.

Also since $\inf_{A_{ij}\in\mathcal{A}_{ij}} (\sum_{j\neq i} A_{ij}\bar{x}^j - v_i\,e^i) \geq 0$, letting $\varepsilon \downarrow 0$ in the inequality $0 \leq (\bar{x}^i)^t\,[\inf_{A_{ij}}(\sum_{j\neq i} A_{ij}\bar{x}^j - v_i\,e^i)] \leq (\bar{x}^i)^t[\sum_{j\neq i} M_{ij}(\varepsilon)\bar{x}^j - v_i\,e^i]$ we see that $(\bar{x}^i)^t\,[\inf_{A_{ij}}(\sum_{j\neq i} A_{ij}\bar{x}^j - v_i\,e^i)] = 0$, so that $\bar{x}^i \wedge [\inf_{A_{ij}\in\mathcal{A}_{ij}} (\sum_{j\neq i} A_{ij}\bar{x}^j - v_i\,e^i)] = 0$.

Thus we have $F(z) = 0$. ∎

LEMMA 6.3.2 *Assume that $\mathcal{A}_{ij}$, $i \neq j$ are nonempty sets of matrices in $R^{m_i \times m_j}$ which are (componentwise) uniformly bounded below by the positive number L and above by the positive number U. Let $p^i \in R^{m_i}$ be nonnegative. Then*

(a) $F_{p^1,p^2,\ldots,p^n}$ *is continuous on* R^{m+n}.

(b) *The set of solutions to the equation* $F_{p^1,p^2,\ldots,p^n}(z) = 0$ *is bounded.*

(c) *For the function* F *there is a bounded set* Ω *in* R^{m+n} *such that* $\deg(F,\Omega,0) \neq 0$. *In particular,* $F(z) = 0$ *has a solution on* Ω.

Proof of (a). Note that it is enough to show that $\phi^i_{p^i}$ and ψ^i are continuous functions. Since ψ^i are linear functions, they are continuous. Let $||y||$ be the l_1-norm of y, i.e., $||y|| = \Sigma|y_i|$ for any $y \in R^s$ for any positive integer s. Now

$$\phi^i_{p^i}(u^1,u^2,\ldots,u^{i-1},u^{i+1},\ldots,u^n,v_i) = \inf_{A_{ij}\in\mathcal{A}_{ij}} [\sum_{j\neq i} A_{ij}u^j - v_i e^i] + p^i.$$

For any coordinate r,

$$[\phi^i_{p^i}(u^1,u^2,\ldots,u^{i-1},u^{i+1},\ldots,u^n,v_i)]_r = \inf_{A_{ij}\in\mathcal{A}_{ij}} [(\sum_{j\neq i} A_{ij}u^j)_r - v_i] + p^i_r$$

$$= \inf_{A_{ij}\in\mathcal{A}_{ij}} [(\sum_{j\neq i} A_{ij}x^j)_r - v_i^* + [\sum_{j\neq i} A_{ij}(u^j - x^j)]_r - (v_i - v_i^*) + p^i_r]$$
$$\leq [\phi^i_{p^i}(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n,v_i^*)]_r + U||u-x|| + |v_i - v_i^*|, \text{ where } u,x \in R^m.$$

By symmetry we also have

$$[\phi^i_{p^i}(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n,v_i^*)]_r \leq [\phi^i_{p^i}(u^1,\ldots,u^{i-1},u^{i+1},\ldots,u^n,v_i)]_r$$
$$+U||u-x|| + |v_i - v_i^*|.$$

Hence,

$$|[\phi^i_{p^i}(u^1,\ldots,u^{i-1},u^{i+1},\ldots,u^n,v_i) - \phi^i_{p^i}(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n,v_i^*)]_r|$$
$$\leq U||u-x|| + |v_i - v_i^*| \text{ for all } r \text{ and for all } i.$$

This inequality shows that the functions $\phi^i_{p^i}$ are continuous.

Proof of (b). Suppose $F_{p^1,p^2,\ldots,p^n}(\bar{z}) = 0$,
where $\bar{z} = ((\bar{x}^1)^t,\ldots,(\bar{x}^n)^t,\bar{v}_1,\ldots,\bar{v}_n)^t$.
Since $\bar{x}^i \wedge \phi^i_{p^i}(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n,\bar{v}_i) = 0$, it is clear that $\bar{x}^i \geq 0, \ \forall\ i$. Further, as $\bar{v}_i \wedge \psi^i(\bar{x}^i) = 0$, we have $\sum_j \bar{x}^i_j \geq 1$, $\bar{v}_i \geq 0$. If $\bar{v}_i = 0$

for some i, then $\sum_r \bar{x}^i_r \geq 1$ and $\phi^i_{p^i} = \inf[\sum_{i \neq j} A_{ij}\bar{x}^i_j] + p^i > 0$. However, $\bar{x}^i \wedge \phi^i_{p^i}(\bar{x}^1, \ldots, \bar{x}^{i-1}, \bar{x}^{i+1}, \ldots, \bar{x}^n, \bar{v}_i) \neq 0$, as there exists an r such that $\bar{x}^i_r > 0$ and $\phi^i_{p^i}(\bar{x}^1, \ldots, \bar{x}^{i-1}, \bar{x}^{i+1}, \ldots, \bar{x}^n, \bar{v}_i) > 0$. Therefore, $\bar{v}_i > 0$ and $\sum_r \bar{x}^i_r = 1, \forall i$.
Further, we have

$\phi^i_{p^i}(\bar{x}^1, \ldots, \bar{x}^{i-1}, \bar{x}^{i+1}, \ldots, \bar{x}^n, \bar{v}_i) \geq 0 \Rightarrow \sum_{j \neq i} A_{ij}\bar{x}^j - \bar{v}_i e^i + p^i \geq 0, \ \forall A_{ij} \in \mathcal{A}_{ij}$.

Hence, $\sum_{j \neq i} A_{ij}\bar{x}^j + p^i \geq \bar{v}_i e^i, \ \forall A_{ij} \in \mathcal{A}_{ij}$

or, $\bar{v}_i e^i \leq \sum_{j \neq i} A_{ij}\bar{x}^j + p^i \leq U \sum_{j \neq i} E^{ij}\bar{x}^j + p^i$

$= (n-1)Ue^i + p^i$ where E^{ij} is the matrix of 1's of order $m_i \times m_j$.

Therefore, $\bar{v}_i \leq (n-1)U + \min_{1 \leq r \leq m_i} p^i_r$. Hence the $\bar{v}_i$'s are bounded.

Hence $\bar{z}$ is a solution to $F_{p^1,\ldots,p^n}(z) = 0$ implies that $0 \leq \bar{z}_i \leq 1$ for $1 \leq i \leq m$ and $0 \leq \bar{v}_i \leq (n-1)U + \max_{1 \leq k \leq n} \min_{1 \leq r \leq m_k} p^k_r, \ i \in \{1, \ldots, n\}$.

Thus the set of solutions to $F_{p^1,\ldots,p^n}(z) = 0$ is bounded.

Proof of (c). Let E^{ij} be the matrix of 1's of order $m_i \times m_j$.

Let $\forall\ i\ \alpha^i = ((\alpha^i_1)^t, (\alpha^i_2)^t, \ldots, (\alpha^i_{m_i})^t)^t$ where the real numbers α^i_s's satisfy $0 < \alpha^i_1 < \alpha^i_2 < \ldots < \alpha^i_{m_i}$ and let

$$(H_t(z))_i = \bar{x}^i \wedge \inf_{A_{ij} \in \mathcal{A}_{ij}} \{t \sum_{j \neq i} A_{ij}\bar{x}^j + (1-t) \sum_{j \neq i} E^{ij}\bar{x}^j - (tv_i + (1-t)v^*_i)e^i + (1-t)\alpha^i\}, \ 1 \leq i \leq n.$$

$$\text{Further, } H_t(z) = \begin{bmatrix} (H_t(z))_1 \\ \vdots \\ (H_t(z))_n \\ (tv_1 + (1-t)v^*_1) \wedge \psi^1(\bar{x}^1) \\ \vdots \\ (tv_n + (1-t)v^*_n) \wedge \psi^n(\bar{x}^n) \end{bmatrix}.$$

Note that by (a) H_t is a continuous function and H_t sets up a homotopy between $H_1 = F_{0,0,\ldots,0}$ and G where G is H_0. We also note that by (b) the solution set of $H_t(z) = 0$ is bounded for each $0 \leq t \leq 1$.

Let $S = \{z \mid H_t(z) = 0 \text{ for some } t \in [0,1]\}$. Note that S is bounded. Now let Ω be an open bounded set in R^{m+n} containing S. By Property 2 (the homotopy invariance property) of the degree we have

$$\deg(F, \Omega, 0) = \deg(G, \Omega, 0).$$

We shall now calculate the $\deg(G, \Omega, 0)$ and show that it is nonzero (it is in fact ± 1).

Consider the equation $G(z) = 0$. Let

$$\bar{x}^i = (1, 0, \ldots, 0)^t, \quad \forall\, 1 \le i \le n$$

$$v_i^* = (n-1) + \alpha_1^i, \quad \forall\, 1 \le i \le n.$$

Note that $v_i^* > 0$, $\forall\, 1 \le i \le n$. Let $\bar{z} = ((\bar{x}^1)^t, (\bar{x}^2)^t, \ldots, (\bar{x}^n)^t, v_1^*, \ldots, v_n^*)^t$. Then $\forall\, 1 \le i \le n$,

$$\begin{aligned}(G(\bar{z}))_i &= (H_0(\bar{z}))_i \\ &= \bar{x}^i \wedge (\sum_{j \ne i} E^{ij} \bar{x}^j - v_i^* e^i + \alpha^i).\end{aligned}$$

Now

$$\begin{aligned}\sum_{j \ne i} (E^{ij} \bar{x}^j - v_i^* e^i + \alpha^i) &= (n-1)e^i - v_i^* e^i + \alpha^i \\ &= (n-1)e^i - (n-1)e^i - \alpha_1^i e^i + \alpha^i \\ &= \alpha^i - \alpha_1^i e^i.\end{aligned}$$

Note that $(\alpha^i - \alpha_1^i e^i)_1 = 0$ and $(\alpha^i - \alpha_1^i e^i)_r > 0$ for all $r > 1$ by our choice of the numbers α_s^i. Therefore, $\inf \bar{x}^i \wedge (\alpha^i - \alpha_1^i e^i) = 0$. This holds for all i. Further $\psi^i(\bar{x}^i) = 0$, $\forall\, i$. Hence, $v_i^* \wedge \psi^i(\bar{x}^i) = 0$, $\forall i$. Thus $G(\bar{z}) = 0$.

Now suppose $\hat{z} = ((\hat{x}^1)^t, \ldots, (\hat{x}^n)^t, \hat{v}_1, \ldots, \hat{v}_n)^t$ is another solution to $G(z) = 0$. Suppose $\hat{v}_i = 0$. Note that $\sum_r \hat{x}_r^j \ge 1$, $\forall\, j$ since $\hat{v}_j \wedge \psi^j(\hat{x}^j) = 0$.

Hence $\sum_{j\neq i} E^{ij}\hat{x}^j + \alpha^i - \hat{v}_i e^i = \sum_{j\neq i} E^{ij}\hat{x}^j + \alpha^i > 0$ as $\hat{x}^j$ is nonzero for each j and E^{ij} is the matrix of all 1's. This contradicts our hypothesis that

$$\hat{x}^i \wedge (\sum_{j\neq i} E^{ij}\hat{x}^j - \hat{v}_i e^i + \alpha^i) = 0.$$

Hence $\hat{v}_i > 0$, $\forall i$ and therefore $\sum_r \hat{x}^i_r = 1$, $\forall i$.

Now $\sum_{j\neq i}(E^{ij}\hat{x}^j - \hat{v}_i e^i + \alpha^i) = (n-1)e^i - \hat{v}_i e^i + \alpha^i$.

As $\hat{x}^i \wedge ((n-1)e^i - \hat{v}_i e^i + \alpha^i) = 0$, it follows that $(n-1)e^i - \hat{v}_i e^i + \alpha^i \geq 0$, which implies $\hat{v}_i \leq (n-1) + \alpha^i_r$, $\forall\, r$. Therefore, $\hat{v}_i \leq (n-1) + \alpha^i_1$. If $\hat{v}_i < (n-1) + \alpha^i_1$, then $(n-1)e^i - \hat{v}_i e^i + \alpha^i > 0$, which contradicts our hypothesis that $\hat{x}^i \wedge ((n-1)e^i - \hat{v}_i e^i + \alpha^i) = 0$, as $\hat{x}^i \neq 0$. Hence $\hat{v}_i = (n-1) + \alpha^i_1$ and this holds for all i. It also follows that

$$(\sum_{j\neq i} E^{ij}\hat{x}^j - \hat{v}_i e^i + \alpha^i)_r = [(n-1)e^i - \hat{v}_i e^i + \alpha^i]_r > 0 \text{ for } r \neq 1.$$

Hence $\hat{x}^i_r = 0$ for $r \neq 1$. Thus $\hat{v}_i = v^*_i$ and $\hat{x}^i = \bar{x}^i$. Therefore, the solution is unique.

Let $\mathcal{I} = \{i \mid \bar{z}_i > 0\}$ and let $Q_{\mathcal{I}\mathcal{I}}$ denote the submatrix of rows and columns of Q from $\mathcal{I}$, where

$$Q = \begin{bmatrix} 0 & E^{12} & \cdots & E^{1n} & -e^1 & 0 & \cdots & 0 \\ E^{21} & 0 & \cdots & E^{2n} & 0 & -e^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E^{n1} & E^{n2} & \cdots & 0 & 0 & 0 & \cdots & -e^n \\ (e^1)^t & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & (e^2)^t & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (e^n)^t & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is easy to note that $Q_{\mathcal{I}\mathcal{I}}$ is of the form

$$\begin{bmatrix} X_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

where $X = ((X_{ij}))$ with $x_{ii} = 0$ and $x_{ij} = 1$, $i \neq j$. It is easy to see that the matrix Q_{II} has determinant ± 1. Thus by Property 6 of degree, $\deg(G, \Omega, 0)$ is nonzero (± 1). From here it follows that $\deg(F, \Omega, 0)$ is nonzero and F must have a zero in the open set Ω. This completes the proof of Lemma 6.3.2. ■

We then have the following theorem.

THEOREM 6.3.1 *Suppose that the nonempty sets $\mathcal{A}_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$ are bounded in $R^{m_i \times m_j}$. Then there exists a set $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ of probability vectors, where $\bar{x}^i$ is of order $m_i \times 1$ with the following property:*

For every $\varepsilon > 0$, there exists a row representative $M_{ij}(\varepsilon)$ of $\mathcal{A}_{ij}$ such that

$$(\bar{x}^i)^t \sum_{j \neq i} M_{ij}(\varepsilon)\bar{x}^j \leq (u^i)^t \sum_{j \neq i} R_{ij}\,\bar{x}^j + \varepsilon, \;\; 1 \leq i \leq n, 1 \leq j \leq n \qquad (6.3.5)$$

for all probability vectors u^i and for all row representatives R_{ij} of $\mathcal{A}_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$.

When all the $\mathcal{A}_{ij}$'s are compact sets, the above holds with $\varepsilon = 0$.

Proof. The conclusion of the theorem remains the same when a constant matrix is added to all the matrices in $\mathcal{A}_{ij}$. Thus, without loss of generality we may assume that all the matrices in $\mathcal{A}_{ij}$ have uniform positive lower bounds.

The theorem now follows from Lemma 6.3.1 and Lemma 6.3.2. If the sets $\mathcal{A}_{ij}$ are compact then for each $\varepsilon > 0$, we get an $M_{ij}(\varepsilon)$, $i \neq j$ and as the $M_{ij}(\varepsilon)$'s are from a compact set as $\varepsilon \to 0$, $M_{ij}(\varepsilon) \to M_{ij}(0)$, which is the required row representative of $\mathcal{A}_{ij}$ and the inequalities hold with $\varepsilon = 0$, for the $M_{ij}(0)$'s. The conclusion follows. ■

6.4 Solution Stability for the Generalized Polymatrix Games

DEFINITION 6.4.1 An isolated equilibrium set $\bar{X}$ of the polymatrix game with sets $\mathcal{A}_{ij}$, $i \neq j$ is called *stable* if $\forall\ \delta > 0$ there exists an $\varepsilon > 0$ such that for any polymatrix game with sets $\mathcal{A}'_{ij}$, $i \neq j$, each being compact in $R^{m_i \times m_j}$ and $\sum_{i \neq j} d_H(\mathcal{A}_{ij}, \mathcal{A}'_{ij}) < \delta$, we have

$$\mathcal{E}(\mathcal{A}'_{ij}) \cap B_\varepsilon(\bar{X}) \neq \emptyset$$

where $\mathcal{E}(\mathcal{A}_{ij})$ is the collection of equilibrium sets of the polymatrix game with sets $\mathcal{A}_{ij}$ and $B_\varepsilon(\bar{X})$ is the open ε-ball around $\bar{X}$.

THEOREM 6.4.1 *If for compact $\mathcal{A}_{ij}$, $i \neq j$, the polymatrix game with sets $\mathcal{A}_{ij}$ has a finite number of equilibrium points, then it has a stable equilibrium point.*

Proof. Let $\mathcal{E}(\mathcal{A}_{ij})$ denote the set of equilibrium points corresponding to $\mathcal{A}_{ij}$. Thus, $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \in \mathcal{E}(\mathcal{A}_{ij}) \Rightarrow F_{0,0,\ldots,0}(z) = 0$ where $z = ((\bar{x}^1)^t, \ldots, (\bar{x}^n)^t, v_1 \ldots, v_n)^t$. Let $s^\ell = (((\bar{x}^1)^t, \ldots, (\bar{x}^n)^t)^t)^\ell$, $\ell = 1, 2, \ldots, r$ be the equilibrium points, i.e., the enumeration of $\mathcal{E}(\mathcal{A}_{ij})$. Then we can find U^ℓ, $\ell = 1, 2, \ldots, r$ such that $s^\ell \in U^\ell$, U^ℓ's are disjoint, $U^\ell \subseteq R^m$ and U^ℓ is a $\bar{\delta}$-neighbourhood of s^ℓ. From Lemma 6.3.2.c and by Property 4 (the domain decomposition property) of degree we derive

$$0 \neq \deg(F, \cup_{\ell=1}^r U^\ell, 0) = \sum_{\ell=1}^r \deg(F, U^\ell, 0).$$

Therefore, there exists an index ℓ, say $\ell = 1$, such that $\deg(F, U^1, 0) \neq 0$.

We claim that $s^1 = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)^1$ is a stable equilibrium set. Take an arbitrary $\delta > 0$ and let $\mathcal{A}'_{ij} \subseteq R^{m_i \times m_j}$ be compact for $i \neq j$ such that

$$\sum_{i \neq j} d_H(\mathcal{A}_{ij}, \mathcal{A}'_{ij}) < \delta.$$

For a suitable $\varepsilon > 0$ we can find an open set V_1 containing s^1 such that $V_1 \subseteq U^1$ and $V_1 \subseteq B_\varepsilon(s^1)$ and satisfying

$$\sup_{s \in V_1} \|F(s) - F'(s)\| < \operatorname{dist}(0, F(\partial V_1))$$

where $F' = F'_{0,\ldots,0}$ is the function defined in Section 6.3 corresponding to $\mathcal{A}'_{ij}$. Now $\deg(F, V_1, 0) = \deg(F, U^1, 0) \neq 0$ by Property 4 (the domain decomposition property) and Property 5 (excision property) of the degree.

By Property 3 (the nearness property) of degree there exists a solution $s^0 \in V_1$ of the equation $F'(z) = 0$. Clearly $s^0 \in \mathcal{E}(\mathcal{A}'_{ij})$ and also belongs to $B_\varepsilon(s^1)$. This concludes the proof. ∎

In addition to the existence of a stable equilibrium point, the following global stability property also holds.

THEOREM 6.4.2 *Let $\mathcal{A}_{ij}$ be compact $\forall\, i \neq j$ and let $\mathcal{B} \subseteq R^m$ be the unit ball around* 0. *Then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that*

$$\mathcal{E}(\mathcal{A}'_{ij},\, i \neq j) \cap [\mathcal{E}(\mathcal{A}_{ij},\, i \neq j) + \varepsilon\mathcal{B}] \neq \emptyset,$$

$\forall\, \mathcal{A}'_{ij}$ *for which* $\sum_{i \neq j} d_H(\mathcal{A}_{ij}, \mathcal{A}'_{ij}) < \delta$.

Proof. We have already shown that $\mathcal{E}(\mathcal{A}_{ij},\, i \neq j)$ is nonempty and bounded, since $\mathcal{A}_{ij}$'s are compact for $i \neq j$. Consider the open set $\mathcal{D} = \mathcal{E}(\mathcal{A}_{ij},\, i \neq j) + \varepsilon\mathcal{B}$. Without loss of generality, we may assume that $\mathcal{D} \subset \Omega$, where Ω is an open set containing $\mathcal{E}(\mathcal{A}_{ij},\, i \neq j)$ such that $\deg(F, \Omega, 0) \neq 0$ and F is the function introduced in (6.3.1). Since there are no solutions of $F(z) = 0$ in $\Omega \backslash \mathcal{D}$, by Property 5 (the excision property) of the degree it follows that

$$\deg(F, \mathcal{D}, 0) = \deg(F, \Omega, 0) \neq 0.$$

Now for a suitable $\delta > 0$

$$\sup_{\mathcal{D}} \|F(s) - F'(s)\| < \operatorname{dist}(0, F(\partial\mathcal{D}))$$

where F' is the function defined in (6.3.1) corresponding to $\mathcal{A}'_{ij}$ for all $\mathcal{A}'_{ij}$ such that

$$\sum_{i\neq j} d_H(\mathcal{A}_{ij}, \mathcal{A}'_{ij}) < \delta.$$

By Property 3 (the nearness property) of the degree, $\deg(F', \mathcal{D}, 0) = \deg(F, \mathcal{D}, 0) \neq 0$. Hence the equation $F'(z) = 0$ has a solution in $\mathcal{D}$. The conclusion of the theorem follows. ∎

6.5 The VLCP Associated with a Generalized Polymatrix Game

Suppose the given $\mathcal{A}_{ij}$'s are finite sets of matrices. We shall show in this section that the problem of finding an equilibrium set of strategies $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$, where each $\bar{x}^i$ is a probability vector, for the polymatrix game with the sets $\mathcal{A}_{ij}$, $j \neq i$ can be formulated as a vertical linear complementarity problem.

Suppose the set $\mathcal{A}_{ij}$ contains s_i matrices of order $m_i \times m_j$. Define the matrices C^r_{ij} of order $s_i \times m_j$ as follows: For $i \neq j$,

$$(C^r_{ij})_{p\cdot} = (A^p_{ij})_{r\cdot}, \ \ 1 \leq p \leq s_i$$

where $\{A^p_{ij} \mid p = 1, 2, \ldots, s_i\} = \mathcal{A}_{ij}$ and for $i = j$, take

$$(C^r_{ij})_{p\cdot} = 0, \ \ 1 \leq p \leq s_i.$$

In this section, let e^{s_k} denote the vector of order $s_k \times 1$ each of whose coordinate is 1 and e^{m_r} denote the vector of order $m_r \times 1$ each of whose coordinates is 1. Let $\rho_1 = 0$ and $\rho_k = \sum_{j=1}^{k-1} m_j$. For $1 \leq k \leq n$, let $\mathcal{N}^{\rho_k}$ denote the m_k blocks of matrices $\mathcal{N}^{\rho_k + j}$ defined as

$$\mathcal{N}^{\rho_k+j} = \begin{bmatrix} C^j_{k1} & C^j_{k2} & \cdots & C^j_{k(k-1)} & 0 & C^j_{k(k+1)} & \cdots & C^j_{kn} & 0 & 0 & -e^{s_k} & 0 & \cdots & 0 \end{bmatrix}$$

for $1 \leq j \leq m_k$.

For $1 \leq r \leq n$, let $\mathcal{N}^{m+r}$ consists of exactly one row and $m+n$ columns which is given as follows:

$$\mathcal{N}^{m+r} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (e^{m_r})^t & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now consider the vertical block matrix $\mathcal{N}$ of order $(\sum_{i=1}^{n} s_i m_i + n) \times (m+n)$ given by

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}^{\rho_1} \\ \mathcal{N}^{\rho_2} \\ \vdots \\ \mathcal{N}^{\rho_n} \\ \mathcal{N}^{m+1} \\ \vdots \\ \mathcal{N}^{m+n} \end{bmatrix} \quad \text{where } \mathcal{N}^{\rho_k} \text{ is the set of blocks } \mathcal{N}^{\rho_k+j},\ 1 \leq j \leq m_k.$$

The total number of blocks in $\mathcal{N}$ is $\sum_{i=1}^{n} m_i + n = m+n$.

Let q be the vector of order $(\sum_{i=1}^{n} s_i m_i + n) \times 1$ whose first $\sum_{i=1}^{n} s_i m_i$ coordinates are 0's and the last n coordinates are -1's. Then in view of Lemma 6.3.1, it is easy to see that an equilibrium set of probability vectors $\bar{X} = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\}$ and a corresponding choice of representative matrices from $\mathcal{A}_{ij}$ can be obtained by solving the vertical block VLCP$(q, \mathcal{N})$ where q and $\mathcal{N}$ are as defined above.

The equivalent matrix M (see Section 2.3) which is of order $(\sum_{i=1}^{n} s_i m_i + n) \times (\sum_{i=1}^{n} s_i m_i + n)$ has the same structure as the matrix that occurs in the linear complementarity formulation of a polymatrix game. Therefore, Howson's scheme [37] can be applied to the equivalent LCP(q, M) and if we skip the trivial pivots in this algorithm, we can obtain a modified Howson algorithm for the VLCP$(q, \mathcal{N})$ associated with the given generalized polymatrix game.

EXAMPLE 6.5.1 Suppose the sets $\mathcal{A}_{ij}$ are as follows:

$$\mathcal{A}_{12} = \left\{ \begin{bmatrix} 2 & 0 & 4 \\ 1 & 6 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 0 \end{bmatrix} \right\}, \quad \mathcal{A}_{13} = \left\{ \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \right\},$$

$$\mathcal{A}_{21} = \left\{ \begin{bmatrix} 3 & 4 \\ 8 & 3 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 3 & 7 \\ 6 & 1 \end{bmatrix} \right\}, \quad \mathcal{A}_{23} = \left\{ \begin{bmatrix} 2 & 6 \\ 6 & 5 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 2 & 6 \end{bmatrix} \right\},$$

$$\mathcal{A}_{31} = \left\{ \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 9 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix} \right\},$$

$$\mathcal{A}_{32} = \left\{ \begin{bmatrix} 4 & 1 & 3 \\ 2 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 & 6 \\ 5 & 4 & 8 \end{bmatrix}, \begin{bmatrix} 7 & 2 & 5 \\ 1 & 6 & 4 \end{bmatrix} \right\}.$$

The related matrices C_{ij}^r are as follows:

$$C_{11}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{12}^1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 4 & 7 \end{bmatrix}, \quad C_{13}^1 = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix},$$

$$C_{11}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{12}^2 = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 3 & 0 \end{bmatrix}, \quad C_{13}^2 = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix},$$

$$C_{21}^1 = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}, \quad C_{22}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{23}^1 = \begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix},$$

$$C_{21}^2 = \begin{bmatrix} 8 & 3 \\ 3 & 7 \end{bmatrix}, \quad C_{22}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{23}^2 = \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix},$$

$$C_{21}^3 = \begin{bmatrix} 4 & 6 \\ 6 & 1 \end{bmatrix}, \quad C_{22}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{23}^3 = \begin{bmatrix} 5 & 4 \\ 2 & 6 \end{bmatrix},$$

$$C_{31}^1 = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 3 & 5 \end{bmatrix}, \quad C_{32}^1 = \begin{bmatrix} 4 & 1 & 3 \\ 3 & 5 & 6 \\ 7 & 2 & 5 \end{bmatrix}, \quad C_{33}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_{31}^2 = \begin{bmatrix} 3 & 6 \\ 9 & 7 \\ 6 & 2 \end{bmatrix}, \quad C_{32}^2 = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 4 & 8 \\ 1 & 6 & 4 \end{bmatrix}, \quad C_{33}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The vertical block matrix $\mathcal{N}$ of type $(\eta_1, \eta_2, \ldots, \eta_{10})$ is given by

$$\mathcal{N} = \begin{bmatrix} 0 & 0 & 2 & 0 & 4 & 2 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 7 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 6 & 5 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 4 & 2 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 2 & 6 & 0 & -1 & 0 \\ 4 & 4 & 0 & 0 & 0 & 7 & 1 & 0 & -1 & 0 \\ 8 & 3 & 0 & 0 & 0 & 6 & 5 & 0 & -1 & 0 \\ 3 & 7 & 0 & 0 & 0 & 5 & 5 & 0 & -1 & 0 \\ 4 & 6 & 0 & 0 & 0 & 5 & 4 & 0 & -1 & 0 \\ 6 & 1 & 0 & 0 & 0 & 2 & 6 & 0 & -1 & 0 \\ 4 & 3 & 4 & 1 & 3 & 0 & 0 & 0 & 0 & -1 \\ 8 & 6 & 3 & 5 & 6 & 0 & 0 & 0 & 0 & -1 \\ 3 & 5 & 7 & 2 & 5 & 0 & 0 & 0 & 0 & -1 \\ 3 & 6 & 2 & 5 & 1 & 0 & 0 & 0 & 0 & -1 \\ 9 & 7 & 5 & 4 & 8 & 0 & 0 & 0 & 0 & -1 \\ 6 & 2 & 1 & 6 & 4 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

where $\eta_1 = s_1 = 2$, $\eta_2 = s_1 = 2$, $\eta_3 = s_2 = 2$, $\eta_4 = s_2 = 2, \eta_5 = s_2 = 2$, $\eta_6 = s_3 = 3$, $\eta_7 = s_3 = 3$, $\eta_8 = 1$, $\eta_9 = 1$, $\eta_{10} = 1$.

The matrix M associated with the equivalent LCP is given by

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 4 & 4 & 2 & 2 & 2 & 3 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 4 & 4 & 7 & 7 & 1 & 1 & 1 & 3 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 6 & 6 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 0 & 0 & 4 & 4 & 4 & 2 & 2 & 2 & -1 & 0 & 0 \\ 3 & 3 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 6 & 6 & 6 & 0 & -1 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 7 & 1 & 1 & 1 & 0 & -1 & 0 \\ 8 & 8 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 5 & 5 & 5 & 0 & -1 & 0 \\ 3 & 3 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & -1 & 0 \\ 4 & 4 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 & 4 & 4 & 4 & 0 & -1 & 0 \\ 6 & 6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 6 & 6 & 6 & 0 & -1 & 0 \\ 4 & 4 & 3 & 3 & 4 & 4 & 1 & 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 8 & 8 & 6 & 6 & 3 & 3 & 5 & 5 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 3 & 3 & 5 & 5 & 7 & 7 & 2 & 2 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 9 & 9 & 7 & 7 & 5 & 5 & 4 & 4 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 6 & 6 & 2 & 2 & 1 & 1 & 6 & 6 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

REMARK 6.5.1 Given a square matrix M, we say that a given $q \in R^n$ is *non-degenerate* with respect to M if (w, z) is a solution to LCP(q, M) implies that $w + z > 0$. We say that a generalized polymatrix game with n players and finite sets $\mathcal{A}_{ij}$, $i \neq j$ is *nondegenerate* if the associated equivalent LCP(q, M) is such that q is nondegenerate with respect to M. From the theory linear complementarity, it is known that if LCP$(0, M)$ has a unique solution then LCP(q, M) has the same parity of (either even or odd) number of solutions for all nondegenerate $q \in R^n$. This property is known as the constant parity property. As can be seen easily the VLCP$(q, \mathcal{N})$ has a unique solution $x^i = 0$, $v_i = 0$, $\forall\, i$, when $q = 0$. Hence the constant parity property holds. It follows that if the game is nondegenerate, then it has an odd number of equilibria.

REMARK 6.5.2 We also note that the equivalent LCP(q, M) associated with the VLCP$(q, \mathcal{N})$ has the same form as noted in (1.5.5) with $B \geq 0$. Thus LCP(q, M) can be solved by Garcia's scheme or by Miller-Zucker's scheme. See Section 1.5. It also follows that the algorithm given by Cottle and Dantzig [4] can be used to solve the VLCP$(q, \mathcal{N})$ by generalizing Garcia's scheme or Miller-Zucker's scheme.

Chapter 7

The Horizontal Linear Complementarity Problem

7.1 Introduction

In this chapter, we answer affirmatively the following question posed by Sznajder and Gowda in [89]. Is it possible to rewrite an HLCP equivalently as an LCP ? Sznajder and Gowda [89] list some situations where HLCP can be transformed as an LCP. We settle the question completely by providing an equivalent formulation.

In Section 7.2, we provide the necessary definitions and notations used throughout the chapter. An equivalent formulation of HLCP is given in Section 7.3. In Section 7.4, we show that if $B \in E(d)$, the equivalent matrix $\mathcal{M} \in E(d)$.

7.2 Preliminaries

Given a pair of $n \times n$ matrices A and B and a vector $q \in R^n$, let $F(q, A, B) = \{(x, y) \in R^n \times R^n \mid x \geq 0, y \geq 0, Ax - By = q\}$ and the solution set $S(q, A, B) = \{(x, y) \in F(q, A, B) \mid x^t y = 0\}$.

DEFINITION 7.2.1 A pair $(x, y) \in F(q, A, B)$ is called a *feasible solution* for the HLCP(q, A, B). A vector $q \in R^n$ is said to be feasible if $F(q, A, B) \neq \emptyset$.

DEFINITION 7.2.2 A vector $q \in R^n$ is said to be *solvable* for (A, B) if the solution set $S(q, A, B) \neq \emptyset$.

DEFINITION 7.2.3 We say that (A, B) is a *Q-pair* if $\forall\, q \in R^n$, HLCP(q, A, B) has a solution.

DEFINITION 7.2.4 We say that (A, B) is a Q_0*-pair* if for any $q \in R^n$, $F(q, A, B) \neq \emptyset \Rightarrow S(q, A, B) \neq \emptyset$.

7.3 An Equivalent Formulation of HLCP

In this section, we show that the HLCP can be formulated as an LCP. Let $A, B \in R^{n \times n}$ and I be the identity matrix of order n. Also, let $e \in R^n$ be a vector of all 1's. We can express HLCP(q, A, B) as an equivalent LCP$(\tilde{q}, \mathcal{M})$. The equivalent LCP is given below:

Find $w, u, v, z, x, y \in R^n$ such that

$$\begin{bmatrix} w \\ u \\ v \end{bmatrix} = \begin{bmatrix} q \\ -e \\ 2e \end{bmatrix} + \begin{bmatrix} 0 & -A & B \\ I & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \end{bmatrix}, \quad \begin{bmatrix} w \\ u \\ v \end{bmatrix}^t \begin{bmatrix} z \\ x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} w \\ u \\ v \end{bmatrix}, \begin{bmatrix} z \\ x \\ y \end{bmatrix} \geq 0.$$

Here the equivalent LCP matrix

$$\mathcal{M} = \begin{bmatrix} 0 & -A & B \\ I & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \text{ is of order } 3n \times 3n \text{ and } \bar{q} = \begin{bmatrix} q \\ -e \\ 2e \end{bmatrix}.$$

Note that $\mathcal{M}$ is singular whatever be the pair (A, B). We call the matrix $\mathcal{M}$ constructed in this manner *the equivalent LCP matrix of the pair* (A, B) and the equivalent LCP is denoted as LCP$(\bar{q}, \mathcal{M})$.

LEMMA 7.3.1 *Given the HLCP(q, A, B), let $\mathcal{M} \in R^{3n\times 3n}$ be the equivalent LCP matrix of the pair (A, B). HLCP(q, A, B) has a solution if and only if LCP$(\bar{q}, \mathcal{M})$ has a solution.*

Proof. Let (η, ξ), $\xi \in R^{3n}$, $\eta \in R^{3n}$ be a solution to LCP$(\bar{q}, \mathcal{M})$ where

$$\eta = \begin{bmatrix} w \\ u \\ v \end{bmatrix} \text{ and } \xi = \begin{bmatrix} z \\ x \\ y \end{bmatrix}.$$

Note that $u_j + v_j = 1$ and $z_j \geq 1 \Rightarrow w_j = 0 \ \forall \ j$. So (1.3.4) is always satisfied. Since $y_j > 0 \Rightarrow v_j = 0 \Rightarrow u_j = 1 \Rightarrow x_j = 0$ and $x_j > 0 \Rightarrow u_j = 0 \Rightarrow v_j = 1 \Rightarrow y_j = 0$. Hence (x, y) solves HLCP(q, A, B).

Conversely, suppose (x, y) solves HLCP(q, A, B). Choose z_j as follows:

$$z_j = \begin{cases} 1 & \text{if } x_j > 0 \text{ and } y_j = 0 \\ 2 & \text{if } x_j = 0 \text{ and } y_j > 0 \\ z_j^*, & 1 < z_j^* < 2 \text{ if } x_j = 0 \text{ and } y_j = 0. \end{cases}$$

Also choose $w = 0$. Now it follows that $w^t z = 0$, $u^t x = 0$ and $v^t y = 0$. Therefore, (η, ξ) solves LCP$(\bar{q}, \mathcal{M})$ where ξ and η are defined as above. ■

7.4 Some Results on HLCP

LEMMA 7.4.1 *(i) (A, B) is a Q-pair if and only if LCP($\bar{q}, \mathcal{M}$) has a solution $\forall\ q \in R^n$.*

(ii) (A, B) is a Q_0-pair if and only if LCP($\bar{q}, \mathcal{M}$) has a solution for any $q \in R^n$ such that $F(q, A, B) \neq \emptyset$.

Proof. This follows from Lemma 7.3.1. ■

THEOREM 7.4.1 *Given the pair (A, B), let $B \in \mathcal{L}_1$ then*

$$\mathcal{M} = \begin{bmatrix} 0 & -A & B \\ I & 0 & 0 \\ -I & 0 & 0 \end{bmatrix} \in \mathcal{L}_1.$$

Proof. Let $\alpha_1 = \{1, \ldots, n\}, \alpha_2 = \{(n+1), \ldots, 2n\}$ and $\alpha_3 = \{(2n+1), \ldots, 3n\}$.

Suppose $\bar{\xi} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$.

Case I. $0 \neq z \geq 0$, $0 \neq x \geq 0$ and $0 \neq y \geq 0$. In this case $\exists$ an index $i \in \alpha_2$, $x_{i-n} > 0$ and $(\mathcal{M}\bar{\xi})_i = z_{i-n} \geq 0$.

Case II. $0 \neq z \geq 0, 0 \neq x \geq 0$ and $y = 0$. Then proceed as in Case I.

Case III. $0 \neq z \geq 0$, $0 \neq y \geq 0$ and $x = 0$. Note that since $B \in \mathcal{L}_1$ $\exists$ an $i \in \alpha_1$, such that $y_i > 0$ and $(By)_i \geq 0$. If now $z_i > 0$, note that $\bar{\xi}_i > 0$ and $(\mathcal{M}\bar{\xi})_i = (By)_i \geq 0$. However if $z_i = 0$, note that $\bar{\xi}_{2n+i} > 0$ and $(\mathcal{M}\bar{\xi})_{2n+i} = 0$.

Case IV. $0 \neq x \geq 0$, $0 \neq y \geq 0$ and $z = 0$. In this case $\exists$ an index $i \in \alpha_2 \cup \alpha_3$, $\bar{\xi}_i > 0$ and $(\mathcal{M}\bar{\xi})_i \geq 0$.

The other cases **(V)** $0 \neq z \geq 0$, $x = 0$, $y = 0$, **(VI)** $z = 0$, $0 \neq x \geq 0$, $y = 0$ and **(VII)** $z = 0$, $x = 0$, $0 \neq y \geq 0$ are easy to verify.

Hence $\mathcal{M} \in \mathcal{L}_1$. This concludes the proof. ■

In fact, we can prove the following result.

THEOREM 7.4.2 *If $B \in E(\tilde{d})$ for some vector $\tilde{d} > 0$ then the equivalent matrix $\mathcal{M} \in E(\hat{d})$ where* $\hat{d} = \begin{bmatrix} \tilde{d} \\ e \\ e \end{bmatrix}$.

Proof. To see this suppose $B \in E(\tilde{d})$ for some $\tilde{d} > 0$.
Let $\left(\begin{bmatrix} w \\ u \\ v \end{bmatrix}, \begin{bmatrix} z \\ x \\ y \end{bmatrix} \right)$ be a solution to LCP$(\hat{d}, \mathcal{M})$ where $\hat{d} = \begin{bmatrix} \tilde{d} \\ e \\ e \end{bmatrix}$.
Now considering the equation $u = e + z$, it follows that $u > 0$ as $z \geq 0$. Hence $x = 0$. Thus we note that (w, y) satisfies the following system of equations and inequalities

$$w = \tilde{d} + By, \;\; w \geq 0, \, y \geq 0.$$

Further, if $y_i > 0$, we have $v_i = 0$ which implies $z_i > 0$. Thus $w_i = 0$. From here it follows that $y^t w = 0$. Thus (w, y) solves the LCP$(\tilde{d}, B)$. Since $B \in E(\tilde{d})$, it follows that $y = 0$, $w = \tilde{d}$. Also since $x = 0$, $y = 0$, the only solution to LCP$(\hat{d}, \mathcal{M})$ is $w = \tilde{d}$, $u = e$, $v = e$. Thus $\mathcal{M} \in E(\hat{d})$. ∎

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