QUANTUM STOCHASTIC CALCULUS WITH INFINITE DEGREES OF FREEDOM AND ITS APPLICATIONS

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TO MY PARENTS

Who taught me to honour truth

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Contents

Introduction

Based on the canonical commutation relations between the creation, conservation (number) and annihilation operators of a free field on a boson Fock space, Hudson and Parthasarathy [21] have developed a quantum stochastic calculus, of which a detailed exposition may be found in Meyer [30] and Parthasarathy [38]. In order to handle the problem of dilating uniformly continuous quantum dynamical semigroups in the algebra of all bouded operators in a Hilbert space, Hudson and Parthasarathy [22] had already noted the importance of extending their calculus when an infinite number of independent creation and annihilation processes are used as integrators. Also, any attempt to extend the results of Meyer [29] and Parthasarathy and Sinha [37] on the realisation of classical Markov processes in the Accardi-Frigerio-Lewis's framework of quantum stochastic processes involves the investigation of quantum stochastic differential equations (q.s.d.e 's) with infinite degrees of freedom.

The aim of the present thesis is to present a brief exposition of the Hudson-Parthasarathy calculus in Fock space when a possibly infinite number of basic integrators are involved and apply it to the study of the following two basic problems:
(a) Under what conditions on $Z \equiv \{Z^i_j,\ i,j \in \overline{\mathcal{S}}\}$ the following quantum stochastic differential equation

$$dV(t) = \sum_{k \in \mathcal{S}} V(t) Z^i_j \Lambda^j_i(t),\ V(0) = I \qquad (0.1)$$

where $\Lambda^i_j(t),\ i,j \in \overline{\mathcal{S}} = \mathcal{S} \cup \{0\}$ are the basic integrators in the boson-Fock space $\Gamma(L\!\!\!L^2(I\!\!R_+, \mathcal{K}))$ with respect to an orthonormal basis $\{e_i,\ i \in \mathcal{S}\}$ for the Hilbert space $\mathcal{K}$ and $Z \equiv \{Z^i_j,\ i,j \in \overline{\mathcal{S}}\}$ is a family of densely defined operators in the initial Hilbert space $\mathcal{H}_0$, admits a contractive, isometric, co-isometric or unitary operator valued adapted process $V \equiv \{V(t),\ t \geq 0\}$ as a solution ?
(b) Given a family $\Theta \equiv \{\theta^i_j,\ i,j \in \overline{\mathcal{S}}\}$ of structure maps on an initial $*$-algebra $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$ under what conditions, a quantum stochastic flow $\mathcal{J} \equiv \{\mathcal{J}_t,\ t \geq 0\}$ in the sense of Evans and Hudson [12] exists and satisfies a q.s.d.e of the

form:

$$\mathcal{J}_0(x) = x, \quad d\mathcal{J}_t(x) = \sum_{i,j\in\overline{S}} \mathcal{J}_t(\theta^i_j(x))d\Lambda^j_i(t) \tag{0.2}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ for all $x \in \mathcal{A}_0$?

Here is a brief summary of our results:

In Theorem 2.12 of Section 2 the first problem is given a complete solution in the form of necessary and sufficient conditions when the family $Z \equiv \{Z^i_j,\ i,j \in \overline{S}\}$ satisfies the inequalities

$$\sum_{i\in\overline{S}}(||Z^i_j f||^2 + ||(Z^j_i)^* f||^2) \leq c_j||f|^2$$

for all $f \in \mathcal{H}_0, j \in \overline{S}$, c_j being a positive constants. This generalises and sharpens the previously known results of Hudson- Parthasarathy [21], Mohari-Sinha [33] and Mohari-Parthasarathy [31]. In particular we prove that (0.1) admits a unique contractive solution if and only if $Z \in \mathcal{Z}^-_R$ where

$$\mathcal{Z}^-_R \equiv \{Z,\ ((Z^i_j + (Z^j_i)^* + \sum_{k\in S}(Z^k_i)^* Z^k_j\ ))_{i,j\in S'} \leq 0, \text{ for all } S' \subset \overline{S}\ , \#S' < \infty\}.$$

This class plays an important role in dealing with (0.1), in the spirit of semigroup theory developed as in Yosida [39], for unbounded dissipative coefficients (See Section 5). As in Mohari-Parthasarathy [32] the role of Journé's time reversal principle (See Theorem 2.11) in the proof seems to be of special interest.

When $V \equiv \{V(t),\ t \geq 0\}$ is the unitary solution in the discussion above the quantum stochastic process $j_t(x) = V(t)(x \otimes I)V(t)^*,\ t \geq 0$ satisfies (0.2) for a family of bounded structure maps $\mathcal{L} \equiv \{\mathcal{L}^i_j,\ i,j \in \overline{S}\}$ with $\mathcal{B}(\mathcal{H}_0)$ as the initial algebra. The expressions for the $\mathcal{L}^i_j$ are presented in order to motivate applications to classical Markov processes in Section 3.

In Section 3, following Mohari-Sinha [33] we present a theory for quantum stochastic (QS) flows with countably infinite degrees of freedom for the noise when $\Theta \equiv \{\theta^i_j,\ i,j \in \overline{S}\}$ is a family of bounded srtructure maps obeying the regularity conditions

$$\sum_{i\in\overline{S}}||\theta^i_j(x)f||^2 \leq \sum_{i\in\mathcal{J}_j}||xD^i_j f||^2,$$

$$\sum_{i\in\mathcal{J}_j} ||D^i_j f||^2 \leq \alpha_j^2 ||f||^2$$

for all $x \in \mathcal{A}_0,\ f \in \mathcal{H}_0$ where for each $j \in \overline{\mathcal{S}},\ \mathcal{J}_j$ is a countable index set and $\{D^i_j,\ i \in \mathcal{J}_j\}$ is a family of bounded operators in $\mathcal{H}_0$. As in Parthasarathy and Sinha [37] we show that the family $\{\mathcal{J}_t(x),\ x \in \mathcal{A}_0,\ t \geq 0\}$ is commutative whenever $\mathcal{A}_0$ is abelian. This enables us to conclude that in the vacuum state of the Fock space, $\{\mathcal{J}_t,\ t \geq 0\}$ describes a classical Markov process with the bounded infinitesimal generator θ^0_0. Finally we apply this theory to show that continuous time Markov chains with countablely many state space can be understood as commutative QS flows on the abelian algebra of functions on the state space. This extends the previous studies by Meyer [29] and Parthasarathy and Sinha [37].

In Section 4, following Mohari [34] we consider (0.1) for the class of unbounded coefficients Z which admits a sequence of regular elements $Z(n) \in \mathcal{Z}^-_R,\ n \geq 1$ such that $Z^i_j(n)f \to Z^i_j f,\ i,j \in \overline{\mathcal{S}}$ as $n \to \infty$ for all $f \in \mathcal{D}$, where $\mathcal{D}$ is a dense linear manifold in $\mathcal{H}_0$. We exploit Frigerio's equicontinuity method as outlined in Fagnola [13] to ensure the existence of a contractive operator valued process satisfying (0.1) with Z as its coefficients. The approximating sequence of evolutions being non-commutative, this method only guarantees the existence of a contractive solution as a 'weak operator limit' of a subsequence of the evolutions associated with $Z(n),\ n \geq 1$. Such a construction does not help in examining the analytical properties of the limiting contractive processes. In this context the analyticity of exponential vectors (Wiener chaos expansion) plays an important role in setting up an inductive procedure to get a sufficient condition for the solution to be unique or isometric. Analysing the dual process we also obtain a sufficient condition for the evolution to be co-isometric. It is worth noting that the condition for the evolution to be isometric (co-isometric) is similar to that of Feller's resolvent condition for the minimal process, associated with a Kolmogorov's differential equation, to be faithful. In Section 5 these results are further strengthened.

In the spirit of semigroup theory developed as in Yosida [39] we build, in

Section 5, a theory for (0.1) whenever the entries in Z satisfy the following:

$$Z^i_j = \begin{cases} S^i_j - \delta^i_j & , \quad i,j \in \mathcal{S}, \\ Z_i & , \quad i \in \mathcal{S}, j = 0, \\ -\sum_{k \in \mathcal{S}} Z^*_k S^k_j & , \quad i = 0, j \in \mathcal{S}, \\ Y & , \quad i = 0 = j \end{cases}$$

where Y is the generator of a contractive C_0 semigroup and $\{Z_k,\ k \in \mathcal{S}\}$ is a family of densely defined operators such that $\mathcal{D}(Z_k) \supset \mathcal{D}(Y)$ and

$$< f, Yf > + < Yf, f > + \sum_{k \in \mathcal{S}} < Z_k f, Z_k f > \leq 0, f \in \mathcal{D}(Y) \tag{0.3}$$

and $S = ((\ S^i_j\ ))$ is a contractive operator in $\mathcal{H}_0 \otimes l_2(\mathcal{S})$ such that $S^i_j(\mathcal{D}) \subset \mathcal{D}(Z^*_k)$ where $\mathcal{D}$ is a core for Y. In the spirit of Yosida's approximation method [10] we exihibit an approximating family $Z(\lambda),\ \lambda > 0$ of regular dissipative elements which enables us to exploit the theory developed in Section 4.

In this context we also deal with the dilation problem associated with quantum mechanical Fokker-Planck equation written formally as

$$\rho(0) = \rho,\ \rho(t)' = Y\rho(t)\ + \rho(t)Y^* + \sum_{k \in \mathcal{S}} Z_k \rho(t) Z^*_k \tag{0.4}$$

subject to (0.3) for $\rho \in \mathcal{T}$, the Banach space of all trace class operators in $\mathcal{H}_0$. Davies [9] employed a special perturbation method outlined in Kato [25] to prove the existence of the minimal semigroup σ^{min} such that $\rho(t) = \sigma^{min}_t(\rho)$ is a solution for (0.3)-(0.4). Here the dual semigroup $\tau^{min}_t = (\sigma^{min})^*$ on $\mathcal{B}(\mathcal{H}_0)$ has been realised as the vacumm expectation of a (dissipative) quantum stochastic process. In this context the stability of $\mathcal{Z}^-_R$ under a specific perturbation plays a crucial role in the approximation procedure. A necessary and sufficient condition for the process to be conservative is also obtained. In such a case the 'Feller's resolvent condition ' described in Section 4 is also necessary for the evolution to be isometric.

As an illustration of the general theory of dilations we conclude our exposition with an investigation of Markov processes with countablely many states.

In Section 7 we construct a class of non-commutative operator valued processes, on the algebra of functions on state space, which dilates Feller's minimal solution in Fock space. Feller's condition is still necessary and sufficient for the dilation to be a conservative quantum stochastic process in the sense of [2]. This extends the previous studies by Fagnola [14].

In Section 8, we continue the programme begun in Meyer [29]. In a series of papers (Parthasarathy-Sinha [37], Mohari-Sinha [33]) it has been shown how to realise a classical Markov process with countable state space as a commutative QS flow. But it was restricted only to processes with bounded Markov generators. Here we consider the general situation and realise Feller's minimal solution as a commutative QS flow. We introduce a special sequence of commutative QS flows which approximates the induced QS flow on a suitable algebra in the strong operator topology. A necessary and sufficient condition for the flow to be conservative is also obtained. A notion of quantum exit stop time is introduced. It is a commutative adapted family of strongly continuous increasing projections on Fock space. Feller's exit stop time is realised as the vacuum expectation of these projections. Finally imposing a weak hypothesis on the Markov generator, we show that the dilation constructed by the apprximation procedure satisfies the diffusion equation (0.2) on the abelian algebra of finitely supported functions on the state space. This section is adapted from Mohari [34].

1 Notations and Preliminaries

All the Hilbert spaces that appear here are assumed to be complex and separable with inner product $< \cdot, \cdot >$ linear in the second variable. For any Hilbert space H, a symmetric Fock space over H is a pair $(\Gamma(H), e)$, where $\Gamma(H)$ is a Hilbert space and $e : H \to \Gamma(H)$ satisfies the following:
(i) $\varepsilon(H)$, the span of $\{e(u), u \in H\}$, is dense in $\Gamma(H)$;
(ii) $< e(u), e(v) >= exp(< u, v >)$.

The elements $e(u)$, $u \in H$ are called the **exponential** or **coherent vectors** . The family $\{e(u) : u \in \mathcal{M}\}$ is total for any dense linear manifold $\mathcal{M}$ in H and linearly independent in $\Gamma(H)$. So operators may be defined densely on $\Gamma(H)$ by giving their action on each $e(u)$, $u \in \mathcal{M}$. If $(\Gamma'(H), e')$ is another Fock space over H then $\Gamma(H)$ and $\Gamma'(H)$ are naturally isomorphic under exchange of exponential vectors $e(u) \to e'(u)$, $u \in H$. Here in the following paragraph we outline Fock's construction of symmetric Fock space over H.

For each $n \geq 1$ we define a projection P_n on $H \otimes H \otimes \cdot \cdot \otimes H$, n- fold by linear extension of

$$P_n(u_1 \otimes \cdot \cdot \otimes u_n) = \sum_{\pi} \frac{1}{n!} u_{\pi(1)} \otimes \cdot \cdot u_{\pi(n)}$$

where the sum ranges over all permutations π of the set $\{1, 2, ...n\}$. Put

$$\Gamma(H) = \mathbb{C} \oplus \sum_{n \geq 1} S_n(H \otimes H \cdot \cdot \otimes^{(n)} H).$$

Define $e : H \to \Gamma(H)$ by

$$e(u) = \oplus_{n \geq 0} u^{(n)}$$

where

$$u^{(n)} = \begin{cases} 1 & ; \quad n = 0 \\ \frac{1}{\sqrt{n!}} u^{\otimes^n} & ; \quad n \geq 1 \end{cases}$$

For $u, v \in H$ we have

$$\begin{aligned} < e(u), e(v) > &= \sum_{n \geq 0} < u^{(n)}, v^{(n)} > \\ &= \sum_{n \geq 0} \frac{1}{n!} (< u, v >)^n \\ &= exp(< u, v >). \end{aligned}$$

For any $u_k \in H$, $1 \leq k \leq n$ the map $(z_1, z_2, \cdot, z_n) \rightarrow e(\sum_{1 \leq k \leq n} z_k u_k)$ is analytic. Hence

$$\frac{d}{dz} e(u + zv)_{|z=0}$$

exists for any $u, v \in H$. In particular it can be shown that $e(u)$, $u \in H$ span a dense subspace of $\Gamma(H)$, so that $\Gamma(H)$ is a Fock space over H.

We also denote by $\mathcal{B}(H)$ the C^* algebra of all bounded linear operators in H. When C is a bounded operator on H and u is an element of H, the second quantized $\Gamma(C)$ of C and the Weyl operator $W(u)$ are determined uniquely by the relations:

$$\begin{aligned} \Gamma(C)e(v) &= e(Cv) \\ W(u)e(v) &= \exp\{-\frac{1}{2}||u||^2 - < u, v >\} e(u + v) \end{aligned}$$

for all $v \in H$. The von Neumann algebra generated by the family $\{W(u),\ u \in \mathcal{M}\}$ is $\mathcal{B}(H)$, whenever $\mathcal{M}$ is a dense subspace of H.

The **gauge, creation** and **annihilation** operators [21] $\lambda(A)$, $a^\dagger(u)$, $a(u)$ are defined on $\varepsilon(H)$ by the actions

$$\begin{aligned} \lambda(A)e(v) &= \frac{d}{d\alpha} e(e^{\alpha A} v)_{|\alpha=0}, \\ a^\dagger(u)e(v) &= \frac{d}{d\alpha} e(v + \alpha u)_{|\alpha=0}, \\ a(u)e(v) &= < u, v > e(v),\ v \in H \end{aligned}$$

where $A \in \mathcal{B}(H))$ and $u \in H$.

Suppose now that $\Gamma(H)$ is a Fock space over H where $H = H_1 \oplus H_2$. Let $\Gamma(H_k)$ be the closure of the span of $\{e(u),\ u \in H_k\},\ k = 1,2$. Then $\Gamma(H_k)$ is a Fock space over $H_k,\ k = 1,2$. For $u, v \in H$ with $u = u_1 + u_2,\ v = v_1 + v_2,\ u_k, v_k \in H_k,\ k = 1,2$, we have

$$\begin{aligned} < e(u_1) \otimes e(u_2), e(v_1) \otimes e(v_2) > &= < e(u_1), e(v_1) > < e(u_2), e(v_2) > \\ &= exp(< u_1, v_1 >) exp(< u_2, v_2 >) \\ &= exp(< u, v >) \end{aligned}$$

so that $\Gamma(H_1) \otimes \Gamma(H_2)$ is another Fock space over H with exponential map $u \rightarrow e(u_1) \otimes e(u_2)$. Thus we have

$$\Gamma(H_1) \otimes \Gamma(H_2) \cong \Gamma(H)$$

In future we shall frequently use this isomormhism, interchanging $\Gamma(H_1) \otimes \Gamma(H_2)$ and $\Gamma(H_1^{\text{`}} \oplus H_2)$ without any comment.

Fix two Hilbert spaces $\mathcal{H}_0$ and $\mathcal{K}$ and write $\Gamma_+, \Gamma_{s,t}$ for $\Gamma(H)$ when $H = L^2(I, \mathcal{K})$ and $I = \mathbb{R}_+, [s,t)$ respectively. Set

$$\tilde{H} = \mathcal{H}_0 \otimes \Gamma_+, \tilde{H}_{t]} = \mathcal{H}_0 \otimes \Gamma_{0,t}, \tilde{H}_{[t} = \Gamma_{t,\infty} .$$

we have the decomposition $\tilde{H} = \tilde{H}_{t]} \otimes \tilde{H}_{[t}$. The Hilbert space $\tilde{H}_{t]}$ will be identified with the subspace $\tilde{H}_{t]} \otimes \Phi_{[t}$ of $\tilde{H}$ where $\Phi_{[t}$ is the vacuum vector in $H_{[t}$. Every operator defined on a tensorial factor of $\tilde{H}$ will be identified with its canonical ampliation to the whole space and denoted by the same symbol.

Fix dense linear manifolds $\mathcal{D}$ in $\mathcal{H}_0$ and $\mathcal{M}$ in $L^2(\mathbb{R}_+, \mathcal{K})$. The algebraic tensor product $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$ is dense in $\tilde{H}$, where $\varepsilon(\mathcal{M})$ is the linear manifold generated by the vectors $e(u) : u \in \mathcal{M}$.

Definition 1.1 [21]: A family $X \equiv \{X(t) : t \geq 0\}$ of operators in $\tilde{H}$ is called an *adapted process* with respect to $(\mathcal{D}, \mathcal{M})$ if

(a) $\mathcal{D}(X(t) \supseteq \mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$;

(b) $X(t)fe(u\chi_{[0,t]}) \in \tilde{H}_{t]}$ and $X(t)fe(u) = \{X(t)fe(u\chi_{[0,t)})\}e(u\chi_{[t,\infty)})$ for all $t \geq 0, f \in \mathcal{D}, u \in \mathcal{M}$.

It is said to be *regular*, if in addition, the map $t \to X(t)fe(u)$ from $\mathbb{R}_+$ into $\tilde{H}$ is continuous for each $f \in \mathcal{D}, u \in \mathcal{M}$. An adapted process is called *bounded, contractive, isometric, co-isometric* or *unitary* according as the operators $X(t)$ are bounded, contractive, isometric, co-isometric or unitary for every $t \geq 0$.

For $0 \leq s \leq t$ denote by $a_{s,t}$ the von-Neumann subalgebra of $a \equiv B(\Gamma_+)$ given by

$$\{W(u) : \text{supp } u \subseteq [s,t]\}$$

This is simply $I_{0,s} \otimes B(\Gamma_{s,t}) \otimes I_{t,\infty}$. The family $\{N_{s,t} = a_0 \otimes a_{s,t}; 0 \leq s \leq t\}$ forms a filtration of the von-Neumann algebra $N := a_0 \otimes a$ where $a_0 := B(\mathcal{H}_0)$. Vacuum conditional expectations $\{\mathbb{E}_{s,t} : 0 \leq s \leq t\}$ on each of these subalgebras exist and are characterized by

$$\mathbb{E}_{s,t}[B \otimes W(u)] =< e(0), W(u\chi_{[s,t]^C})e(0) > B \otimes W(u\chi_{[s,t]})$$

where $[s,t]^c = \mathbb{R}_+ \backslash [s,t]$. They satisfy the relations:

$$\mathbb{E}_{s,t} o \mathbb{E}_{s',t'} = \mathbb{E}_{s,t}$$

where $[s,t] \subseteq [s',t']$. We also write $\mathbb{E}_s$ for $\mathbb{E}_{0,s}$.

Definition 1.2 [35]: A $(\mathcal{D}, \mathcal{M})$- adapted process $X \equiv \{X(t) : t \geq 0\}$ is said to be a *martingale* if

$$< fe(u\chi_{[0,s]}), X(t)ge(v\chi_{[0,s]}) >=< fe(u\chi_{[0,s]}), X(s)ge(v\chi_{[0,s]}) >$$

for all $0 \leq s \leq t, f, g \in \mathcal{D},\ u, v \in \mathcal{M}$. A bounded martingale is said to be *regular* if there is a Radon measure μ on $\mathbb{R}_+$ for which

$$||[X(t) - X(s)]\psi||^2 + ||[X(t)^* - X(s)^*]\psi||^2 \leq \mu([s,t])||\psi||^2$$

whenever $0 \leq s \leq t$ and $\psi \in \Gamma_{0,s} \otimes \Phi_{[s}$.

We fix an orthonormal basis $\{e_i : i \in \mathcal{S}\}$ in $\mathcal{K}$ and set $E^i_j = |e_j >< e_i| : i, j \in \mathcal{S}$. The basic quantum stochastic processes $\{\Lambda^i_j : i, j \in \bar{\mathcal{S}} := \mathcal{S} \cup \{0\}\}$ are defined in $\tilde{H}$ by

$$\Lambda^i_j(t) = \begin{cases} I_0 \otimes \lambda(\chi_{[0,t]} \otimes E^i_j) & ; \quad i, j \in \mathcal{S} \\ I_0 \otimes a(\chi_{[0,t]} \otimes e_i) & ; \quad i \in \mathcal{S}, j = 0 \\ I_0 \otimes a^\dagger(\chi_{[0,t]} \otimes e_j) & ; \quad i = 0, j \in \mathcal{S} \\ tI & ; \quad i = 0 = j \end{cases}$$

These basic processes are $(\mathcal{H}_0, \mathcal{K})$ - adapted and increaments in disjoint interval commute for any pair of the basic processes. Except Λ^0_0 these are martingale.

We denote by $u^j(s) =< e_j, u(s) >, u_j(s) = \overline{u^j(s)}$ for $j \in \mathcal{S}$ and $u_0(s) = u^0(s) = 1$. Choose $\mathcal{M} \equiv \{u \in H : u^j(\cdot) = 0$ for all but finitely many $j \in \mathcal{S}\}$ and set $N(u) = \{j; u^j(\cdot) \not\equiv 0\}$. So $\# N(u) < \infty$ for $u \in \mathcal{M}$.

Definition 1.3: An adapted process $L \equiv \{L(s),\ s \geq 0\}$ is called simple if it is piecewise constant, taking the following form:

$$L(s) = \sum_{0 \leq k < \infty} L(s_k) \chi_{[s_k, s_{k+1})}(t)$$

where $0 = s_0 < s_1 < s_2 < \dots. < s_n \to \infty$. We shall denote by $\mathbb{L}^0(\mathcal{D}, \mathcal{M})$ the class of simple $(\mathcal{D}, \mathcal{M})$- adapted processes.

For $L \in \mathbb{L}^0(\mathcal{S}, \mathcal{M})$, we define, for $t \geq 0$ the quantum stochastic integral by

$$\int_0^t L(s) d\Lambda(s) = \sum_{k \geq 0} L(s_k)(\Lambda(t \wedge s_{k+1}) - \Lambda(t \wedge s_k))$$

where Λ represents any one of the basic processes and $t \wedge s$ is the minimam of t and s. For any simple adapted process $L^i_j,\ i,j \in \overline{S}$ we set

$$\Lambda_{L^i_j}(t) = \int_0^t L^i_j(s) d\Lambda^j_i(s)$$

Lemma 1.4 : Fix any pair $L^i_j, M^l_k \in \mathbb{L}^0(\mathcal{D},\mathcal{M})$,where $i,j,l,k \in \overline{S}$. Then for $t \geq 0, f,g \in \mathcal{D},\ u,v \in \mathcal{K}$ the following hold:

$$< fe(u), \Lambda_{L^i_j}(t) ge(v) >= \int_0^t dsu_i(s)v^j(s) < fe(u), L^i_j(s) ge(v) >; \qquad (1.1)$$

$$\begin{aligned} < \Lambda_{L^i_j}(t) fe(u) \quad , \quad \Lambda_{M^l_k}(t) ge(v) >= & \int_0^t \{ dsu_k(s)v^l(s) < \Lambda_{L^i_j}(s) fe(u), L^k_l(s) ge(v) > \\ + & \ dsu_j(s)v^i(s) < L^i_j(s) fe(u), \Lambda_{M^k_l}(s) ge(v) > \\ + & \ \hat{\delta}^i_l dsu_k(s)v^j(s) < L^i_j(s) fe(u),\ L^k_l(s) ge(v) > \}; \qquad (1.2) \end{aligned}$$

where

$$\hat{\delta}^i_l = \begin{cases} 0 & \text{if} \quad l = 0 \text{ or } i = 0 \\ \delta^i_l & \quad \text{otherwise;} \end{cases}$$

and

$$||\Lambda_{L^i_j}(t) fe(u)||^2 \leq 2\exp(\ \nu_u(t)) \int_0^t ||L^i_j(s) fe(u)||^2 d\nu_u(s) \qquad (1.3)$$

where

$$\nu_u(t) = \int_0^t (1 + ||u(s)||^2) ds.$$

Proof : For the proof of this fundamental Lemma in bosonic calculus developed in Hudson-Parthasarathy [21] the reader is refered Parthasathy [38]. In these notational framework it is available in Evans [11]. ∎

Quantum Ito's formula (1.2) can be expressed as:

$$d\Lambda^i_j d\Lambda^k_l = \hat{\delta}^i_l d\Lambda^k_j \qquad (1.4)$$

for all $i,j,k,l \in \bar{S}$.

Proposition 1.5 : [21] Let $L \equiv \{L(s),\ s \geq 0\}$ be a $(\mathcal{D}, \mathcal{M})$ - adapted measurable process satisfying the following:

$$\int_0^t ||L(s)fe(u)||^2 d\nu_u(s) < \infty \tag{1.5}$$

for all $f \in \mathcal{D}, u \in \mathcal{M}$. Then there exists a sequence of elements $L^{(n)} \in \mathbb{L}^0(\mathcal{D}, \mathcal{M})$ such that

(i) $\lim_{n\to\infty} \int_0^t ||[L(s)fe(u) - L^{(n)}(s)]fe(u)||^2 d\nu_u(s) = 0$;

(ii) There exists a unique regular $(\mathcal{D}, \mathcal{M})$ - adapted Λ_L process such that

$$\lim_{n\to\infty} ||[\Lambda_L(t) - \Lambda_{L^{(n)}}(t)]fe(u)|| = 0$$

where Λ is one of the basic processes;

(iii) For any pair of process L^i_j and M^l_k satisfying (1.4) the unique process $\Lambda_{L^i_j}$ and $\Lambda_{M^k_l}$ defined as in (ii) satisfy (1.1),(1.2) and (1.3).

Proof: See Hudson-Parthasarathy [21].

In the present exposition we shall deal with quantum stochastic calculus where countably many noise components are present. To this end we introduce the following class of processes.

Definition 1.6 : [21, 33] $L \equiv \{L^i_j(s) : i, j \in \overline{S}\}$ is said to be a $(\mathcal{D}, \mathcal{M})$ adapted ***square integrable*** family of processes if each L^i_j is $(\mathcal{D}, \mathcal{M})$ adapted ,measurable and for each $j \in \overline{S}, f \in \mathcal{D}, u \in \mathcal{M}$ and $t \geq 0$

$$\sum_{i \in \overline{S}} \int_0^t \|L^i_j(s)fe(u)\|^2 d\nu_u(s) < \infty. \tag{1.6}$$

We shall denote by $\mathbb{L}(\mathcal{D}, \mathcal{M})$ the class of all such square integrable families. Note that $\mathbb{L}^0(\mathcal{D}, \mathcal{M}) \subset \mathbb{L}(\mathcal{D}, \mathcal{M})$.

For further details on these definitions and quantum Ito's formula the reader is refered to Hudson-Parthasarathy [21], Evans [11] and Mohari-Sinha [33]. A complete account is available in Parthasarathy [38].

Theorem 1.7 : [21, 33] Suppose $L \in \mathbb{L}(\mathcal{D}, \mathcal{M})$. Then

$$X(t) = \sum_{i,j \in \overline{S}} \int_0^t L^i_j(s) d\Lambda^j_i(s)$$

exists in the strong sense on $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$ and defines a regular adapted process satisfying for $f, g \in \mathcal{D}$ and $u, v \in \mathcal{M}$

$$< fe(u), X(t)ge(v) >= \sum_{i,j \in \overline{S}} \int_0^t ds u_i(s) v^j(s) < fe(u), L^i_j(s) ge(v) > \quad (1.7)$$

$$||X(t)fe(u)||^2 \leq 2 \, \exp{(\nu_u(t))} \sum_{i \in \overline{S}, j \in N(u)} \int_0^t ||L^i_j(s) fe(u)||^2 d\nu_u(s) \quad (1.8)$$

Moreover for a regular process L, $X(t) = 0$ for all $t \geq 0$ if and only if $L^i_j(t) = 0$ for all $t \geq 0$.

If M is another element in $\mathbb{L}(\mathcal{D}, \mathcal{M})$ and

$$Y(t) = \sum_{i,j \in \overline{S}} \int_0^t M^i_j(s) d\Lambda^j_i(s)$$

then

$$< Y(t)fe(u), X(t)ge(v) >= \sum_{i,j \in \overline{S}} \int_0^t ds u_i(s) v^j(s) \{ < Y(s)fe(u), L^i_j(s) g$$

$$e(v) > + < M^j_i(s) fe(u), X(s) ge(v) > + \sum_{k \in S} < M^k_i(s) fe(u), L^k_j(s) ge(v) > \} \quad (1.9)$$

Proof: Fix any increasing sequence $\mathcal{S}^n$, $n \geq 1$ of subsets of $\mathcal{S}$ such that $\#\mathcal{S}^n < \infty$ and $\cup_{n \geq 1} \mathcal{S}^n = \mathcal{S}$. Also set $\overline{\mathcal{S}^n} := \mathcal{S}^n \cup 0$. Define the sequence of regular $(\mathcal{D}, \mathcal{M})$ -adapted processes X_n, Y_n; $n \geq 1$ by

$$X_n(t) = \sum_{i,j \in \overline{S}^n} \int_0^t L^i_j(s) d\Lambda^j_i(s),$$

$$Y_n(t) = \sum_{i,j \in \overline{S}^n} \int_0^t M^i_j(s) d\Lambda^j_i(s).$$

on $\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$. So we have

$$< fe(u), X_n(t)ge(v) >= \int_0^t \sum_{i,j\in\overline{S}^n} u_i(s)v^j(s) < fe(u), L_j^i(s)ge(v) > ds, \tag{1.10}$$

$$\begin{aligned}
&< X_n(t)fe(u), Y_n(t)ge(v) > \\
&= \int_0^t \sum_{i,j\in\overline{S}^n} < X_n(s)fe(u), M_j^i(s)ge(v) > u_i(s)v^j(s)ds \\
&\quad + \int_0^t \sum_{i,j\in\overline{S}^n} < L_j^i(s)fe(u), Y_n(s)ge(v) > u_j(s)v^i(s)ds \\
&\quad + \int_0^t \sum_{i\in S^n} \sum_{j,k\in\overline{S}^n} < L_j^i(s)fe(u), M_k^i(s)ge(v) > u_j(s)v^k(s)ds \quad (1.11)
\end{aligned}$$

for each $n \geq 0$, $f, g \in \mathcal{D}$, $u, v \in \mathcal{M}$. Now taking $Y_n = X_n$, $g = f$, $v = u$ in (1.7) we have for $N(u) \subseteq \overline{S}^n$

$$\begin{aligned}
&||X_n(t)fe(u)||^2 \\
&= \int_0^t 2\,Re \sum_{i,j\in N(u)} < X_n(s)fe(u), L_j^i(s)fe(u) > u_i(s)u^j(s)ds \\
&\quad + \int_0^t \sum_{i\in S^n} \sum_{j,k\in N(u)} < L_j^i(s)fe(u), L_k^i(s)fe(u) > u_j(s)u^k(s)ds \\
&= \int_0^t \{2\,Re \sum_{i\in N(u)} < u^i(s)X_n(s)fe(u), \sum_{j\in N(u)} u^j(s)L_j^i(s)fe(u) > \\
&\quad + \sum_{i\in S^n} || \sum_{j\in N(u)} u^j(s)L_j^i(s)fe(u)||^2\}ds \\
&\leq \int_0^t \{2 \sum_{i\in N(u)} ||u^i(s)X_n(s)fe(u)||\;|| \sum_{j\in N(u)} u^j(s)L_j^i(s)fe(u)|| \\
&\quad + \sum_{i\in S^n} || \sum_{j\in N(u)} u^j(s)L_j^i(s)fe(u)||^2\}ds \\
&\leq \int_0^t \{||X_n(s)fe(u)||^2 + 2 \sum_{j\in N(u)} \sum_{i\in\overline{S}^n} ||L_j^i(s)fe(u)||^2\}d\nu_u(s).
\end{aligned}$$

where Cauchy-Schwartz inequality has been used to get the inequalities. By

Gronwall inequality we conclude that

$$||X_n(t)fe(u)||^2 \leq 2e^{\nu_u(t)} \int_0^t \{ \sum_{j\in N(u)} \sum_{i\in \overline{S}^n} ||L_j^i(s)fe(u)||^2\} d\nu_u(s). \tag{1.12}$$

Since for $n \geq m$

$$d(X_n - X_m) = \sum_{i,j \in \overline{S}^n \setminus \overline{S}^m} L_j^i d\Lambda_i^j$$

the same argument shows that for all $n \geq m$ and $N(u) \subseteq \overline{S}^m$

$$\begin{aligned} &||\{X_n(t) - X_m(t)\} fe(u)||^2 \\ &\leq 2e^{\nu_u(t)} \int_0^t \{ \sum_{j\in N(u)} \sum_{i\in \overline{S}^n - \overline{S}^m} ||L_j^i(s)fe(u)||^2\} d\nu_u(s). \end{aligned}$$

Define the $(\mathcal{D}, \mathcal{M})$ - adapted process X by

$$X(s)fe(u) = \lim_{n\to\infty} X_n(s)fe(u);$$

Also observe that for $0 \leq s \leq t$

$$||X(t) - X(s)fe(u)||^2 \leq 2e^{\nu_u(t)} \sum_{j\in N(u)} \int_s^t \sum_{i\in S} ||L_j^i(s)fe(u)||^2 d\nu_u(s)$$

So X is a regular $(\mathcal{D}, \mathcal{M})$ -adapted process. Now taking limit $n \to \infty$ in (1.10), (1.11) and (1.12) we conclude (1.7), (1.8) and (1.9). That for regular L, $X(t) = 0$, $t \geq 0$ implies that $L \equiv 0$ is immediate from (1.7). This completes the proof. ∎

If

$$X(t) = X(0) + \int_0^t \sum_{i,j\in\overline{S}} L_j^i(s) d\Lambda_i^j(s)$$

for all $t \geq 0$ we write

$$dX = \sum_{i,j\in S} L_j^i(s) d\Lambda_i^j(s).$$

Following elementary lemmas will play an importent role in dealing with countably many noise components.

Lemma 1.8 Let $\{A_k\}, k \geq 1$ be bounded operators in $\mathcal{B}(h)$ such that for some constant $c > 0$

$$\sum_k ||L_k f||^2 \leq c||f||^2$$

holds for each $f \in \mathcal{H}$. Then $\sum_k L_k^* L_k$ converges in strong operator topology. The same is true for their ampliations in $\mathcal{H} \otimes \mathcal{K}$, where $\mathcal{K}$ is a Hilbert space.

Proof Let $M_n = \sum_{1 \leq k \leq n} A_k^* A_k$. Then $\{M_n\}$ is an increasing sequence of positive operators. So it suffices to show that $M = w.lim_{n \to \infty} M_n$ exists and $||M|| \leq c$. For any $f, g \in \mathcal{H}$, $n \geq m$ we have

$$\begin{aligned} | < f, (M_n - M_m)g > |^2 &\leq \{ \sum_{m+1 \leq k \leq n} ||A_k f|| \, ||A_k g||\}^2 \\ &\leq \{ \sum_{m+1 \leq k \leq n} ||A_k f||^2\}\{ \sum_{m+1 \leq k \leq n} ||A_k g||^2\}. \end{aligned}$$

Also observe that $||M_n|| \leq c$ for all $n \geq 1$. By uniform boundedness principle we conclude the required results. To prove the second part observe that

$$\sum_k ||A_k \otimes 1\psi||^2 = < \psi, \{(\sum_k A_k^* A_k) \otimes I\}\psi > \leq ||M|| \, ||\psi||^2.$$

■

Lemma 1.9 Let $\{A_k\}, \{B_k\}, k \geq 1$ be bounded operators in $\mathcal{H}$ such that the sums $\sum_k A_k^* A_k$ and $\sum_k B_k^* B_k$ converge strongly. Then $\sum_k A_k^* B_k$ converges strongly.

Proof: By our hypothesis we have for each $f \in \mathcal{H}$ and some positive constant c

$$\sum_{k \geq 1} ||A_k f||^2 \leq c||f||^2.$$

Hence for all $1 \leq m \leq n$ $f, g \in \mathcal{H}$

$$| < f, \sum_{k=m}^{n} A_k^* B_k g > |^2 \leq (\sum_{k=m}^{n} ||A_k f|| \, ||B_k g||)^2$$

$$\leq \ (\sum_{k\geq 1}||A_k f||^2)\sum_{k=m}^{n}||B_k g||^2$$
$$\leq \ c||f||^2 \sum_{k=m}^{n}||B_k g||^2.$$

Taking superemum over all unit vectors f we have

$$||\sum_{k=m}^{n} A_k^* B_k g||^2 \leq c\sum_{k=m}^{n}||B_k g||^2$$

and the right hand side tends to 0 as $m, n \to \infty$. ∎

Notes and Remarks:

This section is a brief account of the basic tools of boson Fock stochastic integration theory, based on Hudson-Parthasarathy [21]. This exposition is based on the notation of Evans [11] and adapted from Mohari-Sinha [33]. For further details we refer to Parthasarathy [38].

2 A class of quantum stochastic differential equations with bounded coefficients:

Denote by $\mathcal{Z}_R$ the class of elements $L \equiv (L^i_j \in \mathcal{B}(\mathcal{H}_0),\ i,j \in \overline{S})$ such that for each $j \in \overline{S}$ there exists non-negative constant (depending on L) c_j, satisfying

$$\sum_{i\in\overline{S}} \|L^i_j f\|^2 \leq c_j^2 \|f\|^2 \tag{2.1}$$

for all $f \in \mathcal{H}_0$.

Theorem 2.1 [21,33] : Suppose $L \in \mathcal{Z}_R$. Then there exists a unique regular $(\mathcal{H}_0, \mathcal{M})$-adapted process $X \equiv \{X(t),\ t \geq 0\}$ satisfying

$$dX = \sum_{i,j\in\overline{S}} L^i_j d\Lambda^j_i(t) X(t),\ X(0) = X_0 \otimes I \tag{2.2}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$,where $X_0 \in \mathcal{B}(\mathcal{H}_0)$.

Proof : First we set up the iterative scheme of regular $(\mathcal{H}_0, \mathcal{M})$ -adapted processes :

$$\begin{aligned} X_{-1}(t) &\equiv 0 \\ X_0(t) &\equiv X_0 \otimes I \\ X_n(t) &= X_0 \otimes I + \int_0^t \sum_{i,j\in\overline{S}} L^i_j X_{n-1}(s) d\Lambda^j_i(s), n \geq 1. \end{aligned} \tag{2.3}$$

Our aim is to show that for each $n \geq 0, X_n$ is well defined, $\{L^i_j X_n,\ i,j \in \overline{S}\} \in \mathbb{L}(\mathcal{H}_0, \mathcal{M})$ and for any fixed $T \geq 0$

$$\|(X_n(t) - X_{n-1}(t)) f e(u)\|^2 \leq \frac{\{\beta_u(T)\nu_u(t)\}^n}{n!} \|X_0\|^2 \|f\|^2 \|e(u)\|^2, 0 \leq t \leq T \tag{2.4}$$

where

$$\beta_u(T) = 2e^{\nu_u(T)} \sum_{j\in N(u)} c_j^2, f \in \mathcal{H}_0, u \in \mathcal{M}.$$

For $n = 0$ these are immediate. Now suppose that these are verified for $0 \leq n \leq k$. By Lamma 1.5 and (2.4), X_{k+1} is well defined and from (1.4) in Theorem 1.4 we get

$$\begin{aligned} &||\{X_{k+1}(t) - X_k(t)\}fe(u)||^2 \\ \leq\ & 2e^{\nu_u(T)}\sum_{j=0}^{N(u)}\int_0^t \sum_{i\geq 0}||L_j^i\{X_k(s) - X_{k-1}(s)\}fe(u)||^2 d\nu_u(s)) \end{aligned}$$

which by (2.1), Lemma 1.8 and induction hypothesis implies

$$\begin{aligned} &||\{X_{k+1}(t) - X_k(t)\}fe(u)||^2 \\ \leq\ & 2e^{\nu_u(T)}\sum_{j=0}^{N(u)} c_j^2 \int_0^t ||\{X_k(s) - X_{k-1}(s)\}fe(u)||^2 d\nu_u(s) \\ \leq\ & \frac{\beta_u(T)^{k+1}}{k+1!}\nu_u(t)^{k+1}||X_0||^2||f||^2||e(u)||^2, 0 \leq t \leq T. \end{aligned}$$

This proves that the iterative scheme (2.3) is well defined and (2.4) holds for all n . In particular, $X(t)fe(u) = s.lim_{n\to\infty}X_n(t)fe(u)$ exists for all $f \in \mathcal{H}_0, u \in \mathcal{M}$ and defines a $(\mathcal{H}_0, \mathcal{M})$ -adapted process. From (2.4) we have the following inequalities:

$$||X(t)fe(u)|| \leq \alpha(u,T)||X||\ ||f|| \tag{2.5}$$

where

$$\alpha(u,T) = \sum_{n\geq 0}\frac{\{\beta_u(T)\nu_u(t)\}^{\frac{n}{2}}}{n!^{\frac{1}{2}}}||e(u)||.$$

These in particular imply that $\{L_j^i X(t),\ i,j \in \overline{S}\} \in \mathbb{L}(\mathcal{H}_0, \mathcal{M})$. Each X_n being a regular process, (2.5) implies that X is also a regular process. Now for any $f \in \mathcal{H}_0, u \in \mathcal{M}, n \geq 0$ by triangle inequality and (1.4) we have

$$\begin{aligned} ||X(t)\ &-\ X_0 \otimes I - \int_0^t \sum_{i,j\in\overline{S}} L_j^i d\Lambda_i^j(s)X(s)fe(u)||^2 \\ &\leq\ 2\{||[X(t) - X_n(t)]fe(u)||^2 + \beta_u(T)\int_0^t ||[X(s) - X_n(s)]fe(u)||^2 d\nu_u(s)\}. \end{aligned}$$

That X satisfies (2.2) is immediate from the above inequality once we take limit as $n \to \infty$.

Let X' be another solution of (2.2) with $X'(0) = X_0 \otimes 1$. By (1.4) in Theorem 1.7 and Lemma 1.8 we have for all $0 \le t \le T, f \in \mathcal{H}_0, u \in \mathcal{M}$

$$||(X(t) - X'(t))fe(u)||^2 \le \beta_u(T) \int_0^t ||(X(s) - X'(s))fe(u)||^2 d\nu_u(s)$$

By Gronwall inequality we conclude the required identity. ∎

Before we procced for the next result we introduce some more terminology. Denote by θ_t the right shift on $L^2(\mathbb{R}_+, \mathcal{K})$ so that for all $t \ge 0$

$$(\theta_t u)(x) = \begin{cases} u(x-t) & , \quad x \ge t, \\ 0 & , \quad 0 \le x \le t, \end{cases}$$

Definition 2.2: [1,19,32] A family $X = \{X(s,t), 0 \le s \le t < \infty\}$ of bounded operators in $\mathcal{H}$ is said to be a *right cocycle* if the following are fulfilled.

(a)$X(s,\cdot)$ is a regular $(\mathcal{H}_0, \mathcal{M})$-adapted process in $[s, \infty)$ for every s;

(b) For all $0 \le r \le s \le t < \infty$

$$X(s,s) = I, \ X(r,t) = X(s,t)X(r,s); \tag{2.6}$$

(c) For all $r \ge 0, 0 \le s \le t < \infty$

$$\Gamma(\theta_r^*)X(s+r, t+r)\Gamma(\theta_r) = X(s,t). \tag{2.7}$$

X is said to be a *strongly continuous* right cocycle if the map $(s,t) \to X(s,t)$ is continuous in strong operator topology.

A family $V = \{V(s,t), 0 \le s \le t < \infty\}$ of bounded operators in $\mathcal{H}$ is called a *left cocycle* if their adjoints $\{V(s,t)^*, 0 \le s \le t < \infty\}$ constitute a right cocycle.

For any bounded operator A in $\tilde{H}, \Gamma(\theta_t)A\Gamma(\theta_t^*)$ takes $\mathcal{H}_0 \otimes \Phi_{t]} \otimes \tilde{\mathcal{H}}_{[t}$ into itself. Denote by $\overline{\Gamma(\theta_t)A\Gamma(\theta_t^*)}$ the canonical ampliation to the whole space $\tilde{H}$.

An adapted bounded process $V \equiv \{V(t) : t \geq 0\}$ is said to be a *bar-cocycle* [23] if for all $s, t \geq 0$

$$V(t+s) = V(t)\overline{\Gamma(\theta_t)V(s)\Gamma(\theta_t^*)} \tag{2.8}$$

For a left cocycle $V = \{V(s,t), 0 \leq s \leq t < \infty\}$ the family $\{V(0,t), 0 \leq t < \infty\}$ is a bar-cocycle. Conversely if $V \equiv \{V(t),\ t \geq 0\}$ is a bar-cocycle then $V(s,t) := \overline{\Gamma(\theta_s)V(t)\Gamma(\theta_s^*)}$, $0 \leq s \leq t < \infty$ is a left cocycle. It is worth noting that $\eta \equiv \{\eta(t) := I - V(t)V(t)^*,\ t \geq 0\}$ is an increasing family of positive operators whenever V is a contractive bar-cocycle.

We quote the following theorem without proof.

Theorem 2.3 [1,19]: Suppose $V \equiv \{V(t) : t \geq 0\}$ is a strongly continuous contractive bar-cocycle. Then

(i) there exists a stronly continuous one parameter contraction semigroup $P \equiv \{P_t,\ t \geq 0$ in $\mathcal{H}_0$ such that

$$P_t := \mathbb{E}_0[V(t)]$$

(ii) there exist two weakly * continuous semigroups $\tau \equiv \{\tau_t : t \geq 0\}, \tilde{\tau} = \{\tilde{\tau}_t : t \geq 0\}$ of completely positive contractive maps on $\mathcal{B}(\mathcal{H}_0)$ such that

$$\begin{aligned} \tau_t(B) &= \mathbb{E}_0[V(t)^*(B \otimes I)V(t)] \\ \tilde{\tau}_t(B) &= \mathbb{E}_0[V(t)(B \otimes I)V(t)^*] \end{aligned} \tag{2.9}$$

for all $B \in \mathcal{B}(\mathcal{H}_0)$.

A bar-cocycle V is said to be regular if $P_t,\ t \geq 0$ is norm continuous. In such a case the associated family X of operators is called a regular right cocycle. It is easy to varify that $X(s,t) := W(\chi_{[s,t)}u)$ is an unitary regular right cocycle for any fixed $u \in \mathbb{L}_2(\mathbb{R}_+, \mathcal{K})$. Here our aim is to construct a class of regular right cocycles.

Proposition 2.4 : Suppose $L \in \mathcal{Z}_R$. Then for any $s \geq 0$ there exists a unique regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process $X(s,.)$ in the time interval $[s,\infty)$ satisfying the q.s.d.e

$$dX(s,t) = \sum_{i,j\in\overline{S}} L^i_j d\Lambda^j_i X(s,t),\ X(s,s) = I \tag{2.10}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$. Moreover

$$\Gamma(\theta^*_r)X(s+r,t+r)\Gamma(\theta_r) = X(s,t) \tag{2.11}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ for all $0 \leq s \leq t < \infty$ and $r \geq 0$.

Proof: Fix any $s \geq 0$. 'First part' is nothing but a restatement of Proposition 2.1 with $\mathcal{H}_0$ and L replaced by $\mathcal{H}_0 \otimes \Gamma_s$ and it's cannonical ampliation. Since (2.10) admits a unique solution identity (2.11) follows once we verify that

$$X'(s,t) := \Gamma(\theta_r)X(s+r,t+r)\Gamma(\theta^*_r)$$

is also a solution of (2.10). From (1.3) and (2.10) we have

$$\begin{aligned}
& < fe(u), X'(s,t)ge(v) > - < fe(u), ge(v) > \\
= & \int_{r+s}^{r+t} \sum_{i,j\in\overline{S}} (\theta_r u)_i(\tau)(\theta_r v)^j(\tau) < fe(u), L^i_j X'(s+r,t+r)ge(v) > d\tau \\
= & \int_s^t \sum_{i,j\in\overline{S}} u_i(\tau)v^j(\tau) < fe(u), L^i_j X'(s+r,t+r)ge(v) > d\tau
\end{aligned}$$

whenever $f, g \in \mathcal{H}_0,\ u, v \in \mathcal{M}$. Hence we get the required identity. ∎

Corollary 2.5 : Consider the family $\{X(s,t), 0 \le s \le t < \infty\}$ of operators defined as in Proposition 2.4. If X has a bounded extension then

(i) X is a bounded right cocycle;

(ii) In such a case the map $(s,t) \to X(s,t)$ is continuous in strong operator topology if and only if

$$||X(s,t)|| \le \gamma e^{\lambda(t-s)}, \quad \text{for all } 0 \le s \le t < \infty$$

for some $\lambda > 0, \gamma \ge 1$.

Proof: Note that $X(r,\cdot)$ and $X(s,\cdot)X(r,s)$ are both solutions of (2.10) in the interval $[s,\infty)$ with initial value $X(r,s)$. It now follows from the uniqueness property that $X(r,t) = X(s,t)X(r,s)$ for all $t \ge s \ge r$. Identities (2.7) follow from (2.11). This completes the proof of (i).

X being a right cocycle (2.7) implies that

$$X(s,t) = \overline{\Gamma(\theta_s)X(0,t-s)\Gamma(\theta_s^*)}, 0 \le s \le t < \infty.$$

Also observe that the map $t \to \Gamma(\theta_t)$ is continuous in strong topology. Hence it is enough to varify (ii) for $s = 0$. Also note that (2.6) and (2.7) imply that

$$||X(t_1+t_2)|| \le ||X(t_1, t_1+t_2)||\ ||X(t_1)|| = ||X(t_2)||\ ||X(t_1)||, t_1, t_2 \ge 0$$

where $X(t) := X(0,t)$, $t \ge 0$. **(ii)'only if ':** The map $t \to X(t)$ being continuous in strong topology by uniform boundedness principle we have

$$||X(t)|| \le C, 0 \le t \le 1$$

for some $C \ge 1$. If $[t]$ is the integer part of $t \ge 0$ we conclude that

$$||X(t)|| \le C^{[t]+1} \le C^{t+1} = \gamma e^{\lambda t}.$$

'if': $X(0,t), t \geq$ being a regular process and uniformly bounded on compacta we conclude strong continuity by a standerd method in real analysis. This completes the proof of (ii). ∎

We set $\mathcal{M}_c \equiv \{u \in \mathcal{M}, u \text{ is continuous }\}$. It is evident that $\varepsilon(\mathcal{M}_c)$ is dense in Γ_+.

Proposition 2.6 : Consider the family $\{X(s,t), 0 \leq s \leq t < \infty\}$ of operators defined as in Proposition 2.4. Then for any fixed $u, v \in \mathcal{M}_c$, there exists a unique family $\{N(s,t), 0 \leq s \leq t < \infty\}$ of bounded operators on $\mathcal{H}_0$ satisfying the following:

(i) $< fe(u), X(s,t)ge(v) > \;=\; < f, N(s,t)g >$ for all $f, g \in h$;

(ii) $\frac{\partial}{\partial t} N(s,t) = L_{u(t),v(t)} N(s,t)$,

(iii) $N(s,t) \doteq < e(u), e(v) > \{I+$
$\sum_{n=1}^{\infty} \int_{s<t_1<\cdots<t_n<t} L_{u(t_n),v(t_n)} L_{u(t_{n-1}),v(t_{n-1})} \cdots L_{u(t_1),v(t_1)} dt_1 dt_2 ... dt_n ; \}$

(iv) $\frac{\partial N(s,t)}{\partial s} = -N(s,t) L_{u(s),v(s)}$,

where the series in (iii) converges in operator norm operator topology and

$$L_{u(t),v(t)} = \sum_{i,j \in \overline{S}} u_i(t) v^j(t) L^i_j.$$

Proof : By Proposition 2.4 and (1.3) we have

$$< fe(u), X(s,t)ge(v) > = < fe(u), ge(v) > + \int_s^t < fe(u), L_{u(\tau),v(\tau)} X(s,\tau) ge(v) > d\tau. \tag{2.12}$$

Now consider the ordinary differential equation

$$\frac{\partial N(s,t)}{\partial t} = L_{u(t),v(t)} N(s,t), t \geq s \tag{2.13}$$

with the initial condition $N(s,s) =< e(u), e(v) > I$ in $\mathcal{B}(\mathcal{H}_0)$. Since $\sup_{t \geq 0} ||L_{u(t),v(t)}|| < \infty$ it follows that (2.13) admits a unique bounded solution given by the infinite series in (iii) of the proposition. Now a comparison with (2.12) shows that property (i) of the proposition holds. (iv) follows from (iii) by straightforward differentiation with respect to s. ∎

Our aim is now to study the quantum stochastic evolution satisfying (2.10) in detail. To this end we introduce the following notations.

For any $L \in \mathcal{Z}_R$ define the family of bounded linear operators $\{\mathcal{L}^i_j(X),\ i,j \in \overline{\mathcal{S}}, X \in \mathcal{B}(\tilde{H})\}$ on $\tilde{H}$) by

$$\mathcal{L}^i_j(X) = XL^i_j + (L^j_i)^* X + \sum_{k\in\mathcal{S}} (L^k_i)^* X L^k_j \tag{2.14}$$

where the necessary convergence follows from (2.1) and Lemma 1.9. Observe that for all $i,j \in \overline{\mathcal{S}}$, $\mathcal{L}^i_j(X)^* = \mathcal{L}^j_i(X^*)$. In particular for any bounded self-adjoint operator X,

$$\mathcal{L}_{\mathcal{S}'}(X) \equiv ((\ \mathcal{L}^i_j(X)\ ))_{i,j\in\mathcal{S}'}$$

is a self-adjoint operator on the Hilbert space $\tilde{H} \otimes l_2(\mathcal{S}')$, for any finite subset $\mathcal{S}'$ of $\overline{\mathcal{S}}$. We set

$$\mathcal{Z}^-_R \equiv \{L,\ \mathcal{L}_{\mathcal{S}'}(I)\ \le\ 0, \text{ for all } \mathcal{S}' \subset \mathcal{S}\ , \#\mathcal{S}' < \infty\};$$

$$\mathcal{Z}^+_R \equiv \{L,\ \mathcal{L}_{\mathcal{S}'}(I)\ \ge\ 0, \text{ for all } \mathcal{S}' \subset \mathcal{S}\ , \#\mathcal{S}' < \infty\}$$

and $\mathcal{I}_R$ for the class of elements $L \in \mathcal{Z}_R$ satisfying $\mathcal{L}^i_j(I) = 0$ for all $i,j \in \overline{\mathcal{S}}$. Hence $\mathcal{I}_R \subset \mathcal{Z}^{\mp}_R \subset \mathcal{Z}_R$ and $\mathcal{Z}^-_R \bigcap \mathcal{Z}^+_R = \mathcal{I}_R$.

The following Lemma of independent interest will help us in describing various classes of operators introduced in this section. For any bounded semi-definite operator in a Hilbert space we denote $A > 0$ if there exists a $\delta > 0$ such that $A \geq \delta I$.

Lemma 2.7 : For bounded operators $A > 0,\ B \geq 0$ acting respectively on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and C acting from $\mathcal{K}$ into $\mathcal{H}$, the bounded self-adjoint operator

$$D = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{K}$ is positive semi-definite if and only if $B \geq C^* A^{-1} C$.

Proof : Without loss of generality assume that $B > 0$, otherwise replace B by $B + \varepsilon$, $\varepsilon > 0$ and adopt a limiting argument. In such a case we claim that the following statements are equivalent:

(a) D is positive semi-definite;

(b) $| < f, Ck > |^2 \leq < f, Af >< k, Bk >$ for all $f \in \mathcal{H}$, $k \in \mathcal{K}$;

(c) $| < f, A^{-1/2} C B^{-1/2} k > | \leq ||f|| \, ||k||$ for all $f \in \mathcal{H}$, $k \in \mathcal{K}$;

(d) $||A^{-1/2} C B^{-1/2}|| \leq 1$,

(e) $C^* A^{-1} C \leq B$.

Details of the proof are omited. ∎

Corollay 2.8 : Consider bounded self- adjoint operators A, B and C, D acting as in Lemma 2.6. Then following hold:

(i) If $B > 0$ then there exists a positive constant λ such that the bounded self-adjoint operator

$$D_\lambda = \begin{bmatrix} A + \lambda & C \\ C^* & B \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{K}$ is positive definite (> 0);

(ii) If $D \geq 0$ and $Bk = 0$ for some $k \in \mathcal{K}$ then $Ck = 0$.

Proof : (i): Since $B > 0$ there exists a $\delta > 0$ such that $< k, Bk > \geq 2\delta ||k||^2$ for all $k \in \mathcal{H}$. Also observe that $< k, C^*(A + \lambda)^{-1} Ck > \leq ||C||^2 ||(A + \lambda)^{-1}|| \, ||k||^2$. Since $\lim_{\lambda \to \infty} ||(A + \lambda)^{-1}|| = 0$, there exist $\lambda > 0$ such that $A + \lambda - \delta > 0$ and $< k, C^*(A + \lambda - \delta)^{-1} Ck > \leq \delta ||k||^2$ for all $k \in \mathcal{K}$. Now combine these two inequalities to conclude that $C^*(A + \lambda - \delta)^{-1} C \leq B - \delta$. Hence by Lamma 2.7 we have $D_\lambda \geq \delta I$. This completes the proof of (i).

(ii): For any $f \in \mathcal{H}$ and scalar λ we have

$$< f, (A + 1) f > + \lambda < f, Ck > + \overline{\lambda} < k, C^* f > \geq 0.$$

If $< f, Ck > \neq 0$ choose $\lambda = -n\frac{|<f,Ck>|}{<f,Ck>}, n \geq 1$. So we have $-2n| < f, Ck > | + < f, (A+1)f > \geq 0$ for all $n \geq 1$. Hence $< f, Ck >= 0$ for all $f \in \mathcal{H}$. This completes the proof of (ii). ∎

For any fixed $L \in \mathcal{Z}_R$ define operators S^i_j, $i, j \in \mathcal{S}$ on $\mathcal{H}_0$ by

$$S^i_j = L^i_j + \delta^i_j$$

and B, S on $\mathcal{H}_0 \otimes \mathcal{K}$ by

$$\begin{aligned} B &= \sum_{i,j \in \mathcal{S}} \mathcal{L}^i_j(I) \otimes |e_i >< e_j|; \\ S &= \sum_{i,j \in \mathcal{S}} S^i_j \otimes |e_i >< e_j| \end{aligned}$$

A simple computation shows that the family $\mathcal{L}(I) \equiv \{\mathcal{L}^i_j(I),\ i, j \in \overline{\mathcal{S}}\}$ on $\mathcal{H}_0$ takes the following form:

$$\mathcal{L}^i_j(I) = \begin{cases} \sum_{k \in \mathcal{S}} (S^k_i)^* S^k_j - \delta^i_j & , \ i, j \in \mathcal{S} \\ (L^j_i)^* + \sum_{k \in \mathcal{S}} (S^k_i)^* L^k_j & , \ i \in \mathcal{S}, j = 0 \\ L^i_j + \sum_{k \in \mathcal{S}} (L^k_i)^* S^k_j & , \ i = 0, j \in \mathcal{S} \\ L^i_j + (L^j_i)^* + \sum_{k \in \mathcal{S}} (L^k_i)^* L^k_j & , \ i = 0 = j \end{cases}$$

It is clear that $B = S^*S - I$. Hence $B \leq 0$ if and only if $||S|| \leq 1$. If $L \in \mathcal{Z}^{\mp}_R$ and $S^*S = I$ then by Corollary 2.8 (ii) we have $\mathcal{L}^i_j(I) = 0$ whenever $i$or $j \in \mathcal{S}$.

Proposition 2.9: Consider the family $X \equiv \{X(s,t), 0 \leq s \leq t < \infty\}$ of operators defined as in Proposition 2.4. The following statements are valid:

(i) X has a contractive extension if and only if $L \in \mathcal{Z}^-_R$; In such a case X is a strongly continuous right cocycle;

(ii) $||X(s,t)\psi|| \geq ||\psi||, 0 \leq s \leq t < \infty, \psi \in \mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ if and only if $L \in \mathcal{Z}^+_R$;

(iii) If $B < 0$ then X has a strongly continuous bounded extension;

(iv) If $\mathcal{L}_{S'}(I) > 0$, for some non empty $S' \subset S$ then $X(s,t)$ is unbounded whenever $t \neq s$;

(v) If $B > 0$ then $X(s,t)$ is unbounded whenever $s \neq t$;

(vi) X has an isometric extension if and only if $L \in \mathcal{I}_I$.

Proof: By Proposition 2.4 and (2.10) we have

$$< X(s,t)fe(u), X(s,t)ge(v) > - < X(s,t')fe(u), X(s,t')ge(v) >$$

$$= \int_{t'}^{t} < X(s,\tau)fe(u), \sum_{i,j\in\overline{S}} u_i(\tau)v^j(\tau)\mathcal{L}^i_j(I)X(s,\tau)ge(v) > d\tau,\ 0 \le s \le t' \le t \tag{2.15}$$

for all $f, g \in \mathcal{H}_0, u, v \in \mathcal{M}$. For any finitely many vectors $f_\alpha \in \mathcal{H}_0$, ${}^\alpha u \in \mathcal{M}$ set the vector $\psi := \sum_\alpha f_\alpha e({}^\alpha u)||e({}^\alpha u)||^{-1}$. From (2.15) we have

$$\frac{\partial}{\partial t}||X(s,t)\psi||^2 = < \tilde{\psi}(s,t), \tilde{\mathcal{L}}(I)\tilde{\psi}(s,t) > \tag{2.16}$$

where $\tilde{\psi}(s,t), \tilde{\mathcal{L}}(I)$ are the vectors and bounded operator in the Hilbert space $\oplus_\alpha \tilde{H}_\alpha$, $\tilde{H}_\alpha = \oplus_{j\in N({}^\alpha u)}\tilde{H}$ such that $\tilde{\mathcal{L}}(I)^\alpha_\beta = \mathcal{L}(I)$ and $\tilde{\psi}(s,t) = \oplus_\alpha \psi_{f_\alpha, {}^\alpha u}(s,t)$ and $\psi_{f,u}(s,t) = \oplus_{j\in N(u)} u^j(t)X(s,t)fe(u)||e(u)||^{-1}$.

Also observe that $\mp\mathcal{L}(I)$ is positive semi-definite if and only if $\mp\tilde{\mathcal{L}}(I)$ is so. Hence from (2.16) it is clear that for any fixed $s \ge 0$, the map $t \to ||X(s,t)\psi||, t \ge s$ is decreasing or increasing according as $L \in \mathcal{Z}^-_R$ or $L \in \mathcal{Z}^+_R$.

(i) 'if': Since $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ is a dense linear manifold in $\tilde{H}$ 'if part' of (i) is immediate from (2.16). **(i) 'only if':** By Corollay 2.5 X is a contractive cocycle. So for any $\psi \in \tilde{H}, ||X(s,t)\psi|| \le ||\psi||$ whenever $0 \le s \le t \le \infty$. Hence $\frac{\partial}{\partial t}||X(s,t)\psi||^2_{|s=t} \le 0$. Now fix any finite set of vectors $g_\alpha \in \mathcal{H}_0, \alpha \in S'$, where $\mathcal{S}' \subset \overline{\mathcal{S}}, \#\mathcal{S}' < \infty$. Taking ${}^{\alpha}u \in \mathcal{M}_c$ in (2.16) where ${}^{\alpha}u^j(0) = \delta_j^{\alpha}$ and

$$f_\alpha = \begin{cases} g_\alpha & , \quad \text{if } \alpha \neq 0, \\ g_0 - \sum_\beta f_\beta & , \quad \text{if } 0 \in \mathcal{S}', \alpha = 0 \\ -\sum_\beta f_\beta & , \quad \text{if } 0 \notin \mathcal{S}', \alpha = 0 \end{cases}$$

we have

$$\sum_{\alpha,\beta \in \mathcal{S}'} < g_\alpha, \mathcal{L}_\beta^\alpha(I) g_\beta > \leq 0$$

Hence $L \in \mathcal{Z}_R^-$. Strong continuity follows from Corollay 2.5. This completes the proof of (i).

(ii) follows by a similar method employed in (i).

Define for any real scalar λ the regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process $Y_\lambda \equiv \{Y_\lambda(s,t), 0 \leq s \leq t < \infty\}$ by

$$Y_\lambda(s,t) = e^{-\lambda(t-s)} X(s,t)$$

Also observe that Y_λ is the unique solution for (2.10) where coefficients ${}^{\lambda}L_j^i$ are same as that of L_j^i except ${}^{\lambda}L_0^0 = L_0^0 - \lambda$.

Assume $B < 0$. By Corollay 2.8 there exists a $\lambda \leq 0$ such that ${}^{\lambda}\mathcal{L}(I) \leq 0$ where ${}^{\lambda}\mathcal{L}_j^i$ are identical to that of $\mathcal{L}_j^i$ except ${}^{\lambda}\mathcal{L}_0^0 = \mathcal{L}_0^0 + 2\lambda$. Hence by (i) conclude that Y_λ has a strongly continuous contrative extension. This completes the proof of (iii).

Asumme $\mathcal{L}_{\mathcal{S}'}(I) > 0$ for some non empty $\mathcal{S}' \subset \mathcal{S}$. So there exist constants $\delta > 0, \lambda \geq 0$ such that ${}^{\lambda}\mathcal{L}_{\overline{\mathcal{S}'}}(I) \geq \delta I$, where ${}^{\lambda}\mathcal{L}_j^i$ are identical to that of $\mathcal{L}_j^i$ except ${}^{\lambda}\mathcal{L}_0^0 = \mathcal{L}_0^0 + 2\lambda$. Hence for any $f \in \mathcal{H}_0, u \in \mathcal{M},\ u_j = 0,$ if $j \notin \mathcal{S}'$ (2.15) and (ii) implies that

$$||Y_\lambda(s,t) f e(u)||^2 \geq ||f e(u)||^2 + \delta \int_s^t (1 + ||u(\tau)||^2)\ ||Y_\lambda(s,\tau) f e(u)||^2$$

So in particular $||Y_\lambda(s,t)fe(u)||^2 \geq ||fe(u)||^2$. Hence $||Y_\lambda(s,t)fe(u)||^2||fe(u)||^{-2} \geq 1 + \delta\int_s^t(1+||u(\tau)||^2)$. Since $\sup_{u\in\mathcal{M}_{S'}}\int_s^t(1+||u(\tau)||^2) = \infty$ whenever $s \neq t$ we conclude that $Y_\lambda(s,t)$ is an unbounded operator whenever $s \neq t$. This completes the proof of (iv). (v) is immediate from (iv).

To prove (vi) first observe that $\mathcal{Z}_R^- \cap \mathcal{Z}_R^+ = \mathcal{I}_R$. Hence (vi) is immediate from (i) and (ii). ∎

Now our aim is to dualise the process considered in Proposition 2.4. and arrive at a counter part of Proposition 2.9. To this end we introduce some more notation. For any $L \equiv (L_j^i : i,j \in \overline{S})$ elements in $\mathcal{B}(\mathcal{H}_0)$ we define $\tilde{L} \equiv \{\tilde{L}_j^i : i,j \in \overline{S}\}$ by

$$\tilde{L}_j^i = (L_i^j)^*,\ i,j \in \overline{S}.$$

and set

$$\tilde{\mathcal{Z}}_R \equiv \{L,\ \tilde{L} \in \mathcal{Z}_R\},\ \tilde{\mathcal{Z}}^{\mp} \equiv \{L,\ \tilde{L} \in \mathcal{Z}^{\mp}\}, \text{and },\tilde{\mathcal{I}}_R \equiv \{L,\ \tilde{L} \in \mathcal{I}_R\}.$$

We define $R_t, t \geq 0$ the time reversal operators on $\mathbb{L}^2(\mathbb{R}_+, k)$ by

$$(R_t u)(x) = \begin{cases} u(t-x) & , \quad 0 \leq x \leq t \\ u(x) & , \quad t < x \end{cases}$$

and $\mathcal{U}_t := \Gamma(R_t)$.

Definition 2.10 : [23,32] Let $X \equiv \{X(s,t), 0 \leq s \leq t < \infty\}$ be a family of operators defined on a common domain $\mathcal{H}_0\underline{\otimes}\varepsilon(\mathcal{M})$ such that for each $s \geq 0$, $X(s,t)$, $t \geq s$ is a $(\mathcal{H}_0, \mathcal{M})-$ adapted process. The family $\tilde{X} \equiv \{\tilde{X}(s,t),\ 0 \leq s \leq t < \infty\}$ of operators is said to be the dual process [23] of X if for each $0 \leq s \leq t < \infty$ $\mathcal{H}_0\underline{\otimes}\varepsilon(\mathcal{M}) \subseteq \mathcal{D}(X(s,t)^*)$ and the following equality holds

$$\tilde{X}(s,t) = \mathcal{U}_t X(0,t-s)^* \mathcal{U}_t \tag{2.17}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$.

It is worth noting that $(\tilde{\tilde{L}}) = L$ and for each X, $\tilde{X}(s,t)$, $t \geq s$ is also a $(\mathcal{H}_0, \mathcal{M})-$ adapted process and $(\tilde{\tilde{X}}) = X$. The following theorem relates these two identities.

Theorem 2.11 [32] : (Journé's time reversal principle [23]).

Let $L \in \mathcal{Z}_R \cap \tilde{\mathcal{Z}}_R$. Consider the family $\{X(s,t), 0 \leq s \leq t < \infty\}$ of operators defined as in Proposition 2.4. Then the dual process $\tilde{X}$ exists and for each $s \geq 0$, $\tilde{X}(s,)$, $s \leq t < \infty$ is a regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process satisfying

$$d\tilde{X}(s,t) = \sum_{i,j \in \overline{S}} \tilde{L}^i_j d\Lambda^j_i \tilde{X}(s,t), \tilde{X}(s,s) = I$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ for $t \geq s$.
Moreover X is a right cocycle if and only if $\tilde{X}$ is a right cocycle.

Proof : Let $Y = \{Y(s,t),\ 0 \leq s \leq t < \infty\}$ be the unique solution of (2.10) as described in Proposition 2.4 with L replaced by $\tilde{L}$. Fix $T \geq 0$, $u, v \in \mathcal{M}_c$ and define operators $M(s,t), N(s,t)$, $0 \leq s \leq t \leq T$ in $\mathcal{H}_0$ by the relations

$$< f, M(s,t)g > \; = < X(T-t, T-s)\mathcal{U}_T fe(u), \mathcal{U}_T ge(v) >, \qquad (2.18)$$

$$< f, N(s,t)g > \; =< fe(u), Y(s,t)ge(v) > \qquad (2.19)$$

for all $f, g \in \mathcal{H}_0$. By Proposition 2.6 (iv), $N(s,t)$ is a bounded operator and

$$\frac{dN(s,t)}{ds} = -N(s,t)\tilde{L}_{u(s),v(s)}. \qquad (2.20)$$

The definitions of X and Proposition 2.6 (ii) imply through a change of variables

$$\begin{aligned} < M(s,t)g, f > &- < ge(v), fe(u) > \\ &= < ge(R_T v), X(T-t, T-s)fe(R_T u) > - < ge(v), fe(u) > \end{aligned}$$

$$= \int_{T-t}^{T-s} < ge(R_T v), L_{v(T-\tau),u(T-\tau)} X(T-t,\tau) fe(R_T u) > d\tau$$
$$= \int_{s}^{t} < g(e(R_T v), L_{v(\tau),u(\tau)} X(T-t,T-\tau) fu(R_T) > d\tau$$

for all $0 \leq s \leq t \leq T$. Thus

$$< f, M(s,t)g > = =< fe(u), ge(v) > + \int_s^t < f, M(\tau,t)\tilde{L}_{u(\tau),v(\tau)} g > d\tau.$$

Hence for all $0 \leq s \leq t \leq T$ we have

$$\frac{dM(s,t)}{ds} = -M(s,t)\tilde{L}_{u(s),v(s)}.$$

Comparing with (2.20) we conclude that $M(s,t) = N(s,t)$ for all $0 \leq s \leq t \leq T$. Since u, v are arbitrary but subject to being in $\mathcal{M}_c$ only it follows that $Y_T(s,t) = Y(s,t)$ on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M}_c)$ for all $0 \leq s \leq t \leq T$, where

$$Y_T(s,t) = \mathcal{U}_T X(T-t, T-s)^* \mathcal{U}_T,\ 0 \leq s \leq t \leq T\,. \tag{2.21}$$

Since for each $0 \leq s \leq t < \infty, \mathcal{D}(Y(s,t)) \supseteq \mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ $Y_T(s,t)$ has an extension to $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$. In particular this also implies that Y_T does not depend on $T \geq t$. Letting T descend to t we see that $\tilde{X}$ satisfies the required conditions. Now the second part of the theorem follows from Corollay 2.5. ■

Theorem 2.12 : Let $L \in \mathcal{Z}_R \bigcap \tilde{\mathcal{Z}}_R$. There exists a family $X = \{X(s,t), 0 \leq s \leq t < \infty\}$ of operators in $\mathcal{H}$ such that $X(s,\cdot)$ is a regular $(\mathcal{H}_0, \mathcal{M})$-adapted process in $[s, \infty)$ for each $s \geq 0$, satisfying

$$dX(s,t) = \sum_{i,j \in \overline{S}} L^i_j d\Lambda^j_i(t) X(s,t),\ X(s,s) = I \tag{2.22}$$

on $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$. Moreover the following hold:

(i) The following statements are equivalent:

(a) X has a contractive extension;

(b) $L \in \mathcal{Z}_R^-$;

(c) $L \in \tilde{\mathcal{Z}}_R^-$.

In such a case X is a strongly continuous right cocycle.

(ii) X has an isometric extension if and only if $L \in \mathcal{I}_R$;

(iii) X has a co-isometric extension if and only if $L \in \tilde{\mathcal{I}}_R$;

(iv) X has an unitary extension if and only if $L \in \mathcal{I}_R \bigcap \tilde{\mathcal{I}}_R$.

Proof :(i): (a) $\Leftrightarrow$ (b) is nothing but a restatement of Proposition 2.9 (i). For (a) $\Leftrightarrow$ (c) consider the dual process described in Theorem 2.11 and conclude the result by Proposition 2.9. That X is a strongly continuous right cocycle follows from Corollay 2.5.

(ii): is nothing but a restatement of Proposition 2.9 (vi). **(iii):** It follows from (ii) and Theorem 2.11 once we note that X is co-isometric if and only if $\tilde{X}$ is isometric.

(iii) is immediate from (i) and (ii). ∎

Remark 2.13 : For $L, M \in \mathcal{Z}_R$ we define $M \star L = \{(M \star L)^i_j\}$ elements in $\mathcal{B}(\mathcal{H}_0)$ by

$$(M \star L)^i_j = L^i_j + (M^j_i)^* + \sum_{k \in S} (M^k_i)^* L^k_j$$

where the necessary convergence follows from (2.1) and Lemma 1.9. It is worth noting that $(\mathcal{Z}_R, \star)$ forms an associative non-commutative unital semi-group with identity 0. Moreover L is an invertible element if and only if S admits a bounded inverse. Set $\hat{L}$ for it's inverse and Y for the regular adapted process satisfying (2.10) with $\hat{L}$ as it's coefficients we have

$$< Y(s,t)\psi, X(s,t)\psi' > = < \psi, \psi' >$$

for all $\psi, \psi' \in \mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$. This seems to suggest that if $L \in \mathcal{Z}_R^+$ then Y is a contractive process.

Proposition 2.14 : Fix $L \in \mathcal{Z}_R^- \cap \tilde{\mathcal{Z}}_R^-$ and consider the family of maps

$$j_t(x) = X(0,t)^* x \otimes 1 X(0,t),\ t \geq 0, x \in \mathcal{B}(\mathcal{H}_0) \tag{2.23}$$

where X is the the contractive operator valued process satisfying (2.22). Then $\{j_t(x),\ t \geq 0\}$ satisfies the following stochastic differential equation:

$$dj_t(x) = \sum_{i,j \in \overline{S}} j_t(\mathcal{L}_j^i(x)) d\Lambda_i^j(t) \tag{2.24}$$

on $\mathcal{H}_0 \otimes \varepsilon(\mathcal{M})$ for all $x \in \mathcal{B}(\mathcal{H}_0)$, where the family $\mathcal{L} \equiv \{\mathcal{L}_j^i,\ i,j \in \overline{S}\}$ of bounded linear maps on $\mathcal{B}(\mathcal{H}_0)$ are as follows:

$$\mathcal{L}_j^i(x) = \begin{cases} \sum_{k \in \mathcal{S}} (S_i^k)^* x S_j^k - \delta_j^i x & ,\ i,j \in \mathcal{S}; \\ (L_i^j)^* x + \sum_{k \in \mathcal{S}} (S_i^k)^* x L_j^k & ,\ i \in \mathcal{S}, j = 0; \\ x L_j^i + \sum_{k \in \mathcal{S}} (L_i^k)^* x S_j^k & ,\ i = 0, j \in \mathcal{S}; \\ x L_0^0 + (L_0^0)^* x + \sum_{k \in \mathcal{S}} (L_k)^* x L_k & ,\ i = 0 = j \end{cases} \tag{2.25}$$

Proof : This result follows from quantum Ito's formula (1.4) once we have shown that $\{j_t(\mathcal{L}_j^i(x)),\ i,j \in \overline{\mathcal{S}}, t \geq 0\} \in \mathbb{L}(\mathcal{H}_0, \mathcal{M})$. We claim that for each $j \in \overline{\mathcal{S}}$ there exists a constant $\alpha_j \geq 0$ such that

$$\sum_{i \in \overline{S}} ||\mathcal{L}_j^i(x)\psi||^2 \leq \alpha_j ||x||^2 ||\psi||^2 \tag{2.26}$$

for all $x \in \mathcal{H}_0,\ \psi \in \tilde{H}$. Since $\tilde{S}$ is a contraction on $\mathcal{H}_0 \otimes l_2(\mathcal{S})$ we have

$$\begin{aligned} \sum_{i \in \mathcal{S}} ||\mathcal{L}_j^i(x)\psi||^2 &\leq 2\{||x\psi||^2 + \sum_{k,k' \in \mathcal{S}} < x S_j^{k'} \psi, (\sum_{i \in \mathcal{S}} S_i^{k'} S_i^k) x S_j^k \psi >\} \\ &\leq 2\{||x\psi||^2 + < \tilde{\psi}, S^* S \tilde{\psi} >\} \\ &\leq 2\{||x\psi||^2 + ||\tilde{\psi}||^2\} \\ &\leq 2\{||x\psi||^2 + \sum_{k \in \mathcal{S}} ||x S_j^k \psi||^2\} \\ &\leq 4||x||^2 ||\psi||^2. \end{aligned} \tag{2.27}$$

for all $j \in \mathcal{S}$, $\psi \in \tilde{H}, x \in \mathcal{B}(\mathcal{H}_0)$, where $\tilde{\psi} = \oplus_{k \in \mathcal{S}} x S_j^k \psi$. A similar computation also shows that

$$\sum_{i \in \mathcal{S}} ||\mathcal{L}_0^i(x)\psi||^2 \leq 2\{\sum_{i \in \mathcal{S}} ||(L_i^0)^* x\psi||^2 + \sum_{k \in \mathcal{S}} ||x L_j^k \psi||^2\} \tag{2.28}$$

Now combining these two inequalities with the assumption that $L \in \mathcal{Z}_{\bar{R}} \cap \tilde{\mathcal{Z}}_{\bar{R}}$ we conclude (2.26). $X(0,t), t \geq 0$ being a contractive process ,for each $j \in \overline{\mathcal{S}}$ we have

$$\begin{aligned} \sum_{i \in \overline{\mathcal{S}}} ||j_t(\mathcal{L}_j^i(x)\psi||^2 &\leq \sum_{i \in \overline{\mathcal{S}}} ||\mathcal{L}_j^i(x)\ X(0,t)\psi||^2 \\ &\leq \alpha_j ||x||^2 ||X(0,t)\psi||^2 \leq \alpha_j ||x||^2 ||\psi||^2. \end{aligned}$$

for all $\psi \in \mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M}), x \in \mathcal{B}(\mathcal{H}_0)$ This completes the proof. ∎

Corollay 2.15 : Let Y be the generator of a norm continuous one parameter contraction semigroup and $\{Z_k,\ k \in \mathcal{S}\}$ be a family of bounded operators in $\mathcal{H}_0$ such that

$$Y + Y^* + \sum_{k \in \mathcal{S}} Z_k^* Z_k \leq 0 \tag{2.29}$$

where the series converges in strong operator topology and $W = ((\ W_j^i\ ))_{i,j \in \mathcal{S}}$ be a contractive operator in $\mathcal{H}_0 \otimes l_2(\mathcal{S})$. Then there exists a unique strongly continuous contractive bar-cocycle $V \equiv \{V(t),\ t \geq 0\}$ satisfying

$$dV(t) = \sum_{k \in \mathcal{S}} V(t) Z_j^i \Lambda_i^j(t),\ V(0) = I \tag{2.30}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ where

$$Z_j^i = \begin{cases} S_j^i - \delta_j^i & , \quad i,j \in \mathcal{S}, \\ Z_i & , \quad i \in \mathcal{S}, j = 0, \\ -\sum_{k \in \mathcal{S}} Z_k^* S_j^k & , \quad i = 0, j \in \mathcal{S}, \\ Y & , \quad i = 0 = j. \end{cases}$$

Moreover the following statements are valid:

(i) Y is the generator of the semigroup $P_t := E_0(V(t))$;

(ii) Consider the family of contractive maps

$$j_t(x) = \tilde{V}(t)(x \otimes 1)\tilde{V}(t)^*,\ t \geq 0, x \in \mathcal{B}(\mathcal{H}_0)$$

where $\tilde{V} := \mathcal{U}_t V(t)^* \mathcal{U}_t$, the contractive bar-cocycle associated with V. Then $\{j_t(x),\ t \geq 0\}$ satisfies the following quantum stochastic differential equation:

$$dj_t(x) = \sum_{i,j \in \overline{\mathcal{S}}} j_t(\mathcal{L}^i_j(x)) d\Lambda^j_i(t) \tag{2.31}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ for all $x \in \mathcal{B}(\mathcal{H}_0)$, where $\mathcal{L} \equiv \{\mathcal{L}^i_j,\ i,j \in \overline{\mathcal{S}}\}$ is a family of bounded linear maps on $\mathcal{B}(\mathcal{H}_0)$ described as follows:

$$\mathcal{L}^i_j(x) = \begin{cases} \sum_{k \in \mathcal{S}} (S^k_i)^* x S^k_j - \delta^i_j x & ,\ i,j \in \mathcal{S}, \\ \sum_{k \in \mathcal{S}} (S^i_k)^* [x,\ Z_k] & ,\ i \in \mathcal{S}, j = 0, \\ \sum_{k \in \mathcal{S}} [Z^*_k,\ x] S^k_j & ,\ i = 0, j \in \mathcal{S}, \\ Y^* x\ +\ xY\ + \sum_{k \in \mathcal{S}} Z^*_k x Z_k & ,\ i = 0 = j. \end{cases} \tag{2.32}$$

(iii) In such a case $\mathcal{L}^0_0$ is the generator of the one parameter semigroup of completely positive maps τ_t, $t \geq 0$ defined as in (2.9) associated with the contractive bar-cocycle V. For all $t \geq 0, \tau_t(I) = I$ if and only if equality holds in (2.29).

(iv) V is isomertric, co-isometric, or unitary if and only if equality holds in (2.29) and S is isomertric, co-isometric, or unitary.

Proof : A simple computation shows that $\mathcal{L}^i_j(I) = 0$ whenever $i \in \mathcal{S}, j = 0$, $\mathcal{L}^0_0(I) = Z\ + Z^* + \sum_{k \in \mathcal{S}} Z^*_k Z_k \leq 0$ and $B = S^* S - I \leq 0$. Hence $Z \in \mathcal{Z}^-_R$. That Z is also in $\tilde{\mathcal{Z}}^-_R$ follows by Theorem 2.12(i) once we varify that $Z \in \tilde{\mathcal{Z}}_R$. Note that it is enough to show that the series $\sum_{j\in\mathcal{S}} L_j^* L_j$ converges in strong operator topology, where $L_j = (Z_j^0)^*$, $j \in \mathcal{S}$. For any fixed $f \in \mathcal{H}_0$ we have

$$\begin{aligned}\sum_{j\in\mathcal{S}} \|L_j f\|^2 &= \sum_{k,k'\in\mathcal{S}} < L_k f, \sum_{j\in\mathcal{S}} S_j^k (S_j^{k'})^* L'_k f > \\ &= \|S\tilde{f}\| \le \sum_{k\in\mathcal{S}} \|Z_k f\|^2\end{aligned}$$

where $\tilde{f} := \oplus_{k\in\mathcal{S}} Z'_k f$ an element in $\mathcal{H}_0 \otimes l_2(\mathcal{S})$. Now appeal to Theorem 2.12 to conclude the first part of the Proposition.

Since V is contractive bar-cocycle (i) is evident from Theorem 2.3(i) and (1.1). For (ii) we consider the unique contractive right cocycle Y satisfying (2.22) where $L := \tilde{Z}$. Then by Theorem 2.11 we get that the dual cocycle $\tilde{Y}$ satisfies (2.22) with Z as its coefficients. So $V(t) = Y(0,t)^*$, $t \ge 0$, hence $\tilde{V} = \tilde{Y}(0,t)$ by (2.17). Now take $X(0,t) = \tilde{Y}(0,t)$ in Proposition 2.14 to conclude (ii).

Since $\mathcal{U}_t f e(0) = f e(0)$ for all $f \in \mathcal{H}_0$ we have $< f, \tau_t(x) g > = < f, \tilde{V}(t) x \tilde{V}(t)^* g >$ for all $f, g \in \mathcal{H}_0$, hence (iii) is evident from (2.31).

For (iv) we appeal to Theorem 2.12(ii)-(iv). This compltes the proof. ∎

Remark 2.16 : One of our central aims of this exposition is to construct contractive bar-cocycles satisfying the quantum stochastic differential equation (2.30) with coefficients Z where Y is the generator of a strongly continuous contractive semigroup and $\{L_k, k \in \mathcal{S}\}$ is a family of densely defined operators so that $\mathcal{D}(Y) \subset \mathcal{D}(L_k), k \in \mathcal{S}$ and (2.29) holds as a bilinear form on $\mathcal{D}(Y)$. The class $\mathcal{Z}_R^- \cap \tilde{\mathcal{Z}}_R^-$ of operators is stable under a specific perturbation which will play a crucial role in dilating the associated one parameter semigroup of completely positive maps (See Section 5).

Notes and Remarks:

Theorem 2.1 occurs in Hudson-Parthasarathy [21]. The proof here is based on Mohari-Sinha [33]. Theorem 2.12 (iv) is the main result in q.s.d.e. with bounded coefficients, proved in Hudson-Parthasarathy [21]. Conversely, Hudson and Lindsay [20] have shown that X satisfies (2.22) with coefficients $L \in \mathcal{I}_R \cap \tilde{\mathcal{I}}_R$, whenever X is a regular unitary right-cocyle. This characterization of a regular unitary right-cocycle can be exploited to prove a weaker version of 'Journé's time reversal principle' (Theorem 2.11). For further details we refer to Mohari [34]. The proof of Theorem 2.11 is based on Mohari-Parthasarathy [32].

3 Quantum stochastic flows:

Let $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$ be a $\star$ subalgebra.

Definition 3.1 : [2,12] A family $\{\mathcal{J}_t,\ t \geq 0\}$ of $\star$ homomorphisms from $\mathcal{A}_0$ into $\mathcal{B}(\tilde{H})$ is said to be a *quantum stochastic flow*(QS flow) with *initial algebra* $\mathcal{A}_0$ if for each $x \in \mathcal{A}_0$ the following are fulfilled:

(i) $j_0(x) = x \otimes I$;

(ii) $\mathcal{J}_t(x) \in \mathcal{B}(\Gamma_t)$ for all $t \geq 0$;

(iii) There exist maps $\theta^i_j : \mathcal{A}_0 \to \mathcal{A}_0, i, j \in \overline{S}$ such that $\{\mathcal{J}_t(x),\ t \geq 0\}$ is a regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process satisfying the quantum stochastic differential equations:

$$d\mathcal{J}_t(x) = \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(x)) d\Lambda^j_i(t) \tag{3.1}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$.

The QS flow will be called *conservative* if it can be extended to a unital $\star$ algebra $\mathcal{A} \supseteq \mathcal{A}_0$ satisfying (i), (ii) and

(iv) for all $t \geq 0,\ \mathcal{J}_t(I) = I$.

$\Theta \equiv \{\theta^i_j, \in \overline{S}\}$ is called the family of *structure maps* associated with the QS flow $\{\mathcal{J}_t,\ t \geq 0\}$ with $\mathcal{A}_0$ as initial algebra. For further details of this notion and the motivation from various point of view we refer to Accardi-Frigerio-Lewis [2].

Proposition 3.2 : Let $\{\mathcal{J}_t,\ t \geq 0\}$ be a QS flow with initial algebra $\mathcal{A}_0$ and structure maps $\{\theta^i_j,\ i, j \in \overline{S}\}$. Then for all $i, j \in \overline{S}$

(i) θ^i_j is linear on $\mathcal{A}_0$;

(ii) $\theta^i_j(x)^* = \theta^j_i(x^*)$;

(iii)If for each $l \in \overline{S}, x \in \mathcal{A},\ \sum_{k \in S} \theta^l_k(x)\theta^k_l(x)$ converges in strong operator topology and $(t, x) \to \mathcal{J}_t(x)$ is jointly continous in the strong operator topology of $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$ then for all $x, y \in \mathcal{A}_0$

$$\theta^i_j(xy) = \theta^i_j(x)y + x\theta^i_j(y) + \sum_{kS} \theta^i_k(x)\theta^k_j(y);$$

(iv) if the QS flow is conservative then $\theta^i_j(I) = 0$;

Proof : $\mathcal{J}_t$ being a $\star$ homomorphism, for any $x, y \in \mathcal{A}_0$ and scalars α we have

$$\begin{aligned}
0 &= d[\mathcal{J}_t(\alpha x + y) - \alpha\mathcal{J}_t(x) - \mathcal{J}_t(y)] \\
&= \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(\alpha x + y) - \alpha\theta^i_j(x) - \theta^i_j(y))d\Lambda^j_i(t); \\
0 &= d[\mathcal{J}_t(x^*) - \mathcal{J}_t(x)^*] \\
&= \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(x^*) - \theta^j_i(x)^*)d\Lambda^j_i(t)
\end{aligned}$$

and

$$\begin{aligned}
0 &= d\{\mathcal{J}_t(xy) - \mathcal{J}_t(x)\mathcal{J}_t(y)\} \\
&= \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(xy) - \theta^i_j(x)y - x\theta^i_j(y)) - \sum_{k \in S} \mathcal{J}_t(\theta^i_k(x)\theta^k_j(y))d\Lambda^j_i(t). \\
&= \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(xy) - \theta^i_j(x)y - x\theta^i_j(y) - \sum_{k \in S} \theta^i_k(x)\theta^k_j(y))d\Lambda^j_i(t).
\end{aligned}$$

where quantum Ito's formula (1.1) and Lemma 2.6 have been used to get the last equality. By Theorem 1.4 these three equalities imply (i), (ii) and (iii) respectively. If $\mathcal{J}_t, t \geq 0$ is conservative then

$$0 = d\mathcal{J}_t(I) = \sum_{i,j \in \overline{S}} \mathcal{J}_t(\theta^i_j(I))d\Lambda^j_i(t),$$

Once again by Theorem 1.4 we conclude (iv). ∎

Definition 3.3 : [11,33] Let $\mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}_0)$ be a $\star$ algebra. A family $\Theta \equiv \{\theta^i_j,\ i,j \in \overline{S}\}$ of maps from $\mathcal{A}_0$ into itself is said to obey the *structure equations* if the following holds : for all $i,j \in \overline{S}$ and $x,y \in \mathcal{A}_0$

(1) θ^i_j is linear on $\mathcal{A}_0$;

(2) $\theta^i_j(x^*) = \theta^j_i(x)^*$;

(3) $\theta^i_j(xy) = \theta^i_j(x)y + x\theta^i_j(y) + \sum\limits_{k \in S}\theta^i_k(x)\theta^k_j(y)$

where the series converges in the strong operator topology.

(4) If $I \in \mathcal{A}_0$ then $\theta^i_j(I) = 0$;

It is said to be *regular* if for each $j \in \overline{S}$ there exist constants $\alpha_j > 0$, a countable index sets $\mathcal{J}_j$ and a family $\{D^i_j,\ i \in \mathcal{J}_j\} \subset \mathcal{B}(\mathcal{H}_0)$ such that for all $f \in \mathcal{H}_0, x \in \mathcal{A}_0$

$$\sum_{i \in \overline{S}} ||\theta^i_j(x)f||^2 \le \sum_{i \in \mathcal{J}_j} ||xD^i_jf||^2 \tag{3.2}$$

where

$$\sum_{i \in \mathcal{J}_j} ||D^i_jf||^2 \le \alpha_j^2 ||f||^2,$$

It is to be noted that (3.2), Lemma 2.5 and Lemma 2.6 imply that $\sum\limits_{k \in S}\theta^i_k(x)\theta^k_j(y) = \sum\limits_{k \in S}\theta^k_i(x^*)^*\theta^k_j(y)$ is, indeed, a strongly convergent sum. It is worth noting that if $\mathcal{A}'$ is a unital subalgebra of $\mathcal{A}$ such that $\theta^i_j(\mathcal{A}') \subset \mathcal{A}'$ for all $i,j \in S$ then Θ is also a regular structure maps with $\mathcal{A}'$ as its initial algebra.

The central aim of this section is to establish the existence of a QS flow with structure maps $\{\theta^i_j,\ i,j \in \overline{S}\}$ whenever the structure equations and the regularity condition are fulfilled.

Fix a family $\Theta \equiv \{\theta^i_j,\ i,j \in \overline{S}\}$ of linear maps on a unital $\star$ algebra $\mathcal{A}_0$ satisfying the regularity condition (3.2). Fix $T > 0$ and for any $f \in \mathcal{H}_0, u \in$

$\mathcal{M}$ and $x \in \mathcal{A}_0$ set

$$\begin{aligned} K_0(x,f,u) &= \|xf\|^2 \\ K_n(x,f,u) &= [2e^{\nu_u(T)}]^n \sum_{\substack{i_k \in \mathcal{J}_{j_k}, j_k \in N(u) \\ 1 \le k \le n}} \|xD_{j_n}^{i_n} D_{j_{n-1}}^{i_{n-1}} \cdots D_{j_1}^{i_1} f\|^2 \qquad (3.3) \\ K(T,u) &= 2e^{\nu_u(T)} \sum_{j \in N(u)} \alpha_j^2. \qquad (3.4) \end{aligned}$$

By (3.2) and (3.3) we have

$$2e^{\nu_u(T)} \sum_{\substack{i \in \overline{S}, \\ j \in N(u)}} K_n(\theta_j^i(x), f, u) \le K_{n+1}(x,f,u), \qquad (3.5)$$

$$K_n(x,f,u) \le K(T,u)^n \|x\|^2 \|f\|^2. \qquad (3.6)$$

We write

$$S_n(x,f,u) = [\sum_{k=0}^{n} \frac{K_k(x,f,u)}{\sqrt{k!}}][\sum_{k=0}^{n} \frac{\nu_u(T)^k}{\sqrt{k!}}] \qquad (3.7)$$

and note that (3.7) implies

$$S(x,f,u) := \lim_{n\to\infty} S_n(x,f,u) \le [\sum \frac{K(T,u)^k}{\sqrt{k!}} \sum \frac{\nu_u(T)^k}{\sqrt{k!}}] \|x\|^2 \|f\|^2. \qquad (3.8)$$

Proposition 3.4 : For every $x \in \mathcal{A}_0$ there exists a sequence $\{\mathcal{J}_t^{(n)}(x),\ t \ge 0\}, n \ge -1$ of regular $(\mathcal{H}_0, \mathcal{M})$-adapted processes satisfying

$$\begin{aligned} \mathcal{J}_t^{(-1)}(x) &= 0, \\ \mathcal{J}_t^{(n)}(x) &= x + \int_0^t \sum_{i,j \in \overline{S}} \mathcal{J}_s^{(n-1)}(\theta_j^i(x)) d\Lambda_i^j(s) \qquad (3.9) \end{aligned}$$

$$\|(\mathcal{J}_t^{(n)}(x) - \mathcal{J}_t^{(n-1)}(x)) f e(u)\|^2 \le \frac{K_n(x,f,u)\nu_u(t)^n}{n!} \|e(u)\|^2, \qquad (3.10)$$

$$\|\mathcal{J}_t^{(n)}(x) f e(u)\|^2 \le S_n(x,f,u) \|e(u)\|^2 \qquad (3.11)$$

for all $n \ge 0,\ 0 \le t \le T, f \in \mathcal{H}_0, u \in \mathcal{M}$, where K_n and S_n are as in (3.6) and (3.7).

Proof : The proof is along the lines of Theorem 2.1. Our aim is to show that for each $n \geq 0$ (3.9) is well defined and it satisfies (3.10) and (3.11). For $n = 0$ (3.9)-(3.11) are immediate. Suppose that (3.9)-(3.11) have been proved for $0 \leq n \leq k$. Then by (3.9) and (3.11) for each $j \in \overline{S}$ we have

$$\begin{aligned}\sum_{i\in\overline{S}} \|\mathcal{J}_t^{(k)}(\theta_j^i(x))fe(u)\|^2 &\leq S_k(\sum_{i\in\overline{S}}\theta_i^j(x^*)\theta_j^i(x), f, e(u))\|e(u)\|^2 \\ &\leq \alpha(T,u)\|e(u)\|^2\|\sum_{i\in\overline{S}}\theta_i^j(x^*)\theta_j^i(x)\|^2\ \|f\|^2\end{aligned}$$

where

$$\alpha(T,u) = [\sum_{n=0}^{\infty}\frac{K(T,u)^n}{\sqrt{n!}}][\sum_{n=0}^{\infty}\frac{\nu_u(T)^n}{\sqrt{n!}}]. \tag{3.12}$$

Thus $\{\mathcal{J}_t^{(k)}(\theta_j^i(x))\} \in \mathbb{L}(\mathcal{H}_0, \mathcal{M})$ and once again by Theorem 1.4 $\mathcal{J}_t^{(k+1)}, t \geq 0$ is well defined and from (1.4) we have

$$\begin{aligned}&\|(\mathcal{J}_t^{(k+1)}(x) - \mathcal{J}_t^{(k)}(x))fe(u)\|^2 \\ &= \|\int_0^t \sum_{i,j\in\overline{S}}\{\mathcal{J}_s^{(k)}(\theta_j^i(x)) - \mathcal{J}_s^{(k-1)}(\theta_j^i(x))\}d\Lambda_i^j(s)fe(u)\|^2 \\ &\leq 2e^{\nu_u(t)}\sum_{j\in N(u)}\int_0^t\sum_{i\in\overline{S}}\|[\mathcal{J}_s^{(k)}(\theta_j^i(x)) - \mathcal{J}_s^{(k-1)}(\theta_j^i(x))]fe(u)\|^2 d\nu_u(s) \\ &\leq 2e^{\nu_u(T)}\sum_{i,j\in\overline{S}}K_k(\theta_j^i(x), f, u)\frac{\nu_u(t)^{k+1}}{k+1!}\|e(u)\|^2 \\ &\leq K_{k+1}(x,f,u)\frac{\nu_u(t)^{k+1}}{k+1!}\|e(u)\|^2.\end{aligned}$$

This proves (3.10). To complete the proof of (3.11) observe that

$$\begin{aligned}\|\mathcal{J}_t^{(k+1)}(x)fe(u)\|^2 &\leq (\|\mathcal{J}_t^{(k)}(x)fe(u)\| + \|\mathcal{J}_t^{(k+1)}(x)fe(u) - \mathcal{J}_t^{(k)}(x)fe(u)\|)^2 \\ &\leq (\|\mathcal{J}_t^{(k)}(x)fe(u)\| + \left[\frac{K_{k+1}(x,f,u)\nu_u(t)^{k+1}}{k+1!}\right]^{1/2}\|e(u)\|)^2,\end{aligned}$$

use (3.11) for $n = k$, Schwarz's inequality and proceed exactly as in the case $k = 0$. ∎

We set for fixed $f,g \in \mathcal{H}_0,\ u,v \in \mathcal{M},\ x,y \in \mathcal{A}_0$

$$\begin{aligned} R_0(x,f,u) &= S(x,f,u) \\ R_n(x,f,u) &= \sum_{i\in\overline{S}, j\in N(u)} R_{n-1}(\theta^i_j(x),f,u) \end{aligned}$$

Observe that

$$\begin{aligned} R_n(x,f,u) &= \sum_{\substack{i_k\in\overline{S}, j_k\in N(u)\\ 1\le k\le n}} S(\theta^{i_1}_{j_1}\cdots\theta^{i_n}_{j_n}(x),f,u) \\ &\le \sum_{k\ge 0}\frac{\nu_u(T)^k}{\sqrt{k!}}\sum\frac{(2e^{\nu_u(T)})^k}{\sqrt{k!}}(\sum_{j\in N(u))}\alpha_j^2)^{n+k}||x||^2||f||^2 \\ &= \alpha(T,u)(\sum_{j\in N(u)}\alpha_j^2)^n||x||^2||f||^2. \end{aligned} \tag{3.13}$$

Proposition 3.5 : Let $\{\mathcal{J}_t^{(n)}(x),\ t\ge 0, x\in\mathcal{A}_0\}$ be as in Proposition 3.4. Then there exists a family $\{\mathcal{J}_t(x),\ t\ge 0\}, x\in\mathcal{A}_0$ of regular $(\mathcal{H}_0,\mathcal{M})$ - adapted processes satisfying the following : for each $x\in\mathcal{A}_0,\ f\in\mathcal{H}_0,\ u\in\mathcal{M}$ and $0\le t\le T$

(i) $\mathcal{J}_t(x)fe(u) = \lim_{n\to\infty}\mathcal{J}_t^{(n)}(x)fe(u)$ and the map $x\to\mathcal{J}_t(x)fe(u)$ is linear ;

(ii) $||\mathcal{J}_t(x)fe(u)||^2 \le S(x,f,u)||e(u)||^2 \le \alpha(T,u)||x||^2||f||^2||e(u)||^2$;

(iii) $||(\mathcal{J}_t(x)-\mathcal{J}_t^{(n)}(x))fe(u)|| \le \sum_{k=n+1}^{\infty}\left(\frac{K_k(x,f,u)\nu_u(T)^k}{k!}\right)^{1/2}||e(u)||$;

(iv) $\mathcal{J}_t(x) = x + \int_0^t\sum_{i,j\in\overline{S}}\mathcal{J}_s(\theta^i_j(x))d\Lambda^j_i(s)$;

(v) the map $(t,x)\to\mathcal{J}_t(x)fe(u)$ is strongly continuous with respect to the strong operator topology of $\mathcal{A}_0\subseteq\mathcal{B}(\mathcal{H}_0)$;

(vi) if for each $i,j\in\overline{S},\ x\in\mathcal{A}_0\ \theta^i_j(x)^* = \theta^j_i(x^*)$ then

$$< ge(v), B\mathcal{J}_t(x^*)fe(u) > = < \mathcal{J}_t(x)ge(v), Bfe(u) >,\ t\ge 0,\ g\in\mathcal{H}_0, v\in\mathcal{M}$$

whenever B is an element in the commutant of $\mathcal{A}_0\otimes a_{0,t}$.

Proof : From (3.10) we have

$$||(\mathcal{J}_t^{(m)}(x)-\mathcal{J}_t^{(n)}(x))fe(u)|| \le \sum_{k=n+1}^{m}||(\mathcal{J}_t^{(k)}(x)-\mathcal{J}_t^{(k-1)}(x))fe(u)||$$

$$\leq \sum_{k=n+1}^{m} (\frac{K_k(X,f,u)\nu_u(t)}{k!})^{1/2}||e(u)||.$$

for any $n \geq n$, $x \in \mathcal{A}_0$, $f \in \mathcal{H}_0$, $u \in \mathcal{M}$. So the right hand side of (i) exists and determines $\mathcal{J}_t(x)$ as a regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process. Letting $m \to \infty$ and using (i) we get (ii) and (iii).

Now first observe that (ii) and (3.13) guarantee that $\{\mathcal{J}_t(\theta^i_j(x), i, j \in \overline{S}\} \in I\!L(\mathcal{H}_0, \mathcal{M})$. Hence by Theorem 1.4 we have

$$\begin{aligned} &||(\mathcal{J}_t(x) - x - \int_0^t \sum_{i,j\in\overline{S}} \mathcal{J}_s(\theta^i_j(x))d\Lambda^j_i)fe(u)||^2 \\ \leq\ & 2||(\mathcal{J}_t(x) - \mathcal{J}^{(n)}_t(x))fe(u)||^2 \\ &+2||\int_0^t \sum_{i,j\in\overline{S}} \{\mathcal{J}_s(\theta^i_j(x)) - \mathcal{J}^{(n-1)}_s(\theta^i_j(x))\}d\Lambda^j_i(s)fe(u)||^2. \quad (3.14)\end{aligned}$$

By Theorem 1.4 and (iii) we conclude that the second term on the right hand side of this inequality does not exceed

$$\begin{aligned} &4e^{\nu_u(t)} \sum_{j\in N(u)} \int_0^t \sum_{i\in\overline{S}} ||\{\mathcal{J}_s(\theta^i_j(x)) - \mathcal{J}^{(n-1)}_s(\theta^i_j(x))\}fe(u)||^2 d\nu_u(s) \\ \leq\ & 4e^{\nu_u(t)}\nu_u(t) \sum_{i\in\overline{S}, j\in N(u)} (\sum_{k=n}^{\infty}[\frac{K_k(\theta^i_j(x), f, u)\nu_u(T)^k}{k!}]^{1/2})^2||e(u)||^2.\end{aligned}$$

By Schwarz's inequality in the summation over k, (3.5) and (3.6) the right hand side of this inequality does not exceed

$$2\nu_u(t)\sum_{k=n}^{\infty}\frac{\nu_u(T)^k}{k!}\sum_{k=n}^{\infty}\frac{K(T,u)^{k+1}}{k!}||x||^2||f||^2||e(u)||^2.$$

These observations, (3.14) and (i) imply (iv). To prove (v) we first observe that by (iv), Theorem 1.4, (ii) and (3.8)

$$\begin{aligned} &||(\mathcal{J}_t(x) - \mathcal{J}_s(x))fe(u)||^2 = ||\int_s^t \sum_{i,j} \mathcal{J}_s(\theta^i_j(X))d\Lambda^j_i fe(u)||^2 \\ \leq\ & 2e^{\nu_u(T)} \int_{t_1}^{t_2} \sum_{j=0}^{N(u)} \sum_{i\in\overline{S}} S(\theta^i_j(x), f, u)||e(u)||^2 d\nu_u(s)\end{aligned}$$

$$
\begin{aligned}
&\leq\ 2e^{\nu_u(T)}[\nu_u(t)-\nu_u(s)]\,||e(u)||^2\sum_{k=0}^{\infty}\frac{\nu_u(T)^k}{\sqrt{k!}}\sum_{k=0}^{\infty}\sum_{i\in\overline{S},j\in N(u)}\frac{K_k(\theta^i_j(x),f,u)}{\sqrt{k!}}\\
&\leq\ [\nu_u(t)-\nu_u(s)]||e(u)||^2\sum_{k\geq 0}\frac{\nu_u(T)^k}{\sqrt{k!}}\sum_{k\geq 0}\frac{K(t,u)^{k+1}}{\sqrt{k!}}||x||^2||f||^2. \qquad (3.15)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
||\mathcal{J}_t(x-y)fe(u)||^2 &\leq\ S(x-y,f,u)||e(u)||^2\\
&\leq\ ||e(u)||^2\sum_{n=0}^{\infty}\frac{\nu_u(T)^n}{\sqrt{n!}}\sum_{n=0}^{\infty}\{\frac{(2e^{\nu_u(T)})^n}{\sqrt{n!}}\\
&\quad\times\sum_{\substack{i_k\in\mathcal{J}_{j_k},j_k\in N(u)\\ 1\leq k\leq n}}||(x-y)D^{i_n}_{j_n}\cdots D^{i_1}_{j_1}f||^2\} \qquad (3.16)
\end{aligned}
$$

Inequalities (3.15), (3.16) and dominated convergence theorem imply (v).

(vi) follows from (i) once we have shown that for all $n\geq 0, x\in\mathcal{A}_0 f, g\in\mathcal{H}_0,\ u,v\in\mathcal{M}$

$$
< ge(v), B\mathcal{J}^{(n)}_t(x^*)fe(u) > = < \mathcal{J}^{(n)}_t(x)ge(v), Bfe(u) > \qquad (3.17)
$$

whenever B is of the form $x'\otimes W(\chi_{(t,T]})$, where $T\geq t$ and x' in the commutant of $\mathcal{A}_0$. For $n=0$ (3.17) is immediate. Assume (3.17) for $n-1, n\geq 1$. By (1.3) and structure equations we have

$$
\begin{aligned}
&< ge(v), B\mathcal{J}^{(n)}_t(x^*)fe(u) > = < xB^*ge(v), fe(u) >\\
+&\ < B^*ge(v), \int_0^t\sum_{i,j\in\overline{S}}\mathcal{J}^{(n-1)}_s(\theta^i_j(x^*))d\Lambda^j_i(s)fe(u) >\\
=&\ < B^*xge(v), fe(u) > + \int_0^t\sum_{i,j\in\overline{S}}v_i(s)u^j(s) < B^*ge(v), \mathcal{J}^{(n-1)}_s(\theta^i_j(x^*))fe(u) > ds\\
=&\ < xge(v), Bfe(u) > + \int_0^t\sum_{i,j\in\overline{S}}v_i(s)u^j(s) < \mathcal{J}^{(n-1)}_s(\theta^j_i(x))ge(v), Bfe(u) > ds\\
=&\ < \mathcal{J}^{(n)}_t(x)ge(v), Bfe(u) > .
\end{aligned}
$$

■

Proposition 3:6 : Let $k_t, 0 \leq t \leq T$ be a family of linear maps from $\mathcal{A}_0$ into $\mathcal{B}(\tilde{H})$ such that for each $x \in \mathcal{A}_0$, $k_t(x)$ $0 \leq t \leq T$ is a regular $(\mathcal{H}_0, \mathcal{M})$ adapted process satisfying

$$dk_t(x) = \sum_{i,j \in \overline{S}} k_t(\theta^i_j(x)) d\Lambda^j_i(t), \; k_0(x) = 0 \tag{3.18}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$. If there exists a constant C such that $\sup_{0 \leq t \leq T} ||k_t(x)|| \leq C||x||$ for all $x \in \mathcal{A}_0$ then $k_t(x) = 0$ for all $0 \leq t \leq T$.

Proof : For any fixed $f, g \in \mathcal{H}_0$, $u, v \in \mathcal{M}_c, t \geq 0$ we have

$$< fe(u), k_t(x) ge(v) >= \int_0^t < fe(u), k_s(\theta^{v(s)}_{u(s)}(x)) ge(v) > ds \tag{3.19}$$

where

$$\theta^{v(s)}_{u(s)}(x) = \sum_{i,j \in \overline{S}} u_i(s) v^j(s) \theta^i_j(x).$$

Note that $\theta^{v(s)}_{u(s)} x) \in \mathcal{A}_0$ and for any fixed $T \geq 0$ there exists a constant $\gamma \geq 0$ such that $||\theta^{v(t)}_{u(t)}(x)|| \leq \gamma ||x||$ whenever $0 \leq t \leq T$. Now iterating (3.19) n times we have

$$< fe(u), k_t(x) ge(v) >= \int_{0 \leq t_n \leq \cdot \cdot \leq t_1 \leq t} < fe(u), k_s(\theta^{v(t_1)}_{u(t_1)}(\cdot \cdot (\theta^{v(t_n)}_{u(t_n)}(x) \cdot \cdot)) ge(v) > dt_n \cdot \cdot dt_1.$$

Hence $| < fe(u), k_t(x) ge(v) > | \leq C_T \frac{\gamma^n T^n}{n!}$ whenever $0 \leq t \leq T$. Taking limit as $n \to \infty$ in the above inequality we conclude that $< fe(u), k_t(x) ge(v) >= 0$. Since $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M}_c)$ is dense in $\tilde{H}$ this completes the proof. ∎

Corolary 3.7 : Let $\Theta \equiv \{\theta^i_j, i, j \in \overline{S}\}$ be a family of linear maps on $\mathcal{A}_0$ satisfying (3.2). Then there exists atmost one regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process $\{\mathcal{J}_t(x), t \geq 0\}$ satisfying (3.1) on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ and $||\mathcal{J}_t(x)|| \leq ||x||$ for each $x \in \mathcal{A}_0$.

Proof : If $\mathcal{J}_t(x)$, $t \geq 0$ and $j'_t(x), t \geq 0$ are two contractive regular $(\mathcal{H}_0, \mathcal{M})$ -adapted processes satisfying (3.1) then $k_t(x) = \mathcal{J}_t(x) \; - j'_t(x)$ is a solution of (3.18) and $||k_t(x)|| \leq 2||x||$ for all $t \geq 0$. Hence the result follows once we appeal to Proposition 3.6. ∎

Proposition 3.8 : Let $\{\mathcal{J}_t,\ t \geq 0\}$ be as in Proposition 3.5. If $\Theta \equiv \{\theta^i_j,\ i, j \in \overline{S}\}$ is a family of structure maps with initial algebra $\mathcal{A}_0$ then for any $f, g \in \mathcal{H}_0, x, y \in \mathcal{A}_0, u, v \in \mathcal{M}$

$$< fe(u), \mathcal{J}_t(xy)ge(v) > = < \mathcal{J}_t(x^*)fe(u), \mathcal{J}_t(y)ge(v) > .$$

Proof : For any $n \geq 0$ set

$$B^{(n)}_t(x,y) =< \mathcal{J}^{(n)}_t(x^*)fe(u), \mathcal{J}^{(n)}_t(y)ge(v) > - < fe(u), \mathcal{J}^{(n)}_t(xy)ge(v) > \tag{3.20}$$

where $\mathcal{J}^{(n)}_t$ are as in Proposition 3.5. We claim that

$$\begin{aligned} |B^{(n)}_t(x,y)| \leq \sum_{k=1}^{n} \frac{\nu(T)^{\frac{n-k+1}{2}} \nu_{u,v}(T)^k}{k!\sqrt{n-k+1!}} \{\sum [G^{i_1 \cdots i_k}_{j_1 \cdots j_k}(x, f, u; y, g, v)^{1/2} \\ + G^{i_1 \cdots i_k}_{j_1 \cdots j_k}(y^*, g, v; x^*, f, u)^{1/2}]\} \end{aligned} \tag{3.21}$$

where $\nu(T) = max\{\nu_u(T), \nu_v(T)\}$,

$$\nu_{u,v}(T) = \int_0^t [(1 + ||u(s)||^2)(1 + ||v(s)||^2)]^{1/2} ds,$$

$$\begin{aligned} & G^{i_1 \cdots i_k}_{j_1 \cdots j_k}(x, f, u; y, g, v) \\ = \ & [2e^{\nu_u(T)}]^{\frac{-\sum^k i_r}{2}} K_{n-k+\sum_1^k i_r}(x^*, f, u) \\ & \times R_{\sum_1^k j_r}(y, g, v)||e(u)||^2||e(v)||^2 \end{aligned} \tag{3.22}$$

and the second summation in (3.21) is over $i_r's$ and $j_r's$ subject to the constraint $i_1 = j_1, 1 \leq i_r + j_r \leq 2, 2 \leq r \leq k$.

We prove the claim by induction. When $n = 0, B_t^{(0)}(x,y) = 0$ and the claim holds trivially. By (1.3), (1.4) and Proposition 3.5 we have

$$B_t^{(n)}(x,y) = \int_0^t \sum_{i\in N(u), j\in N(v)} u_i(s)v^j(s)\{B_s^{(n-1)}(x,\theta_j^i(y)) + B_s^{(n-1)}(\theta_j^i(x),y) + \sum_{k\in S} B_s^{(n-1)}(\theta_k^i(x),\theta_j^k(y))\}ds + \rho_t^{(n)}(x,y) \tag{3.23}$$

where

$$\rho_t^{(n)}(x,y) = \int_0^t \sum_{i\in N(u), j\in N(v)} u_i(s)v^j(s)\{< (\mathcal{J}_s^{(n)}(x^*) - \mathcal{J}_s^{(n-1)}(x^*))fe(u), \mathcal{J}_s^{(n-1)}(\theta_j^i(y))ge(v) > + < \mathcal{J}_s^{(n-1)}(\theta_j^i(x^*))fe(u), (\mathcal{J}_s^{(n)}(y) - \mathcal{J}_s^{(n-1)}(y))ge(v) >\}ds. \tag{3.24}$$

By (3.10), (3.11) and Schwarz's inequality we have

$$|\rho_t^{(n)}(x,y)| \leq \{[K_n(x^*,f,u)R_1(y,g,v)]^{1/2} + [K_n(y,g,v)R_1(x^*,f,u)]^{1/2}\} \times \frac{\nu(T)^{n/2}\nu_{u,v}(T)}{\sqrt{n!}}||e(u)||\,||e(v)||. \tag{3.25}$$

When $n = 1, B_t^{(1)}(x,y) = \rho_t^{(1)}(x,y)$ and (3.25) implies (3.21). An elementary application of Cauchy-Schwarz's inequality shows that

$$\int_0^t\{\sum_{i\in N(u), j\in N(v)} |u_i(s)v^j(s)|G_{j_1\cdots j_k}^{i_1\cdots i_k}(x,f,u;\theta_j^i(y),g,v)^{1/2}\}ds \leq G_{j_1,\cdots j_k,1}^{i_1,\cdots,i_k,0}(x,f,u;y,g,v)\nu_{u,v}(t); \tag{3.26}$$

$$\int_0^t\{\sum_{i\in N(u), j\in N(v)} |u_i(s)v^j(s)|G_{j_1\cdots j_k}^{i_1\cdots i_k}(\theta_j^i(x),f,u;y,g,v)^{1/2}\}ds \leq G_{j_1,\cdots j_k,0}^{i_1,\cdots,i_k,1}(x,f,u;y,g,v)\nu_{u,v}(t); \tag{3.27}$$

$$\int_0^t\{\sum_{i\in N(u), j\in N(v)} |u_i(s)v^j(s)|\sum_{r\geq 1} G_{j_1\cdots j_k}^{i_1\cdots i_k}(\theta_r^i(x),f,u;\theta_j^r(x),g,v)^{1/2}\}ds \leq G_{j_1,\cdots j_k,1}^{i_1,\cdots,i_k,1}(x,f,u;y,g,v)\nu_{u,v}(t); \tag{3.28}$$

Now assume that (3.21) holds for $n-1$. Using triangle inequality in (3.23), (3.24) and the estimates (3.26) - (3.28) we obtain (3.21) for n. This proves the claim (3.21) for all n. By (3.6) and (3.13) we have

$$\begin{aligned} & |G^{i_1,\cdots,i_k}_{j_1,\cdots,j_k}(x,f,u;y,g,v)| \\ \leq \;& c\alpha(T,v)(\sum_{j\in N(v)}(\alpha_j^2)^{\sum j_r}[2e^{\nu_u(T)}]^{-\sum i_r}K(T,u)^{n-k+\sum i_r} \end{aligned}$$

where $c = ||x||^2||y||^2||f||^2||g||^2||e(u)||^2||e(v)||^2$. The right hand side does not exceed

$$\begin{aligned} & c\alpha(T,v)(\sum_{j\in N(v)}\alpha_j^2)^{\sum j_r}(\sum_{j\in N(u)}\alpha_j^2)^{\sum i_r}K(T,u)^{n-k} \\ \leq \;& c\alpha(T,v)\{max(\sum_{j\in N(u)}\alpha_j^2, \sum_{j\in N(v)}\alpha_j^2, 1)\}^{2k}K(T,u)^{n-k}. \end{aligned}$$

Thus by (3.21)

$$|B_t^{(n)}(x,y)| \leq C\sum_{k=1}^{n}\frac{a^{n-k}b^k}{k!\sqrt{n-k+1!}}$$

where C, a, b are constants,

$$a = [\nu(T)K(T,u)]^{1/2}, b = 3\nu(T)\max(\sum_{j\in N(u)}\alpha_j^2, K(T,u), K(T,v), 1).$$

Since

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{a^{n-k}b^k}{k!\sqrt{(n-k+1)!}} \leq \lim_{n\to\infty}\frac{(a+b)^n}{\sqrt{n!}} = 0$$

we conclude that $B_t^{(n)}(x,y) \to 0$ as $n \to \infty$. By Proposition 3.5 we conclude the required result. ∎

Proposition 3.9 [11] Let $\{\mathcal{J}_t,\ t \geq 0\}$ be as in Proposition 3.5. If Θ is a family of structure maps then for every $t \geq 0, x \in \mathcal{A}_0$, $\mathcal{J}_t(x)$ extends uniquely to a bounded operator on $\tilde{H}$. Denote this extension by the same symbol $\mathcal{J}_t(x)$. Then for all $t \geq 0, x \in \mathcal{A}_0$

(i) $||\mathcal{J}_t(x)|| \leq ||x||$;

(ii) If $\mathcal{A}_0$ is a von-Neumann algebra then $\mathcal{J}_t(x) \in \mathcal{A}_0 \otimes a_{0,t}$.

Proof: Choose any element $\psi \in \tilde{H}$ of the form $\psi = \sum_{i=1}^{k} f_i e(u_i)$ where $f_i \in \mathcal{H}_0$ and $u_i \in \mathcal{M}$. By repeated use of Proposition 3.8 we have

$$\begin{aligned}
||\mathcal{J}_t(x)\psi||^2 &= <\psi, \mathcal{J}_t(x^*x)\psi> \\
&\leq ||\psi||\, ||\mathcal{J}_t(x^*x)\psi|| \\
&\leq ||\psi||^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}} ||\mathcal{J}_t((x^*x)^{2^n})\psi||^{\frac{1}{2^n}} \\
&\leq ||\psi||^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}} \{\sum_{i=1}^{k} ||\mathcal{J}_t((x^*x)^{2^n}) f_i e(u_i)||\}^{\frac{1}{2^n}}
\end{aligned}$$

Using (ii) in Proposition 3.5 we now get

$$||\mathcal{J}_t(x)\psi||^2 \leq ||\psi||^{1+\frac{1}{2}+\cdots+\frac{1}{2^n}} ||x^*x|| (\sum_{i=1}^{k} \alpha(T, u_i) ||f_i||\, ||e(u_i)||)^{\frac{1}{2^n}}$$

for all $t \leq T$. Letting $n \to \infty$ on the right hand side we have

$$||\mathcal{J}_t(x)\psi|| \leq ||x||\, ||\psi||.$$

Since vectors of the form ψ are dense in $\tilde{H}$ the proof of (i) is complete. Now (ii) is immediate from (i) and Proposition 3.5 (vi). ∎

The following theorem summarises the results we have obtained so far.

Theorem 3.10: [11,33] Let $\Theta \equiv \{\theta^i_j,\ i,j \in \mathcal{S}\}$ be a family of regular structure maps on a $\star$ algebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H}_0)$. Then there exists a unique contractive QS flow $\{\mathcal{J}_t,\ t \geq 0\}$ with initial algebra $\mathcal{A}_0$. Moreover the map $(t,x) \to \mathcal{J}_t(x)$ is continuous in the strong operator topology of $\mathcal{A}_0 \subset \mathcal{B}(\tilde{H})$. If $\mathcal{A}_0$ is a unital $\star$ algebra then the QS flow is conservative.

Proof : Proposition 3.5, Proposition 3.8 - 3.9 imply the existence of a QS flow satisfying the required equation in any bounded interval. The existence and uniqueness of the QS flow in $\mathbb{R}_+$ will follow if uniqueness is established in every bounded interval. Suppose $\{\mathcal{J}'_t, t \geq 0\}$ is another contractive QS flow satisfying the same equation in $[0,T]$. Then taking $k_t := \mathcal{J}_t - \mathcal{J}'_t$ in Proposition 3.6 we conclude that $\mathcal{J}_t = \mathcal{J}'_t$.

Strong continuity of the map follows by the density of $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M}_c)$ in $\tilde{H}$ once we appeal to Proposition 3.5(v) and Proposition 3.7(i). The last assertion follows from Proposition 3.2(iv). ∎

Theorem 3.11 : [37] In Theorem 3.10 suppose that $\mathcal{A}_0$ is abelian. Then

$$[\mathcal{J}_s(x), \mathcal{J}_t(y)] = 0 \text{ for all } s, t \geq 0 \text{ and } x, y \in \mathcal{A}_0.$$

Proof : Without loss of generality we assume $s < t$. Since $\mathcal{J}_s$ is a homomorphism and $\mathcal{A}_0$ is abelian we have

$$\mathcal{J}_s(x)\mathcal{J}_s(y) = \mathcal{J}_s(xy) = \mathcal{J}_s(yx) = \mathcal{J}_s(y)\mathcal{J}_s(x).$$

Now for any fixed $y \in \mathcal{A}_0$ define the family of maps $k_t, t \geq s$ from $\mathcal{A}_0$ into $\tilde{H}$ by

$$k_t(x) = \mathcal{J}_t(x)\mathcal{J}_s(y) - \mathcal{J}_s(y)\mathcal{J}_t(x).$$

By Theorem 3.9

$$\mathcal{J}_t(x) = \mathcal{J}_s(x) + \int_s^t \sum_{i,j \in \overline{S}} \mathcal{J}_\alpha(\theta^i_j(x)) d\Lambda^j_i(\alpha).$$

Since $\{\mathcal{J}_t(x),\ t \geq 0\}$ is adapted and $\mathcal{J}_s(y)$ commutes with the increments of Λ^j_i in $[s, \infty)$ we conclude that $k_t,\ t \geq s$ satisfies (3.18) with $k_s(x) = 0$ for all $t \geq s$. Also note that $||k_t(x)|| \leq 2||y||\ ||x||$ whenever $t \geq s$ Hence by a simple variation of Proposition 3.6 we conclude the required identity. This completes the proof. ∎

Corollary 3.12 : Let $L \in \mathcal{Z}^-_R \cap \tilde{\mathcal{Z}}^-_R$. Consider the family $j_t : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\tilde{H}),\ t \geq 0$ of contractive maps defined as in (2.24). Then the following hold:

(i) The family $\mathcal{L} \equiv \{\mathcal{L}^i_j,\ i,j \subset \overline{\mathcal{S}}\}$ of maps on $\mathcal{B}(\mathcal{H}_0)$ defined as in (2.25) satisfies the regularity condition (3.2). The family $j_t,\ t \geq 0$ is the unique contractive map satisfying (2.24);

(ii) If $L \in \mathcal{I}_R \cap \tilde{\mathcal{I}}_R$ then $\mathcal{L}$ is a family of regular structure maps with initial algebra $\mathcal{B}(\mathcal{H}_0)$. In such a case $\mathcal{L}$ takes the followig form:

$$\mathcal{L}^i_j(x) = \begin{cases} \sum\limits_{k\in\overline{\mathcal{S}}}(S^k_i)^* x S^k_j - \delta^i_j x & ,\quad i,j \in \mathcal{S}, \\ \sum\limits_{k\in\overline{\mathcal{S}}}(S^k_i)^*[x,\ L_k] & ,\quad i \in \mathcal{S}, j = 0, \\ \sum\limits_{k\in\overline{\mathcal{S}}}[L^*_k,\ x]S^k_j & ,\quad i = 0, j \in \mathcal{S}, \\ iH - \frac{1}{2}\sum\limits_{k\in\overline{\mathcal{S}}}\{L^*_k L_k x + x L^*_k L_k - 2L^*_k x L_k\} & ,\quad i = 0 = j. \end{cases} \tag{3.29}$$

where $S \equiv \{S^i_j,\ i,j \in \mathcal{S}\}$ is a unitary operator on $\mathcal{H}_0 \otimes l_2(\mathcal{S})$, H is a bounded self-adjoint operator on $\mathcal{H}_0$ and $L_k,\ k \in \mathcal{S}$ is a family of bounded operator in $\mathcal{H}_0$ so that the series $\sum\limits_{k\in\mathcal{S}}(L_k)^* L_k$ converges in the strong operator topology. In such a case $\{j_t,\ t \geq 0\}$ is the unique QS flows satisfing (3.1) with $\mathcal{L}$ as its structure maps on $\mathcal{B}(\mathcal{H}_0)$.

Proof : For any fixed $i,j \in \overline{\mathcal{S}},\ f \in \mathcal{H}_0,\ x \in \mathcal{B}(\mathcal{H}_0)$ we have

$$||\mathcal{L}^i_j(x)f||^2 \leq 4\{||xL^i_j f||^2 + ||(L^j_i)^* x f||^2 + ||\sum_{k\in\mathcal{S}}(L^k_i)^* x L^k_j f||^2\} \tag{3.30}$$

For any $g \in \mathcal{H}_0$ we have

$$\begin{aligned} |<g, \sum_{k\in\mathcal{S}}(L^k_i)^* x L^k_j f>|^2 &\leq \{\sum_{k\in\mathcal{S}}||L^k_i)g||\ ||xL^k_j f||\}^2 \\ &\leq (\sum_{k\in\mathcal{S}}||L^k_i g||^2)(\sum_{k\in\mathcal{S}}||xL^k_j f||^2) \\ &\leq ||\sum_{k\in\mathcal{S}}(L^k_j)^* L^k_i||\ ||g||^2(\sum_{k\in\mathcal{S}}||xL^k_j f||^2) \end{aligned}$$

Taking supremum over all $g \in \mathcal{H}_0, ||g|| = 1$ we conclude that the last expression in (3.30) is

$$\leq ||\sum_{k\in S}(L_j^k)^* L_i^k|| \, (\sum_{k\in S}||xL_j^k f||^2).$$

That $\mathcal{L}$ satisfies the regular condition (3.2) is now immediate once we appeal to (2.27) and (2.28). Uniqueness of the contractive process satisfying (2.24) follows from Corollary 3.7. This completes the proof of (i).

Since $L \in \mathcal{I}_R \cap \tilde{\mathcal{I}}_R$ by Theorem 2.12(iv) X is unitary. Hence that $\mathcal{L}$ satisfy structure relations follows by Proposition 3.2 once we note that $j_t(x) = X(0,t)^*(x \otimes I)X(0,t)$ is a QS flow satisfying (3.1) with $\mathcal{L}$ as its structure maps. Since $\mathcal{L}(I) = 0$ we conclude that S is an isometric operator and (2.25) implies the above form of the structure maps. Since $L \in \tilde{\mathcal{I}}_R$ the operator S is also co-isometric. This completes the proof. ∎

Example 3.13 : [29,37,33] Let $P(t),\ t \geq 0$ be a transition probability matrix $P(t) \equiv \{P_{ab}(t),\ a,b \in \mathcal{E}\}\ t \geq 0$ for a Markov process with denumerable state space $\mathcal{E}$. So the family $\alpha_t(\phi) := P(t)\phi,\ \phi \in l_\infty(\mathcal{E})$ of operators forms a one parameter contraction semigroup in the Banach space $l_\infty(\mathcal{E})$. Let α_t be continuous in norm operator topology and $\Omega \equiv \{\Omega_{ab},\ a,b \in \mathcal{E}\}$ be the generator of the semigroup i.e.

$$\frac{d}{dt}P_{ab}(t)_{|t=0} = \Omega_{ab}$$

Then $\Omega_{ab} \geq 0$ if $a \neq b$, $\Omega_{aa} = -\sum_{b\neq a}\Omega_{ab}$ and $\sup_{a\in\mathcal{E}}|\Omega_{aa}| = ||\Omega|| < \infty$. We shall now realise the Markov process as a commutative QS flow. Put any group structure on $\mathcal{E}$ so that $G = \mathcal{E}$, μ is the counting measure and G acts on itself by left translation. Define the unitary representation S_a of G in $\mathbb{L}_2(G)$ by

$$(S_a u)(b) = u(a^{-1}b),\ u \in \mathbb{L}^2(\mu)$$

Choose a matrix $((\ m(a,b)\ )), a,b \in G$ such that $\Omega(a,b) = |m(a,b)|^2$ if $a \neq$

$b, -|m(a,a)|^2$ if $a = b$ and set bounded operators $L_a := S_a M_a,\ a \neq e$ where

$$(M_a\phi)(b) = m(b, ab)\phi(b),\ b \in G$$

Also observe that

$$\begin{aligned} < f, \sum_{a\neq e} L_a^* L_a f > &= \sum_{a\neq e}\sum_{b\in G} \Omega(b, ab)|f(b)|^2 \\ &= -\sum_{b\in G} \Omega(b,b)|f(b)|^2 \\ &\leq \delta||f||^2. \end{aligned}$$

Now with the elements of $\mathcal{S} := G\backslash e$ as indices and $S_b^a := \delta_{ab}S_a,\ a, b \in \mathcal{S}$ we consider the structure maps Θ the restriction of $\mathcal{L}$ defined as in (3.29) on the abelian von-Numann algerba $L^\infty(\mu) \subset \mathcal{B}(L^2(\mu))$ and $S_b^a := \delta_{ab}S_a,\ a, b \in \mathcal{S}$:

$$\begin{aligned} \theta_a^a(\phi)(b) &= \delta_{ab}[\phi(ab) - \phi(b)] \\ \theta_0^a(\phi)(b) &= m(b, ab)[\phi(ab) - \phi(b)] \\ \theta_a^0(\phi)(b) &= \overline{m(b, ab)}[\phi(ab) - \phi(b)] \\ \theta_0^0(\phi)(b) &= \sum_{a\in S} |m(b, ab)|^2[(\phi(ab) - \phi(b)]. \end{aligned}$$

It follows from Therem 3.11 that there a commutative QS flow $\{j_t,\ t \geq 0\}$ with initial algebra $L^\infty(\mu)$ satisfying the following q.s.d.e:

$$dj_t(\phi) = \sum_{a,b\in\overline{S}} j_t(\theta_b^a(\phi))d\Lambda_a^b,\ j_0(\phi) = \phi.$$

Since $\theta_0^0(\phi)(b) = \sum_{a\in S} \Omega(b, a)\ \phi(a)$ we conclude that $\alpha_t(\phi) = \mathbb{E}_0[j_t(\phi)],\ t \geq 0$.

Notes and Remarks :

The notion of quantum stochastic process is introduced by Accardi-Frigerio -Lewis [2]. For a general theory of Markov dilation we refer to Kummerer [26]. The construction of quantum stochastic flows based on structure maps is due to Evans-Hudson [12], Evans [11]. The present exposition is adapted from Mohari-Sinha [33] where the theory has been extended to deal with QS flows with infinite degrees of freedom. If $\#\mathcal{S} < \infty$ then the regularity condition (3.2) is equivalent to the following condition:

$$||\theta^i_j(x)|| \leq \gamma ||x||,\ x \in \mathcal{A},\ i,j \in \overline{\mathcal{S}}$$

for some constant $\gamma > 0$. The construction in [11] is based on this as a regularity condition on the family of structure maps Θ. The proof of Proposition 3.8 is different from [11] and in a sense direct. Theorem 3.11 is adapted from Pathasarathy-Sinha [37]. It worth noting that for any central projection Π (i.e.$\Pi \in \mathcal{A} \cap \mathcal{A}'$) the family $\Pi\theta^i_j(x)$ is also a regular structure maps. We exploit this observation in Section 8 where we shall deal with the dilation problem associated with Markov process.

4 A Quantum stochastic differential equation (qsde) with unbounded coefficients :

In this section we shall consider the class of contractive evolutions $V \equiv \{V(t) : t \geq 0\}$ *satisfying the following qsde:*

$$dV(t) = \sum_{i,j \in \overline{S}} V(t) Z^i_j d\Lambda^j_i(t) \ \ ; V(0) = I \tag{4.1}$$

on $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$, where $\mathcal{D}$ is a common dense domain of the family $Z \equiv \{Z^i_j,\ i,j \in \overline{S}\}$ of operators in the initial Hilbert space $\mathcal{H}_0$.

The following proposition will play a crucial role to guarantee the existence of a contractive solution for a certain class of coefficients Z.

Proposition 4.1 [13,31] : Suppose $Z \in \mathcal{Z}^-_R \cap \tilde{\mathcal{Z}}^-_R$. Then there exists a unique strongly continuous regular $(\mathcal{H}_0, \mathcal{M})$ -adapted contractive operator valued process $V \equiv \{V(t) : t \geq 0\}$ satisfying (4.1) on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ and for all $f \in \mathcal{H}_0, u \in \mathcal{M}, 0 \leq s \leq t < T$

$$\|[V(t) - V(s)] f e(u)\|^2 \leq K_T(f,u)[\nu_u(t) - \nu_u(s)] \tag{4.2}$$

where

$$K_T(f,u) = 2\ \exp(\nu_u(T)) \|e(u)\|^2 \sum_{i \in \overline{S}, j \in N(u)} \|Z^i_j f\|^2.$$

Moreover the dual contractive bar-cocycle $\tilde{V} \equiv \{\tilde{V}(t) := \mathcal{U}_t V(t)^* \mathcal{U}_t,\ t \geq 0\}$ satifies (4.1) and (4.2) with Z replaced by $\tilde{Z}$.

Proof *: The first part is essentially a restatement of* Theorem 2.12, where $V(t) = (X(0,t))^*, t \geq 0, L = \tilde{Z}$ except (4.2) which follows from the basic estimate (1.4) and the fact that $\|V(t)\| \leq 1$ for all $t \geq 0$.

For the second part follows from Theorem 2.11 once observe that for all $t \geq 0$, $\tilde{X}(0,t) = \tilde{V}(t)$, where $\tilde{X}$ is the dual cocycle associated with X. ∎

For a dense linear manifold $\mathcal{D}$ in $\mathcal{H}_0$, we denote by $\mathcal{Z}^-(\mathcal{D})$ the class of densely defined operators $Z \equiv (Z^i_j : i,j \in \overline{S})$ satisfying

$$\text{(a)} \quad \mathcal{D} \subseteq \mathcal{D}(Z^i_j); (i,j \in \overline{S}); \tag{4.3}$$

(b) There exists a sequence $Z(n) \in \mathcal{Z}^-_R \cap \tilde{\mathcal{Z}}^-_r, n \geq 1$ so that for all $f \in \mathcal{D}, i,j \in \overline{S}$

$$\text{s-lim}_{n\to\infty} Z^i_j(n)f = Z^i_j f \tag{4.4}$$

Lemma 4.2 : Let $Z \equiv (Z^i_j : i,j \in \overline{S})$ be a family of densely defined operators satisfying (4.3) and (4.4) Then for each $f \in \mathcal{D},\ j \in \overline{S}$ there exists a constant $c_j(f) \geq 0$ such that

$$\sup_{n\geq 1} \sum_{i\in\overline{S}} \|Z^i_j(n)f\|^2 \leq c_j(f)$$

and

$$\sum_{i\in\overline{S}} \|Z^i_j f\|^2 \leq c_j(f). \tag{4.5}$$

Proof : $Z(n) \in \mathcal{Z}^-_R$ implies that for each fixed $j \in \overline{S}$

$$\begin{aligned} \sum_{i\in\overline{S}} \|Z^i_j(n)f\|^2 &= \|Z^0_j(n)f\|^2 - < Z^j_j(n)f, f > - < f, Z^j_j(n)f > \\ &\leq \|Z^0_j(n)f\|^2 + 2\|f\|\ \|Z^j_j(n)f\| \end{aligned} \tag{4.6}$$

Now the required inequality follows once we apply (4.4) in (4.6). A simple application of Fatou's Lemma in (4.6) and (4.2) establishes (4.5). ∎

Fix $Z \in \mathcal{Z}^-(\mathcal{D})$ and $Z(n) \in \mathcal{Z}^-_R$ satisfying (4.3) and (4.4). We denote by $V^{(n)} \equiv \{V^{(n)}(t) : t \geq 0\}$ the unique regular $(\mathcal{H}_0, \mathcal{M})$ adapted contractive process satisfying (4.1) with $Z(n)$ as it's coefficients (Proposition 4.1).

Following an idea of Frigerio as outlined in Fagnola [13] and Mohari-Parthasarathy [31] we shall investigate the asymptotic behaviour of $\{V^{(n)}\}$ as $n \to \infty$.

Proposition 4.3 : The sequence $\{V^{(n)}\}$ admits a subsequence $\{V^{(n_k)}\}$ satisfying the following:

$$\text{(i)w-lim}_{k\to\infty}V^{(n_k)}(t) = V(t) \text{ exists for all } t \geq 0; \tag{4.7}$$

(ii) $V \equiv \{V(t) : t \geq 0\}$ is a contractive $(\mathcal{H}_0, \mathcal{M})$-adapted process for which

$$\lim_{k\to\infty} \sup_{t\leq T} | < \psi, [V^{(n_k)}(t) - V(t)]fe(u) > | = 0$$

for $0 \leq T < \infty, \psi \in \tilde{H}, f \in \mathcal{D}, u \in \mathcal{M}$;

(iii) For each $0 \leq T < \infty, f \in \mathcal{D}, u \in \mathcal{M}$ there exists a constant $c = c(f, u, T)$ such that

$$\|[V(t) - V(s)]fe(u)\| \leq c[\nu_u(t) - \nu_u(s)]^{1/2}; 0 \leq s \leq t \leq T; \tag{4.8}$$

(iv) $V = \{V(t) : t \geq 0\}$ is a strongly continuous $(\mathcal{H}_0, \mathcal{M})$ adapted process, $\{V(t)Z^i_j\} \in \mathbb{L}(\mathcal{D}, \mathcal{M})$ and

$$dV(t) = \sum_{i,j\in\overline{S}} V(t)Z^i_j d\Lambda^j_i(t) \;\; V(0) = I \tag{4.9}$$

holds on$\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$;

(v) If (4.9) admits a unique contractive solution then V is a cocycle and

$$w. \lim_{n\to\infty} V^{(n)}(t) = V(t)(t \geq 0)$$

Proof: As in [13,31] consider the sequence $\{\rho_n\}$ of continuous functions on $\mathbb{R}_+$ defined by

$$\rho_n(t) =< \psi, V^{(n)}(t)fe(u) >$$

where $\psi \in \tilde{\mathcal{H}}, f \in \mathcal{D}, u \in \mathcal{M}$ are fixed. By (4.2) and (4.5) we have for $0 \leq s \leq t < T$

$$|\rho_n(t) - \rho_n(s)| \leq \|\psi\| \|[V^{(n)}(t) - V^{(n)}(s)]fe(u)\|$$

$$\leq \|\psi\| c(f,u,T)[\nu_u(t)-\nu_u(s)]^{1/2}$$

where $c(f,u,T)$ is a non-negative constant independent of n. Furthermore $|\rho_n(t)| \leq \|\psi\|\|fe(u)\|$ for all $t \geq 0$ and $n \geq 1$. Hence by Arzela-Ascoli theorem $\{\rho_n\}$ is conditionally compact in the topology of uniform convergence on compacta. Using the seperabality of the spaces involved and usual diagonalisation procedure extract a subsequence $\{V^{(n_k)}\}$ satisfying (i) and (ii). For (iii) observe that for any $\psi \in \mathcal{H}, f \in \mathcal{D}, u \in \mathcal{M}$

$$\begin{aligned} |<\psi,[V(t)-V(s)]fe(u)>| &= \lim_{k\to\infty} |<\psi,[V^{(n_k)}(t)-V^{(n_k)}(s)]fe(u)>| \\ &\leq \|\psi\| c(f,u,T)[\nu_u(t)-\nu_u(s)]^{1/2}. \end{aligned}$$

So taking supremum over all unit vectors ψ we get

$$\|[V(t)-V(s)]fe(u)\| \leq c(f,u,T)[\nu_u(t)-\nu_u(s)]^{1/2}.$$

$V \equiv \{V(t) : t \geq 0\}$ being contractive, strong continuity follows from (4.8). Lemma 4.2 implies that $\{V(t)Z^i_j\} \in I\!\!L(\mathcal{D},\mathcal{M})$. Now by (1.3) and (4.6) we have for each $f,g \in \mathcal{D}, u,v \in \mathcal{M}$ and $t \geq 0$

$$<fe(u),V(t)ge(v)>= \lim_{k\to\infty} <fe(u),V^{(n_k)}(t)ge(v)>$$

$$=<fe(u),ge(v)>+\lim_{k\to\infty}\sum_{i,j\in\overline{S}}\int_0^t ds u_i(s)v^j(s) <fe(u),V^{(n_k)}(s)Z^i_j(n_k)ge(v)>$$

$$=<fe(u),ge(v)>+\sum_{i,j\in\overline{S}}\int_0^t ds u_i(s)v^j(s) <fe(u),V(s)Z^i_j ge(v)>$$

which implies (4.9) and proves (iv).

Fix any $s \geq 0$ and define as in [20] the contractive adapted process $V_s = (V_s(t)\,;t \geq 0)$ by

$$V_s(t) = \begin{cases} V(t) & ,0 \leq t \leq s, \\ V(s)\overline{\Gamma(\theta_s)V(t-s)\Gamma(\theta^*_s)} & ,t \geq s. \end{cases}$$

The proof of the first part of (v) is complete once we have shown that V_s is also a solution of (4.9). V being a solution of (4.9), the following holds for $t \geq s$:

$$dV_s(t) = V(s)\overline{\Gamma(\theta_s)V(t-s)\{\sum_{i,j\in\overline{S}} Z^i_j d\Lambda^j_i(t-s)\Gamma(\theta^*_s)\}}$$

on $\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$. Also observe that $\overline{\Gamma(\theta_s)d\Lambda^j_i(t-s)\Gamma(\theta^*_s)} = d\Lambda^j_i(t)$ and $\Gamma(\theta^*_s)\Gamma(\theta_s) = I$. So

$$dV_s(t) = V(s)\overline{\Gamma(\theta_s)V(t-s)\Gamma(\theta^*_s)}\{\sum_{i,j\in\overline{S}} Z^i_j\overline{\Gamma(\theta_s)d\Lambda^j_i(t-s)\Gamma(\theta^*_s)}\}.$$

Hence we obtain the required result. The 'second part' of (v) follows by a standard subsequence argument. ■

The following extension of Proposition 4.3 will be required in Section 5 where we will deal with the dilation of minimal quantum dynamical semigroup.

Proposition 4.4 : Let for each $n \geq 1$ $Z(n) \equiv \{Z^i_j(n), i,j \in \mathcal{S}\}$ be a family of operators and $\mathcal{D}$ be a dense linear manifold such that $\mathcal{D} \subset \mathcal{D}(Z^i_j(n))$, $i,j \in \overline{\mathcal{S}}$. Suppose there exists a regular contractive $(\mathcal{H}_0, \mathcal{M})$- adapted process $V^{(n)}$ satifying (4.9) on $\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$ with $Z(n)$ as its coefficients. If (4.3) and (4.5) hold on $\mathcal{D}$ for a densely defined family of operators $Z \equiv \{Z^i_j,\ i,j \in \overline{\mathcal{S}}\}$ then Proposition 4.3 holds as well for the sequence $V^{(n)}$.

Proof : It follows without difficulty once we adopt the method employed for the proof of Proposition 4.3. ■

Lemma 4.5: Suppose $X \equiv \{X(t) : t \geq 0\}$ is a strongly continuous bounded operator valued $(\mathcal{H}_0, \mathcal{M})$ adapted process satisfying

$$dX(t) = \sum_{i,j\in\overline{S}} X(t)Z^i_j d\Lambda^j_i(t),\ X(0) = 0 \tag{4.10}$$

on $\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$. Then for all $m,n \geq 0$, $f,g \in \mathcal{D}, u,v \in \mathcal{M}$ and $t \geq 0$ the following holds:

$$< fu^{(m)}, X(t)gv^{(n)} >= \sum_{i,j \in \overline{S}} \int_0^t ds u_i(s) v^j(s) < fu^{(m_i)}, X(s)gv^{(n_j)} > \quad (4.11)$$

where $u^{(-1)} = 0$ and for any $n \geq 0$

$$n_i = \begin{cases} n & \text{if} i = 0, \\ n-1 & \text{if} i \in S. \end{cases}$$

Proof: X being strongly continuous, for any $T \geq 0$, $\sup_{0 \leq t \leq T} \|X(t)\| < \infty$. Now use the fact that $s \to e(su)$ is real analytic for any fixed $u \in \mathcal{M}$ and dominated convergence theorem to get (4.11) from (4.10). ■

Lemma 4.6: Suppose $T \equiv (T(t) : t \geq 0)$ is a family of strongly continuous operators in $\mathcal{H}_0$ such that $\sup_{t \geq 0} \|T(t)\| < \infty$ and

$$dT(t) = T(t)K dt, T(0) = 0 \quad (4.12)$$

holds on $\mathcal{D}$. If K is the generator of a contraction C_0-semigroup with $\mathcal{D}$ as a core then $T(t) = 0$ for all $t \geq 0$.

Proof: $\mathcal{D}$ being a core for K, for all $\lambda > 0$ we get

$$\overline{(K-\lambda)\mathcal{D}} = \mathcal{H}_0. \quad (4.13)$$

Define bounded operators $R_\lambda; \lambda > 0$ by

$$R_\lambda = \int_0^\infty e^{-\lambda t} T(t) dt$$

and from (4.12) observe that

$$\lambda R_\lambda = R_\lambda K \quad (4.14)$$

on $\mathcal{D}$. Hence by (4.13) and (4.14) we have $R_\lambda = 0$ for all $\lambda > 0$, so $T(t) = 0$ for all $t \geq 0$.

Proposition 4.7 : [34] If Z_0^0 is the generator of a contractive C_0-semigroup with $\mathcal{D}$ as a core then equation (4.9) has a unique contractive solution.

Proof: Let $V' \equiv \{V'(t) : t \geq 0\}$ be an another contractive process satisfying (4.9). Using the basic estimate (1.4) and (4.5) observe that V' also satisfies (4.8). Hence V' is strongly continuous. Define $X(t) = V(t) - V'(t)(t \geq 0)$. To show that $X(t) \equiv 0(t \geq 0)$ it is enough to show that for any $u, v \in \mathcal{M}$

$$T_{u^{(m)},v^{(n)}}(t) = 0 \tag{4.15}$$

where $T_{u^{(m)},v^{(n)}}(t) \in \mathcal{B}(\mathcal{H}_0)$ is defined by

$$< f, T_{u^{(m)},v^{(n)}}(t) g >=< f u^{(m)}, X(t) g v^{(n)} >$$

In view of Lemma 4.6 ,we are to show that $T_{u^{(m)},v^{(n)}}(t)$ satisfies (4.12). We shall do this by induction on $m, n \geq 0$. For $m = 0 = n$ it is immediate from (4.11) with $u = v = 0$. Assume that (4.15) holds for all $u, v \in \mathcal{M}$ and $m, n \geq 0$ such that $m + n \leq k$. Then by induction hypothesis and (4.11) observe that $T_{u^{(m)},v^{(n)}}(t)$ satisfies (4.12) for all $u, v \in \mathcal{M}$ and $m, n \geq 0$ where $m + n = k + 1$. Now an application of Lemma 4.6 completes the proof. ∎

For any $X \in \mathcal{B}(\tilde{\mathcal{H}})$ we define the bilinear forms $\mathcal{L}^i_j(X)(i, j \in \overline{S})$ on $\mathcal{D} \underline{\otimes} \mathcal{E}(\mathcal{M})$ by

$$\begin{aligned} < fe(u), \mathcal{L}^i_j(X) ge(v) &= < fe(u), X Z^i_j ge(v) > + < Z^j_i fe(u), X ge(v) > \\ &+ \sum_{k \in S} < Z^k_i fe(u), X Z^k_j ge(v) > \end{aligned} \tag{4.16}$$

where the necessary convergence follows from (4.5) and Cauchy - Schwarz inequality. In order that the solution $V \equiv \{V(t) : t \geq 0\}$ of (4.9) be isometric it is necessary that $\mathcal{L}^i_j(I) = 0(i,j \in \overline{S})$. Here our aim is to get a sufficient condition for $V \equiv \{V(t) : t \geq 0\}$ to be isometric. To this end we introduce a few more notations:

$$\mathcal{I} \equiv \{Z \in \mathcal{Z}(\mathcal{D}) : \mathcal{L}^i_j(I) = 0; i,j \in \overline{S}\}$$

and for $\lambda > 0$

$$\beta_\lambda \equiv \{B \geq 0 : B \in \mathcal{B}(\mathcal{H}_0); \mathcal{L}^0_0(B) = \lambda B\}$$

Lemma 4.8: If $Z \in \mathcal{I}$ and $X(t) = I - V(t)^*V(t)$, $t \geq 0$. then for all $m, n \geq 0, f, g \in \mathcal{D}, u, v \in \mathcal{M}$ and $t \geq 0$

$$< fu^{(m)}, X(t)gv^{(n)} >= \sum_{i,j \in \overline{S}} \int_0^t ds v_i(s) v^j(s) < fu^{(m_i)}, \mathcal{L}^i_j(X(t))gv^{(n_j)} > \tag{4.17}$$

where $m_i, n_j(i,j \in \overline{S})$ are as in (4.11).

Proof: $Z \in \mathcal{I}$ and quantum Ito's formula implies that for all $f, g \in \mathcal{D}, u, v \in \mathcal{M}$ and $t \geq 0$

$$< fe(u), X(t)ge(v) >= \sum_{i,j \in \overline{S}} \int_0^t ds u_i(s) v^j(s) < e(u), \mathcal{L}^i_j(X(t))ge(v) > \tag{4.18}$$

We obtain (4.17) from (4.18) and analyticity of the map $s \rightarrow e(sv)$ (for any $v \in \mathcal{M}$), where the necessary convergence follows from (4.5). ∎

Proposition 4.9: *If $Z \in \mathcal{I}$ and $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$ then the solution $V \equiv \{V(t) : t \geq 0\}$ of (4.9) is isometric.*

Proof: Note that $0 \leq X(t) \leq I, X(0) = 0$. Define non-negative operators $Y_\lambda \in \mathcal{B}(\tilde{H})$ and $B^{(n)}_\lambda(u) \in \mathcal{B}(\mathcal{H}_0)(\lambda > 0, n \geq 0, u \in \mathcal{M})$ by

$$Y_\lambda = \int_0^\infty e^{-\lambda t} X(t) dt$$

and

$$< f, B_\lambda^{(n)}(u)g > = < fu^{(n)}, Y_\lambda g u^{(n)} > .$$

Observe that for any fixed $n \geq 0, u \in \mathcal{M}, B_\lambda^{(n)}(u) = 0$ for some $\lambda > 0$ if and only if $X(t)fu^{(n)} = 0$ for all $f \in \mathcal{H}_0$ and $t \geq 0$. We shall show by induction on $n \geq 0$ that for all $f \in \mathcal{H}_0, u \in \mathcal{M}, t \geq 0$

$$X(t)fu^{(n)} = 0. \tag{4.19}$$

Taking $u = 0 = v$ in (4.17) observe that $B_\lambda^{(0)}(0) \in \beta_\lambda$. So (4.19) follows for $n = 0$ by our earlier observation and the assumption that $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. Now assuming (4.19) for $n - 1 (n \geq 1)$ we get for $(i, j) \neq (0, 0)$ and $t \geq 0$

$$< fu^{(n_i)}, \mathcal{L}_j^i(X(t)) g u^{(n_j)} > = 0.$$

Hence (4.17) implies that $B_\lambda^{(n)}(u) \in \beta_\lambda$ for all $u \in \mathcal{M}, \lambda > 0$, so $B_\lambda^{(n)}(u) = 0$ for some $\lambda > 0$, which by the observation made earlier implies (4.19) and completes the proof. ∎

Now our aim is to exploit the time reversal principle to obtain a sufficient condition for $V \equiv \{V(t) : t \geq 0\}$ to be co-isometric. To this end we impose some additional conditions on Z.

Assumption 4.10: *For the triad $(\mathcal{D}, Z, Z(n); n \geq 1)$ satisfying (4.3)-(4.4) and $Z(n) \in \mathcal{Z}_R^- \cap \tilde{\mathcal{Z}}_R^-$ there exists a dense linear manifold $\tilde{D}$ in $\mathcal{H}_0$ such that $(\tilde{D}, \tilde{Z}, \tilde{Z}(n); n \geq 1)$ also satisfies (4.3) and (4.4).*

If Z satisfies Assumption 4.10, Lemma 4.1 implies that $Z \in \mathcal{Z}(\mathcal{D})$ and $\tilde{Z} \in Z(\tilde{\mathcal{D}})$. For any $X \in \mathcal{B}(\tilde{H})$ define the bilinear forms $\tilde{\mathcal{L}}_j^i(X)$ $(i, j \in \overline{S})$ on $\tilde{D} \underline{\otimes} \varepsilon(\mathcal{M})$ as in (4.16) with Z replaced by $\tilde{Z}$ and set

$$\tilde{\mathcal{I}} \equiv \{Z : \tilde{\mathcal{L}}_j^i(I) = 0; i, j \in \overline{S}\}$$

and for $\lambda > 0$

$$\tilde{\beta}_\lambda \equiv \{B \geq 0 : B \in B(\mathcal{H}_0) : \tilde{\mathcal{L}}_0^0(B) = \lambda B\}.$$

Since for each $n \geq 1$, $Z(n) \in \mathcal{Z}_R^- \cap \tilde{\mathcal{Z}}_R^-$ by Theorem 4.1 there exists a unique regular $(\mathcal{H}_0, \mathcal{M})$ -adapted process $V^{(n)} \equiv \{V^{(n)}(t) : t \geq 0\}$ satisfying (4.1) with $Z(n)$ as its coefficients. Moreover the dual contractive bar-cocycle $\tilde{V}^{(n)} \equiv \{\tilde{V}^{(n)}(t) : t \geq 0\}$ satisfies (4,1) with $\tilde{Z}(n)$ as its coefficients. It is evident from (4.7) that

$$w.\lim_{k\to\infty} \tilde{V}^{(n_k)}(t) = \tilde{V}(t)(t \geq 0) \tag{4.20}$$

where

$$\tilde{V}(t) = \mathcal{U}_t V(t)^* \mathcal{U}_t^*(t \geq 0).$$

Proposition 4.11 : [34] Let for Z Assumption 4.10 be valid. Then

(i) $\tilde{V} \equiv \{\tilde{V}(t) : t \geq 0\}$ is a strongly continuous $(\mathcal{H}_0, \mathcal{M})$ adapted process, $\{\tilde{V}(t)\tilde{Z}_j^i\} \in \mathbb{L}(\tilde{\mathcal{D}}, \mathcal{M})$ and

$$d\tilde{V}(t) = \sum_{i,j\in\overline{S}} \tilde{V}(t)\tilde{Z}_j^i d\Lambda_i^j(t) : \tilde{V}(0) = I$$

holds on $\tilde{\mathcal{D}}\underline{\otimes}\varepsilon(\mathcal{M})$.

(ii) $V^* \equiv \{V(t)^* : t \geq 0\}$ is strongly continuous.

(iii) If V is co-isometric then $Z \in \tilde{\mathcal{I}}$

(iv) If $Z \in \tilde{\mathcal{I}}$ and $\tilde{\beta}_\lambda \equiv \{0\}$ for some $\lambda > 0$ then V is co-isometric.

Proof: $\tilde{Z} \in \mathcal{Z}(\tilde{D})$, so i) is immediate from Proposition 4.3 and (4.20). ii) follows from i) because $t \to \mathcal{U}_t$ is continuous in strong operator topology. For

iii) and iv) observe that V is co-isometric if and only if $\tilde{V}$ is isometric. Hence the required *results follow from* Proposition 4.9 and i) . ∎

Notes and Remarks :

In the context of the charactization problem associated with a strongly continuous bar-cocycle Journé [23] established a class of quantum stochastic differential equation with unbounded coefficients. The ' time reversal principle' of markovian cocycle is also indicated in [23]. Frigerio's equicontinuity method is employed in Fagnola [13] to guarantee a contractive evolution satisfying a q.s.d.e. with unbounded coefficients , associated with pure birth (pure death) process (Proposition 4.3). In this constext a necessary and sufficient condition for the evolution to be conservative is also obtained. The present form of Proposition 4.3 is an improvement of the result obtained in Mohari-Parthasarathy [31] and Mohari [34]. *Proposition* 4.7, Proposition 4.9 and Proposition 4.11 are adapted from Mohari [34].

5 Minimal quantum dynamical semigroup and its dilation in Boson-Fock space :

We consider the quantum mechanical Fokker-Plank equation writen formally as

$$\rho(0) = \rho, \ \rho(t)' = Y\rho(t) \ + \rho(t)Y^* \ + \sum_{k\in\mathcal{S}} Z_k\rho(t)Z_k^* \tag{5.1}$$

subject to

$$Y \ + Y^* \ + \sum_{k\in\mathcal{S}} Z_k^* Z_k \leq 0 \tag{5.2}$$

for $\rho \in \mathcal{T}_h$, where $Y, Z_k, \ k \in \mathcal{S}$ are densely defined operators in $\mathcal{H}_0$ and $\mathcal{T}_h$ is the real Banach space of all self-adjoint trace class operators in $\mathcal{H}_0$. When Y is a bounded operator (5.2) implies that $\{Z_k, k \in \mathcal{S}\}$ is a family of bounded operators and the series $\sum_{k\in\mathcal{S}} Z_k^* Z_k$ converges in strong operator topology. In such a case, for each ρ, (5.1) admits a unique $\mathcal{T}_h$ operator valued solution $\rho(t), \ t \geq 0$ and the map $\rho \to \sigma_t(\rho) = \rho(t), t \geq 0$ is a one parameter contraction semigroup in the Banach space ($\mathcal{T}_h, || \cdot ||_{tr}$). On the other hand there exists a unique regular $(\mathcal{H}_0, \mathcal{M})$ -adapted contractive operator valued process $V \equiv \{V(t), \ t \geq 0\}$ satisfying

$$dV(t) = \sum_{k\in\mathcal{S}} V(t) Z^i_j \Lambda^j_i(t), \ V(0) = I \tag{5.3}$$

on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$ where

$$Z^i_j = \begin{cases} S^i_j - \delta^i_j & , \quad i, j \in \mathcal{S}, \\ Z_i & , \quad i \in \mathcal{S}, j = 0, \\ -\sum_{k\in\mathcal{S}} Z_k^* S^k_j & , \quad i = 0, j \in \mathcal{S}, \\ Y & , \quad i = 0 = j \end{cases} \tag{5.4}$$

and $S = ((\ S^i_j\ ))$ is a contractive operator in $\mathcal{H}_0 \otimes l_2(\mathcal{S})$. The contractive one parameter semigroup $\tau_t := \mathbb{E}_0[\ V(t)^*(x \otimes I)V(t)], t \geq 0$ of completely positive maps and $\sigma_t,\ t \geq 0$ satisfy the following relation

$$tr(x\sigma_t(\rho)) = tr(\rho\tau_t(x))$$

whenever $t \geq 0, \rho \in \mathcal{T}_h,\ x \in \mathcal{B}(\mathcal{H}_0)$. For further details we refer to Corollary 2.15.

Here our aim is to deal with the dilation problem associated with Fokker-Planck equation (5.1)-(5.2) when operators $Y, Z_k,\ k \in S$ are not necessarily bounded operators.

Definition 5.1: [18,28] A one parameter family of completely positive maps $\tau \equiv \{\tau_t,\ t \geq 0\}$ on $\mathcal{B}(\mathcal{H}_0)$ is said to be a *quantum dynamical semigroup* if the following hold:

(i) $\tau_0(x) = x,\ \ \tau_t(\tau_s(x)) = \tau_{s+t}(x),\ s,t \geq 0,\ x \in \mathcal{B}(\mathcal{H}_0)$;

(ii) $||\tau_t|| \leq 1, t \geq 0$;

(iii) The map $t \rightarrow tr(\rho\tau_t(x))$ is continuous for any fixed $x \in \mathcal{B}(\mathcal{H}_0)$ and $\rho \in \mathcal{T}$, the trace class operators in $\mathcal{H}_0$.

(iv) For each $t \geq 0$ the map $x \rightarrow \tau_t(x)$ is continuous in the ultra-weak operator topology.

For a dynamical semigroup τ we define a one parameter family of maps $\sigma \equiv \{\sigma_t,\ t \geq 0\}$ on the pre-dual space of all trace class operators $\mathcal{T}$ so that

$$tr(x\sigma_t(\rho)) = tr(\rho\tau_t(x)) \tag{5.5}$$

whenever $t \geq 0, \rho \in \mathcal{T},\ x \in \mathcal{B}(\mathcal{H}_0)$. Note that the family σ is uniquely determined if (5.5) holds for $\rho := |f >< g|,\ f, g \in \mathcal{H}_0$. It is also evident that σ is a strongly continuous one parameter semigroup in the Banach space $(\mathcal{T},\ ||\cdot||_{tr})$. Conversely for a strongly continuous one parameter semigroup σ (5.5) determines a unique dynamical semigroup τ. Moreover for any $t \geq 0,\ tr(\sigma_(\rho)) = tr(\rho), \rho \in \mathcal{T}_h$ if and only if $\tau_t(I) = I$.

The central aim of this section is to exploit the theory developed in Section 4 and the construction of the minimal quantum dynamical semigroup , as outlined in Davies [9], in dilating the minimal semigroup in a boson-Fock space.

Before we proceed to the next result we state the following simple but useful lemmas:

Lemma 5.2 : Let $s.\lim_{n\to\infty}A_n = A$ and $s.\lim_{n\to\infty}B_n = B$. Then $\lim_{n\to\infty}A_n\rho B_n^* = A\rho B^*$ in $||\cdot||_{tr}$ topology whenever $\rho \in \mathcal{T}$.

Proof : It suffices to show for $A = B = 0$, $\rho \geq 0$. In such a case there exists a complete orthonormal set $f_k, k \geq 1$ in $\mathcal{H}$ and a sequnce $c_k \geq 0, k \geq 1, \sum_{k\geq 1} c_k < \infty$ such that $\rho = \sum_{k\geq 1} c_k|f_k><f_k|$ and for each $n \geq 1$

$$\begin{aligned} ||A_n\rho B_n^*||_{tr} &\leq \sum_{k\geq 1} c_k|| \, |A_n f_k><B_n f_k| \, ||_{tr} \\ &\leq \sum_{k\geq 1} c_k ||A_n f_k|| \, ||B_n f_k|| \end{aligned}$$

Now use uniform boundedness principle and dominated convergence theorem to conclude the required result. ∎

Lemma 5.3 : Let $A_k, k \geq 1$ and $B_k, k \geq 1$ be two families of bounded operators such that both the series $\sum_{k\geq 1} A_k^* A_k$ and $\sum_{k\geq 1} B_k^* B_k$ converge in strong operator topology. Then for each $\rho \in \mathcal{T}_h$ the series $\sum_{k\geq 1} B_k \rho A_k^*$ converges in $||\cdot||_{tr}$ norm topology.

Proof : Lemma 1.9 implies that the series $\sum_{k\in 1} A_k^* B_k$ converges in strong operator topology. Also note that for any bounded operators A, B, ρ the following polarization identity holds:

$$4\, A^*\rho B = \sum_{0\leq k\leq 3} (-i)^k (A + i^k B)^* \rho (A + i^k B).$$

Hence without loss of generality we assume that $A_k = B_k,\ k \geq 1$. In such a case for any $m \geq n \geq 1, \rho \geq 0$ we have

$$\begin{aligned} \|\sum_{n<k\leq m} A_k \rho A_k^*\|_{tr} &= \sum_{n<k\leq m} tr(A_k \rho A_k^*) \\ &= tr(\rho \sum_{n<k\leq m} A_k^* A_k) \end{aligned}$$

Hence the result follows from the above equality once we appeal to Lemma 5.2. ■

Following Davies [9], let Y be the generator of a strongly continuous contractive semigroup in $\mathcal{H}_0$ and $Z_k,\ k \in \mathcal{S}$ be a family of densely defined operators on $\mathcal{H}_0$ such that

$$\mathcal{D}(Y) \subseteq \mathcal{D}(Z_k),\ k \in \mathcal{S} \tag{5.6}$$

and

$$< f, Yf > + < Yf, f > + \sum_{k \in \mathcal{S}} < Z_k f, Z_k f > \leq 0 \tag{5.7}$$

for all $f \in \mathcal{D}(Y)$.

In view of Lemma 5.2 the following relation

$$\kappa_t(\rho) = e^{tY} \rho e^{tY*}$$

defines a strongly continuous, positive, one parameter, contraction semigroup on $\mathcal{T}_h$, whose generator G is given formally by

$$G(\rho) = Y\rho + \rho Y^*. \tag{5.8}$$

We introduce the positive one to one map π on $\mathcal{T}_h$ defined by

$$\pi(\rho) = (1 - Y)^{-1} \rho (1 - Y^*)^{-1}.$$

Set $\pi(\mathcal{T}_h) = \{\pi(\rho),\ \rho \in \mathcal{T}_h\}$.

As in [9] we define the positive linear map $\mathcal{J} : \pi(\mathcal{T}_h) \to \mathcal{T}_h$ by

$$\mathcal{J}(\rho) = \sum_{k \in \mathcal{S}} Z_k \rho Z_k^* \tag{5.9}$$

where the convergence follows from (5.7) and Lemma 5.3.

We quote the following proposition without proof.

Proposition 5.4 Consider the family $Y,\ Z_k,\ k \in \mathcal{S}$ of operators satisfying (5.6) and (5.7). Then the following hold:

(i) $\pi(\mathcal{T}_h)$ is a dense core for G and (5.8) is valid for all $\rho \in \pi(\mathcal{T}_h)$.

(ii) The map $\mathcal{J}$ has a positive extension $\mathcal{J}'$ on $\mathcal{D}(G)$ such that

$$tr(G(\rho) + \mathcal{J}'(\rho)) \leq 0 \tag{5.10}$$

whenever $\rho \in \mathcal{D}(G)$. Moreover equality holds in (5.11) if and only if equality holds in (5.7).

(iii) For each fixed $\lambda > 0,\ \mathcal{J}'(\lambda - G)^{-1}$ is a map from $\pi(\mathcal{T}_h)$ into $\mathcal{T}_h$ and has a unique bounded positive extension A_λ in $\mathcal{T}_h$ such that $\|A_\lambda\| \leq 1$ and $J'(\rho) = A_1[1 - G](\rho)$ for all $\rho \in \mathcal{D}(G)$;

(iv) For any fixed $0 \leq r < 1,\ \pi(\mathcal{T}_h)$ is a core for the operator $W^{(r)} = G + r\mathcal{J}'$ defined on $\mathcal{D}(G)$. Moreover $W^{(r)}$ is the generator of a strongly continuous positive one parameter contraction semigroup σ_t^r, whose resolvent at $\lambda > 0$ is given by

$$(\lambda - W^{(r)})^{-1} = (\lambda - G)^{-1} \sum_{k \geq 0} r^k\, A_\lambda^k, \tag{5.11}$$

where the series converges in trace norm.

(v) For each $\rho \geq 0,\ t \geq 0$ the map $r \to \sigma_t^{(r)}(\rho),\ r \in [0,1)$ is increasing and continuous.

(vi) There exists a positive one parameter strongly continuous contraction semigroup σ_t^{min} on $\mathcal{T}_h$ such that

$$\lim_{r \uparrow 1} \sigma_t^r(\rho) = \sigma_t^{min}(\rho)$$

for all $\rho \in \mathcal{T}_h$.

(vii) For each $\lambda > 0, R^{(n)}(\lambda) := (\lambda - G)^{-1} \sum_{0 \leq k \leq n} A_\lambda^k \rightarrow R(\lambda)$ strongly as $n \rightarrow \infty$, where $R(\lambda) = (\lambda - W)^{-1}$, W is the generator of σ_t^{min};

Proof : For (i)-(vi) see Davies [9]. Now for (vii) we follow Kato [25] (Lemma 7). For each $\lambda > 0, 0 \leq r < 1$ we have

$$R_r^{(n)}(\lambda) := (\lambda - G)^{-1} \sum_{0 \leq k \leq n} r^k A_\lambda^k \leq R_r(\lambda) \leq R(\lambda).$$

Letting $r \uparrow 1$ we get $R^{(n)}(\lambda) \leq R(\lambda)$. But as $R^{(n)}(\lambda)$ is incresing with n, $\text{s.lim}_{n \rightarrow \infty} R^{(n)}(\lambda) = R'(\lambda)$ exists and $R'(\lambda) \leq R(\lambda)$. We also have $R_r^{(n)}(\lambda) \leq R^{(n)}(\lambda) \leq R'(\lambda)$. Hence $R_r(\lambda) = \lim_{n \rightarrow \infty} R_r^{(n)}(\lambda) \leq R'(\lambda)$, $R(\lambda) = \lim_{r \uparrow 1} R_r(\lambda) \leq R'(\lambda)$ by (vi). This completes the proof. ∎

Now our aim is to obtain a necessary and sufficient condition for σ to be trace preserving. It is evident that equality in (5.7) is necessary. Still following Kato [25] we obtain the following theorem.

Theorem 5.5 : Consider the semigroup σ_t^{min}, $t \geq 0$ defined as in Proposition 5.4. Let equality in (5.7) be valid then the following statements are equivalent:

(i) $tr(\sigma_t^{min}(\rho)) = tr(\rho)$ for all $t \geq 0$, $\rho \in \mathcal{T}_h$;

(ii) for each fixed $\lambda > 0$, $A_\lambda^n \rightarrow 0$ strongly as $n \rightarrow \infty$;

(iii) for each fixed $\lambda > 0$, $(\lambda - W_0)(\pi(\mathcal{T}_h))$ is dense in $\mathcal{T}_h$;

(iv) for each fixed $\lambda > 0$, the characteristic equation $W_0^*(x) = \lambda x$ has no non-zero solution in $\mathcal{B}(\mathcal{H}_0)$,

where $W_0 = G + \mathcal{J}'$ with domain $\pi(\mathcal{T}_h)$ and W_0^* is the adjoint of W_0;

(v) for any fixed $\lambda > 0$, there is no non-zero $x \in \mathcal{B}(\mathcal{H}_0)$ so that

$$< f, xYg > + < Yf, xg > + \sum_{k \in \mathcal{S}} < Z_k f, xZ_k g > = \lambda < f, g > \qquad (5.12)$$

hold for all $f, g \in \mathcal{D}(Y)$.

Proof : The proof is exactly along the line of Theorem 3 in [25]. We write $\sigma = \sigma^{min}$. As in [25] in this context we note that

$$||R(\lambda)(\rho)||_{tr} = \int_0^\infty exp(-\lambda t)||\sigma_t(\rho)||_{tr} dt \tag{5.13}$$

for all $\rho \geq 0$, which follows from the basic resolvent formula $R(\lambda) = \int_0^\infty exp(-\lambda t)\sigma_t dt$, $\lambda > 0$. As a simple consequence of the following identity

$$I + \mathcal{J}' R^{(n)}(\lambda) = (\lambda I - G)R^{(n)}(\lambda) + A_\lambda^{n+1} \tag{5.14}$$

and Proposition 5.4(ii) we get $tr(\rho) = \lambda \; tr(R^{(n)}(\lambda)(\rho)) + tr(A_\lambda^{n+1}(\rho))$ for $\rho \in \mathcal{T}$. Since $R^{(n)}(\lambda)(\mathcal{T}_+) \subset \mathcal{T}_+$ we have

$$||\rho|| = \lambda ||R^{(n)}(\lambda)(\rho)||_{tr} + ||A_\lambda^{n+1}(\rho)||_{tr} \tag{5.15}$$

for all $\rho \geq 0$. Now taking limit as $n \to \infty$ in (5.16) by Proposition 5.4(vii) we get

$$\begin{aligned} \lim_{n\to\infty} ||A_\lambda^{n+1}(\rho)||_{tr} &= ||\rho|| - \lambda ||R(\lambda)(\rho)||_{tr} \\ &= \lambda \int_0^\infty exp(-\lambda t)(||\rho||_{tr} - \sigma_t(\rho)||_{tr}) \end{aligned} \tag{5.16}$$

for all $\rho \geq 0$, where we have used (5.13) in the second equality. Since for each fixed $\rho \in \mathcal{T}$ the map $t \to ||\sigma_t(\rho)||_{tr}$ is continuous and $||\sigma_t(\rho)||_{tr} \leq ||\rho||_{tr}, t \geq 0$ from (5.15) we conclude that (i) and (ii) are equivalent.

Our next aim is show that (ii) and (iii) are equivalent for any fixed $\lambda > 0$. From (5.14) we note that (ii) is equivalent to equivalent to

$$\lim_{n\to\infty} [\lambda - G - \mathcal{J}'] R^{(n)}(\lambda)(\rho) = \rho$$

for all $\rho \in \mathcal{T}$. Since $R^{(n)}(\lambda)(\rho) \in \mathcal{D}(G)$ we conclude that $[\lambda - G - \mathcal{J}'](\mathcal{D}(G))$ is dense in $\mathcal{T}$. Since $\pi(\mathcal{T}_h)$ is a core for G, for any fixed $\rho \in \mathcal{D}(G)$ we choose a sequence $\rho_n \in \pi(\mathcal{T}_h)$ such that $\rho_n \to \rho$ and $G(\rho_n) \to G(\rho)$ as $n \to \infty$. By Proposition 5.4(ii) we have

$$||\mathcal{J}'(\rho)||_{tr} = ||A_1[1 - G)](\rho)||_{tr} \leq ||[1 - G](\rho)||_{tr} \leq ||\rho||_{tr} + ||G(\rho)||_{tr}$$

for all ρ in $\mathcal{D}(G)$, hence $\mathcal{J}'(\rho) = \lim_{n\to\infty}\mathcal{J}'(\rho_n)$. Thus it is evident that

$$(\lambda - G - \mathcal{J}')(\rho) = \lim_{n\to\infty}(\lambda - G - \mathcal{J}')(\rho_n)$$

Hence $[\lambda - G - \mathcal{J}'](\pi(\mathcal{T}_h))$ is dense in $\mathcal{T}_h$.

Conversely let (iii) be valid. Since $[I - A_\lambda](\mathcal{T}_h) = [I - A_\lambda][\lambda - G](\mathcal{D}(G)) = [\lambda - G - \mathcal{J}'][\pi(\mathcal{D}(G)] \supset [\lambda - G - \mathcal{J}'][\pi(\mathcal{T}_h)]$, $[I - A_\lambda](\mathcal{T}_h)$ is dense in $\mathcal{T}_h$. Set $C_\lambda^{(n)} = \frac{1}{n+1}\sum\limits_{0\le k\le n} A_\lambda^k$, which is a uniformly bounded by $||C_\lambda^{(n)}|| \le 1, n \ge 0$. That $\lim_{n\to\infty}C_\lambda^{(n)} = 0$ is now an easy consequence of $C_\lambda^{(n)}[I - A_\lambda] = \frac{1}{n+1}[I - A_\lambda^{n+1}]$. On the other hand, A_λ being a contractive positive map, $||A_\lambda^m|| \le ||A_\lambda^n||$ whenever $m \ge n$, hence

$$||C_\lambda^{(n)}(\rho)||_{tr} = \frac{1}{n+1}\sum_{0\le k\le n} ||A_\lambda^k(\rho)||_{tr} \ge ||A_\lambda^n(\rho)||_{tr}$$

whenever $\rho \ge 0$. Thus we have $A_\lambda^n(\rho) \to 0$ as $n \to \infty$. This shows that (ii) and (iii) are equivalent.

That (iii) and (iv) are equivalent follows by the definition of adjoint of a densely defined operator and Hann-Banach theorem.

The proof is complete once we show (iv) and (v) are equivalent. We claim that an element $x \in \mathcal{D}(W_0^*)$ satifies $W_0^*(x) = \lambda x$ if and only if x satisfies (5.12). For any fixed $f, g \in \mathcal{H}_0$ and $x \in \mathcal{D}(W_0^*)$ we have

$$\begin{aligned} tr(\pi(|f><g|)W_0^*(x)) &= \ <Y(1-Y)^{-1}f, x(1-Y)^{-1}g> \\ &+ \ <(1-Y)^{-1}f, xY(1-Y)^{-1}> \\ &+ \ \sum_{k\in S} <Z_k(1-Y)^{-1}f, xZ_k(1-Y)^{-1}g> \quad (5.17) \end{aligned}$$

Since $\mathcal{R}((1-Y)^{-1}) = \mathcal{D}(G)$ the claim follows from (5.17). That (iv) and (v) are eqivalent is a simple consequence of the claim. ∎

We use the same symbol for the linear canonical extension of a bounded map that appeared in Proposition 5.4 to the Banach space of all trace class operators. In the case of an unbounded operator, say G we extend to $\mathcal{D}(G)+i\mathcal{D}(G)$ by linearity.

The family of maps $\tau_t^{min} := (\sigma_t^{min})^*$ on the dual space $\mathcal{B}(\mathcal{H}_0)$ is called *minimal dynamical semigroup*. For further details we refer to [9].

Our aim is to deal with the dilation problem associated with Fokker-Planck equation (5.1) whenever the operators $Y,\ Z_k,\ k \in \mathcal{S}$ satisfy the following assumption.

Assumption 5.7: Y is the generator of a strongly continuous semigroup on $\mathcal{H}_0$ and $Z_k,\ k \in \mathcal{S}$ is a family of densely defined operators satisfying (5.6) and (5.7). There exists a dense linear manifold $\mathcal{D}$ in $\mathcal{H}_0$ so that it is a core for Y and

$$S_j^k(\mathcal{D}) \subseteq \mathcal{D}(Z_k^*),\ k, j \in \mathcal{S}$$

where $S \equiv ((S_j^k,\ k, j \in \mathcal{S}))$ is a contractive operator on $\mathcal{H}_0 \otimes l_2(\mathcal{S})$ such that for any fixed $j \in \mathcal{S}$, $S_j^i \neq 0$ for finitely many $i \in \mathcal{S}$.

For any $\lambda > 0$ we define bounded operators $Y_\lambda,\ L_k^\lambda,\ k \in \mathcal{S}$ by

$$Y_\lambda = \lambda^2(\lambda - Y^*)^{-1}Y(\lambda - Y)^{-1},\ Z_k^\lambda = \lambda Z_k(\lambda - Y)^{-1}$$

where boundedness of $L_k^\lambda,\ k \in \mathcal{S}$ follows from (5.7). Moreover for each $\lambda > 0,\ Y_\lambda,\ L_k^\lambda,\ k \in \mathcal{S}$ satisfies (5.2), hence the series $\sum_{k\in\mathcal{S}}(L_k^\lambda)^* L_k^\lambda$ converges in strong operator topology. On the other hand for each $g \in \mathcal{D}(Y)$ we have $Y_\lambda g \to Yg$ as $\lambda \to \infty$. Taking $f = (I - \lambda(\lambda - Y)^{-1})g,\ g \in \mathcal{D}(Y)$ in (5.7) we get

$$||Z_k(I - \lambda(\lambda - Y)^{-1})g|| \leq 2||(I - \lambda(\lambda - Y)^{-1})g||||Y(I - \lambda(\lambda - Y)^{-1})g||$$

Hence $Z_k^\lambda g \to Z_k g$ as $\lambda \to \infty$ for all $g \in \mathcal{D}(Y)$.

For any $\lambda > 0, 0 \leq r \leq 1$ we define bounded operators $Z(\lambda, r) \equiv \{Z^i_j(\lambda, r),\ i,j \in \overline{S}\}$ as in (5.4) with $Y,\ Z_k, k \in \mathcal{S}$ replaced by $Y_\lambda,\ r^{1/2}Z^\lambda_k, k \in \mathcal{S}$ respectively. So for each $0 \leq r \leq 1, \lambda > 0,\ Z(\lambda, r) \in \mathcal{Z}^-_R \cap \tilde{\mathcal{Z}}^-_R$. We denote $V^{(\lambda,r)} = \{V^{(\lambda,r)}(t),\ t \geq 0\}$ for the unique regular $(\mathcal{H}_0, \mathcal{M})-$ adapted contractive process satisfying (5.3) with $Z(\lambda, r)$ as its coefficients.

We also define operators $Z(r)$ on $\mathcal{D}$ as in (5.4) with $Z_k,\ k \in \mathcal{S}$ replaced by $r^{1/2}Z_k,\ k \in \mathcal{S}$. When $r = 1$ we also write $Z(\lambda,\ r) = Z(\lambda)$, and $Z(r) = Z$. For each $0 \leq r \leq 1$ it is evident that

$$\lim_{\lambda\to\infty} Z^i_j(r,\lambda)f = Z^i_j(r)f,\ f \in \mathcal{D}.$$

for all $i,j \in \overline{\mathcal{S}}$.

Proposition 5.8 : Consider the operators $Y,\ Z_k,\ k \in \mathcal{S}$ satisfying Assumption 5.7. Then the following hold:

(i) For each $0 \leq r \leq 1,\quad w.\lim_{\lambda\to\infty} V^{(\lambda,r)}(t) = V^{(r)}(t)$ exists for all $t \geq 0$ and $V^{(r)} = \{V^{(r)}(t),\ t \geq 0\}$ is the unique regular $(\mathcal{H}_0, \mathcal{M})$ -adapted contractive operator valued process satisfying (5.3) on $\mathcal{D}\underline{\otimes}\varepsilon(\mathcal{M})$ with $Z(r) \equiv \{Z^i_j(r),\ i,j \in \overline{\mathcal{S}}\}$ as its coefficients. Moreover $V^{(r)}$ is a strongly continuous contractive bar-cocycle;

(ii) For each $t \geq 0$ the map $r \to V^{(r)}(t),\ 0 \leq r \leq 1$ is continuous in weak operator topology.

Proof : For each $0 \leq r \leq 1,\ \lambda > 0,\ Z(\lambda, r) \in \mathcal{Z}^-_R \cap \tilde{\mathcal{Z}}^-_R$ and the triad $Z(r), Z(\lambda, r), \lambda > 0,\ \mathcal{D}$ satisfy (4.1) and (4.2). By our hypothesis $\mathcal{D}$ is also a core for Y. Hence we conclude (i) from Proposition 4.3 and Proposition 4.6.

Fix $0 \leq r, r_n \leq 1,\ n \geq 1$ such that $r_n \to r$ as $n \to \infty$. By (i) we note the triad $(Z(r), Z(r_n), n \geq 1, \mathcal{D})$ satisfies (4.1) and (4.2) on $\mathcal{D}$. Since $Z^i_j(r_n)f \to Z^i_j(r)f$, as $r_n \to \infty$ for any $f \in \mathcal{D}$ and $V^{(r)}$ being the unique contractive solution of (5.3) with $Z(r)$ as its coefficients, Proposition 4.4 implies that $w.\lim_{n\to\infty} V^{(r_n)}(t) = V^{(r)}(t)$, $0 \le t < \infty$. This completes the proof. ∎

For each $\lambda, \mu > 0$, $0 \le r, s \le 1$, we define one parameter family of semigroups $\tau^{(\lambda,\mu,r,s)}$ on $\mathcal{B}(\mathcal{H}_0)$ by

$$\tau^{(\lambda,\mu,r)}(x) = \mathbb{E}_0[V^{(\lambda,r)}(t)^* x V^{(\mu,s)}(t)],\ t \ge 0,$$

where semigroup property follows from bar-cocycle property of the contractive processes $V^{(\lambda,r)}$. The associated pre-dual semigroup $\sigma^{(\lambda,\mu,r,s)}$ on $\mathcal{T}$ is defined as in (5.5) whose bounded generator $\mathcal{L}_*^{(\lambda,\mu,r,s)}$ is given by

$$\mathcal{L}_*^{(\lambda,\mu,r,s)}(\rho) = Y_\mu \rho\ + \rho Y_\lambda^*\ + \sqrt{rs}\sum_{k\in S} Z_k^\mu \rho (Z_k^\lambda)^*, \quad \rho \in \mathcal{T}.$$

For each $0 \le r < 1$ we also have

$$W^{(r)}(\rho) = Y\rho\ + \rho Y^*\ + r\sum_{k\in S} Z_k \rho Z_k^*, \quad \rho \in \pi(\mathcal{T}),$$

where $W^{(r)}$ is described in Proposition 5.4.

We also write $\tau^{(\lambda,r)}$, $\tau^{(\lambda,\mu,r)}$ and for $\tau^{(\lambda,\lambda,r,r)}$, $\tau^{(\lambda,\mu,r,r)}$ respectively. Similarly for their pre-dual maps on $\mathcal{T}$. When $r = 1$ we omit the symbol r.

For each $0 \le r, s \le 1$ we also set one parameter semigroup

$$\tau^{(r,s)} := \mathbb{E}_0[V^{(r)}(t)^* x V^{(s)}(t)], t \ge 0$$

on $\mathcal{B}(\mathcal{H}_0)$. When $r = 1$ we omit the symbol r.

Our aim is to show that $\sigma_t^{(min)}$ is the pre-dual map of τ^t, $t \ge 0$ for all, where σ^{min} is defined as in Proposition 5.4. Before we proceed for the next result we state the following Lemma.

Lemma 5.9 : Let $A_k, k \ge 1$ and $B_k, k \ge 1$ be two families of bounded operators such that the series $\sum_{n\ge1} A_k^* A_k$ converge in strong operator topology and $s.\lim_{n\to\infty} B_n = B$. Then for each $\rho \in \mathcal{T}_h$ the sequence $C(m,n) =$

$\sum_{k\in S} A_k B_m\ \rho\ B_n^* A_k^*$ converges in $||\cdot||_{tr}$ topology to $C = \sum_{k\in S} A_k B\ \rho\ B^* A_k^*$ as $(m,n) \to \infty$, independent of the order of limiting variables.

Proof : Lemma 5.3 implies that $C, C(m,n),\ m,n \geq 1$ are elements in $\mathcal{T}$. For any fixed $m,n \geq 1$ and $\rho \in \mathcal{T}$ we have

$$\begin{aligned} ||C(m,n) - C||_{tr} &\leq \sum_{k\geq 1}\{||A_k(B_m - B)\rho(A_k B_n)^*||_{tr} \\ &- ||A_k B\rho(A_k(B_n - B))^*||_{tr}.\} \end{aligned}$$

Hence for $\rho = |f><g|$ we have

$$\begin{aligned} ||C(m,n) - C||_{tr} &\leq \sum_{k\geq 1}\{||A_k(B_m - B)f||\ ||A_k B_n g|| + ||A_k B f||\ ||A_k(B_n - B)g||\} \\ &\leq (\sum_{k\geq 1}||A_k(B_m - B)f||^2)^{\frac{1}{2}}(\sum_{k\geq 1}||A_k B_n g||^2)^{\frac{1}{2}} \\ &+ (\sum_{k\geq 1}||A_k f||^2)^{\frac{1}{2}}(\sum_{k\geq 1}||A_k(B_n - B)g||^2)^{\frac{1}{2}} \\ &\leq \alpha(||(B_m - B)f||\ ||B_n g|| + ||f||\ ||(B_n - B)g||) \leq \beta||f||\ ||g|| \end{aligned}$$

where α, β are some positive constants independent of f, g. Hence the result follows for $\rho = |f><g|,\ f, g \in \mathcal{H}$. For a general $\rho = \sum_i c_i\ |f_i><g_i|,\ ||f_i|| = ||g_i|| = 1, \sum_i |c_i| < \infty$, use dominated convergence theorem to conclude the required result. This completes the proof. ∎

Proposition 5.10 : Cosider the family of operators $\{Y,\ Z_k,\ k \in S\}$ satisfying (5.6) and (5.7). Then for each fixed $0 \leq r < 1$ the following hold:

(i) For each $\lambda, \mu > 0,\ 0 \leq r, s \leq 1$

$$\sigma_t^{(\lambda,\mu,r,s)} = \sigma^{(\lambda,\mu,\sqrt{rs})},\ t \geq 0;$$

(ii) $\lim_{(\lambda,\mu)\to\infty}||\mathcal{L}_*^{(\lambda,\mu,r)}(\rho) - W^{(r)}(\rho)||_{tr} = 0$
whenever $\rho \in \pi(\mathcal{T})$, independent of the order of the limiting variables;

(iii) $\lim_{\ (\lambda,\mu)\to\infty}||\sigma_t^{(\lambda,\mu,r)}(\rho) - \sigma_t^{(r)}(\rho)||_{tr} = 0$ for all $\rho \in \mathcal{T}$, where $\sigma^{(r)}$ is the map defined as in Proposition 5.4;

(iv) The pre-dual map of $\tau_t^{(r)}$ is $\sigma_t^{(r)}$, $t \geq 0$;

(v) For each $0 \leq s < 1$, $\sigma_t^{(r,s)} = \sigma_t^{(\sqrt{rs})}$ for all $t \geq 0$.

Proof : Since for each fixed $\lambda, \mu > 0$, $\mathcal{L}_*^{(\lambda,\mu,r,s)} = \mathcal{L}^{(\lambda,\mu,\sqrt{rs},\sqrt{rs})}$ we conclude (i) by the fact that a bounded generator uniquely determines the semigroup.

Now for (ii) first observe that

$$\begin{aligned}\mathcal{L}_*^{(\lambda,\mu,r)}(\pi(\rho)) &= Y_\mu\pi(\rho) + \pi(\rho)Y_\lambda^* \\ &+ r^2\sum_{k\in\mathcal{S}} Z_k^1(\mu(\mu - Y)^{-1})\rho(\lambda(\lambda - Y)^{-1})^*(Z_k^1)^*\end{aligned}$$

and

$$W^{(r)}(\pi(\rho)) = Y\pi(\rho) \ + \pi(\rho)Y^* + r^2\sum_{k\in\mathcal{S}} Z_k^1\rho(Z_k^1)^*$$

for all $\rho \in \mathcal{T}$ and $Y_\mu\pi(\rho) = \mu^2(\mu - Y^*)^{-1}(\mu - Y)^{-1}\,(Y(1-Y)^{-1}\rho(1-Y^*)^{-1})$. Now (ii) is immediate from Lemma 5.9.

Since $\pi(\mathcal{T})$ is a core for $W^{(r)}$ which is the genrator of a strongly continuous contraction semigroup, (iii) is evident from (ii) once we appeal to a standard result in the semigroup theory [10].

For any fixed $f, g \in \mathcal{H}_0$, $, \lambda, \mu > 0$ we have

$$tr(x\sigma_t^{\lambda,\mu,r}(|f><g|)) = < fe(0), V^{(\lambda,r)}(t)^* x V^{(\mu,r)}(t) ge(0) >$$

Hence (iv) follows from Proposition 5.8(i) and (iii). Also conclude (v) from (i) and (iii). This completes the proof. ∎

The following theorem establishes our central aim set in this exposition.

Theorem 5.11 : Let Y, $Z_k, k \in \mathcal{S}$ be a family of operators satisfying Assumption 5.7. Consider the family $Z \equiv \{Z_j^i,\ i,j \in \mathcal{S}\}$ defined as in (5.4) on $\mathcal{D}$. Then there exists a unique regular $(\mathcal{D},\mathcal{M})$-adapted contractive process $V \equiv \{V(t),\ t \geq 0\}$ satisfying (5.3) on $\mathcal{D}\underline{\otimes}\epsilon(\mathcal{M})$.

Moreover the following hold:

(i) $\tau_t^{min}(x) = \mathbb{E}_0[V(t)^* x V(t)]$, where τ^{min} is the minimal dynamical semigroup on $\mathcal{B}(\mathcal{H}_0)$ associated with (5.1) and (5.2).

(ii) Let equality in (5.7) be valid and S be also isometric operator then $Z \in \mathcal{I}$. In such a case V is isometric if and only if $\beta_\lambda = 0$ for some $\lambda > 0$, where β_λ is defined as in *Proposition* 4.9.

Proof : The first part is nothing but a restatement of Proposition 5.8 (i) for $r = 1$.

In view of Proposition 5.10 it is evident that for all $0 \leq r < 1$ following hold:

$$
\begin{aligned}
tr(\sigma_t^{(\sqrt{r})}(|f><g|)x) &= \lim_{s\uparrow 1} tr(\sigma_t^{(\sqrt{rs},n)}(|f><g|)x) \\
&= \lim_{s\uparrow 1} < fe(0), V^{(r)}(t)^* x V^{(s)}(t) ge(0) > \\
&= < fe(0), V^{(r)}(t)^* x V(t) ge(0) >
\end{aligned}
$$

for any $f, g \in \mathcal{H}_0$. Now taking limit as $r \uparrow 1$ in the above identity we get the required identity for (i) by Proposition 5.4(vi).

That $Z \in \mathcal{I}$ is simple to varify. 'Only if' part of (ii) follows from (i) and Theorem 5.5. For the converse we appeal to Proposition 4.9. This completes the proof. ∎

Now combiniting Theorem 4.11 and Theorem 5.11 we arrive at a necessary and sufficient condition for V to be co-isometric.

Corollary 5.12 : Consider the family $V\{V(t),\ t \geq 0\}$ of operators defined as in Proposition 5.6. Let the family $\{Y^*,\ Z_k,\ k \in \mathcal{S}\}$ of operators also satisfy (5.6) ,(5.7) and $\tilde{\mathcal{D}}$ be a core for Y^* so that $\mathcal{D} \subset \mathcal{D}(Z_k^*)$, $k \in \mathcal{S}$. Then the followig hold:

(i) The dual bar-cocycle $\tilde{V}$ satisfies (5.3) on $\tilde{\mathcal{D}}\underline{\otimes}\varepsilon(\mathcal{M})$ with coefficients Z

replaced by $\tilde{Z}$ where

$$\tilde{Z}^i_j = \begin{cases} (S^j_i)^* - \delta^i_j & , \quad i,j \in \mathcal{S}, \\ -\sum_{k \in \mathcal{S}} (S^k_i)^* Z_k & , \quad i \in \mathcal{S}, j = 0, \\ Z^*_j & , \quad i = 0, j \in \mathcal{S}, \\ Y^* & , \quad i = 0 = j \end{cases}$$

are defined on $\tilde{\mathcal{D}}$;

(ii) Let equality in (5.7) be valid and S be co-isometric operator then $Z \in \tilde{\mathcal{I}}$. In such a case V is co-isometric if and only if $\tilde{\beta}_\lambda = 0$ for some $\lambda > 0$, where $\tilde{\beta}_\lambda$ is defined as in Proposition 4.11.

Proof : S being a contractive operator we observe that

$$\sum_{k \in \mathcal{S}} \|L_k f\|^2 \leq \sum_{k \in \mathcal{S}} \|Z_k f\|^2$$

for each $f \in \mathcal{D}(Z_k), k \in \mathcal{S}$, where $L_k = \sum_{l \in \mathcal{S}} (S^l_k)^* Z_l$. Hence the family $\{Y^*,\ L_k,\ k \in \mathcal{S}\}$ also satisfy (5.6) and (5.7). Thus (i) is immediate from Proposition 4.11. The proof is complete once we note that V is co-isometric if and only if $\tilde{V}$ is isometric and appeal to Theorem 5.11(ii). ∎

Example 5.13 : Let $L_k,\ k \in \mathcal{S}$ be a family of closed operators in $\mathcal{H}_0$ and Y be the generator of a contractive C_0 semigroup satisfying (5.6) and (5.7). For each $k \in \mathcal{S}$ consider the polar decomposition $L_k = S_k|L_k|$, where S_k is the partial isometry with intial subspace as $\overline{\mathcal{R}(|L_k|)}$, hence $S^*_k L_k = |L_k|$. Now with $Z_k = L_k, S^k_j = \delta^k_j S_k,\ k,i \in \mathcal{S}$ define the family of operators $Z \equiv \{Z^i_j,\ i,j \in \overline{\mathcal{S}}\}$ as in (5.4) on $\mathcal{D}(Y)$. It is evident that Assumption 5.7 is valid. For more explicit example we refer to Section 7 and Section 8.

Notes and Remarks:

The notion of minimal semigroup associated with Kolmogorov backward and forward differential equations was introduced by Feller [15]. Kato [25] employed a special perturbation method to construct the minimal semigroup in the spirit of Hille-Yosida semigroup theory. Proposition 5.4 is adapted from Davies [9]. Theorem 5.5 can be regarded as an abstraction of the main result obtained in [25]. The notion of dilation in boson Fock space associated with a dynamical semigroup is introduced in [1]. In Hudson-Parthasarathy [22] the dilation was established when the dynamical semigroup is uniformly continuous. Here we used [9,25] as our guiedline to exploite the class $\mathcal{Z}_R^- \cap \tilde{\mathcal{Z}}_R^-$ of operators introduced in Section 2 and develope a theory in the spirit of semigroup theory as developed in Yosida [39]. Here our choice of the contractive operator S is restricted so that the operators Z_j^0, $j \in \mathcal{S}$ is well defined on $\mathcal{D}$. It is note clear wheather this restriction is necessary. Although the results obtain here can be applied to deal with the dilation problems considered in Section 7 and Section 8 we follow a special method outlined as in Mohari [34]. Some results on the related dilation problem may be found in Chevotarev [6]. The method employed here is different from that in [6].

6 Classical Markov Processes:

Let $(\mathcal{X}, \mathcal{F}, \wp)$ be a probability space and $X(t, \cdot)$, $t \geq 0$ be a time homogeneous Markov process with denumerable state space $\mathcal{E}$. We identify the state space $\mathcal{E}$ with a subset of natural numbers $\mathbb{Z}$. Let $P(t) \equiv \{P_{ij}(t),\ i, j \in \mathcal{E}\}$, $t \geq 0$ be the transition probability matrices defined as follows:

$$P_{ij}(t) = \wp(\omega,\ X(t,\omega) = j | X(0,\omega) = i)$$

It is evident that for all $i, k \in \mathcal{E}$ the following hold:

(a) $P_{ik}(0) = \delta^i_k$;

(b) $P_{ik}(s+t) = \sum_{j \in \mathcal{E}} P_{ij}(s) P_{jk}(t)$ for all $s, t \geq 0$;

(c) $P_{ik}(t) \geq 0$ and $\sum_{j \in \mathcal{E}} P_{ij}(t) \leq 1$ for all $t \geq 0$;

where the inequality indicates that the process may go out of the state space with positive probability. The family $P(t)$, $t \geq 0$ is said to be strictly stochastic if equality holds in (c) for all $i \in \mathcal{E}$, $t \geq 0$.

We assume the standard weak regularity property :

(d) $\lim_{t \downarrow 0} P_{ik}(t) = \delta^i_k$ for all $i, k \in \mathcal{E}$.

In such a case Doob and Kolmogorov proved the following facts:

(i) The limit $\Omega_i = \lim_{t \to 0+} (1 - P_{ii}(t))/t$ exists for all $i \in \mathcal{E}$ but it may be ∞;

(ii) The limit $\Omega_{ik} = \lim_{t \to 0+} P_{ik}(t)/t$ exists and is finite.

(iii) $\Omega_{ik} \geq 0$ for all $i \neq k$. Moreover, for each fixed i

$$\sum_{j: j \neq i} \Omega_{ij} \leq \Omega_i \tag{6.1}$$

If $\Omega_i < \infty$ for all $i \in \mathcal{E}$ and equality in (6.1) holds then $P(t)$, $t \geq 0$ also satisfies the Kolmogorov forward and backward differential equations i.e. the map $t \to P_{ik}(t)$ is continuously differentiable for each $i, k \in \mathcal{E}$ and

(e) $P'_{ik}(t) = \sum_{j\in\mathcal{E}} P_{ij}(t)\Omega_{jk}$;
(f) $P'_{ik}(t) = \sum_{j\in\mathcal{E}} \Omega_{ij}P_{jk}(t)$;
where $\Omega_{ii} = -\Omega_i$.

We say that $\Omega = ((\ \Omega_{ij}\ ))$ is a *Markov matrix* if $\Omega_{ij} \geq 0$ for all $i \neq j$ and equlity holds in (6.1) with $\Omega_i = \Omega_{ii}$ for every i.

For a given Markov matrix Ω Feller [15] proved that there exists a transition probability matrix $F(t),\ t \geq 0$ satisfying (a)-(f). Moreover if a family $P(t) \equiv \{P_{ik}(t),\ t \geq 0\}$ of positive matrices satisfies either (e) or (f), then for all $t \geq 0$ $P_{ik}(t) \geq F_{ik}(t)$. It is evident that the *minimal* solution $F(t),\ t \geq 0$ of Kolmogorov differential equations is uniquely determined. In general (a) - (f) do not determine a unique solution. For an account the reader is refered to Chung [8].

We shall quote without proof the construction of Feller's 'minimal' solution as outlined in Ledermann-Reuter [27].

Definition 6.1 : A family of matrices $\Omega \equiv \{\Omega(t) = (\Omega_{ij}(t) : i,j \in \mathcal{E}); t \geq 0\}$ is said to be *regular Markov* if the following holds:

(a) $\Omega_{ij}(t) \geq 0 (i \neq j)$; $\qquad \Omega_{ii}(t) = -\sum_{j\neq i}\Omega_{ij}(t)$;
(b) the map $t \to \Omega_{ij}(t)$ is continuous for each $i,j \in \mathcal{E}$.

For any $n \geq 1$ denote the family of finite matrices

$$\Omega^{(n)} := \{\Omega^{(n)}(t) = (\Omega_{ij}(t) : i,j \in \mathcal{E}_n); t \geq 0\}$$

and $F^{(n)} := \{F^{(n)}(s,t) = (F^{(n)}_{ij}(s,t) : i,j \in \mathcal{E}_n; 0 \leq s \leq t\}$, the unique solution of

$$\frac{\partial}{\partial t}F^{(n)}(s,t) = F^{(n)}(s,t)\Omega^{(n)}(t), F^{(n)}(s,s) = I; 0 \leq s \leq t$$

where $\mathcal{E}_n, n \geq 1$ is an increasing sequence of finite subsets of $\mathcal{E}$ such that $\mathcal{E}_n \uparrow \mathcal{E}$.

Lemma 6.2 : For all $n \geq 1, 0 \leq s \leq t < \infty, i, k \in \mathcal{E}_n$ the following holds:

$$\text{(i)} \qquad F_{ik}^{(n)}(s,s) = \delta_{ik}; \tag{6.2}$$

$$\text{(ii)} \qquad \frac{\partial}{\partial t} F_{ik}^{(n)}(s,t) = \sum_{j \in \mathcal{E}_n} F_{ij}^{(n)}(s,t)\Omega_{jk}(t); \tag{6.3}$$

$$\text{(iii)} \qquad -\frac{\partial}{\partial s} F_{ik}^{(n)}(s,t) = \sum_{j \in \mathcal{E}_n} \Omega_{ij}(s) F_{jk}^{(n)}(s,t); \tag{6.4}$$

$$\text{(iv)} \qquad F_{ik}^{(n)}(s,t) = \sum_{j \in \mathcal{E}_n} F_{ij}^{(n)}(s,r) F_{jk}^{(n)}(r,t); (s \leq r \leq t); \tag{6.5}$$

$$\text{(v)} \qquad F_{ik}^{(n)}(s,t) \geq 0 \ , \quad \sum_{j \in \mathcal{E}_n} F_{ij}^{(n)}(s,t) \leq 1; \tag{6.6}$$

$$\text{(vi)} \qquad F_{ik}^{(n+1)}(s,t) \geq F_{ik}^{(n)}(s,t); \tag{6.7}$$

(vii) If $\Omega(t) \equiv \Omega$, set $F_{ik}^{(n)}(t) = F_{ik}^{(n)}(0,t)$, then

$$F_{ik}^{(n)}(s,t) = F_{ik}^{(n)}(t-s). \tag{6.8}$$

So as $n \to \infty$, $F_{ik}^{(n)}(s,t)$ tends to a limit say $F_{ik}(s,t)$. From Lemma 6.2 we have the following theorem.

Theorem 6.3 : For any fixed $s \geq 0$. $F_{ik}(s,t)$ is absolutely continuous in t, and for any fixed $t \geq 0, F_{ik}(s,t)$ is continuously differentiable in s. For all $0 \leq s \leq t < \infty$ and $i, k \in \mathcal{E}$ the following holds:

$$\text{(i)} \qquad F_{ik}(s,s) = \delta_{ik}; \tag{6.9}$$

$$\text{(ii)} \qquad \frac{\partial}{\partial t} F_{ik}(s,t) = \sum_j F_{ij}(s,t)\Omega_{jk}(t) \tag{6.10}$$

for almost all $t \geq s$ (s held fixed);

$$\text{(iii)} \qquad -\frac{\partial}{\partial s} F_{ik}(s,t) = \sum_j \Omega_{ij}(s) F_{jk}(s,t); \tag{6.11}$$

$$\text{(iv)} \qquad F_{ik}(s,t) = \sum_j F_{ij}(s,r)F_{jk}(r,t); \tag{6.12}$$

$$\text{(v)} \qquad F_{ik}(s,t) \geq 0, \sum_j F_{ij}(s,t) \leq 1; \tag{6.13}$$

(vi) If $\Omega(t) \equiv \Omega$ as in Lemma 6.2 (vii), set $F_{ik}(t) = F_{ik}(0,t)$, then

$$F_{ik}(s,t) = F_{ik}(t-s) \tag{6.14}$$

and (6.10) is valid for all $t \geq s$.

Theorem 6.4 : If a family of matrices $P(s,t) \equiv \{P_{ik}(s,t) : i,k \in \mathcal{E}\}, 0 \leq s \leq t < \infty$ satisfies

$$P_{ik}(s,s) = \delta_{ik}; \qquad P_{ik}(s,t) \geq 0$$

and either (6.11) or (6.12) then

$$P_{ik}(s,t) \geq F_{ik}(s,t) \tag{6.15}$$

for all $0 \leq s \leq t < \infty$.

Proof : For a complete account of these results see Ledermann-Reuter [27]. ■

Consider the situation when $\Omega(t) \equiv \Omega$ and set $F_{ik}(t) : t \geq 0$ as in Theorem 6.3(vi). It is clear from (6.14) that for all $t \geq 0$

$$\sum_k F_{ik}(t) \leq 1. \tag{6.16}$$

The following theorem indicates a necessary and sufficient condition for equality in (6.16).

Theorem 6.5 For all $i \in \mathcal{E}$ and $t \geq 0$ equality holds in (6.16) if and only if $B_\lambda \equiv \{0\}$ for some $\lambda > 0$ where

$$B_\lambda \equiv \{x \geq 0, x \in \ell_\infty(\mathcal{E}) : \Omega x = \lambda x\}$$

Proof : See Feller [15]. ∎

Notes and Remarks:

In general Feller's resolvent condition for the minimal process to be strictly stochastic is difficult to varify. However in pure birth process this is equivalent to $\sum_{k\geq 0}\lambda_k^{-1} = \infty$, where $\Omega_{k,k+1} = -\Omega_{kk} = \lambda_k,\ k \geq 1$, otherwise 0. For a more explicit description of Feller's condition for birth and death processes, the reader is refered to Karlin-McGregor[24]. Also see Kato [25].

7 A class of non-commutative Markov processes :

In this section we shall deal with a class of quantum stochastic evolutions initiated by Fagnola [14]. Some resuts in this direction will be found in [7]. Here we consider a class of non-commutative Markov processes which inparticular includes the related result obtained in [14] and and improves some unsatisfactory parts in Chebotarev-Fagnola-Frigerio [7].

Fix a Markov matrix $\Omega \equiv (\Omega_{ij}; i, j \in \mathbb{Z})$ and choose complex numbers $m_{ij}(i, j \in \mathbb{Z})$ such that

$$\Omega_{ij} = \begin{cases} |m_{ij}|^2 & , \; i \neq j \\ -|m_{ii}|^2 & , \; i = j \end{cases} \tag{7.1}$$

and $\mathcal{S} \subseteq \mathbb{Z} \backslash \{0\}$ so that for all $k \in \mathbb{Z}, i \notin \overline{\mathcal{S}}$

$$m_{k,k+i} = 0.$$

So for each $i \in \mathbb{Z}, -\Omega_{ii} = \sum\limits_{j \in \mathcal{S}} \Omega_{ij}$ holds. Also fix an orthonormal basis $\{f_k : k \in \mathbb{Z}\}$ for $\mathcal{H}_0$ and denote by $\mathcal{D}$ the linear manifold generated by the basis vectors. Define unitary operators $S_i, (i \in \mathcal{S})$ and projections $\phi_k(k \in \mathbb{Z}), \Pi_n(n \geq 1)$ in $\mathcal{H}_0$ by

$$S_i f_k = f_{k+i},$$

$$\phi_k = |f_k >< f_k|, \tag{7.2}$$

$$\Pi_n = \sum_{|k| \leq n} \phi_k$$

and denote by $\mathcal{A}$ the commutative von-Neumann algebra generated by $\{\phi_k \, ; k \in \mathbb{Z}\}$. Also consider the normal operators $Z_i (i \in \mathcal{S})$ satisfying

$$Z_i f_k = m_{k,k+i} f_k.$$

Observe that for each $f \in \mathcal{D}$ there exists a constant $c(f) \geq 0$ such that

$$\sum_{i \in S} ||Z_i f||^2 \leq c(f) \tag{7.3}$$

Now consider operators $Z \equiv (Z^i_j; i, j \in \bar{S})$ defined by

$$Z^i_j \equiv \begin{cases} 0 & , \quad i, j \in S, \\ -S_i Z_i & , \quad i \in S, j = 0, \\ Z^*_j S^*_j & , \quad i = 0, j \in S, \\ -\frac{1}{2}\sum_{k \in S} Z^*_k Z_k & , \quad i = 0 = j. \end{cases} \tag{7.4}$$

Taking $Z(n)(n \geq 1)$ as in (7.4) with $Z_i (i \in S)$ replaced by $Z^{(n)}_i := Z_i \Pi_n$ a routine verification shows that the triad $(\mathcal{D}, Z, Z(n), \geq 1)$ satisfies Assumption 4.10 and $\mathcal{D}$ is a core for Z^0_0 which is the generator of a contractive C_o-semigroup. For each $n \geq 1$ set regular $(\mathcal{H}_0, \mathcal{M})$ -adapted contractive process $V^{(n)}$ as in Proposition 4.3. It is also evident that $Z \in \mathcal{I} \cap \tilde{\mathcal{I}}$. Now exploiting the results proved in Section 4 we obtain the following theorem.

Theorem 7.1 : Consider the family of operators $Z \equiv (Z^i_j,\ i, j \in \bar{S})$ defined as in (7.4). Then

(i) there exists a unique strongly continuous $(\mathcal{H}_0, \mathcal{M})$ adapted contractive evolution $V \equiv \{V(t) : t \geq 0\}$ satisfying

$$dV(t) = \sum_{i,j \in \bar{S}} V(t) Z^i_j d\Lambda^j_i(t); \quad V(0) = I$$

on $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$

(ii) V is a cocycle and for all $i, j \in \mathbb{Z}, t \geq 0$ the following holds:

(a) $< f_i, \tau_t(\phi_j) f_i >= F_{ij}(t)$

(b) $< f_i, \tilde{\tau}_t(\phi_j) f_i >= F_{ij}(t)$

where $\tau \equiv (\tau_{t \geq 0} : t \geq 0)$ and $\tilde{\tau} \equiv (\tilde{\tau}_t : t \geq 0)$ are as in (2.9) and $F(t) \equiv (F_{ij}(t) : i, j \in \mathbb{Z})$ is the minimal solution for the Markov matrix Ω.

(iii) The following statements are equivalent:

(a) $V \equiv \{V(t) : t \geq 0\}$ is isometric.

(b) $V \equiv \{V(t) : t \geq 0\}$ is co-isometric.

(c) $B_\lambda = 0$ for some $\lambda > 0$.

where $B_\lambda (\lambda > 0)$ are defined as in Theorem 6.5.

Proof : (i) is immediate from Proposition 4.3 and Proposition 4.7. For (ii) set matrices $P^{(m,n)}(t) \equiv \{P_{ij}^{(m,n)}(t) : -n \leq i, j \leq n\}; m \geq n$ defined by

$$P_{ij}^{(m,n)}(t) = < f_i e(0), V^{(m)}(t)^* \phi_j V^{(n)}(t) f_i e(0) >$$

We shall show that for each $n \geq 1$ and $m \geq n$

$$P^{(m,n)}(t) = F^{(n)}(t) \tag{7.5}$$

where $F^{(n)}(t)(t \geq 0)$ is described in Lemma 6.2 (vii). To show this first observe that (7.5) is true for $t = 0$. Quantum Ito's formula (1.4) implies that

$$P^{(m,n)}(0) = 0, \ \frac{d}{dt} P^{(m,n)}(t) = \Omega^{(n)} P^{(m,n)}(t), \ t \geq 0 \tag{7.6}$$

where $\Omega^{(n)} \equiv (\Omega_{ij} : -n \leq i, j \leq n)$.

But (7.6) admits a unique solution, so (7.5) is immediate. Proposition 4.3(vi) implies that w-$\lim_{n \to \infty} V^{(n)}(t) = V(t)(t \geq 0)$. Hence for all $t \geq 0, i, j \in \mathbb{Z}$ we have

$$\lim_{n \to \infty} F_{ij}^{(n)}(t) = \lim_{n \to \infty} \lim_{m \to \infty} P_{ij}^{(m,n)}(t) = < f_i, \tilde{\tau}_t(\phi_j) f_i >$$

Hence (b) in (ii) follows from Theorem 6.3(vi). (a) in (ii) follows by an identical method and we omit the details. For (iii) we shall show that (a) $\Longleftrightarrow$ (c), a similar method will yield (b) $\Longleftrightarrow$ (c). For (a) $\Longrightarrow$ (c), observe that $V \equiv \{V(t) : t \geq 0\}$ being an isometric process we have from (ii), for each $j \in \mathbb{Z}, t \geq 0, \ \sum_j F_{ij}(t) = 1$ Hence by Theorem 6.5 we get $B_\lambda \equiv \{0\}$ for some

$\lambda > 0$. To show the converse recall the sufficient condition for $V \equiv \{V(t) \geq 0\}$ to be isometric, described in Proposition 4.9. Let $B \in \beta_\lambda$ for some $\lambda > 0$. Denote $x \equiv (x(k) : k \in \mathbb{Z})$ defined by

$$x(k) = < f_k, Bf_k >$$

A simple computation shows that $x \in B_\lambda$. Hence by our hypothesis $x = 0, B$ being a non-negative element we have $B = 0$. Hence $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. This completes the proof. ∎

Notes and Remarks:

In [14], improving the basic inequalities concerning iterative integrals, a sufficient condition on the coefficients is obtained to guarantee the existence of a unitary evolution. In particular it successfully deals with the quantum harmonic oscillator. This section has been reproduced from Mohari [34]. It is known (see [7]) that $\{\alpha_t(\phi) := V(t)\phi V(t)^* \;\; ; t \geq \;\; 0; \phi \in \mathcal{A}\}$ is a non-commutative family of bounded operators. By Therom 7.1, α_t is an identity preserving *homomorphism if and only if $B_\lambda = 0$ for some $\lambda \; > \; 0$. For an unbounded Markov generator it is not clear whether it satisfies a diffusion equation in the sense of [12].

8 A class of commutative quantum Markov processes

As in Section 7, Ω is a Markov matrix and operators $Z_i, S_i (i \in S)$ and $\Pi_n (n \geq 1)$ are as in (7.1) - (7.3). Now consider operators $Z = (Z^i_j : i, j \in \overline{S})$ defined by

$$Z^i_j \equiv \begin{cases} (S^*_i - I)\delta_{ij} & , \quad i, j \in S, \\ -Z_i & , \quad i \in S, j = 0, \\ Z^*_j S^*_j & , \quad i = 0, j \in S, \\ -\frac{1}{2}\sum_{k \in S} Z^*_k Z_k & , \quad i = 0 = j. \end{cases} \tag{8.1}$$

Taking $Z(n)(n \geq 1)$ as in (8.1) with $Z_i(i \in S)$ replaced by $Z^{(n)}_i = Z_i \Pi_n$ a routine verification shows that Z satisfies Assumption 4.10 and $Z \in \mathcal{I} \cap \tilde{\mathcal{I}}$. Moreover $\mathcal{D}$ is a core for Z^0_0 which is the generator of a contractive C_0-semigroup. Exploiting the results proved in Section 4 we arrive at the following theorem.

Theorem 8.1 : Suppose the operators $Z \equiv (Z^i_j; i, j \in \overline{S})$ are as in (8.1). Then

(i) There exists a unique strongly continuous $(\mathcal{H}_0, \mathcal{M})$ adapted isometric evolution $V \equiv \{V(t)\} : t \geq 0\}$ satisfying

$$dV(t) = \sum_{i,j \in \overline{S}} V(t) Z^i_j d\Lambda^j_i(t); V(0) = I \tag{8.2}$$

on $\mathcal{D} \underline{\otimes} \varepsilon(\mathcal{M})$.

(ii) V is a bar-cocycle and for all $i, j \in \mathbb{Z}, t \geq 0$

$$< f_i, \tau_t(\phi_j) f_i >= F_{ij}(t)$$

where $\tau \equiv (\tau_t : t \geq 0)$ is defined as in (2.9) and $F(t) \equiv (F_{ij}(t) : i, j \in \mathbb{Z})$ is the minimal solution associated with the Markov matrix Ω.

(iii) $\eta \equiv (\eta(t) := I - V(t)V(t)^* \,; t \geq 0)$ is a strongly continuous increasing projection valued commutative adapted process.

(iv) $V \equiv \{V(t) : t \geq 0\}$ is co-isometric if and only if $B_\lambda = \{0\}$ for some $\lambda > 0$.

Proof: (i) is immediate from Proposition 4.3 and Proposition 4.7 except that V is isometric which follows once we verify the sufficient condition indicated in Proposition 4.9. To this end let $B \in \beta_\lambda$ and set $x(k) :=< f_k, Bf_k >$ $(k \in \mathbb{Z})$. B being an element in β_λ we have from (3.16)

$$\begin{aligned}\lambda x(k) &= -\frac{1}{2}|m_{kk}|^2 x(k) - \frac{1}{2}|m_{kk}|^2 x(k) + \sum_{j \in S} |m_{k,k+j}|^2 x(k) \\ &= (\sum_{i \in \overline{S}} \Omega_{ki}) x(k) = 0.\end{aligned}$$

Hence $\beta_\lambda \equiv \{0\}$ for all $\lambda > 0$. This completes the proof of (i).

(ii) follows by a similar method employed for the proof of (b) in Theorem 7.1(ii).

V being a contactive bar-cocyle η is an increasing positive operator valued process. Hence (iii) follows once we use the fact that $V(t)$ is an isometry for each $t \geq 0$.

Now for the 'only if' part in (iv) use (ii) and Theorem 6.5. For the converse recall the sufficient condition indicated in Proposition 4.11 for $V \equiv \{V(t) : t \geq 0\}$ to be co-isometric and observe that it is the same as that for $V \equiv \{V(t) : t \geq 0\}$ in Theorem 7.1 to be co-isometric. So $B_\lambda = \{0\}$ for some $\lambda > 0$ implies $\beta_\lambda \equiv \{0\}$ for some $\lambda > 0$. Hence this completes the proof of (iv). ∎

Consider the family of maps $\alpha = \{\alpha_t; t \geq \geq 0\}$ defined by

$$\alpha_t(\phi) = V(t)\phi V(t)^* \quad (\phi \in \mathcal{A}) \tag{8.3}$$

If Ω is a bounded Markov generator i.e. $\sup_i|\Omega_{ii}| < \infty$, then $\alpha = \{\alpha_t; t \geq 0\}$ is the unique family of strongly continuous identity preserving * homomorphisms satisfying:

$$d\alpha_t(\phi) = \sum_{i,j\in\overline{S}} \alpha_t(\theta^i_j(\phi))d\Lambda^j_i(t); \alpha_0(\phi) = \phi \tag{8.4}$$

on $\mathcal{H}_0 \otimes \varepsilon(\mathcal{M})$,where $\theta \equiv (\theta^i_j : i,j \in \overline{S})$ is a family of regular structure maps on $\mathcal{A}$ given by

$$\theta^i_j(\phi_k) = \begin{cases} (\phi_{k-i} - \phi_k)\delta_{ij} & , \quad i,j \in S, \\ m_{k-i,k}\phi_{k-i} - m_{k,k+i}\phi_k & , \quad i \in S, j = 0, \\ \overline{m}_{k-j,k}\phi_{k-j} - \overline{m}_{k,k+j}\phi_k & , \quad i = 0, j \in S, \\ \sum_{r\in S}|m_{k-r,k}|^2\phi_{k-r} - |m_{kk}|^2\phi_k & , \quad i = 0 = j. \end{cases} \tag{8.5}$$

Furthermore $\{\alpha_t(\phi) : t \geq 0, \phi \in \mathcal{A}\}$ is a commutative family of bounded operators. For further details we refer to Example 3.13.

Here our aim is to drop the boundedness assumption on Ω and investigate the family $\alpha \equiv \{\alpha_t : t \geq 0\}$ in detail.

By Theorem 8.1 observe that $\alpha \equiv \{\alpha_t : t \geq 0\}$ is a family of strongly continuous * homomorphisms. It preserves identity if and only if $B_\lambda \equiv \{0\}$ for some $\lambda > 0$.

In [31] the asymptotic behaviour of the induced maps $j^{(n)}_t(\phi) := V^{(n)}(t)\phi V^{(n)}(t)^*(t \geq 0, \phi \in \mathcal{A})$ as $n \to \infty$ has been investigated but it is not clear whether it approximates the process $\alpha \equiv \{\alpha_t : t \geq 0\}$ in a reasonable topology. Here we shall modify the approximating sequence to ensure it and conclude some properties of $\alpha \equiv \{\alpha_t : t \geq 0\}$ from that of the approximating sequence. In particular, we shall show the commutativity of the process and prove that the differential equation (8.4) is satisfied in weak sense. Finally with an additional hypothesis on Ω, we shall show that it satisfies (8.4) in strong sense. To this end we introduce some notations.

Define bounded operators $S_i^{(n)}(i \in S), Z(n) \equiv (Z_j^i(n) : i, j \in \overline{S})(n \geq 1)$ by

$$S_i^{(n)} = \begin{cases} S_i \Pi_{[-n} + I - \Pi_{[-n} & , \ i > 0 \\ S_i \Pi_{n]} + I - \Pi_{n]} & , \ i < 0 \end{cases}$$

where

$$\Pi_{[-n} = \sum_{k \geq -n} \phi_k, \ \ \Pi_{n]} = \sum_{k \leq n} \phi_k$$

and

$$Z_j^i(n) \equiv \begin{cases} ((S_i^{(n)})^* - I)\delta_{ij} & , \ i, j \in S, \\ -Z_i^{(n)} & , \ i \in S, j = 0, \\ (Z_j^{(n)})^*(S_j^{(n)})^* & , \ i = 0, j \in S, \\ -\frac{1}{2}\sum_{k \in S}(Z_k^{(n)})^* Z_k^{(n)} & , \ i = 0 = j. \end{cases} \tag{8.6}$$

For each $n \geq 1$ a simple computation shows that for all $i, j \in \overline{S}$

$$Z_j^i(n) + Z_i^j(n)^* + \sum_{k \in S} Z_i^k(n)^* Z_j^k(n) = \begin{cases} (S_i^{(n)}(S_i^{(n)})^* - I)\delta_{ij} & , \ i, j \in S, \\ 0 & , \ \text{otherwise.} \end{cases}$$

So for each $j \in \overline{S}$ and $f \in \mathcal{D}$ we have

$$\sum_{i \in \overline{S}} \|Z_j^i(n) f\|^2 \leq \|f\|^2 + \|Z_j^0(n) f\|^2 + 2\|f\| \ \|Z_j^j(n) f\|$$

These show that $(Z, Z(n), n \geq 1, \mathcal{D})$ satisfy (4.3) and (4.4), where Z is defined as in (8.1). Denote $C^{(n)} \equiv \{C^{(n)}(t) : t \geq 0\}$ the unique co-isometric solution of (4.1) with coefficients $Z(n)$ $(n \geq 1)$ defined as in (8.6). So by Proposition 4.3 and Theorem 8.1 we have

$$\text{s-lim}_{n \to \infty} C^{(n)}(t) = V(t)(t \geq 0). \tag{8.7}$$

Now consider the maps $\alpha^{(m,n)} = (\alpha_t^{(m,n)} : t \geq 0)m, n \geq 1$ defined by

$$\alpha_t^{(m,n)}(\phi) = C^{(m)}(t)\phi C^{(n)}(t)^*, \ \ \phi \in \mathcal{A}. \tag{8.8}$$

We also write $\alpha^{(n)}$ for $\alpha^{(n,n)}(n \geq 1)$.

A simple application of quantum Ito's formula (1.4) shows that

$$d\alpha_t^{(m,n)}(\phi) = \sum_{i,j\in\overline{S}} \alpha_t^{(m,n)}({}^{(m,n)}\mu_j^i(\phi))d\Lambda_i^j(t); \quad \alpha_0^{(m,n)}(\phi) = \phi \tag{8.9}$$

where

$$ {}^{(m,n)}\mu_j^i(\phi) = \begin{cases} (\sigma_i(\phi) - \phi)\delta_{ij} & , \ i,j \in S, \\ \sigma_i(\phi)Z_i^{(n)} - Z_i^{(m)}\phi & , \ i \in S, j = 0, \\ (Z_j^{(m)})^*\sigma_j(\phi) - \phi(Z_j^{(n)})^* & , \ i = 0, j \in S, \\ \sum_{k\in S}\{(Z_k^{(m)})^*\sigma_k(\phi)Z_k^{(n)} - \frac{1}{2}(Z_k^{(m)})^*Z_k^{(m)}\phi & \\ -\frac{1}{2}\phi(Z_k^{(n)})^*Z_k^{(n)}\} & , \ i = 0 = j \end{cases} \tag{8.10}$$

and

$$\sigma_k(\phi) = (S_k^{(m)})^*\phi S_k^{(n)}, k \in S.$$

We also write ${}^{(n)}\mu$ for ${}^{(n,n)}\mu$ for each $n \geq 1$. For $n \geq 1$ denote ${}^{(n)}\theta \equiv \{{}^{(n)}\theta_j^i : i,j \in \overline{S}\}$ the regular structure maps defined by (8.10) where $m = n$ and $\sigma_k(\phi) = S_k^*\phi S_k (k \in S)$. Some algebraic relations among these maps are listed in the following Lemma.

Lemma 8.2 : Fix any $n \geq 1$ .The following holds for all $i,j \in \overline{S}$:

(i) for $\phi \in \mathcal{A}$

$$ {}^{(n)}\mu_j^i(\phi) = \begin{cases} \Pi_{[-n} \ {}^{(n)}\theta_j^i(\phi) & ; \ i,j \geq 0, \\ \Pi_{n]} \ {}^{(n)}\theta_j^i(\phi) & ; \ \text{otherwise} \end{cases}$$

(ii) for $|k| \leq n \leq m$

$$ {}^{(n)}\mu_j^i(\phi_k) = {}^{(m,n)}\mu_j^i(\phi_k).$$

Proof : Note that for all $i \in S, n \geq 1$

(a) $S_i^{(n)}Z_i^{(n)} = S_iZ_i^{(n)}$;

(b) for $k \in \mathbb{Z}$

$$(S_i^{(n)})^* \phi_k S_i^{(n)} - \phi_k = \begin{cases} \Pi_{[-n}(\phi_{k-i} - \phi_k) & ; \; i > 0, \\ \Pi_{n]}(\phi_{k-i} - \phi_k) & ; \; i < 0; \end{cases}$$

(c) for $|k| \leq n \leq m$

$$(S_i^{(m)})^* \phi_k S_i^{(n)} = (S_i^{(n)})^* \phi_k S_i^{(n)}.$$

With these observations a routine computation implies (i) and (ii). ∎

Let $\mathcal{A}_0$ be the linear manifold generated by $\{\phi_k : k \in \mathbb{Z}\}$. So $\mathcal{A}_0$ is weakly dense in $\mathcal{A}$.

Proposition 8.3 : For any $n \geq 1$

(i) $\alpha^{(n)} = \{\alpha_t^{(n)} : t \geq 0\}$ is a family of * homomorphisms from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\Gamma_+)$ and the family $\{\alpha_t^{(n)}(\phi) : t \geq 0, \phi \in \mathcal{A}\}$ is commutative;

(ii) for $|k| \leq n \leq m, t \geq 0$

$$\alpha_t^{(n)}(\phi_k) = \alpha_t^{(m,n)}(\phi_k); \tag{8.11}$$

(iii) for $\phi \in \mathcal{A}_0, t \geq 0$

$$\text{s-lim}_{n \to \infty} \alpha_t^{(n)}(\phi) = \alpha_t(\phi); \tag{8.12}$$

(iv) $\alpha = \{\alpha_t : t \geq 0\}$ is a family of * homomorphisms from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\Gamma_+)$ and the family $\{\alpha_t(\phi) : t \geq 0, \phi \in \mathcal{A}\}$ is commutative.

Proof: Since ${}^{(n)}\theta$ is a family of regular structure maps Lemma 8.2(i) implies that ${}^{(n)}\mu$ is also a family of regular structure maps on $\mathcal{A}$. Hence (i) follows from Theorem 3.10 and Theorem 3.11.

For any fixed $f, g \in \mathcal{H}_0, u, v \in \mathcal{M}, n \geq 1$ denote $x^{(m)}(t) \equiv \{x_k^{(m)}(t) : |k| \leq n\}; t \geq 0, m \geq n$ defined by

$$x_k^{(m)}(t) = < fe(u), \alpha_t^{(m,n)}(\phi_k) ge(v) > .$$

From (8.9) we get for $m \geq n$

$$\frac{d}{dt}x^{(m)}(t) = x^{(m)}(t)\Omega^{(n)}(t)(t \geq 0) \tag{8.13}$$

where $\Omega^{(n)}(t) = \{\Omega_{ij}(t) : -n \leq i, j \leq n\}; t \geq 0$ are defined by

$$\Omega_{ij}(t) = \begin{cases} (u_{j-i}(t) + m_{ij})(v^{j-i}(t) + \overline{m}_{ij}) & , \quad i \neq j, \\ -\sum\limits_{r \neq i} \Omega_{ir}(t) & , \quad i = j. \end{cases}$$

Also observe that $x^{(m)}(0)$ is independent of $m \geq n$. Since (8.13) admits a unique solution we have for all $m \geq n \geq |k|, f, g \in \mathcal{H}_0, u, v \in \mathcal{M}$ and $t \geq 0$

$$< fe(u), \alpha_t^{(n)}(\phi_k)ge(v) >=< fe(u), \alpha_t^{(m,n)}(\phi_k)ge(v) > .$$

Now a standard argument implies (ii). For (iii) it is enough to show (8.12) for $\phi = \phi_k, k \in \mathbb{Z}$. From (8.7) and (8.11) we have for each $n \geq |k|$

$$\alpha_t^{(n)}(\phi_k) = \text{w-lim}_{m \to \infty} \alpha_t^{(m,n)}(\phi_k) = V(t)\phi_k C^{(n)}(t)^*(t \geq 0). \tag{8.14}$$

Hence we get applying (8.7) once more in (8.14)

$$\text{w-lim}_{n \to \infty} \alpha_t^{(n)}(\phi_k) = \alpha_t(\phi_k) \quad (t \geq 0).$$

Since $\alpha_t^{(n)} : n \geq 1$ and $\alpha_t(t \geq 0)$ are * homomorphisms, (8.12) follows. This completes the proof of (iii). For (iv) use (i) and (ii) to show that $\{\alpha_t(\phi) : t \geq 0, \phi \in \mathcal{A}_0\}$ is a commutative family. Since $\mathcal{A}_0$ is strongly dense in $\mathcal{A}$, (iv) follows by a standard approximation argument. ∎

We shall show that $\alpha = \{\alpha_t : t \geq 0\}$ is indeed, a quantum analogue of Feller's minimal solution. To this end we introduce a few notations. For any fixed $u \in \mathcal{M}_c$ consider the family of matrices $P(s,t) \equiv \{P_{ij}(s,t) : -\infty < i, j < \infty\}, P^{(n)}(s,t) \equiv \{P_{ij}^{(n)}(s,t) : -n \leq i, j \leq n\}; 0 \leq s \leq t$ and $n \geq 1, \Omega^{(n)}(t) = \{\Omega_{ij}(t) : n \leq i, j \leq n\}; t \geq 0$ where

$$P_{ij}^{(n)}(s,t) =< f_i e(u), C^{(n)}(s)^* C^{(n)}(t)\phi_j C^{(n)}(t)^* C^{(n)}(s) f_i e(u) > ||e(u)||^{-2}$$

$$P_{ij}(s,t) =< f_i e(u), V(s)^* V(t)\phi_j V(t)^* V(s) f_i e(u) > \|e(u)\|^{-2}$$

$$\Omega_{ij}(t) = \begin{cases} |m_{ij} + u_{j-i}(t)|^2 & , \ i \neq j \\ -\sum_{k \neq i} \Omega_{ik}(t) & , \ i = j. \end{cases}$$

Proposition 8.4 : For any fixed $u \in \mathcal{M}_c$ the following holds :

(i) $\lim_{n\to\infty} P_{ij}^{(n)}(s,t) = P_{ij}(s,t) \quad (0 \le s \le t < \infty)$;

(ii) $\{P_{ij}(s,t),\ 0 \le s \le t < \infty\}$ is the minimal solution described as in Theorem 6.3 associated with the family of Markov regular matrices $\Omega \equiv \{\Omega(t) = (\Omega_{ij}(t),\ t \ge 0)\}$;

Proof : (i) follows from (8.7) and (8.12). Using quantum Ito's formula (1.4) we have for $0 \le s \le t < \infty$ and $n \ge 1$

$$\frac{\partial}{\partial t} P^{(n)}(s,t) = P^{(n)}(s,t)\Omega^{(n)}(t). \tag{8.15}$$

Since (8.15) admits a unique solution, we have for any $i,j \in \mathbb{Z}$ and $n \ge \max(|i|,|j|)$

$$P_{ij}^{(n)}(s,t) = \sum_{|k|\le n} P_{ik}^{(n)}(s,s) F_{kj}^{(n)}(s;t) \tag{8.16}$$

where $F^{(n)}(s,t)$ is the unique solution of (6.3). Now taking limit as $n \to \infty$ in (8.16) we get for all $i,j \in \mathbb{Z}$

$$P_{ij}(s,t) = F_{ij}(s,t)$$

where (6.7) and (6.13) have been used to employ dominated convergence theorem. Hence (ii) follows by Theorem 6.3. ■

For the rest of this section we shall impose the following hypothesis on the Markov matrix Ω:

$(\mathcal{H})$ for each $j \in \mathbb{Z}, \sup_i \Omega_{ij} < \infty$.

Observe that for Ω satisfying $(\mathcal{H}), \theta \equiv \{\theta^i_j : i, j \in \bar{S}\}$ described as in (8.5) possesses the property that each θ^i_j maps $\mathcal{A}_0$ into $\mathcal{A}$. Furthermore we have the following Lemma.

Lemma 8.5 : Let $(\mathcal{H})$ be valid. Then for $\phi \in \mathcal{A}_0$ the following holds:

(i)

$$\sum_i \theta^i_j(\phi)^* \theta^i_j(\phi) \tag{8.17}$$

is convergent in strong operator topology for $j \in \overline{S}$.

(ii)

$$\text{w-lim}_{n\to\infty} \alpha^{(n)}_t({}^{(n)}\mu^i_j(\phi)) = \alpha_t(\theta^i_j(\phi)) \tag{8.18}$$

for $t \geq 0,\ i, j \in \overline{S}$.

Proof : In view of Lemma 1.9 to show (i) it is enough to verify (8.17) for $\phi = \phi_k,\ k \in \mathbb{Z}$. For $j \in S$ (8.17) is always valid since only finitely many terms are non-zero. For $j = 0, i \in S$ we have

$$\theta^i_0(\phi_k)^* \theta^i_0(\phi_k) = \Omega_{k-i,k}\phi_{k-i} + \Omega_{k,k+i}\phi_k,$$

so for each $f \in \mathcal{H}_0$

$$\sum_{i \in \overline{S}} \|\theta^i_0(\phi_k) f\|^2 \leq 2(|\Omega_{kk}| + \sup_{i:i\neq k} \Omega_{ik})\|f\|^2.$$

Hence this completes the proof of (i). For (ii) note that it suffices to verify (8.18) for $\phi = \phi_k,\ k \in \mathbb{Z}$. For $(i, j) \neq (0, 0)$,${}^{(n)}\mu^i_j(\phi_k)$ being equal to $\theta^i_j(\phi_k)$ for sufficiently large n, (8.18) follows from (8.12). Proof of (ii) will be complete once we verify (8.18) for $i = 0 = j$. To show this observe the following:

(a) ${}^{(n)}\mu^0_0(\phi_k)$ being an element in the linear span of $\{\phi_r : |r| \leq n\}$ (8.14) implies that

$$\alpha^{(n)}_t({}^{(n)}\mu^0_0(\phi_k)) = V(t)({}^{(n)}\mu^0_0(\phi_k))C^{(n)}(t)^*(t \geq 0)$$

(b) ${}^{(n)}\mu_0^0(\phi_k) : n \geq |k|$ is a sequence of self-adjoint operators and

$$\text{s-lim}^{(n)}\mu_0^0(\phi_k) = \theta_0^0(\phi_k)\,.$$

A standard argument coupled with these observations and (8.7) lead us to the required result. This completes the proof.

∎

Theorem 8.6 : Consider the family of maps $\alpha \equiv (\alpha_t : t \geq 0)$ defined as in (8.3). Then the following holds:

(i) $\alpha_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma_+); t \geq 0$ is a family of strongly continuous * homomorphisms and $\{\alpha_t(\phi) : t \geq 0, \phi \in \mathcal{A}\}$ is a commutative family of bounded operators;

(ii) α is identity preserving if and only if $B_\lambda = \{0\}$ for some $\lambda > 0$;

(iii) If $(\mathcal{H})$ holds then for all $\phi \in \mathcal{A}_0$

$$\alpha_0(\phi) = \phi, d\alpha_t(\phi) = \sum_{i,j\in\overline{S}} \alpha_t(\theta_j^i(\phi))d\Lambda_i^j(t) \quad (t \geq 0) \tag{8.19}$$

holds on $\mathcal{H}_0 \underline{\otimes} \varepsilon(\mathcal{M})$;

(iv) For any $f \in \mathcal{H}_0, u \in \mathcal{M}, t \geq 0, \phi \geq 0$ and a positivity preserving bounded process $j = \{j_t(\phi) : t \geq 0, \phi \in \mathcal{A}\}$ satisfying (8.19) the following inequality holds:

$$< fe(u), j_t(\phi)fe(u) > \geq < fe(u), \alpha_t(\phi)fe(u) > .$$

Proof: By Proposition 4.11(ii) observe that V^* is strongly continuous, hence Theorem 8.1 implies the first part of (i). For the rest of (i) appeal to Proposition 8.3(iv). Note that (ii) follows from Theorem 8.1(iv).

First observe that for all $f, g \in \mathcal{H}_0, u, v \in \mathcal{M}, \phi \in \mathcal{A}_0$ and $t \geq 0$
$< fe(u), \alpha_t(\phi)ge(v) >= \lim_{n\to\infty} < fe(u), \alpha_t^{(n)}(\phi)ge(v) >$

$$=< fe(u), \phi ge(v) > + \sum_{i,j\in\overline{S}} \lim_{n\to\infty} \int_0^t ds u_i(s) v^j(s) < fe(u), \alpha_s^{(n)}({}^{(n)}\mu_j^i(\phi)) ge(v) >$$

$$=< fe(u), \phi ge(v) > + \sum_{i,j\in\overline{S}} \int_0^t ds u_i(s) v^j(s) < fe(u), \alpha_s(\theta_j^i(\phi)) ge(v) >$$

where (8.12), $u, v \in \mathcal{M}$ and (8.18) have been used in the first, second and last equality respectively. Now for (iii) it is enough to show for each $\phi \in \mathcal{A}_0, \{\alpha_t(\theta_j^i(\phi))\} \in \mathbb{L}(\mathcal{H}_0, \mathcal{M})$. Adaptedness of the processes is clear from Theorem 8.1(i) and for each $\phi \in \mathcal{A}_0, j \in \overline{S}, \alpha_t$ being a homomorphism we get from (8.17)

$$\sum_{i\in\overline{S}} \alpha_t(\theta_j^i(\phi))^* \alpha_t(\theta_j^i(\phi)) = \alpha_t(\sum_{i\in\overline{S}} \theta_j^i(\phi)^* \theta_j^i(\phi)). \tag{8.20}$$

where the series converge in strong operator topology. α_t being a contractive map for each $t \geq 0$, we get the required result from (8.20). This completes the proof of (iii).

For (iv) we need to show for each $f \in \mathcal{H}_0, u \in \mathcal{M}$ and $|k| \leq n$

$$y_k(t) \geq x_k^{(n)}(t) \quad (t \geq 0)$$

where

$$y_k(t) =< fe(u), j_t(\phi_k) fe(u) >$$

and

$$x_k^{(n)}(t) =< fe(u), \alpha_t^{(n)}(\phi_k) fe(u) > .$$

Fix any $n \geq 1$ and observe by our assumption on $j \equiv \{j_t : t \geq 0\}$

$$\frac{d}{dt} y^{(n)}(t) = y^{(n)}(t) \Omega^{(n)}(t) + z^{(n)}(t) (t \geq 0) \tag{8.21}$$

where $y^{(n)}(t) = \{y_k(t) : -n \leq k \leq n\}$ and $z^{(n)}(t) = \{z_k^{(n)}(t) : -n \leq k \leq n\}$ is given by

$$z_k^{(n)}(t) = \sum_{|j|>n} y_j(t) \Omega_{jk}(t) \quad (t \geq 0)$$

and $z^{(n)}(t) \geq 0$. Also note that $x^{(n)}(t) = \{x_k^{(n)}(t) : -n \leq k \leq n\}$ is the unique solution of (8.21) where $z^{(n)}(t) \equiv 0$. With these observations we get the required inequality by integrating the differential equation. This completes the proof. ∎.

Notes and Remarks:

The programme was initiated by Meyer [29], where it was indicated how to realise Markov a chain as a QS flow. Parthasarathy-Sinha [37] proved that the QS flow restricted to a suitable commutative algebra is indeed a commutative prosses. This theory has been subsequently generalised in Mohari-Sinha [33] to deal with the dilation problem associated with a countable state Markov process having a bounded generator. Fagnola [13] initiated the programe when the Markov generator is unbounded and deal with quantum stochastic evolution associated with pure birth (pure death) process. (See Theorem 8.1). In Mohari-Parthasarathy [31] the theory has been extended in the context of a more general state space and a class of unbounded Markov generators. The present exposition is reproduced from Mohari [34]. In analogy with the classical Feller minimal process, we expect an operator inequality in Theorem 8.6(iv). However, with an additional assumption on j_t, namely if for all $i \neq j$ and $u, v \in \mathcal{M}$ $< f_i e(u), j_t(\phi) f_j e(v) >= 0$ for all $t \geq 0$ then

$$j_t(\phi) \geq \alpha_t(\phi)$$

whenever $\phi \geq 0$. It remains an open question whether Feller's condition is also sufficient for the existence of a unique positivity preserving contractive flow satisfying (8.19).

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