Oscillating Multipliers And Bochner-Riesz Means

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Contents

Introduction

Suppose P is a differential operator of degree d on a Riemannian manifold M, which is self adjoint and formally non-negative. Let

$$Pf = \int_0^\infty \lambda dE_\lambda f$$

be the spectral resolution of P. Given a bounded function $m(\lambda)$ we can define the operator $m(P)$ by

$$m(P) = \int_0^\infty m(\lambda) dE_\lambda.$$

Such operators are always bounded on $L^2(M)$. However, some smoothness assumptions are needed on $m(\lambda)$ to ensure that $m(P) : L^p(M) \to L^p(M)$ is bounded for $p \neq 2$. It is a basic problem in Harmonic Analysis to find sufficient conditions on m so the the operator $m(P)$ will be bounded on $L^p(M)$. There is a universal multiplier theorem due to Stein [36], which guarantees that $m(P)$ is bounded on $L^p(M)$, $1 < p < \infty$. His condition on $m(\lambda)$ requires that $m(\lambda)$ is in the symbol class $S_1^0(R)$.

Recall that the symbol classes $S_\rho^\alpha(R)$, $\alpha \in R, 0 \leq \rho \leq 1$ are defined to be the class of all C^∞ functions on R which satisfy the estimates

$$|m^{(j)}(\lambda)| \leq C_j(1 + |\lambda|)^{\alpha - \rho j}$$

for $j = 0, 1, \ldots$. Sharp results under weaker regularity assumptions are known in many particular cases. When $P = -\Delta$ on R^n the corresponding result is the classical theorem of Marcinkiewicz-Mihlin-Hormander and one requires the above estimate to hold only for $j = 0, 1, \ldots N$ where N is the smallest integer bigger than $\frac{n}{2}$. The case of compact Riemannian manifolds has been studied by Seeger and Sogge [34]. In most of the particular cases optimal conditions on $m(\lambda)$ which guarantee boundedness of $m(P)$ on a given L^p have been obtained.

Operators of the form $m(P)$ with m coming from $S_\rho^{-\alpha}(R), 0 \leq \rho < 1$ are also important as they occur naturally in applications. For example, the solution to the Cauchy problem

$$\partial_t^2 u(x,t) + P\, u(x,t) = 0, u(x,0) = 0, \partial_t u(x,0) = f(x)$$

is given by

$$u(x,t) = \frac{\sin t\sqrt{P}}{\sqrt{P}} f(x)$$

and the function $m(\lambda) = \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}$ comes from the symbol class $S_{\frac{1}{2}}^{-\frac{1}{2}}(R)$. The boundedness properties of this operator have been investigated in various contexts.

When $P = -\Delta$ on R^n, Miyachi [19] and Peral [30] have shown that $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ is bounded on $L^p(R^n)$ for $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$. The case of the sublaplacian on the Heisenberg group H^n has been recently settled by Muller and Stein [25]. Operators of the form $P = -\Delta + V(x)$ where V is a non negative potential have been studied in the thesis of Zhong [49]. More generally, multipliers of the form

$$m_{\alpha\beta}(\lambda) = |\lambda|^{-\beta} e^{i|\lambda|^\alpha} \psi(\lambda), \qquad \mathrm{Re}\beta \geq 0,\ \alpha \geq 0$$

where ψ denotes a $C^\infty(\mathbb{R})$ function which vanishes for $|\lambda| \leq \frac{1}{2}$ and equals 1 if $|\lambda| \geq 1$ have attracted much interest. For the Euclidean case see the works of Hirschman [9] , Wainger [48] , Miyachi [20] , Schonbek [33] and others. Multipliers of the above type on non-compact symmetric spaces have been studied by Giulini and Meda [6]. Results for the sublaplacian on stratified groups have been obtained by Mauceri and Meda [17].

In this thesis we are mainly interested in the case of the sublaplacian $\mathcal{L}$ on the Heisenberg group H^n and operators related to $\mathcal{L}$ such as the twisted sublaplacian (also known as the special Hermite operator) L on $\mathbb{C}^n$ and the Hermite operator H on $\mathbb{R}^n$. We prove optimal results for multipliers of these operators by making use of the explicit spectral theory of these operators. Another tool we use is a general multiplier theorem for operators P for which the associated Bochner-Riesz kernel $s_R^\delta(x,y)$ satisfies a good pointwise estimate for large values of δ.

We first recall the definition of the Bochner- Riesz means associated to P. These means are defined for $\delta \geq 0$ by the equation

$$S_R^\delta f = \int_0^R (1-\frac{\lambda}{R})^\delta \, dE_\lambda f.$$

Let $s_R^\delta(x,y)$ be the kernel of S_R^δ defined by the equation

$$S_R^\delta f(x) = \int s_R^\delta(x,y)\, f(y)\, dy.$$

We consider operators P for which the kernels $s_R^\delta(x,y)$ satisfy an estimate of the form

$$|s_R^\delta(x,y)| \leq \; CR^{\frac{n}{d}}(1+R^{\frac{1}{d}}|x-y|)^{-a\delta+\beta}$$

where $a > 0$ and β are fixed constants.

Estimates of the above type are known in various special cases such as the Laplacian $-\Delta$ on R^n, the Hermite operator $H = -\Delta + |x|^2$ on R^n, the special Hermite operator $L = -\Delta + \frac{1}{4}|z|^2 - i\sum(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j})$ on C^n and so on. Another class of operators for which pointwise estimates for the Bochner-Riesz kernels are known is the class of Rockland operators on stratified nilpotent Lie groups. If L is a Rockland operator of homogeneous degree d on a stratified nilpotent group G, then the estimate

$$|s_R^\delta(x)| \leq CR^{\frac{Q}{d}}(1+R^{\frac{1}{d}}|x|)^{-\frac{\delta}{3}+\beta}.$$

has been proved in Hulanicki [11] . Here Q is the homogeneous dimension of G.

Using a heat kernel estimate proved in [2] and modifying a method in Hebisch [7] one can prove the following. If L is a Rockland operator of homogeneous degree 2, then the Riesz kernel associated to L satisfies

$$|s_R^\delta(x)| \leq CR^{\frac{Q}{2}}(1+R^{\frac{1}{2}}|x|)^{-\delta+\beta}.$$

In [16] Mauceri studied operators of the form $p(iT, \mathcal{L})$ on the Heisenberg group H^n where $T = \partial_t$, $\mathcal{L}$ is the sublaplacian and p is a homogeneous polynomial of degree d with certain properties. We remark that they fall under the category of Rockland operators. Among other things he has proved that the Riesz kernel satisfies the estimate

$$|s_R^\delta(g)| \leq CR^{\frac{Q}{d}}(1 + R^{\frac{1}{d}}|g|)^{-\delta-1}.$$

In the first chapter we study multipliers of the form $m(P)$ for m coming from $S_\rho^{-\alpha}$ and P is a differential operator whose associated Bochner-Riesz kernel satisfies an estimate of the above type.

In the second chapter we specialise to certain eigenfunction expansions and study some multipliers in detail. First we consider the special Hermite operator L on $\mathbb{C}^n$. We remark that L is related to the sublaplacian on the Heisenberg group H^n . The Heisenberg group H^n is the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, with the group operation

$$(z,t).(w,s) = (z + w, t + s + \frac{1}{2} Im z.\overline{w}).$$

The sublaplacian $\mathcal{L}$ is explicitly given by

$$\mathcal{L} = -\Delta - \frac{1}{4}|z|^2 \, \partial_t^2 - N\partial_t$$

where N is the rotation operator $\sum_{j=1}^n (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j})$. The special Hermite operator L and the sublaplacian $\mathcal{L}$ are related by $\mathcal{L}(e^{it}f(z)) = e^{it}Lf(z)$. For this reason L is called the twisted Laplacian. Spectral decomposition associated to L is given by the special Hermite expansions, namely

$$Lf(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k + n) f \times \varphi_k(z).$$

Here $\varphi_k(z)$ are the Laguerre functions of type $(n-1)$,

$$\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2},$$

and the twisted convolution $f \times g$ of two functions on $\mathbb{C}^n$ is given by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}Im z.\overline{w}}dw.$$

We remark that this is related to the group convolution on H^n.

Given a bounded function m on $\mathbb{R}$ define the operator

$$m(L)f = (2\pi)^{-n}\sum_{k=0}^{\infty} m(2k+n)f \times \varphi_k.$$

Multipliers for the special Hermite expansions have been extensively studied in recent times, see the works [44] and [46] and the references given there. Consider the multipliers given by

$$m_\alpha(\lambda) = \sqrt{t}\ \lambda^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{\lambda})$$

Observe that when $\alpha = \frac{1}{2}$

$$m_\alpha(\lambda) = \sqrt{\frac{2}{\pi}}\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}$$

and so, $\sqrt{\frac{\pi}{2}}\ t^{\frac{1}{2}}L^{-\frac{1}{4}}J_{\frac{1}{2}}(t\sqrt{L})f(z) = u(z,t)$ solves the wave equation

$$(\partial_t^2 + L)u(z,t) = 0, u(z,0) = 0, \partial_t u(z,0) = f(z).$$

The function m_α belongs to $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{1}{4}}(\mathbb{R})$ and so the above mentioned general multiplier theorem will show that $L^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{L}) : L^p(\mathbb{C}^n) \to L^p(\mathbb{C}^n)$ is bounded for $\alpha > 2n|\frac{1}{p}-\frac{1}{2}| - \frac{1}{2}$. However, this result can be improved to show that the operators $L^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{L})$ satisfy the estimates

$$||L^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{L})f||_p \leq C_t||f||_p \quad , f \in L^p(\mathbb{C}^n)$$

for $\alpha > (2n-1)|\frac{1}{p}-\frac{1}{2}| - \frac{1}{2}$ *and* $1 \leq p \leq \infty$.

As a corollary we obtain the boundedness of the solution to the wave equation associated to the special Hermite operator. The solution $u(z,t) = \frac{\sin t\sqrt{L}}{\sqrt{L}}f(z)$ of the Cauchy problem

$$(\partial_t^2 + L)u(z,t) = 0, u(z,0) = 0, \partial_t u(z,0) = f(z).$$

satisfies $||u(.,t)||_p \leq C_t||f||_p$ provided $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n-1}$.

The above result will be proved by studying an analytic family of operators. Let

$$\psi_k^\alpha(\lambda) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(\frac{1}{2}\lambda^2)e^{-\frac{1}{4}\lambda^2}$$

be the Laguerre functions of type α defined for all Re $\alpha > -\frac{1}{2}$. Consider the family

$$T_t{}^\alpha f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \psi_k^\alpha(t)\ f \times \varphi_k(z).$$

We will use the facts that $T_t{}^\alpha : L^1(\mathbb{C}^n) \to L^1(\mathbb{C}^n)$ is bounded if Re $\alpha > n-1$ and $T_t{}^\alpha : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is bounded if Re $\alpha > -\frac{1}{2}$. Analytic interpolation will prove that $T_t{}^\alpha : L^p \to L^p$ if $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Using a Hilb type asymptotic formula for the Laguerre functions $\psi_k^\alpha(t)$, we will compare the operators $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ and $T_t{}^\alpha$ which will complete the proof.

The above results have analogues for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. We remark that H and the sublaplacian $\mathcal{L}$ on the Heisenberg group are related via the group Fourier transform. Recall that for each $\lambda \in \mathbb{R}$ there is an irreducible unitary representation π_λ of H^n realised on $L^2(\mathbb{R}^n)$ which is explicitly given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x.\xi + \frac{1}{2}x.y)}\phi(\xi + y),$$

$\phi \in L^2(\mathbb{R}^n)$. The group Fourier transform of a function $f \in L^1(H^n)$ is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{H^n} \pi_\lambda(z,t) f(z,t)dzdt.$$

Then it is known that (see [46])

$$(\hat{\mathcal{L}f})(\lambda) = \hat{f}(\lambda)H(\lambda)$$

where $H(\lambda) = -\Delta + \lambda^2|x|^2$. In particular $(\hat{\mathcal{L}f})(1) = \hat{f}(1)H$.

The spectral decomposition of H is given by the Hermite expansions. Let $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^n$ be the normalised Hermite functions which are eigenfunctions of H with eigenvalues $(2|\alpha| + n)$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $P_k f$ be the projections defined by

$$P_k f(x) = \sum_{|\alpha|=k} (f, \Phi_\alpha)\Phi_\alpha(x).$$

Then the spectral decomposition of H is given by

$$Hf = \sum_{k=0}^{\infty} (2k+n)\ P_k f.$$

For various properties of the Hermite expansions we refer to the monograph of Thangavelu [44].

We start with a study of the following analytic family of operators defined by

$$S_t^\alpha f(x) = \sum_{k=0}^{\infty} \psi_k^\alpha(t) P_k f(x).$$

The operators S_t^α and $T_t{}^\alpha$ are related to each other via the Weyl transform. The representation π_1 of H^n defines a projective representation π of $\mathbb{C}^n$ by the prescription $\pi(z) = \pi_1(z, 0)$. The integrated representation

$$W(f) = \int_{\mathbb{C}^n} \pi(z)\ f(z)\ dz\ .$$

is then called the Weyl transform, which takes functions f on $\mathbb{C}^n$ into bounded operators acting on $L^2(\mathbb{R}^n)$. For $f \in L^1(\mathbb{C}^n)$ we have the relation

$$W(T_t{}^\alpha f) = S_t^\alpha W(f)\ .$$

Using this connection we will prove prove that S_t^α is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ whenever $\alpha > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.

Using the explicit expressions obtained for the kernels of the S_t^α we study the maximal operator $\sup_{0<t\le 1} |S_t^\alpha f|$ and improve the almost everywhere convergence obtained in [32].

As before comparing $H^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{H})$ with S_t^α we obtain corresponding results for the operators $H^{-\frac{\alpha}{2}}J_\alpha(t\sqrt{H})$ and by taking $\alpha = \frac{1}{2}$ we get the estimate $||u(.,t)||_p \leq C_t \; ||f||_p$ provided $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n}$, for the solution $u(x,t)$ of the Cauchy problem:

$$(\partial_t^2 + H)u(x,t) = 0, u(x,0) = 0, \; \partial_t u(x,0) = f(x).$$

Unlike the case of L or the standard Laplacian $-\Delta$, the above is the best one can get. That is, the above estimate for the solution $u(x,t)$ to the wave equation cannot be extended to the bigger range $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$. This has already been observed in [49] where such estimates for the operators $-\Delta + V(x)$ have been obtained. However, if we consider only radial functions it is possible to improve the above result, see Theorem 2.2.7.

In the third chapter we take up a study of multipliers associated to the sublaplacian $\mathcal{L}$ on the Heisenberg group. Boundedness of $m(\mathcal{L})$ has been studied by several authors. See the works [13], [15], [17] and [43]. The optimal result has been proved in Muller-Stein [24] and Hebisch [8]. Note that $\mathcal{L}$ and the operator $T = \frac{\partial}{\partial t}$ commute with each other and so they admit a joint spectral decomposition which can be written down explicitly. Define

$$e_k^\lambda(z,t) = e^{i\lambda t}\varphi_k^\lambda(z) = e^{i\lambda t}\varphi_k(\sqrt{|\lambda|}z)$$

for $\lambda \in I\!R$, $\lambda \neq 0$. Then $e_k^\lambda(z,t)$ are joint eigenfunctions of $\mathcal{L}$ and T :

$$\mathcal{L}e_k^\lambda(z,t) = (2k+n)|\lambda|e_k^\lambda(z,t) \; , \;\; Te_k^\lambda(z,t) = i\lambda e_k^\lambda(z,t).$$

The explicit spectral decomposition of $\mathcal{L}$ and T studied in great details by Strichartz [40] and [41] is then written as

$$f(z,t) = c_n \int_{-\infty}^{\infty} (\sum_{k=0}^{\infty} \; f * e_k^\lambda(z,t)) \; |\lambda|^n \; d\lambda.$$

Given a bounded function $m(\xi, \eta)$ of two variables we can consider the operator

$$Mf(z,t) = c_n \int_{-\infty}^{\infty} \; (\sum_{k=0}^{\infty} \; m(k,\lambda) \; f * e_k^\lambda(z,t)) \; |\lambda|^n \; d\lambda.$$

One can naturally ask for conditions on $m(k, \lambda)$ so that M extends to a bounded operator on $L^p(H^n)$.

Recently this problem has received considerable attention. In the papers[22] and [23] Muller, Ricci and Stein have obtained sufficient conditions on $m(\xi, \eta)$ so that M is bounded on $L^p(H^n)$. More precisely, if $m(\xi, \eta)$ satisfies the Marcinkiewicz type conditions

$$|(\xi\partial_\xi)^\alpha \ (\eta\partial_\eta)^\beta \ m(\xi, \eta)| \leq C_{\alpha,\beta}$$

for sufficiently many derivatives, then M is bounded on $L^p(H^n)$ $1 < p < \infty$. In [23] the authors have obtained a sharp Marcinkiewicz multiplier theorem where the above conditions are required to hold only for an optimal number of derivatives.

When $m(k, \lambda) = m((2k+n)|\lambda|)$ the operator$M$ is nothing but $m(\mathcal{L})$ and Marcinkiewicz conditions hold when $m \in S_1^0(I\!R)$. In the general case, when $m \in S_1^0(I\!R^2)$ the corresponding operator M is bounded on $L^p(H^n)$, $1 < p < \infty$ as is proved in [23]. Boundedness properties of the solution to the wave equation associated to $\mathcal{L}$ have been studied recently by Stein and Muller, where they have proved that for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$, the operator $\frac{\sin s\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}}$ is bounded on $L^p(H^n)$. Our aim is to improve this result to the bigger range $|\frac{1}{p} - \frac{1}{p}| < \frac{1}{2n-1}$ in the case when f is band limited in the central variable. We also explain a method to obtain pointwise estimates on Bochner-Riesz kernels using estimates on heat kernels.

Last chapter is devoted to a comparative study of Bochner-Riesz means associated to the Hermite and Fourier expansions. Using a recent result of Stempak and Zienkiewicz we also study boundedness of Bochner-Riesz means associated to Hermite expansions for polyradial functions on $I\!R^{2n}$. Recall that the Bochner- Riesz means associated to the Fourier transform on $I\!R^n$ are defined by

$$S_t^\delta f(x) = (2\pi)^{-n/2} \int_{|y|\leq t} e^{ix\cdot y} \left(1 - \frac{|y|^2}{t^2}\right)^\delta \hat{f}(y) \, dy$$

where

$$\hat{f}(y) = (2\pi)^{-n/2} \int_{I\!R^n} e^{-ix.y} f(x) dx$$

is the Fourier transform on $I\!R^n$. The Bochner-Riesz means for of Hermite expansions are given by

$$S_R^\delta f(x) = \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f(x).$$

For the properties of Hermite functions and related results see [44].

In our study of the Bochner-Riesz means associated to Hermite and special Hermite expansions we make use of a transplantation theorem of Kenig-Stanton -Tomas [12]. From the transplantation theorem it follows that local results on boundedness of Bochner-Riesz means associated to Hermite expansions imply global results for the Bochner-Riesz means associated to the Fourier transform on the Euclidean space. At this point a natural question arises, to what extent the converse is true?. We answer this question in the affirmative in dimensions one and two and partially in higher dimensions. We also study the equi-summability of the special Hermite expansions. In this case we show that the local uniform boundedness of the Bochner-Riesz means for the special Hermite operator is equivalent to the uniform boundedness of S_t^δ on $I\!R^{2n}$.

Chapter 1

Bochner-Riesz kernels and Multipliers

In this chapter we study multipliers associated to a class of differential operators on $\mathbb{R}^n$. Let P be a self-adjoint non-negative differential operator on $\mathbb{R}^n$ of degree d. Let

$$P = \int_0^{\infty} \lambda dE_\lambda$$

be its spectral resolution. If m is a bounded function on $\mathbb{R}$ then $m(P)$ will stand for the operator defined by $\int_0^\infty m(\lambda)\, dE_\lambda$. Clearly $m(P)$ is bounded on L^2. We study boundedness properties of $m(P)$ on L^p spaces for m coming from the symbol classes $S_\rho^\alpha(\mathbb{R})$.

1.1 A General Multiplier Theorem

We assume that Bochner-Riesz kernels $s_R^\delta(x,y)$ associated to P satisfy estimates of the form

$$|s_R^\delta(x,y)| \leq CR^{\frac{n}{d}} \left(1 + R^{\frac{1}{d}}|x-y|\right)^{-a\delta+\beta} \qquad (1.1.1)$$

for some constants $a > 0$ and β, for all large δ. For this class of operators we have the following.

Theorem 1.1.1 *Let $m \in S_\rho^{-\alpha}(\mathbb{R})$ $0 \le \rho \le 1$ and $1 < p < \infty$. Assume that the spectral measure of P has no mass at the origin. If the Bochner-Riesz kernel $s_R^\delta(x,y)$ associated to P satisfies the estimates (1.1.1) then $m(P) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded whenever $\alpha > \frac{n(1-\rho)}{a}|\frac{1}{p} - \frac{1}{2}|$.*

For proving Theorem 1.1.1 we start with a simple proposition.

Proposition 1.1.2 *Let m be a smooth compactly supported function on $\mathbb{R}$ and $1 \le p \le \infty$. If the Bochner-Riesz kernel $s_R^\delta(x,y)$ associated to P satisfies the estimates (1.1.1) then $m(P) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.*

Proof: Let $K(x,y)$ be the kernel of the operator $m(P)$. Thus,

$$K(x,y) = \int_0^\infty m(\lambda)\ dE_\lambda(x,y) = \int_0^\infty m(\lambda)\ \partial_\lambda S_\lambda^0(x,y).$$

Integrating by parts and making use of the identity

$$\frac{d}{d\lambda}(\lambda^l S_\lambda^l(x,y)) = l\lambda^{l-1} S_\lambda^{l-1}(x,y)$$

we get

$$K(x,y) = C_l \int_0^\infty m^{(l+1)}(\lambda)\ \lambda^l\ S_\lambda^l(x,y)\ d\lambda.$$

Now using the estimate (1.1.1) we have

$$|K(x,y)| \le C_l \int_0^\infty m^{(l+1)}(\lambda)\ \lambda^{l+\frac{n}{a}}\ (1 + \lambda^{\frac{1}{a}}(|x-y|)^{-al+\beta}.$$

From the above expression it is clear that, if l is large enough then we have

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|\ dx < \infty$$

which proves the proposition. □

In view of the above proposition, to prove Theorem 1.1.1 it is enough to prove the following.

Theorem 1.1.3 *Let $m \in S_\rho^{-\alpha}(I\!R)$, $0 \leq \rho \leq 1$ be such that $m(\lambda) = 0$ for $|\lambda| \leq 1$ and $1 < p < \infty$. If the Bochner-Riesz kernel $s_R^\delta(x,y)$ associated to P satisfies the estimates (1.1.1) then $m(P) : L^p(I\!R^n) \to L^p(I\!R^n)$ is bounded whenever $\alpha > \frac{n(1-\rho)}{a}|\frac{1}{p} - \frac{1}{2}|$.*

Proof: Let $\varphi \in C_0^\infty(\frac{1}{2} \leq t \leq 2)$ be such that $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}t) = 1$ for every $t \neq 0$. Let $m_j(t) = m(t)\varphi(2^{-j}t)$ and $m_j(P)$ be the corresponding operator, that is

$$m_j(P)f = \int_0^\infty m_j(\lambda)dE_\lambda f.$$

We then have $m(P) = \sum_{j=0}^{\infty} m_j(P)$ since $m(\lambda)$ vanishes for $|\lambda| \leq 1$. Under the hypothesis of Theorem 1.1.3 we will show that there exists a $\delta > 0$ such that

$$||m_j(P)f||_p \leq C\ 2^{-\delta j}||f||_p \qquad (1.1.2)$$

for all $f \in L^p(I\!R^n)$. Theorem 1.1.3 will then easily follow by summing a geometric series.

In order to get the estimate (1.1.2) we look at the kernel $k_j(x,y)$ of $m_j(P)$ which is given by

$$k_j(x,y) = \int_0^\infty m_j(\lambda)dE_\lambda(x,y).$$

Let $1 < p \leq 2$. Since $\alpha > \frac{n(1-\rho)}{a}(\frac{1}{p} - \frac{1}{2})$ we can choose $\epsilon > 0$ such that $\alpha > n(\frac{1-\rho}{a} + \epsilon)(\frac{1}{p} - \frac{1}{2})$. Let $\gamma = \frac{1-\rho}{a} + \epsilon - \frac{1}{d}$ so that $\alpha > n(\gamma + \frac{1}{d})(\frac{1}{p} - \frac{1}{2})$. We write

$$k_j(x,y) = k_{j,1}(x,y) + k_{j,2}(x,y)$$

where $k_{j,1}(x,y) = k_j(x,y)$ if $|x-y| \leq 2^{\gamma j}$ and 0 elsewhere. We first consider the operator given by

$$\mathcal{K}_{j,2}f(x) = \int_{I\!R^n} k_{j,2}(x,y)f(y)dy$$

Proposition 1.1.4 *Under the hypothesis of the Theorem 1.1.3 the following holds. For some* $\delta > 0$

$$\int_{\mathbb{R}^n} |\mathcal{K}_{j,2} f(x)|^p dx \leq C 2^{-\delta j p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

for all $f \in L^p(\mathbb{R}^n)$ *and* $j = 1, 2, \ldots$.

In order to prove the above proposition we will make use of the following estimate on the kernel $k_j(x, y)$.

Proposition 1.1.5 *Let* m *be as in Theorem 1.1.3. Then we have*

$$|k_j(x,y)| \leq C_l \;\; 2^{\,j[l(1-\rho-\frac{a}{d})+\frac{n+\beta+a}{d}]} \;\; |x-y|^{-al+\beta+a}$$

for all positive integers l.

We will assume this for a moment and complete the proof of Proposition 1.1.4. We only need to show that

$$\sup_y \int_{\mathbb{R}^n} |k_{j,2}(x,y)| dx \leq C 2^{-\delta j}.$$

for some $\delta > 0$. In view of the above estimate on the kernel $k_j(x, y)$ we have

$$\int_{\mathbb{R}^n} |k_{j,2}(x,y)| dx \leq C_l \; 2^{j[l(1-\rho-\frac{a}{d})+\frac{n+\beta+a}{d}]} \int_{2^{\gamma j}}^{\infty} t^{-al+\beta+a+n-1} dt$$

$$\leq C_l \; 2^{jl[1-\rho-a(\gamma+\frac{1}{d})]} 2^{j(n+\beta+a)(\gamma+\frac{1}{d})}$$

Since $1 - \rho - a(\gamma + \frac{1}{d}) < 0$ choosing l large enough we can get the required decay.

To prove Proposition 1.1.5 we need to use the estimate (1.1.1). Since

$$k_j(x,y) = \int_0^\infty m_j(\lambda) dE_\lambda(x,y)$$

and $E_\lambda(x, y) = S_\lambda^0(x, y)$, integrating by parts and making use of the identity

$$\frac{d}{d\lambda}(\lambda^l S_\lambda^l(x,y)) = l\lambda^{l-1} S_\lambda^{l-1}(x,y)$$

we get

$$k_j(x,y) = C_l \int_0^\infty \lambda^{l-1} S_\lambda^{l-1}(x,y) \partial_\lambda^l (m_j(\lambda)) d\lambda.$$

As $m_j \in S_\rho^{-\alpha}$ and is supported in $2^{j-1} \leq t \leq 2^{j+1}$ we have

$$|k_j(x,y)| \leq C_l \int_{2^{j-1}}^{2^{j+1}} \lambda^{l-1} \lambda^{-\rho l} \lambda^{\frac{n}{d}} (1 + \lambda^{\frac{1}{d}} |x-y|)^{-al+a+\beta} d\lambda$$

$$\leq C_l \, |x-y|^{-al+a+\beta} \; 2^{j[l(1-\rho-\frac{a}{d}) + \frac{n+\beta+a}{d}]}.$$

This completes the proof of the Proposition 1.1.5. □

Thus we have taken care of $k_{j,2}(x,y)$. To deal with $k_{j,1}$,we proceed as follows. First we prove the following analogue of the Hardy-Littlewood-Sobolev theorem for the operator P which has been proved in [47]. However for the sake of completeness we state and prove it here.

Theorem 1.1.6 *Let $0 < \alpha < n, 1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then we have*

$$||(1+P)^{-\frac{\alpha}{d}} f||_q \leq C ||f||_p.$$

Proof: By spectral theorem

$$(1+P)^{-\frac{\alpha}{d}} f = \int_0^\infty (1+\lambda)^{-\frac{\alpha}{d}} dE_\lambda f$$

and so the kernel of $(1+P)^{-\frac{\alpha}{d}}$ is given by

$$k_\alpha(x,y) = \int_0^\infty (1+\lambda)^{-\frac{\alpha}{d}} dE_\lambda(x,y).$$

As above let φ be a smooth function supported in $(\frac{1}{2}, 2)$ such that $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\lambda) = 1$ for every $\lambda \neq 0$. Let $k_{\alpha,j}(x,y)$ be the kernel of $\varphi(2^{-j}P)(1+P)^{-\frac{\alpha}{d}}$. Then

$$k_{\alpha,j}(x,y) = \int_{2^{j-1}}^{2^{j+1}} m_{\alpha,j}(\lambda) \; dE_\lambda(x,y)$$

where $m_{\alpha,j}(\lambda) = \varphi(2^{-j}\lambda)(1+\lambda)^{-\frac{\alpha}{d}}$.

Integrating by parts we get

$$k_{\alpha,j}(x,y) = c_{\alpha,l}\int_{2^{j-1}}^{2^{j+1}} \partial_\lambda^l(m_{\alpha,j})(\lambda)\lambda^{l-1}S_\lambda^{l-1}(x,y)d\lambda.$$

It is easy to see that $|\partial_\lambda^l(m_{\alpha,j})(\lambda)| \leq C\,\lambda^{-l}$ with C depending only on α and l. We use the estimate (1.1.1) to get

$$|k_{\alpha,j}(x,y)| \leq C\int_{2^{j-1}}^{2^{j+1}} \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}(1+\lambda^{\frac{1}{d}}|x-y|)^{-al+a+\beta}d\lambda$$

which is bounded by

$$C|x-y|^{\alpha-n}\int_{2^{j-1}|x-y|^d}^{2^{j+1}|x-y|^d} \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}(1+\lambda^{\frac{1}{d}})^{-al+a+\beta}d\lambda.$$

Since for any $t > 0$ at most two of the intervals of the type $(2^{j-1}t, 2^{j+1}t)$ can intersect we have,

$$|k_\alpha(x,y)| \leq C\sum_{j=-\infty}^{\infty} |k_{\alpha,j}(x,y)| \leq C\;|x-y|^{\alpha-n}\int_0^\infty \lambda^{-\frac{\alpha}{d}+\frac{n}{d}-1}\;(1+\lambda^{\frac{1}{d}})^{-al+a+\beta}.$$

The last integral converges if l is large enough since $0 < \alpha < n$ and we have

$$|k_\alpha(x,y)| \leq C|x-y|^{\alpha-n}\,.$$

Now it is a routine matter to prove the proposition. See, for example, the proof of the Hardy-Littlewood-Sobolev theorem in Stein [38] . □

Using the above theorem,we prove the following result. Let B be any ball of radius $2^{\gamma j}$.

Proposition 1.1.7 *There is a $\delta > 0$ such that*

$$\left(\int_B |m_j(P)f(x)|^p dx\right)^{\frac{1}{p}} \leq C2^{-\delta j}\left(\int |f(x)|^p dx\right)^{\frac{1}{p}}.$$

for all $f \in L^p(\mathbb{R}^n), 1 \leq p \leq 2$.

Proof: By Holder's inequality,

$$\left(\int_B |m_j(P)f(x)|^p dx\right)^{\frac{1}{p}} \le |B|^{\frac{1}{p}-\frac{1}{2}} \left(\int |m_j(P)f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Now by spectral theorem

$$||m_j(P)f||_2^2 = \int_{2^{j-1}}^{2^{j+1}} |m_j(\lambda)|^2 d(E_\lambda f, f).$$

Since $m_j \in S_\rho^{-\alpha}$, the above is bounded by

$$\begin{aligned} &\int_{2^{j-1}}^{2^{j+1}} (1+\lambda)^{-2\alpha} d(E_\lambda f, f) \\ &\le \int_{2^{j-1}}^{2^{j+1}} \lambda^{-2\alpha+\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})}(1+\lambda)^{-\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})} d(E_\lambda f, f) \\ &\le C 2^{j[-2\alpha+\frac{2n}{d}(\frac{1}{p}-\frac{1}{2})]} ||(1+P)^{-\frac{t}{d}} f||_2^2. \end{aligned}$$

with $t = n(\frac{1}{p}-\frac{1}{2})$. Using the result of Proposition 1.1.6 we obtain

$$\begin{aligned} \left(\int_B |m_j(P)f(x)|^p dx\right)^{\frac{1}{p}} &\le C 2^{n\gamma j(\frac{1}{p}-\frac{1}{2})} 2^{j[-\alpha+\frac{n}{d}(\frac{1}{p}-\frac{1}{2})]} ||f||_p \\ &= 2^{-j[\alpha-(\gamma+\frac{1}{d})n(\frac{1}{p}-\frac{1}{2})]} ||f||_p \end{aligned}$$

which completes the proof by the choice of γ. □

We are now in a position to complete the proof of Theorem 1.1.3. Let $\mathcal{K}_{j,1}$ be the operator defined by

$$\mathcal{K}_{j,1} f(x) = \int_{\mathbb{R}^n} k_{j,1}(x,y)\ f(y)\ dy.$$

To deal with this operator we decompose f into three parts. Let $\xi \in \mathbb{R}^n$ and define

$$f_1(x) = f(x)\chi\left(|x-\xi| \le \frac{3}{4} 2^{j\gamma}\right)$$

$$f_2(x) = f(x)\chi\left(\frac{3}{4}2^{j\gamma} < |x-\xi| \le \frac{5}{4} 2^{j\gamma}\right)$$

and $f_3 = f - f_1 - f_2$. Let $B(\xi)$ be the ball $|x-\xi| \le \frac{1}{4}2^{j\gamma}$. We will show that

$$\int_{B(\xi)} |\mathcal{K}_{j,1}f(x)|^p\, dx \le C2^{-\epsilon jp} \int_{|x-\xi|\le\frac{5}{4}2^{j\gamma}} |f(x)|^p\, dx$$

for some $\epsilon > 0$. Integration with respect to ξ will prove

$$\int |\mathcal{K}_{j,1}f(x)|^p\, dx \le C2^{-\epsilon jp}\|f\|_p^p.$$

When $|x-\xi| \le \frac{1}{4}2^{j\gamma}$ and y belonging to the support of f_3 it follows that $|x-y| > 2^{j\gamma}$ and consequently $\mathcal{K}_{j,1}f_3 = 0$. When $|x-\xi| \le \frac{1}{4}2^{j\gamma}$ and y belonging to the support of f_2 one has $|x-y| > \frac{1}{2}2^{j\gamma}$ and we can repeat the proof of the Proposition 1.1.4 to conclude that

$$\int_{B(\xi)} |\mathcal{K}_{j,1}f_2(x)|^p\, dx \le C2^{-\epsilon jp}\int |f_2(x)|^p\, dx.$$

Finally applying Proposition 1.1.7 we obtain the estimate

$$\int_{B(\xi)} |\mathcal{K}_{j,1}f_1(x)|^p\, dx \le C2^{-\epsilon jp}\int |f_1(x)|^p\, dx.$$

Putting together all the above estimates we prove Theorem 1.1.3.

We finish this chapter with the following observations. Above method can be applied to certain differential operators on homogeneous groups. Let G be a stratified nilpotent Lie group. Let Q stand for its homogeneous dimension. If P is a positive Rockland operator (see [5] for definition and properties of Rockland operators) which is homogeneous of degree d then it is known that the Bochner-Riesz kernel $s_R^\delta(g)$ associated to P satisfies the estimate (see [11])

$$|s_R^\delta(g)| \le CR^{\frac{Q}{d}}(1 + R^{\frac{1}{d}}|g|)^{-\frac{\delta}{3}+\beta}.$$

Here $|.|$ stands for a homogeneous norm on G. Assuming that the spectral measure of P has no mass at the origin and proceeding as in the above theorem we can prove that, if $m \in S_\rho^{-\alpha}(\mathbb{R})$ then $m(P)$ is bounded on $L^p(G)$ provided $\alpha > 3Q(1-\rho)|\frac{1}{p}-\frac{1}{2}|$. But when P is of homogeneous degree 2, this can be improved. We have the following.

Theorem 1.1.8 *Let P be a Rockland operator of homogeneous degree 2. Let $m \in S_\rho^{-\alpha}$ and $1 < p < \infty$. Then $m(P)$ is bounded on $L^p(G)$ provided $\alpha > Q(1-\rho)|\frac{1}{p} - \frac{1}{2}|$.*

To prove the above theorem it is enough to get the following estimate on the Bochner-Riesz kernels:

$$|s_R^\delta(g)| \leq CR^{\frac{Q}{2}}(1 + R^{\frac{1}{2}}|g|)^{-\delta+\beta}. \tag{1.1.3}$$

This can be achieved by following a method by W. Hebisch [7] and using the heat kernel estimates proved in [2]. This will be explained in more details in Chapter 3 (see Propositions 3.1.1 and 3.1.2) where we study certain multipliers associated to the sublaplacian on H^n.

Chapter 2

Special Hermite and Hermite Expansions

Main objective of this chapter is to study certain multipliers associated to the special Hermite operator L and the Hermite operator H. We consider the wave equation associated to these operators and study the L^p boundedness properties of the solutions. In the course of the proofs we make use of Stein's analytic interpolation theorem, a Hilb type asymptotic formula which connects Laguerre functions and Bessel functions and the general multiplier theorem proved in the previous chapter.

2.1 Wave equation for the special Hermite operator

Before we proceed to study the wave equation, some remarks are in order. Let S_R^δ be the Bochner-Riesz operator associated to the special Hermite operator. Then

$$S_R^\delta f(z) = f \times s_R^\delta(z)$$

where the kernel s_R^δ is given by

$$s_R^\delta(z) = \sum_{k=0}^{\infty} \left(1 - \frac{2k+n}{R}\right)_+^\delta \varphi_k(z).$$

Here $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)\ e^{-\frac{1}{4}|z|^2}$. Then it is known that $s_R^\delta(z)$ satisfies the estimate,

$$|s_R^\delta(z)| \leq CR^n(1 + R^{\frac{1}{2}}|z|)^{-\delta-n-\frac{1}{3}}\ ,$$

see Proposition 2.5.1 in [44]. So Theorem 1.1.1 will imply that operators $m(L)$, m coming from $S_\rho^{-\alpha}(I\!R)$ are bounded on $L^p(\mathbb{C}^n)$ provided $\alpha > 2n(1-\rho)|\frac{1}{p} - \frac{1}{2}|, 1 < p < \infty$. We remark that the above result can be improved to include the case $p = 1$ as well. To prove this we proceed as follows. A close examination of the proof of Theorem 1.1.1 reveals that we only need to prove the following.

Let B be any ball of radius $2^{\gamma j}$ where γ is as in the proof of Theorem 1.1.3 and let k_j be the kernel of $m_j(L)$.

Proposition 2.1.1 *There is a $\delta > 0$ such that*

$$\left(\int_B |f \times k_j(z)|dz\right) \leq C2^{-\delta j} \int |f(z)|dz$$

for all $f \in L^1(\mathbb{C}^n)$.

Proof : By Holder's inequality

$$\int_B |f \times k_j(z)|dz \leq C|B|^{\frac{1}{2}} \left(\int |f \times k_j(z)|^2 dz\right)^{\frac{1}{2}}.$$

By the Plancherel theorem for the special Hermite expansion,we have

$$\int_{\mathbb{C}^n} |f \times k_j(z)|^2 dz = (2\pi)^{-2n} \sum_{k=0}^{\infty} |m_j(2k+n)|^2 \|f \times \varphi_k\|_2^2$$

which is dominated by

$$\sum_{2^{j-1} \leq 2k+n \leq 2^{j+1}} (2k+n)^{-2\alpha} \|f \times \varphi_k\|_2^2.$$

Since $||\varphi_k||_2 \leq Ck^{\frac{n-1}{2}}$ (see [44]), we have

$$||f \times \varphi_k||_2 \leq ||\varphi_k||_2 \; ||f||_1 \leq Ck^{\frac{n-1}{2}}||f||_1$$

and therefore the above is dominated by $C2^{j(-2\alpha+n)}||f||_1^2$. Now the proof can be completed as in the proof of Theorem 1.1.1. □

Next we turn our attention towards the Cauchy problem,

$$(\partial_t^2 + L)u(z,t) = 0, u(z,0) = 0, \partial_t u(z,0) = f(z). \qquad (2.1.1)$$

Solution to this problem is given by the multiplier operator $L^{-\frac{1}{2}} \sin t\sqrt{L} f(z)$. Note that $\frac{\sin t\sqrt{s}}{\sqrt{s}} = (\frac{t\pi}{2})^{\frac{1}{2}} \frac{J_{\frac{1}{2}}(t\sqrt{s})}{(\sqrt{s})^{\frac{1}{2}}}$. So more generally one can look at the operators $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$. The functions $\frac{J_\alpha(t\sqrt{s})}{(\sqrt{s})^\alpha}$ belong to $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{1}{4}}$ and so from Theorem 1.1.1 it follows that $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ is bounded on $L^p(\mathbb{C}^n)$ provided $\alpha > 2n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. But this can be improved. We have the following result.

Theorem 2.1.2 *Let $1 \leq p \leq \infty$. The operators $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ are bounded on $L^p(\mathbb{C}^n)$ whenever $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.*

Taking $\alpha = \frac{1}{2}$ we get the following estimate for the solution to the Cauchy problem (2.1.1).

Corollary 2.1.3 *Let $u(z,t)$ be the solution to the Cauchy problem (2.1.1). Then we have $||u(.,t)||_p \leq C_t||f||_p$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n-1}$.*

To study the boundedness properties of $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$ we consider the analytic family of operators defined for $\mathrm{Re}\alpha > -\frac{1}{2}$ by

$$T_t^\alpha f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \psi_k^\alpha(t) \; f \times \varphi_k(z).$$

Here ψ_k^α are the Laguerre functions

$$\psi_k^\alpha(t) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(\frac{1}{2}t^2)e^{-\frac{1}{4}t^2}.$$

We require the estimate

$$\sup_{0<t\leq 1} |\psi_k^\alpha(t)| \leq C \qquad \text{for} \quad \alpha \geq -\frac{1}{2} \tag{2.1.2}$$

for which we refer to Szego[42]. We also make use of the following proposition.

Proposition 2.1.4 *Let μ_t stand for the normalised surface measure on the sphere $S_t = \{|z| = t\}$ in $\mathbb{C}^n$. Then*

$$f \times \mu_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\, \varphi_k(t)\; f \times \varphi_k(z)\;.$$

Proof: See Theorem 2.4.4 in [46].

Using analytic interpolation we establish the following result.

Theorem 2.1.5 *Let $1 \leq p \leq \infty$ and $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Then for $f \in L^p(\mathbb{C}^n)$, $||T_t^\alpha f||_p \leq C||f||_p$ uniformly in t, $0 < t \leq 1$.*

Proof: In view of Proposition 2.1.4 we know that T_t^{n-1} is bounded on $L^p(\mathbb{C}^n)$, $1 \leq p \leq \infty$ uniformly in $0 < t \leq 1$. From the estimate (2.1.2) it follows that $T_t^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{C}^n)$ uniformly in $0 < t \leq 1$. For the analytic interpolation we need to consider T_t^α when α is complex. We make use of the following formula connecting Laguerre polynomials of different type :

$$L_k^{\alpha+\beta}(t) = \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta)\;\Gamma(k+\alpha+1)} \int_0^1 s^\alpha(1-s)^{\beta-1} L_k^\alpha(st)ds$$

which is valid for $\mathrm{Re}\alpha > -1$ and $\mathrm{Re}\,\beta > 0$.

Given $\alpha = n-1+\delta+i\sigma, \delta > 0$ we can write

$$\psi_k^\alpha(t) = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(n)} \int_0^1 s^{n-1}\,(1-s)^{\delta+i\sigma-1}\, e^{-\frac{1}{4}(1-s)t^2}\, \psi_k^{n-1}(t\sqrt{s})\; ds\;,$$

so that $T_t{}^\alpha$ is expressible in terms of T_t^{n-1} as

$$T_t{}^\alpha = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\Gamma(n)} \int_0^1 s^{n-1}\ (1-s)^{\delta+i\sigma-1}\ e^{-\frac{1}{4}(1-s)t^2}\ T_{t\sqrt{s}}^{n-1}\ ds\ .$$

From this it follows that $T_t{}^\alpha$ is bounded on $L^p(\mathbb{C}^n)$ $1 \le p \le \infty$ for $\mathrm{Re}\alpha > n-1$. Similarly when $\alpha = -\frac{1}{2}+\delta+i\sigma$ it can be shown that $T_t{}^\alpha$ is bounded on $L^2(\mathbb{C}^n)$. Using estimates for the gamma functions ,we can easily check that the family $T_t{}^\alpha$ is admissible in the sense of Stein [37]. Applying Stein's analytic interpolation theorem we obtain Theorem 2.1.5. □

We now consider operators of the form $L^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{L})$, with the corresponding multiplier $(2k+n)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+n})$. We first remark that it is enough to consider the multiplier $(2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+\alpha+1})$. To see this let us assume $t=1$ and look at their difference

$$m(k) = (2k+n)^{-\frac{\alpha}{2}} J_\alpha(\sqrt{2k+n}) - (2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(\sqrt{2k+\alpha+1}).$$

Writing $F(s) = s^{-\alpha} J_\alpha(s)$ we have

$$\begin{aligned} m(k) &= F(\sqrt{2k+n}) - F(\sqrt{2k+\alpha+1}) \\ &= \int_{\alpha+1}^{n} \frac{F'(\sqrt{s+2k})}{2\sqrt{s+2k}} ds. \end{aligned}$$

Since $F'(s) = -s^{-\alpha} J_{\alpha+1}(s)$ we have the expression

$$m(k) = -\frac{1}{2}\int_{\alpha+1}^{n} \frac{J_{\alpha+1}(\sqrt{s+2k})}{(\sqrt{s+2k})^{\alpha+1}} ds$$

which clearly shows that $m \in S_{\frac{1}{2}}^{-\frac{2\alpha+3}{4}}(\mathbb{R})$. Therefore $m(k)$ will define an L^p multiplier provided $\frac{\alpha}{2}+\frac{3}{4} > n(\frac{1}{p}-\frac{1}{2})$ which is clearly satisfied when $\alpha > (2n-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$.

Thus it is enough to consider the multiplier $(2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+\alpha+1})$. Recall that we are assuming $0 < t \le 1$. We compare this multiplier with $\psi_k^\alpha(t)$ using a Hilb type asymptotic formula for the Laguerre polynomials; see Szego [42]. More precisely formula (8.64.3) on page 217 of [42] gives,

$$\psi_k^\alpha(t) = 2^\alpha\ \Gamma(\alpha+1)\ \frac{J_\alpha(t\sqrt{2k+\alpha+1})}{(t\sqrt{2k+\alpha+1})^\alpha} + m(k,\alpha,t) \qquad (2.1.3)$$

where

$$m(k,\alpha,t) = \frac{C(\alpha)}{\sin\alpha\pi}\, t^4$$
$$\int_0^1 \{J_\alpha(t\sqrt{N})J_{-\alpha}(ts\sqrt{N}) - J_{-\alpha}(t\sqrt{N})J_\alpha(ts\sqrt{N})\}\; s^{\alpha+3}\; \psi_k^\alpha(ts)\; ds.$$

where $N = 2k+\alpha+1$ and $C(\alpha)$ a constant which depends only on α. In the above formula if α is an integer $J_{-\alpha}$ must be replaced by the modified Bessel function Y_α and $\sin\alpha\pi$ by -1. Now define $m_\alpha(\lambda) = \lambda^{-\frac{\alpha}{2}} J_\alpha(\sqrt{\lambda})$ and

$$a_\alpha(\lambda,t,s) = \left(J_\alpha(t\sqrt{\lambda})J_{-\alpha}(ts\sqrt{\lambda}) - J_{-\alpha}(t\sqrt{\lambda})J_\alpha(ts\sqrt{\lambda})\right)\; s^{\alpha+3}t^4.$$

For the symbols a_α we prove the following estimates.

Lemma 2.1.6 *For $0 \le r\ , s\ \le 1$ we have the estimates*

$$|\partial_\lambda^k\; a_\alpha(\lambda,r,s)| \le C_k\; (1+\lambda)^{-\frac{k}{2}-\frac{1}{2}}$$

valid for all $\lambda > 0\ , k \ge 0$. More precisely ,

$$|\partial_\lambda^k a_\alpha(\lambda,r,s)| \le\; C\, r^3\; s^{\frac{5}{2}}(1+\lambda)^{-\frac{1}{2}(k+1)}$$
$$\{(1+r^2\lambda)^{-\frac{\alpha}{2}}\; (1+r^2s^2\lambda)^{\frac{\alpha}{2}} + s^{2\alpha}(1+r^2\lambda)^{\frac{\alpha}{2}}\; (1+r^2s^2\lambda)^{-\frac{\alpha}{2}}\}$$

Proof: Let $B_\alpha(\lambda) = \lambda^{-\frac{1}{2}\alpha}J_\alpha(\sqrt{\lambda})$ and when α is a negative integer replace J_α by Y_α. Then B_α satisfies the equation

$$\frac{d}{d\lambda}B_\alpha(\lambda) = -\frac{1}{2}B_{\alpha+1}(\lambda).$$

The asymptotic properties of the Bessel function give us the estimates

$$|(\frac{d}{d\lambda})^k\; B_\alpha(\lambda)| \le\; C(1+\lambda)^{-\frac{1}{2}(\alpha+k+\frac{1}{2})}.$$

Consider the first term in $a_\alpha(\lambda,t,s)$ which is equal to $B_\alpha(r^2\lambda)B_{-\alpha}(r^2s^2\lambda)s^3r^4$. The k^{th} derivative of this term is a linear combination of terms of the form

$$r^{2j+4}\; B_{\alpha+j}(r^2\lambda)\; (r^2\; s^2)^{k-j}\; B_{-\alpha+k-j}(r^2\; s^2\; \lambda)\; s^3$$

which is bounded by a constant times

$$r^{2k+4}\ s^{2k-2j+3}\ (1+r^2\ \lambda)^{-\frac{1}{2}(\alpha+j+\frac{1}{2})}\ (1+r^2\ s^2\ \lambda)^{-\frac{1}{2}(-\alpha+k-j+\frac{1}{2})}.$$

As $0 \le r\ , s\ \le 1$, the above is bounded by a constant times

$$r^3 s^{\frac{5}{2}}\ (1+\lambda)^{-\frac{1}{2}(k+1)}\ (1+r^2\lambda)^{-\frac{\alpha}{2}}\ (1+r^2 s^2\lambda)^{\frac{\alpha}{2}}$$

which is bounded by $C\ (1+\lambda)^{-\frac{1}{2}(k+1)}$. Similarly, the k^{th} derivative of the second term is bounded by

$$C\ r^3\ s^{2\alpha+\frac{5}{2}}\ (1+\lambda)^{-\frac{1}{2}-\frac{k}{2}}\ (1+r^2\lambda)^{\frac{\alpha}{2}}\ (1+r^2 s^2\lambda)^{-\frac{\alpha}{2}}$$

which in turn is bounded by $C\ (1+\lambda)^{-\frac{1}{2}(k+1)}$. This proves the lemma. □

Iterating in the formula (2.1.3) we have

$$\psi_k^\alpha(t) = 2^\alpha\ \Gamma(\alpha+1)\frac{J_\alpha(t\ \sqrt{2k+\alpha+1})}{(t\sqrt{2k+\alpha+1})^\alpha} + m_1(\sqrt{2k+\alpha+1}, t) + e(\sqrt{2k+\alpha+1}, t)$$

where

$$m_1(\sqrt{2k+\alpha+1}, t) = C_1(\alpha)\ \int_0^1\ a_\alpha(N,t,s) m_\alpha(t^2 s^2 N)\ ds.$$

From the above expression it is easy to check that $m_1(\sqrt{\lambda}, t) \in S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{3}{4}}$ and so when $\alpha > (2n-1)(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2}, m_1(\sqrt{2k+\alpha+1}, t)$ defines an L^p multiplier. Further iteration produces better and better terms. We can write

$$\psi_k^\alpha(t) = 2^\alpha\ \Gamma(\alpha+1)\frac{J_\alpha(t\sqrt{2k+\alpha+1})}{(t\sqrt{2k+\alpha+1})^\alpha} + \sum_{j=1}^{l} m_j(\sqrt{2k+\alpha+1}, t) + e_l(\sqrt{2k+\alpha+1}, t)$$

where the error term $e_l(\sqrt{2k+\alpha+1}, t)$ can be written as

$$\begin{aligned} & e_l(\sqrt{2k+\alpha+1}, t) \\ = \ & C_l(\alpha) \int_0^1 \cdots \int_0^1 a_\alpha(N,t,s_1)\ a_\alpha(N,ts_1,s_2)\cdots a_\alpha(N, ts_1 s_2\cdots s_{l-1}, s_l) \\ & \qquad \psi_k^\alpha(ts_1\cdots s_l)\ ds_1\ \cdots ds_l. \end{aligned}$$

Here $N = 2k + \alpha + 1$.

All the sequences $m_j(\sqrt{2k+\alpha+1}, t)$, $j = 1, 2 \cdots l$ will define bounded multipliers on $L^p(\mathbb{C}^n)$ when $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Now it follows from Lemma 2.1.6 that the multiplier $a_\alpha(\lambda, t, s_1)a_\alpha(\lambda, ts_1, s_2) \cdots a_\alpha(\lambda, ts_1s_2 \cdots s_{l-1}, s_l)$ belongs to $S_{\frac{1}{2}}^{-\frac{l}{2}}$ with estimates uniform in $s_1, s_2, \cdots s_l$. Hence using Theorem 1.1.1 and Theorem 2.1.5 we get that the operator defined by the sequence $e_l(\sqrt{2k+\alpha+1}, t)$ is also bounded on the same L^p if $\frac{\alpha}{2} + \frac{3}{4} \leq \frac{l}{2}$. Thus the difference

$$\psi_k^\alpha(t) - 2^\alpha\ \Gamma(\alpha+1)(t\sqrt{2k+\alpha+1})^{-\alpha} J_\alpha(t\sqrt{2k+\alpha+1})$$

defines a bounded L^p multiplier for $\alpha > (2n-1)(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. As $\psi_k^\alpha(t)$ defines an L^p multiplier this implies that $(2k+\alpha+1)^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{2k+\alpha+1})$ also defines a bounded L^p multiplier which completes the proof of Theorem 2.1.2 .

By taking $\alpha = \frac{1}{2}$ in the theorem and noting that $\sqrt{\frac{2}{\pi}} \frac{\sin\sqrt{t}}{\sqrt{t}} = \frac{J_{\frac{1}{2}}(\sqrt{t})}{(\sqrt{t})^{\frac{1}{2}}}$ we infer that $u(z,t) = \frac{\sin t\sqrt{L}}{\sqrt{L}} f(z)$ satisfies the estimate $||u(.,t)||_p \leq C_t ||f||_p$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n-1}$. This proves Corollary 2.1.3 .

We finish this section with a few comments on the above corollary. As in the Euclidean case it is natural to expect that the above is valid in the range $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{2n-1}$. Before we describe the difficulties involved, let us briefly explain the methods used in [30] by Peral to get the end point result. First he considers a different analytic family of multiplier operators given by the functions $J_{\frac{n}{2}-1}(|\xi|)|\xi|^\alpha$. Using the boundedness of $(-\Delta)^{i\beta}$, $\beta \in \mathbb{R}$ and the Riesz transforms on the Hardy space H^1, it is proved that the family of operators given by the above multipliers and their derivatives are all bounded from H^1 to L^1. Now expanding the Bessel functions in terms of sines and cosines, taking proper linear combinations and using analytic interpolation he gets the result at the end point $\frac{1}{p} - \frac{1}{2} = \frac{1}{n-1}$ as well. We make an attempt to adapt these methods to our situation. To begin with, we have to study boundedness properties of $L^{i\beta}$ on spaces which are analogues to the usual Hardy spaces. The natural space to look at is the twisted Hardy Space $\mathcal{H}^1$. These spaces were introduced and studied in [18].

Let ψ be a C^∞-function on $\mathbb{C}^n$ with compact support such that $\psi = 1$ in a neighborhood of zero. Define

$$R_j(z) = \frac{z_j}{|z|^{2n+1}}\psi(z), \quad \overline{R_j}(z) = \frac{\overline{z_j}}{|z|^{2n+1}}\psi(z)$$

for $j = 1, \cdots, n$. Then $\mathcal{H}^1$ can be defined as the set of all $f \in L^1$ for which $R_j \times f$ and $\overline{R_j} \times f$ are in L^1 for all j. Norm on $\mathcal{H}^1$ is given by

$$||f||_{\mathcal{H}^1} = ||f||_1 + \sum_{j=1}^{n} ||R_j \times f||_1 + \sum_{j=1}^{n} ||\overline{R_j} \times f||_1.$$

Basic properties such as atomic decomposition and boundedness of singular integral operators etc were studied in [18]. We will make use of the following theorem proved in [18].

Theorem 2.1.7 *Let K be a function with compact support such that*

$$\int_{|z|>2|w|} |K(z-w) - K(z)|\ dz \leq A,$$

and assume either $||K \times f||_2 \leq B||f||_2$ or $|\hat{K}(\xi)| \leq B$. Then $\mathrm{K}f = K \times f$ is a bounded operator from $\mathcal{H}^1$ into itself.

Now we proceed to study the boundedness of $L^{i\beta}$ on $\mathcal{H}^1$. We have the following result.

Theorem 2.1.8 *The operator $L^{i\beta}$ is bounded from $\mathcal{H}^1$ to L^1 for all $\beta \in \mathbb{R}$.*

Note that $L^{i\beta} = m(L)$ where $m(t) = t^{i\beta}$. Let ϕ be a C^∞ function on $\mathbb{R}$ such that supp $\phi \subset (\frac{1}{2}, 2)$ and $\sum_{j=-\infty}^{\infty} \phi(2^j t) = 1$ for every $t \neq 0$. Let $m_j(t) = \phi(2^{-j}t)m(t)$. Then we have $m(L) = \sum_{j=0}^{\infty} m_j(L)$. Let $k_j(z)$ be the kernel of $m_j(L)$. Then

$$k_j(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} m_j(2k+n)\ \varphi_k(z).$$

We first obtain estimates for the kernels k_j away from origin. We need the following proposition. Let Δ_+ and Δ_- denote the forward and backward finite difference operators defined by

$$\Delta_+\psi(k) = \psi(k+1) - \psi(k-1), \quad \Delta_-\psi(k) = \psi(k) - \psi(k-1)$$

and let Δ stand for the operator $\Delta\psi(k) = -(k\Delta_-\Delta_+\psi(k) + n\Delta_-\psi(k))$.

Proposition 2.1.9 *If $M_\psi(z) = \sum_{k=0}^{\infty}\psi(k)\varphi_k(z)$ then we have*

$$\frac{1}{2}|z|^2 M_\psi(z) = \sum_{k=0}^{\infty} \Delta\psi(k)\ \varphi_k(z).$$

Proof: See Lemma 2.4.2 in [44]

Proposition 2.1.10 *Let $\alpha(z)$ be a C_c^∞ function such that $\alpha = 1$ in a neighborhood of the origin. Then there exists a $\delta > 0$ such that*

$$\int_{\mathbb{C}^n} |\ (1-\alpha(z))k_j(z)|\ dz \leq C\ 2^{-\delta j}$$

with C independent of j.

Proof: A repeated application of the Proposition 2.1.9 gives

$$(\frac{1}{2}|z|^2)^N k_j(z) = \sum_{k=0}^{\infty}\ \Delta^N m_j(2k+n)\varphi_k(z).$$

Hence

$$|k_j(z)| \leq C|z|^{-2N}\ |\sum_{k=0}^{\infty}\ \Delta^N m_j(2k+n)\varphi_k(z)|.$$

Note that the function $m(t) = t^{i\beta}$ satisfies the estimates $|m^{(j)}(t)| \leq C|t|^{-j}$ for every j. So $|\Delta^N m_j(2k+n)| \leq C_N(2k+n)^{-N}$ where C_N depends only on N. Hence using Cauchy-Schwarz inequality and the orthogonality of φ_k we have

$$\int_{\mathbb{C}^n} |(1-\alpha(z))k_j(z)|\ dz \leq C_N \left(\sum_{2^{j-1}\leq 2k+n \leq 2^{j+1}} (2k+n)^{-2N}\ \|\varphi_k\|_2^2 \right)^{\frac{1}{2}}$$

where C_N depends only on N. Since $||\varphi_k||_2^2 \leq C\ k^{n-1}$, choosing N large enough we get the proposition. □

This takes care of the part at infinity. To deal with the local part we look at the operators $L^{-\epsilon+i\beta}$ for $0 < \epsilon < 1$. Let p_t and K_ϵ stand for the kernels of the operators e^{-tL} and $L^{-\epsilon+i\beta}$ respectively. Then

$$p_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-t(2k+n)} \varphi_k(z) = (4\pi)^{-n} (\sinh t)^{-n}\ e^{-\frac{1}{4}|z|^2(\coth t)}.$$

Using the the identity

$$L^{-\epsilon+i\beta} = \frac{1}{\Gamma(\epsilon - i\beta)} \int_0^\infty t^{\epsilon - i\beta}\ e^{-tL} dt$$

we have

$$K_\epsilon(z) = C \frac{1}{\Gamma(\epsilon - i\beta)} \int_0^\infty t^{\epsilon - i\beta} e^{-\frac{1}{4}|z|^2(\coth t)}\ (\sinh t)^{-n}\ dt.$$

An easy computation shows that

$$|\alpha(z) K_\epsilon(z)| \leq C\ |z|^{-2n},$$

$$|\nabla(\alpha K_\epsilon)(z)| \leq C|z|^{-2n-1}$$

with C independent of $0 < \epsilon < 1$.

Now we are in a position to prove Theorem 2.1.8. Let $K(z)$ stand for the kernel of the operator $L^{i\beta}$. From Proposition 2.1.10 it follows that $(1 - \alpha(z))K(z) \in L^1$ and so the operator $f \to f \times (1-\alpha)K$ is bounded from $\mathcal{H}^1$ to L^1. Note that the operators $f \to f \times K_\epsilon$ are all bounded on $L^2(\mathbb{C}^n)$, uniformly in $\epsilon \geq 0$. Proceeding as in Proposition 2.1.10 we can easily show that the kernels $(1 - \alpha(z))K_\epsilon(z)$ are in $L^1(\mathbb{C}^n)$ with norms uniformly bounded in $0 < \epsilon \leq 1$. Hence it follows that the operators $f \to f \times \alpha K_\epsilon$ are all bounded on $L^2(\mathbb{C}^n)$ uniformly in $0 < \epsilon \leq 1$. Now using the above observations and Theorem 2.1.7 we have the operators $f \to f \times \alpha K_\epsilon$ are bounded from $\mathcal{H}^1$ to L^1 uniformly in $0 < \epsilon < 1$. Letting $\epsilon \to 0$

we have the boundedness of the operator $f \to f \times \alpha K$. Putting together we get Theorem 2.1.8. □

Now let us define another analytic family of operators by setting $G_t^\alpha = L^{-\frac{\alpha}{2}} J_{n-1}(t\sqrt{L})$. When $\alpha = n-1+i\beta$ the operator G_t^α is the composition of the operators $L^{-\frac{(n-1)}{2}} J_{n-1}(t\sqrt{L})$ and $L^{-\frac{i\beta}{2}}$. Comparing $\sqrt{L}^{n-1} J_{n-1}(t\sqrt{L})$ with T_t^{n-1} as in the proof of Theorem 2.1.2 we get that the operator $\sqrt{L}^{n-1} J_{n-1}(t\sqrt{L})$ is bounded on $L^1(\mathbb{C}^n)$. Using Theorem 2.1.8 we have G_t^α is bounded from $\mathcal{H}^1$ to L^1. When Re $\alpha = -\frac{1}{2}$, G_t^α is bounded on L^2 as $s^{-\frac{\alpha}{2}} J_{n-1}(t\sqrt{s})$ is a bounded function. Now we need to apply analytic interpolation to the above family. For $\mathcal{H}^1$ replaced by the usual Hardy space H^1 this is a famous theorem of Fefferman and Stein [3]. The same proof can be modified to deal with the present situation. The sharp maximal function has to be replaced by the twisted sharp maximal function

$$f_\tau^*(z) = \sup \frac{1}{|Q|} \int_Q |f(w) - \tilde{f}_Q \, e^{\frac{i}{2}\mathrm{Im} z.\overline{w}}| \, dw$$

where $\tilde{f}_Q = \frac{1}{|Q|} \int_Q f(w) e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} \, dw$. Here Q is a cube centered at z. In order to complete the proof of analytic interpolation theorem we need the fact that

$$C_1 \, \|f\|_p \leq \|f_\tau^*\|_p \leq C_2 \|f\|_p.$$

This has already been proved in Phong- Stein [31] and so we have the following interpolation theorem in the twisted setup.

Theorem 2.1.11 *Let T^α be an admissible analytic family of operators, defined on the strip $\{z,\ 0 \leq Re\ z \leq 1\}$ such that, when Re $\alpha = 0$, T^α is bounded from $\mathcal{H}^1$ to L^1 and when Re $\alpha = 1$, T^α is bounded on L^2. Then for $0 < \alpha < 1$, T^α is bounded on L^p for $1 < p < 2$ where p is defined by $1 - \frac{\alpha}{2} = \frac{1}{p}$.*

Applying analytic interpolation to the family G_t^α we have that G_t^α is bounded on L^p whenever $\alpha \geq (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. If we could get the same results for the family $L^{-\frac{\alpha}{2}} J_n(t\sqrt{L})$

then by taking appropriate linear combinations we would have the Corollary 2.1.3 in the range $|\frac{1}{p}-\frac{1}{2}| \leq \frac{1}{2n-1}$. Methods in [30] are not suitable at this point.

2.2 Multipliers for the Hermite operator

In this section we study certain multipliers associated to the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. Bochner-Riesz kernels associated to H are given by the expression

$$s_R^\delta(x,y) = \sum_{k=0}^{\infty} \left(1 - \frac{2k+n}{R}\right)_+^\delta \Phi_k(x,y)$$

where

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y).$$

They satisfy the estimate (see [47])

$$|s_R^\delta(x,y)| \leq CR^{\frac{n}{2}}(1 + R^{\frac{1}{2}}|x-y|)^{-\delta+\beta+n+2}.$$

So Theorem 1.1.1 implies that $m(H)$ is bounded on $L^p(\mathbb{R}^n)$ for m coming from $S_\rho^{-\alpha}(\mathbb{R})$ provided $\alpha > n(1-\rho)|\frac{1}{p}-\frac{1}{2}|$, $1 < p < \infty$. We remark that as in the previous section,using the estimate $||P_k f||_2 \leq Ck^{\frac{n}{2}-\frac{1}{4}}||f||_1$, the above can be improved to include the case $p = 1$ as well. We start with a study of the following analytic family defined by

$$S_t^\alpha f(x) = \sum_{k=0}^{\infty} \psi_k^\alpha(t)\ P_k f(x)$$

where

$$\psi_k^\alpha(t) = \frac{\Gamma(k+1)\ \Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(\frac{1}{2}t^2)e^{-\frac{1}{4}t^2}.$$

For this family we prove the following.

Theorem 2.2.1 *Let S_t^α be defined as above. Then for $1 \leq p \leq \infty$, $S_t^\alpha : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded whenever $\alpha > n|\frac{1}{p}-\frac{1}{2}| - \frac{1}{2}$.*

Note that when $\alpha = n-1$, S_t^{n-1} is precisely the Weyl transform of the surface measure μ_t on the sphere $\{z : |z| = t\}$ in $\mathbb{C}^n$. That is

$$S_t^{n-1} f = \int_{|z|=t} \pi(z) f \, d\mu_t$$

where $\pi(x)$ is the projective representation of $\mathbb{C}^n$ defined by $\pi(z)f(\xi) = e^{ix.\xi + \frac{1}{2}x.y} f(\xi + y)$. Here $z = x + iy$.

First we express S_t^α for Re $\alpha > n-1$ as the Weyl transform of a function. Let

$$g_t^\alpha(z) = 2\, \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)} w_{2n-1}^{-1} t^{-2n} \left(1 - \frac{|z|^2}{t^2}\right)_+^{\alpha-1} e^{-\frac{1}{4}(t^2 - |z|^2)}$$

where w_{2n-1} is the measure of $\{z : |z| = 1\}$. Then for Re $\alpha > 0$, g_t^α is an integrable function.

Proposition 2.2.2 *Let S_t^α and g_t^α be defined as above. Then for Re $\alpha > 0$ we have the relation $S_t^{\alpha+n-1} f = W(g_t^\alpha) f$, $f \in L^2(\mathbb{R}^n)$.*

Proof: We use the following formula which connects Laguerre polynomials of different types :

$$\frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(k+\mu+\nu+1)} L_k^{\mu+\nu}(\frac{t^2}{2}) e^{-\frac{t^2}{4}} =$$
$$\frac{\Gamma(k+1)\Gamma(\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \int_0^1 s^\mu (1-s)^{\nu-1} e^{-\frac{t^2}{4}(1-s)} L_k^\mu(\frac{st^2}{2}) e^{-\frac{st^2}{4}} ds$$

which is valid for Re $\mu > -1$, *Re* $\nu > 0$. In the above take $\mu = n-1$ and $\nu = \alpha$. Then after a change of variables we have

$$\begin{aligned}
&\frac{\Gamma(k+1)\Gamma(\alpha+n)}{\Gamma(k+\alpha+n)} L_k^{\alpha+n-1}(\frac{t^2}{2}) e^{-\frac{t^2}{4}} \\
&= 2\, \frac{\Gamma(k+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(k+n)} \int_0^1 s^{2n-1}(1-s^2)^{\alpha-1} e^{-\frac{t^2}{4}(1-s^2)} L_k^{n-1}(\frac{s^2t^2}{2}) e^{-\frac{s^2t^2}{4}} ds \\
&= 2\, w_{2n-1}^{-1} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n)} \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} (1-|z|^2)_+^{\alpha-1} e^{-\frac{t^2}{4}(1-|z|^2)} \varphi_k(tz)\, dz \\
&= \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} g_t^\alpha(z)\, \varphi_k(z)\, dz.
\end{aligned}$$

Thus we have the relation

$$\psi_k^{\alpha+n-1}(t) = \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} g_t^\alpha(z)\, \varphi_k(z)\, dz.$$

Since $g_t^\alpha(z)$ is a radial function it can be expanded in terms of $\varphi_k(z)$ and we have the expansion

$$g_t^\alpha(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \left(\int_{\mathbb{C}^n} g_t^\alpha(w)\, \varphi_k(w)\, dw \right) \varphi_k(z).$$

which leads to the formula

$$g_t^\alpha(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \psi_k^{\alpha+n-1}(t)\, \varphi_k(z).$$

Taking Weyl transform of both sides and noting that $W(\varphi_k) = (2\pi)^n P_k$ we get our proposition. □

We make use of the above proposition in the following way. The function g_t^α is integrable and hence $W(g_t^\alpha)$ is a bounded operator on $L^2(\mathbb{R}^n)$, whenever Re $\alpha > 0$. We will express $W(g_t^\alpha)$ as an integral operator with an explicit kernel K_t^α which has an analytic continuation for Re $\alpha > -\frac{n}{2} + \frac{1}{2}$. Using this, we will analytically continue $W(g_t^\alpha)$ to the region Re $\alpha > -\frac{n}{2} + \frac{1}{2}$ with an integrable kernel. Now the above proposition will imply that this continuation has to agree with S_t^α on that range. Thus we obtain expressions for the kernels of the operators S_t^α which proves that these operators are bounded on $L^1(\mathbb{R}^n)$ whenever Re $\alpha > -\frac{n}{2} + \frac{1}{2}$. Using estimates on the Laguerre functions we can easily prove that S_t^α is bounded on L^2 whenever Re $\alpha > -\frac{1}{2}$. Analytic interpolation will prove Theorem 2.2.1.

In the next proposition we compute the kernel of $W(g_t^\alpha)$. Let us set $G_\alpha(r) = 2^\alpha r^{-\alpha} J_\alpha(r)$.

Proposition 2.2.3 *Let $K_t^\alpha(\xi, y)$ stand for the kernel of $W(g_t^\alpha)$. Then we have*

$$K_t^\alpha(\xi, y) = c_n \Gamma(\alpha+n) t^{-n}$$
$$\sum_{j=0}^{\infty} \frac{(-1)^j\, t^{2j}}{4^j\, j!} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \left(1 - \frac{|y-\xi|^2}{t^2}\right)_+^{\alpha+\frac{n}{2}+j-1} G_{\alpha+\frac{n}{2}+j-1}\left(\frac{1}{2} ts|y+\xi|\right).$$

Proof: Recalling the definition of the Weyl transform given in the introduction, we have the explicit formula

$$W(g)\phi(\xi) = \int_{\mathbb{C}^n} e^{i(x.\xi+\frac{1}{2}x.y)}\ g(x,y)\ \phi(\xi+y)\ dxdy$$

where $g(x,y)$ stands for $g(x+iy)$. Thus $W(g)$ is an integral operator with kernel

$$K_g(\xi,y) = \int_{\mathbb{R}^n} g(x,y-\xi)\ e^{\frac{i}{2}(\xi+y).x}\ dx.$$

In view of this formula, the kernel K_t^α of $W(g_t^\alpha)$ is given by

$$K_t^\alpha(\xi,y) = \int_{\mathbb{R}^n} g_t^\alpha(x,y-\xi)\ e^{\frac{i}{2}(\xi+y).x}\ dx.$$

In order to evaluate this integral we expand the exponential factor in the definition of g_t^α into an infinite series getting

$$g_t^\alpha(z) = 2\,\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)}\,w_{2n-1}^{-1}\ t^{-2n}\sum_{j=0}^{\infty}\frac{(-1)^j\ t^{2j}}{4^j\ j!}\ (1-\frac{|z|^2}{t^2})_+^{\alpha+j-1}.$$

We now define k_t^α to be the kernel

$$k_t^\alpha(\xi,y) = \frac{t^{-2n}}{\Gamma(\alpha)}\int_{\mathbb{R}^n}\left(1-\frac{|x|^2}{t^2}-\frac{|y-\xi|^2}{t^2}\right)_+^{\alpha-1} e^{\frac{i}{2}x.(\xi+y)}\ dx$$

so that K_t^α is expressed as

$$K_t^\alpha(\xi,y) = 2\,\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(n)}w_{2n-1}^{-1}\sum_{j=0}^{\infty}\frac{(-1)^j\ t^{2j}}{4^j\ j!}\ \Gamma(\alpha+j)\ k_t^{\alpha+j}(\xi,y).$$

Note that $k_t^\alpha(\xi,y)$ vanishes for $|y-\xi|\geq t$. Therefore , by putting $s^2 = 1-\frac{|y-\xi|^2}{t^2}$ and making a change of variables in the definition of k_t^α we get

$$\begin{aligned} k_t^\alpha(\xi,y) &= \frac{t^{-2n}}{\Gamma(\alpha)}\int_{\mathbb{R}^n}(s^2-\frac{|x|^2}{t^2})_+^{\alpha-1}\ e^{\frac{i}{2}x.(\xi+y)}\ dx \\ &= \frac{t^{-n}}{\Gamma(\alpha)}\ s^{2\alpha+n-2}\int_{\mathbb{R}^n}(1-|x|^2)_+^{\alpha-1}\ e^{\frac{i}{2}tsx.(y+\xi)}\ dx. \end{aligned}$$

The last integral is a constant multiple of the Bessel function (see Theorem 4.15 in Stein-Weiss [37]) :

$$\int_{I\!R^n} (1-|x|^2)_+^{\alpha-1} e^{ix.\xi}\, dx = \pi^{\frac{n}{2}}\, 2^{\alpha+\frac{n}{2}-1}\, \Gamma(\alpha)\, J_{\alpha+\frac{n}{2}-1}(|\xi|)\, |\xi|^{-\alpha+\frac{n}{2}-1}.$$

Therefore, we have the formula

$$k_t^\alpha(\xi,y) = \pi^{\frac{n}{2}}\, t^{-n} \left(1-\frac{|y-\xi|^2}{t^2}\right)_+^{\alpha+\frac{n}{2}-1} G_{\alpha+\frac{n}{2}-1}(\frac{1}{2}ts|y+\xi|)$$

Putting this back in the expression for K_t^α we obtain

$$K_t^\alpha(\xi,y) = c_n\Gamma(\alpha+n)t^{-n}$$
$$\sum_{j=0}^{\infty} \frac{(-1)^j\, t^{2j}}{4^j\, j!}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \left(1-\frac{|y-\xi|^2}{t^2}\right)_+^{\alpha+\frac{n}{2}+j-1} G_{\alpha+\frac{n}{2}+j-1}(\frac{1}{2}ts|y+\xi|)$$

which ends the proof of the proposition. □

Note that for fixed y and ξ each term in the sum is holomorphic in α as long as Re $\alpha > -(\frac{n-1}{2})$. We also note that

$$\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots\cdots(\alpha+j-1)$$

is an entire function of α.

For the kernel K_t^α we now prove the following estimate.

Proposition 2.2.4 *Assume that Re $\alpha > -(\frac{n-1}{2})$. Then*

$$|K_t^\alpha(\xi,y)| \le C_\alpha\, e^{\frac{t^2}{4}}\, t^{-n} \left(1-\frac{|y-\xi|^2}{t^2}\right)_+^{-\frac{1}{2}}$$

where C_α is of admissible growth as a function of Im α.

Proof: We only need to check that

$$\sup_{j\ge 0}\left|\frac{\Gamma(\alpha+n)\,\Gamma(\alpha+j)}{\Gamma(\alpha)}\, G_{\alpha+\frac{n}{2}+j-1}(r)\right| \le C_\alpha$$

for all values of $r > 0$, when Re $\alpha > -(\frac{n-1}{2})$. The Bessel function $J_\alpha(r)$, for Re $\alpha > -\frac{1}{2}$ is defined by the integral

$$J_\alpha(r) = \frac{2^{-\alpha}\, r^\alpha}{\Gamma(\alpha+\frac{1}{2})\,\sqrt{\pi}} \int_{-1}^{1} e^{irs}\,(1-s^2\,)^{\alpha-\frac{1}{2}}\,ds.$$

Therefore, $|G_\alpha(r)| \leq A|\Gamma(\alpha+\frac{1}{2})|^{-1}$ where A depends only on Re α. So we have to show that

$$\left|\frac{\Gamma(\alpha+n)\;\Gamma(\alpha+j)}{\Gamma(\alpha)\;\Gamma(\alpha+j+\frac{n-1}{2})}\right| \leq C_\alpha.$$

When $n = 2m+1$, the left hand side reduces to $\frac{\alpha(\alpha+1)\cdots\cdots(\alpha+n-1)}{(\alpha+j)(\alpha+j+1)\cdots\cdots(\alpha+j+n-1)}$ which is certainly bounded by a constant C_α of admissible growth. When n is even we can use Stirling's formula to arrive at the same conclusion. □

We can now complete the proof of Theorem 2.2.1. Consider the family of operators

$$\mathcal{K}_t^\alpha f(x) = \int_{\mathbb{R}^n} K_t^\alpha(x,y)\; f(y)\; dy.$$

Note that in view of proposition 2.2.4 this is an admissible analytic family of operators for Re $\alpha > -(\frac{n-1}{2})$ which are bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ uniformly in $0 < t \leq 1$. By the result of Proposition 2.2.2 we know that S_t^α agrees with $\mathcal{K}_t^{\alpha-(n-1)}$ for Re $\alpha > n-1$. But S_t^α is analytic in the bigger range Re $\alpha > -\frac{1}{2}$ and so we can think of S_t^α as an analytic continuation of $\mathcal{K}_t^{\alpha-(n-1)}$.

As in the proof of Theorem 2.1.5 we now have the estimate $||S_t^\alpha f||_2 \leq C||f||_2$ for Re $\alpha > -\frac{1}{2}$, C being independent of t, for $0 < t \leq 1$. For $\alpha = \frac{n-1}{2} + \delta + i\gamma$, $S_t^\alpha = \mathcal{K}_t^{-(\frac{n-1}{2})+\delta+i\gamma}$ is bounded on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. By analytic interpolation we get

$$||S_t^\alpha f||_p \leq C||f||_p, \qquad \alpha > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$$

where C is independent of t, $0 < t \leq 1$. This completes the proof of Theorem 2.2.1. □

The operators S_t^α were studied in [32]. Among other things it was proved in [32] that $S_t^{n-1} f(x)$ converges to $f(x)$ as $t \to 0$ for almost every x in $\mathbb{R}^n$ and for $f \in L^p$, $p > \frac{2n}{2n-1}$.

Using the explicit expressions we obtained for the kernels of S_t^α we improve this to get the following.

Theorem 2.2.5 *The maximal operator* $\sup_{0<t\leq 1}|S_t^\alpha f(x)|$ *is bounded on* $L^p(\mathbb{R}^n)$, $1<p<\infty$ *for* $\alpha \geq \frac{n}{2}$. *Consequently, for* $f \in L^p(\mathbb{R}^n)$

$$\lim_{t\to 0} S_t^\alpha f(x) = f(x)$$

for almost every x *in* $\mathbb{R}^n$.

Proof: When $\alpha > \frac{n-1}{2}$, S_t^α is given by the kernel $K_t^{\alpha-(n-1)}$. It is enough to prove the theorem for $\alpha = \frac{n}{2}$. Proceeding as in the proof of Proposition 2.2.4 it is easy to show that

$$\sup_{0<t\leq 1} |S_t^{\frac{n}{2}} f(x)| \leq C \sup_{0<t<\infty} t^{-n} \int_{|x-y|\leq t} |f(y)|\, dy.$$

The right hand side is just the Hardy-Littlewood maximal function and hence

$$\int \sup_{0<t\leq 1} |S_t^{\frac{n}{2}} f(x)|^p\, dx \leq C \int |f(x)|^p\, dx$$

for $1<p<\infty$. Hence $S_t^{\frac{n}{2}} f(x)$ converges to $f(x)$ almost everywhere as $t \to 0$. □

Next we consider the following Cauchy problem for the Hermite operator.

$$\partial_t^2 v(x,t) + Hv(x,t) = 0, \quad v(x,0) = f(x) \quad \partial_t v(x,0) = 0.$$

Formally the solution is given by

$$v(x,t) = \cos t\sqrt{H} f(x).$$

We consider what may be called the Riesz means of the solution v. These are defined by

$$v^\alpha(x,t) = \int_0^t \left(1 - \frac{s^2}{t^2}\right)^{\alpha-\frac{1}{2}} v(x,s)\, ds.$$

As in [32], we can prove the pointwise convergence of Riesz means to the initial data in small dimensions. That is, for $n = 1$ and 2, we have

$$\lim_{t \to 0} \frac{v^{\frac{n}{2}}(x,t)}{t} = f(x)$$

for almost all x in $\mathbb{R}^n$ provided $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. We omit the details.

Next we study the boundedness properties of the operator $H^{-\frac{1}{2}} \sin t\sqrt{H}$. Proceeding as in the previous section we can compare S_t^α with the operators $H^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{H})$ and prove that the operators $H^{-\frac{\alpha}{2}} J_\alpha(t\sqrt{H})$ are bounded on $L^p(\mathbb{R}^n)$ as long as $\alpha > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. But we note that these results follow from a direct application of the general multiplier theorem proved in the first chapter. Taking $\alpha = \frac{1}{2}$ we get the following estimate for the solution to the wave equation associated to the Hermite operator.

Corollary 2.2.6 *For $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n}$ we have the estimate*

$$\|\frac{\sin t\sqrt{H}}{\sqrt{H}} f\|_p \leq C_t \|f\|_p.$$

If we wish to extend the above to the bigger range $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n-1}$, in the proof of Theorem 2.2.1 we have to interpolate between Re $\alpha > \frac{n}{2} - 1$ and Re $\alpha > -\frac{1}{2}$. But a close look at the expression obtained for the kernels K_t^α of S_t^α shows that, K_t^α are not integrable for Re $\alpha \leq -\frac{n}{2} + \frac{1}{2}$. Hence the above corollary seems to be the optimal result one can obtain about the L^p boundedness of the operators $\frac{\sin t\sqrt{H}}{\sqrt{H}}$. This has been already proved by Zhong [49], where he has considered Schrodinger operators with nonnegative potentials. However if we restrict ourselves to radial functions it is possible to extend the above result to the bigger range $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n-1}$. We prove

Theorem 2.2.7 *Assume $f \in L^p(\mathbb{R}^n)$ is radial. Then*

$$\|\frac{\sin t\sqrt{H}}{\sqrt{H}} f\|_p \leq C\|f\|_p \quad \text{for} \quad |\frac{1}{p} - \frac{1}{2}| < \frac{1}{n-1}.$$

To prove the above we have to recall several facts about Hermite expansions and Laguerre translations. We refer to the monograph [44] for more details. When f is a radial function Hermite expansion reduces to a Laguerre expansion. Then we can view the operators S_t^α as operators on the space $L^p([0,\infty), r^{n-1}dr)$. When $\alpha = \frac{n}{2} - 1$, S_t^α becomes a Laguerre convolution operator which is bounded on all L^p, for $1 \le p \le \infty$ and when $\alpha = -\frac{1}{2}$, S_t^α becomes a bounded operator on L^2. Analytic interpolation then will prove Theorem 2.2.7. First we recall the necessary results:

Theorem 2.2.8 *If f is a radial function then $P_{2k+1}f = 0$ and*

$$P_{2k}f(x) = R_k^{\frac{n}{2}-1}(f)\ L_k^{\frac{n}{2}-1}(|x|^2)\ e^{-\frac{1}{2}|x|^2}$$

where

$$R_k^{\frac{n}{2}-1}(f) = 2\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})}\int_0^\infty f(s)\ L_k^{\frac{n}{2}-1}(s^2)\ e^{-\frac{1}{2}s^2}\ s^{n-1}\ ds.$$

Proof: See [44] (Theorem 3.4.1).

We also need the following facts about the Laguerre translations. The Laguerre translations $T_x^\alpha f(y)$ of a function f on $\mathbb{R}_+ = [0,\infty)$ for $\alpha \ge 0$ is defined by

$$T_x^\alpha f(y) = \frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{2\pi}}\int_0^\pi f\left((x^2+y^2+2xy\cos\theta)^{\frac{1}{2}}\right) j_{\alpha-\frac{1}{2}}(xy\sin\theta)\ \sin^{2\alpha}\theta\ d\theta.$$

where we have written $j_\alpha(t) = t^{-\alpha}J_\alpha(t)$. The following results are proved in [44]. (see pages 139 and 141.)

Lemma 2.2.9 *If $\psi_k^\alpha(x) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(x^2)\ e^{-\frac{1}{2}x^2}$ then $T_x^\alpha\psi_k^\alpha(y) = \psi_k^\alpha(x)\psi_k^\alpha(y)$.*

Let $d\mu(x) = x^{2\alpha+1}dx$. Let us denote the norm in $L^p(R_+, d\mu)$ by $||f||_{p,\mu}$. Thus $||f||_{p,\mu}^p = \int_0^\infty |f(x)|^p\ x^{2\alpha+1}dx$.

Lemma 2.2.10 *For $\alpha \ge 0$ and $1 \le p \le \infty$ we have $||T_x^\alpha f||_{p,\mu} \le ||f||_{p,\mu}$.*

We refer to the monograph [44] for more properties of Laguerre translations. Now we proceed to prove Theorem 2.2.7. Define the analytic family of operators

$$B_t^\alpha f = \sum_{k=0}^{\infty} \psi_k^\alpha(t) P_{2k} f$$

where $\psi_k^\alpha(t) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha(t^2)\ e^{-\frac{1}{2}t^2}$. Here we are assuming f is a radial function. For this family we have the following results.

Proposition 2.2.11 *i)* $||B_t^\alpha f||_1 \leq C(\alpha)||f||_1$, *for* $Re\ \alpha > \frac{n}{2} - 1$

ii) $||B_t^\alpha f||_2 \leq C(\alpha)||f||_2$, *for* $Re\ \alpha > -\frac{1}{2}$

Proof: Since f is a radial function we have by Theorem 2.2.8

$$B_t^\alpha f(x) = \sum_{k=0}^{\infty} \psi_k^\alpha(t)\ R_k^{\frac{n}{2}-1}(f)\ L_k^{\frac{n}{2}-1}(|x|^2)\ e^{-\frac{1}{2}|x|^2}.$$

Hence by Lemma 2.2.9

$$B_t^{\frac{n}{2}-1} f(x) = T_t^{\frac{n}{2}-1} f(|x|)$$

and by Lemma 2.2.10 we have

$$||B_t^{\frac{n}{2}-1}||_{L^1(\mathbb{R}^n)} = ||B_t^{\frac{n}{2}-1} f||_{1,\mu} \leq C\ ||f||_{1,\mu} = C\ ||f||_{L^1(\mathbb{R}^n)}.$$

Now as in the earlier sections we can prove (i). Similarly we can prove (ii) as the sequence $\psi_k^{-\frac{1}{2}}(t)$ forms a bounded sequence and so the operator $B_t^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}^n)$. It can be checked that B_t^α forms an admissible analytic family of operators in the sense of Stein. By analytic interpolation theorem we get the following result.

Theorem 2.2.12 *Assume that* $f \in L^p(\mathbb{R}^n)$ *is radial. Then we have the estimate* $||B_t^\alpha f||_p \leq C_t||f||_p$, *for* $\alpha > (n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. □

Again as before comparing $J_\alpha(t)\, t^{-\alpha}$ with $\psi_k^\alpha(t)$ we get the same result for the multiplier $\frac{J_\alpha(t\sqrt{2k+n})}{(\sqrt{2k+n})^\alpha}$; that is

$$||\frac{J_\alpha(t\sqrt{H})}{\sqrt{H}^\alpha} f||_p \leq C||f||_p$$

for all radial functions when $\alpha > (n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$. Taking $\alpha = \frac{1}{2}$ we get *Theorem* 2.2.7.

We finish this chapter with the following remarks. More generally one can consider Schrodinger operators on $\mathbb{R}^n$ with non negative potentials. Let $P = -\Delta + V(x)$ where the potential $V(x)$ is non negative and continuous. Let $e^{-tP}(x,y)$ denote the kernel of the operator e^{-tP}. By the Feynman-Kac formula we see that [7]

$$0 \leq e^{-tP}(x,y) \leq p_t(x,y) \qquad (\ 2.2.4\ )$$

where $p_t(x,y) = (4\pi t)^{-\frac{n}{2}}\ e^{-\frac{|x-y|^2}{4t}}$. Now, using the above and modifying the method in [7] one can obtain pointwise estimates for the Bochner-Riesz kernel $s_R^\delta(x,y)$ associated to P. This will be described in details in Chapter 3 (see Propositions 3.1.1 and 3.1.2). We have

$$|s_R^\delta(x,y)| \leq C\ R^{\frac{n}{2}}(1 + R^{\frac{1}{2}}|x-y|)^{-\delta+\beta} \qquad (\ 2.2.5\ )$$

for some constant β. Hence, Theorem 1.1.1 is applicable to the above operator. As a consequence, we get the following result.

Theorem 2.2.13 *Let P be as above and $f \in L^p(\mathbb{R}^n)$. Then the estimate $||\frac{\sin t\sqrt{P}}{\sqrt{P}} f||_p \leq C_t\ ||f||_p$, holds for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{n}$* □

This theorem has been proved by Zhong [49] using a different method.

Chapter 3

Oscillating Multipliers on the Heisenberg Group

In this chapter we deal with the sublaplacian $\mathcal{L}$ on the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$. The sublaplacian can be explicitly written as $\mathcal{L} = -\Delta - \frac{1}{4}|z|^2\partial_t^2 - N\partial_t$. Wave equation associated to $\mathcal{L}$ has been studied recently by Stein and Muller. We slightly improve their result on the boundedness of the operator $\mathcal{L}^{-\frac{1}{2}} \sin s\sqrt{\mathcal{L}}$ in the case of band limited functions. Before we proceed to study the operator $\mathcal{L}^{-\frac{1}{2}} \sin s\sqrt{\mathcal{L}}$, we first explain how to obtain Bochner-Riesz kernel estimates using known estimates on heat kernels and modifying a method used by Hebisch in [7]. We will be considering the operators $P_a = \mathcal{L} + iaT$, where $T = \frac{\partial}{\partial t}$. We obtain uniform estimates on the Bochner-Riesz kernels of these operators as long as $|a| \leq n - \epsilon$, $\epsilon > 0$. We remark that this method is applicable to a wider class of operators such as Rockland operators on stratified nilpotent groups and Schroedinger operators on $\mathbb{R}^n$ and so on.

3.1 Heat kernels and Estimates on Bochner-Riesz kernels

For $a \in I\!R$ consider the operator $P_a = \mathcal{L} + iaT$ which is a Rockland operator on H^n as long as a is admissible, that is when $|a| < n$. We have the following uniform estimates for the heat kernels of these operators.

Proposition 3.1.1 *Let $\epsilon > 0$ be fixed. Let $p_{s,a}(z,t)$ be the kernel of the operator e^{-sP_a}, $s > 0$. Then we have*

$$p_{s,a}(z,t) \leq C\ s^{-\frac{Q}{2}}\ e^{-\frac{A}{s}(|z|^2+|t|)}$$

where A and C are independent of a for $|a| \leq n - \epsilon$.

Proof: Note that P_a is homogeneous of degree 2 with respect to the Heisenberg dilations. So it is enough to consider $s = 1$. Let us write $p_{1,a}(z,t) = K_a(z,t)$. It is easily seen that the kernel is given by the formula

$$K_a(z,t) = c_n \int_{-\infty}^{\infty} k_a(z,t,\lambda)\ d\lambda$$

where

$$k_a(z,t,\lambda) = e^{-a\lambda} \left(\frac{\lambda}{\sinh\lambda}\right)^n e^{-\frac{\lambda}{4}(\coth\lambda)|z|^2}\ e^{i\lambda t}.$$

Note that $k_a(z,t,\lambda)$ extends to a holomorphic function of λ in the strip $|\text{Im}\ \lambda| < \frac{\pi}{2}$. Hence by Cauchy's theorem

$$K_a(z,t) = \lim_{R\to\infty} \{ \int_0^{\frac{\pi}{4}} k_a(z,t,-R+i\sigma)\ d\sigma + \int_{-R}^{R} k_a(z,t,\lambda + i\frac{\pi}{4})\ d\lambda - \int_0^{\frac{\pi}{4}} k_a(z,t,R+i\sigma)\ d\sigma \}.$$

In the above the first and last integrals go to zero uniformly in a as $R \to \infty$, provided $|a| \leq n - \epsilon$. Then we get

$$K_a(z,t) = c_n \int_{-\infty}^{\infty} k_a(z,t,\lambda + i\frac{\pi}{4}) d\lambda$$

and from this we obtain

$$|K_a(z,t)| \leq C\, e^{-\frac{\pi}{4}|t|}\ ,\ t > 0 \qquad (3.1.1)$$

where C is independent of a. The same estimate holds for $t < 0$ as well. As $\coth\lambda$ behaves like λ^{-1} for λ small we easily get the estimate

$$|K_a(z,t)| \leq C\, e^{-\frac{1}{4}|z|^2}. \qquad (3.1.2)$$

The estimates (3.1.1) and (3.1.2) put together prove the proposition. □

Using the heat kernel estimates proved above and following a method of Hebisch[7] we can obtain uniform estimates on the Bochner-Riesz kernels associated to P_a. Let us write $w = (z,t)$ and $|w|$ be the homogeneous norm defined by $|w|^4 = |z|^4 + |t|^2$.

Proposition 3.1.2 *Let $S^{\delta}_{R,a}(w)$ be the kernel of the Bochner-Riesz means associated to P_a. Then for $|a| \leq n - \epsilon$ and for all large δ,*

$$|S^{\delta}_{R,a}(w)| \leq C\, R^{\frac{Q}{2}}(1 + R^{\frac{1}{2}}|w|)^{-\delta+\beta}$$

where C is independent of a, R and β is a fixed constant.

Proof: Due to homogeneity of the operators P_a it is enough to consider $R = 1$. Following Hebisch we let $E^a_n(w)$ be the kernel of the operator $e^{inK}\ K$ with $K = e^{-P_a}$. By appealing to Theorem 3.1 in [7] we get the estimate

$$\int_{H^n} |E^a_n(w)|\ (1 + |w|)^{\gamma}\ dw \leq C(1 + |n|)^{\gamma+\frac{Q}{2}}$$

for every $\gamma \geq 0$ and C independent of a. Defining e^a_n to be the kernel of $e^{inK}\ K^2$ we have

$$e^a_n(w) = E^a_n * p_{1,a}(w).$$

Here $*$ denotes the convolution on H^n. Using the L^1 estimate of E^a_n and the heat kernel estimate of P_a we easily get the estimate

$$|e^a_n(w)| \leq C\ (1 + |w|)^{-\gamma}\ (1 + |n|)^{\gamma+\frac{Q}{2}} \qquad (3.1.3)$$

for all $\gamma \geq 0$ with C independent of a.

We can now make use of the functional calculus developed in [7] to get estimates of the Bochner-Riesz kernel. For the sake of completeness we briefly indicate the method. Let $F(\lambda) = (1-\lambda)_+^\delta\ \psi(\lambda)$ where $\psi \in C^\infty$ be such that $\psi(\lambda) = 1$ for $\lambda \geq 0$ and $\psi(\lambda) = 0$ for $\lambda \leq -e^{-1}$. Let $G(\lambda) = \lambda^{-2}F(-log\lambda)$ for $\lambda > 0$; $G(\lambda) = 0$ otherwise. Then $G(\lambda)$ is supported in $[0,e]$ and $F(P_a) = G(e^{-P_a})\ e^{-2P_a}$. Expanding $G(\lambda)$ into Fourier series as $G(\lambda) = \sum_{n=-\infty}^{\infty} \hat{G}(n)\ e^{in\lambda}$ we get

$$F(P_a) = \sum_{n=-\infty}^{\infty} \hat{G}(n)\ e^{inK}\ K^2$$

where, as before, $K = e^{-P_a}$.

Using the estimate (3.1.3) we get

$$|S_{1,a}^\delta(x,y)| \leq\ C\ (1+|w|)^{-\gamma} \sum_{n=-\infty}^{\infty} |\hat{G}(n)|(1+|n|)^{\gamma+\frac{Q}{2}}.$$

The coefficients $\hat{G}(n)$ are given by

$$\hat{G}(n) = \frac{1}{2\pi}\int_0^e\ G(\lambda)\ e^{-in\lambda}\ d\lambda.$$

Making a change of variables we get

$$\hat{G}(n) = \frac{1}{2\pi}\int_{-e^{-1}}^{1}\ F(t)\ e^t\ e^{-ine^{-t}}\ dt.$$

As $F(t) = (1-t)_+^\delta \psi(t)$ we easily get the estimate

$$|\hat{G}(n)| \leq C(1+|n|)^{-l}$$

provided $\delta > l-1$. Taking $\delta = \gamma + \frac{Q}{2} + 2$ we have $|\hat{G}(n)| \leq C(1+|n|)^{-\gamma-\frac{Q}{2}-2}$ and hence

$$|S_{1,a}^\delta(w)| \leq\ C\ (1+|w|)^{-\delta+\frac{Q}{2}+2}$$

where C is independent of a. This completes the proof of the proposition. □

We remark that, as a consequence of the above and Theorem 1.1.8 in Chapter 1 we have the following.

Theorem 3.1.3 *Let $m \in S_\rho^{-\alpha}$ and $1 < p < \infty$. Then the operators $m(P_a)$ are uniformly bounded on $L^p(H^n)$ for $|a| \leq n - \epsilon$ whenever $\alpha > (2n+2)(1-\rho)|\frac{1}{p} - \frac{1}{2}|$.* □

3.2 Wave equation for the sublaplacian

Next we return to the wave equation, with s as the time variable,

$$\partial_s^2 u(z,t,s) = \mathcal{L}u(z,t,s), \quad u(z,t,0) = 0, \quad \partial_s u(z,t,0) = f(z,t). \tag{3.2.4}$$

In [25] Stein and Muller have studied the L^p boundedness properties of the solution to the above problem. They have proved

Theorem 3.2.1 *(Muller-Stein) For $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$, the operator $\mathcal{L}^{-\frac{1}{2}} \sin s\sqrt{\mathcal{L}}$ extends to a bounded operator on $L^p(H^n)$.*

We improve this result slightly in the case when f is band limited in the t variable. Let $L^p_B(H^n)$ stand for all f in $L^p(H^n)$ for which the partial Fourier transform $f^\lambda(z)$ in the t-variable is supported in $|\lambda| \leq B$. On this space we have the following improvement of Theorem 3.2.1.

Theorem 3.2.2 *Let $n \geq 2$. The operator $\mathcal{L}^{-\frac{1}{2}} \sin\sqrt{\mathcal{L}}$ is bounded from $L^p_B(H^n)$ to $L^p_B(H^n)$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n-1}$.*

More generally we can consider operators of the form $\mathcal{L}^{-\frac{\alpha}{2}} J_\alpha(\sqrt{\mathcal{L}})$. We have the following result for these operators.

Theorem 3.2.3 *The operator $\mathcal{L}^{-\frac{\alpha}{2}} J_\alpha(\sqrt{\mathcal{L}})$ is bounded on $L^p_B(H^n)$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{2\alpha+1}{4n-2}$ provided $6\alpha \leq 4n-5$. Otherwise it is bounded on $L^p_B(H^n)$ in the smaller range $|\frac{1}{p} - \frac{1}{2}| < \frac{2\alpha+3}{4n+4}$.*

Note that Theorem 3.2.2 follows from the above when $\alpha = \frac{1}{2}$. As in the earlier chapters we first consider the operators

$$T_r^\alpha f(z,t) = c_n \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \psi_k^\alpha(r\sqrt{|\lambda|}) f * e_k^\lambda(z,t) \right) |\lambda|^n \, d\lambda.$$

For this family we prove the following result.

Theorem 3.2.4 *i) When $\alpha > (2n-1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$, T_r^α are uniformly bounded on $L_B^p(H^n)$ for $0 < r \leq 1$.*
ii) When $\alpha > (2n - \frac{4}{3})|\frac{1}{p} - \frac{1}{2}| - \frac{1}{3}$, T_r^α are uniformly bounded on $L^p(H^n)$ for all $r > 0$.

Proof: Let μ_r be the normalised surface measure on the sphere $S_r = \{(z,0);\ |z| = r\}$. Then it is well known (see [29]) that

$$f * \mu_r = c_n \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|}\, r)\, f * e_k^\lambda \right) |\lambda|^n \, d\lambda. \qquad (3.2.5)$$

Now Laguerre functions of different type are related by the formula

$$L_k^{\alpha+\beta}(r) = \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta)\,\Gamma(k+\alpha+1)} \int_0^1 s^\alpha (1-s)^{\beta-1} L_k^\alpha(sr)\, ds$$

which is valid for Re $\alpha > -1$ and Re $\beta > 0$. Using this we can write when $\alpha = n - 1 + \delta + i\sigma$

$$\psi_k^\alpha(r) = \frac{\Gamma(k+n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\,\Gamma(k+n)} \int_0^1 s^{n-1} (1-s)^{\delta+i\sigma-1} e^{-\frac{1}{4}(1-s)r^2} \psi_k^{n-1}(r\sqrt{s})\, ds. \qquad (3.2.6)$$

Let us define an operator $A_r f$ by

$$(A_r f)^\lambda(z) = e^{-\frac{1}{4}r|\lambda|} f^\lambda(z)$$

where $f^\lambda(z)$ is the partial inverse Fourier transform of $f(z,t)$ in the t-variable. For $\alpha = n - 1 + \delta + i\sigma$ we then have

$$T_r^\alpha f = \frac{\Gamma(n+\delta+i\sigma)}{\Gamma(\delta+i\sigma)\,\Gamma(n)} \int_0^1 s^{n-1}(1-s)^{\delta+i\sigma-1} T_{r\sqrt{s}}^{n-1} A_{(1-s)r^2} f \; ds \qquad (3.2.7)$$

Similarly when $\alpha = -\frac{1}{2} + \delta + i\sigma$ we have

$$T_r^\alpha f = \frac{\Gamma(-\frac{1}{2} + \delta + i\sigma)}{\Gamma(\delta + i\sigma)\Gamma(-\frac{1}{2})} \int_0^1 s^{-\frac{1}{2}}(1-s)^{\delta+i\sigma-1} T_{r\sqrt{s}}^{-\frac{1}{2}} A_{(1-s)r^2} f \; ds. \qquad (3.2.8)$$

The operators $A_r f$ are nothing but the Poisson integrals in the t-variable and so they are uniformly bounded on $L^p(H^n)$ for all $1 \leq p \leq \infty$. Therefore, from (3.2.7) we see that

$$||T_r^\alpha f||_p \leq C(\sigma)||f||_p \, , \; 1 \leq p \leq \infty$$

when $\alpha = n - 1 + \delta + i\sigma$. When $\alpha = -\frac{1}{2}$, the Laguerre functions $\psi_k^{-\frac{1}{2}}(r)$ are uniformly bounded in k as long as r remains bounded. Let $\chi \in C_0^\infty(|\lambda| \leq (B+1))$ be such that $\chi(\lambda) = 1$ for $|\lambda| \leq B$. Define $\chi(i\partial_t)$ to be the operator

$$(\chi(i\partial_t)f)^\lambda(z) = \chi(\lambda)f^\lambda(z).$$

Then the multiplier corresponding to $T_r^\alpha \chi(i\partial_t)$ is $\psi_k^\alpha(\sqrt{|\lambda|}\; r)\chi(\lambda)$ which is uniformly bounded. That is ,

$$|\psi_k^\alpha(\sqrt{|\lambda|}\; r)\chi(\lambda)| \leq C$$

for all $\lambda \in I\!R$, $k = 0, 1, \cdots$ and $0 \leq r \leq 1$. Therefore, by Plancherel theorem

$$||T_r^\alpha \chi(i\partial_t)f||_2 \leq C_B(\sigma)||f||_2$$

when $\alpha = -\frac{1}{2} + \delta + i\sigma$. Using Stirling's formula for the gamma function we can check that $C(\sigma)$ and $C_B(\sigma)$ are of admissible growth.

By appealing to Stein's analytic interpolation theorem we obtain

$$||T_r^\alpha \chi(i\partial_t)f||_p \leq C||f||_p$$

for $\alpha > (2n-1)(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. This proves part (i) of Theorem 3.2.4. To prove the other part we use the uniform estimate $|\psi_k^{-\frac{1}{3}}(t)| \leq C$ which is valid for all $t > 0$ and $k = 0, 1, \cdots$(see Szego [42]). As before, analytic interpolation will prove part (ii). □

Now we will use the above theorem to study multipliers of the form $m((2k+\alpha+1)|\lambda|)$. However, to prove Theorem 3.2.2 we need to treat multipliers of the form $m((2k+n)|\lambda|)$. This can be achieved by comparing these two multipliers. Taking $m(t) = t^{-\frac{\alpha}{2}} J_\alpha(\sqrt{t})$ we have the equation

$$m((2k+n)|\lambda|) - m((2k+\alpha+1)|\lambda|) = |\lambda| \int_{\alpha+1}^{n} m'((2k+t)|\lambda|)dt.$$

Since $m'(t) = -\frac{1}{2} t^{-\frac{\alpha+1}{2}} J_{\alpha+1}(\sqrt{t})$ we have

$$m((2k+n)|\lambda|) - m((2k+\alpha+1)|\lambda|) = -\frac{1}{2}\,|\lambda| \int_{\alpha+1}^{n} \frac{J_{\alpha+1}(\sqrt{(2k+t)|\lambda|}\,)}{(\sqrt{(2k+t)|\lambda|})^{\alpha+1}} dt. \qquad (3.2.9)$$

Note that $\lambda^{-\frac{\alpha+1}{2}} J_{\alpha+1}(\sqrt{\lambda})$ belongs to the symbol class $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{3}{4}}(\mathbb{R})$ whereas $\lambda^{-\frac{\alpha}{2}} J_\alpha(\sqrt{\lambda})$ belongs to $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{1}{4}}(\mathbb{R})$.

Therefore, if we can show that the operators $J_r^\alpha f$ defined by

$$J_r^\alpha f = \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \frac{J_{\alpha+1}(\sqrt{(2k+r)|\lambda|})}{(\sqrt{(2k+r)|\lambda|})^{\alpha+1}} f * e_k^\lambda \Big) |\lambda|^n \, d\lambda$$

are uniformly bounded on $L^p(H^n)$ for $2\alpha+3 > 2Q(\frac{1}{p}-\frac{1}{2})$, $\alpha+1 \leq r \leq n$, then from (3.2.9) it will follow that $m(\mathcal{L})$ is bounded on $L_B^p(H^n)$ when the multiplier $m((2k+\alpha+1)|\lambda|)$ defines a bounded operator on $L_B^p(H^n)$. Thus we require the following.

Theorem 3.2.5 *Let $m \in S_\rho^{-\alpha}(\mathbb{R})$ and let M_r be the operator given by the multiplier $m((2k+r)|\lambda|)$ where $0 < \epsilon < r < 2n-\epsilon$. Then M_r are uniformly bounded on $L^p(H^n)$ when $\alpha > Q(1-\rho)|\frac{1}{p}-\frac{1}{2}|$.*

Proof: Let Hf be the Hilbert transform of f in the t-variable defined by

$$(Hf)^\lambda(z) = -isgn\lambda \; f^\lambda(z).$$

Write $g = \frac{1}{2}(f+iHf)$ and $h = \frac{1}{2}(f-iHf)$ so that $f = g+h$ and $||g||_p \leq C||f||_p$, $||h||_p \leq C||f||_p$. Note that $g^\lambda(z)$ vanishes for $\lambda < 0$ and $h^\lambda(z)$ for $\lambda > 0$. Now

$$M_r g = c_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} m((2k+n)|\lambda| + (r-n)\lambda) \; g * e_k^\lambda \Big) \, |\lambda|^n \, d\lambda$$

which is nothing but $m(\mathcal{L}+i(n-r)T)g$. Similarly $M_r h = m(\mathcal{L}-i(n-r)T)h$. Now from Theorem 3.1.3 the operators M_r are uniformly bounded on $L^p(H^n)$. As $M_r f = M_r g + M_r h$ we see that M_r is bounded on $L^p(H^n)$. This finishes the proof of Theorem 3.2.5. □

In view of above theorem and the remarks preceding it, it is enough to consider the operator M_r^α given by the multiplier $m_r^\alpha(k,\lambda)$ where

$$m_r^\alpha(k,\lambda) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(r\sqrt{(2k+\alpha+1)|\lambda|})}{(r\sqrt{(2k+\alpha+1)|\lambda|})^\alpha}.$$

As in the previous chapter we compare the multipliers $m_r^\alpha(k,\lambda)$ and $\psi_k^\alpha(\sqrt{|\lambda|}\, r)$ by using a Hilb type asymptotic formula for the Laguerre polynomials. Formula 8.64.3 on page 217 of Szego [42] gives

$$\psi_k^\alpha(r) = m_r^\alpha(k,1) + e(k,\alpha,r) \qquad (3.2.10)$$

where $e(k,\alpha,r)$ is given by the integral

$$\frac{\pi}{2^3}\frac{r^4}{\sin\alpha\pi}\int_0^1 (J_\alpha(r\sqrt{K})J_{-\alpha}(rs\sqrt{K}) - J_\alpha(rs\sqrt{K})\, J_{-\alpha}(r\sqrt{K}))\, s^{\alpha+3}\psi_k^\alpha(rs)\, ds$$

In the above formula $K = 2k+\alpha+1$. When α is an integer, $\sin\alpha\pi$ in the above formula has to be replaced by -1 and $J_{-\alpha}$ by the modified Bessel function Y_α.

Let us define $a_\alpha(\lambda,r,s)$ for $\lambda>0$ by

$$a_\alpha(\lambda,r,s) = (\, J_\alpha(r\sqrt{\lambda})\, J_{-\alpha}(rs\sqrt{\lambda}) - J_{-\alpha}(r\sqrt{\lambda})\, J_\alpha(rs\sqrt{\lambda})\,)s^{\alpha+3}r^4$$

and let $A_\alpha(r,s)$ be the operator whose multiplier is $a_\alpha(\,(2k+n)|\lambda|,r,s)$. Let χ and $\chi(i\partial_t)$ be as before. From (3.2.10) it follows that

$$T_r^\alpha\,\chi(i\partial_t)f = M_r^\alpha\chi(i\partial_t)f + c_1\int_0^1 A_\alpha(r,s)\,T_{rs}^\alpha\,\chi_1(i\partial_t)\,f\,ds$$

where $\chi_1(\lambda) = \lambda^2\chi(\lambda)$ and c_1 is some constant. Another iteration produces the formula

$$\begin{aligned} M_r^\alpha\chi(i\,\partial_t)f &= T_r^\alpha\chi(i\,\partial_t)f + c_1\int_0^1 A_\alpha(r,s)\,M_{rs}^\alpha\chi_1(i\,\partial_t)f\,ds \\ &+ c_2\int_0^1\int_0^1 A_\alpha(r,s)\,A_\alpha(rs,s')\,T_{rss'}^\alpha\,\chi_2(i\,\partial_t)f\,ds\,ds' \qquad (3.2.11)\end{aligned}$$

where $\chi_2(\lambda) = \lambda^4\chi(\lambda)$ and c_1 , c_2 are constants.

We are now in a position to prove Theorem 3.2.2 . From Theorem 3.2.4 we know that $T_1^\alpha\chi(i\partial_t)$ is bounded on $L^p(H^n)$ for $|\frac{1}{p} - \frac{1}{2}| < \frac{2\alpha+1}{4n-2}$. If $6\alpha \leq 4n - 5$, then $\frac{2\alpha+1}{4n-2} \leq \frac{2\alpha+3}{4n+4}$ and consequently $\frac{\alpha}{2} + \frac{3}{4} > \frac{Q}{2}|\frac{1}{p} - \frac{1}{2}|$ whenever $|\frac{1}{p} - \frac{1}{2}| < \frac{2\alpha+1}{4n-2}$. The multiplier corresponding to the product $A_\alpha(1,s)\ M_s^\alpha$ is given by the symbol

$$m(\lambda, s) = a_\alpha(\lambda, 1, s)\ B_\alpha(s^2\ \lambda)$$

where $B_\alpha(t) = t^{-\frac{\alpha}{2}}J_\alpha(\sqrt{t})$, which belongs to the class $S_{\frac{1}{2}}^{-\frac{\alpha}{2}-\frac{3}{4}}(I\!R)$. Using Lemma 2.1.6 in Chapter 2 we can show that

$$|\partial_\lambda^k m(\lambda, s)| \leq C\ (1+\lambda)^{-\frac{1}{2}(\alpha+\frac{3}{2}+k)}$$

where C is uniform for $0 \leq s \leq 1$. Since $\frac{\alpha}{2} + \frac{3}{4} > \frac{Q}{2}|\frac{1}{p} - \frac{1}{2}|$, from Theorem 3.2.5 we conclude that

$$||A_\alpha(1,s)\ M_s^\alpha f||_p \leq C\ ||f||_p$$

where C is independent of s. Therefore, the operator

$$\int_0^1 A_\alpha(1,s)\ M_s^\alpha\chi(i\ \partial_t)f\ ds$$

is bounded on $L^p(H^n)$.

For the third term in (3.2.11), the symbol of the operator $A_\alpha(1,s)\ A_\alpha(s,s')$ comes from $S_{\frac{1}{2}}^{-1}(I\!R)$ and the derivatives satisfy uniform estimates for $0 \leq s$, $s' \leq 1$ in view of Lemma 2.1.6 in Chapter 2. If $0 \leq \alpha \leq \frac{1}{2}$ we can conclude that the operator

$$\int_0^1 \int_0^1 A_\alpha(1,s)\ A_\alpha(s,s')\ T_{ss'}^\alpha\ \chi_2(i\ \partial_t)f\ ds\ ds'$$

is also bounded on $L^p(H^n)$. Therefore, from (3.2.11) we see that $M_1^\alpha\ \chi(i\ \partial_t)$ is bounded on $L^p(H^n)$. If $\alpha > \frac{1}{2}$, we can perform further iterations and then the symbol of

$$A_\alpha(1,s_1)\ A_\alpha(s_1,s_2)\ \cdots\ A_\alpha(s_1s_2\cdots s_{l-1},s_l)$$

will come from $S_{\frac{1}{2}}^{-\frac{l}{2}}(I\!R)$ with estimates uniformly in $s_1, s_2, \cdots s_l$. We can choose l large enough so that $\frac{\alpha}{2} + \frac{3}{4} \leq \frac{l}{2}$ and appealing to Theorem 3.2.5 we get the boundedness of M_1^α in the case when $6\alpha \leq 4n - 5$. If $6\alpha > 4n - 5$ then we need to assume the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{2\alpha+3}{4n+4}$ so that $\frac{\alpha}{2} + \frac{3}{4} > \frac{Q}{2}|\frac{1}{p} - \frac{1}{2}|$. We then proceed as before to complete the proof. □

Chapter 4

Bochner-Riesz Means and Equisummability

This chapter is devoted to a study of the Bochner-Riesz means associated to various eigenfunction expansions. From a transplantation Theorem of Kenig- Stanton-Tomas it follows that local uniform boundedness of Bochner-Riesz means associated to Hermite or special Hermite expansions imply corresponding global results for the Euclidean Fourier expansions. Here we are interested in the converse of this result. In the first section we prove that uniform boundedness of Bochner-Riesz means associated to Fourier transform on $\mathbb{R}^n$ is equivalent to the local uniform boundedness of Bochner-Riesz means associated to the Hermite expansions in small dimensions and partially in higher dimensions. Next, by comparing Bochner-Riesz means associated to special Hermite expansions on $\mathbb{C}^n$ with that of the Fourier expansions on $\mathbb{R}^{2n}$ we prove the equivalence of local boundedness for special Hermite expansions and uniform boundedness of Fourier expansions. We also study the Bochner-Riesz means associated to the Hermite expansions on $\mathbb{R}^{2n}$ for functions having some homogeneity.

4.1 Hermite expansions

In our study of the Bochner-Riesz means associated to Hermite and special Hermite expansions we make use of a transplantation theorem of Kenig-Stanton -Tomas [12]. Let us briefly recall their result. Let P be a differential operator acting on $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ which is self adjoint. Let

$$P = \int \lambda dE_\lambda$$

be the spectral resolution of P. Let m be a bounded function on $\mathbb{R}$ and define

$$m_R(P) = \int m(\frac{\lambda}{R})\ dE_\lambda.$$

Let K be a subset of $\mathbb{R}^n$ with positive measure and define the projection operator χ_K on $L^2(\mathbb{R}^n)$ by

$$\chi_K f(x) = \chi_K(x)\ f(x).$$

where $\chi_K(x)$ is the characteristic function of K. Let $p(x,\xi)$ be the principal symbol of P. Since P is symmetric p is real valued. In [12] the following transplantation theorem is proved.

Theorem 4.1.1 *Assume $1 \leq p \leq \infty$ and that there is a set of positive measure K_0 for which the operators $\chi_{K_0} m_R(P) \chi_{K_0}$ are bounded on $L^p(\mathbb{R}^n)$ uniformly in R. If x_0 in K_0 is any point of density, then $m(p(x_0,\xi))$ is a Fourier multiplier of $L^p(\mathbb{R}^n)$.*

Let B be any compact set in $\mathbb{R}^n$ containing origin as a point of density. Let S_R^δ be the Bochner-Riesz operator associated to Hermite expansions on $\mathbb{R}^n$ and let S_t^δ be the Bochner-Riesz operator associated to the Fourier transform on $\mathbb{R}^n$. Then from Theorem 4.1.1 it follows that the uniform boundedness of $\chi_B S_R^\delta \chi_B$ on $L^p(\mathbb{R}^n)$ implies the uniform boundedness of S_t^δ on $L^p(\mathbb{R}^n)$. Thus once we have the local summability theorem for Hermite expansions then a global result is true for the Fourier transform.

In the higher dimensions it is convenient to work with Cesaro means rather than Riesz means. These are defined by

$$\sigma_N^\delta f(x) = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta P_k f(x)$$

where A_k^δ are the binomial coefficients defined by $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$. It is well known that σ_N^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ iff S_R^δ are uniformly bounded. Let E stand for the operator

$$Ef(x) = e^{-\frac{1}{2}|x|^2} f(x).$$

We then have the following equisummability result.

Theorem 4.1.2 *$E\sigma_N^\delta E$ are uniformly bounded on $L^p(R^n)$ iff S_t^δ are uniformly bounded on the same $L^p(\mathbb{R}^n)$, provided $\delta \geq max\ \{0, \frac{n}{2} - 1\}$.*

Let $S_N f(x) = \sum_{k=0}^{N} (f, h_k) h_k(x)$ be the partial sums associated to the one dimensional Hermite expansions. In 1965 Askey-Wainger [1] proved the following celebrated theorem

Theorem 4.1.3 *$S_N f \to f$ in the L^p norm iff $\frac{4}{3} < p < 4$.*

Note that this is in sharp contrast with the well known result, S_t^δ are uniformly bounded on L^p, $1 < p < \infty$ for all $\delta \geq 0$. In particular when $\delta = 0$ we have, the partial sum operators S_t associated to Fourier transform on $\mathbb{R}^n$ are uniformly bounded. As a corollary we have the following.

Corollary 4.1.4 *Let $1 < p < \infty$. Then for the partial sum operators associated to the one dimensional Hermite expansion we have the uniform estimate*

$$\int |S_N f(x)|^p e^{-\frac{p}{2}x^2} dx \leq C \int |f(y)|^p e^{\frac{p}{2}y^2} dy.$$

Thus for $f \in L^p(e^{\frac{p}{2}y^2} dy)$, $1 < p < \infty$ the partial sums converge to f in $L^p(e^{-\frac{p}{2}x^2} dx)$.

For a general weighted norm inequality for Hermite expansions see Muckenhopt's paper [21].

The celebrated theorem of Carleson-Sjolin for the Fourier expansion on $\mathbb{R}^2$ says that if $\delta > 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$, $1 \le p < \frac{4}{3}$ then S_t^δ are uniformly bounded on $L^p(\mathbb{R}^2)$. As a corollary to this we obtain the following result for the Cesaro means σ_N^δ on $\mathbb{R}^2$.

Corollary 4.1.5 *Let $n = 2$, $1 \le p < \frac{4}{3}$ and $\delta > 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then for $f \in L^p(\mathbb{R}^2)$*

$$\int |\sigma_N^\delta f(x)|^p e^{-\frac{p}{2}|x|^2} dx \le C \int |f(y)|^p e^{\frac{p}{2}|y|^2} dy.$$

It is an interesting and more difficult problem to establish the above without the exponential factors.

We now proceed to prove Theorem 4.1.2. It is a trivial matter to see that uniform boundedness of $E\sigma_N^\delta E$ implies the same for $\chi_B \sigma_N^\delta \chi_B$ for any compact subset B of $\mathbb{R}^n$. In fact, if $E\sigma_N^\delta E$ are uniformly bounded, then

$$\begin{aligned}
&\int |\chi_B \sigma_N^\delta \chi_B f|^p \, dx \\
&= \int_B e^{-\frac{p}{2}|x|^2} \, e^{\frac{p}{2}|x|^2} |\sigma_N^\delta \left(e^{-\frac{1}{2}|y|^2}(\chi_B f(y) \, e^{\frac{1}{2}|y|^2})\right)|^p dx \\
&\le C \int |E\sigma_N^\delta E(\chi_B f(y) \, e^{\frac{1}{2}|y|^2})|^p \, dx \\
&\le C \int |f(x)|^p \, dx.
\end{aligned}$$

In view of the transplantation theorem this proves one way implication. To prove the converse we proceed as follows. Let

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y)$$

be the kernel of the projection operator P_k. Then the kernel $\sigma_N^\delta(x,y)$ of the Cesaro means is given by

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \Phi_k(x,y).$$

We first obtain a usable expression for this kernel in terms of certain Laguerre functions. Let $L_k^\alpha(t)$ be the Laguerre polynomials of type $\alpha > -1$ defined for $t > 0$ by

$$e^{-t}t^\alpha L_k^\alpha(t) = (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}(e^{-t}t^{k+\alpha}).$$

We have the following expression.

Proposition 4.1.6

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta}\sum_{k=0}^{N}(-1)^k L_{N-k}^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)e^{-\frac{1}{4}|x-y|^2}\ L_k^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)e^{-\frac{1}{4}|x+y|^2}.$$

Proof : The generating function identity for the projection kernels $\Phi_k(x,y)$ reads (see [44])

$$\sum_{k=0}^{\infty} r^k\Phi_k(x,y) = \pi^{-\frac{n}{2}}(1-r^2)^{-\frac{n}{2}}\ e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

Since

$$(1-r)^{-\delta-1} = \sum_{k=0}^{\infty} A_k^\delta r^k$$

the generating function for $\sigma_k^\delta(x,y)$ is given by

$$\sum_{k=0}^{\infty} r^k A_k^\delta \sigma_k^\delta(x,y) = (1-r)^{-\delta-\frac{n}{2}-1}(1+r)^{-\frac{n}{2}}e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

The right hand side of the above expression can be written as

$$(1-r)^{-\delta-\frac{n}{2}-1}e^{-\frac{1}{4}\frac{1+r}{1-r}|x-y|^2}(1+r)^{-\frac{n}{2}}e^{-\frac{1}{4}\frac{1-r}{1+r}|x+y|^2}.$$

Now the generating function for the Laguerre polynomials L_k^α is (see Szego [42])

$$\sum_{k=0}^{\infty} r^k L_k^\alpha\left(\frac{1}{2}t^2\right)e^{-\frac{1}{4}t^2} = (1-r)^{-\alpha-1}e^{-\frac{1}{4}\frac{1+r}{1-r}t^2}.$$

Therefore, we have

$$\begin{aligned}\sum_{k=0}^{\infty} r^k A_k^\delta\sigma_k^\delta(x,y) &= \left(\sum_{j=0}^{\infty} r^j L_j^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)e^{-\frac{1}{4}|x-y|^2}\right)\\ &\quad\left(\sum_{i=0}^{\infty}(-r)^i L_i^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)e^{-\frac{1}{4}|x+y|^2}\right).\end{aligned}$$

Equating the coefficients of r^k on both sides we obtain the proposition. □

The Laguerre functions L_k^α are expressible in terms of Bessel functions J_α. More precisely, we have the formula (see [14])

$$e^{-x}x^{\frac{\alpha}{2}}L_k^\alpha(x) = \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-t}t^{k+\frac{\alpha}{2}}J_\alpha(2\sqrt{tx})dt.$$

Using this, the kernel $e^{-\frac{1}{2}|x|^2}\sigma_N^\delta(x,y)e^{-\frac{1}{2}|y|^2}$ of the operator $E\sigma_N^\delta E$ is given by the integral

$$\frac{C}{A_N^\delta}\int_0^\infty\int_0^\infty e^{-t}e^{-s}\frac{(t-s)^N}{N!}\, t^\delta s^{\frac{n}{2}-1}\, t^{\frac{n}{2}}\frac{J_{\delta+\frac{n}{2}}(\sqrt{2t}|x-y|)}{(\sqrt{2t}|x-y|)^{\delta+\frac{n}{2}}}\ \frac{J_{\frac{n}{2}-1}(\sqrt{2s}|x+y|)}{(\sqrt{2s}|x+y|)^{\frac{n}{2}-1}}dtds$$

where C depends only on δ. Now the kernel of the Bochner-Riesz means S_t^δ on $I\!R^n$ is given by

$$S_t^\delta(x,y) = t^n\frac{J_{\delta+\frac{n}{2}}(t|x-y|)}{(t|x-y|)^{\delta+\frac{n}{2}}}.$$

When n =1 we have $\pi^{\frac{1}{2}}\ t^{\frac{1}{2}}\ J_{-\frac{1}{2}}(t) = 2^{\frac{1}{2}}\ \cos t$ and hence

$$E\sigma_N^\delta Ef(x) = C\frac{1}{A_N^\delta}\int_0^\infty\int_0^\infty e^{-t}e^{-s}\frac{(t-s)^N}{N!}\ t^\delta\ s^{-\frac{1}{2}}\ T_{t,s}^\delta f(x)dt\ ds,$$

where

$$T_{t,s}^\delta f(x) = \int_{I\!R} S_{\sqrt{2t}}^\delta(x,y)\cos(\sqrt{2s}|x+y|)\ f(y)\ dy$$

and C an absolute constant. Writing $\cos\sqrt{2s}(x+y) = \frac{1}{2}(e^{i\sqrt{2s}(x+y)} + e^{-i\sqrt{2s}(x+y)}$ it is easy to see that uniform boundedness of S_t^δ implies uniform boundedness of $T_{t,s}^\delta$. By Minkowski's integral inequality we get

$$\begin{aligned}||E\sigma_N^\delta Ef||_p &\leq C\frac{1}{N^\delta}\int_0^\infty\int_0^\infty e^{-t}e^{-s}\frac{|t-s|^N}{N!}\ t^\delta\ s^{-\frac{1}{2}}||T_{t,s}^\delta f||_p\, dt\ ds\\ &\leq C||f||_p\end{aligned}$$

since

$$\int_0^\infty\int_0^\infty e^{-t}\ e^{-s}\ |t-s|^N\ t^\delta\ s^{-\frac{1}{2}}\ dt\ ds$$

$$
\begin{aligned}
&\leq \int_0^\infty e^{-t}\, t^\delta \left(\int_0^t e^{-s}\, t^N s^{-\frac{1}{2}}\, ds + \int_t^\infty e^{-s}\, s^N\, s^{-\frac{1}{2}}\, ds \right) dt \\
&\leq C \int_0^\infty e^{-t}\, t^\delta\, t^N\, dt \;+\; \Gamma\left(N + \frac{1}{2}\right) \int_0^\infty e^{-t}\, t^\delta\, dt \\
&\leq C N!\, N^\delta
\end{aligned}
$$

which proves the theorem in one dimension.

When $n \geq 2$ we have the Bessel functions $J_{\frac{n}{2}-1}$ inside the integral. If $d\mu$ is the surface measure on the unit circle $|x| = 1$ in $\mathbb{R}^n$ then we have

$$
c_n \frac{J_{\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} = \int_{|y|=1} e^{ix.y} d\mu(y).
$$

where c_n is an absolute constant. If we use this in the above we get $E\sigma_N^\delta E f(x)$ equals

$$
\frac{c_n}{A_N^\delta} \int_{|\xi|=1} \int_0^\infty \int_0^\infty e^{-t} e^{-s} \frac{(t-s)^N}{N!}\, t^\delta\, s^{\frac{n}{2}-1} S_{\sqrt{2t}}^\delta F_{\xi,s}(x)\; e^{i\sqrt{2s}x.\xi}\; dt ds d\mu(\xi),
$$

where $F_{\xi,s}(x) = f(x) e^{i\sqrt{2s}x.\xi}$. As before, using Minkowski's inequality we get

$$
||E\sigma_N^\delta E f||_p \leq C ||f||_p
$$

since

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty e^{-t}\, e^{-s}\, |t-s|^N\, t^\delta\, s^{\frac{n}{2}-1}\, dt\, ds \\
&\qquad \leq \int_0^\infty e^{-t}\, t^\delta \left(\int_0^t e^{-s}\, t^N s^{\frac{n}{2}-1}\, ds + \int_t^\infty e^{-s}\, s^{N+\frac{n}{2}-1}\, ds \right) dt \\
&\qquad \leq C \int_0^\infty e^{-t}\, t^\delta\, t^N\, dt \;+\; \Gamma\left(N + \frac{n}{2}\right) \int_0^\infty e^{-t}\, t^\delta\, dt \\
&\qquad \leq C\Gamma(N + \delta + 1)
\end{aligned}
$$

provided $\delta \geq \frac{n}{2} - 1$. This completes the proof. □

4.2 Special Hermite Expansions

Recall that the Cesaro means associated to special Hermite expansions are defined by

$$\sigma_N^\delta f(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \, f \times \varphi_k(z).$$

In this section we prove the following Theorem 4.2.1. Throughout this section S_t^δ will stand for the Bochner-Riesz means for the Fourier transform on $\mathbb{R}^{2n} = \mathbb{C}^n$.

Theorem 4.2.1 *Let B be any compact subset of $\mathbb{C}^n$ containing the origin. Then for a fixed δ, $\chi_B \sigma_N^\delta \chi_B$ are uniformly bounded on L^p, $1 \leq p \leq \infty$ if and only if S_t^δ are uniformly bounded on the same L^p.*

Proof : The kernel $\sigma_N^\delta(z)$ of σ_N^δ is given by

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \varphi_k(z).$$

Using the formula (see Szego [42])

$$\sum_{k=0}^{N} A_{N-k}^\delta L_k^\alpha(t) = L_N^{\alpha+\delta+1}(t)$$

we have

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} L_N^{\delta+n}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2}.$$

As in the previous section we can express the Laguerre function in the terms of the Bessel functions, thus getting

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} \frac{1}{\Gamma(N+1)} e^{\frac{1}{4}|z|^2} \int_0^\infty e^{-t} t^{\delta+N+n} \frac{J_{\delta+n}(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^{\delta+n}} \, dt.$$

Now, $\sigma_N^\delta f = f \times \sigma_N^\delta$ so that

$$\sigma_N^\delta f(z) = \int_{\mathbb{C}^n} \sigma_N^\delta(z,w) \, f(w) dw$$

where

$$\sigma_N^\delta(z,w) = e^{\frac{i}{2} Im z.\overline{w}}\, \sigma_N^\delta(z-w).$$

Writing $|z-w|^2 = |z|^2 + |w|^2 - 2Re z.\overline{w}$ we have

$$\sigma_N^\delta(z,w) = e^{\frac{1}{4}|z|^2} e^{-\frac{1}{2}z.\overline{w}} \frac{1}{A_N^\delta \Gamma(N+1)} \int_0^\infty e^{-t} t^{\delta+n+N} \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}}\, e^{\frac{1}{4}|w|^2} dt$$

$$= \left(\sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|} \frac{(z.\overline{w})^\alpha}{\alpha!}\right) e^{\frac{1}{4}|z|^2} \frac{1}{A_N^\delta \Gamma(N+1)} \int_0^\infty e^{-t} t^{\delta+n+N} \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}} e^{\frac{1}{4}|w|^2} dt$$

where $(z.\overline{w})^\alpha$ stands for $(z_1\overline{w_1})^{\alpha_1}\cdots(z_n\overline{w_n})^{\alpha_n}$. Therefore,

$$\chi_B \sigma_N^\delta \chi_B f(z) = \frac{1}{A_N^\delta \Gamma(N+1)} \sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|} \frac{1}{\alpha!} \int_0^\infty e^{-t} t^{\delta+N}\, T_{\alpha,\delta}^t f(z) dt$$

where

$$T_{\alpha,\delta}^t f(z) = \chi_B(z)\, z^\alpha e^{\frac{1}{4}|z|^2} \int_{\mathbb{C}^n} t^n \frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}} \chi_B(w)\, \overline{w}^\alpha e^{\frac{1}{4}|w|^2} f(w) dw.$$

If we assume that S_t^δ are uniformly bounded we get

$$||T_{\alpha,\delta}^t f||_p \le C R^{2|\alpha|} ||f||_p,$$

when B is contained in the ball $\{z : |z| \le R\}$. Using this in the above equation we get

$$||\chi_B \sigma_N^\delta \chi_B f||_p \le C_B ||f||_p.$$

The converse is the transplantation theorem of Kenig-Stanton-Tomas. □

In [45] the following local estimates for the Cesaro means were established

Theorem 4.2.2 *Let* $\frac{2(2n+1)}{2n-1} < p \le \infty$ *and* $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. *Then for any compact subset B of* $\mathbb{C}^n$

$$\int_B |\sigma_N^\delta f(z)|^p dz \le C_B \int_{\mathbb{C}^n} |f(z)|^p dz.$$

Recently K.Stempak and J. Zienkiewicz [39] have proved the global estimate

$$\int_{\mathbb{C}^n} |\sigma_N^\delta f(z)|^p dz \le C \int_{\mathbb{C}^n} |f(z)|^p dz$$

for the range $\frac{2(2n+1)}{2n-1} < p \le \infty$. The key point is the restriction theorem namely, the estimate

$$||f \times \varphi_k||_2 \le C k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} ||f||_p$$

which they established in the range $1 \le p < \frac{2(2n+1)}{2n+3}$. In the next section we use this restriction theorem in order to prove a positive result for the Hermite expansions on $\mathbb{R}^{2n}$.

4.3 Hermite expansions on $\mathbb{R}^{2n}$

In this section we consider the operator $-\Delta + \frac{1}{4}|z|^2$ rather than the operator $-\Delta + |z|^2$. If $\Phi_\mu(x,y)$, $\mu \in N^{2n}$ are the eigenfunctions of the operator $-\Delta + |z|^2$ then $\Psi_\mu(z) = \Phi_\mu(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})$ are the eigenfunctions of $-\Delta + \frac{1}{4}|z|^2$ with eigenvalues $(|\mu| + n)$. The operator $-\Delta + \frac{1}{4}|z|^2$ has another family of eigenfunctions namely the special Hermite functions. In fact, $\Phi_{\alpha\beta}$ are eigenfunctions of the operator $-\Delta + \frac{1}{4}|z|^2$ with eigenvalue $(|\alpha| + |\beta| + n)$; here $\alpha, \beta \in N^n$

In this section we study the expansion in terms of Ψ_μ for functions having some homogeneity. The torus $T(n) = \{(e^{i\theta_1}, e^{i\theta_2} \cdots e^{i\theta_n}) \ : \ \theta \in \mathbb{R}^n\}$ acts on functions on $\mathbb{C}^n$ by $\tau_\theta f(z) = f(e^{i\theta} z)$ where $e^{i\theta} z = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \cdots e^{i\theta_n} z_n)$. We say that a function is m-homogeneous if $\tau_\theta f(z) = e^{im.\theta} f(z)$, here $m \in Z^n$ and $m.\theta = m_1.\theta_1 + \cdots + m_n.\theta_n$ It is a fact that $\Phi_{\alpha\beta}$ is $(\beta - \alpha)$ homogeneous. 0-homogeneous functions are also called polyradial.

The operator $-\Delta + \frac{1}{4}|z|^2$ commutes with τ_θ for all θ, therefore $P_k \tau_\theta f = \tau_\theta P_k f$ which shows that $P_k f$ is m-homogeneous if f is . In particular, $P_k f$ is polyradial if f is. Therefore, for such functions $L(P_k f) = (-\Delta + \frac{1}{4}|z|^2) P_k f = (k+n) P_k f$. This shows that $P_k f$ is an eigenfunction of L with eigenvalue $k + n$. But the spectrum of L is $\{2k + n \ : \ k = 0, 1, ..\}$ which forces $P_k f = 0$ when k is odd.

Proposition 4.3.1 *Let f be polyradial on $\mathbb{C}^n$. Then $P_{2k+1}f = 0$ and $P_{2k}f = f \times \varphi_k$.*

Proof : We show that when f is polyradial the operators $f \to P_{2k}f$ and $f \to f \times \varphi_k$ have the same kernel. Let

$$\Psi_k(z,w) = \sum_{|\mu|=k} \Psi_\mu(z)\Psi_\mu(w)$$

be the kernel of P_k. Then by Mehler's formula

$$\sum_{k=0}^{\infty} t^k \Psi_k(z,w) = \pi^{-n}(1-t^2)^{-n} e^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}Re(z.\overline{w})}.$$

so that

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \pi^{-n}(1-t^2)^{-n} \int_{\mathbb{C}^n} e^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}Re(z.\overline{w})} f(w)dw.$$

Let $w_j = u_j + iv_j = r_j e^{i\theta_j}$. When f is polyradial $f(w) = f_0(r_1, r_2, \cdots r_n)$ and so we have

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \int_0^{\infty} \cdots \int_0^{\infty} \Psi(s,r)\ f_0(r_1,\cdots r_n) r_1 r_2 \cdots r_n\ dr_1\, dr_2 \cdots dr_n$$

where $s = (s_1, s_2 \cdots s_n)$, $s_j = |z_j|$ and Ψ is given by

$$\Psi(s,r) = (1-t^2)^{-n} \int_{[0,2\pi]^n} e^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|r|^2+|s|^2)} e^{\frac{t}{1-t^2}Rez.\overline{w}}\ d\theta_1 d\theta_2 \cdots d\theta_n.$$

Now Re $z_j.\overline{w}_j = r_j s_j \cos(\theta_j - \varphi_j)$ where $z_j = s_j e^{i\varphi_j}, w_j = r_j e^{i\theta_j}$. Consider the integral

$$\int_0^{2\pi} e^{\frac{t}{1-t^2} r_j s_j \cos(\theta_j - \varphi_j)} d\theta_j$$

which equals, if we recall the definition of the Bessel functions, $J_0(\frac{it}{1-t^2} r_j s_j)$. Thus we have proved

$$\Psi(s,r) = (1-t^2)^{-n} e^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(r^2+s^2)}\ \Pi_{j=1}^n J_0(\frac{it}{1-t^2} r_j s_j).$$

On the other hand when f is polyradial $f \times \varphi_k$ reduces to the finite sum

$$f \times \varphi_k = \sum_{|\alpha|=k} (f, \Phi_{\alpha\alpha})\Phi_{\alpha\alpha}(z)$$

$$= \sum_{|\alpha|=k} \left(\int_0^{\infty} \cdots \int_0^{\infty} f_0(r_1, \cdots r_n)\ \Phi_{\alpha\alpha}(r_1, \cdots r_n)\ r_1 \cdots r_n\ dr_1 \cdots dr_n \right)\ \Phi_{\alpha\alpha}(s_1, \cdots s_n)$$

where we have written $\Phi_{\alpha\alpha}(z) = \Phi_{\alpha\alpha}(r_1, \cdots r_n)$ as it is polyradial. Then $f \times \varphi_k$ is given by the integral

$$\int_0^{\infty} \cdots \int_0^{\infty} \left(\sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(r_1, \cdots r_n) \Phi_{\alpha,\alpha}(s_1, \cdots s_n) \right) f_0(r_1, \cdots r_n)\ r_1 \cdots r_n\ dr_1 \cdots dr_n.$$

We have the formula (see[44])

$$\Phi_{\mu\mu}(z) = (2\pi)^{-\frac{n}{2}} \Pi_{j=1}^n L_{\mu_j}(\frac{1}{2}|z_j|^2) e^{-\frac{1}{4}|z_j|^2}.$$

Recalling the generating function identity for the Laguerre polynomials of type 0,

$$\sum_{k=0}^{\infty} L_k(x) L_k(y) w^k = (1-w)^{-1} e^{-\frac{w}{1-w}(x+y)} J_0 \left(\frac{2(-xyw)^{\frac{1}{2}}}{1-w} \right)$$

we get, if $S_k(r,s)$ is the kernel for $f \times \varphi_k$

$$\sum_{k=0}^{\infty} t^k S_k(r,s) = (1-t)^{-n} e^{-\frac{1}{4}\frac{1+t}{1-t}(r^2+s^2)}\ \Pi_{j=1}^n J_0 \left(\frac{i\sqrt{t}}{1-t} r_j s_j \right).$$

Comparing the two generating functions we see that

$$\sum_{k=0}^{\infty} t^{2k} S_k(r,s) = \sum_{k=0}^{\infty} t^k \Psi_k(r,s)$$

from which follows $\Psi_{2k}(r,s) = S_k(r,s)$ and this proves the proposition. □

Consider now the Bochner-Riesz means associated to the expansions in terms of $\Psi_\mu(z)$ defined by

$$S_R^\delta f(z) = \sum_\mu \left(1 - \frac{(|\mu|+n)}{R} \right)_+^\delta (f, \Psi_\mu) \Psi_\mu(z).$$

For these means we have the following result.

Theorem 4.3.2 *Let* $1 \le p < 2(\frac{2n+1}{2n+3})$, $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ *and let* $f \in L^p(\mathbb{C}^n)$ *be polyradial. Then*

$$||S_R^\delta f||_p \le C||f||_p$$

where C *is independent of* f *and* R.

The key ingredient in proving the above theorem is the $L^p - L^2$ estimates

$$||P_k f||_2 \leq C k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} ||f||_p$$

which now follows from the corresponding estimates for $f \times \varphi_k$. We omit the details.

We conclude this chapter with the following remarks. As we have observed, $P_k f$ is m-homogeneous whenever f is and so $P_k f$ can be obtained in terms of $f \times \varphi_k$ when f is m-homogeneous. So an analogue of the above theorem is true for all m-homogeneous functions. More generally, let us call a function f of type N if it has the Fourier expansion

$$f(z) = \sum_{|m| \leq N} f_m(z)$$

where

$$f_m(z) = \int f(e^{i\theta} z) e^{-im.\theta} d\theta_1 \cdots d\theta_n.$$

Note that f_m is m-homogeneous. We can show that when f is of type N then

$$||S_R^\delta f||_p \leq C_N ||f||_p,$$

under the conditions of the above theorem on p and δ where now C_N depends on N. We omit the details. It is an interesting problem to see if the theorem is true for all functions.

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