The Fixed Point Index as a Local Lefschetz Number

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CHAPTER 1

INTRODUCTION

In this thesis we define a class of self maps of connected compact polyhedra — those which *preserve expanding directions* — and define the fixed point indices of such maps at an isolated set of fixed points of the map as a *local Lefschetz number.* Our definition uses simplicial approximations of the given map in the spirit of O'Neill ([19] and Fournier [7]) and is intrinsic so that it is "computable".

Let X be a connected compact polyhedron and $f : X \longrightarrow X$ be a map on X. The Lefschetz number $L(f)$ of f is then defined to be ([13]),

$$L(f) = \sum_{j \geq 0} (-1)^j \text{Trace}\{f_* : H_j(X, \mathbb{Q}) \longrightarrow H_j(X, \mathbb{Q})\}$$

By a Lefschetz number we will always mean the alternating sum of the traces of a chain (cochain) map on a chain (cochain) complex and not just the corresponding number of a graded endomorphism.

The fixed point set of f is,

$$\text{Fix}\, f \doteq \{x \in X : f(x) = x\}.$$

Theorem [The Lefschetz Fixed Point Theorem, [14], [15]]
If $L(f) \neq 0$ the fixed point set of any map homotopic to f is nonempty.

Let U be an open subset of X such that $\partial U \cap \text{Fix}\, f = \emptyset$, where ∂U is the frontier of U in X. Then,

Theorem *It is possible to canonically associate an integer $i(f, U)$ with U called the fixed point index of f at U, which can be characterized by a set of simple axioms. (see Brown [4], or Dold [5], [6]).*

Definition A set of fixed points $C \subset \text{Fix}\, f$ is called an *isolated set* of fixed points if C is compact and open in $\text{Fix}\, f$.

Let C be an isolated set of fixed points of f and W be any open neighbourhood of C in $|K|$ such that, $\overline{W} \cap \text{Fix}\, f = W \cap \text{Fix}\, f = C$. Then the *fixed point index* $i(f, C)$, of f at C is defined to be, $i(f, C) = i(f, W)$ (see Jiang [12]). This definition of $i(f, C)$ is independant of the choice of the open neighbourhood of C.
Since $\text{Fix}\, f$ is a compact subset of X, it is clear that any collection of distinct isolated sets of fixed points of f is finite.

Theorem *Let $C_1, \ldots, C_k$ be a collection of isolated sets of fixed points of f such that $\text{Fix}\, f = \sum_{j=1}^{k} C_j$. Then,*

$$L(f) = \sum_{j=1}^{k} i(f, C_j) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\star)$$

If one can express $i(f, C)$ explicitly in terms of *local data* concerning f in a neighbourhood of C, then $(\star)$ is called a *Lefschetz Fixed Point Formula* or LFPF for short. Of particular interest is the case when the map f has finitely many isolated fixed points, say $\{p_k\}$: then $i(f, \{p_k\})$ is thought of as the *algebraic multiplicity* of the fixed point p_k.

For a smooth map $f : X \longrightarrow X$ which is transverse to the diagonal, the index at a transversal fixed point is the sign of $\det(1 - f_*(p))$, (see [1]) where $f_*(p) : X_p \longrightarrow X_p$ is the derivative of f at the fixed point p and 1 is the identity map. (This is the simplest and oldest example of a LFPF). This formula can be easily reorganized so as to exhibit the index $i(f, p)$ as a *local Lefschetz Number*.

Observation ([2]) Let $A : V \longrightarrow V$ be any linear map of a finite dimensional linear space and $\lambda^i A : \lambda^i V \longrightarrow \lambda^i V$ the induced map on the i^{th} exterior power of V. Then,

$$\det(1_V - A) = \sum_{i>0} \mathrm{Trace}\,(\lambda^i A)$$

Proof: The result is trivial for diagonalizable endomorphisms; the result follows since both sides of the equation are continuous functions of A and since diagonalizable endomorphisms form a dense subset of End V. ∎

Hence ($\star$) may be rewritten in the form,

$$i(f;p) = \frac{1}{|\det(1 - f_*(p))|} L(f_*)$$

that is, the fixed point index at a transversal fixed point p is, *modulo* the change in "volume" arising from the map $1 - f$, the Lefschetz number of the cochain transformation induced by f on the sheaf of germs of differential forms at p. In other words, $i(f,p)$ is a "normalized *local Lefschetz number*".

Over the past thirty years much attention has been devoted to obtaining Lefschetz Fixed Point Formulas, ([1], [2],[8], [9], [23]) particularly expressions of the fixed point index as a local Lefschetz Number, in studying the Lefschetz Fixed Point Theorem in its analytic and geometric *avatars*, but topological variants of such results have been lacking except for a few simple cases such as the following result which we show in Section 2 of Chapter 5,

> *the fixed point index of a simplicial map $f : K \longrightarrow K$ at an isolated set of fixed points C is the Euler characteristic of C.*

The best approach to obtaining an intrinsic and computable definition of the fixed point index at an isolated set of fixed points as a local Lefschetz number which yields the LFPF for maps on compact connected polyhedra seems to be to consider the simplicial approximations of a given map. Unfortunately, simplicial approximations are quite "coarse" in the sense that given a map $\varphi : |K| \longrightarrow |K|$ on a compact connected polyhedron $|K|$, the fixed point set of φ may differ drastically from the fixed point set of any of its simplicial approximations. Thus in general, computation of the fixed point index at an isolated set of fixed points (in contrast with the fixed point index on an open set) may not be posssible by looking at simplicial approximations. Nevertheless this approach proves profitable for a large class of maps .

In Chapter 2 of the thesis, we discuss a few classical definitions of the fixed point index and recall its properties.

A finite simplicial complex K such that $|K|$ is connected, will be referred to as a finite connected simplicial complex K.
The notations used in this thesis including those used in this chapter are listed in Section 2 of Chapter 3.
It is a well known fact that barycentric subdivision is a covariant functor on the category of finite simplicial complexes and simplicial maps. We study the effect of this functor on the fixed point set of simplicial maps in Chapter 3 : the results discussed in this chapter might be well known to experts but we were unable to find them in the literature. We observe the following interesting fact about the fixed point set of a simplicial map,

Proposition 1.1 *Each path component of the fixed point set of a simplicial map $g : \mathrm{sd}^n K \longrightarrow K, n \geq 0$, is an isolated set of fixed points of $|K|$. Moreover, if $n > 0$ then the fixed point set of g is a finite set of points of $|K|$.*

This is clearly not true in general. For example, it is possible to define a map on the closed unit interval in $\mathbb{R}$ whose fixed point set is precisely, $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, ([21]). Then 0 is a path component of the fixed point set of the map but is clearly not an isolated fixed point.

Let $\sigma_{[0]}$ be a simplex of a finite connected simplicial complex K and $\sigma_{[m]}$ be a simplex of $\mathrm{sd}^m K$ such that $\dim \sigma_{[m]} = \dim \sigma_{[0]}$ and $\langle \sigma_{[m]} \rangle \subset \langle \sigma_{[0]} \rangle$. In Chapter 4 we define a *radial* retraction,

$$\rho : \mathrm{sd}^m \{\overline{\mathrm{st}}\,(\sigma_{[0]}, K)\} \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[m]}, \mathrm{sd}^m K),$$

which depends only on the simplicial structure of K near $\sigma_{[0]}$.

In Chapter 5 we discuss the problem of establishing the LFPF for simplicial maps $f : \mathrm{sd}^n K \longrightarrow K, n \geq 0$ on a finite connected simplicial complex K. As mentioned earlier the case of simplicial maps $f : K \longrightarrow K$ is the simplest. More for motivational reasons than because of any actual difficulty involved, we prove the LFPF for this case.

The case of simplicial maps, $f : \mathrm{sd}^m K \longrightarrow K, m > 0$, is more complicated. Since f is simplicial, $\mathrm{Fix}\, f$ is a set of isolated points of $|K|$. Let $x \in \mathrm{Fix}\, f$ and let the carrier of x in $\mathrm{sd}^p K, p \geq 0$ be $\sigma_{[p]}$. Then,

$$f : \overline{\mathrm{st}}\,(\sigma_{[m]}, \mathrm{sd}^m K) \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[0]}, K).$$

Also, $H_p(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)) \xrightarrow{\cong} H_p(\overline{\mathrm{st}}\,(\sigma_{[m]}, \mathrm{sd}^m K))$, the isomorphism being induced by a retraction,

$$|\overline{\mathrm{st}}\,(\sigma_{[0]}, K)| \xrightarrow{\theta} |\overline{\mathrm{st}}\,(\sigma_{[m]}, \mathrm{sd}^m K)|.$$

Since $\mathrm{Fix}\,(\theta \circ f)$ need not be equal to $\mathrm{Fix}\, f$, $L(\theta \circ f)$ does not necessarily define $i(f, x)$, even though, $f = \theta \circ f$, on $|\overline{\mathrm{st}}\,(\sigma_{[m+1]}, \mathrm{sd}^{m+1} K)|$. If θ' is another retraction then at the homology level, $\theta_* = \theta'_*$, so, the Lefschetz number of the map $\theta \circ f$ is independant of θ. We choose θ to be the retraction ρ defined in Chapter 4.

We define a class of simplicial maps, those which *preserve expanding directions*,

Definition 1.2 Let $f : \mathrm{sd}^n K \longrightarrow K, n \geq 1$ be a simplicial map and $x \in \mathrm{Fix}\, f$. Let the carrier of x in $\mathrm{sd}^p K, p \geq 0$ be $\sigma_{[p]}$. Then f *preserves expanding directions at* x if there is a subcomplex $M(x) = M$ of $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ satisfying,

(a) $\tau \in \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ such that, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$, implies that $\tau \in M$.

(b) $\tau \in M$ and $f(\tau) \prec \sigma_{[0]}$ implies that $\tau \prec \sigma_{[n]}$.

(c) $\tau \in M$ implies that,
$$\mathrm{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]} \subset M.$$

A subcomplex of $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ is a *subcomplex at x expanded by the map f* if it satisfies (a),(b) and (c).
The map f *preserves expanding directions* if it preserves expanding directions at each of its fixed points.

We show,

Theorem 1.3 *Let $f : \mathrm{sd}^n K' \longrightarrow K, n \geq 1$, preserve expanding directions and $x \in Fix\, f$. Let the carrier of x in $\mathrm{sd}^p K, p \geq 0$ be $\sigma_{[p]}$ and the subcomplex at x expanded by f be M. Then the relative Lefschetz number of the map $|\rho||f|$ on the pair $(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K), M)$ is the fixed point index $i(f, x)$ of f at x. Also, $L(f) = \sum_{x \in Fix\, f} i(f, x)$.*

Goresky and MacPherson ([9]) have defined the fixed point indices at isolated sets of fixed points of a map on a compact stratified space as local Lefschetz numbers provided the map is *weakly hyperbolic.*

Definition A map $f : X \longrightarrow X$ on a compact stratified space is weakly hyperbolic if for every isolated set of fixed points C of f there is an open neighbourhood W of C in X and an "indicator map",

$$t : W \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$$

such that, $t^{-1}((0,0)) = C$ and for all points x of $W \cap f^{-1}(W)$,

$$t_1(f(x)) \geq t_1(x), \quad t_2(f(x)) \leq t_2(x).$$

where, $t_j(x)$ is the j^{th} coordinate of $t(x)$.

The notion of weakly hyperbolic maps extends immedietly to maps on compact, connected polyhedra. We show that,

Theorem 1.4 *A weakly hyperbolic simplicial map preserves expanding directions.*

We compute the local indices for a few continuous maps using our methods and show by examples that,

- *a simplicial map which preserves expanding directions need not be weakly hyperbolic.*
- *not all simplicial maps preserve expanding directions.*

Helga Schirmer ([21]) has shown that for any closed set C of $|K|$ it is possible to define a map on $|K|$ whose fixed point set is precisely C. We avoid this generality in what follows and consider only those maps whose fixed point set is a subpolyhedron of $|K|$.
In Chapter 6 we introduce the notion of *fp-equivalent* simplicial approximations.

Definition 1.5 Let $f : |K| \longrightarrow |K|$ be a map. A simplicial approximation $g : \mathrm{sd}^m K \longrightarrow K$ to f is *fp-equivalent* to f if $\mathrm{Fix}\, g \subset \mathrm{Fix}\, f$ and for any fixed point component C of f,

$$i(f, C) = \sum_{x \in \mathrm{Fix}\, g \cap C} i(g, x).$$

Definition 1.6 A map $f : |K| \longrightarrow |K|$ *preserves expanding directions* if there is an fp-equivalent simplicial approximation to f which preserves expanding directions at each of its fixed points which is also a fixed point of f.

We show by an example that,

not all maps have a fp-equivalent simplicial approximation.

Let L be a subcomplex of a simplicial complex K. We denote the regular neighbourhood of L in K by $N(L, K)$.

Let $f : |K| \longrightarrow |K|$ be a map on a compact connected polyhedron $|K|$. The *proximity set*, ([4]) of f is the set of all points x of $|K|$, such that $f(x)$ belongs to the star of the carrier of x in $|K|$.

Let $f : |K| \longrightarrow |K|$ be a map on a compact connected polyhedron $|K|$ whose fixed point set is a subpolyhedron $|F|$ of $|K|$. Let C be a component of F. If for all $m > 0$ such that $x \notin \mathtt{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, $f(x)$ also does not belong to $\mathtt{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, then x *is said to define a direction expanded* by f with respect to C. The set of all points in $\mathtt{N}(C, K)$ which define a direction expanded by f with respect to C will be denoted by $E(f, C)$. We prove the following result,

Theorem 1.7 *Let F be contained in the interior of the proximity set of f. Suppose that for any component C of F and for any simplex σ of $\mathtt{N}(C, K)$, the following holds :*

$$\langle\sigma\rangle \cap E(f, C) \neq \emptyset \Rightarrow \{|\overline{\sigma}| - |\overline{\sigma} \cap C|\} \subset E(f, C).$$

Then f preserves expanding directions.

This result justifies the nomenclature of such maps as those which preserve expanding directions.

As an immediete consequence we have,

Corollary 1.8 *An "expanding" or "contracting" map on a connected compact polyhedron, whose fixed point set is a subpolyhedron contained in the interior of the proximity set of the map, preserves expanding directions.*

We also show the following,

Theorem 1.9 *Let $f : |K| \longrightarrow |K|$ be a map on a compact connected polyhedron $|K|$ whose fixed point set is a subpolyhedron $|F|$ of $|K|$. Let for all simplex $\langle\sigma\rangle \subset \mathrm{N}(F, K)$, $\sigma' = \sigma \cap F$. Let the proximity set of f be precisely F. If for all simplex $\langle\sigma\rangle \subset \mathrm{N}(F, K)$, $f(\langle\sigma\rangle) \cap \{\dot{\sigma}' * \mathrm{Lk}(\sigma', K)\} = \emptyset$,*
then f preserves expanding directions.

Theorems 1.7 and 1.9 show that a large class of self maps on connected compact polyhedra preserve expanding directions. However, this property of maps on connected compact polyhedra is not invariant under homotopy.

CHAPTER 2

THE FIXED POINT INDEX

2.1 INTRODUCTION

We discuss a few classical definitions of the fixed point index to appreciate the difficulties involved in arriving at a LFPF through most of them. We also recall the properties of the fixed point index in this chapter. Full details are given in [4], [5], [6], [10], [11] and [19].

2.2 CLASSICAL DEFINITIONS OF THE FIXED POINT INDEX

The frontier of an open subset U of X in X will be denoted by ∂U.

One of the most general procedures for evaluating local indices is the degree formula described by Dold ([5], [6]). Let X be an ENR, $f : X \longrightarrow X$ and W be an open subset of X such that $\partial W \cap \operatorname{Fix} f = \emptyset$. There is an embedding $i : X \longrightarrow V$ of X in an open subset V of $\mathbb{R}^n$ as a retract. Let $r : V \longrightarrow X$ be a retraction and $r^{-1}(W) = U, r^{-1}(C) = G$ and, $g = ifr : U \longrightarrow V$. Then $i(f, W)$, the index of f on W is the degree of the map $(j - g) : (U, U - G) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n_*)$ where j is the inclusion map and $\mathbb{R}^n_* = \mathbb{R}^n - \underline{0}$.

Let C be an isolated set of fixed points of f and W be any open neighbourhood of C in X such that, $\overline{W} \cap \operatorname{Fix} f = W \cap \operatorname{Fix} f = C$. Recall then that the fixed point index $i(f, C)$, of f at C is defined to be, $i(f, C) = i(f, W)$, (see Jiang [12]). This definition of $i(f, C)$ is independant of the choice of the open neighbourhood of C.

Dold's beautiful characterization of the fixed point index is *not* a LFPF — the need to imbed X in a Euclidean space and the choice of retract renders it "non-local" and less than satisfactory if one is trying to compute an index at a fixed point component.

A simple illustration of the disadvantages of Dold's procedure can be seen from the following example; an "ancient war-horse" of topological fixed point theory, (see [4],[12]).

Example 2.1 Let $X = A \vee B, A = B = \mathbb{S}^1$ and, $f : X \longrightarrow X$ be defined by $f(z) = z^{-1}, z \in A, f(z) = z^2, z \in B$. Let $w = 1 \in A \subset \mathbb{C}$ be the wedge point and a be the antipode of w in A. Then $\operatorname{Fix} f = \{w, a\}$ and $L(f) = 0$. Since there is a neighbourhood of a isomorphic to $\mathbb{R}$, applying degree formula, it is easy to compute $i(f, a) = 1$.

The fixed point at w is more interesting but no topological formula is readily available for dealing with "non-manifold-like" points and to compute $i(f, w)$ using the degree formula is clearly troublesome. So both Brown and Jiang use the additivity property of the local index ([5]), to compute $i(f, w) = L(f) - i(f, a) = -1$.

As a contrast to this generality we consider the situation when X is a closed connected oriented n-manifold and $f : X \longrightarrow X$ is a map on X (see Vick, [24]). Let $\delta X = \{(x, x) : x \in X\}$. Let $T \in H^n(X \times X, X \times X - \delta X)$ be the Thom class of X and $z \in H_n(X)$ be the fundamental class of X. Let W be an open subset of X such that $\partial W \cap \operatorname{Fix} f = \emptyset$ and V be an open subset of X such that $\operatorname{Fix} f \cap W \subset V \subset \overline{V} \subset W$. Then $H_n(X \times X, X \times X - \delta X)$ is isomorphic to $\mathbb{Q}$, the isomorphism given by $\alpha \mapsto \langle T, \alpha \rangle$, where $\alpha \in H_n(X \times X, X \times X - \delta X)$ and $\langle T, \alpha \rangle$ is the Kronecker pairing ([22]). Then, (i, f, W) is the integer defined by the image of z under the composition,

$$H_n(X) \longrightarrow H_n(X, X - V) \cong H_n(W, W - V) \xrightarrow{(1 \times f)_*} H_n(X \times X, X \times X - \delta X)$$

When considering simplicial maps on a compact polyhedron, using Hopf's construction, ([10], [11]) it is possible to homotope the map to a simplicial map whose fixed point set is a set of isolated points and each fixed point lies in the interior of a principal simplex. We first need to recall the notion of the *barycentric subdivision of a simplicial complex modulo a subcomplex.* The notations used in this chapter are listed in Section 2 of Chapter 3.

Let K be a finite connected simplicial complex and L be a subcomplex of K. The barycentric subdivision of K modulo L is a simplicial complex $(K,L)'$ such that the vertex set of $(K,L)'$ is,

$$V((K,L)') = V(L) \cup \{b(\sigma) : \sigma \in K, \sigma \notin L\}$$

and a p-simplex of $(K,L)'$ is a $p+1$ tuple, $\{v_0, \dots, v_q, b(\tau_{q+1}), \dots, b(\tau_p)\}$ where $\sigma = [v_0, \dots, v_p]$ is a simplex of L and $\sigma \prec \tau_{q+1} \prec \cdots \prec \tau_p \in K$.

Let K be a finite connected simplicial complex and $\widetilde{K}$ be a subdivision of K. Let $\psi : \widetilde{K} \longrightarrow K$ be a simplicial map. Let τ be a simplex of $\widetilde{K}$ such that $\mathrm{Fix}\,\psi \cap \langle\tau\rangle \neq \emptyset$ and τ is not principal. Let $\psi(\tau) = \sigma$. Clearly σ is not a principal simplex of K. Let μ be a principal simplex of $\widetilde{K}$ such that $\tau \prec \mu$ and ν be the principal simplex of K such that $\langle\mu\rangle \subset \langle\nu\rangle$. Then $\sigma \prec \nu$, (for details see [4, Chapter VIII] for instance).

Let $L = \widetilde{K} - \mathbf{st}\,(\tau, \widetilde{K})$ and consider the simplicial complex $(\widetilde{K}, L)'$. Define a simplicial map $\psi' : (\widetilde{K}, L)' \longrightarrow K$ induced by the map on the vertex set of $(\widetilde{K}, L)'$ as follows :

$$\begin{aligned} \psi'(v) &= v && \text{for all } v \in L \\ \psi'(b(\tau')) &\in \sigma && \text{if} \tau' \prec \tau, \tau' \neq \tau \\ \psi(b(\tau)) &\in \nu - \sigma && \end{aligned}$$

Then $\psi'(x) \neq x$ for all $x \in \langle\tau\rangle \cup L$.

Repeating this process a finite number of times we obtain a simplicial map homotopic to ψ whose fixed point set is a set of isolated points and each fixed point lies in the interior of a principal simplex.

The index of a simplicial map at an isolated fixed point which lies in the interior of a principal simplex is given by the degree of the map on the principal simplex. This *is* an intrinsic and computable definition of determining fixed point indices though a problem remains since the number of isolated fixed points may increase arbitrarily by Hopf's construction. The "war-horse" gives an illustration of this as well.

Example 2.2 Let $f : X \longrightarrow X$ be the map defined in Example 2.1. Consider the vertex assignments $g : V(K') \longrightarrow V(K)$ shown in Figure 1, where $K' = K_1 \vee K_2$. The image $g(v) = w$ of any vertex v of K' is shown as (w) in Figure 1. The assignments $v \mapsto g(v)$ indicated in Figure 1 is a simplicial map such that

$$g_1 \doteq g|K_1 : K_1 \longrightarrow K,$$
$$\text{and} \quad g_2 \doteq g|K_2 : K_2 \longrightarrow K$$

are simplicial approximations to

$$f_i = \begin{cases} f|A & \text{if } i = 1, \\ f|B & \text{if } i = 2 \end{cases} \quad \text{respectively.}$$

Also $\operatorname{Fix} g = \{0, 12\}$ and $i(f, w) = i(g, 0)$ and $i(f, a) = i(g, 12)$. In Hopf's

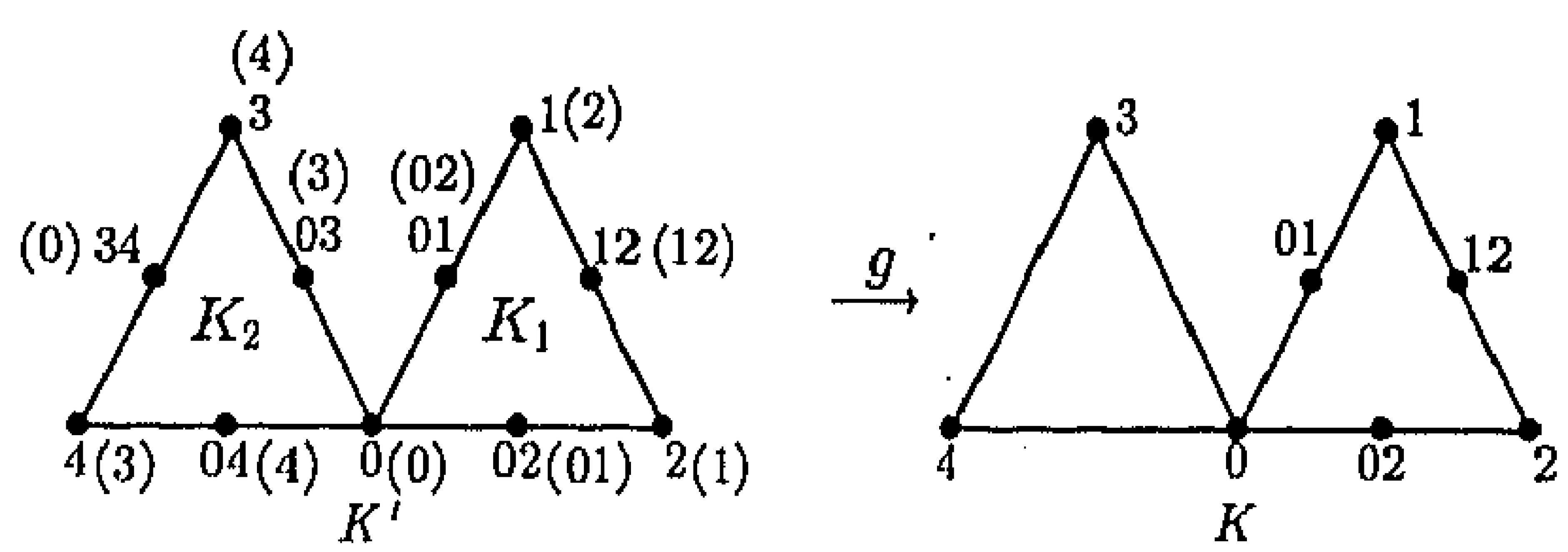


Figure 1: A simplicial approximation of $z^2 \vee z^{-1}$ on $\mathbb{S}^1 \vee \mathbb{S}^1$

construction there is a choice involved. One of the choices could lead to the simplicial approximation $g' : K'' \longrightarrow K$ to f, see Figure 2, where K'' is a subdivision of K'. As above, the image $g'(v) = w$ of any vertex v of K'' is shown as (w) in the figure. The fixed points of g' are x, y, z and v. Also, $i(g, 12) = i(g', v)$ and $i(g, 0) = i(g', x) + i(g', y) + i(g', z)$. Thus we need to compute four integers instead of two. One can show in this example that all possible choices in Hopf's construction for g lead to an increase in the number of fixed points.

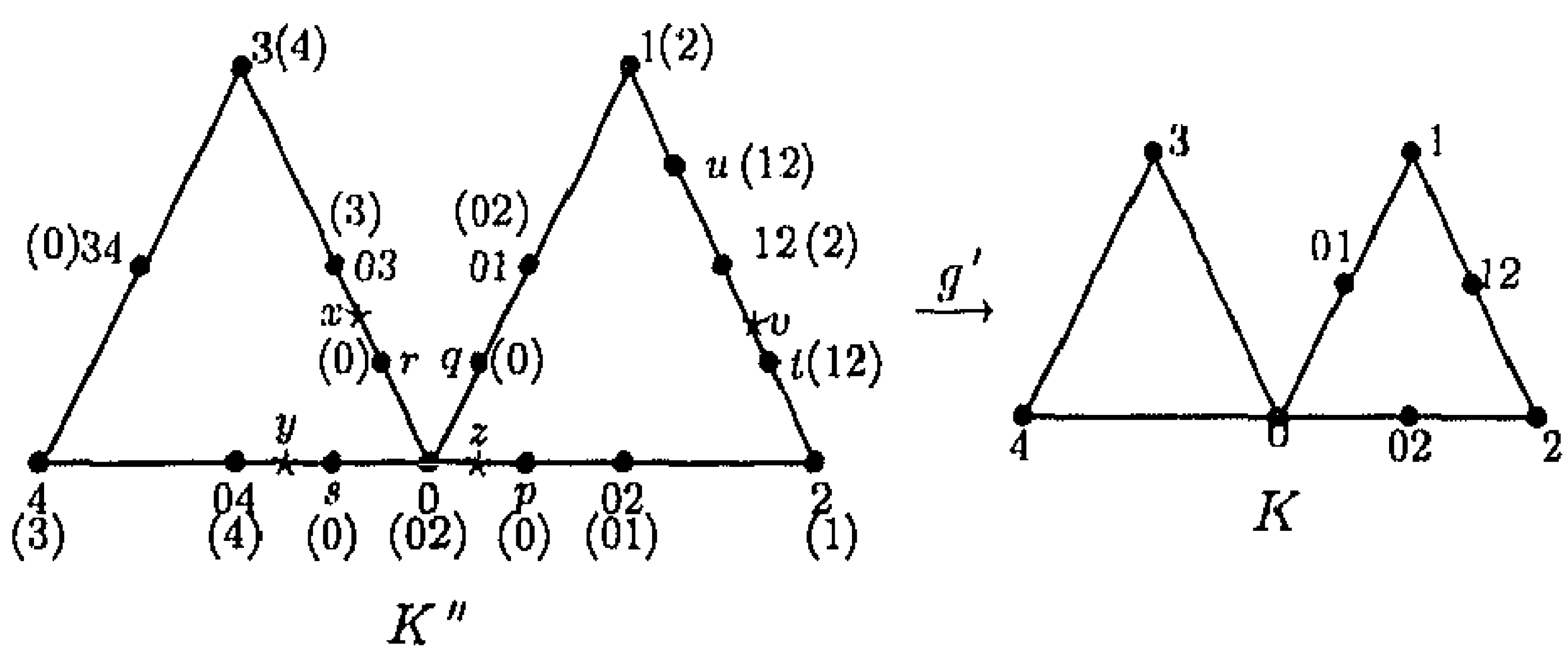


Figure 2: Hopf's construction applied to $z^2 \vee z^{-1}$ on $\mathbb{S}^1 \vee \mathbb{S}^1$

Barrat O'Neill ([19]) used simplicial approximations of a given map on a connected compact polyhedron to give an axiomatic definition of the fixed point indices. Briefly his definition is as follows : let $|K|$ be a triangulation of a connected compact polyhedron X. For any simplex σ of K the elementary cochain with respect to σ is the cochain s_σ given by,

$$s_\sigma(\sigma) = 1, s_\sigma(\tau) = 0, \text{if } \tau \neq \sigma$$

An *inner product* on the vector space of oriented cochains $C^\bullet(K)$ is defined by,

$$\begin{aligned} \langle s_\sigma, s_\tau \rangle &= 1 \quad \text{if } \sigma = \tau \\ \langle s_\sigma, s_\tau \rangle &= 0 \quad \text{if } \sigma \neq \tau \end{aligned}$$

Let $f : X \longrightarrow X$ be a map on a connected compact polyhedron and U be an open subset of X such that $\partial U \cap \operatorname{Fix} f = \emptyset$. Choose a triangulation $|K|$ of X and let $g : \mathrm{sd}^m K \longrightarrow K$ be a simplicial approximation to f.
Let $\lambda : C^\bullet(\mathrm{sd}^m K) \longrightarrow C^\bullet(K)$ be the subdivision operator. Then $f^+ = \lambda g^\bullet$ is a class of cochain *transformations* on $C^\bullet(K)$. The fixed point index of f on U is the integer,

$$L(f^+, U) = \sum_p (-1)^p \sum \{\langle s_\sigma, f^+(s_\sigma)\rangle : \sigma \text{ is a } p\text{-simplex}, \langle\sigma\rangle \cap U \neq \emptyset\}.$$

This definition is independent of the simplicial approximation chosen.
A simplicial complex may have an unmanageable number of simplices and any hopefully practical simplicial result ought to involve the homology groups. The above definition of fixed point indices cannot be realised as

the Lefschetz number of even a graded endomorphism generally.

2.3 THE PROPERTIES OF THE FIXED POINT INDEX

A triple (X, f, W) is *admissible* ([4]) if,

- X is a connected compact polyhedron.
- $f : X \longrightarrow X$ is a map.
- W is open in X.
- there are no fixed points of f on the frontier of W in X.

It is possible to associate with any admissible triple (X, f, W), the *fixed point index* of f on W denoted by $i(f, W)$ (see [4], [5], [6], [19] or above). The fixed point index satisfies the following properties :

Property 1 (Localization) : Let (X, f, U) and (X, g, U) be admissible triples such that for all $x \in U$, $g(x) = f(x)$. Then $i(f, U) = i(g, U)$.

Property 2 (Homotopy) : Let $H : X \times [0, 1] \longrightarrow X$ be a homotopy and define $f_t : X \longrightarrow X, t \in [0, 1]$ by, $f_t(x) = H(x, t)$. If (X, f_t, U) is admissible for all $t \in [0, 1]$, then $i(f_0, U) = i(f_1, U)$.

Property 3 (Additivity) : Let $U_1, \ldots U_r$ be a set of mutually disjoint open subsets of U such that $\operatorname{Fix} f \cap U \subset \bigcup_{j=1}^{r} U_j$. Then $i(f, U) = \sum_{j=1}^{r} i(f, U_j)$.

Property 4 (Normalization) : $i(f, X) = L(f)$.

Property 5 (Commutativity) : Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be maps and let (X, gf, U) be admissible. Then $(Y, fg, g^{-1}(U))$ is admissible and $i(gf, U) = i(fg, g^{-1}(U))$.

CHAPTER 3

THE FIXED POINT SET OF A SIMPLICIAL MAP

3.1 INTRODUCTION

The number of path components of the fixed point set of a continuous map on a compact, connected polyhedron may not be finite. As H. Schirmer ([11]) has shown : given any closed subset C of the polyhedron, it is possible to define a map on the polyhedron, whose fixed point set is precisely C.
The situation simplifies when we consider simplicial maps on finite connected simplicial complexes. Let $g : \mathrm{sd}^n K \longrightarrow K, n \geq 0$, be a simplicial map. We show (Remark 3.14) that if $n > 0$, then the fixed point set of g is a finite set of isolated points. Therefore, the number of path components of the fixed point set of g is finite. Also, if $n = 0$ by Lemma 3.19, there is a subcomplex of $\mathrm{sd}\, K$ whose geometric realization is precisely $\mathrm{Fix}\, g$. Thus the subdivision of all the distinct path components of $\mathrm{Fix}\, g$ are disjoint subcomplexes of $\mathrm{sd}\, K$. Hence, $\mathrm{Fix}\, g$ has finitely many path components and each path component is closed in $\mathrm{Fix}\, g$. This shows that,

> *each path component of the fixed point set of a simplicial map $g : \mathrm{sd}^n K \longrightarrow K, n \geq 0$, is compact and open in $\mathrm{Fix}\, g$.*

We study the *subdivision* of a simplicial map and show that this process does not alter the fixed point set of the map and also that subdivision of a simplicial map is homotopic to the original simplicial map.
The results discussed in this chapter will be used throughout the thesis.

3.2 NOTATIONS

Throughout the thesis, we will be interested only in finite simplicial complexes whose geometric realization is connected. Such simplicial complexes will be referred to as finite connected simplicial complexes or sometimes only as simplicial complexes. This need not lead to any confusion since we will work only with finite connected simplicial complexes.

We shall denote the p^{th} barycentric subdivision of a simplicial complex K by $\mathrm{sd}^p K, p \geq 0$ and the simplices of a simplicial complex by lower case Greek letters : $\sigma, \tau, \ldots$. The notation $\sigma \in K$ will mean that σ is a simplex of the simplicial complex K.

The geometric realization of a simplicial complex K will be denoted by $|K|$. By a simplicial map $f : K \longrightarrow L$ we will either mean the combinatorial map on the abstract simplicial complex K induced by a vertex assignment, $f : V(K) \longrightarrow V(L)$ so that, whenever σ is a simplex of K, $f(\sigma)$ is a simplex of L or its geometric realization $|f| : |K| \longrightarrow |L|$, ([18]).

The dimension of a simplex τ of a simplicial complex K will be denoted by $\dim \tau$.

Let v be a vertex of a simplicial complex K. The barycentric coordinate of a point $x \in |K|$ with respect to v will be denoted by $x(v)$.

The set of all vertices of a simplicial complex K will be denoted by $V(K)$.

The metric d on $|K|$ is given by,

$$d(x,y) = \left[\sum_{v \in V(K)} (x(v) - y(v))^2 \right]^{\frac{1}{2}}, \quad x, y \in |K|.$$

If σ, τ are simplices of K, then :

- $\langle \sigma \rangle$ (resp. $\overline{\sigma} \subset |K|$) will denote the open (resp. closed) simplex corresponding to σ,
- $\mathrm{st}\,(\sigma, K)$ (resp. $\overline{\mathrm{st}}\,(\sigma, K)$) will denote the star (resp. closed star) of σ in K,
- $\tau \prec \sigma$ will mean that τ is a face of σ,
- if $\tau \prec \sigma$, we will write $\{\tau : \sigma\}$ for $\dim \sigma - \dim \tau$.

- if $\sigma = [u_0, \ldots, u_p]$ we shall sometimes write $b(\sigma) = u_0 \cdots u_p$, for the barycenter of σ.

- $\dot{\sigma}$ will denote the boundary of σ. Thus, $\dot{\sigma} = \cup\{\overline{\tau} : \tau \prec \sigma, \text{ but}, \tau \neq \sigma\}$

The join of two subcomplexes M and P of K will be denoted by $M * P$. Recall that, $\overline{\mathtt{st}}\,(\sigma, K) - \mathtt{st}\,(\sigma, K) = \mathtt{Lk}\,(\sigma, K) * \dot{\sigma}$.

Definition 3.1 Let K' be a subdivision of K. A p-simplex σ' of K' is *primitive* with respect to K, if there is a p-simplex σ of K such that $\langle \sigma' \rangle \subset \langle \sigma \rangle$.
We shall also say that σ' is a *primitive simplex* of K' *with respect to* σ if σ' is primitive with respect to K and $\langle \sigma' \rangle \subset \langle \sigma \rangle$.

A primitive simplex of $\mathtt{sd}^m K$ with respect to $\tau \in K$ will be denoted by $\tau_{|m|}$. If more than one such primitive simplex is being discussed, then we shall denote them as $\tau_{|m|}(1), \tau_{|m|}(2), \ldots$.
We shall denote the *carrier* of a point x of $|\mathtt{sd}^p K|$ by $\sigma_p(x)$. Recall that this means that $x \in \langle \sigma_p(x) \rangle$. If the carrier of a point x of $|\mathtt{sd}^p K|$ is a primitive simplex then we will denote it by $\sigma_{|p|}(x)$.
Let L be a subcomplex of K. The regular neighbourhood of L in K will be denoted by $\mathtt{N}\,(L, K)$. Thus,

$$\begin{aligned} \mathtt{N}\,(L, K) &= \cup\{\mathtt{st}\,(\sigma, K) : \sigma \in L\} \\ \overline{\mathtt{N}}\,(L, K) &= \cup\{\overline{\mathtt{st}}\,(\sigma, K) : \sigma \in L\} \\ \mathtt{Lk}\,(L, K) &= \overline{\mathtt{N}}\,(L, K) - \mathtt{N}\,(L, K) \end{aligned}$$

Let C be a subset of $|K|$. The smallest subcomplex of K containing C is,

$$[K]_C = \{\tau \in K : \tau \prec \sigma; \langle \sigma \rangle \cap C \neq \emptyset\}.$$

Let (K, L) be a simplicial pair. Then $C_*(K, L)$ will denote the chain complex of oriented simplicial chains of (K, L).
Homology will always be with coefficients in $\mathbb{Q}$.

Let (K, L) be a simplicial pair and

$$\varphi_* : H_*(K, L) \longrightarrow H_*(K, L)$$

be a sequence of homomorphisms. The Lefschetz number of φ_* is,

$$L(\varphi, K, L) = \sum_{i \geq 0}(-1)^i \text{Trace } \{\varphi_i : H_i(K, L) \longrightarrow H_i(K, L)\}.$$

Whenever K, L are clear from the context, we will write $L(\varphi)$ to mean $L(\varphi, K, L)$. We recall a classical result.

Lemma 3.2 (Hopf) *Let (K, L) be a simplicial pair. For all $i \geq 0$, let φ_i^L, φ_i^K, and $\varphi_i^{(K,L)}$ be homomorphisms such that the following diagram commutes,*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_p(L) & \xrightarrow{i_*} & H_p(K) & \xrightarrow{j_*} & H_p(K, L) & \xrightarrow{\partial_*} & H_{p-1}(L) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_p^L} & & \downarrow{\scriptstyle \varphi_p^K} & & \downarrow{\scriptstyle \varphi_p^{(K,L)}} & & \downarrow{\scriptstyle \varphi_{p-1}^L} & & \\ \cdots & \longrightarrow & H_p(L) & \xrightarrow{i_*} & H_p(K) & \xrightarrow{j_*} & H_p(K, L) & \xrightarrow{\partial_*} & H_{p-1}(L) & \longrightarrow & \cdots \end{array}$$

Diagram 3.2

where the horizontal rows are the exact sequence of the pair (K, L). Then, $L(\varphi^{(K,L)}) = L(\varphi^K) - L(\varphi^L)$.

Proof: The homology exact sequence of the pair (K, L) is an acyclic chain complex. The lemma now follows by Hopf Trace Theorem ([22]).

3.3 Depiction of Simplicial Maps Through Figures

We shall use figures to describe simplicial maps. To define a simplicial map $g : \mathrm{sd}^n K \longrightarrow K, n \geq 0$, we will draw the figures showing the geometric realizations of K and $g(\mathrm{sd}^n K)$. A vertex shown as v on $g(\mathrm{sd}^n K)$ will thus be the image of a vertex u of $\mathrm{sd}^n K$ by the map g. For example let K be the standard 2-simplex with vertex set, $V(K) = \{1, 2, 3\}$. Let us denote the vertex $b([1, 2, 3]) = 123$ of $\mathrm{sd}\, K$ by $\underline{3}$. Let $g : \mathrm{sd}\, K \longrightarrow K$ be the simplicial map given on the vertices by,

$$g(1) = 1, \; g(12) = 1, \; g(2) = 2,$$
$$g(23) = 2, \; g(3) = 3, \; g(13) = 1, \; g(\underline{3}) = 1.$$

Then the figure would be,

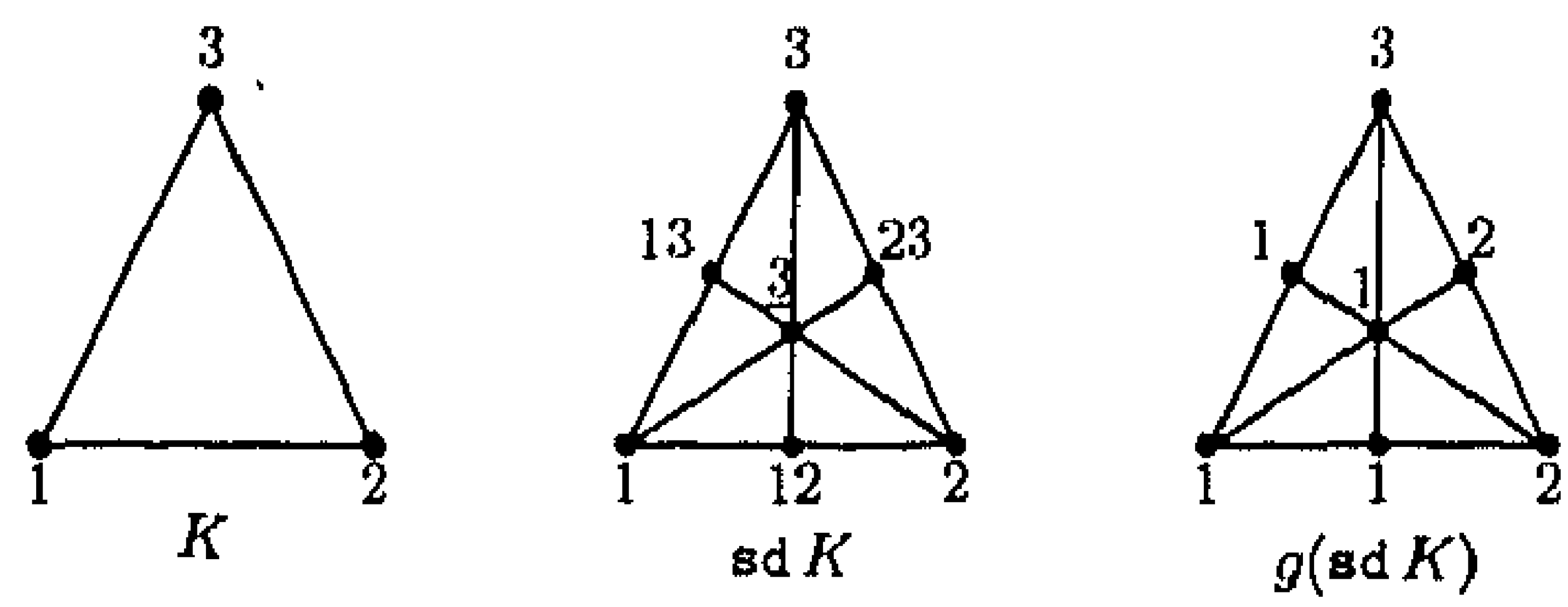


Figure 3: Simplicial Maps described through Figures

Thus in Figure 3, the vertex description of $g(\mathrm{sd}\,K)$ is as follows : the vertices labelled by 1 are the images of the vertices $1, 12, 13$ and $\underline{3}$ of $\mathrm{sd}\,K$, the vertices labelled by 2 are the images of the vertices 2 and 23 of $\mathrm{sd}\,K$, the vertex labelled by 3 is the image of the vertex 3 of $\mathrm{sd}\,K$.
In actual practice we will omit the figure showing the geometric realization of $\mathrm{sd}\,K$ and draw only the figures showing the geometric realization of K and $g(\mathrm{sd}\,K)$.

3.4 Preliminary Results on Simplicial Complexes

Let L be a simplicial complex. For $m \geq 1$, there is a partial order $(\leq)$ defined on $V(\mathrm{sd}^m L)$ as follows : for all $b(\sigma), b(\tau) \in V(\mathrm{sd}^m L), b(\sigma) \leq b(\tau)$ if and only if $\sigma \prec \tau$. Then, any $S \subset V(\mathrm{sd}^m L), m \geq 1$ is a simplex of $\mathrm{sd}^m L$ if and only if it is linearly ordered. We assume from now on that $K = \mathrm{sd}\,L$ for some simplicial complex L. Thus we can talk of the "largest" vertex of a simplex.

Lemma 3.3 *Let $\sigma = \sigma_{[0]}$ be a simplex of K and $\sigma_{[1]}$ be a primitive simplex of $\mathrm{sd}\,K$ with respect to $\sigma_{[0]}$. A vertex $b(\mu)$ of $\mathrm{sd}\,K$ is a vertex of $\mathrm{Lk}\,(\sigma_{[1]}, \mathrm{sd}\,K)$ if and only if $\langle \mu \rangle \subset \mathrm{st}\,(\sigma_{[0]}, K)$.*

Proof: Let $\sigma_{[1]} = [b(\tau_0), \ldots, b(\tau_p)]$, where, $\tau_0 \prec \cdots \prec \tau_p = \sigma_{[0]}$ and for all $0 \leq i \neq j \leq p, \tau_i \neq \tau_j$. Let $b(\mu)$ be a vertex of $\mathtt{Lk}(\sigma_{[1]}, \mathtt{sd}\, K)$. Then $\sigma_{[1]} \cup b(\mu)$ is a simplex and hence, there is a $j, 0 \leq j < p$ such that,

$$\tau_j \prec \mu \prec \tau_{j+1}.$$

Since $\sigma_{[1]}$ is a primitive simplex, $\dim \sigma_{[1]} = p = \dim \sigma_{[0]} = \dim \tau_p$. Therefore, each τ_j is a j-simplex. Hence $\sigma_{[0]} \prec \mu$ i.e. $\langle \mu \rangle \subset \mathtt{st}\,(\sigma_{[0]}, K)$.
On the other hand, $\langle \mu \rangle \subset \mathtt{st}\,(\sigma_{[0]}, K)$ implies that $\sigma_{[0]} \prec \mu$ and hence that $\sigma_{[1]} \cup b(\mu)$ is a simplex of $\mathtt{sd}\, K$. So $b(\mu) \in \mathtt{Lk}\,(\sigma_{[1]}, \mathtt{sd}\, K)$. ∎

Observation 3.4 Let M and P be two subcomplexes of K. Then,

$$\mathtt{sd}\, M \cap \mathtt{sd}\, P = \mathtt{sd}\,(M \cap P), \text{ and, } \mathtt{sd}\, M \cup \mathtt{sd}\, P = \mathtt{sd}\,(M \cup P).$$

Lemma 3.5 *Let C_1 and C_2 be two disjoint closed subsets of K. Then there is a $m \geq 0$ such that,*

$$\overline{\mathtt{N}}\,([\mathtt{sd}^m K]_{C_1}, \mathtt{sd}^m K) \cap \overline{\mathtt{N}}\,([\mathtt{sd}^m K]_{C_2}, \mathtt{sd}^m K) = \emptyset.$$

Proof: For any $m \geq 0$ and for $i = 1$ or $2, \mathtt{N}\,([\mathtt{sd}^m K]_{C_i}, \mathtt{sd}^m K)$ is an open neighbourhood of C_i in $|K|$. Since C_1 and C_2 are disjoint closed subsets of $|K|$, there are open neighbourhoods U_1, U_2 in $|K|$ of C_1, C_2 respectively such that, $\overline{U_1} \cap \overline{U_2} = \emptyset$. Choose $m \geq 0$ such that $\mathtt{N}\,([\mathtt{sd}^m K]_{C_i}, \mathtt{sd}^m K) \subset U_i$ for $i = 1$ and 2. The result follows. ∎

Lemma 3.6 *Let $\sigma = \sigma_{[0]}$ be a simplex of K and for any $m \geq 0$ let $\sigma_{[m]}$ be a primitive simplex of $\mathtt{sd}^m K$ with respect to $\sigma_{[0]}$. Then,*

$$\overline{\mathtt{st}}\,(\sigma_{[m]}, \mathtt{sd}^m K) \subset \mathtt{sd}^m\{\overline{\mathtt{st}}\,(\sigma_{[0]}, K)\}.$$

Proof: Let for all $1 \leq j \leq m$, $\sigma_{[j]}$ be a primitive simplex of $\mathrm{sd}^j K$ with respect to $\sigma_{[j-1]}$. Let $0 \leq j < n$ and $\mu = [b(\mu_0), \dots, b(\mu_q)]$ be a simplex of $\overline{\mathtt{st}}\,(\sigma_{[j+1]}, \mathrm{sd}^{j+1}K)$ such that $\langle \mu \rangle \subset \mathtt{st}\,(\sigma_{[j+1]}, \mathrm{sd}^{j+1}K)$. Then $\sigma_{[j+1]} \prec \mu$. Let $\sigma_{[j+1]} = [b(\mu_{i_0}), \dots, b(\mu_{i_p})]$. Then $\langle \sigma_{[j+1]} \rangle \subset \langle \mu_{i_p} \rangle$ and hence, $\mu_{i_p} = \sigma_{[j]}$. Also $\langle \mu \rangle \subset \langle \mu_q \rangle$ and $\mu_{i_p} \prec \mu_q$. Therefore μ_q is a simplex of $\overline{\mathtt{st}}\,(\sigma_{[j]}, \mathrm{sd}^j K)$ and hence, μ is a simplex of $\mathrm{sd}\,\overline{\mu_q}$ which in its turn is a subcomplex of $\mathrm{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[j]}, \mathrm{sd}^j K)\}$. Thus, for all $0 \leq j \leq m-1$,

$$\overline{\mathtt{st}}\,(\sigma_{[j+1]}, \mathrm{sd}^{j+1}K) \subset \mathrm{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[j]}, \mathrm{sd}^j K)\}.$$

Now by Observation 3.4,

$$\overline{\mathtt{st}}\,(\sigma_{[m]}, \mathrm{sd}^m K) \subset \mathrm{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[m-1]}, \mathrm{sd}^{m-1}K)\} \subset \cdots \subset \mathrm{sd}^m\{\overline{\mathtt{st}}\,(\sigma_{[0]}, K)\}.$$ ■

Lemma 3.7 *Let $g : K \longrightarrow L$ be a simplicial map, and let M be a subcomplex of K, P be a subcomplex of L, such that, $g(M) \subset P$. Then,*

$$g(\overline{\mathtt{N}}\,(M, K)) \subset \overline{\mathtt{N}}\,(P, L)$$

Proof: Let σ be a simplex of K such that $\langle \sigma \rangle \subset \mathtt{N}\,(M, K)$. Then there is a simplex τ of M such that $\tau \prec \sigma$. Since $g(\tau)$ is a simplex of P and $g(\tau) \prec g(\sigma)$, it follows that $\langle g(\sigma) \rangle \subset \mathtt{N}\,(P, L)$. Therefore,

$$g(\mathtt{N}\,(M, K)) \subset \mathtt{N}\,(P, L).$$

The result follows. ■

Lemma 3.8 *Let $\tau \in K$ and $\sigma \in \mathrm{sd}^p K, p \geq 0$ be such that $\langle \sigma \rangle \subset \langle \tau \rangle$. Then the largest vertex of σ belongs to $\langle \tau \rangle$.*

Proof: Let $\tau \in K$ and $\sigma \in \mathrm{sd}\,K$ be such that $\langle \sigma \rangle \subset \langle \tau \rangle$.
Let $\sigma = [b(\tau_0), \dots, b(\tau_r)]$ where, $\tau_0 \prec \cdots \prec \tau_r$ are simplices of K. So the largest vertex of σ is $b(\tau_r)$ and $\langle \sigma \rangle \subset \langle \tau_r \rangle$. Therefore $\tau_r = \tau$. Hence $b(\tau_r) \in \langle \tau \rangle$.

Thus, for any simplicial complex L and any simplex μ of L, if $\nu \in \mathrm{sd}\, L$ is such that $\langle\nu\rangle \subset \langle\mu\rangle$, then the largest vertex of ν belongs to $\langle\mu\rangle$.
Let $\sigma \in \mathrm{sd}^k K, k \geq 2$ such that $\langle\sigma\rangle \subset \langle\tau\rangle$. There is a simplex μ of $\mathrm{sd}^{k-1}K$, such that $\langle\sigma\rangle \subset \langle\mu\rangle \subset \langle\tau\rangle$. Then the largest vertex of σ belongs to $\langle\mu\rangle$ by applying what we have just now proved to $\mathrm{sd}^{k-1}K$. Hence the largest vertex of σ belongs to $\langle\tau\rangle$. ∎

3.5 The fixed point set of a simplicial map

Let K' be a subdivision of K and $g : K' \longrightarrow K$ be a simplicial map. Let σ be a p-simplex of K'.

Lemma 3.9 *$Fix\, g \cap \langle\sigma\rangle \neq \emptyset$ implies that $\langle\sigma\rangle \subset \langle g(\sigma)\rangle$.*

Proof: Let $x \in \mathrm{Fix}\, g \cap \langle\sigma\rangle$ and $\tau \in K$ such that $\langle\sigma\rangle \subset \langle\tau\rangle$. Then $x \in \langle\tau\rangle$ and $x = g(x) \in \langle g(\sigma)\rangle$. Hence $g(\sigma) = \tau$. ∎

Remark 3.10 It is clear that if $K' = K$, then, $\mathrm{Fix}\, g \cap \langle\sigma\rangle \neq \emptyset$ implies that $\sigma = g(\sigma)$. Of course, if $\sigma = g(\sigma)$ and w is a vertex of σ then $g(w)$ may not be equal to w.

Lemma 3.11 *Let $x \in Fix\, g$ and $\sigma'(x)$ be the carrier of x in K'. Then, $\sigma'(x)$ is a primitive simplex.*

Proof: The carrier of x in K is trivially a primitive simplex. We denote it by $\sigma_{[0]}(x)$. Since $\langle\sigma'(x)\rangle \subset \langle\sigma_{[0]}(x)\rangle, \dim\sigma'(x) \leq \dim\sigma_{[0]}(x)$. Also, $x \in \mathrm{Fix}\, g$, implies by Lemma 3.9 that $g(\sigma'(x)) = \sigma_{[0]}(x)$. Therefore $\dim\sigma_{[0]}(x) \leq \dim\sigma'(x)$. Hence $\dim\sigma_{[0]}(x) = \dim\sigma'(x)$ and so $\sigma'(x)$ is a primitive simplex. ∎

Lemma 3.12 *Let* $\langle\sigma\rangle \subset \langle g(\sigma)\rangle$. *Then* $Fix\, g \cap |\overline{\sigma}| \neq \emptyset$.

Proof: The degree of the map $g : (\overline{\sigma}, \dot{\sigma}) \longrightarrow (g(\overline{\sigma}), g(\dot{\sigma}))$ is non zero. Since the inclusion map $i : \overline{\sigma} \longrightarrow g(\overline{\sigma})$ is inessential, g has a coincidence with i ([17, Theorem 2.2]). This is a generalisation of Brouwer's fixed point theorem (also see [3], [20]).

We prove this result independently as follows. Assume that g has no coincidence with i. For any point x of $|\overline{\sigma}|$ let x' be the intersection of the straight line path from $i(x)$ to $g(x)$ with $g(\dot{\sigma})$. Define a map $\varphi : |\overline{\sigma}| \longrightarrow g(\dot{\sigma})$ by $\varphi(x) = x'$. Clearly φ is continuous and is an extension of $g|\,\dot{\sigma}$. But this contradicts the fact that $g|\,\dot{\sigma}$ has non zero degree. Therefore g has a coincidence with i. In other words, $\text{Fix}\, g \cap |\overline{\sigma}| \neq \emptyset$. ∎

Proposition 3.13 *Let K' be fine enough subdivision of K such that, for any $\sigma \in K'$ and $\tau \in K$, if $\langle\sigma\rangle \subset \langle\tau\rangle$, then diameter $\overline{\sigma} < d(b(\tau), \dot{\tau})$. Then $Fix\, g$ is a finite set of points of $|K|$.*

Proof: Let σ be a simplex of K' such that $\langle\sigma\rangle \subset \langle g(\sigma)\rangle$. Then for any $x, y \in |\overline{\sigma}|, d(x,y) < d(g(x), g(y))$ ([12, Chapter III, Lemma 3.5(i)]). This can be seen also as follows by considering the barycentric coordinate representation of the points.

Let $\sigma = [u_0 < \cdots < u_p]$ and

$$x = \sum_{i=0}^{p} t_i u_i, \quad y = \sum_{i=0}^{p} r_i u_i$$

Let $u_i = \sum_{j=0}^{p} s_{ij} g(u_j)$. Then for all $1 \leq i \leq p$ and $0 \leq j \leq p, 0 \leq s_{ij} < 1$.

Now,

$$x = \sum_{j=0}^{p} \left\{ \sum_{i=0}^{p} t_i s_{ij} \right\} g(v_j), \quad y = \sum_{j=0}^{p} \left\{ \sum_{i=0}^{p} r_i s_{ij} \right\} g(v_j)$$

Therefore,

$$
\begin{aligned}
d(x,y) &= \left[\sum_{j=0}^{p}\left\{\sum_{i=0}^{p}(t_i - r_i)^2 s_{ij}^2\right\}\right]^{\frac{1}{2}} = \left[\sum_{i=0}^{p}(t_i - r_i)^2\left\{\sum_{j=0}^{p}s_{ij}^2\right\}\right]^{\frac{1}{2}} \\
&< \left[\sum_{i=0}^{p}(t_i - r_i)^2\left\{\sum_{j=0}^{p}s_{ij}\right\}\right]^{\frac{1}{2}} = \left[\sum_{i=0}^{p}(t_i - r_i)^2\right]^{\frac{1}{2}} \\
&= d(g(x), g(y)).
\end{aligned}
$$

So $g(\overline{\sigma})$ has at most one fixed point of g. Therefore if σ is a simplex of K' such that $\text{Fix}\,g \cap \langle\sigma\rangle = \{x\}$, then $\text{st}\,(\sigma, K') \cap \text{Fix}\,g = \{x\}$. ∎

Remark 3.14 If $K' = \text{sd}^n K, n > 0$, then the hypothesis in Proposition 3.13 holds. Hence for $n > 0$, the fixed point set of a simplicial map $g : \text{sd}^n K \longrightarrow K$ is a finite set of points of $|K|$.

3.6 Subdivision of a simplicial map

Let K' be a subdivision of K and $g : K' \longrightarrow K$ be a simplicial map. The 1^{st} subdivision of g is the simplicial map,

$$\text{sd}\,g : \text{sd}\,K' \longrightarrow \text{sd}\,K$$

defined on the vertices by,

$$\text{sd}\,g(b(\sigma)) = b(g(\sigma)), \text{ for all } \sigma \in K'.$$

The n^{th} subdivision of g for $n \geq 1$ is defined inductively to be,

$$\text{sd}^n\,g = \text{sd}\,(\text{sd}^{n-1}\,g) : \text{sd}^n K' \longrightarrow \text{sd}^n K.$$

Let K' be a subdivision of K and $g : K' \longrightarrow K$ be a simplicial map. Let σ be a p-simplex of K'.

Lemma 3.15 *Let $g(\sigma)$ be a p-simplex of K. Then for all $x \in |\overline{\sigma}|$, $g(x) = \text{sd}\,g(x)$.*

Proof: Since $g(\sigma)$ is a p-simplex of K, for all $\tau \prec \sigma$, $g(b(\tau)) = b(g(\tau))$ and by definition, $\operatorname{sd} g(b(\tau)) = b(g(\tau))$. Therefore, $g : \operatorname{sd}\overline{\sigma} \longrightarrow g(\operatorname{sd}\overline{\sigma})$ is also simplicial and for all vertex w of $\operatorname{sd}\overline{\sigma}$, $g(w) = \operatorname{sd} g(w)$. Hence the result follows. ∎

Remark 3.16 The result is not true if $g(\sigma)$ is not a p-simplex of K. This can be seen from the following example. Let $\sigma = [u_0, \dots u_p]$ be a simplex such that $g(u_0) = g(u_1)$ and $g([u_2, \dots, u_p])$ is a $(p-2)$-simplex. Then,

$$g(b(\sigma)) = \frac{1}{p+1}\sum_{i=2}^{p} g(u_i) + \frac{2}{p+1} g(u_1).$$

On the other hand, $\operatorname{sd} g(b(\sigma) = b(g(\sigma)) = \frac{1}{p}\sum_{i=1}^{p} g(u_i)$.

Lemma 3.17 *Let $K' = K$. Then $Fix\, g = Fix \operatorname{sd} g$.*

Proof: Let $x \in \operatorname{Fix} g$ and the carrier of x in K be $\sigma_{[0]}$. Then by Lemma 3.9, $\sigma_{[0]} = g(\sigma_{[0]})$ and hence by Lemma 3.15, $g(x) = \operatorname{sd} g(x) = x$.
Conversely let $x \in \operatorname{Fix}\operatorname{sd} g$. Let the carrier of x in K be $\sigma_{[0]} = [u_0, \dots, u_p]$. Let $\tau_j = [u_{i_0}, \dots, u_{i_{r_j}}]$, and $\tau_0 \prec \cdots \prec \tau_q = \sigma_{[0]}$. Let the carrier of x in $\operatorname{sd} K$ be $\sigma_{[1]} = [b(\tau_0), \dots, b(\tau_q)]$.
By Lemma 3.9, $\operatorname{sd} g(\sigma_{[1]}) = \sigma_{[1]}$. Let $\operatorname{sd} g(b(\tau_k)) = b(\tau_l)$. Then $g(\tau_k) = \tau_l$. This implies that $l \leq k$. Therefore $g(\tau_0) = \tau_0$. If possible, let $g(\tau_1) = \tau_0$. Then $\operatorname{sd} g(\sigma_{[1]})$ is a proper face of $\sigma_{[1]}$, a contradiction.
Therefore $g(\tau_1) = \tau_1$. Proceeding similarly, we see that $g(\tau_q) = \tau_q$, i.e. $g(\sigma_{[0]}) = \sigma_{[0]}$. Therefore by Lemma 3.15, $g(x) = \operatorname{sd} g(x) = x$. ∎

Remark 3.18 From the above proof it is clear that if $K' = K$ then for all vertices $b(\tau_j)$ of $\sigma_{[1]}$, $\operatorname{sd} g(b(\tau_j)) = b(\tau_j)$.

Lemma 3.19 *Let $K' = K$. Then $Fix \operatorname{sd} g$ is a subcomplex of $\operatorname{sd} K$.*

Proof: Let $F = \{\mu \in \mathrm{sd}\, K : \mu \prec \tau, \langle\tau\rangle \cap \mathrm{Fix}\,\mathrm{sd}\, g \neq \emptyset\}$. It follows by Remark 3.18 that if $\langle\tau\rangle \cap \mathrm{Fix}\,\mathrm{sd}\, g \neq \emptyset$, then for all vertex w of $\overline{\tau}$, $\mathrm{sd}\, g(w) = w$. Therefore $\overline{\tau} \subset \mathrm{Fix}\,\mathrm{sd}\, g$. So $F = \mathrm{Fix}\,\mathrm{sd}\, g$. ∎

Thus when $K' = K$, the path components of $\mathrm{Fix}\, g$ are precisely the geometric realizations of the distinct subcomplexes of $\mathrm{sd}\, K$ whose union is $|\,\mathrm{Fix}\, g|$. Hence the following result follows from Proposition 3.13 and Lemma 3.19,

Proposition 3.20 *Each path component of the fixed point set of a simplicial map* $f : \mathrm{sd}^n K \longrightarrow K, n \geq 0$, *is compact and open in* $\mathit{Fix}\, f$.

Lemma 3.21 *Let* K' *be finer than* $\mathrm{sd}\, K$. *Then* $\mathit{Fix}\, g = \mathit{Fix}\,\mathrm{sd}\, g$.

Proof: Let $x \in \mathrm{Fix}\, g$. Then the carrier of x in K' is a primitive simplex. Let $\sigma'_{[0]}$ be the carrier of x in K'. Then by Lemma 3.9,

$$\langle\sigma'_{[0]}\rangle \subset \langle g(\sigma'_{[0]})\rangle$$

and hence by Lemma 3.15 $g(x) = \mathrm{sd}\, g(x) = x$. Thus $x \in \mathrm{Fix}\,\mathrm{sd}\, g$. Conversely let $x \in \mathrm{Fix}\,\mathrm{sd}\, g$. Then by Lemma 3.11, the carrier of x in $\mathrm{sd}\, K'$ is a primitive simplex, say, $\sigma'_{[1]}$ with respect to $\mathrm{sd}\, K$. So $\dim \sigma'_{[1]} = \dim \sigma_{[1]}$ where the carrier of x in $\mathrm{sd}\, K$ is $\sigma_{[1]}$. Since K' is finer than $\mathrm{sd}\, K$,

$$\dim \sigma'_{[1]} \leq \dim \sigma'_{[0]} \leq \dim \sigma_{[1]}.$$

Therefore $\dim \sigma'_{[1]} = \dim \sigma'_{[0]}$. Let $\dim \sigma'_{[1]} = \sigma'_{[0]} = p$. If $g(\sigma'_{[0]})$ is not a p-simplex of K, there are two vertices $u_0 \neq u_1$ of $\sigma'_{[0]}$ such that $g(u_0) = g(u_1)$. Let $\sigma'_{[0]} = [u_0, \ldots, u_p]$ and $\tau_0 \prec \cdots \prec \tau_p = \sigma'_{[0]}$ be such that,

$$\sigma'_{[1]} = [b(\tau_0), \ldots, b(\tau_p)].$$

Then τ_j is a j-simplex of K'. Let for some i, where $0 \leq i \leq p$, τ_i be such that only one of u_0 or u_1 is a vertex of τ_i and both u_0 and u_1 are vertices of τ_{i+1}. Then $g(\tau_i) = g(\tau_{i+1})$ and hence $\mathrm{sd}\, g(\sigma'_{[1]})$ is not a p-simplex of $\mathrm{sd}\, K$, a contradiction. Hence $g(\sigma)$ is a p-simplex of K. Then by Lemma 3.15, $g(x) = \mathrm{sd}\, g(x) = x$. Thus $x \in \mathrm{Fix}\, g$. ∎

Lemma 3.22 *For any subcomplex L of K,* $\mathtt{sd}\, g(\mathtt{sd}\, L) = \mathtt{sd}\,(g(L))$.

Proof: Let τ be a simplex of $\mathtt{sd}\, L$ and let σ be a simplex of L such that $\langle\tau\rangle \subset \langle\sigma\rangle$. Then τ is a simplex of $\mathtt{sd}\, \overline{\sigma}$. Let $\sigma = [u_0, \ldots, u_p]$ and for all $0 \leq j \leq p$, let $\mu_j = [u_0, \ldots, u_j]$ and $\mu = [b(\mu_0), \ldots, b(\mu_p)]$. Without loss of generality we can assume that $\tau \prec \mu$. Then $\mathtt{sd}\, g(\tau) \prec \mathtt{sd}\, g(\mu)$ and $\mathtt{sd}\, g(\mu) \in \mathtt{sd}\,\{g(\overline{\sigma})\} \subset \mathtt{sd}\,(g(L))$. Therefore $\mathtt{sd}\, g(\mathtt{sd}\, L) \subset \mathtt{sd}\,(g(L))$.

Conversely let $\tau \in \mathtt{sd}\,(g(L))$. Then there is a $\sigma \in L$ such that τ is a simplex of $\mathtt{sd}\,\{g(\overline{\sigma})\}$. Let $\sigma = [u_0, \ldots, u_p]$, for all $0 \leq j \leq p$, $\mu_j = [g(u_0), \ldots, g(u_j)]$ and $\mu = [b(\mu_0), \ldots, b(\mu_p)]$. Without loss of generality we can assume that $\tau \prec \mu$. Let for all $0 \leq j \leq p, \sigma_j = [u_0, \ldots, u_j]$ and $\nu = [b(\sigma_0), \ldots, b(\sigma_p)]$. Then, $\nu \in \mathtt{sd}\, L$ and $\mu = \mathtt{sd}\, g(\nu)$. So μ is a simplex of $\mathtt{sd}\, g(\mathtt{sd}\, L)$. Hence, $\mathtt{sd}\,(g(L)) \subset \mathtt{sd}\, g(\mathtt{sd}\, L)$. ∎

Lemma 3.23 $\mathtt{sd}\, g$ *is homotopic to* g.

Proof: Consider the map $f = |\mathtt{sd}\, g| : |K'| \longrightarrow |K|$. Then g is a simplicial approximation to f. This can be seen by verifying the star condition. Let v be a vertex of K'. We wish to show that $f(\mathtt{st}\,(v, K')) \subset \mathtt{st}\,(g(v), K)$. Let $\langle\sigma\rangle \subset \mathtt{st}\,(v, K')$ and let $\tau \in \mathtt{sd}\, K'$ such that $\langle\tau\rangle \subset \langle\sigma\rangle$. By the definition of $\mathtt{sd}\, g$, $\langle \mathtt{sd}\, g(\tau)\rangle \subset \langle g(\sigma)\rangle \subset \mathtt{st}\,(g(v), K)$. Since this is true for all $\tau \in \mathtt{sd}\, K'$ such that $\langle\tau\rangle \subset \langle\sigma\rangle$, it follows that,

$$f(\langle\sigma\rangle) \subset \langle g(\sigma)\rangle \subset \mathtt{st}\,(g(v), K).$$

Since g is a simplicial approximation to $\mathtt{sd}\, g$, it follows that $\mathtt{sd}\, g$ is homotopic to g ([22]).

Let the homotopy be $H : |K| \times I \longrightarrow |K|$. If L is a subcomplex of K' and $g(L) = M$, then, $H : |L| \times I \longrightarrow |M|$. ∎

Remark 3.24 $\mathtt{sd}\, g$ is not a simplicial approximation to g. This can be seen by an example. Let K be the standard 2-simplex with $V(K) = \{0, 1, 2\}$. Define a simplicial map $g : K \longrightarrow K$ with values on the vertices :

$$g(0) = g(1) = 1 \text{ and } g(2) = 2.$$

Let x be the point $x = \frac{1}{4}2 + \frac{3}{4}012$. Then $x \in \mathtt{st}\,(2, \mathtt{sd}\, K)$. Also,

$$x = \tfrac{1}{4}(0+1) + \tfrac{1}{2}2.$$

Therefore $g(x) = \frac{1}{2}(1+2) = 12 \notin \mathtt{st}\,(2, \mathtt{sd}\, K)$. Thus, $\mathtt{sd}\, g(2) = 2$, but $g(\mathtt{st}\,(2, \mathtt{sd}\, K) \not\subset \mathtt{st}\,(2, \mathtt{sd}\, K)$.

Remark 3.25 Let $f : |K| \longrightarrow |K|$ be a map and let $g : \mathtt{sd}^n K \longrightarrow K$ be a simplicial approximation to f. Then $\mathtt{sd}\, g$ need not be a simplicial approximation to f. This can be seen from an example. Let $g : \mathtt{sd}\, K \longrightarrow K$ be the simplicial approximation to identity defined by,

$$g(b([u_0 < \cdots < u_p])) = u_p,$$

for all simplex $[u_0, \ldots, u_p]$ of K, where, $\leq$ is the ordering on $V(K)$ mentioned in Section 4 of this chapter. Then for any simplex τ of K such that, $\dim \tau > 0$, $\mathtt{sd}\, g(b(\tau)) \neq b(\tau)$ and hence $\mathtt{sd}\, g$ is not a simplicial approximation to the identity map on $|K|$.

Remark 3.26 The choice of a collection of isolated sets of fixed points is clearly not unique as can be seen from a simple example. It is possible to define a map on the closed unit interval in $\mathbb{R}$ whose fixed point set is precisely, $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then for any $m \geq 1$ a class of isolated sets of fixed points could be defined to be,

$$C_0 = \{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{N}, n \geq m+1\}, \quad C_j = \{\tfrac{1}{j}\}, 1 \leq j \leq m.$$

For simplicial maps we choose the class of isolated sets of fixed points of the map to be the distinct path components of the fixed point set.

Chapter 4

A Retraction of Star Neighbourhoods

4.1 Introduction

Let K be a simplicial complex and $\sigma_{[0]}$ be a simplex of K. Let $\sigma_{[n]}$ be a primitive simplex of $\mathrm{sd}^n K$ with respect to $\sigma_{[0]}$. Then $\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)$ is a subcomplex of $\mathrm{sd}^n(\overline{\mathrm{st}}\,(\sigma_{[0]}, K))$ (Lemma 3.6) and $|\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)|$ is a deformation retract of $|\overline{\mathrm{st}}\,(\sigma_{[0]}, K)|$.

In this chapter we define a simplicial map,

$$\rho_{0,n}(\sigma_{[n]}) : \mathrm{sd}^n(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)) \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)$$

which is a deformation retraction and discuss a few properties of this retraction which will be relevant to us. This retraction plays a major role in the remainder of the thesis. The results of this chapter are technical to state but the ideas involved are quite geometric.

4.2 Definition of the deformation retraction

Let K be a simplicial complex and $\sigma_{[0]}$ be a simplex of K. Let $\sigma_{[n]}$ be a primitive simplex of $\mathrm{sd}^n K$ with respect to $\sigma_{[0]}$. Let for all $1 \leq j \leq n, \sigma_{[j]}$ be a primitive simplex of $\mathrm{sd}^j K$ with respect to $\sigma_{[j-1]}$. Clearly then for all $0 \leq j \leq n, \sigma_{[j]}$ is a primitive simplex of $\mathrm{sd}^j K$ with respect to $\sigma_{[0]}$.

Let $s \geq 0$. Let $\sigma_{[s]} = [v_0, \ldots, v_p]$, and $\sigma_{[s+1]} = [b(\mu_0), \ldots, b(\mu_p)]$, where, $\mu_j = [v_0, \ldots, v_j]$. Define,

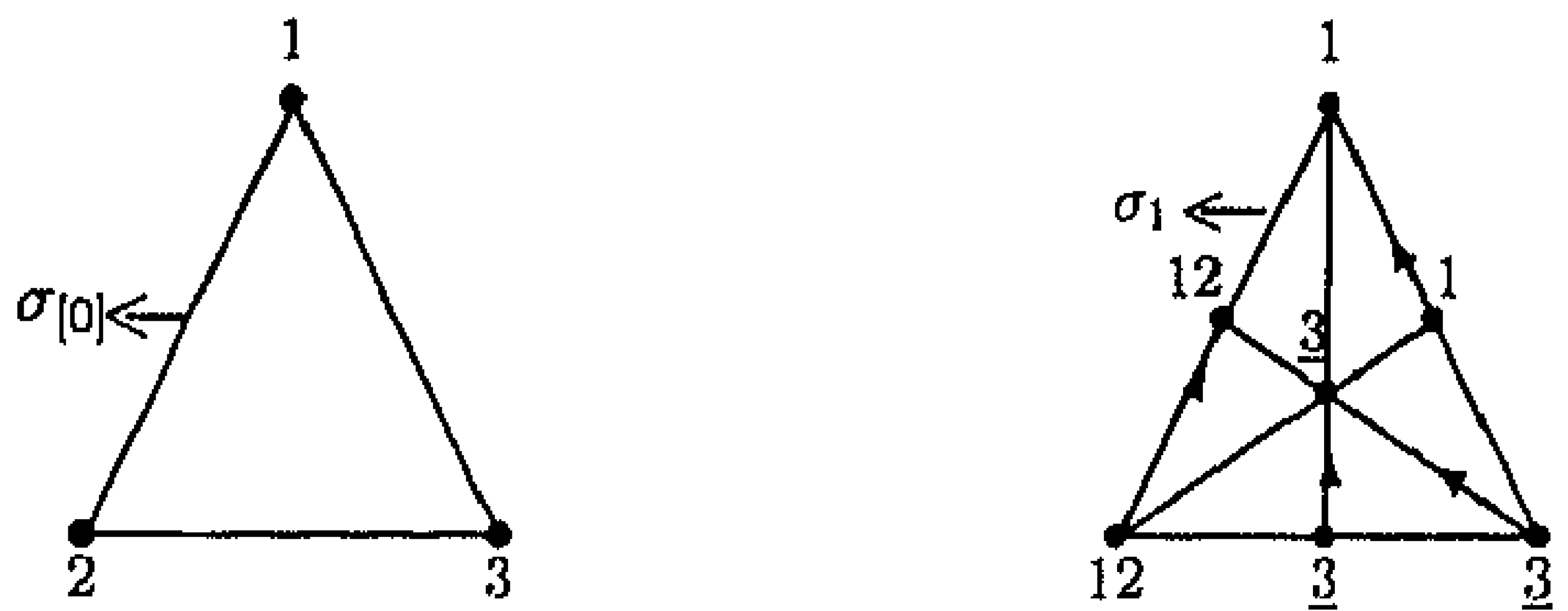
$$\rho_s : \mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^s K)\} \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1} K)$$

by,

$$
\begin{array}{lll}
\rho_s(b(\tau)) & = b(\sigma_{[s]} \cup \tau) & \text{if } \mu_j \prec \tau \Rightarrow \mu_{j+1} \prec \tau, \text{ for all } 0 \leq j \leq p-1 \\
\rho_s(b(\tau)) & = b(\mu_j) & \text{if } \mu_j \prec \tau \text{ and } \mu_{j+1} \not\prec \tau, \text{for some } 0 \leq j \leq p-1
\end{array}
$$

This definition can be best understood by an example. Let K be the standard 2-simplex with $V(K) = \{1, 2, 3\}$ and let $\sigma_{[0]}$ be the 1-simplex $[1, 2]$ of K and $\sigma_{[1]}$ be the 1-simplex $[1, 12]$ of $\mathsf{sd}\, K$. Let us denote the vertex $b([1, 2, 3]) = 123$ of $\mathsf{sd}\, K$ by $\underline{3}$.

Figure 4 depicts the map $\rho_1 : \mathsf{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[0]}, K)\} \longrightarrow \overline{\mathtt{st}}\,(\sigma_{[1]}, \mathsf{sd}\, K)$, (see Section 3 of Chapter 3 for the convention regarding the description of a simplicial map through figures).



$\overline{\mathtt{st}}\,(\sigma_{[0]}, K)$ $\qquad$ $\rho(\mathsf{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[0]}, K)\})$

Figure 4: The retraction ρ_1

We fix the notations as above unless stated otherwise.

Lemma 4.1 *ρ_s is a simplicial map.*

Proof: Let $\tau = [b(\tau_0) < \cdots < b(\tau_q)]$ be a simplex of $\mathsf{sd}\,\{\overline{\mathtt{st}}\,(\sigma_{[s]}, \mathsf{sd}^s K)\}$. Let $\{b(\tau_r) < \cdots < b(\tau_s)\}$ be the set of vertices of τ such that, for all $0 \leq r \leq k \leq t \leq q$, there exist a vertex $b(\mu_{j_k})$ of $\sigma_{[s+1]}$, such that,

$$\mu_{j_k} \prec \tau_k \text{ and}, \mu_{j_k+1} \not\prec \tau_k.$$

Since for all $0 \leq r \leq k \leq t \leq q$, $b(\mu_{j_k})$ are vertices of $\sigma_{[s+1]}$, either $\mu_{j_k} \prec \mu_{j_{k+1}}$

or $\mu_{j_{k+1}} \prec \mu_{j_k}$. Let if possible $\mu_{j_{k+1}} \prec \mu_{j_k}$. Again $b(\mu_{j_k})$ and $b(\mu_{j_k+1})$ are vertices of $\sigma_{[s+1]}$, hence either $\mu_{j_k} \prec \mu_{j_k+1}$ or $\mu_{j_k+1} \prec \mu_{j_k}$. Since $\{\mu_j : \mu_{j+1}\} = 1$ for all $0 \leq j \leq p$, it follows that,

$$\mu_{j_{k+1}+1} \prec \mu_{j_k} \prec \tau_k \prec \tau_{k+1}$$

This is a contradiction, since $\mu_{j_{k+1}+1} \not\prec \tau_{k+1}$. Therefore, $\mu_{j_r} \prec \cdots \prec \mu_{j_t}$. Also, for all $t+1 \leq k \leq q$, $\sigma_{[s]}$ is a face of τ_k. Thus,

$$\mu_{j_r} \prec \cdots \prec \mu_{j_t}(\prec \sigma_{[s]}) \prec \tau_0 \cup \sigma_{[s]} \prec \cdots \prec \tau_{r-1} \cup \sigma_{[s]} \prec \tau_{t+1} \prec \cdots \prec \tau_q$$

and hence,

$$\rho_s(\tau) = [b(\tau_0 \cup \sigma_{[s]}), \ldots, b(\tau_{r-1} \cup \sigma_{[s]}), b(\mu_{j_r}), \ldots, b(\mu_{j_t}), b(\tau_{t+1}), \ldots, b(\tau_q)]$$

is a simplex of $\overline{\mathtt{st}}\,(\sigma_{[s+1]}, \mathsf{sd}^{s+1}K)$. ■

Let $\sigma_{[s]} = \sigma_{[s+1]} = v$, a vertex of $\mathsf{sd}^s K$. Then for all $0 \leq j \leq p-1$,

$$\mu_j = \mu_{j+1} = v.$$

Thus for any simplex τ of $\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)$, if $\mu_j \prec \tau$, then $\mu_{j+1} \prec \tau$. Thus, the map $\rho_s : \mathsf{sd}\,\{\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)\} \longrightarrow \overline{\mathtt{st}}\,(v, \mathsf{sd}^{s+1}K$ is defined by

$$\boxed{\rho_s(b(\tau)) = b(\tau \cup v)}$$

for all simplex τ of $\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)$.

Thus, if $\sigma_{[s]} = \sigma_{[s+1]} = v$, the map ρ_s is *radial* i.e. for any vertex $w \neq v$ of $\mathsf{sd}\,\{\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)\}$, $\rho_s(w)$ is the point of intersection of the line segment joining w and v with $\mathsf{Lk}\,(v, \mathsf{sd}^{s+1}K)$. This can be seen as follows. Let $\tau = [u_0, \ldots, u_p]$ be a p-simplex of $\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)$ and consider the vertex $b(\tau)$ of $\mathsf{sd}\,\{\overline{\mathtt{st}}\,(v, \mathsf{sd}^s K)\}$. If $\langle\tau\rangle \subset \mathtt{st}\,(v, \mathsf{sd}^s K)$, then v is a vertex of τ and hence $\rho_s(b(\tau)) = b(\tau)$. Thus trivially $\rho_s(b(\tau))$ lies on the line segment joining v and $b(\tau)$. If $\tau \in \mathsf{Lk}\,(v, \mathsf{sd}^s K)$, then $\rho_s(b(\tau)) = b(\tau \cup v)$. Hence,

$$\rho_s(b(\tau)) = \frac{p+1}{p+2} b(\tau) + \frac{1}{p+2} v.$$

So, $\rho_s(b(\tau))$ lies on the line segment joining v and $b(\tau)$.

Lemma 4.2 (i) $\rho_s|\overline{\mathtt{st}}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$ *is the identity map.*

(ii) $\rho_s : \mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\} \longrightarrow \mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$ *is a simplicial homeomorphism.*

(iii) $\rho_s(\mathtt{sd}\ \dot{\sigma}_{|s|}) \subset \dot{\sigma}_{|s+1|}$

(iv) $\rho_s(\mathtt{sd}\,(\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K) * \dot{\sigma}_{|s|})) \subset \mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K) * \dot{\sigma}_{|s+1|}$

Proof: (i) Let $b(\tau)$ be a vertex of $\mathtt{sd}\,(\overline{\mathtt{st}}\,(\sigma_{|s|}, \mathtt{sd}^{s}K))$. Then τ is a simplex of $\overline{\mathtt{st}}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)$. By Lemma 3.3, $b(\tau)$ is a vertex of $\mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$ if and only if, $\sigma_{|s|} \prec \tau$. Therefore, if $b(\tau)$ is a vertex of $\mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$, then $\rho_s(b(\tau)) = b(\tau \cup \sigma_{|s|}) = b(\tau)$. If $b(\tau)$ is a vertex of $\sigma_{|s+1|}$, then $\tau = \mu_j$ for some $0 \le j \le p$. Hence $\rho_s(b(\tau)) = \rho_s(b(\mu_j)) = b(\mu_j) = b(\tau)$. Thus, $\rho_s|\overline{\mathtt{st}}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$ is the identity map.

(ii) If $\tau \in \mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)$, then $\overline{\tau} \cap \overline{\sigma}_{|s|} = \emptyset$. So $\rho_s(b(\tau)) = b(\tau \cup \sigma_{|s|})$. Since $\tau \cup \sigma_{|s|}$ has $\sigma_{|s|}$ as a proper face, $b(\tau \cup \sigma_{|s|}) \in \mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$.
Let $\mu = [b(\tau_0) < \cdots < b(\tau_r)]$ be a simplex of $\mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\}$. Then,

$$\sigma_{|s|} \prec \tau_0 \cup \sigma_{|s|} \prec \cdots \prec \tau_r \cup \sigma_{|s|}$$

and hence, $\rho_s(\mu)$ is a simplex of $\mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$. Thus,

$$\rho_s(\mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\}) \subset \mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K).$$

We show that $\rho_s|\mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\}$ is a simplicial homeomorphism.
It is enough to show that ρ_s is a one to one correspondence. Let $b(\tau)$ and $b(\tau')$ be vertices of $\mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\}$ such that $\rho_s(b(\tau)) = \rho_s(b(\tau'))$. This implies that $b(\tau \cup \sigma_{|s|}) = b(\tau' \cup \sigma_{|s|})$ i.e. $\tau \cup \sigma_{|s|} = \tau' \cup \sigma_{|s|}$. Since $\overline{\tau} \cap \overline{\sigma}_{|s|} = \emptyset = \overline{\tau}' \cap \overline{\sigma}_{|s|}$, it follows that $\tau = \tau'$. Hence $b(\tau) = b(\tau')$. Thus $\rho_s|\mathtt{sd}\,\{\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)\}$ is one to one. Let $b(\tau) \in \mathtt{Lk}\,(\sigma_{|s+1|}, \mathtt{sd}^{s+1}K)$. Then $\sigma_{|s|}$ is a proper face of τ. Let $\tau - \sigma_{|s|} = \nu$. Then ν is a simplex of $\mathtt{Lk}\,(\sigma_{|s|}, \mathtt{sd}^{s}K)$ and $\rho_s(b(\nu)) = b(\tau)$. Hence ρ_s is onto.

Therefore for any subcomplex L of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$, $\rho_s|\mathrm{sd}\,L$ is a simplicial homeomorphism.

(iii) Let τ be a simplex of $\dot{\sigma}_{[s]}$. Then $\tau \prec \sigma_{[s]}$. If there is a $\mu_j, 0 \le j \le p-1$, such that $\mu_j \prec \tau$ and $\mu_{j+1} \not\prec \tau$, then, $\rho_s(b(\tau)) = b(\mu_j) \in \sigma_{[s+1]}$. If there is no such μ_j then, $\rho_s(b(\tau)) = b(\tau \cup \sigma_{[s]}) = b(\sigma_{[s]})$. Hence, in either case, $\rho_s(b(\tau))$ is a vertex of $\sigma_{[s+1]}$.

Let ν be a simplex of sd $\dot{\sigma}_{[s]}$. Since $\overline{\sigma}_{[s+1]}$ is a full subcomplex and all vertices of $\rho_s(\nu)$ belong to $\dot{\sigma}_{[s+1]}$, it follows that $\rho_s(\nu)$ is a face of $\sigma_{[s+1]}$. Also, $\nu \in$ sd $\dot{\sigma}_{[s]}$ implies that $\dim \nu \le p-1$. Therefore $\rho_s(\nu)$ is a proper face of $\sigma_{[s+1]}$. In other words, $\rho_s(\nu)$ is a simplex of $\dot{\sigma}_{[s+1]}$. Hence,

$$\rho_s(\mathrm{sd}\ \dot{\sigma}_{[s]}) \subset \dot{\sigma}_{[s+1]}$$

(iv) Let $\nu = [b(\tau_0) < \cdots < b(\tau_q)]$ be a simplex of $\mathrm{sd}\,\{\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K) * \dot{\sigma}_{[s]}\}$. Then $\sigma_{[s]}$ is not a face of τ_q. Therefore, if $\nu \prec \sigma_{[s+1]}$, then it is a proper face of $\sigma_{[s+1]}$. Let $\tau_t \prec \cdots \prec \tau_q, 0 \le t \le q$ be such that, for all $t \le k \le q$,

$$\mu_{j_k} \prec \tau_k, \text{ and } \mu_{j_k+1} \not\prec \tau_k.$$

Then,

$$\rho_s(\nu) = [b(\tau_0 \cup \sigma_{[s]}), \ldots, b(\tau_{t-1} \cup \sigma_{[s]}), b(\mu_{j_t}), \ldots, b(\mu_{j_q})].$$

We show that $\sigma_{[s+1]}$ is not a face of $\rho_s(\nu)$.

Let for some $k, 0 \le k \le p-1$, there exist a μ_k such that for any $\tau_i, \mu_k \prec \tau_i$ implies that $\mu_{k+1} \prec \tau_i$. Then $b(\mu_k)$ does not belong to $\rho_s(\nu)$.

On the other hand, if there does not exist such a μ_k, then $q-t = p-1$ and $[v_0, \ldots, v_{p-1}] \prec \tau_q$. If v_p is a vertex of τ_j for some $t \le j \le q$ then $\sigma_{[s]} \prec \tau_q$, a contradiction. Therefore, for all $t \le j \le q, v_p \notin \tau_j$. Also $v_0 \in \tau_t$.

Let if possible $v_k \in \tau_t$, for some $1 \le k \le p-1$. Then for all $j \ge t$,

$$[v_0, \ldots, v_{k-1}] \prec \tau_j \text{ implies that, } [v_0, \ldots, v_k] \prec \tau_j.$$

Hence $b(\mu_{k-1}) \notin \rho_s(\nu)$, a contradiction. Therefore for all $1 \le k \le p, v_k \notin \tau_t$.

Let $t > 0$. Since for all $0 \le j \le t-1, \tau_j \prec \tau_t$ it follows that, for all $0 \le k \le p, v_k \notin \tau_j$. Therefore $\tau_j \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$, for all $0 \le k \le p$ and

hence $b(\sigma_{[s]}) \notin \rho_s(\mu)$.
If $t = 0$ then, $\rho_s(\mu) = \sigma_{[s+1]} - b(\sigma_{[s]})$. So in any case $\sigma_{[s+1]}$ is not a face of $\rho_s(\mu)$ and hence, $\rho_s(\mu)$ is a simplex of $\mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \dot{\sigma}_{[s+1]}$.
Therefore, $\rho_s(\mathrm{sd}\,(\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]})) \subset \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \dot{\sigma}_{[s+1]}$. ∎

Let $j : \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) \longrightarrow \mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)\}$ be the inclusion map.

Lemma 4.3 *$j\rho_s$ is homotopic to the identity map.*

Proof: It follows from above that for any vertex $b(\tau)$ of $\mathrm{sd}\,(\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K))$, $j\rho_s(b(\tau)) \in |\overline{\tau} * \overline{\sigma}_{[s]}|$. Thus for any $x \in |\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)|$, there is a simplex μ of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)$ such that, $x, j\rho_s(x) \in |\overline{\mu}|$. Hence we can define the straight line homotopy,

$$H : |\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)| \times I \longrightarrow |\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)|$$

such that, $H(x,0) = x, H(x,1) = j\rho_s(x)$, for all $x \in |\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)|$.

Hence, ρ_s is a deformation retraction.

Lemma 4.4 *$\mathrm{sd}\,\rho_s$ is a deformation retraction.*

Proof: By Lemma 3.22,

$$\begin{aligned} & \mathrm{sd}\,\rho_s[\mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\}] \\ = \; & \mathrm{sd}\,[\rho_s\{\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\}] \\ = \; & \mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\}. \end{aligned}$$

So $\mathrm{sd}\,\rho_s|\mathrm{sd}\,(\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K))$ is the identity map. Also, $|\mathrm{sd}\,\rho_s|$ is homotopic to ρ_s by Lemma 3.23. Hence $|j| \circ |\mathrm{sd}\,\rho_s|$ is homotopic to $|j\rho_s|$ which is homotopic to the identity map on $|\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)|$. ∎

Let the composition,

$$\rho_{n-1} \circ \mathrm{sd}\,\rho_{n-2} \circ \cdots \circ \mathrm{sd}^{\,n-1}\rho_0 \quad : \quad \mathrm{sd}^{\,n}\{\overline{\mathrm{st}}\,(\sigma_{[0]}, K)\} \longrightarrow \overline{\mathrm{st}}\,(\sigma, \mathrm{sd}^{\,n}K).$$

be denoted by $\rho_{0,n}(\sigma_{[n]})$.

For any $m < n$ we define similarly,

$$\rho_{m,n}(\sigma_{[n]}) : \mathrm{sd}^{\,n-m}\{\overline{\mathrm{st}}\,(\sigma_m, \mathrm{sd}^{\,m}K)\} \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K).$$

Since the composition of deformation retractions is a deformation retraction ([22]), $\rho_{m,n}(\sigma_{[n]})$ is a deformation retraction.
This is one of the several deformation retractions that can be defined on the star neighbourhoods. When there is no point of confusion we shall simply write $\rho_{m,n}$ for $\rho_{m,n}(\sigma_{[n]})$.

4.3 Some special properties of ρ

We let the notations be as in the previous section.

Lemma 4.5 *Let $\mu \in \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]}$. There is a simplex ν' of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}$ such that, $\mu \in \rho_s(\mathrm{sd}\,\overline{\nu}')$ and $\langle\mu\rangle \subset \langle\sigma_{[s]} \cup \nu'\rangle$.*

Proof: Let μ be a simplex of $\mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]}$. Then there exist simplices $\tau_0 \prec \cdots \prec \tau_q$ of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$ such that,

$$\mu = [b(\tau_0), \ldots, b(\tau_r), b(\tau_{r+1}), \ldots, b(\tau_q)].$$

Assume that $b(\tau_0), \ldots, b(\tau_r)$ are vertices of $\sigma_{[s+1]}$ and $\sigma_{[s]}$ is a proper face of $\tau_{r+1} \prec \cdots \prec \tau_q$. Then $\tau_r \prec \sigma_{[s]}$.
Let τ_r be a proper face of $\sigma_{[s]}$. Let for all $r+1 \le k \le q, \nu_k = \tau_k - \sigma_{[s]}$. Since ν_k is a simplex of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$, for all $r+1 \le k \le q, \sigma_{[s]} \cup \nu_k$ is a simplex of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$ and since for all $0 \le j \le r$, $\tau_j \cup \nu_q \prec \sigma_{[s]} \cup \nu_q$, it follows that, for all $0 \le j \le r$, $\tau_j \cup \nu_q$ is a simplex of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$. Also,

$$\nu_{r+1} \prec \cdots \prec \nu_q \prec \tau_0 \cup \nu_q \prec \cdots \prec \tau_r \cup \nu_q$$

where $\tau_r \cup \nu \in \mathrm{Lk}\,(\sigma_{|s|}, \mathrm{sd}\,^s K) * \dot{\sigma}_{|s|}$. Therefore,

$$\nu = [b(\nu_{r+1}), \ldots, b(\nu_q), \ldots, b(\tau_0 \cup \nu_q), \ldots, b(\tau_r \cup \nu_q)]$$

is a simplex of $\mathrm{sd}\,\{\mathrm{Lk}\,(\sigma_{|s|}, \mathrm{sd}\,^s K) * \dot{\sigma}_{|s|}\}$ and $\rho_s(\nu) = \mu$.

Let $\nu' = \tau_r \cup \nu_q$. Then $\nu \in \mathrm{sd}\,(\overline{\nu}')$. Now $\langle \mu \rangle \subset \langle \tau_q \rangle$, and, $\sigma_{|s|} \cup \nu' = \tau_q$. Therefore, $\langle \mu \rangle \subset \langle \sigma_{|s|} \cup \nu' \rangle$

Next, let $\tau_r = \sigma_{|s|}$. Since $\sigma_{|s+1|} \not\prec \mu$, for some $0 \leq k \leq p$ and for some $0 \leq l \leq r$, $\mu_k \neq \tau_l$. Let $\tau_{j_i} \prec \mu_k \prec \tau_{j_i+1}$. Define,

$$\begin{aligned}
\nu_j &= \tau_j - \sigma_{|s|} && \text{for all } r+1 \leq j \leq q \\
\nu_j &= [v_{k+1}] \cup \nu_q \cup \tau_j && 0 \leq j \leq j_i \\
\nu_j &= \nu_q \cup \tau_j && j_i + 1 \leq j \leq r-1 \\
\nu_r &= \nu_q \cup [v_{k+1}]
\end{aligned}$$

Then,

$$\nu_{r+1} \prec \cdots \prec \nu_q \prec \nu_r \prec \nu_0 \prec \cdots \prec \nu_{j_i} \prec \nu_{j_i+1} \prec \cdots \prec \nu_{r-1}.$$

Also, $\sigma_{|s|}$ is not a face of ν_{r-1}. Therefore,

$$\nu = [b(\nu_{r+1}), \ldots, b(\nu_q), b(\nu_r), b(\nu_0), \ldots, b(\nu_{j_i}), b(\nu_{j_i+1}), \ldots, b(\nu_{r-1})]$$

is a simplex of $\mathrm{sd}\,\{\mathrm{Lk}\,(\sigma_{|s|}, \mathrm{sd}\,^s K) * \dot{\sigma}_{|s|}\}$ and $\rho_s(\nu) = \mu$.

Let $\nu' = \nu_{r-1}$. Then $\nu \in \mathrm{sd}\,(\overline{\nu}')$. Now $\langle \mu \rangle \subset \langle \tau_q \rangle$, and $\sigma_{|s|} \cup \nu' = \tau_q$. Therefore $\langle \mu \rangle \subset \langle \sigma_{|s|} \cup \nu' \rangle$. ∎

Lemma 4.6 *The dimension of a simplex $\tau \in \mathrm{Lk}\,(\sigma_{|s|}, K) * \dot{\sigma}_{|s|}$ is greater than the dimension of $\rho_s(\mathrm{sd}\,\overline{\tau})$ if and only if either of the following two conditions hold :*

(i) there is a l, for some $0 \leq l \leq p-1$ such that $\mu_l \prec \tau$ and $\mu_{l+1} \not\prec \tau$ and $\dim\{(\tau \cap \dot{\sigma}_{|s|}) - \mu_l\} \geq 1$.

(ii) for all $0 \leq l \leq p-1, \mu_l \not\prec \tau$, $\tau \notin \mathrm{Lk}\,(\sigma_{|s|}, \mathrm{sd}\,^s K)$ and $\dim \tau \cap \dot{\sigma}_{|s|} \geq 1$.

Proof: Let there be a $\mu_l \prec \tau$ such that, $\mu_{l+1} \not\prec \tau$ and assume that $\dim\{(\tau \cap \dot{\sigma}_{|s|}) - \mu_l\} < 1$.

Let $\dim\{(\tau \cap \dot{\sigma}_{[s]}) - \mu_l\} = 0$ and $\tau = \mu_l \cup [v_{l+i}] \cup [u_0, \dots u_r]$ where, $\tau' = [u_0, \dots u_r] \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$ and $i > 1$. Then $\dim \tau = l + r + 2$. Let $\nu = [b(v_{l+i}), b([u_0, v_{l+i}]), \dots, b(\tau' \cup v_{l+i}), b(\tau' \cup v_{l+i} \cup v_0), \dots, b(\tau' \cup v_{l+i} \cup \mu_l)]$.

Then $\nu \in \mathrm{sd}\,\overline{\tau}$ and,

$$\rho_s(\nu) = [b(\sigma_{[s]}), b(\sigma_{[s]} \cup u_0), \dots, b(\sigma_{[s]} \cup \tau'), b(v_0), \dots, b(\mu_l)].$$

Clearly $\dim \rho_s(\nu) = l + r + 2 = \dim \tau$.

Let $\dim\{(\tau \cap \dot{\sigma}_{[s]}) - \mu_l\} = -1$ i.e. $(\tau \cap \dot{\sigma}_{[s]}) - \mu_l = \emptyset$ and $\tau = \mu_l \cup [u_0, \dots u_r]$ where, $\tau' = [u_0, \dots u_r] \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$. Then $\dim \tau = l + r + 1$. Let

$$\nu = [b([u_0]), \dots, b(\tau'), b(\tau' \cup v_0), \dots, b(\tau' \cup \mu_l)].$$

Then, $\nu \in \mathrm{sd}\,\overline{\tau}$ and,

$$\rho_s(\nu) = [b(\sigma_{[s]} \cup u_0), \dots, b(\sigma_{[s]} \cup \tau'), b(v_0), \dots, b(\mu_l)].$$

Clearly $\dim \rho_s(\nu) = l + r + 1 = \dim \tau$.

Next, let $\mu_l \not\prec \tau$ for all $0 \le l \le p - 1$ and $\tau \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$.

Since, $\rho_s | \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$ is a simplicial homeomorphism, it follows that $\dim \rho_s(\mathrm{sd}\,\overline{\tau}) = \dim \tau$.

On the other hand, let $\mu_l \not\prec \tau$ for all $0 \le l \le p - 1$ but $\dim \tau \cap \dot{\sigma} = 0$. Let $\tau = [v_i, u_0, \dots, u_r]$, where, $i > 0$ and, $\tau' = [u_0, \dots u_r] \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$. Then $\dim \tau = r + 1$. Let $\nu = [v_i, b([v_i, u_0]), \dots, b(v_i \cup \tau')]$. Then ν is a simplex of $\mathrm{sd}\,\overline{\tau}$ and,

$$\rho_s(\nu) = [b(\sigma_{[s]}), b(\sigma_{[s]} \cup u_0), \dots, b(\sigma_{[s]} \cup \tau')].$$

Clearly, $\dim \rho_s(\nu) = \dim \tau$.

Conversely let there is a $\mu_l \prec \tau$ such that $\mu_{l+1} \not\prec \tau, 0 \le l \le p - 1$ and $\dim\{(\tau \cap \dot{\sigma}_{[s]}) - \mu_l\} \ge 1$. Let ν be a simplex of $\mathrm{sd}\,\overline{\tau}$ whose dimension is the same as the dimension of τ. Then there are vertices $b(\tau'), b(\tau'')$ of ν such that τ' is a face of τ'' and, $\tau'' - \tau' \in (\tau \cap \dot{\sigma}_{[s]}) - \mu_l$. Also $\tau' \cap \dot{\sigma}_{[s]} \neq \mu_l$. Then $\rho_s(b(\tau')) = \rho_s(b(\tau''))$. Therefore $\dim \rho_s(\mathrm{sd}\,\overline{\tau}) < \dim \tau$.

If $\mu_l \not\prec \tau$ for all $0 \le l \le p - 1, \tau \notin \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^s K)$ and, $\dim\{\tau \cap \dot{\sigma}\} \ge 1$, then again let, $\nu \in \mathrm{sd}\,\overline{\tau}$ be such that $\dim \nu = \dim \tau$. There are vertices $b(\tau')$, $b(\tau'')$ of ν such that $\tau' \prec \tau''$ and, $\tau'' - \tau' \in \tau \cap \dot{\sigma}_{[s]}$. Then $\rho_s(b(\tau')) = \rho_s(b(\tau''))$. Therefore, $\dim \rho_s(\mathrm{sd}\,\overline{\tau}) < \dim \tau$. ∎

For any point x of $\mathrm{st}\,(\sigma_{[0]}, K)$, $\rho_{0,n}(x)$ is an interior point of the carrier of x in K and,
$\rho_{0,n} : |\overline{\mathrm{st}}\,(\sigma_{[0]}, K)| - \mathrm{st}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) \longrightarrow |\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) - \mathrm{st}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K)|$.
Precisely speaking we have the following technical lemma,

Lemma 4.7 *For all* $\tau \in \overline{\mathrm{st}}\,(\sigma_{[0]}, K)$,

$$\langle \tau \rangle \subset \mathrm{st}\,(\sigma_{[0]}, K) \Rightarrow \rho_{0,n}\{\mathrm{sd}^{\,n}(\overline{\tau})\} \subset \mathrm{sd}^{\,n}(\overline{\tau}) \cap \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K)$$

$$\tau \in \mathrm{Lk}\,(\sigma_{[0]}, K) * \dot{\sigma}_{[0]} \Rightarrow \rho_{0,n}\{\mathrm{sd}^{\,n}(\overline{\tau})\} \subset \mathrm{sd}^{\,n}(\overline{\sigma}_{[0]} * \overline{\tau}) \cap \{\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) * \dot{\sigma}_{[n]}\}$$

Proof: Let for all $1 \leq j \leq n, \sigma_{[j]}$ be a primitive simplex of $\mathrm{sd}^{\,j}K$ with respect to $\sigma_{[j-1]}$. Then for all $0 \leq j \leq n, \sigma_{[j]}$ is a primitive simplex of $\mathrm{sd}^{\,j}K$ with respect to $\sigma_{[0]}$.
Let $s \geq 0$. Let $\sigma_{[s]} = [v_0, \ldots, v_p]$, and, $\sigma_{[s+1]} = [b(\mu_0), \ldots, b(\mu_p)]$, where $\mu_j = [v_0, \ldots, v_j]$. We show that for all $\langle \tau \rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$,

$$\rho_s(\mathrm{sd}\,\overline{\tau}) = \mathrm{sd}\,\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) \tag{1}$$

Let $\tau \in \mathrm{sd}^{\,s}K$ be such that $\langle \tau \rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$. Then $\sigma_{[s]} \prec \tau$. Let $\nu \in \mathrm{sd}\,\overline{\tau}$ and let $\nu = [b(\nu_0), \ldots, b(\nu_q)]$.
Then for all $0 \leq j \leq q, \nu_j \prec \tau$. Assume that $\nu_t \prec \cdots \prec \nu_r$ are such that for all $t \leq k \leq r$,

$$\mu_{j_k} \prec \nu_k, \quad \mu_{j_k+1} \not\prec \nu_k.$$

Then,
$\rho_s(\nu) = [b(\mu_{j_t}), \ldots, b(\mu_{j_r}), b(\nu_0 \cup \sigma_{[s]}), \ldots, b(\nu_{t-1} \cup \sigma_{[s]}), b(\nu_{r+1}), \ldots, b(\nu_q)]$
The following holds :

$$\nu_j \prec \tau \quad \text{for all } r+1 \leq j \leq q,$$
$$\nu_j \cup \sigma_{[s]} \prec \tau \quad \text{for all } 0 \leq j \leq t-1,$$
$$\mu_{j_k} \prec \nu_k \prec \tau \quad \text{for all } t \leq k \leq r.$$

Therefore each vertex of $\rho_s(\mu)$ belongs to $\mathrm{sd}\,\overline{\tau}$. Also $\mathrm{sd}\,\overline{\tau}$ is a full subcomplex of $\mathrm{sd}^{s+1}K$ ([S]). So $\rho_s(\mu) \in \mathrm{sd}\,\overline{\tau}$.

Also $\rho_s(\mu) \in \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)$. Hence for all $\langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{s}K)$,

$$\rho_s(\mathrm{sd}\,\overline{\tau}) \subset \mathrm{sd}\,\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)$$

If $b(\nu) \in \mathrm{sd}\,\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)$, then $\rho_s(b(\nu)) = b(\nu)$ and hence, $b(\nu)$ is a vertex of $\rho_s(\mathrm{sd}\,\overline{\tau})$. Thus Equation 1 holds.

We next show that for all $\tau \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$,

$$\rho_s(\mathrm{sd}\,\overline{\tau}) \subset \mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \dot{\sigma}_{[s+1]} \tag{2}$$

Let τ be a simplex of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$ and $\nu \in \mathrm{sd}\,\overline{\tau}$. Then $\rho_s(\nu)$ is a simplex of $\mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \dot{\sigma}_{[s+1]}$.

Let $\nu = [b(\nu_0), \ldots, b(\nu_q)]$. Then for all $0 \leq j \leq q, \nu_j \prec \tau \prec \tau \cup \sigma_{[s]}$ and hence, for all $0 \leq k \leq p, 0 \leq j \leq q, \nu_j \cup \mu_k \prec \tau \cup \sigma_{[s]}$. Thus every vertex of $\rho_s(\nu)$ belongs to $\mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau})$ and $\mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau})$ is a full subcomplex of $\mathrm{sd}^{s+1}K$. Thus $\rho_s(\nu) \in \mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau})$. Also if $b(\mu_k)$ is a vertex of $\rho_s(\nu)$, then either, $\mu_k \prec \nu_j \prec \tau$, for some $0 \leq j \leq q$ or, $\mu_k = \sigma_{[s]}$. Therefore,

$$\rho_s(\mathrm{sd}\,\overline{\tau}) \subset \mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \{\,\dot{\sigma}_{[s+1]} \cap (\mathrm{sd}\,\overline{\tau} \cup b(\sigma_{[s]}))\}.$$

Conversely let $\tau' \in \mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \{\,\dot{\sigma}_{[s+1]} \cap (\mathrm{sd}\,\overline{\tau} \cup b(\sigma_{[s]}))\}$. By Lemma 4.5, there is a $\tau'' \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$ such that, $\tau' \in \rho_s(\mathrm{sd}\,\overline{\tau}'')$ and $\langle\tau'\rangle \subset \langle\tau'' \cup \sigma_{[s]}\rangle$. Then $\tau'' \cup \sigma_{[s]} \prec \tau \cup \sigma_{[s]}$. From the proof of Lemma 4.5, it is clear that $\tau'' \prec \tau$. Hence for all $\tau \in \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$,

$$\rho_s(\mathrm{sd}\,\overline{\tau}) = \mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \{\,\dot{\sigma}_{[s+1]} \cap (\mathrm{sd}\,\overline{\tau} \cup b(\sigma_{[s]}))\} \tag{3}$$

Therefore Equation 2 holds.

Now if L is a subcomplex of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$,

$$\rho_s(\mathrm{sd}\,L) = \rho_s(\cup_{\tau \in L}(\mathrm{sd}\,\overline{\tau})) = \cup_{\tau \in L}\rho_s(\mathrm{sd}\,\overline{\tau})$$

$$\subset \cup_{\tau \in L}\{\mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]}\}$$

by Equation 2. Therefore,

$$\rho_s(\mathrm{sd}\,L) \subset \{\cup_{\tau \in L}\mathrm{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]}$$

Thus, if L is a subcomplex of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}$,

$$\rho_s(\mathrm{sd}\,L) \subset \mathrm{sd}\,(\overline{\sigma}_{[s]} * L) \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]} \tag{4}$$

If L is a subcomplex of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$ but not of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}$ then,

$$\mathrm{sd}\,L = \mathrm{sd}\,\{L \cap \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]} \cup \{\overline{\tau} : \tau \in L, \langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)\}\}$$

So,

$$\begin{aligned}\rho_s(\mathrm{sd}\,L) &= \rho_s[\mathrm{sd}\,\{L \cap \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}\}] \\ &\quad \cup\{\rho_s(\mathrm{sd}\,\overline{\tau}) : \tau \in L, \langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)\}\end{aligned}$$

Therefore by Equations 1 and 4,

$$\begin{aligned}\rho_s(\mathrm{sd}\,L) \subset\ & \mathrm{sd}\,\{(L \cap \mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}) * \overline{\sigma}_s)\} \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]} \\ & \cup[(\cup\{\mathrm{sd}\,(\overline{\tau}) : \tau \in L, \langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)\}) \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K)],\end{aligned}$$

and hence,

$$\rho_s(\mathrm{sd}\,L) \subset \{\mathrm{sd}\,L \cap \mathrm{Lk}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) * \dot{\sigma}_{[s+1]}\} \cup \{\mathrm{sd}\,L \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K)\}.$$

Hence, if L is a subcomplex of $\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K)$ but not of $\mathrm{Lk}\,(\sigma_{[s]}, \mathrm{sd}^{\,s}K) * \dot{\sigma}_{[s]}$ then,

$$\rho_s(\mathrm{sd}\,L) \subset \mathrm{sd}\,L \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{\,s+1}K) \tag{5}$$

Now let $\langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[0]}, K)$. Then $\mathrm{sd}^{\,n-1}\rho_0(\mathrm{sd}^{\,n}\overline{\tau}) = \mathrm{sd}^{\,n-1}[\rho_0(\mathrm{sd}\,\overline{\tau})]$; by Lemma 3.22. So $\mathrm{sd}^{\,n-1}\rho_0(\mathrm{sd}\,\overline{\tau}) \subset \mathrm{sd}^{\,n-1}\{(\mathrm{sd}\,\overline{\tau}) \cap \overline{\mathrm{st}}\,(\sigma_{[1]}, \mathrm{sd}\,K)\}$ by Equation 1 and hence by Observation 3.4

$$\mathrm{sd}^{\,n-1}\rho_0(\mathrm{sd}^{\,n}\overline{\tau}) \subset \mathrm{sd}^{\,n}(\overline{\tau}) \cap \mathrm{sd}^{\,n-1}\{\overline{\mathrm{st}}\,(\sigma_{[1]}, \mathrm{sd}\,K)\}.$$

Let for any $0 \le r \le s-1$,

$$\mathrm{sd}^{n-r-1}\rho_r \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \\ \mathrm{sd}^n\overline{\tau} \cap \mathrm{sd}^{n-r-1}\{\overline{\mathrm{st}}\,(\sigma_{[r+1]}, \mathrm{sd}^{r+1}K)\} \tag{6}$$

Then,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \\ \subset \mathrm{sd}^{n-s-1}\rho_s\{\mathrm{sd}^n\overline{\tau} \cap \mathrm{sd}^{n-s}\{\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^sK)\}\}$$

by Equation 6. Again, by Observation 3.4,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \\ \subset \mathrm{sd}^{n-s-1}\rho_s\{\mathrm{sd}^{n-s}[\mathrm{sd}^s\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^sK)]\}$$

and by Lemma 3.22,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \\ \subset \mathrm{sd}^{n-s-1}\,[\rho_s\{\mathrm{sd}\,[\mathrm{sd}^s\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^sK)]\}]$$

Hence by Equation 5 it follows that,

$$\begin{aligned} & \mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \\ \subset\ & \mathrm{sd}^{n-s-1}\,[\mathrm{sd}\,\{\mathrm{sd}^s\overline{\tau} \cap \overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^sK)\} \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)] \\ =\ & \mathrm{sd}^n\overline{\tau} \cap \mathrm{sd}^{n-s-1}[\mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[s]}, \mathrm{sd}^sK)\} \cap \overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)] \\ =\ & \mathrm{sd}^n\overline{\tau} \cap \mathrm{sd}^{n-s-1}\{\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\}. \end{aligned}$$

Thus for all $s \ge 0$ and for all τ such that $\langle\tau\rangle \subset \mathrm{st}\,(\sigma_{[0]}, K)$,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \\ \mathrm{sd}^n\overline{\tau} \cap \mathrm{sd}^{n-s-1}\{\overline{\mathrm{st}}\,(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\} \tag{7}$$

Let $\tau \in \mathrm{Lk}\,(\sigma_{[0]}, K) * \dot{\sigma}_{[0]}$. Then $\mathrm{sd}^{n-1}\rho_0(\mathrm{sd}^n\overline{\tau}) = \mathrm{sd}^{n-1}[\rho_0(\mathrm{sd}\,\overline{\tau})]$ by Lemma 3.22. Therefore,

$$\mathrm{sd}^{n-1}\rho_0(\mathrm{sd}^n\overline{\tau}) \subset \mathrm{sd}^{n-1}[\mathrm{sd}\,(\overline{\sigma}_{[0]} * \overline{\tau}) \cap \mathrm{Lk}\,(\sigma_{[1]}, \mathrm{sd}\,K) * \dot{\sigma}_{[1]}] \\ = \mathrm{sd}^n(\overline{\sigma}_{[0]} * \overline{\tau}) \cap \mathrm{sd}^{n-1}(\mathrm{Lk}\,(\sigma_{[1]}, \mathrm{sd}\,K) * \dot{\sigma}_{[1]}),$$

by Equation 2 and Observation 3.4. Let for all $0 \leq r \leq s-1$,

$$\mathrm{sd}^{n-r-1}\rho_r \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \mathrm{sd}^n\{\overline{\sigma}_{[0]} * \overline{\tau}\} \cap \mathrm{sd}^{n-r-1}\{\mathrm{Lk}(\sigma_{[r+1]}, \mathrm{sd}^{r+1}K) * \dot{\sigma}_{[r+1]}\} \quad (8)$$

Then,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \mathrm{sd}^{n-s-1}\rho_s[\mathrm{sd}^n\{\overline{\sigma}_{[0]} * \overline{\tau}\} \cap \mathrm{sd}^{n-s}\{\mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}\}]$$

by Equation 8. Again,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \mathrm{sd}^{n-s-1}\rho_s[\mathrm{sd}^{n-s}\{\mathrm{sd}^s\{\overline{\sigma}_{[0]} * \overline{\tau}\} \cap \mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}\}]$$

Hence by Lemma 3.22,

$$\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \subset \mathrm{sd}^{n-s-1}[\rho_s\{\mathrm{sd}\,\{\mathrm{sd}^s(\overline{\sigma}_{[0]} * \overline{\tau}) \cap \mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}\}\}]$$

Now by Equation 4,

$$\begin{aligned} &\mathrm{sd}^{n-s-1}\rho_s \circ \cdots \circ \mathrm{sd}^{n-1}\rho_0\{\mathrm{sd}^n(\overline{\tau})\} \\ \subset\ &\mathrm{sd}^{n-s-1}[\mathrm{sd}\,\{(\mathrm{sd}^s(\overline{\sigma}_{[0]} * \overline{\tau}) \cap \mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}) * \overline{\sigma}_{[s]}\}] \\ \cap\ &\mathrm{sd}^{n-s-1}[\mathrm{Lk}(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K) * \dot{\sigma}_{[s+1]}] \end{aligned}$$

Let M be a subcomplex of $\overline{\mathrm{st}}(\sigma_{[s]}, \mathrm{sd}^{s}K), s \geq 0$ such that $\sigma_{[s]} \in M$. Let τ be a simplex of $M \cap \mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}$. Then $\overline{\tau} * \overline{\sigma}_{[s]} \subset M * \overline{\sigma}_{[s]} \subset M$, since, $\sigma_{[s]} \in M$. Also $\overline{\tau} * \overline{\sigma}_{[s]} \subset \overline{\mathrm{st}}(\sigma_{[s]}, \mathrm{sd}^{s}K)$. Therefore the following holds,

$$\{M \cap \mathrm{Lk}(\sigma_{[s]}, \mathrm{sd}^{s}K) * \dot{\sigma}_{[s]}\} * \overline{\sigma}_{[s]} \subset M \cap \overline{\mathrm{st}}(\sigma_{[s]}, \mathrm{sd}^{s}K)$$

Also $\sigma_{[s]} \in \mathrm{sd}^s\overline{\sigma}_{[0]} \in \mathrm{sd}^s(\overline{\sigma}_{[0]} * \overline{\tau})$.
Hence for all $s \geq 0$ and for all $\tau \in \mathrm{Lk}(\sigma_{[0]}, K) * \dot{\sigma}_{[0]}$,

$$(\mathrm{sd}^{n-s-1}\rho_s) \circ \cdots \circ (\mathrm{sd}^{n-1}\rho_0)\{\mathrm{sd}^n(\overline{\tau})\} \subset \mathrm{sd}^n\{\overline{\sigma}_{[0]} * \overline{\tau}\} \cap \mathrm{sd}^{n-s-1}\{\mathrm{Lk}(\sigma_{[s+1]}, \mathrm{sd}^{s+1}K)\} * \dot{\sigma}_{[s+1]} \quad (9)$$

Taking $s = n - 1$ in Equations 7 and 9, the result follows. ∎

Remark 4.8 Let $\dot{\sigma}_{[0]} = \emptyset$. Then $\sigma_{[0]} = v$ is a vertex of K and,

$$\rho_{0,n} : \mathtt{sd}^n\{\overline{\mathtt{st}}\,(v, K)\} \longrightarrow \overline{\mathtt{st}}\,(v, \mathtt{sd}^n K).$$

It follows that,

$$\langle\tau\rangle \subset \mathtt{st}\,(v, K) \Rightarrow \rho_{0,n}\{\mathtt{sd}^n(\overline{\tau})\} = \mathtt{sd}^n(\overline{\tau}) \cap \overline{\mathtt{st}}\,(v, \mathtt{sd}^n K)$$

$$\tau \in \mathrm{Lk}\,(v, K) \Rightarrow \rho_{0,n}\{\mathtt{sd}^n(\overline{\tau})\} = \mathtt{sd}^n(v * \overline{\tau}) \cap \mathrm{Lk}\,(v, \mathtt{sd}^n K)$$

Proof: In this case, for all $0 \leq j \leq n$, $\sigma_{[j]} = v$. For all $s \geq 0$,

$$\rho_s : \mathtt{sd}\,\{\overline{\mathtt{st}}\,(v, \mathtt{sd}^s K)\} \longrightarrow \overline{\mathtt{st}}\,(v, \mathtt{sd}^{s+1} K)$$

is defined as follows :

$$\rho_s(b(\tau)) = b(v \cup \tau), \text{ for all } \tau \in \overline{\mathtt{st}}\,(v, \mathtt{sd}^s K)$$

Then for all $s \geq 0$ Equation 1 reduces to,

$$\rho_s(\mathtt{sd}\,\{\overline{\tau}\}) = \mathtt{sd}\,\overline{\tau} \cap \overline{\mathtt{st}}\,(v, \mathtt{sd}^{s+1} K) \tag{10}$$

Since $\dot{v} = \emptyset$, it follows from Equation 3 that,

$$\rho_s(\mathtt{sd}\,\overline{\tau}) = \mathtt{sd}\,(\overline{\sigma}_{[s]} * \overline{\tau}) \cap \mathrm{Lk}\,(v, \mathtt{sd}^{s+1} K) \tag{11}$$

Now proceeding on similar lines as in the proof of Lemma 4.7, we get the desired result. ∎

Chapter 5

LFPF For Simplicial Maps

5.1 Introduction

Let $f : \mathrm{sd}^n K \longrightarrow K, n \geq 0$ be a simplicial map. A path component of $\mathrm{Fix}\, f$ will be called a *fixed point component* of f. We have noted in Chapter 3 that the number of fixed point components of f is finite and the fixed point components form a class of isolated sets of fixed points of f.
In this chapter, we discuss the problem of defining the fixed point index of a simplicial map $f : \mathrm{sd}^n K \longrightarrow K, n \geq 0$ at a fixed point component of f as a local Lefschetz number and establish the Lefschetz fixed point formula (LFPF). We show that the fixed point index of $f : K \longrightarrow K$ at a fixed point component is the Euler characteristic of the fixed point component.
For $n \geq 1$ we define a class of simplicial maps $f : \mathrm{sd}^n K \longrightarrow K$ — those which *preserve expanding directions* — and define the fixed point index of such maps at a fixed point as a local Lefschetz number and establish the LFPF for this case.
We extend the definition of weakly hyperbolic maps to polyhedra and show that a weakly hyperbolic simplicial map preserves expanding directions.

5.2 The fixed point index of $f : K \longrightarrow K$

Let $f : K \longrightarrow K$ be a simplicial map. Let $C_1, \ldots, C_r$ be the distinct fixed point components of f. We shall denote the subcomplex of K associated with C_j by $[K]_j$ for brevity.
Let

$$\mathrm{st}\,([K]_j, K) = \cup\{\mathrm{st}\,(\sigma, K) : \langle\sigma\rangle \cap C_j \neq \emptyset\}.$$
$$\overline{\mathrm{st}}\,([K]_j, K) = \cup\{\overline{\mathrm{st}}\,(\sigma, K) : \langle\sigma\rangle \cap C_j \neq \emptyset\}.$$

Clearly, $\mathtt{st}([K]_j, K)$ is an open set of $|K|$ and, $\overline{\mathtt{st}}([K]_j, K)$ is a subcomplex of $\overline{\mathtt{N}}([K]_j, K)$. We show that if $i \neq j$, then,

$$\mathtt{st}([K]_i, K) \cap \mathtt{st}([K]_j, K) = \emptyset.$$

Let if possible for some $i \neq j$, σ be a simplex of K such that, $\langle\sigma\rangle \cap C_i \neq \emptyset$ and $\langle\sigma\rangle \cap C_j \neq \emptyset$. Let $x \in \langle\sigma\rangle \cap C_i$ and $y \in \langle\sigma\rangle \cap C_j$. Then for all $t \in [0,1], tx + (1-t)y \in \langle\sigma\rangle \cap \operatorname{Fix} f$ and, $tx + (1-t)y \in C_i \cap C_j$, a contradiction to the fact that these are distinct components of $\operatorname{Fix} f$.

Now let τ be a simplex of K such that, there are faces μ, ν of τ such that $\langle\mu\rangle \cap C_i \neq \emptyset$, and $\langle\nu\rangle \cap C_j \neq \emptyset$, where, $i \neq j$. Again, if $x \in \langle\mu\rangle \cap C_i$ and $y \in \langle\nu\rangle \cap C_j$, then for all $t \in [0,1], tx + (1-t)y \in \overline{\tau} \cap \operatorname{Fix} f$ and $tx + (1-t)y \in C_i \cap C_j$, a contradiction to the fact that these are distinct components of $\operatorname{Fix} f$. Thus,

$$\mathtt{st}([K]_i, K) \cap \mathtt{st}([K]_j, K) = \emptyset.$$

Take any $\sigma \in K$ such that, $\langle\sigma\rangle \cap C_j \neq \emptyset$. Then clearly $\langle f(\sigma)\rangle \cap C_j \neq \emptyset$ and hence, $f : [K]_j \longrightarrow [K]_j$. So, by Lemma 3.7,

$$f(\overline{\mathtt{N}}([K]_j, K)) \subset \overline{\mathtt{N}}([K]_j, K).$$

It in fact follows from the proof of Lemma 3.7 that,

$$f : \overline{\mathtt{st}}([K]_j, K) \longrightarrow \overline{\mathtt{st}}([K]_j, K).$$

For all $1 \leq j \leq r$, we define,

$$I(f, C_j) = L(f|\overline{\mathtt{st}}([K]_j, K))$$

Lemma 5.1 *For all* $1 \leq j \leq r, I(f, C_j) = I(\mathtt{sd}\, f, C_j)$.

Proof: Let $[\mathtt{sd}\, K]_j$ be the smallest subcomplex of $\mathtt{sd}\, K$ containing C_j. Let σ be a simplex of $\mathtt{sd}\, K$ such that, $\langle\sigma\rangle \cap C_j \neq \emptyset$. Let τ be a simplex of K such that $\langle\sigma\rangle \subset \langle\tau\rangle$. Then $\langle\tau\rangle \cap C_j \neq \emptyset$.

Hence, σ is a simplex of $\mathtt{sd}([K]_j)$ and hence, $[\mathtt{sd}\, K]_j \subset \mathtt{sd}([K]_j)$.

We can show as above that,

$$\mathsf{sd}\, f : \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K) \longrightarrow \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K).$$

Let $i : \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K) \longrightarrow \mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\}$ be the inclusion map. Then the following diagram is commutative,

$$\begin{array}{ccc}
\mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\} & \xleftarrow{\;i\;} & \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K) \\
\Big\uparrow{\scriptstyle \mathsf{sd}\, f} & & \Big\uparrow{\scriptstyle \mathsf{sd}\, f} \\
\mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\} & \xleftarrow[\;i\;]{} & \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K)
\end{array}$$

Diagram 5.1

Let σ be a simplex of $\mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\}$ such that $\sigma \notin \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K)$. Then, $\overline{\sigma} \cap \operatorname{Fix} \mathsf{sd}\, f = \emptyset$, by Lemma 3.17. By Brouwer's fixed point theorem, $\sigma \neq \mathsf{sd}\, f(\sigma)$. Hence for all $p \geq 0$,

$$\begin{aligned}
&\operatorname{Trace}[\mathsf{sd}\, f : C_p(\mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\}) \longrightarrow C_p(\mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\})] \\
&= \operatorname{Trace}[\mathsf{sd}\, f : C_p(\overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K)) \longrightarrow C_p(\overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K))]
\end{aligned}$$

Therefore by Hopf Trace Theorem ([22]),

$$L(\mathsf{sd}\, f | \mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\}) = L(\mathsf{sd}\, f | \overline{\mathtt{st}}\,([\mathsf{sd}\, K]_j, \mathsf{sd}\, K).$$

Since f and $\mathsf{sd}\, f$ are homotopic maps,

$$L(f | \overline{\mathtt{st}}\,([K]_j, K)) = L(\mathsf{sd}\, f | \mathsf{sd}\,\{\overline{\mathtt{st}}\,([K]_j, K)\}).$$

Therefore $I(f, C_j) = I(\mathsf{sd}\, f, C_j)$. ∎

Theorem 5.2 $L(f) = \sum_{j=1}^{r} I(f, C_j)$.

Proof: Let $\mathsf{sd}^m K = P$ be such that, $\overline{\mathtt{st}}\,([P]_i, P) \cap \overline{\mathtt{st}}\,([P]_j, P) = \emptyset, \text{if}\, i \neq j$ where for all $1 \leq i \leq r$, $[P]_i$ is the smallest subcomplex of P containing C_i. Let $\mathsf{sd}^m f = g$. By Lemma 3.17, $\operatorname{Fix} f = \operatorname{Fix} g$. Also by Lemma 5.1, for any $1 \leq j \leq r$, $I(f, C_j) = I(g, C_j)$. Since f and g are homotopic maps, $L(f) = L(g)$ and hence to prove the result it is enough to establish that,

$$L(g) = \sum_{j=1}^{r} I(g, C_j).$$

Let $[P]_{\operatorname{Fix} g} = [P]_g$. Then clearly $[P]_g = \cup_{j=1}^{r} [P]_j$ and,

$$g : \overline{\mathtt{st}}\,([P]_g, P) \longrightarrow \overline{\mathtt{st}}\,([P]_g, P).$$

Also by the relative Mayer-Vietoris sequence,

$$H_p(\overline{\mathtt{st}}\,([P]_g, P)) \cong \oplus_{j=1}^{r} H_p(\overline{\mathtt{st}}\,([P]_j, P)).$$

Therefore, $L(g, \overline{\mathtt{st}}\,([P]_g, P), \emptyset) = \sum_{j=1}^{r} I(g, C_j)$. The theorem follows if we can show that $L(g) = L(g, \overline{\mathtt{st}}\,([P]_g, P), \emptyset)$. Let $i : \overline{\mathtt{st}}\,([P]_g, P) \longrightarrow P$ be the inclusion map. The following diagram is commutative,

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & P \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ \overline{\mathtt{st}}\,([P]_g, P) & \xrightarrow[g]{} & \overline{\mathtt{st}}\,([P]_g, P) \end{array}$$

Diagram 5.2

The result follows as in Lemma 5.1. If $\sigma \in P$ is not a simplex of $\overline{\mathtt{st}}\,([P]_g, P)$, then $\operatorname{Fix} g \cap \overline{\sigma} = \emptyset$ which implies by Lemma 3.17 that, $\sigma \neq g(\sigma)$. Therefore, for all $p \geq 0$,

$$\begin{aligned} &\operatorname{Trace}\{g_p : C_p(P) \longrightarrow C_p(P)\} \\ = &\operatorname{Trace}\{g_p : C_p(\overline{\mathtt{st}}\,([P]_g, P)) \longrightarrow C_p(\overline{\mathtt{st}}\,([P]_g, P))\}. \end{aligned}$$

Hence by Hopf Trace Theorem, $L(g) = L(g, \overline{\mathtt{st}}\,([P]_g, P), \emptyset)$. ∎

Remark 5.3 For all $1 \leq j \leq r$, $\mathtt{st}\,([K]_j, K)$ is an open neighbourhood of C_j in $|K|$ such that $\mathtt{st}\,([K]_j, K) \cap \operatorname{Fix} f = C_j$ and,

$$f : \mathtt{st}\,([K]_j, K) \longrightarrow \mathtt{st}\,([K]_j, K).$$

Let us denote the fixed point index of f on $\mathtt{st}\,([K]_j, K)$ as defined by the degree formula, ([5], [6]) by $i(f, \mathtt{st}\,([K]_j, K))$. Then by the localization and normalization property of the fixed point index, (see Chapter 2 of this thesis)

$$L(f|\mathtt{st}\,([K]_j, K)) = i(f, \mathtt{st}\,([K]_j, K)).$$

Also we have seen that, $L(f|\overline{\mathtt{st}}\,([K]_j, K)) = L(f|\mathtt{st}\,([K]_j, K))$. Therefore,

$$\begin{aligned} I(f, C_j) &= L(f|\overline{\mathtt{st}}\,([K]_j, K)) &= L(f|\mathtt{st}\,([K]_j, K)) \\ &= i(f, \mathtt{st}\,([K]_j, K)) &= i(f, C_j) \end{aligned}$$

Remark 5.4 We noted in Lemma 3.19 that $\operatorname{Fix} f$ is the geometric realisation of a subcomplex of $\mathtt{sd}\,K$. From the proof of Lemma 3.19 it is clear that for all $1 \leq j \leq r$, the geometric realisation of $[P]_j$ is C_j. Then,

$$I(f, C_j) = I(g, [P]_j) = L(g|\overline{\mathtt{st}}\,([P]_j, P)).$$

Now it is clear from Remark 3.9 that any simplex of $\overline{\mathtt{st}}\,([P]_j, P)$ contributes to the trace of g only if it is a simplex of $[P]_j$. Also $g|[P]_j$ is the identity map. Hence, $I(f, C_j)$ is the Euler characteristic of $[P]_j$ i.e. of C_j.

5.3 Simplicial Maps which Preserve Expanding Directions

Let $f : \mathtt{sd}^n K \longrightarrow K, n \geq 1$ be a simplicial map. Then $\operatorname{Fix} f$ is a finite set of points of $|K|$ (Remark 3.14). Let $x \in \operatorname{Fix} f$.

Proposition 5.5 *The carrier of x in $\mathtt{sd}^p K, p \geq 0$ is a primitive simplex with respect to K.*

Proof: Let us *a priori* denote the carrier of x in $\mathtt{sd}^p K, p \geq 0$ by $\sigma_{[p]}$. Since x is a fixed point of f, it follows by Lemma 3.11 that the carrier of x in $\mathtt{sd}^p K, 0 \leq p \leq n$ is a primitive simplex with respect to K. Assume that $\sigma_{[p]}$ is a primitive simplex with respect to $\sigma_{[0]}$ for all $0 \leq p \leq mn, m \geq 1$. For all $0 \leq q \leq n$, consider the map,

$$\mathtt{sd}^{(m-1)n+q} f : \mathtt{sd}^{mn+q} K \longrightarrow \mathtt{sd}^{(m-1)n+q} K$$

By Lemma 3.21, x is a fixed point of $\mathrm{sd}^{(m-1)n+q} f$. Thus $\sigma_{[mn+q]}$ is a primitive simplex with respect to $\sigma_{[(m-1)n+q]}$ by Lemma 3.11. By the induction hypothesis $\sigma_{[(m-1)n+q]}$ is a primitive simplex with respect to $\sigma_{[0]}$. Hence $\sigma_{[mn+q]}$ is a primitive simplex with respect to $\sigma_{[0]}$. Thus $\sigma_{[p]}$ is a primitive simplex with respect to K for all $0 \leq p \leq (m+1)n$. The claim now holds by mathematical induction. ■

Since x is a fixed point of f, $f(\sigma_{[n]}) = \sigma_{[0]}$. It easily follows that,

$$f : \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K) \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[0]}, K).$$

Also there is a deformation retraction (see Chapter 4 of this thesis),

$$\rho_{0,n} = \rho : \mathrm{sd}^n(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)) \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K).$$

Let $\lambda_p : C_p(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)) \longrightarrow C_p(\mathrm{sd}^n\{\overline{\mathrm{st}}\,(\sigma_{[0]}, K)\})$ be the subdivision operator. The following is a chain map,

$$C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)) \xrightarrow{f_p} C_p(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)) \xrightarrow{\lambda_p} C_p(\mathrm{sd}^n(\overline{\mathrm{st}}\,(\sigma_{[0]}, K)))$$
$$\xrightarrow{\rho_p} C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K))$$

Let,

$$\tilde{f}_p = \rho_p \circ \lambda_p \circ f_p : C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)) \longrightarrow C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)).$$

It is clear that Lefschetz number of $\tilde{f}_*$ is the Lefschetz number of the map $|\rho| \circ |f|$. The retraction ρ might produce new fixed points of the map $|\rho| \circ |f|$ on $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$. Therefore as mentioned earlier, the Lefschetz number of $\tilde{f}_*$ does not give the index of the map f at x. If there is a subcomplex M of $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ which contains all the fixed points of the map $|\rho| \circ |f|$ on $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ and which maps into itself by the map $|\rho||f|$, then, the map $\tilde{f}_*$ is a chain map on the relative homology groups of $(\overline{\mathtt{st}}\,(\sigma_{|n|}, \mathtt{sd}^n K), M)$. The relative Lefschetz number of $\tilde{f}$, for the pair $(\overline{\mathtt{st}}\,(\sigma_{|n|}, \mathtt{sd}^n K), M)$ then gives the index of f at x by a principle analogous to the additivity property of fixed point indices ([5], [6]).

We now formalize these ideas.

Let $f : \mathtt{sd}^n K \longrightarrow K, n \geq 1$ be a simplicial map and $x \in \mathrm{Fix}\, f$. Let the carrier of x in $\mathtt{sd}^p K, p \geq 0$ be $\sigma_{|p|}$.

Definition 5.6 The map f *preserves expanding directions at* x if there is a subcomplex $M(x) = M$ of $\mathrm{Lk}\,(\sigma_{|n|}, \mathtt{sd}^n K) * \dot{\sigma}_{|n|}$ satisfying,

(a) $\tau \in \mathrm{Lk}\,(\sigma_{|n|}, \mathtt{sd}^n K) * \dot{\sigma}_{|n|}$, such that, $\langle \sigma_{|n|} \cup \tau \rangle \subset \langle \sigma_{|0|} \cup f(\tau) \rangle$, implies that $\tau \in M$.

(b) $\tau \in M$ and $f(\tau) \prec \sigma_{|0|}$ implies that $\tau \prec \sigma_{|n|}$.

(c) $\tau \in M$ implies that,

$$\mathtt{sd}^n\{\overline{\sigma}_{|0|} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{|n|}, \mathtt{sd}^n K) * \dot{\sigma}_{|n|} \subset M.$$

A subcomplex of $\mathrm{Lk}\,(\sigma_{|n|}, \mathtt{sd}^n K) * \dot{\sigma}_{|n|}$ is a *subcomplex at x expanded by the map f* if it satisfies (a),(b) and (c).
The map f *preserves expanding directions* if it preserves expanding directions at each of its fixed points.

In the following discussion, unless the contrary is stated,

$$f : \mathtt{sd}^n K \longrightarrow K, n \geq 1$$

will be a simplicial map which preserves expanding directions, x will be a fixed point of f and the subcomplex at x expanded by f will be M. The carrier of x in $\mathtt{sd}^p K, p \geq 0$ will be denoted by $\sigma_{|p|}$. The retraction,

$$\rho_{0,n} : \mathtt{sd}^n(\overline{\mathtt{st}}\,(\sigma_{|0|}, K)) \longrightarrow \overline{\mathtt{st}}\,(\sigma_{|n|}, \mathtt{sd}^n K)$$

will be denoted by ρ.

Observation 5.7 $f : M \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) \longrightarrow \mathrm{Lk}\,(\sigma_{[0]}, K)$.

Proof: Let $\tau \in M$. If possible let $\sigma_{[0]} \prec f(\tau)$. Let $\mu \prec \tau$ such that, $f(\mu) = \sigma_{[0]}$. Since $\mu \in M$, by property (b) of M it follows that μ is a face of $\sigma_{[n]}$. Also $\sigma_{[n]}$ is a primitive simplex. Therefore, it follows that $\mu = \sigma_{[n]}$, a contradiction to the fact that μ is a simplex of $M \subset \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) * \dot{\sigma}_{[n]}$. Therefore, $f : M \longrightarrow \mathrm{Lk}\,(\sigma_{[0]}, K) * \dot{\sigma}_{[0]}$. It is also clear from this that, $f : M \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) \longrightarrow \mathrm{Lk}\,(\sigma_{[0]}, K)$. ■

Lemma 5.8 *A simplex* τ *of* $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) * \dot{\sigma}_{[n]}$ *is such that,*

$$\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$$

if and only if $\tau \in \mathrm{sd}^{\,n}\{\overline{\sigma_{[0]}} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) * \dot{\sigma}_{[n]}$.

Proof: Let, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$. Then, $\sigma_{[n]} \cup \tau \in \mathrm{sd}^{\,n}\{\overline{\sigma}_{[0]} * f(\overline{\tau})\}$. Since $\tau \prec \sigma_{[n]} \cup \tau$, the result follows.

Conversely, let τ be a simplex of $\mathrm{sd}^{\,n}\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K) * \dot{\sigma}_{[n]}$. Now $\sigma_{[n]} \in \mathrm{sd}^{\,n}\overline{\sigma}_{[0]} \subset \mathrm{sd}^{\,n}\{\overline{\sigma}_{[0]} * f(\overline{\tau})\}$. Since $\mathrm{sd}^{\,n}\{\overline{\sigma}_{[0]} * f(\overline{\tau})\}$ is a full subcomplex, $\sigma_{[n]} \cup \tau \in \mathrm{sd}^{\,n}\{\overline{\sigma}_{[0]} * f(\overline{\tau})\}$. Then $\dim\{\sigma_{[n]} \cup \tau\} \leq \dim\{\sigma_{[0]} \cup f(\tau)\}$. But,

$$\begin{aligned} \dim\{\sigma_{[n]} \cup \tau\} &= \dim \sigma_{[n]} + \dim \tau + 1 = \dim \sigma_{[0]} + \dim \tau + 1 \\ &\geq \dim \sigma_{[0]} + \dim f(\tau) + 1 = \dim\{\sigma_{[0]} \cup f(\tau)\}. \end{aligned}$$

Therefore, $\dim\{\sigma_{[0]} \cup f(\tau)\} = \dim\{\sigma_{[n]} \cup \tau\}$. Now, let μ be a simplex of K such that, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \mu \rangle$. Then, $\mu \prec \sigma_{[0]} \cup f(\tau)$.

Also, $\dim\{\sigma_{[n]} \cup \tau\} \leq \dim \mu \leq \dim\{\sigma_{[0]} \cup f(\tau)\}$. Hence, $\mu = \sigma_{[0]} \cup f(\tau)$. ■

Observation 5.9 If the support of a chain $c \in C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K))$ is contained in M, then the support $||\tilde{f}_p(c)||$ of the chain $\tilde{f}_p(c)$ is contained in M.

Proof: Let $c = \sum_\tau n_\tau \tau$. Then, $n_\tau \neq 0$ implies that $\tau \in M$. Also,

$$\begin{aligned} ||\tilde{f}_p(c)|| &\subset \cup_{n_\tau \neq 0} |\rho_{0,n}(\mathtt{sd}^n(f(\overline{\tau})))| \\ &\subset \cup_{n_\tau \neq 0} \mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]} \end{aligned}$$

by Lemma 5.7 and Lemma 4.7. It now follows by property (c) of M that, $||\tilde{f}_p(c)||$ is contained in M. ∎

Hence there is a chain homomorphism,

$$\tilde{f}_p : C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M) \longrightarrow C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M).$$

Define,

$$\boxed{I(f,x) = L(\tilde{f}, \overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M).}$$

Lemma 5.10 *$I(f,x)$ is independant of the choice of the subcomplex M at x expanded by f.*

Proof: Let M' be another subcomplex at x which is expanded by f. Let $M'' = M \cap M'$. We show that M'' is also a subcomplex at x expanded by f.

Let $\tau \in \mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]}$ be such that, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$. Then $\tau \in M$ and $\tau \in M'$. So, $\tau \in M''$.

On the other hand, if $\tau \in M''$, then $\tau \in M$ and, $\tau \in M'$. Therefore $f(\tau) \prec \sigma_{[0]}$ if and only if $\tau \prec \sigma_{[n]}$. Also,

$$\mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]} \subset M$$

and,

$$\mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]} \subset M'.$$

Therefore,

$$\mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]} \subset M''.$$

Hence M'' is a subcomplex at x expanded by f. Let,

$$i : M'' \longrightarrow M,$$
$$k : M'' \longrightarrow M'$$

be the inclusion maps. As above, there are chain homomorphisms,

$$\tilde{f}_p : C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M') \longrightarrow C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M')$$
$$\tilde{f}_p : C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M'') \longrightarrow C_p(\overline{\mathtt{st}}\,(\sigma_{[n]}, \mathtt{sd}^n K), M'')$$

The following diagram is commutative,

$$\begin{array}{ccccc} C_p(M) & \xleftarrow{i_p} & C_p(M'') & \xrightarrow{k_p} & C_p(M') \\ \downarrow \tilde{f}_p & & \downarrow \tilde{f}_p & & \downarrow \tilde{f}_p \\ C_p(M) & \xleftarrow[i_p]{} & C_p(M'') & \xrightarrow[k_p]{} & C_p(M') \end{array}$$

Diagram 5.10

Let $\tau \in M$ (respectively M') be such that, $|\overline{\tau}| \subset ||\tilde{f}_p(\tau)||$. Then τ is a simplex of $\rho[\mathtt{sd}^n\{f(\overline{\tau})\}]$. By Lemma 5.7 and Lemma 4.7,

$$\rho[\mathtt{sd}^n\{f(\overline{\tau})\}] \subset \mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \{\mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]}\}.$$

Therefore,

$$\tau \in \mathtt{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\tau})\} \cap \{\mathrm{Lk}\,(\sigma_{[n]}, \mathtt{sd}^n K) * \dot{\sigma}_{[n]}\}.$$

Hence by Lemma 5.8, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup f(\tau) \rangle$. Thus by property (a) of M' (respectively M), it follows that, $\tau \in M'$ (respectively M). Therefore, $\tau \in M''$.

Hence,

$$\begin{aligned} & \mathrm{Trace}\,\{\tilde{f}_p : C_p(M) \longrightarrow C_p(M)\} \\ = \; & \mathrm{Trace}\,\{\tilde{f}_p : C_p(M'') \longrightarrow C_p(M'')\} \\ = \; & \mathrm{Trace}\,\{\tilde{f}_p : C_p(M') \longrightarrow C_p(M')\}. \end{aligned}$$

The lemma now follows from Hopf Trace Theorem ([22]) and Lemma 3.2.

■

By Lemma 3.21, x is a fixed point of $\mathsf{sd}\, f$.

Lemma 5.11 *: Let $\rho_n : \mathsf{sd}\{\overline{\mathsf{st}}\,(\sigma_{[n]}, \mathsf{sd}^n K)\} \longrightarrow \overline{\mathsf{st}}\,(\sigma_{[n+1]}, \mathsf{sd}^{n+1} K)$ be the retraction defined in Chapter 4. Let $\widetilde{M} = \rho_n(\mathsf{sd}\, M)$. Then $\widetilde{M}$ is a subcomplex at x expanded by $\mathsf{sd}\, f$.*

Proof: We first show that $\widetilde{M}$ satisfies property (a) of Definition 5.6. Let τ be a simplex of $\mathsf{Lk}\,(\sigma_{[n+1]}, \mathsf{sd}^{n+1} K) * \dot{\sigma}_{n+1}$ such that,

$$\langle \sigma_{[n+1]} \cup \tau \rangle \subset \langle \sigma_{[1]} \cup \mathsf{sd}\, f(\tau) \rangle.$$

By Lemma 5.8, this implies that,

$$\tau \in \mathsf{sd}^n\{\overline{\sigma}_{[1]} * \mathsf{sd}\, f(\overline{\tau})\} \cap \mathsf{Lk}\,(\sigma_{[n+1]}, \mathsf{sd}^{n+1} K) * \dot{\sigma}_{[n+1]} \qquad (12)$$

Also, it follows from Lemma 4.5, that there is $\mu \in \mathsf{Lk}\,(\sigma_{[n]}, \mathsf{sd}^n K) * \dot{\sigma}_{[n]}$, such that, $\tau \in \rho_n(\mathsf{sd}\, \overline{\mu})$ and, $\langle \tau \rangle \subset \langle \sigma_{[n]} \cup \mu \rangle$. This in turn implies by Equation 2 that,

$$\tau \in \mathsf{sd}\{\overline{\sigma}_{[n]} * \overline{\mu}\} \cap \mathsf{Lk}\,(\sigma_{[n+1]}, \mathsf{sd}^{n+1} K) * \dot{\sigma}_{n+1} \subset \mathsf{sd}\{\overline{\sigma}_{[n]} * \overline{\mu}\}.$$

Therefore, $\mathsf{sd}\, f(\tau)$ is a simplex of $\mathsf{sd}\, f(\mathsf{sd}\{\overline{\sigma}_{[n]} * \overline{\mu}\})$ and hence by Lemma 3.22, $\mathsf{sd}\, f(\tau)$ is a simplex of $\mathsf{sd}\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$. Also,

$$\sigma_{[1]} \in \mathsf{sd}\, \overline{\sigma}_{[0]} \subset \mathsf{sd}\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}.$$

Since $\mathsf{sd}\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$ is a full subcomplex, this implies that $\sigma_{[1]} \cup \mathsf{sd}\, f(\tau)$ is a simplex of $\mathsf{sd}\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$. Therefore by Equation 12, τ is a simplex of $\mathsf{sd}^{n+1}\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$. Also, $\langle \tau \rangle \subset \langle \sigma_{[n]} \cup \mu \rangle$. Therefore $\sigma_{[n]} \cup \mu$ is a simplex of $\mathsf{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$. Hence, $\mu \in \mathsf{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\mu})\}$. Thus,

$$\mu \in \mathsf{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\mu})\} \cap \mathsf{Lk}\,(\sigma_{[n]}, \mathsf{sd}^n K) * \dot{\sigma}_{[n]}.$$

This implies from Lemma 5.8 that, $\langle \sigma_{[n]} \cup \mu \rangle \subset \langle \sigma_{[0]} \cup f(\mu) \rangle$. Thus by property (a) of M, $\mu \in M$. Since $\tau \in \rho_n(\mathsf{sd}\, \overline{\mu})$, this implies that $\tau \in \widetilde{M}$. Hence, $\widetilde{M}$ satisfies property (a) of Definition 5.6.

We next show that $\widetilde{M}$ satisfies property (b) of Definition 5.6.

Let $\tau \in \widetilde{M}$. Let $\nu \in M$ be such that, $\tau \in \rho_n(\mathrm{sd}\,\overline{\nu})$. Let ν' be a simplex of $\mathrm{sd}\,\overline{\nu}$ such that $\rho_n(\nu') = \tau$. Let $\sigma_{[n+1]} = [b(\mu_0), \ldots, b(\mu_p)]$, where, $\mu_p = \sigma_{[n]}$ and, $\nu' = [b(\nu_0), \ldots, b(\nu_q)] \in \mathrm{sd}\,\overline{\nu}$, where, $b(\nu_s), \ldots, b(\nu_r)$, for $0 \leq s \leq r \leq q$ are such that, there is a $\mu_{j_k} \prec \nu_k$ and, $\mu_{j_k+1} \not\prec \nu_k$. Then,

$$\begin{aligned}\tau &= [b(\nu_0 \cup \sigma_{[n]}), \ldots, b(\nu_{s-1} \cup \sigma_{[n]}), b(\mu_{j_s}), \ldots, b(\mu_{j_r}),\\ &\quad b(\nu_{r+1} \cup \sigma_{[n]}), \ldots, b(\nu_q \cup \sigma_{[n]})].\end{aligned}$$

and,

$$\begin{aligned}\mathrm{sd}\, f(\tau) &= [b(f(\nu_0) \cup \sigma_{[0]}), \ldots, b(f(\nu_{s-1}) \cup \sigma_{[0]}), b(f(\mu_{j_s})), \ldots, b(f(\mu_{j_r})),\\ &\quad b(f(\nu_{r+1}) \cup \sigma_{[0]}), \ldots, b(f(\nu_q) \cup \sigma_{[0]})].\end{aligned}$$

Note that for $0 \leq j \leq s-1$ or $r+1 \leq j \leq q$, if $b(f(\nu_j) \cup \sigma_{[0]})$ is a vertex of $\sigma_{[1]}$ then, $f(\nu_j) \cup \sigma_{[0]}$ is a face of $\sigma_{[0]}$ which implies that $f(\nu_j)$ is a face of $\sigma_{[0]}$. Since $\nu_j \prec \nu \in M$ it follows by property (b) of M that $\nu_j \prec \sigma_{[n]}$, contrary to our assumption. Thus for $0 \leq j \leq s-1$ or $r+1 \leq j \leq q$, $b(f(\nu_j) \cup \sigma_{[0]})$ is not a vertex of $\sigma_{[1]}$. Let $\mathrm{sd}\, f(\tau) \prec \sigma_{[1]}$. Then by the above discussion it follows that $s = 0$ and $r = q$, which implies that, $\tau = [b(\mu_{j_0}), \ldots, b(\mu_{j_q})] \prec \sigma_{[n+1]}$. Thus $\widetilde{M}$ satisfies property (b) of Definition 5.6.

Finally we show that $\widetilde{M}$ satisfies property (c) of Definition 5.6.

Let $\tau \in \widetilde{M}$. Let $\nu \in M$ be such that $\tau \in \rho_n(\mathrm{sd}\,\overline{\nu})$. Let τ' be a simplex of $\mathrm{sd}^n\{\overline{\sigma}_{[1]} * \mathrm{sd}\, f(\overline{\tau})\} \cap \mathrm{Lk}\,(\sigma_{[n+1]}, \mathrm{sd}^{n+1}K) * \dot{\sigma}_{[n+1]}$. By Lemma 4.5, there is a $\tau'' \in \mathrm{sd}\,\{\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}\}$, such that, $\rho_n(\tau'') = \tau'$ and,

$$\langle \tau' \rangle \subset \langle \sigma_{[n]} \cup \tau'' \rangle.$$

Since τ' is a simplex of $\mathrm{sd}^n\{\overline{\sigma}_{[1]} * \mathrm{sd}\, f(\overline{\tau})\}$, it follows that,

$$\tau'' \in \mathrm{sd}^n\{\overline{\sigma}_{[1]} * \mathrm{sd}\, f(\overline{\tau})\} \subset \mathrm{sd}^{n+1}\{\overline{\sigma}_{[0]} * f(\overline{\nu})\}.$$

Therefore,

$$\tau'' \in \mathrm{sd}\,\{\mathrm{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\nu})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}\}.$$

Also ν is a simplex of M, so, by property (c) of M,

$$\mathrm{sd}^n\{\overline{\sigma}_{[0]} * f(\overline{\nu})\} \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]} \subset M.$$

So τ'' is a simplex of $\operatorname{sd} M$ and hence, $\tau' \in \rho_n(\operatorname{sd} M) = \widetilde{M}$. Thus,

$$\operatorname{sd}^n\{\overline{\sigma}_{[1]} * \operatorname{sd} f(\overline{\tau})\} \cap \operatorname{Lk}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K) * \dot{\sigma}_{n+1} \subset \widetilde{M}.$$

Hence $\widetilde{M}$ satisfies property (c) of Definition 5.6 and thus $\widetilde{M}$ is a subcomplex at x expanded by $\operatorname{sd} f$. ∎

Lemma 5.12 $I(f, x) = I(\operatorname{sd} f, x)$.

Proof: Let $\rho' : \operatorname{sd}^n\{\overline{\operatorname{st}}(\sigma_{[1]}, \operatorname{sd} K)\} \longrightarrow \overline{\operatorname{st}}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K)$ be the composition, $\rho' = \rho_n \circ \operatorname{sd} \rho_{n-1} \circ \cdots \circ \operatorname{sd}^{n-1}\rho_1$ and let,

$$\lambda'_p : C_p(\overline{\operatorname{st}}(\sigma_{[1]}, \operatorname{sd} K)) \longrightarrow C_p(\operatorname{sd}^n\{\overline{\operatorname{st}}(\sigma_{[1]}, \operatorname{sd} K)\})$$

be the subdivision operator. The following is a chain map,

$$\begin{aligned} &C_p(\overline{\operatorname{st}}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K)) \overset{(\operatorname{sd} f)_p}{\longrightarrow} C_p(\overline{\operatorname{st}}(\sigma_{[1]}, \operatorname{sd} K)) \\ &\overset{\lambda'_p}{\longrightarrow} C_p(\operatorname{sd}^n(\overline{\operatorname{st}}(\sigma_{[1]}, \operatorname{sd} K))) \overset{\rho'_p}{\longrightarrow} C_p(\overline{\operatorname{st}}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K)) \end{aligned}$$

Then,

$$\begin{aligned} \widetilde{\operatorname{sd} f}_p &= \rho'_p \circ \lambda'_p \circ (\operatorname{sd} f)_p : \\ C_p(\overline{\operatorname{st}}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K)) &\longrightarrow C_p(\overline{\operatorname{st}}(\sigma_{[n+1]}, \operatorname{sd}^{n+1}K)). \end{aligned}$$

Let $\widetilde{M} = \rho_n(\operatorname{sd} M)$. Then by Lemma 5.11, $\widetilde{M}$ is expanded by $\operatorname{sd} f$ and,

$$\widetilde{\operatorname{sd} f}_p : C_p(\widetilde{M}) \longrightarrow C_p(\widetilde{M}).$$

We show that the following diagram is commutative,

$$\begin{array}{ccc} \operatorname{sd}\{\operatorname{Lk}(\sigma_{[0]}, K) * \dot{\sigma}_{[0]}\} & \xrightarrow{\rho_0} & \operatorname{Lk}(\sigma_{[1]}, \operatorname{sd} K) * \dot{\sigma}_{[1]} \\ \uparrow{\scriptstyle \operatorname{sd} f} & & \uparrow{\scriptstyle \operatorname{sd} f} \\ \operatorname{sd} M & \xrightarrow{\rho_n} & \widetilde{M} \end{array}$$

Diagram 5.12(a)

Let $\sigma_{[n]} = [v_0, \ldots, v_p]$ and for all $0 \leq j \leq p$, $\mu_j = [v_0, \ldots, v_j]$ so that, $\sigma_{[n+1]} = [b(\mu_0), \ldots, b(\mu_p)]$.
Then $\sigma_{[0]} = [f(v_0), \ldots, f(v_p)]$ and, $\sigma_{[1]} = [b(f(\mu_0)), \ldots, b(f(\mu_p))]$.
Let $b(\tau) \in \mathrm{sd}\, M$.
First let for some $0 \leq k \leq p-1$, $\mu_k \prec \tau$, and $\mu_{k+1} \not\prec \tau$. Then,

$$\mathrm{sd}\, f \circ \rho_n(b(\tau)) = \mathrm{sd}\, f(b(\mu_k)) = b(f(\mu_k)).$$

Also, $\rho_0 \circ \mathrm{sd}\, f(b(\tau)) = \rho_0(b(f(\tau))$.
Now $f(\mu_k) \prec f(\tau)$. Let if possible, $f(\mu_{k+1}) \prec f(\tau)$. Then, there is a face τ' of τ, such that, $f(\tau') = f(\mu_{k+1}) - f(\mu_k) \prec \sigma_{[0]}$. Since, $\tau' \in M, \tau' \prec \sigma_{[n]}$. But then, $\tau' = \mu_{k+1} - \mu_k$, which contradicts the fact that, $\mu_{k+1} \not\prec \tau$. Therefore, $f(\mu_{k+1}) \not\prec f(\tau)$. Hence,

$$\rho_0 \circ \mathrm{sd}\, f(b(\tau)) = \rho_0(b(f(\tau)) = b(f(\mu_k)) = \mathrm{sd}\, f \circ \rho_n(b(\tau)).$$

Now let for all $0 \leq k \leq p-1$, $\mu_k \prec \tau$ imply that, $\mu_{k+1} \prec \tau$. Then,

$$\mathrm{sd}\, f \circ \rho_n(b(\tau)) = \mathrm{sd}\, f(b(\tau \cup \sigma_{[n]})) = b(f(\tau) \cup \sigma_{[0]}).$$

Also, if there is a k, $0 \leq k \leq p-1$ such that,

$$f(\mu_k) \prec f(\tau), \text{ and, } f(\mu_{k+1}) \not\prec f(\tau),$$

then there is a $\tau'' \prec \tau$ such that, $f(\tau'') = f(\mu_k) \prec \sigma_{[0]}$. Since, τ'' is a simplex of $M, \tau'' \prec \sigma_{[n]}$. But then, $\tau'' = \mu_k$, which implies by the hypothesis on τ that, $\mu_{k+1} \prec \tau$ a contradiction to the fact that, $f(\mu_{k+1}) \not\prec f(\tau)$. Hence there is no such μ_k, and, so,

$$\rho_0 \circ \mathrm{sd}\, f(b(\tau)) = \rho_0(b(f(\tau))) = b(f(\tau) \cup \sigma_{[0]}) = \mathrm{sd}\, f \circ \rho_n(b(\tau)).$$

Thus Diagram 5.12(a) is commutative.
Let

$$\lambda''_p : C_p(\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)) \longrightarrow C_p(\mathrm{sd}\,\{\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)\}),$$

be the subdivision operator. The following diagram commutes,

$$\begin{array}{ccc} C_p(\widetilde{M}) & \xrightarrow{(\widetilde{\mathrm{sd}\, f})_p} & C_p(\widetilde{M}) \\ \uparrow{\scriptstyle (\rho_n)_p\lambda''_p} & & \uparrow{\scriptstyle (\rho_n)_p\lambda''_p} \\ C_p(M) & \xrightarrow[\tilde{f}_p]{} & C_p(M) \end{array}$$

Diagram 5.12(b)

Let $\tau \in M$. To show that Diagram 5.12(b) commutes, it is enough to show that,

$$\rho_n[\mathrm{sd}\,\{\rho(\mathrm{sd}^n(f(\bar{\tau})))\}] = \rho'[\mathrm{sd}^n\{\mathrm{sd}\, f \circ \rho_n(\mathrm{sd}\,(\bar{\tau}))\}] \qquad (13)$$

The left hand side of Equation 13 is,

$$\rho_n[\mathrm{sd}\,\rho\{\mathrm{sd}^{n+1}(f(\bar{\tau}))\}] =$$
$$\rho_n \circ \mathrm{sd}\,\rho_{n-1} \circ \mathrm{sd}^2\rho_{n-2} \circ \cdots \circ \mathrm{sd}^n\rho_0[\mathrm{sd}^{n+1}\{f(\bar{\tau})\}]$$

Therefore by Lemma 3.22,

$$\begin{aligned} \rho_n[\mathrm{sd}\,\rho\{\mathrm{sd}^{n+1}(f(\bar{\tau}))\}] &= \rho' \circ \mathrm{sd}^n\rho_0[\mathrm{sd}^{n+1}\{f(\bar{\tau})\}] \\ &= \rho'[\mathrm{sd}^n\{\rho_0(\mathrm{sd}\,(f(\bar{\tau})))\}] \\ &= \rho'[\mathrm{sd}^n\{\rho_0\mathrm{sd}\, f(\mathrm{sd}\,\bar{\tau})\}] \\ &= \rho'[\mathrm{sd}^n\{\mathrm{sd}\, f\rho_n(\mathrm{sd}\,\bar{\tau})\}] \end{aligned}$$

by Diagram 5.12(a). Therefore Equation 13 holds.

A basis of $C_p(M)$ consists of all oriented p-simplices of M. Let τ be a p-simplex of M. Then $(\rho_n)_p\lambda''_p(\tau) = 0$ implies that, $\dim \rho_n(\mathrm{sd}\,\bar{\tau}) < p$. Therefore by Lemma 4.6, $\tau \not< \sigma_{[n]}$ and $\bar{\tau} \cap \dot{\sigma}_{[n]}$ is a simplex of dimension at least one. Since $\tau \in M$, this implies that, $f(\tau) \not< \sigma_{[0]}$ and $f(\bar{\tau}) \cap \dot{\sigma}_{[n]}$ is a simplex of dimension at least one. Again, by Lemma 4.6, this implies that, $\dim \rho_{0,n}(\mathrm{sd}^n\{f(\bar{\tau})\}) < p$ and hence, τ does not contribute to the trace of $\tilde{f}_p$. This implies that, any element in the kernel of $(\rho_n)_p\lambda''_p$ does not contribute to the trace of $\tilde{f}_p$.

It then follows that,

$$\begin{aligned} &\operatorname{Trace} \tilde{f}_p : C_p(M) \longrightarrow C_p(M) \\ &= \operatorname{Trace} \widetilde{\mathrm{sd}\, f}_p : C_p(\widetilde{M}) \longrightarrow C_p(\widetilde{M}). \end{aligned}$$

Therefore, by Hopf Trace Theorem ([22]),

$$\begin{aligned} &\operatorname{Trace} (\tilde{f}_p)_* : H_p(M) \longrightarrow H_p(M) \\ &= \operatorname{Trace} (\widetilde{\mathrm{sd}\, f}_p)_* : H_p(\widetilde{M}) \longrightarrow H_p(\widetilde{M}). \end{aligned}$$

Hence, $L(\tilde{f}, M) = L(\widetilde{\mathrm{sd}\, f}, M)$.

As in the proof of Lemma 5.1 it can be shown that,

$$L(\tilde{f}, \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^{\,n}K)) = L(\widetilde{\mathrm{sd}\, f}, \overline{\mathrm{st}}\,(\sigma_{[n+1]}, \mathrm{sd}^{\,n+1}K)).$$

Now the result follows from Lemma 3.2. ∎

Theorem 5.13 $L(f) = \sum_{x \in Fix f} I(f, x)$

Proof: Let $\mathrm{Fix}\, f = \{x_1, \ldots, x_k\}$. Let the carrier of x_j in $\mathrm{sd}^p K, p \geq 0$ be $\sigma_{[p]}(x_j)$ and the subcomplex at x_j expanded by f be $M(x_j)$ for all $1 \leq j \leq k$. Let $m \geq 0$ be such that,

$$\overline{\mathrm{st}}\,(\sigma_{[m]}(x_i), \mathrm{sd}^{\,m}K) \cap \overline{\mathrm{st}}\,(\sigma_{[m]}(x_j), \mathrm{sd}^{\,m}K) = \emptyset, \text{ for all } 1 \leq i \neq j \leq k$$

Let $P = \mathrm{sd}^{\,m}K$ and $g = \mathrm{sd}^{\,m} f$. Since f and g are homotopic, $L(f) = L(g)$ and also, $\mathrm{Fix}\, f = \mathrm{Fix}\, g$.

For all $1 \leq i \leq k$, and for all $p \geq 0$ denote,

$$\tau_{[p]}(x_i) = \sigma_{[p+m]}(x_i).$$

Let $\rho_{n,n+m} : \mathrm{sd}^{\,m}\{\overline{\mathrm{st}}\,(\sigma_{[n]}(x_j), \mathrm{sd}^{\,n}K)\} \longrightarrow \overline{\mathrm{st}}\,(\sigma_{[m+n]}(x_j), \mathrm{sd}^{\,m+n}K)$ be the retraction defined in Chapter 4. For any $x_j \in \mathrm{Fix}\, f$, we choose a subcomplex at x_j which is expanded by g to be $\widetilde{M}(x_j) = \rho_{n,n+m}[\mathrm{sd}^{\,m}\{M(x_j)\}]$.

By Lemma 5.12 we see that, $I(g, x_j) = I(f, x_j)$, for all $1 \leq j \leq r$. Therefore, to establish the result it is enough to show that,

$$L(g) = \sum_{j=1}^{k} I(g, x_j).$$

Let $C_q(P) \xrightarrow{\lambda'_q} C_q(\mathrm{sd}^n P)$ be the subdivision operator. Then by Hopf Trace Theorem ([22]),

$$L(g) = \sum_{q \geq 0} (-1)^q \mathrm{Trace}\, \{C_q(\mathrm{sd}^n P) \xrightarrow{g_q} C_q(P) \xrightarrow{\lambda'_q} C_q(\mathrm{sd}^n P)\}.$$

For all q-simplex μ of $\mathrm{sd}^n P$, let, $\lambda'_q g_q(\mu) = \sum n_{\nu\mu} \nu$ where the sum is over all q-simplex $\nu \in \mathrm{sd}^n P$. Then,

$$L(g) = \sum_{q \geq 0} (-1)^q (\sum_{\mu \in \mathrm{sd}^n P} n_{\mu\mu}).$$

Let,

$$F = \cup_{j=1}^{k} \overline{\tau}_{[n]}(x_j) \text{ and, } F' = \cup_{j=1}^{k} \overline{\tau}_{[0]}(x_j).$$

Now, $||\lambda'_q g_q(\mu)|| \subset |g(\overline{\mu})|$ and by Lemma 3.12, if μ is a simplex of $\mathrm{sd}^n P - \mathrm{N}\,(F, \mathrm{sd}^n P)$, then $|\overline{\mu}| \not\subset |g(\overline{\mu})|$. So $\mu \in \mathrm{sd}^n P - \mathrm{N}\,(F, \mathrm{sd}^n P)$ implies that, $n_{\mu\mu} = 0$. Hence,

$$L(g) = \sum_{q \geq 0} (-1)^q (\sum \{n_{\mu\mu} : \mu \in \mathrm{sd}^n P; \langle \mu \rangle \in \mathrm{N}\,(F, \mathrm{sd}^n P)\}) \qquad (14)$$

Also, $\overline{\mathrm{st}}\,(\tau_{[0]}(x_j), P) \cap \overline{\mathrm{st}}\,(\tau_{[0]}(x_i), P) = \emptyset$ if $1 \leq i \neq j \leq k$. Therefore,

$$\overline{\mathrm{N}}\,(F', P) = \cup_{j=1}^{k} \overline{\mathrm{st}}\,(\tau_{[0]}(x_j), P).$$

So, we can define,

$$\rho : \mathrm{sd}^n(\overline{\mathrm{N}}\,(F', P)) \longrightarrow \overline{\mathrm{N}}\,(F, \mathrm{sd}^n P)$$

to be, $\rho|\overline{\mathrm{st}}\,(\tau_{[0]}(x_j), P) = \rho_{m,n+m}(\sigma_{[n+m]}(x_j)) = \rho(j)$.
Let $M = \cup_{j=1}^{k} \widetilde{M}(x_j)$. There is a chain homomorphism,

$$\widetilde{g}_q : C_q(\overline{\mathrm{N}}\,(F, \mathrm{sd}^n P), M) \longrightarrow C_q(\overline{\mathrm{N}}\,(F, \mathrm{sd}^n P), M).$$

By relative Mayer-Vietoris sequence,

$$H_q(\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P), M) \cong \oplus_{j=1}^{k} H_q(\overline{\mathtt{st}}\,(\tau_{|n|}(x_j), \mathtt{sd}\,^n P), \widetilde{M}(x_j))$$

Therefore,

$$L(\tilde{g}, \overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P), M) = \sum_{j=1}^{k} I(g, x_j) \tag{15}$$

It follows from Lemma 3.2 that,

$$L(\tilde{g}, \overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P), M) = L(\tilde{g}, \overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P), \emptyset) - L(\tilde{g}, M, \emptyset) \tag{16}$$

Let for all q-simplex μ of $\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P); \tilde{g}_q(\mu) = \sum m_{\nu\mu}\nu$, where the sum is over all p-simplices ν of $\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P)$. Then,

$$\begin{aligned}\text{Trace}\ \{\tilde{g}_q : C_q(\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P)) \longrightarrow C_q(\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P))\} &= \\ \textstyle\sum\{m_{\mu\mu} : \mu \in \overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P)\}.\end{aligned}$$

Now $\rho|\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P)$ is the identity map, and,

$$\mathtt{sd}\,^n\{\overline{\mathtt{N}}\,(F', P)\} - \mathtt{N}\,(F, \mathtt{sd}\,^n P) \xrightarrow{\rho} \cup_{j=1}^{k} \mathrm{Lk}\,(\tau_{|n|}(x_j), \mathtt{sd}\,^n P) * \dot{\tau}_{|n|}(x_j).$$

Also,

$$\tilde{g}_q(\mu) = \rho_* \lambda'_q g_q(\mu) = \rho_*(\textstyle\sum n_{\nu\mu}\nu) = \sum n_{\nu\mu}\rho(\nu).$$

So for all μ such that, $\langle\mu\rangle \subset \mathtt{N}\,(F, \mathtt{sd}\,^n P); n_{\mu\mu} = m_{\mu\mu}$. Hence,

$$\begin{aligned}\text{Trace}\ \{\tilde{g}_q : C_q(\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P)) \longrightarrow C_q(\overline{\mathtt{N}}\,(F, \mathtt{sd}\,^n P))\} &= \\ \textstyle\sum\{n_{\mu\mu} : \langle\mu\rangle \subset \mathtt{N}\,(F, \mathtt{sd}\,^n P)\}& \\ + \textstyle\sum\{m_{\mu\mu} : \mu \in \cup_{j=1}^{k} \mathrm{Lk}\,(\tau_{|n|}(x_j), \mathtt{sd}\,^n P) * \dot{\tau}_{|n|}(x_j), |\overline{\mu}| \subset ||\tilde{g}_q(\mu)||\}&\end{aligned} \tag{17}$$

Let $\mu \in \mathrm{Lk}\,(\tau_{|n|}(x_j), \mathtt{sd}\,^n P) * \dot{\tau}_{|n|}(x_j)$. Suppose that $\langle g(\mu)\rangle \subset \mathtt{st}\,(\tau_{|0|}(x_j), P)$ and $|\overline{\mu}| \subset ||\tilde{g}_q(\mu)||$. Let ν be a simplex of P such that $\langle\mu\rangle \subset \langle\nu\rangle$. Then, $\dim\mu \leq \dim\nu$. Now, $||\tilde{g}_q(\mu)|| \subset |\rho(j)\{\mathtt{sd}\,^n(g(\overline{\mu}))\}|$ and by Lemma 4.7,

$$|\rho(j)\{\mathtt{sd}\,^n(g(\overline{\mu}))\}| \subset |g(\overline{\mu})| \cap |\overline{\mathtt{st}}\,(\tau_{|n|}(x_j), \mathtt{sd}\,^n P)|.$$

This implies that $\nu \prec g(\mu)$ and hence, $\langle\mu\rangle \subset \langle g(\mu)\rangle$. Thus by Lemma 3.12 $\overline{\mu} \cap \operatorname{Fix} g \neq \emptyset$, which is a contradiction.
So, $\mu \in \operatorname{Lk}(\tau_{|n|}(x_j), \operatorname{sd}^n P) * \dot{\tau}_{|n|}(x_j)$, and $|\overline{\mu}| \subset ||\tilde{g}_q(\mu)||$ implies that $g(\mu)$ is a simplex of $\operatorname{Lk}(\tau_{|0|}(x_j), P) * \dot{\tau}_{|0|}(x_j)$. This implies by Lemma 4.7 that,

$$|\rho(j)\{\operatorname{sd}^n(g(\overline{\mu}))\}| \subset |\tau_{|0|}(x_j) * g(\overline{\mu})| \cap |\operatorname{Lk}(\tau_{|n|}(x_j), \operatorname{sd}^n P) * \dot{\tau}_{|n|}(x_j)|.$$

Also, $||\tilde{g}_q(\mu)|| \subset |\rho(j)\{\operatorname{sd}^n(g(\overline{\mu}))\}|$. Thus,

$$|\overline{\mu}| \subset |\tau_{|0|}(x_j) * g(\overline{\mu})| \cap |\operatorname{Lk}(\tau_{|n|}(x_j), \operatorname{sd}^n P) * \dot{\tau}_{|n|}(x_j)|$$

which implies that, $\langle\tau_{|n|}(x_j) \cup \mu\rangle \subset \langle\tau_{|0|}(x_j) \cup g(\mu)\rangle$, by Lemma 5.8.
It now follows by the property (a) of $\widetilde{M}_j$ that, $\mu \in \widetilde{M}_j \subset M$. Thus,

> *Any simplex* τ *of* $\cup_{j=1}^k\{\operatorname{Lk}(\tau_{|n|}(x_j), \operatorname{sd}^n P) * \dot{\tau}_{|n|}(x_j)\}$, *such that* $|\overline{\tau}| \subset ||\tilde{g}_q(\tau)||$, *is a simplex of* M.

Now, by Equations 14, 15, 16, and 17, it follows that, $L(g) = \sum_{j=1}^{k} I(g, x_j)$.

∎

Remark 5.14 It is also clear from the above discussion that,

$$I(g, x_j) = \sum_{q \geq 0}(-1)^q \sum\{n_{\mu\mu} : \langle\mu\rangle \subset \operatorname{st}(\tau_{|n|}(x_j), \operatorname{sd}^n P), \mu \text{ a } q\text{-simplex}\}$$

This reduces to the definition given by O'Neill ([19]) for the fixed point index of g on the open neighbourhood $\operatorname{st}(\tau_{|n|}(x_j), \operatorname{sd}^n P)$ of x_j.

5.4 Weakly Hyperbolic Simplicial Maps

Definition 5.15 A map $f : |K| \longrightarrow |K|$ on a connected compact polyhedron is *weakly hyperbolic* if for every fixed point component C of f there is an open neighbourhood W of C in $|K|$ and an indicator map, $t : W \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that, $t^{-1}((0,0)) = C$ and for all $x \in W \cap f^{-1}(W)$,

$$t_1(f(x)) \geq t_1(x), t_2(f(x)) \leq t_2(x).$$

where $t_j(x)$ is the j^{th} coordinate of $t(x)$.

Let $f : X \longrightarrow X$ be a map on a connected compact polyhedron. Let $A \subset X$ and W be a neighbourhood of A in X. Define,

$$\begin{aligned}[A]^-_{\overline{W}} &= \{y \in \overline{W} : \exists \{y_n\}_{n\geq 0} \subset \overline{W}, \text{where}, y_0 = y, g(y_n) = y_{n-1} \\ &\quad \text{and}, \lim_{n\to\infty} y_n \subset A\}.\end{aligned}$$

We need the following lemma. (Compare ([8, Proposition on Page 12, Section 3]).

Lemma 5.16 *: Let $\varphi : X \longrightarrow X$ be a weakly hyperbolic map on a polyhedron and F be a fixed point component of φ. Let for any open neighbourhood W of F in X,*

$$[F]^+_W = \{x \in W : \varphi^n(x) \in W, \textit{ for all } n \geq 1, \textit{and} \lim_{n\to\infty} \varphi^n(x) \in F\}$$

Let $t : W \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ be an indicator map. Then, $[F]^+_W \cap [F]^-_W = F$.

Proof: Let $x \in [F]^+_W \cap [F]^-_W$. Then, $\lim_{n\to\infty} \varphi^n(x) \in F$. Since $t_1(F) = 0$, it follows that, $\lim_{n\to\infty} t_1(\varphi^n(x)) = 0$. Now by the property of the indicator function, $0 \leq t_1(x) \leq t_1(\varphi(x)) \leq t_1(\varphi^2(x)) \leq \cdots$ Therefore, $t_1(x) = 0$.
Also, there is a sequence, $\{x_n\}$ in W such that, $x_0 = x, \varphi(x_n) = x_{n-1}$ and, $\lim_{n\to\infty} x_n \in F$. Again, since $t_2(F) = 0$, it follows that, $\lim_{n\to\infty} t_2(x_n) = 0$ and by the property of the indicator function, $0 \leq t_2(x_0) \leq t_2(x_1) \leq t_2(x_2) \leq \cdots$ Therefore, $t_2(x_0) = t_2(x) = 0$. Hence, $x \in F$. ∎

Remark 5.17 It is not generally true that $[F]^+_X \cap [F]^-_X = F$. For instance, consider the map $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, defined by,

$$\varphi(e^{i\theta}) = e^{2i\theta}.$$

Then, $\mathrm{Fix}\,\varphi = \{1\}$. The map is expanding in a neighbourhood of 1 and hence is weakly hyperbolic. But any $e^{i\frac{1}{2^n}}, n \in \mathbb{N}$, belongs to $[1]^+_{S^1} \cap [1]^-_{S^1}$.

Proposition 5.18 *Let $g : \mathrm{sd}^n K \longrightarrow K, n \geq 1$ be a simplicial map. If for all fixed points x of g, there is a neighbourhood W of x in $|K|$ such that, $[x]^-_{\overline{W}} \cap g^{-1}(x) = \{x\}$, then g preserves expanding directions.*

Proof: Let x be a fixed point of g. We show that there is a subcomplex M at x which is expanded by g. We denote the carrier of x in $\mathrm{sd}^p K$ by $\sigma_{[p]}$.

We assume without loss of generality that, $W \subset \mathrm{st}\,(\sigma_{[n]}, \mathrm{sd}^n K)$. Let y be a point of $|\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)|$ such that, there is a sequence $\{y_m\}$ in $|\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)|$ with, $y_0 = y, g(y_m) = y_{m-1}$ and, $\lim_{m \to \infty} y_m = x$. Also assume that, $g(y) = x$. Now g is a piecewise linear map. Therefore there is a sequence $\{z_m\} \subset W$ such that, z_m lies on the line segment joining x and y_m and $g(z_m) = z_{m-1}$ and, $\lim_{m \to \infty} z_m = x$. Therefore, $z_0 \in [x]^-_{\overline{W}}$. Also, since z_0 lies on the line segment joining x and y_0, $g(z_0) = x$, a contradiction. Hence,

$$[x]^-_{\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)} \cap g^{-1}(x) = \{x\} \tag{18}$$

Let $\tau \in \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$, be such that, $\langle \sigma_{[n]} \cup \tau \rangle \subset \langle \sigma_{[0]} \cup g(\tau) \rangle$. Take any $y \in |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)$. Then there is a point y_1 of $\sigma_{[n]} \cup \tau$, such that, $g(y_1) = y$. It can be shown inductively that for all $m \geq 1$, there is a $y_m \in \mathrm{st}\,(\sigma_{[mn]}, \mathrm{sd}^{mn} K)$, such that, $g(y_m) = y_{m-1}$. Clearly then, the sequence $\{y_m\}$ converges to x. Thus,

$$|\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap |\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)| \subset [x]^-_{\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)}. \tag{19}$$

Let w be a vertex of $|\sigma_{[0]} \cup g(\tau)| \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K)$. Let if possible $g(w)$ map to $\dot{\sigma}_{[0]}$. Consider the simplex,

$$\mu = \{\sigma_{[n]} - g^{-1}(g(w))\} \cup \{w\}$$

Then, $|\overline{\mu}| \subset |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap |\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)|$. Clearly $x \notin |\overline{\mu}|$.

Also $g(\mu) = \sigma_{[0]}$ implies that there is a point $y \in |\overline{\mu}|$ such that $g(y) = x$. This is a contradiction to Equations 18 and 19. Thus,

$$g : |\sigma_{[0]} \cup g(\tau)| \cap \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) \longrightarrow \mathrm{Lk}\,(\sigma_{[0]}, K)$$

and hence,

$$g : |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap |\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}| \longrightarrow |\mathrm{Lk}\,(\sigma_{[0]}, K) * \dot{\sigma}_{[0]}|.$$

Let

$$M^1(\tau) = \cup\{\overline{\nu} : |\overline{\nu}| \subset |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap |\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}|\}.$$

Then $M^1(\tau)$ satisfies property (b) of Definition 5.6.

If for all $\nu \in M^1(\tau); g(\nu) \prec g(\tau)$, then $M^1(\tau)$ satisfies property (c) of Definition 5.6.

Let for some $\nu \in M^1(\tau), g(\nu) \not\prec g(\tau)$ and $\tau' \in \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ be such that, $|\overline{\tau}'| \subset |\overline{\sigma}_{[0]} * g(\overline{\nu})|$. Now,

$$\begin{aligned} |\overline{\sigma}_{[0]} * g(\overline{\nu})| \cap \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K) &\subset |\overline{\sigma}_{[0]} * g(\overline{\tau})| \cap \overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K) \\ &\subset [x]^{-}_{\overline{\mathrm{st}}\,(\sigma_{[n]}, \mathrm{sd}^n K)}. \end{aligned}$$

Therefore, for all $y \in |\overline{\sigma}_{[0]} * g(\overline{\nu})|$; if $y \neq x$ then, $g(y) \neq x$. Let.

$$\begin{aligned} M^2(\tau) = \cup\{\overline{\tau}' : |\overline{\tau}'| \subset |\sigma_{[0]} \cup g(\nu)|, \\ \text{for all } \nu \in \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}, \text{ such that } |\overline{\nu}| \subset |\overline{\sigma_{[0]}} * g(\overline{\tau})|\}. \end{aligned}$$

Again, $M^2(\tau)$ satisfies property (b) of Definition 5.6.

Since $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ is a finite simplicial complex, repeating this process a finite number of times, it is possible to get a subcomplex $M(\tau)$ of $\mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}$ satisfying property (c) of Definition 5.6. Let,

$$M = \cup\{M(\tau) : \tau \in \mathrm{Lk}\,(\sigma_{[n]}, \mathrm{sd}^n K) * \dot{\sigma}_{[n]}, |\overline{\sigma}_{[n]} * \overline{\tau}| \subset |\overline{\sigma}_{[0]} * g(\overline{\tau})|\}.$$

Then M is a subcomplex at x which is expanded by g. ∎

Definition 5.19 A simplicial map $g : \mathrm{sd}^n K \longrightarrow K, n \geq 1$ will be called *simplicially weakly hyperbolic* if for all fixed points x of g, there is a neighbourhood W of x in $|K|$ such that, $[x]_{\overline{W}}^{-} \cap g^{-1}(x) = \{x\}$.

Remark 5.20 The converse of Proposition 5.18 does not hold as can be seen from the following example :

Let K be a simplicial complex and $g : \mathrm{sd}\, K \longrightarrow K$ be a simplicial map as shown in Figure 5. Then $\mathrm{Fix}\, g = \{1\}$ and $M(1) = \emptyset$. But for any neighbourhood W of 1 in K, there is a point $x \neq 1$ of $\langle [1, 12] \rangle$ which belongs to $[1]_{\overline{W}}^{-} \cap g^{-1}(1)$.

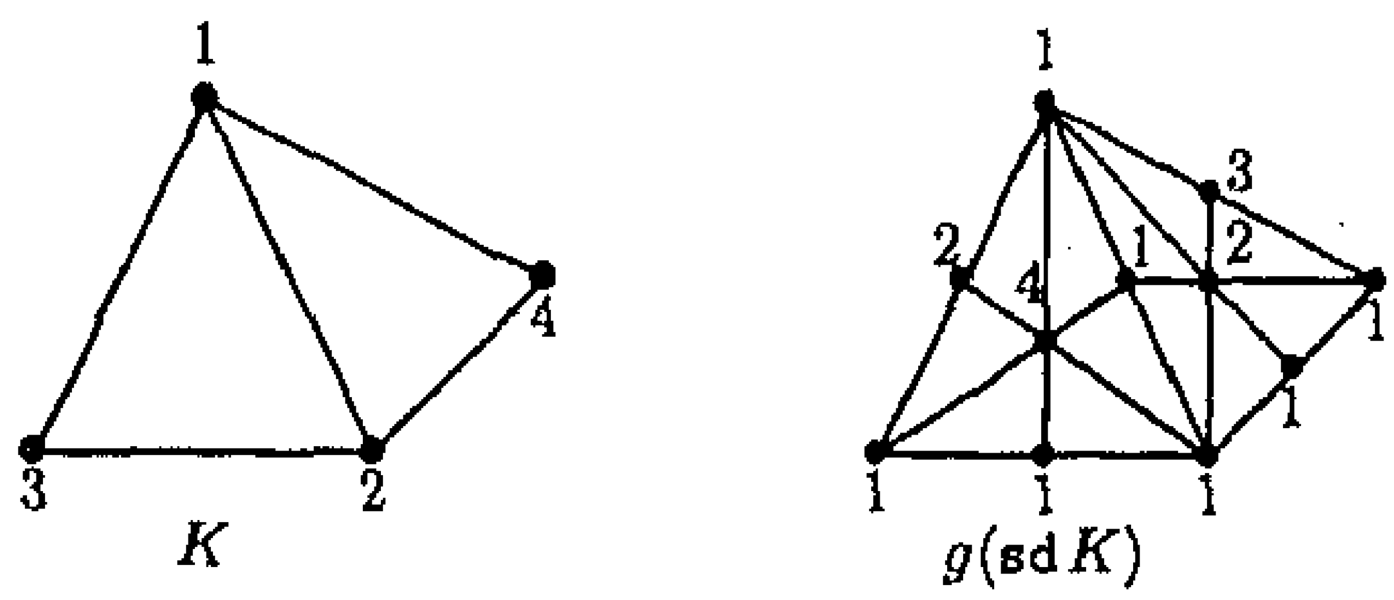


Figure 5: A map which preserves expanding directions but is not simplicially weakly hyperbolic

A weakly hyperbolic simplicial map is clearly simplicially weakly hyperbolic by Lemma 5.16. Thus, the following is an immediate consequence of Proposition 5.18,

Theorem 5.21 *: A weakly hyperbolic simplicial map preserves expanding directions.*

5.5 Examples

Example 5.22 A simplicial map which preserves expanding directions need not be weakly hyperbolic. Let K be the simplicial complex shown in

Figure 6 and $g : \mathrm{sd}\,K \longrightarrow K$ be a simplicial map as described in Figure 6. Then, $\mathrm{Fix}\,g = \{1,3\}$ and, $M(1) = 13, M(3) = \emptyset$, but for any neighbourhood W of 1 in $|K|$, there is a $x \in W \cap \langle [1,15] \rangle$, such that, $x \in [1]^+_W \cap [1]^-_W$.

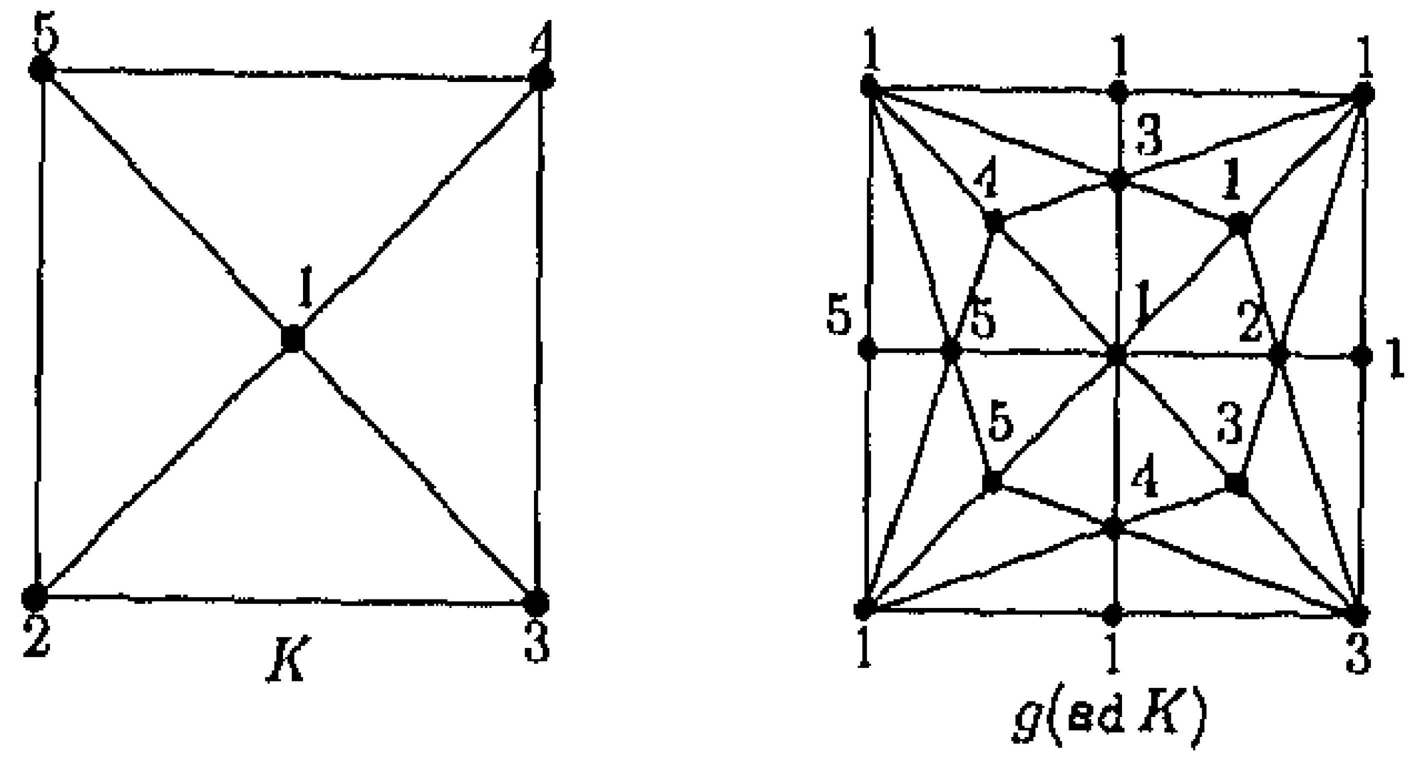


Figure 6: A map which preserves expanding directions but is not weakly hyperbolic

Example 5.23 With notations as above, if for some $x \in \mathrm{Fix}\,g$,

$$g : \mathrm{Lk}\,(\sigma_{|n|}(x), \mathrm{sd}^n K) \longrightarrow \mathrm{Lk}\,(\sigma_{|0|}(x), K),$$

then, a choice of $M(x)$ could be, $M(x) = \mathrm{Lk}\,(\sigma_{|n|}(x), \mathrm{sd}^n K) * \dot{\sigma}_{|n|}(x)$. On the other extreme, if for some $x \in \mathrm{Fix}\,g$,

$$g : \mathrm{Lk}\,(\sigma_{|n|}(x), \mathrm{sd}^n K) * \dot{\sigma}_{|n|}(x) \longrightarrow \overline{\sigma}_{|0|}(x),$$

then, $M(x) = \dot{\sigma}_{|n|}(x)$.

Example 5.24 Let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ and $f : X \longrightarrow X$ be the map $z^2 \vee z^{-1}$ defined in Example 2.1. A simplicial approximation $g : \mathrm{sd}\,K \longrightarrow K$ to f is shown in Figure 7. Then $\mathrm{Fix}\,g = \{0, 12\}$. The maps f and g are

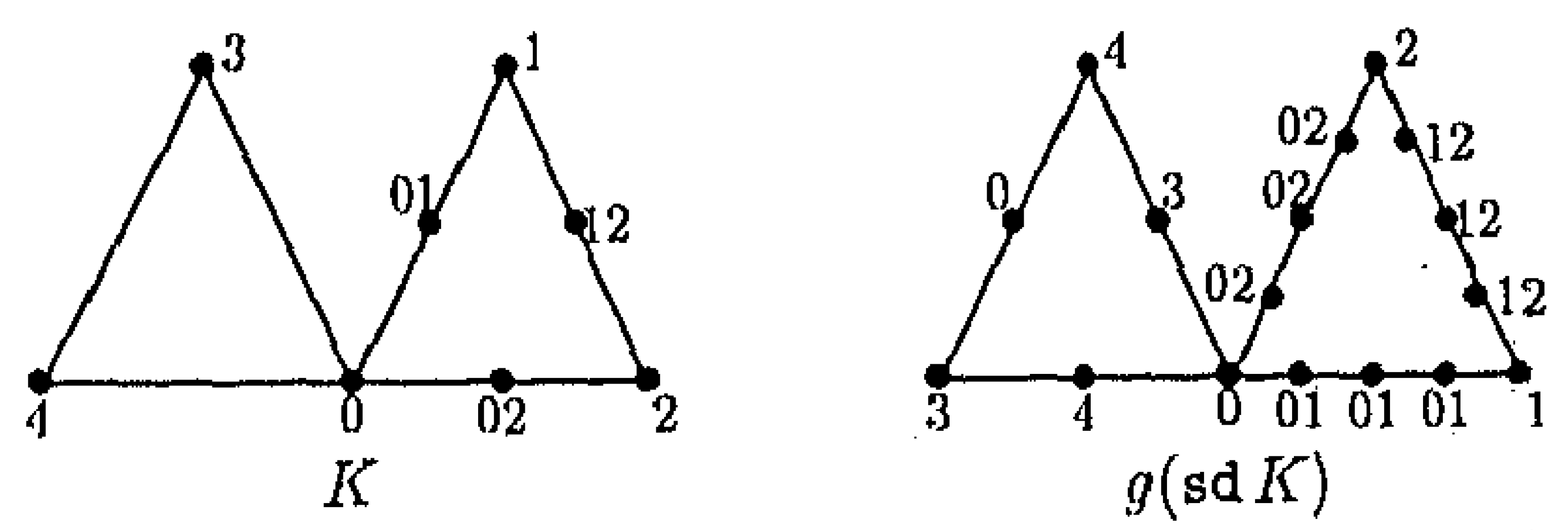


Figure 7: A simplicial approximation to the map $z^2 \vee z^{-1}$ on $\mathbb{S}^1 \vee \mathbb{S}^1$

homotopic by a homotopy which has no fixed points on the boundary of an open neighbourhood of $\mathrm{Fix}\, f$. So,

$$i(f,a) = i(g,12) \text{ and, } i(f,w) = i(g,0).$$

The map g preserves expanding directions. Since $M(12) = \emptyset$, it follows that $i(g,12) = 1$. Also, $M(0) = \mathrm{Lk}(0, \mathrm{sd}\, K)$. The generators of $H_1(\overline{\mathrm{st}}\,(0, \mathrm{sd}\, K), M(0))$ are, $c_1 = [0,001] - [0,002]$, $c_2 = [0,03] - [0,04]$ and, $c_3 = [0,03] - [0,002]$. Then $[0,03] - [0,001] = c_3 - c_1$ and, $\tilde{g}_*$ maps $[c_1]$ to $[-c_1]$, $[c_2]$ to $[c_2]$ and $[c_3]$ to $[c_3] - [c_1]$. Hence, $i(g,0) = -1$.

Example 5.25 Let $X = A \vee B, A = B = \mathbb{S}^2$, where, we think of $\mathbb{S}^2$ as the suspension $S(\mathbb{S}^1)$. Let the wedge point be $(w, \frac{1}{2})$ and the antipode of $(w, \frac{1}{2})$ in B be b. Let $f : X \longrightarrow X$ be the map, $f((z,t)) = (z^2, t)$ for all $(z,t) \in A$ and $f((z,t)) = (z^{-1}, 1-t)$ for all $(z,t) \in B$. Then, $L(f) = 4$ and,

$$\mathrm{Fix}\, f = \{(w,t) \in A : 0 \leq t \leq 1\} \cup \{b\}.$$

A triangulation of $X = |K| = |K_1| \vee |K_2|$ is shown in Figure 8. $\mathrm{Fix}\, f$ is the subcomplex, $\overline{[0,u]} \cup \overline{[0,v]} \cup [2]$. A simplicial approximation, $g : \mathrm{sd}\, K \longrightarrow K$

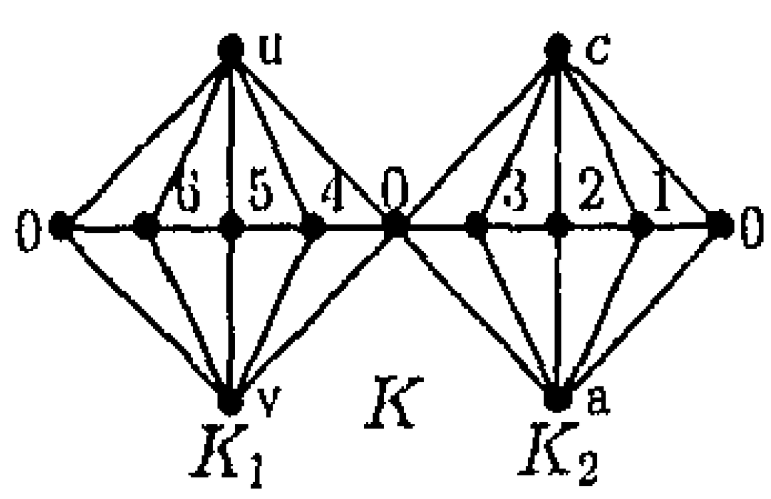


Figure 8: A triangulation of $\mathbb{S}^2 \vee \mathbb{S}^2$

to f is shown in Figure 9. It is clear from Figure 9 that $\mathrm{Fix}\, g = \{0, u, v, 2\}$ and one can take, $M(u) = M(v) = M(2) = \emptyset$. Hence,

$$I(g,u) = I(g,v) = I(g,2) = 1.$$

Also, $M(0) = \mathrm{Lk}\,(0, \mathrm{sd}\, K_1)$. So, $H_i(\overline{\mathrm{st}}\,(0, \mathrm{sd}\, K, M_1(0)) = 0, i \neq 2$ and $H_2(\overline{\mathrm{st}}\,(0, \mathrm{sd}\, K, M_1(0)) \cong \mathbb{Q}$ with the generator being a sum c of all 2 simplices of $\overline{\mathrm{st}}\,(0, \mathrm{sd}\, K_1)$ such that $\partial c = 0$ in $C_1(\overline{\mathrm{st}}\,(0, \mathrm{sd}\, K), M(0))$. Since, $\tilde{g}_*(c) = c, I(g,0) = 1$.

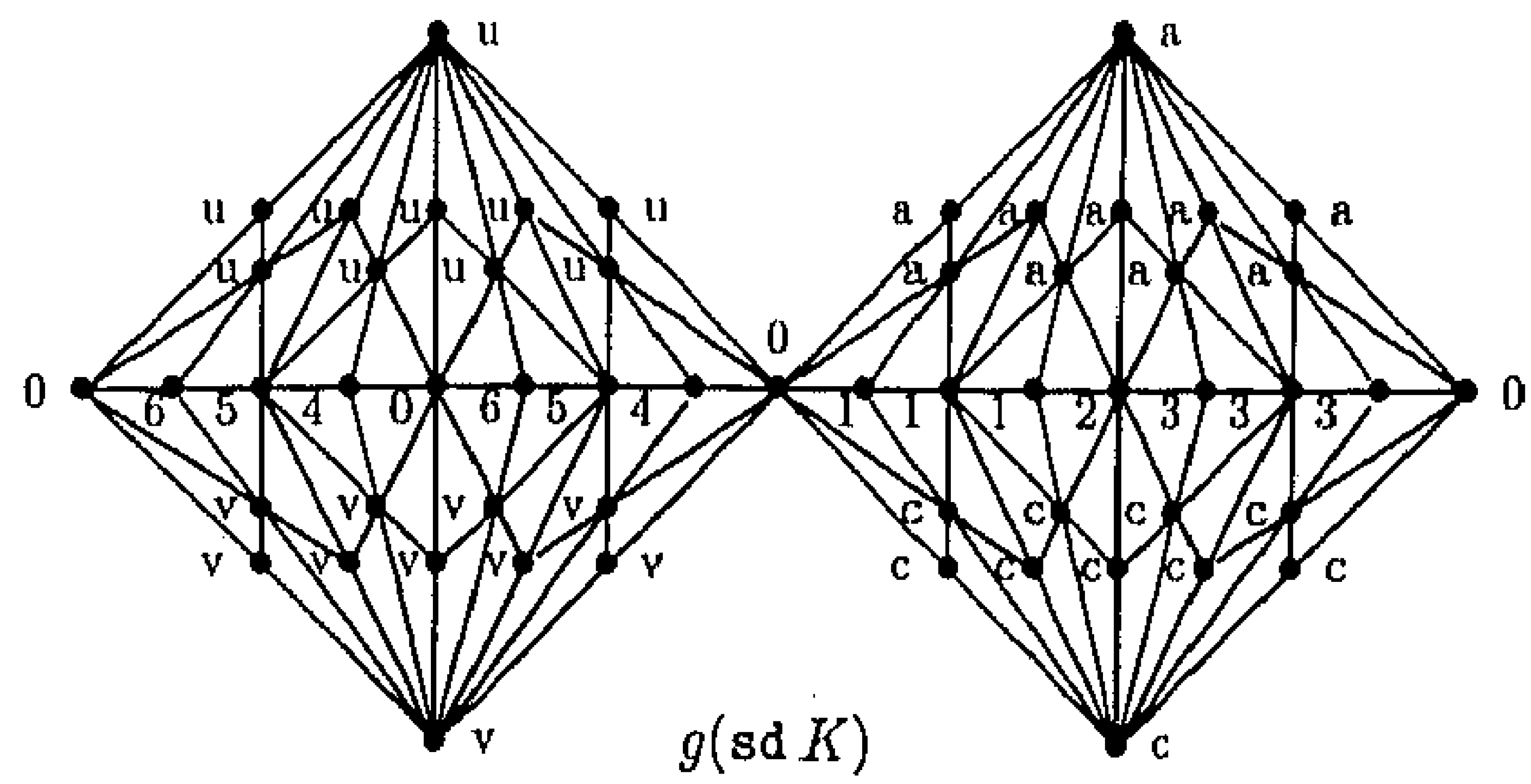


Figure 9: A simplicial approximation to $[z^2, t] \vee [z^{-1}, 1-t]$ on $\mathbb{S}^2 \vee \mathbb{S}^2$

Note that here again, f and g are homotopic maps such that the homotopy has no fixed points on the boundary of a neighbourhood of $\mathrm{Fix}\, f$.
Hence, $i(f, b) = i(g, 2) = 1$ and,

$$i(f, \{(w,t) \in A : t \in [0,1]\}) = i(g,u) + i(g,v) + i(g,0) = 3.$$

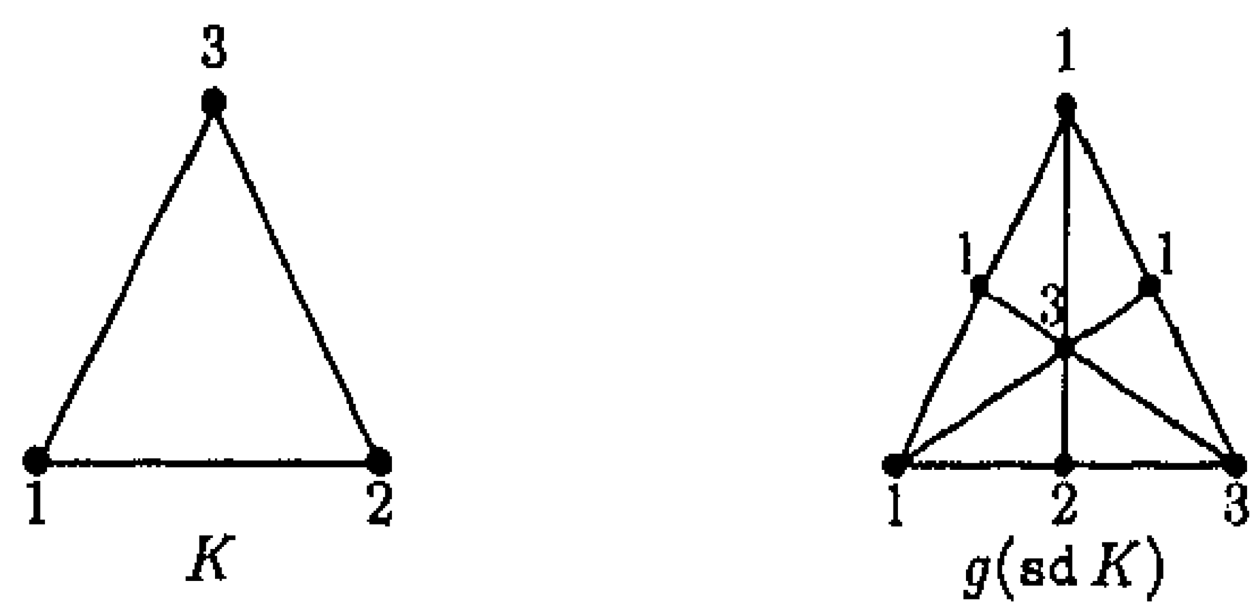


Figure 10: A map which does not preserve expanding directions

Example 5.26 Not all simplicial maps preserve expanding directions. Let, K be the standard 2-simplex with $V(K) = \{1,2,3\}$ and define a simplicial map, $g : \mathrm{sd}\,K \longrightarrow K$ as shown in Figure 10. Then, $\mathrm{Fix}\, g = \{1\}$. Let if possible, $M(1)$ exists. Then, by property (a) of $M(1)$, the simplex $[12, 123]$ must belong to $M(1)$. Then by property (c) of $M(1)$, the simplex $[13, 123]$ must also belong to $M(1)$. This violates property (b) of $M(1)$ since, $g(13) = 1$. Therefore, g does not preserve expanding directions.

CHAPTER 6

FP-EQUIVALENT SIMPLICIAL APPROXIMATIONS

6.1 INTRODUCTION

Examples 5.24 and 5.25, show that for some continuous maps f on a connected compact polyhedron it is possible to obtain a simplicial approximation g to f such that $\mathrm{Fix}\, g \subset \mathrm{Fix}\, f$ and for any fixed point component C of f, $i(f,C) = \sum_{x \in \mathrm{Fix}\, g \cap C} i(g,x)$.

In this chapter we establish sufficient conditions on continuous maps on connected compact polyhedra so that it has a simplicial approximation which satisfies the above properties and also preserves expanding directions. Clearly for such maps one could use the tools developed so far to compute the local indices.

6.2 DEFINITION AND A COUNTEREXAMPLE

Definition 6.1 A simplicial approximation $g : \mathrm{sd}^n K \longrightarrow K, n > 0$ to $f : |K| \longrightarrow |K|$ will be *fixed point equivalent* or in short *fp-equivalent* to f if, there is a neighbourhood W of $\mathrm{Fix}\, f$ such that, $\mathrm{Fix}\, g \cap W \subset \mathrm{Fix}\, f$ and for any fixed point component C of f,

$$i(f,C) = \sum_{x \in \mathrm{Fix}\, g \cap C} i(g,x).$$

It is not always possible to get an fp-equivalent simplicial approximation to f. This can be seen from the following example:

Example 6.2 Let X be the region enclosed by the x axis, y axis and the line $x + y = 1$ in $\mathbb{R}^2$, see Figure 11. Let φ be the curve $\varphi(x) = x^2, x \in [0,1]$.

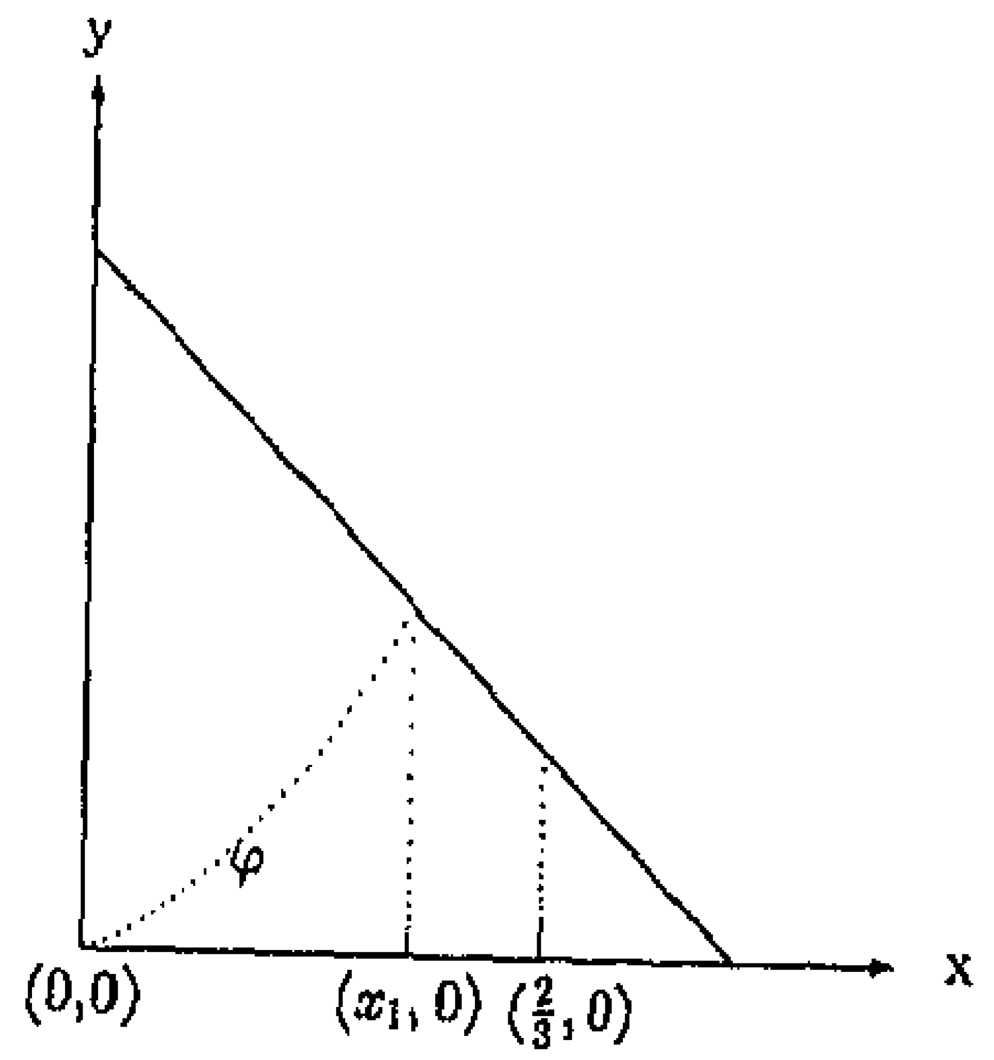


Figure 11: A map with no fp-equivalent simplicial approximation

Let $f : X \longrightarrow X$ be the map defined as follows :

$$f((x,y)) = \begin{cases} (0,0) & \text{if } x^2 \le y \le 1-x \\ (\frac{3}{2}(1-t)x, 0) & \text{if } \begin{cases} y = tx^2, 0 \le t \le 1, \\ 0 \le x \le x_1 = \frac{\sqrt{5}-1}{2} \end{cases} \\ \left(\dfrac{3t(x-x_1)}{2-3x_1} + \dfrac{3}{2}(1-t)x, 0\right) & \text{if } \begin{cases} y = t(1-x), 0 \le t \le 1, \\ x_1 \le x \le \frac{2}{3} \end{cases} \\ (1,0) & \text{if } 0 \le y \le 1-x, \frac{2}{3} \le x \le 1 \end{cases}$$

Then $f : X \longrightarrow \{(x,0) : 0 \le x \le 1\}$ and, $\text{Fix} f = \{(1,0)\} \cup \{(0,0)\}$. Also f is constant in a neighbourhood of $(1,0)$. So $i(f,(1,0)) = 1$. Since $L(f) = 1$, by the additive property of fixed point indices, $i(f,(0,0)) = 0$.

Let K be the standard 2-simplex with $V(K) = \{0,1,2\}$. Then a triangulation of X is the geometric realization of K and the induced triangulation of $\text{Fix} f$ is $[0] \cup [1]$ where, $(0,0)$ corresponds to the geometric realization of $[0]$ and $(1,0)$ to the geometric realization of $[1]$.

Let $f : |\mathrm{sd}^n K| \longrightarrow |K|$ satisfy the star condition. Let w be the vertex of $\mathrm{Lk}\,(0, \mathrm{sd}^n K)$ such that $w \in \langle [0,1] \rangle$. Then the curve $\varphi(t) = t^2$ intersects $\mathrm{st}\,(w, \mathrm{sd}^n K)$ and hence any simplicial approximation to f necessarily maps w to 0. Thus any simplicial approximation to f necessarily maps $\mathrm{Lk}\,(0, \mathrm{sd}^n K)$ to 0. Hence the index of any simplicial approximation to f at 0 has to be 1. Thus no simplicial approximation to f can be fp-equivalent to f.

We will denote the carrier of any point x of $\mathrm{sd}^p K, p \geq 0$ by $\sigma_p(x)$. Recall that if the carrier is a primitive simplex then we denote it be $\sigma_{|p|}(x)$.

Definition 6.3 ([4]) Let $f : |K| \longrightarrow |K|$ be a map. The *proximity set* of f is defined to be,

$$P(f) = \{x \in |K| : f(x) \in \mathrm{st}\,(\sigma_0(x), K)\}$$

A point $x \in P(f)$ will be called a *proximity point* of f.

Remark 6.4 It is possible to use the relative simplicial approximation theorem (see Maunder [16], or Zeeman [25]) to *deform* the map in the above example to obtain one which has a fp-equivalent simplicial approximation. A map $g : |K| \longrightarrow |K|$ is a *fp-deformation of a map* $f : |K| \longrightarrow |K|$ if g is homotopic to f and $\mathrm{Fix}\, g = \mathrm{Fix}\, f$ and the fixed point indices of f and g are the same.

We recall the first step in the construction of the relative simplicial approximation of a map on a polyhedron (see [16] or [25] for full generality and details). Let $f : |K| \longrightarrow |K|$ be a map on a connected compact polyhedron $|K|$ and let L be a subcomplex of K such that $f|_{|L|}$ is simplicial. Let $(K, L)'$ be the barycentric subdivision of K modulo L (see the discussion on Hopf's construction in Section 2 of Chapter 2 of this thesis). Let $\overline{L}$ be the subcomplex of $(K, L)'$ *disjoint* from L i.e. $\overline{L}$ is the subcomplex of $(K, L)'$ consisting of all simplices of $(K, L)'$ which have no face in common with a simplex of L. Let $K^+ = ((K,L)', L \cup \overline{L})'$. Then a vertex of K^+ is either a vertex of $L \cup \overline{L}$ or the barycenter of a simplex of $(K,L)'$ which has a face in L and a face in $\overline{L}$. Define a simplicial approximation $h : K^+ \longrightarrow K$ as follows :

$$\begin{aligned} h(v) &= v \quad \text{if } v \text{ is a vertex of } L \cup \overline{L} \\ h(b(\sigma)) &\subset \overline{\sigma} \cap L \quad \text{otherwise} \end{aligned}$$

The next step is to look at a simplicial approximation of the map $fh : |K^+| \longrightarrow |K|$.

We use this idea in Example 6.2. As above a triangulation of X is the geometric realization of the standard 2-simplex, K, where, $V(K) = \{0,1,2\}$ with $\operatorname{Fix} f = [0] \cup [1]$, where $(0,0)$ corresponds to the geometric realization of $[0]$ and $(1,0)$ to the geometric realization of $[1]$.

Let $L = [K]_{P(f)}$. Then, $L = [0,1]$. Note that $f|_{|L|}$ is not simplicial. Even then we consider the simplicial approximation $h : K^+ \longrightarrow K$ as defined above and described explicitly as follows :

$$\begin{aligned} h(v) &= v \quad \text{if } v \text{ is a vertex of } L \cup \overline{L} \\ h(b(\sigma)) &= 1 \quad \text{if } 1 \in \sigma \\ h(b(\sigma)) &= 0 \quad \text{if } 1 \notin \sigma \text{ and } 0 \in \sigma \end{aligned}$$

Consider the map $f' = fh : |K| \longrightarrow |K|$. Note that $f : |K| \longrightarrow |L|$ and also $f' : |K| \longrightarrow |L|$. Moreover $f = f'$ on $|L|$ and $\operatorname{Fix} f = \operatorname{Fix} f'$. The maps f and f' are homotopic by the straight line homotopy and for all points x of $|K| - \operatorname{Fix} f$, $tf(x) + (1-t)f'(x) \neq x$, for all $0 \leq t \leq 1$. Thus by the homotopy property of the fixed point indices, ([5]) f' is a fp-deformation of f. Note that no point of $\mathbf{st}([0,1], K^+)$ maps to the vertex 0 by the map f'. Thus it is possible to get a fp-equivalent simplicial approximation to f', in fact, we can get such a simplicial approximation with the additional property that it preserves expanding directions.

It is not clear whether this is possible generally or not.

6.3 Stable simplicial approximations

We give a criterion for a simplicial approximation to be fp-equivalent to a given map.

Definition 6.5 A simplicial approximation $g : \mathrm{sd}^n K \longrightarrow K, n > 0$ to $f : |K| \longrightarrow |K|$ is *stable* if, for all component $C \subset |K|$ of $\mathrm{Fix}\, f$, there is a neighbourhood W of C in $|K|$, such that, for all $x \in W - C$,

$$tf(x) + (1-t)g(x) \neq x, t \in [0,1].$$

The neighbourhood W will be called a *stable* neighbourhood of f.

Proposition 6.6 *A simplicial approximation $g : \mathrm{sd}^n K \longrightarrow K, n > 0$ to $f : |K| \longrightarrow |K|$ is stable if and only if, for all component C of $Fix\, f$, there is a neighbourhood W of C in $|K|$ such that, for all points x of $(W - C) \cap P(f)$ there is a vertex v of K satisfying the condition*

$$g(x)(v) - x(v) \neq \frac{|x - g(x)|}{|x - f(x)|}(x(v) - f(x)(v)) \tag{20}$$

Proof: Let, $g : \mathrm{sd}^n K \longrightarrow K, n > 0$ be a simplicial approximation to f which does not satisfy Equation 20 for any neighbourhood of a fixed point component C of f. Then for any neighborhood W of C, there is a $x \in (W - C) \cap P(f)$ such that, for all $v \in V(K)$,

$$g(x)(v) - x(v) = \frac{|x - g(x)|}{|x - f(x)|}(x(v) - f(x)(v))$$

Let $t = 1 + \dfrac{|x - g(x)|}{|x - f(x)|}$. Then $x = \dfrac{1}{t}g(x) + \dfrac{t-1}{t}f(x)$. Hence, g is not a stable simplicial approximation to f.

Conversely, let g not be a stable simplicial approximation to f. Then, there is a fixed point component C of f such that, for all neighbourhood W of C, there is a $x \in W - C$ such that, $x = tf(x) + (1-t)g(x), t \in [0,1)$. This necessarily implies that $x \in P(f)$. If $f(x) = g(x)$, then, $x = f(x)$, a contradiction. So, $f(x) \neq g(x)$. Therefore, $\frac{t}{1-t} = \frac{|x - g(x)|}{|x - f(x)|}$. For all $v \in V(K)$, $x(v) = tf(x)(v) + (1-t)g(x)(v)$ which implies that,

$$g(x)(v) - x(v) = \frac{|x - g(x)|}{|x - f(x)|}(x(v) - f(x)(v)).$$ ■

Corollary 6.7 *Let $g : \mathrm{sd}^n K \longrightarrow K$ be a stable simplicial approximation to f. Then g is fp-equivalent to f.*

Proof: Let W be a stable neighbourhood of $\mathrm{Fix} f$. It is clear from the homotopy property of the local fixed point index ([5]) that, it is enough to show that, $\mathrm{Fix}\, g \cap W \subset \mathrm{Fix}\, f$.

Let $x \in \mathrm{Fix}\, g \cap W$ and if possible, let $x \notin \mathrm{Fix}\, f$. Then, for all $v \in V(K)$,

$$g(x)(v) - x(v) = 0 = \frac{|x - g(x)|}{|x - f(x)|}(x(v) - f(x)(v))$$

This is a contradiction to Proposition 6.6. ■

Remark 6.8 The converse of this result is not true as the following example shows :

Let $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ be the map,

$$f(re^{\imath\theta}) = (r - \varepsilon)e^{\imath\theta}$$
$$f(\underline{0}) = \underline{0}$$

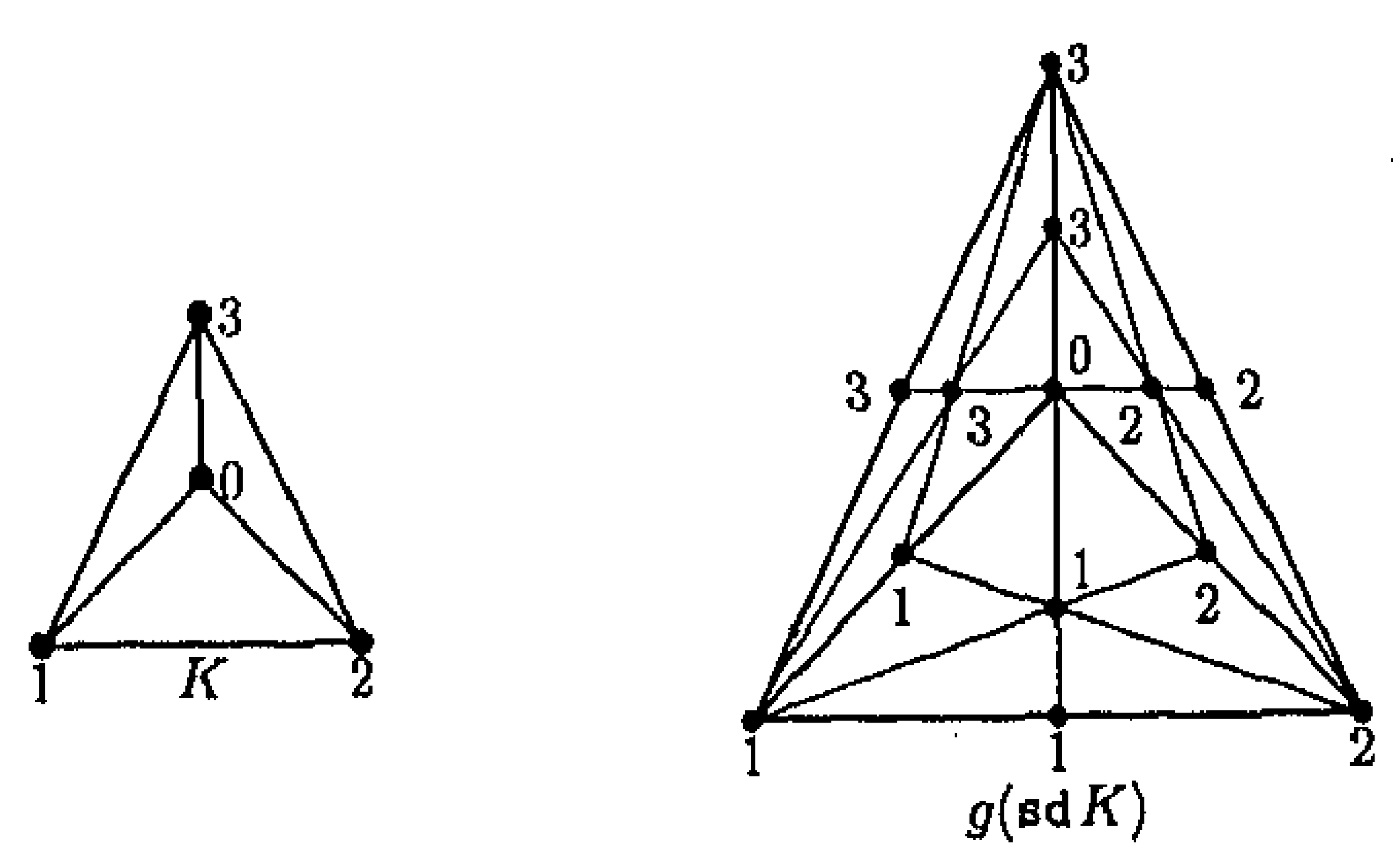


Figure 12: An fp-equivalent simplicial approximation which is not stable

where $\varepsilon > 0$ is sufficiently small, and $\mathrm{Fix}\, f = \{\underline{0}\}$. A triangulation K of $\mathbb{D}^2$ is shown in Figure 12. The vertex 0 is the geometric realization of $\underline{0}$. Let $g : \mathrm{sd}\, K \longrightarrow K$ be a simplicial approximation to f as shown in Figure 12. It is clear from Figure 12 that $\mathrm{Fix}\, g = \{0, 1, 2, 3\}$. If $W = \mathrm{st}\,(0, \mathrm{sd}\, K)$, then $\mathrm{Fix}\, g \cap W = \{0\}$ and also $i(f, 0) = i(g, 0) = 1$. But g is not a stable approximation to f. In fact all points of $\langle [0, 0k] \rangle; k = 1, 2, 3$ are fixed by the homotopy between f and g.

Let, $g : \mathrm{sd}^n K \longrightarrow K$ be a simplicial approximation to $f : |K| \longrightarrow |K|$. Let x be a fixed point of g. Since there is a simplex τ of K such that $f(x) \in \langle \tau \rangle$ and $g(x) \in \overline{\tau}$, it follows that $f(x) \in \mathrm{st}\,(\sigma_0(x), K)$, i.e. x is a proximity point of f.

We have the following sufficient condition on a map to have an fp-equivalent simplicial approximation.

Lemma 6.9 *Let $f : |K| \longrightarrow |K|$ be a map such that $P(f) = \mathit{Fix}\, f$. Then any simplicial approximation to f is fp-equivalent to f.*

Proof: Let $n \geq 0$ be such that for any two distinct fixed point components C and C' of f,

$$\overline{\mathrm{N}}\,([\mathrm{sd}^n K]_C, \mathrm{sd}^n K) \cap \overline{\mathrm{N}}\,([\mathrm{sd}^n K]_{C'}, \mathrm{sd}^n K) = \emptyset$$

and $f : |\mathrm{sd}^n K| \longrightarrow |K|$ satisfies the star condition.
Let $g : \mathrm{sd}^n K \longrightarrow K, n \geq 0$ be a simplicial approximation to f and let C be a fixed point component of f. Let $x \in \mathrm{N}\,([\mathrm{sd}^n K]_C, \mathrm{sd}^n K)$. Suppose that there is a $t \in [0, 1]$ such that $x = tg(x) + (1 - t)f(x)$. This implies that, $x \in \overline{\sigma}_0(f(x))$ or in other words, $f(x) \in \mathrm{st}\,(\sigma_0(x), K)$ i.e. $x \in P(f)$. Since $x \in \mathrm{N}\,([\mathrm{sd}^n K]_C, \mathrm{sd}^n K)$, it follows that $x \in C$. Therefore, g is a stable simplicial approximation to f. Hence, by Corollary 6.7, it follows that g is an fp-equivalent simplicial approximation to f. ∎

Remark 6.10 Let $f : |K| \longrightarrow |K|$ be a map such that $\mathrm{Fix}\, f$ is a set of isolated points, $\mathrm{Fix}\, f = \{x_1, \ldots, x_m\}$ and let $P(f) = \mathrm{Fix}\, f$. Assume that for any $1 \leq i \neq j \leq m$, $\mathtt{st}\,(\sigma_{|0|}(x_i), K) \cap \mathtt{st}\,(\sigma_{|0|}(x_j), K) = \emptyset$.
Let $g : \mathtt{sd}^n K \longrightarrow K$ be a simplicial approximation to f. Then for all x_j, a subcomplex expanded by g is the empty subcomplex. This follows from the following fact : Let τ be a simplex of $\mathrm{Lk}\,(\sigma_{|n|}(x_j), \mathtt{sd}^n K) * \dot{\sigma}_n(x_j)$ such that $\langle \sigma_{|n|}(x_j) \cup \tau \rangle \subset \langle \sigma_{|0|}(x_j) \cup g(\tau) \rangle$ for some $1 \leq j \leq m$. Let y be a point of $\langle \sigma_{|n|}(x_j) \cup \tau \rangle$. Then y as well as $g(y)$ are points of $\langle \sigma_{|0|}(x_j) \cup g(\tau) \rangle$. Hence, $\sigma_0(y) = \sigma_{|0|}(x_j) \cup g(\tau)$ and $f(y) \in \mathtt{st}\,(\sigma_0(y), K)$. Thus any point of $\langle \sigma_{|n|}(x_j) \cup \tau \rangle$ is a proximity point of f. By hypothesis this implies that any point of $\langle \sigma_{|n|}(x_j) \cup \tau \rangle$ is a fixed point of f which leads to a contradiction since $\mathrm{Fix}\, f \cap \mathtt{st}\,(\sigma_{|n|}(x_j), \mathtt{sd}^n K) = \{x_j\}$. Thus if we take $M(x_j) = \emptyset$ then it satisfies all the properties (a), (b) and (c) of Definition 5.6. Therefore it is clear in this case that $i(f, x_j) = 1$ for all $1 \leq j \leq m$.

Remark 6.11 An fp-equivalent simplicial approximation to f need not preserve expanding directions. An example to this is given by Example 5.26. Clearly in that example g is an fp-equivalent simplicial approximation to g but it does not preserve expanding directions.

We wish to find conditions on a map on a polyhedron such that it has an fp-equivalent simplicial approximation which preserves expanding directions. This would enable us to compute the local indices of the map.

From now on we restrict our attention to maps on $|K|$, whose fixed point set is a subpolyhedron of $|K|$.

6.4 A WEAKLY HYPERBOLIC SIMPLICIAL APPROXIMATION TO IDENTITY

Any simplicial approximation to the identity map on $|K|$ is stable, the stable neighbourhood being, $|K|$. Therefore, by Corollary 6.7, any simplicial approximation to the identity map $1|_{|K|}$ on $|K|$ is an fp-equivalent simplicial approximation.

Proposition 6.12 *For all $n \geq 1$, there are simplicial approximations $\varphi : \mathrm{sd}^n K \longrightarrow K$ to the identity map on $|K|$ such that $Fix\varphi = V(K)$.*

Proof: For all simplices σ of K, choose a vertex v_σ of σ. Define,

$$\varphi(w) = v_\sigma, \text{ for all vertex } w \in \mathrm{sd}^n K, \text{ such that } w \in \langle\sigma\rangle$$

The map $\varphi : V(\mathrm{sd}^n K) \longrightarrow V(K)$, clearly extends to a simplicial approximation to $1|_{|K|}$.

Let u be a vertex of K. Then $v_u = u$, and hence, $\varphi(u) = u$. Therefore, $V(K) \subset \mathrm{Fix}\,\varphi$.

Let $x \in |K|$ be a fixed point of φ. Then, the carrier of x in $\mathrm{sd}^n K$ is a primitive simplex with respect to K by Lemma 3.11. Let $\sigma_{[0]}(x) = [u_0, \ldots, u_p]$ and, $\sigma_{[n]}(x) = [w_0, \ldots, w_p]$, where, $w_i \in \langle\tau_i\rangle$, and, $\tau_i \prec \sigma_{[0]}(x), 0 \leq i \leq p$. If possible let there exist a $i, 1 \leq i \leq p$ such that $\{\tau_{i-1} : \tau_i\} \neq 1$. Since $\tau_p = \sigma_{[0]}(x)$, it follows that there is a $j, 0 \leq j \leq p-1$ such that $\tau_j = \tau_{j+1}$. But this implies that $\varphi(w_j) = \varphi(w_{j+1})$ a contradiction.

Hence for all $1 \leq j \leq p, \{\tau_{j-1} : \tau_j\} = 1$. Then, it follows that each τ_i is an i-simplex of K. But then τ_0 is a 0-simplex. Therefore, by Proposition 3.13, it follows that $p = 0$ and $x = \tau_0$.

Therefore, $\mathrm{Fix}\,\varphi \subset V(K)$. Hence, $\mathrm{Fix}\,\varphi = V(K)$. ∎

Remark 6.13 This fp-equivalent simplicial approximation need not preserve expanding directions. A counterexample is as follows :

Let K be the standard 2−simplex with $V(K) = \{1, 2, 3\}$ and define,

$$\varphi : \mathrm{sd}\, K \longrightarrow K$$

as shown in Figure 13. Then, $\mathrm{Fix}\,\varphi = \{1, 2, 3\}$. As in Example 5.26, there is no subcomplex expanded by φ at 1. Therefore, g does not preserve expanding directions.

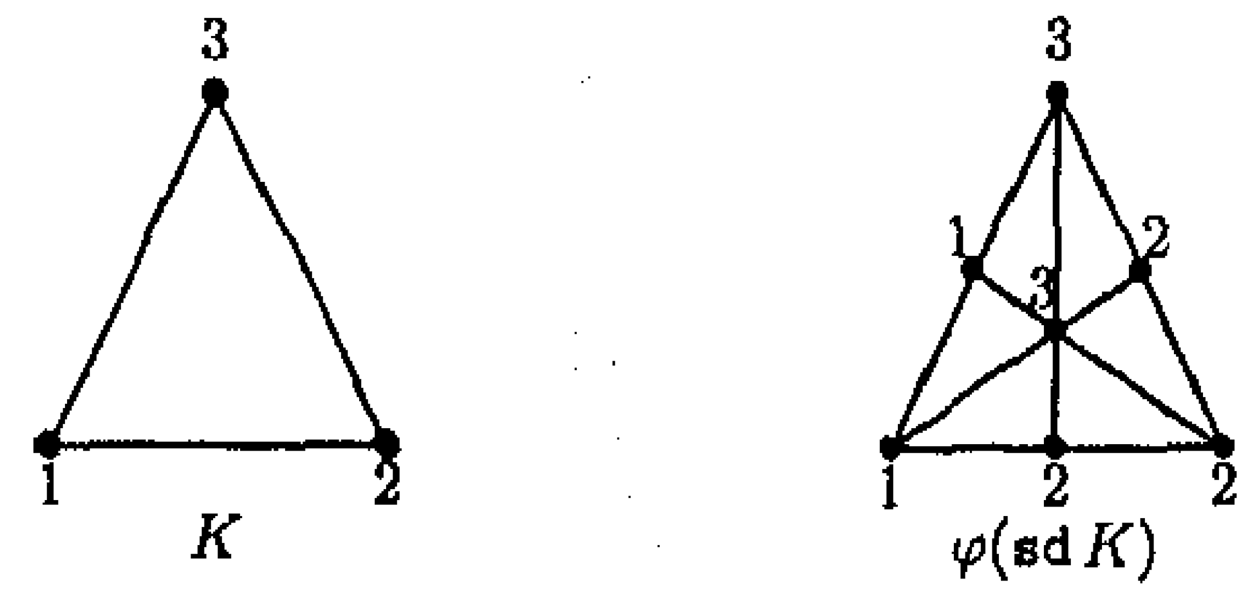


Figure 13: An fp-equivalent simplicial approximation to identity which does not preserve expanding directions

We show that it is possible to get fp-equivalent simplicial approximations, $\mathrm{sd}^n K \longrightarrow K, n \geq 1$ to $1|_{|K|}$ which preserve expanding directions.

Let K be any simplicial complex and $s \geq 0$.
Let $\leq$ be the order on $V(K)$ mentioned in Section 4 of Chapter3.
For any simplex $\tau = [u_0 < \cdots < u_p]$ of $\mathrm{sd}^{s+1}K$ define,

$$\psi_{s+1}(b(\tau)) = u_p$$

Then $\psi_{s+1} : V(\mathrm{sd}^{s+1}K) \longrightarrow V(\mathrm{sd}^s K)$ extends to a simplicial approximation to identity, $\psi_{s+1} : \mathrm{sd}^{s+1}K \longrightarrow \mathrm{sd}^s K$. Clearly ψ_{s+1} is a particular case of the simplicial maps defined in Proposition 6.12 and hence $\mathrm{Fix}\,\psi_{s+1} = V(\mathrm{sd}^s K)$.
Note that for any simplex τ of $\mathrm{sd}^{s+1}K$, if $\sigma \in \mathrm{sd}^s K$ is such that $\langle\tau\rangle \subset \langle\sigma\rangle$, then, $\psi_{s+1}(\tau) \prec \sigma$.

Definition 6.14 Define a simplicial approximation to the identity map on $|K|$ to be,

$$\psi(n) = \psi_1 \circ \cdots \circ \psi_n : \mathrm{sd}^n K \longrightarrow K.$$

It is clear from the definition of $\psi(n)$ that $V(K) \subset \mathrm{Fix}\,\psi(n)$. If we can show that $\psi(n)$ is a particular case of maps defined in Proposition 6.12 then we will have shown that $\mathrm{Fix}\,\psi(n) = V(K)$.

Lemma 6.15 *Let w be a vertex of* $\mathrm{sd}^n K, n > 0$. *Then $\psi(n)(w)$ is the largest vertex of $\sigma_0(w)$.*

Proof: Let $n = 1$. In this case, $w = b(\sigma_0(w))$.
Hence, $\psi(1)(w) = \psi_1(b(\sigma_0(w)))$ = the largest vertex of $\sigma_0(w)$ by definition. Let the result be true for $1 \leq p < n$ and for all simplicial complexes. Let w be a vertex of $\mathrm{sd}^n K$. Then $\psi_n(w)$ = largest vertex of $\sigma_{n-1}(w)$. By Lemma 3.8, $\psi_n(w) \in \langle \sigma_0(w) \rangle$ and hence by the induction hypothesis,

$$\psi(n-1)\{\psi_n((w))\} = \text{largest vertex of } \sigma_0(w)$$

i.e. $\psi(n)(w)$ = largest vertex of $\sigma_0(w)$. ∎

Thus for all simplices σ of K if v_σ is the largest vertex of σ,

$$\psi(n)(w) = v_\sigma, \ \forall \text{ vertex } w \in \mathrm{sd}^n K, w \in \langle \sigma \rangle$$

Hence, $\mathrm{Fix}\,\psi(n) = V(K)$.

Lemma 6.16 *Let v be a vertex of K and a be a vertex of* $\mathrm{Lk}(v, \mathrm{sd}^p K)$. *Let $\sigma_0(a) = [v, w_1, \ldots, w_q]$. Assume that there exist vertices u_j of* $\mathrm{Lk}(v, \mathrm{sd}^p K)$ *such that $\psi(p)(u_j) = w_j$. Then, $\psi(p)(a) \neq v$.*

Proof: By the given hypothesis and Lemma 6.15, it follows that each $w_i, 1 \leq i \leq q$ is the largest vertex of $\sigma_0(u_i)$. Since $u_i \in \mathrm{Lk}(v, \mathrm{sd}^p K)$, v is a vertex of $\sigma_0(u_i)$ for all $1 \leq i \leq q$. Therefore each vertex $w_i, 1 \leq i \leq q$ is larger than v and hence again by Lemma 6.15 it follows that $\psi(p)(a) \neq v$. ∎

Proposition 6.17 *Let $p > 0$. Then $\psi(p)$ is weakly hyperbolic.*

Proof: Let $v \in V(K)$. Define,

$$\begin{aligned}
t(w) &= (1,0) \quad \text{if} \psi(p)(w) \neq v, w \in \mathrm{Lk}(v, \mathrm{sd}^p K) \\
t(w) &= (0,1) \quad \text{if} \psi(p)(w) = v, w \in \mathrm{Lk}(v, \mathrm{sd}^p K) \\
t(v) &= (0,0)
\end{aligned}$$

Extend this linearly over all simplices of $\overline{\mathtt{st}}\,(v, \mathtt{sd}^p K)$ to get a map,

$$t : |\overline{\mathtt{st}}\,(v, \mathtt{sd}^p K)| \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}.$$

Let x be a point of $\mathtt{st}\,(v, \mathtt{sd}^p K)$. Let $\sigma_p(x) = [v, w_1, \ldots, w_q]$ and,

$$x = r_0 v + \sum_{i=1}^{q} r_i w_i, \text{ where for all } 0 \leq i \leq q, 0 \leq r_i \leq 1, \sum_{i=0}^{q} r_i = 1, r_0 \neq 0.$$

Let $\psi(w_1) = \cdots = \psi(w_u) = v, u \leq q$ and for all $q \geq j > u, \psi(w_j) \neq v$. Then,

$$t(x) = \left(\sum_{j=u+1}^{q} r_j \ , \ \sum_{j=1}^{u} r_j \right).$$

So, $t(x) = (0,0)$ implies that, $x = v$.
Also,

$$\psi(x) = \left[\sum_{j=0}^{u} r_j \right] v + \sum_{j=u+1}^{q} r_j \psi(w_j).$$

Let $\psi(x) \in \mathtt{st}\,(v, \mathtt{sd}^p K)$ and let, $\sigma_p(\psi(x)) = [v, a_1, \ldots, a_y]$. Let

$$\psi(x) = r'_0 v + \sum_{j=1}^{y} r'_j a_j, \text{ where for all } 0 \leq j \leq y, 0 \leq r'_j \leq 1, \sum_{j=0}^{y} r'_j = 1, r'_0 \neq 0.$$

Each $a_j \in [v, \psi(w_{u+1}), \cdots, \psi(w_q)]$. Hence by Lemma 6.16, $\psi(a_j) \neq v$ for all $1 \leq j \leq y$. Therefore,

$$t(\psi(x)) = \left(\sum_{j=1}^{y} r'_j \ , \ 0 \right).$$

Thus $t_2(\psi(x)) = 0 \leq t_2(x)$. To show that t is an indicator map on $\mathtt{st}\,(v, \mathtt{sd}^p K)$, all we now need to show is that ,

$$\sum_{j=1}^{y} r'_j \geq \sum_{j=u+1}^{q} r_j.$$

Let $a_i = c_{i0} v + \sum_{j=u+1}^{q} c_{ij} \psi(w_j)$. Then,

$$\psi(x) = \left\{ r'_0 + \sum_{i=1}^{y} r'_i c_{i0} \right\} v + \sum_{j=u+1}^{q} \left\{ \sum_{i=1}^{y} r'_i c_{ij} \right\} \psi(w_j).$$

Thus,

$$r'_0 + \sum_{i=1}^{y} r'_i c_{i0} = \sum_{i=0}^{u} r_i = 1 - \sum_{i=u+1}^{q} r_i.$$

Therefore,

$$\sum_{i=1}^{y} r'_i c_{i0} = 1 - r'_0 - \sum_{i=u+1}^{q} r_i = \sum_{j=1}^{y} r'_j - \sum_{i=u+1}^{q} r_i.$$

The result now follows since $\sum_{i=1}^{y} r'_i c_{i0} \geq 0$. ∎

By Theorem 5.21 it follows that $\psi(p)$ preserves expanding directions.

6.5 Continuous maps which preserve expanding directions

Definition 6.18 A map $f : |K| \longrightarrow |K|$, will be said to *preserve expanding directions* if there is an fp-equivalent simplicial approximation to f which preserves expanding directions at each of its fixed points which is also a fixed point of f.

Theorem 6.19 *Let the proximity set of a map $f : |K| \longrightarrow |K|$ be precisely $\mathrm{Fix}\, f = F$ and let for all simplex σ of $\mathrm{N}(F, K)$, $\sigma' = \sigma \cap F$. If for all simplex σ of $\mathrm{N}(F, K)$,*

$$f(\langle\sigma\rangle) \cap \{\dot{\sigma}' * \mathrm{Lk}(\sigma', K)\} = \emptyset,$$

then f preserves expanding directions.

Proof: By Lemma 6.9, any simplicial approximation to f is a stable simplicial approximation to f.

Let $f : |\mathrm{sd}^n K| \longrightarrow |K|$ satisfy the star condition. Let w be a vertex of $\mathrm{sd}^n F$ and τ be the maximal simplex of $\sigma_0(w)$ such that,

$$f(\mathrm{st}\,(w, \mathrm{sd}^n K)) \subset \mathrm{st}\,(\tau, K).$$

If $\tau \neq \sigma_0(w)$, then there is a point y in $\mathrm{st}\,(\sigma_0(w), K)$ such that $f(y)$ belongs to $\tau * \mathrm{Lk}\,(\sigma_0(w), K)$. This contradicts the assumption on f. Therefore for any vertex w of $\mathrm{sd}^n F$,

$$f(\mathrm{st}\,(w, \mathrm{sd}^n K)) \subset \mathrm{st}\,(\sigma_0(w), K). \qquad (21)$$

We know that the restriction of any simplicial approximation to f to $|F|$ is a simplicial approximation to identity on $|F|$, ([22]). By Equation 21, it follows that we can choose a simplicial approximation to f, such that its restriction to $|F|$ is any simplicial approximation to identity. Hence we can choose a simplicial approximation $g : \mathrm{sd}^n K \longrightarrow K$ to f such that $g|\mathrm{sd}^n F = \psi(n)$ where $\psi(n)$ is the simplicial approximation to identity defined in Definition 6.14. Then in a stable neighbourhood of F,

$$\mathrm{Fix}\, g = V(F)$$

which implies that $\mathrm{Fix}\, g = V(F)$ on $|K|$.

Let v be a vertex of F. Suppose that, τ is a simplex of $\mathrm{Lk}(v, \mathrm{sd}^n K)$ such that, $\langle v \cup \tau \rangle \subset \langle v \cup g(\tau) \rangle$. Then any point of $\langle v \cup \tau \rangle$ is a proximity point of f which implies that τ is a simplex of $\mathrm{sd}^n F$.

~~We saw in Definition 6.14~~ By Proposition 6.17 that $g|\mathrm{sd}^n F$ preserves expanding directions and hence there is a subcomplex $M(v)$ at v expanded by $g|\mathrm{sd}^n F$. By the above discussion it is clear that $M(v)$ is expanded by g also. Thus f preserves expanding directions. ∎

Let $f : |K| \longrightarrow |K|$ be a map whose fixed point set is a subpolyhedron $|F|$ of $|K|$. It is clear then that the number of path components of $\mathrm{Fix} f$ in this case is finite. Let C be a path component of F.

Let $x \in \mathrm{N}(C, K) - |C|$. If for all $m > 0$ such that $x \notin \mathrm{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, $f(x)$ also does not belong to $\mathrm{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, then x *defines a direction expanding* by f with respect to C. The map f is *expanding* if for all components C of F, all points of $|\mathrm{N}(C, K)| - |C|$ define a direction expanding by f with respect to C.

On the other hand, let $x \in \mathrm{N}(C, K) - |C|$. Then there is a $m \geq 1$ such that $x \notin \mathrm{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, but $x \in \mathrm{N}(\mathrm{sd}^{m-1} C, \mathrm{sd}^{m-1} K)$. If for such a m, $f(x)$ is a point of $\mathrm{N}(\mathrm{sd}^m C, \mathrm{sd}^m K)$, then x *defines a direction contracting* by f with respect to C. The map f is *contracting* if for all components C

of F, all points of $|\mathtt{N}\,(C,K)| - |C|$ define a direction contracting by f with respect to C.

The set of all points in $\mathtt{N}\,(C,K)$ which define a direction expanding by f with respect to C will be denoted by $E(f,C)$.

For brevity we shall say that a point x of $\mathtt{N}\,(C,K) - |C|$ is an *expanding point* (*contracting point* respectively) if x defines a direction expanding (contracting respectively) by f with respect to C.

Theorem 6.20 *Let F be contained in the interior of the proximity set of f. Suppose that for any component C of F and for any simplex σ of $\mathtt{N}\,(C,K)$, the following holds :*

$$\langle\sigma\rangle \cap E(f,C) \neq \emptyset \Rightarrow \{|\overline{\sigma}| - |\overline{\sigma} \cap C|\} \subset E(f,C).$$

Then f preserves expanding directions.

Proof: Choose $n > 0$ such that, $f : |\mathtt{sd}^n K| \longrightarrow |K|$ has a simplicial approximation and,

$$\mathtt{N}\,(\overline{\mathtt{N}}\,(\mathtt{sd}^n F, \mathtt{sd}^n K), \mathtt{sd}^n K) \subset \text{interior } P(f).$$

Let w be a vertex of $\overline{\mathtt{N}}\,(\mathtt{sd}^n F, \mathtt{sd}^n K)$. Then w is a proximity point of f. Let $y \in \mathtt{st}\,(w, \mathtt{sd}^n K)$. Then y is a also a proximity point of f and y is a point of $\mathtt{st}\,(\sigma_0(w), K)$. Therefore, $\sigma_0(w) \prec \sigma_0(y)$. Hence,

$$f(y) \in \mathtt{st}\,(\sigma_0(y), K) \subset \mathtt{st}\,(\sigma_0(w), K).$$

Therefore, for all vertices w of $\overline{\mathtt{N}}\,(\mathtt{sd}^n F, \mathtt{sd}^n K)$,

$$f(\mathtt{st}\,(w, \mathtt{sd}^n K)) \subset \mathtt{st}\,(\sigma_0(w), K).$$

We choose a simplicial approximation $g : \mathtt{sd}^n K \longrightarrow K$ to f such that,

$$g|\mathtt{sd}^n F = \psi(n) : \mathtt{sd}^n F \longrightarrow F$$

is the simplicial approximation to identity defined in Definition 6.14 and, for any vertex w of $\mathrm{Lk}\,(\mathrm{sd}^n F, \mathrm{sd}^n K)$, $g(w) \in \mathrm{Lk}\,(F, K)$, if w is an expanding point of f and $g(w)$ is a vertex of F if w is not an expanding point of f. We also ensure that for all expanding vertices w, $g(w)$ is a vertex of $\sigma_0(w)$.

A point x of $|K|$ can lie on the line segment joining $g(x)$ and $f(x)$ only if x is a proximity point of f. Let x be a proximity point of f belonging to $\mathrm{Lk}\,(\mathrm{sd}^n F, \mathrm{sd}^n K)$.

Let x be an expanding point of f. Then by the hypothesis on f all points in the interior of $\sigma_0(x)$ which do not belong to F are expanding points of f. It then follows by the continuity of f that any vertex of $\mathrm{Lk}\,(\mathrm{sd}^n F, \mathrm{sd}^n K)$ which lies on $\overline{\sigma}_0(x)$ has to be an expanding point of f. All vertices of $\sigma_n(x)$ belong to $\overline{\sigma}_0(x)$. Thus by the given hypothesis on f, all vertices of $\sigma_n(x)$ are expanding points of f and hence by the choice of g, $g(x)$ is a point of $\mathrm{Lk}\,(F, K)$. Hence, x does not lie on the line segment joining $f(x)$ and $g(x)$.

Now let x not an expanding point of f. Then again no point of $\sigma_n(x)$ which lies in the interior of $\sigma_0(x)$ is an expanding point of f by the hypothesis on f. Since f is continuous this implies that no vertex of $\sigma_n(x)$ is an expanding point of f. So, $f(x)$ lies in $\mathrm{N}\,(\mathrm{sd}^n F, \mathrm{sd}^n K)$ and $g(x)$ is a point of $|F|$. Thus again, x does not lie on the line segment joining $f(x)$ and $g(x)$.

Thus by the homotopy property of the fixed point indices ([5]), for any component C of F, it follows that,

$$i(f, \mathrm{N}\,(\mathrm{sd}^n C, \mathrm{sd}^n K)) = i(g, \mathrm{N}\,(\mathrm{sd}^n C, \mathrm{sd}^n K)).$$

We now show that for any component C of $\mathrm{Fix}\, f$,

$$\mathrm{Fix}\, g \cap \overline{\mathrm{N}}\,(\mathrm{sd}^n C, \mathrm{sd}^n K) = V(C).$$

Let σ be a simplex of $\mathrm{N}\,(\mathrm{sd}^n C, \mathrm{sd}^n K)$ such that $\langle \sigma \rangle \subset \langle g(\sigma) \rangle$.
Let $\sigma \cap \mathrm{sd}^n C = \sigma'$.

Assume that σ' does not belong to $\mathtt{st}\,(v, \mathtt{sd}^n K)$ for any vertex v of F. Let $\tau = g(\sigma)$ (respectively τ') be the carrier of σ (respectively σ') in K. Let if possible, $\dim \sigma' = \dim \tau'$. Then, there are two distinct vertices of σ' which belong to the interior of a face of τ'. By definition of $\psi(n)$ this implies that σ' collapses by g, a contradiction to the fact that $\langle\sigma\rangle \subset \langle g(\sigma)\rangle$.
Therefore, $\dim \sigma' < \dim \tau'$. This implies that,

$$\dim(\sigma - \sigma') > \dim(\tau - \tau') > \dim \tau'',$$

where $\tau'' = \tau \cap \mathrm{Lk}\,(C, K)$. Since at least one vertex of $\sigma - \sigma'$ maps to a vertex of τ'' by the map g, it follows that all the vertices of $\sigma - \sigma'$ necessarily map to vertices of τ'' by the map g, by the hypothesis on f and the choice of g. This implies that σ collapses by g, again a contradiction.
Thus, $\langle\sigma\rangle \subset \langle g(\sigma)\rangle$ implies that σ' belongs to $\mathtt{st}\,(v, \mathtt{sd}^n K)$ for some vertex v of F. This implies that the only fixed point of g on $\overline{\sigma}$ can be the vertex v, by Proposition 3.13.

Hence g is an fp-equivalent simplicial approximation to f such that,

$$\mathrm{Fix}\, g \cap \overline{\mathrm{N}}\,(\mathtt{sd}^n F, \mathtt{sd}^n K) = V(F).$$

Let C be a component of F. Then,

$$i(f, C) = \sum_{v \in V(C)} i(g, v).$$

Finally we show that g preserves expanding directions at each vertex v of $\mathrm{Fix}\, f$.
Let v be a vertex of C and let a subcomplex at v expanded by $g|\mathtt{sd}^n C$ be $M'(v)$. Let,

$$\begin{aligned} M(v) \;&=\; \cup\{\overline{\sigma} : \sigma \in \mathrm{Lk}\,(v, \mathtt{sd}^n K), \sigma \in M'(v), \\ &\qquad \mathrm{or} \langle\sigma\rangle \cap E(f, C) \neq \emptyset \text{ and } v \notin g(\sigma)\}. \end{aligned}$$

We show that M is a subcomplex expanded by g at v.
Let σ be a simplex of $\mathrm{Lk}\,(v, \mathtt{sd}^n K)$ such that, $\langle v \cup \sigma\rangle \subset \langle v \cup g(\sigma)\rangle$. If $\sigma \in \mathtt{sd}^n C$, then, $\sigma \in M'(v)$ and hence σ is a simplex of $M(v)$. Let σ not

a simplex of $\mathrm{sd}^n C$. If possible let there be a vertex of $\sigma - (\sigma \cap C)$ which maps to a vertex of C. Then, by the hypothesis on f and by the choice of g, it follows that no vertex of σ can be an expanding point of f and hence, $g(\sigma)$ is a simplex of C. But this implies that σ is a simplex of $\mathrm{sd}^n C$, a contradiction. Therefore all vertices of $\sigma - (\sigma \cap C)$ map to $\mathrm{Lk}(C, K)$ and hence $\langle\sigma\rangle \cap E(f, C) \neq \emptyset$. If a vertex of σ maps to v by the map g, then, $\dim(v \cup \sigma) > \dim(v \cup g(\sigma))$, which is a contradiction. So, no vertex of σ maps to v. Thus, σ is a simplex of $M(v)$. Hence, $M(v)$ satisfies property (a) of Definition 5.6.

Let w be a vertex of $M(v)$. If it is a vertex of $M'(v)$, then it does not map to v by g. If it is not a vertex of $M'(v)$ then by the condition on $M(v)$, it does not map to v by g. So, $M(v)$ satisfies property (b) of Definition 5.6.

Finally let σ be a simplex of $M(v)$. If σ is a simplex of $M'(v)$ then,

$$|v \cup g(\sigma)| \cap |\mathrm{Lk}(v, \mathrm{sd}^n K)| \subset |M(v)|.$$

Assume that σ is not a simplex of $M'(v)$. Then, $\langle\sigma\rangle \cap E(f, C) \neq \emptyset$. Let the carrier of σ in K be τ and $\tau \cap C = \tau'$. Then all vertices of $\tau - \tau'$ are expanding points of f and by the choice of g, $g : |\tau - \tau'| \cap |\mathrm{Lk}(v, \mathrm{sd}^n K)| \longrightarrow \overline{\tau}$. It is clear that, $|v \cup g(\sigma)| \cap |\mathrm{Lk}(v, \mathrm{sd}^n K)| \subset |M(v)|$. Hence, $M(v)$ satisfies property (c) of Definition 5.6 also and thus $M(v)$ is a subcomplex at v expanded by g. Thus, g preserves expanding directions and hence, f preserves expanding directions. ∎

Corollary 6.21 *An expanding or contracting map on a connected compact polyhedron, whose fixed point set is a subpolyhedron contained in the interior of the proximity set of the map, preserves expanding directions.*

Note : The above theorem justifies the nomenclature of such maps as those which preserve expanding directions.

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