WIENER TAUBERIAN THEOREMS ON SEMISIMPLE LIE GROUPS

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Contents

Introduction

In their celebrated study of Harmonic analysis on semi-simple Lie groups Ehrenpreis and Mautner [E-M] noticed that the analogue of the classical Wiener Tauberian theorem resting on the unitary dual does not hold for semisimple Lie groups. A simple proof of this fact due to M. Duflo appears in [H]. Ehrenpreis and Mautner went on in [E-M] to formulate the problem on the commutative Banach algebra of the $SO_2(\mathbb{R})$-biinvariant functions in $L^1(SL_2(\mathbb{R}))$, and obtained two different versions of the theorem involving, this time, the dual of the Banach algebra which includes, beside the unitary dual of G, a part of the non-unitary dual as well. They considered the problem of a single function f generating the Banach algebra of bi-invariant functions as a closed ideal. Among the many articles inspired by [E-M] in the intervening years [S1], [S2], and [B-W] are of particular importance in our context. In [S1], Sitaram proved one of the theorems of Ehrenpreis-Mautner, in the setting of a connected semisimple Lie groups G with a finite centre in place of $SL_2(\mathbb{R})$. In [S2] he considers a function f in $L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$ which is of finite type under left $SO_2(\mathbb{R})$ action to get a sufficient conditions for f to generate a dense subspace of $L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$ under left convolution by $L^1(SL_2(\mathbb{R}))$ functions. In a recent paper of Benyamini and Weit [B-W], such sufficient conditions are obtained on a family of bi-invariant functions instead of on a single function, so that the closed ideal generated by them is the entire algebra of bi-invariant functions in $L^1(SL_2(\mathbb{R}))$.

In this thesis we obtain Wiener Tauberian (W-T) theorems for the whole space $L^1(SL_2(\mathbb{R}))$ as well as for $L^p(SL_2(\mathbb{R}))$ for $1 \leq p \leq 2$ and a W-T theorem for $L^p(G/K)$ when G is a connected semisimple Lie group of real rank one with finite centre. Along the way, we examine some related questions as also some of the earlier proofs of the available W-T theorems. Our treatment relies heavily on the results of [B-W] and on the characterization of Fourier transforms of the Schwartz spaces $C^p(G)$ obtained by Trombi [T] and Barker [Ba].

Throughout this thesis G will denote a semisimple Lie group and K will be a maximal compact subgroup of G. We begin with $G = SL_2(\mathbb{R})$ and $K = SO_2(\mathbb{R})$. We denote the characters of K by χ_n, $n \in \mathbb{Z}$. A complex valued function f on G is said to be of left (resp. right) K-type n if $f(kx) = \chi_n(k)f(x)$ (resp. $f(xk) = \chi_n(k)f(x)$), for all $k \in K$ and $x \in G$. For a class of functions $\mathcal{F}$ on G (e.g. $L^p(G)$), $\mathcal{F}_n$ will denote the corresponding subclass of functions of right type n while $\mathcal{F}_{m,n}$ will comprise funtions in $\mathcal{F}_n$ which are also of left type m. The principal series representations of $SL_2(\mathbb{R})$ are parametrized by (σ, λ), where $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{C}$, and the discrete series representations are parametrized by the integers. Principal and discrete parts of the Fourier transform of a function f will be denoted by $\widehat{f}_H$ and $\widehat{f}_B$ respectively. Unless mentioned otherwise p will lie in $[1, 2)$.

For each p let $\gamma = \frac{2}{p} - 1$ and define S^γ by

$$S^\gamma = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq 2/p - 1\}.$$

S^γ_δ denotes the augmented strip $\{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq \gamma + \delta\}$ for $\delta > 0$. Let Γ_n denote the integers between 0 and n of parity opposite to n. Then for $f \in L^p(G)_n$ (equivalently, for an L^p-section of a line bundle over G/K corresponding to n) the natural domain of the continuous part of the Fourier transform $\widehat{f}_H$ is S^γ while that of the discrete part $\widehat{f}_B$ is Γ_n.

$C^p(G)$ is our notation for the L^p-schwartz space and $C^p(\widehat{G})$ is the image of $C^p(G)$ under the Fourier transform. The Schwartz spaces are dense in the respective L^p-spaces. Through the works of Harish-Chandra [H], Trombi-Varadarajan [T-V] and others on semisimple Lie groups these have been projected as the appropriate spaces for harmonic analysis leading to a coherent theory. In this set up $C^2(G)$ becomes the *original* Schwartz space $\mathcal{C}(G)$ introduced by Harish-Chandra, and $C^1(SL_2(\mathbb{R}))_{0,0}$ is the Schwartz space defined in [E-M, p. 415]. Going to the Fourier transforms, in the Plancherel formula on G (which involves the Schwartz space C^2), the Plancherel measure was found to have support on a proper subset of the unitary representations which were called the tempered representations. A Hilbert space representation π is *tempered* if its K-finite matrix coefficients define tempered distributions in the following way:

$$f \mapsto \langle \widehat{f}(\pi)e_m, e_n \rangle,$$

where e_m, e_n are members of an *orthonormal basis* consisting of K-finite vectors of π. In an analogous way one can define L^p-tempered representations as those principal series representations whose K-finite matrix coefficients define L^p-tempered distributions (see Section 2 for definition). A careful observation of the Fourier Inversion Formula reveals that these precisely are the representations responsible for harmonic analysis of C^p functions. In view of this it appears natural that for W-T theorems of L^p functions these representations will be the appropriate objects to consider.

We recall that the characterization of the image under Fourier transform has so far been established for $C^p(G)_{0,0}$ in [T-V], where G is any connected semisimple Lie group with finite center; for $C^p(G : F)$ in [T], where G is of real rank one and $C^p(G : F)$ is the subspace of $C^p(G)$ containing all functions whose K-types are in a finite set F; and also for whole of $C^p(SL_2(\mathbb{R}))$ in [Ba], with $1 < p \leq 2$ in the first two cases and $0 < p \leq 2$ in the last. The isomorphism $C^p(G) \longrightarrow C^p(\widehat{G})$ is of vital importance for us. In fact, we use only the weaker fact that $(C^p(G))_{m,n} \longrightarrow (C^p(\widehat{G}))_{m,n}$ is a topological isomorphism under the Fourier Transform. Equally important is the fact that, for a given n, only finitely many discrete series representations are relevant.

We make an apparent digression from the main stream of the thesis in Section 3. The purpose is, however to get an exact analogue of [E-M, Theorem 7], for the case at hand, namely, that of L^p-functions of nontrivial spherical type.

A basic idea used in [E-M] is to find the conditions on an analytic function on the Helgason-Johnson strip (which coincides with S^1 mentioned above) so that the wave packet with respect to it is an L^1-function on G. We have tried to adapt these arguments to the case of nonspherical L^p-functions. Unlike matrix coefficients of the discrete series representations, matrix coefficients of the (unitary) principal series representations are not *Schwartz space* $(C^2(G))$

functions; but their wave packets with respect to the Schwartz space functions (on the representation space) come down to the Schwartz space $C^2(G)$. Likewise, for any fixed $p \in (0,2)$, matrix coefficients of the principal series representations which dwell inside the closed tube domain S^γ are not in C^p, while matrix coefficients of the discrte series outside the domain are. However, L^p-wave packets with L^p-Schwartz space functions on S^γ are in $C^p(G)$.

A question which occurs at this point is, whether the wave packet of a function $F \in L^p(\widehat{G})$ is in $L^p(G)$. The answer to this is negative. In fact one can find an $F \in L^1(\widehat{G})_{0,0}$ (*e.g.* $F(\lambda) = \frac{1}{\lambda^2 - z^2}$ for $z \in \mathbb{C}$), for which the wave packet does not even exist. Stanton and Tomas [S-T] have shown that the inversion formula for the spherical Fourier transform on a noncompact symmetric space hold a.e. for $f \in L^p(G/K)$, $1 \leq p < 2$, when $\widehat{f} \in L^1(d\mu(\lambda))$. Meaney and Prestini [M-P] found a sharp limit on p for the existence of wave packet. They have shown that only for $\frac{4}{3} < p \leq 2$ the wave packet of every $F \in L^p(\widehat{SL_2(\mathbb{R})})_{0,0}$ converges almost everywhere.

However, sufficient conditions can be obtained for a function F on the Fourier transform space so that its wave packet $\Phi_F^{0,0}$ lands in $L^p(G)_{0,0}$ and also $\widehat{\Phi}_F^{0,0} = F$. Such a sufficient condition involves assumptions on the order of differentiability and the rate of decay of the function F. For $G = SL_2(\mathbb{R})$ and $p = 1$ this has been observed in [E-M] as a part of the proof of the W-T theorem (Theorem 7).

A more elementary question now arises: what are the relations of the order of differentiability and the rate of decay of a function on the Fourier transform space with those of its wave packet? For K-biinvariant L^1 functions of any noncompact connected semisimple Lie group with finite center, this question is answered *qualitatively* in Gangolli-Warner ([G-W]) and in Sitaram ([S1]). It has been shown there that for any r, s, there exist integers m, l so that if F is in C^m and if F and its derivatives upto order m are of the order of reciprocal of a polynomial of degree l, then its wavepacket $\Phi_F^{0,0}$ satisfies $\sup_{x \in G}(1 + \sigma(x))^r \,\Xi(x)^{-2} \Phi_F^{0,0}(g_1, x, g_2) < \infty$, where g_1, g_2 belong to the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and $\deg(g_1) + \deg(g_2) \leq s$.

These results involve methods which call for change of contour of integration in the representation space. An advantage with the biinvariant functions (contrary to the case of functions with nontrivial spherical types) is that, there, the shift of contour does not encounter any pole. However, poles have to be tackled as we step out of the biinvariant case and address the same question. For $SL_2(\mathbb{R})$, we deal with functions of arbitrary K-types and compute the *exact* mutual dependency of the function and its wave packet in terms of rate of decay and order of differentiability.

The results we obtain here show that m, l mentioned above are also dependent on p and the K-types involved. Here we have done a careful tracking of the exact degrees of the polynomials and the order of derivatives which are latent in the results of [Ba].

A corollary to our result will also tell us precisely what condition on a function defined on the strip (tube domain for L^p- functions of $SL_2(\mathbb{R})$) will ensure that its wave packet belongs to $L^p(G)$ and that the Fourier transform of this wave packet goes back to the function we have started with. Let us recall that it would be the fundamental step for getting a W-T theorem for $L^p(G)_{n,n}$ which is a nonspherical analogue of the W-T theorem in [E-M] for biinvariant L^1-functions on $SL_2(\mathbb{R})$. This part of Section 3 may be considered as an exercise to understand some of the intricate methods of Barker [Ba], Campoli [C], Trombi (and Ragozin) [T] and their connection to their precursors in [E-M].

The W-T theorem for $L^p(G)_{n,n}$ is then further extended to $L^p(G)_n$ to give an analogue of the $L^1(G/K)$ case treated in [S2], using the techniques (involving Cororna theorem) used there.

However, these techniques do not work for the algebra $L^p(G)$; it is not difficult to see that one can not generate the whole of $L^p(G)$ starting from a single K-finite function or finitely many of them. Besides, for generating $L^p(G)$ one has to generate $L^p(G)_n$ for every n. And every $L^p(G)_n$ need be generated by using the full strength of the generating function $f \in L^p(G)$, not simply by the right n type projection of f (which may even be zero !). In Section 4 we use the result of [B-W] for an arbitrary family of biinvariant L^1 functions on G as the cornerstone to obtain a stronger W-T theorems for $L^p(G)_{n,n}$ and $L^p(G)_n$ without any K-finite restriction on the generators (Theorem 4.4 and Theorem 4.5). The later is an intermediate step towards the main theorem for $SL_2(\mathbb{R})$.

The main theorem for $SL^2(\mathbb{R})$ shows that if the Fourier transforms of a set of functions in $L^p(G)$ do not vanish simultaneously on any irreducible $L^{p-\varepsilon}$-tempered representation for some $\varepsilon > 0$, and if for each M-type at least one of the matrix coefficients of any of those Fourier transforms does not 'decay too rapidly at ∞' in a certain sense, then this set of functions generate $L^p(G)$ as anb $L^1(G)$-bimodule. This result is on the space of *all* L^p functions, $p \in [1,2)$, of $SL_2(\mathbb{R})$ without any restriction of K-finiteness on the generators.

Our next result, in Section 6, is a Wiener Tauberian (W-T) theorem for Riemannian symmetric spaces G/K of non compact type, where G is one of the following semi simple Lie groups of *real rank* one: $SU(n,1)$, $SO(n,1)$, $SP(n,1)$ or the connected Lie group of real type F_4. From now on G will denote one of these groups.

As mentioed earlier, W-T theorem for symmetric spaces has so far been proved only for L^1 functions on the space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ in [S2] where the generator is *necessarily* K-finite. We have observed above that one of our theorems (Theorem 4.5) improves upon this result by removing the restriction of K-finiteness on the generator. In Section 6 we provide an exact analogue of the theorem for G as above. More precisely, we show that if the Fourier transforms of a set of functions in $L^p(G/K)$ do not vanish simultaneously on any irreducible $L^{p-\varepsilon}$-tempered representations relevant for functions of G/K, for some $\varepsilon > 0$, and if one of these functions has a Fourier transform which does not 'decay too rapidly at ∞' in a certain sense, then this set of functions generate $L^p(G/K)$ as a left $L^1(G)$ module.

In switching over from $SL_2(\mathbb{R})$ to other groups of real rank 1, one encounters a number of difficulties which prevent a straight-forward extension of W-T theorem from $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ to other rank 1 symmetric spaces. To reduce the problem to the biinvariant case (of $SL_2(\mathbb{R})$) [S2] and Section 4 of this thesis have a common way: to find a function g such that the left convolute $g * f$ of the generator f is biinvariant and this convolute has nonvanishing Fourier transform wherever f has the same. But unlike those of $SL_2(\mathbb{R})$, non zero K-types are not in general one dimensional and hence can accommodate more than one M-type. Note that for a function on G/K with a nontrivial $K-$type in the left, the Fourier transform is a matrix valued function. Hence it is posible that two functions f and g of matching K types (*i.e.* the right type of f is the same as left type of g) have non zero Fourier transforms at a certain representation, yet $f * g$ has zero Fourier transform at that representation.

A more subtle difficulty arises from points $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ where the asymptotic expansions of the matrix coefficients of the principal series representations have singularity. As mentioned in Trombi [T], there are linear dependecies among these matrix coefficients. Therefore various matrix coefficients of the Fourier transform of a K-finite function at such a representation are not quite independent of each other. And contrary to what we have experienced in $SL_2(\mathbb{R})$, those relations are *not* "reformulation of embedding of discrete series in principal series" (see [T]).

Singularities of the asymptotic expansions of the matrix coefficients of the principal series representations are the *points of trouble*. Supposing that we locate a function f on G/K having the component f_m of left K-type m with a nonzero Fourier transform $\widehat{f}_m(\lambda_0)$ at a point λ_0 in the strip S^γ. Our strategy is to look for a function g of type $(0, m)$, so that $\widehat{g}(\lambda_0) \neq 0$ and then hope to have $\widehat{g * f} \neq 0$. But if λ_0 is one of those points of trouble, then it is possible that even though $\Phi_{\lambda_0}^{m,0} \neq 0$, we may have $\Phi_{\lambda_0}^{0,m}(x) \equiv 0$, so that no function g of the kind we want can exist.

All these give trouble in tailoring a g which will reduce the generator to a biinvariant function as in the case of $SL_2(\mathbb{R})$. We take help of the results due to Johnson and Wallach [J-W] and Johnson [J] to bypass this difficulty.

Also, we propose a change in the basic step : instead of making a single left convolute of the generator to shoulder the responsibility of having nonvanishing Fourier transform at all points λ, we get, for each point λ of the strip, seperate left convolute $g_\lambda * f$ of the generator f such that Fourier transform of $g_\lambda * f$ is non zero (perhaps only) at the point λ. This sharing of responsibilities over the g_λ's eases the process of finding them and helps us to overcome some of the obstalces encountered. Even in the case of $SL_2(\mathbb{R})$, this strategy works and it avoids the lengthy arguments and constructions used in Section 4. However the proofs for $SL_2(\mathbb{R})$ in Section 4 are more constructive and more elementary in nature than this.

In [S2] Sitaram has used Corona Theorem for a similar extension of W-T theorem from biinvariant L^1-functions to $L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$. A disadvantage of using Corona theorem in this context is that it can handle only finitely many functions and therefore can not be adopted in the above principle of using seperate g_λ for every λ. Here, on the other hand, we have used full force of the W-T theorem for biinvariant functions in [B-W] where the generator set is infinite.

The last two sections, Section 7 and 8 are devoted to a critical examination of two hypotheses we use in our W-T theorems. The first is about the restriction of using a slightly larger strip $S^{\gamma+\delta}$, for the nonvanishing conditions, than what is *necessary*. Slightly larger domain for the Fourier transforms of the generating functions is a common feature in all W-T theorems proved so far. Our theorem inherits this restriction from the W-T theorems for the biinvariant functions. But we claim that our method of extension, contrary to the use of Cororna theorem, does not require the *augmented part of the strip*; if a W-T theorem can be proved for $L^p(G)_{n,n}$ without imposing this condition then the corresponding stronger version of our result will immediately follow. We have demonstrated it by proving a W-T theorem for $L^1(PSL_2(\mathbb{R}))$ with the condition on exact strip extending a recent result for $L^1(SL_2(\mathbb{R}))_{0,0}$ in [B-B-W-H2] .

The last section of this thesis is an attempt to identify the set of functions having Fourier transforms with the required not-too-rapidly-decreasing conditions. We use an uncertainty theorem [S-S] to replace the not-too-rapidly decreasing condition on the Fourier transform of the generator by a decay on the generator function itself.

Finally, we add a few lines about the Fourier transform we work with. For our purpose, instead of the operator valued Fourier transform we can actually use the Fourier coefficients, which are integrals of the function against the matrix coefficients of the principal and discrete series representations. Thus for a suitable function f and a representation π, $\widehat{f}(\pi)$ is only a formal matrix, with entries $\widehat{f}(\pi)_{m,n} = \int f(x)\Phi^{m,n}(x)dx$, where $\Phi^{m,n} = \langle \pi(x)e_m, e_n \rangle$.

We end the introduction with a sectionwise summary of the thesis.

Section 1 contains notation and preliminaries for $SL_2(\mathbb{R})$.

Section 2 is a continuation of the previous section. It establishes the vocabulary used in this thesis, part of which is not so standard. Here we bring together some facts in a form which is uniform with respect to p.

Section 3 establishes the relations of the order of differentiability and the order of a function on S^γ with those of its wave packet when $G = SL_2(\mathbb{R})$. As a corollary we get a sufficient condition on a function on the strip S^γ so that its wave packet falls in $L^p(SL_2(\mathbb{R}))$. A purpose of this whole process is to obtain an analogue of the *original* W-T theorem for biinvariant functions of $L^1(SL_2(\mathbb{R}))$ in [E-M], in the case of $L^p(SL_2(\mathbb{R}))_{n,n}$. We have considered questions which came naturally in this endavour.

Section 4 has a W-T theorem for $L^p(SL_2(\mathbb{R}))$ and for L^p-sections of certain line bundles over $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ without any K-finiteness restriction on the generator.

Section 5 serves extra prerequisites needed for groups of real rank one.

Section 6 contains a W-T theorem for rank one symmetric spaces.

In Section 7 we prove a W-T theorem for $L^1(PSL_2(\mathbb{R}))$ with *exact* non vanishing condition.

Section 8 contains a reformulation of W-T theorems using a 'mathematical uncertainty principle', and also an application of *Hardy's Theorem* for semisimple Lie groups in the W-T theorems.

1 Notation and Preliminaries

Let G be the 2×2 real special linear group $SL_2(\mathbb{R})$ and $\mathfrak{g}$ be its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of 2×2 trace zero matrices. The following elements of $\mathfrak{g}$ are important for harmonic analysis:

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \overline{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Elements in the group G coresponding to X, H and Y are respectively

$$\begin{aligned} k_\theta &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = exp(\theta X), \quad \theta \in \mathbb{R} \\ a_t &= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = exp(2tH), \quad t \in \mathbb{R} \\ n_s &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = exp(sY), \quad s \in \mathbb{R}. \end{aligned}$$

They form the subgroups:

$$K = \{k_\theta | \theta \in \mathbb{R}\}, \; A = \{a_t | t \in \mathbb{R}\} \quad \text{and} \quad N = \{n_s | s \in \mathbb{R}\},$$

among which K is the maximal compact subgroup $SO(2,\mathbb{R})$ of G. The biinvariant normalized Haar measures of these three subgroups are:

$$\begin{aligned} dk &= dk_\theta = d\theta/2\pi, \quad 0 \le \theta \le 2\pi, \\ da &= da_t = dt, \quad t \in \mathbb{R} \\ \text{and} \quad dn &= dn_s = ds, \quad s \in \mathbb{R}. \end{aligned}$$

The Haar measure dx of G breaks up under the Iwasawa decomposition as

$$dx = \alpha(a) dk\, da\, dn, \tag{1}$$

where $\alpha : A \longrightarrow \mathbb{R}^+$ is defined by $\alpha(a_t) = e^{2t}$.

We denote the characters of K by χ_n, $n \in \mathbb{Z}$, where $\chi_n(k_\theta) = e^{in\theta}$. A complex valued function f on G is said to be of *left* (resp. *right*) *K-type n* if $f(kx) = \chi_n(k)f(x)$ (resp. $f(xk) = \chi_n(k)f(x)$), for all $k \in K$ and $x \in G$.

For a class of function $\mathcal{F}$ on G (e.g. $L^p(G)$), $\mathcal{F}_n$ will denote the subclass of functions of right type n while $\mathcal{F}_{m,n}$ will consist of funtions in $\mathcal{F}_n$ which are also of left type m. Functions of left K-type m and right K-type n are also referred to as functions of type (m,n). By $f_{m,n}$ we denote the projection of f in left type m and right type n, which is defined (wherever possible) by

$$f_{m,n} = \int_K \int_K \overline{\chi}_m(k_1)\overline{\chi}_n(k_2) f(k_1 x k_2) dk_1 dk_2 \tag{2}$$

It will also be referred to as the (m,n)-th component of f. A function is called *K-finite* when it has only finitely many such components.

The projection of f in right K-type m is denoted by $f_{-,m}$ and is given by:

$$f_{-,m} = \int_K \overline{\chi}_m(k) f(xk) dk. \tag{3}$$

Similarly the projection of f in left K-type m is denoted by $f_{m,-}$ and is given by:

$$f_{m,-} = \int_K \overline{\chi}_m(k) f(kx) dk. \tag{4}$$

For any class of functions $\mathcal{F}$, the subclass consisting of functions with integral zero will be denoted by $\mathcal{F}^0$.

The complexification of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\mathbb{C}}$ and the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ is denoted by $\mathcal{U}$. The Casimir element Ω of $\mathcal{U}$ is defined by:

$$\Omega = H^2 + H - Y\overline{Y}. \tag{5}$$

The center $\mathcal{Z}$ of $\mathcal{U}$ is generated by Ω. For $g_l, g_r \in \mathcal{U}$, considered as left and right invariant differential operators respectively, $f(g_l, x, g_r)$ denotes their action on f by the following rule:

$$f(X_l; x; X_r) = \frac{d}{dt}\frac{d}{ds} f(\exp(tX_l) x \exp(sX_r)) \,|_{t=0,\, s=0}.$$

where $X_l, X_r \in \mathfrak{g}$.

For $x \in G$, $h(x)$ and $k(x)$ are respectively the A-part and the K-part of x in its Iwasawa decomposition $x = kan$, while $\sigma(x) = |t|$, where t comes from the Cartan decomposition, $x = k_1 a_t k_2$.

For $x \in G$, define

$$\Xi(x) = \int_K \alpha(h(xk))^{-\frac{1}{2}} dk.$$

Then using [Ba, (4.8)] we get $\Xi(x) = \Phi^{0,0}_{\sigma_+,0}$.

For any $p \in (0,2]$ let $\gamma = \frac{2}{p} - 1$ and define S^γ by

$$S^\gamma = \{\lambda \in \mathbb{C} \,|\, |\Re\lambda| \leq 2/p - 1\}.$$

Let S^γ_δ denote the augmented strip $\{\lambda \in \mathbb{C} \,|\, |\Re\lambda| \leq \gamma + \delta\}$ for $\delta > 0$, which is actually $S^{\gamma+\delta}$. It is clear that for $p \in (0,2)$ there is a p' in $(0,p)$ such that the strip corresponding to p' is S^γ_δ. Here $\frac{2}{p'} - 1 = \gamma + \delta$.

Principal series representations: Let M be $\{\pm I\} \subset K$, where I is the identity matrix. Then $P = MAN$ is a parabolic subgroup which may be described as:

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^*, \;\; b \in \mathbb{R} \right\}.$$

Let σ_+ and σ_- denote respectively the trivial and the only nontrivial irreducible representation of M, *i.e.*, $\widehat{M} = \{\sigma_+, \sigma_-\}$. Analogous to the K-types, we may talk of functions f on G being of M-type σ_+ and σ_-. Also, we use the notation:

$$-\sigma_+ = \sigma_- \text{ and } -\sigma_- = \sigma_+.$$

Let

$$\mathbb{Z}^\sigma = \begin{cases} \text{set of even integers} & \text{if } \sigma = \sigma_+ \\ \text{set of odd integers} & \text{if } \sigma = \sigma_- \end{cases}$$

and $\mathbb{Z}^\sigma_+ = \mathbb{Z}^\sigma \cap \mathbb{Z}_+$, $\mathbb{Z}^\sigma_- = \mathbb{Z}^\sigma \cap \mathbb{Z}_-$, where $\mathbb{Z}_+$ and $\mathbb{Z}_-$ are positive and negative integers respectively.

For each $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{C} = \mathfrak{a}^*$ the principal series representation $\pi_{\sigma,\lambda}$ is obtained by parabolic induction through P. For a fixed σ the representations $\pi_{\sigma,\lambda}$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ are realised on the same subspace H_σ of the Hilbert space $L_2(K)$ (*compact picture*). Take the canonical orthonormal basis $\{e_n | n \in \mathbb{Z}\}$ for $L^2(K)$, where $e_n(k_\theta) = e^{in\theta}$, $k_\theta \in K$ and e_0 is the K-fixed vector for the representation. Now $\{e_n | n \in \mathbb{Z}^\sigma\}$ is an orthonormal basis of H_σ. The action of $\pi_{\sigma,\lambda}(x)$ on H_σ is given by:

$$[\pi_{\sigma,\lambda}(x)e_n](k) = \alpha(h(x^{-1}k^{-1}))^{-(\lambda+1)/2} e_n(k(x^{-1}k^{-1}))^{-1} \tag{6}$$

where $x \in G$, $k \in K$ and $n \in \mathbb{Z}^\sigma$. Then $\pi_{\sigma,\lambda}$ is a continuous Hilbert space representation which is unitary if and only if λ is purely imaginary.

It is known that $(\pi_{\sigma,\lambda}, H_\sigma)$ are admissible representations (see [Ba, p.9] for definition). Hence they are also realized as $(\mathfrak{g}, K)$ modules, defined by the action:

$$\pi_{\sigma,\lambda}(A)v = \frac{d}{dt}\pi_{\sigma,\lambda}(\exp tA)v|_{t=0} \tag{7}$$

for $A \in \mathfrak{g}$, $v \in H_\sigma$. Then

$$\pi_{\sigma,\lambda}(X)e_n = ie_n. \tag{8}$$

Define

$$E = 2H + i(Y - \overline{Y}) = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

and

$$F = -2H + i(Y - \overline{Y}) = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}.$$

Then $\{X, E, F\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$ and

$$\pi(E)e_n = (n + \lambda + 1)e_{n+2}, \quad \pi(F)e_n = (n - \lambda - 1)e_{n-2} \tag{9}$$

Subrepresentations of $\pi_{\sigma,\lambda}$:

1. For $k \in \mathbb{Z}_+^{-\sigma}$, $\pi_{\sigma,-k}$ has a single irreducible submodule H_σ^k of dimension k spanned by the basis vectors $\{e_n | n \in \mathbb{Z}^{-\sigma}, |n| < k\}$.

2. For $k \in \mathbb{Z}_+^{-\sigma}$, $\pi_{\sigma,k}$ has two irreducible submodules with the bases $\{e_n | n \in \mathbb{Z}_+^\sigma, n > k\}$ and $\{e_n | n \in \mathbb{Z}_-^\sigma, n < -k\}$; they are denoted by $H_{\sigma,k}$ and $H_{\sigma,-k}$ respectively.

3. The representation $\pi_{\sigma_-,0}$ has two irreducible subrepresentations, the so called *mock discrete series.* We will denote them by D_+ and D_-.

The representation spaces of D_+ and D_- contain, respectively, the basis vectors $e_n \in L^2(K)$ for positive odd n's and negative odd n's.

Barring the above mentioned cases $\pi_{\sigma,\lambda}$ is always irreducible. For $\lambda \notin \mathbb{Z}^{-\sigma}$, $\pi_{\sigma,\lambda}$ and $\pi_{\sigma,-\lambda}$ are equivalent. For more details on the parametrization of the representations $\{\pi_{\sigma,\lambda} | (\sigma, \lambda) \in \widehat{M} \times \mathbb{C}\}$ and their realisation on $L^2(K)$ we refer to [Ba, Section 4].

The matrix coefficients of the principal series representations are taken only with respect to the basis vectors e_n mentioned above. For $m, n \in \mathbb{Z}^\sigma$, the (m, n)-th matrix coefficient $x \mapsto \langle \pi_{\sigma,\lambda}(x) e_m, e_n \rangle$ of the principal series representation $\pi_{\sigma,\lambda}$ is denoted by $\Phi^{m,n}_{\sigma,\lambda}$. It vanishes when either m or n does not belong to $\mathbb{Z}^\sigma$. We here quote from [Ba, Proposition 7.1] the following result for future use:

Proposition 1.1 *(Barker) Let $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $k \in \mathbb{Z}^{-\sigma}$. Then $\Phi^{n,m}_{\sigma,k} \equiv 0$ if and only if $n < -k < m$ or $m < k < n$.*

It is clear from the above proposition that for n and k of opposite parity $\Phi^{n,n}_{\sigma,k}$ is never identically zero. Here n determines σ by $n \in \mathbb{Z}^\sigma$.

Discrete series representation:

Discrete series representations π_k are parametrized by $k \in \mathbb{Z}^*$ and they are subrepresentations of principal series representations $\pi_{\sigma,k}$ for $\pm k \in \mathbb{Z}^{-\sigma}_+$. In fact, $\pi_{\pm k}$ are the restrictions of $\pi_{\sigma,k}$ to $H_{\sigma,\pm k}$ mentioned in (2) above, after renormalisation of the spaces $H_{\sigma,\pm k}$, $k \in \mathbb{Z}^{-\sigma}_+$. Their matrix coefficients are denoted by $\Psi^{m,n}_k$ and $\Psi^{m,n}_{-k}$ respectively. Here these matrix coefficients are taken with respect to vectors e^k_m and e^k_n, which are suitable (positive) multiples of e_m and e_n, so as to have norm 1 in the representation space of π_k (see [Ba, p. 30]).

Also from Proposition 7.3 and Proposition 7.4 of [Ba] we have:

Proposition 1.2 *(Barker) For $k \in \mathbb{Z}^*$ and $n, m \in \mathbb{Z}(k)$,*

$$\Psi^{n,m}_k = \eta^{n,m}(k) \Phi_{\sigma,|k|},$$

$$\Phi^{n,m}_{\sigma,|k|} = \eta^{m,n}(k)\Psi_k,$$

where $k \in \mathbb{Z}^{-\sigma}$ and $\eta^{n,m}(k)$ is a positive constant which satisfies the inequality:

$$\eta^{n,m}(k) \leq M(1+|n|)^{|k|}$$

for all m, n and fixed k where M is a constant.

For a function f, $\widehat{f}_H(D_+) \neq 0$ (resp. $\widehat{f}_H(D_-) \neq 0$) means that $\widehat{f}_H(\sigma_-, 0)$ has a nonzero matrix coefficient $(\widehat{f}_H(\sigma_-, 0))_{m,n}$, where e_m and e_n are in D_+, hence m, n are positive (resp. in D_-, hence m, n are negative).

Let

$$\mathbb{Z}(k) = \begin{cases} \{n \in \mathbb{Z} |\ n > k \text{ and of parity opposite to } k\} & \text{if } k > 0 \\ \{n \in \mathbb{Z} |\ n < k \text{ and of parity opposite to } k\} & \text{if } k < 0 \end{cases}$$

for any $k \in \mathbb{Z}^*$. Then $\{e_n | n \in \mathbb{Z}(k)\}$ denotes the set of vectors forming the basis of the discrete series π_k. Also define

$$\mathbb{Z}^\gamma(n) = \begin{cases} \{k \in \mathbb{Z} |\ \gamma < k < n \text{ and of parity opposite to } n\} & \text{if } n > 0 \\ \{k \in \mathbb{Z} |\ n < k < -\gamma \text{ and of parity opposite to } n\} & \text{if } n < 0 \end{cases}$$

and

$$\Gamma_n = \mathbb{Z}^0(n).$$

Then Γ_n is the set of points parametrizing discrete series representations which are relevant to a function of right or left K-type n, and $\mathbb{Z}^\gamma(n)$ consists of those elements in Γ_n which are outside the strip S^γ. For $\sigma \in \widehat{M}$ and $n, m \in \mathbb{Z}^\sigma$, define

$$L^{m,n}_\sigma = \{k \in \mathbb{Z}^{-\sigma} | 0 < k < \min\{n, m\} \text{ or } \max\{n, m\} < k < 0\} \qquad (10)$$

if $m.n > 0$, and

$L^{m,n}_\sigma$ is empty when $m.n \leq 0$.

Then $L^{m,n}_\sigma$ is the finite set of points parametrizing the discrete series having K-types m, n. It is an important fact for us that only finitely many discrete series, when resticted to K, contain a fixed K type. (This is true in all generality as follows from Vogan's theory of minimal K-types. See [Vo].) We break up the collection $L^{m,n}_\sigma$ into two sets, one parametrizing the points inside the strip and the other parametrizing those outside the strip:

$$L_\sigma^{m,n}(\gamma) = \{l \in L_\sigma^{m,n} \mid |l| \leq \gamma\}$$

and

$$L_\sigma^{m,n}(\gamma)^c = \{l \in L_\sigma^{m,n} \mid |l| > \gamma\}.$$

Note that $L_\sigma^{n,n}(\gamma)^c = Z^\gamma(n)$.

Definition 1.3 For $0 < p \leq 2$ the *L^p-Schwartz space*, $C^p(G)$ is the space of all C^∞ functions f such that $\rho^p_{g_1,g_2;r}(f) < \infty$ for all $g_1, g_2 \in \mathcal{U}$ and $r \in \mathbb{R}$, where the seminorm $\rho^p_{g_1,g_2;r}$ is given by

$$\rho^p_{g_1,g_2;r}(f) = \sup_{x\in G}(1+\sigma(x))^r \Xi(x)^{-\frac{2}{p}} |f(g_1, x, g_2)|. \tag{11}$$

Fourier transforms of C^p-functions and L^p-functions. Though the family of principal series representations $\pi_{\sigma,\lambda}$ are realised on the common subspace H_σ of the Hilbert space $L^2(K)$ for every $\lambda \in \mathbb{C}$, there is difficulty in defining the operator Fourier transform as

$$\widehat{f}(\pi_{\sigma,\lambda}) = \widehat{f}_H(\sigma, \lambda) = \int_G \pi_{\sigma,\lambda}(x^{-1}) f(x)dx.$$

$$\widehat{f}(\pi_n) = \widehat{f}_B(n) = \int_G \pi_n(x^{-1}) f(x)dx.$$

The formula defines a bounded operator valued function for L^1 functions f when $\lambda \in \mathcal{S}^1$ and n is a non-zero integer. For $f \in L^2$, the operator transform can be defined almost everywhere on the imaginary axis and on non-zero integers through the Plancherel theorem, and the definition on this restricted domain can be extended to the case of $f \in L^p(G)$ in the folowing way.

Write $f = f_1 + f_2$, where $f_1 \in L^2(G)$ and $f_2 \in L^1(G)$. For instance, $f_1 = f.\chi_{\{|f|<1\}}$ and $f_2 = f.\chi_{\{|f|\geq 1\}}$. Then $(\widehat{f_2}_H, \widehat{f_2}_B)$ is defined as above. And the corresponding transforms for f_1 are defined by the Plancherel theorem *a.e.* on $\Re\lambda = 0$ and on $k \in \mathbb{Z}$. For $\sigma \in \widehat{M}$, we now define, $\widehat{f}_H(\pi_{\sigma,\lambda}) =$

$\widehat{f_1}_H(\pi_{\sigma,\lambda}) + \widehat{f_2}_H(\pi_{\sigma,\lambda})$ and $\widehat{f}_B(\pi_k) = \widehat{f_1}_B(\pi_k) + \widehat{f_2}_B(\pi_k)$, *a.e.* on $\Re\lambda = 0$ and $k \in \mathbb{Z}$. It can be easily checked that $\widehat{f}_H$ and $\widehat{f}_B$ are independent of the decomposition $f = f_1 + f_2$.

The difficulty in extending the operator Fourier transform beyond this restricted domain forces us to abandon operator Fourier transforms for what can be called formal matrix Fourier transforms with respect to the basis $\{e_n\}$. If $f \in L^p(G)$, $p \in [1,2]$, define:

1. for the principal series representations, $\pi_{\sigma,\lambda}$ with $\sigma \in \widehat{M}$, $\lambda \in S^\gamma$ and $m, n \in \mathbb{Z}^\sigma$,
$$(\widehat{f}_H)_{m,n}(\sigma,\lambda) = F_H^{m,n}(f)(\sigma,\lambda) = \int_G f(x)\Phi_{\sigma,\lambda}^{m,n}(x^{-1})dx$$
2. for the discrete series represenation π_k which has both e_m and e_n, that is, for m, $n \in Z(k)$,
$$(\widehat{f}_B)_{m,n}(k) = F_B^{m,n}(f)(k) = \int_G f(x)\Psi_k^{m,n}(x^{-1})dx$$

These *matrix coefficients* of the Fourier transform are enough for our purpose. We will denote the formal matrices by $\widehat{f}_H$ and $\widehat{f}_B$. It is convenient to state at this point some important properties of functions and their Fourier transforms:

1. For $f \in L^p$, if m, n are of opposite parity then $f_{m,n} \equiv 0$.

2. Let m, n, σ be such that $m, n \in \mathbb{Z}^{-\sigma}$ and let f be an (m, n) type function in L^p. Then $\widehat{f}(\sigma, .) \equiv 0$ on S^γ.

3. For two functions f and g of right type m and left type n respectively with $m \neq n$, $f * g \equiv 0$ whenever their convolution is defined.

4. For an L^p function f, $(\widehat{f}_H)_{m,n}(\sigma,\lambda)$ is a complex analytic function of $\lambda \in S^\gamma$.

5. For two functions f and g, as long as their convolution and Fourier transforms are defined, $\widehat{f * g} = \widehat{f}.\widehat{g}$, where the product on the right hand side is matrix multiplication which remains valid if one of $\widehat{f}$ and $\widehat{g}$ has only finitely many nonzero entries.

6. For $f \in L^p(G)$ and nonzero integer $k \in S^\gamma$,

$$(\widehat{f}_B)_{m,n}(k) = \eta^{m,n}(k)(\widehat{f}_H)_{m,n}(\sigma, |k|)\,,$$

provided $m,\, n \in \mathbf{Z}(k)$, where $\eta^{m,n}(k)$ is a positive number arising from the renormalisation of the representation space. See Proposition 1.2 and [Ba, p.30] for a description of $\eta^{m,n}(k)$. In particular, $(\widehat{f}_B)_{m,n}(k) \neq 0$ if and only if $(\widehat{f}_H)_{m,n}(k) \neq 0$.

In this thesis we denote the Fourier transform of f with respect to the principal series representation $\pi_{\sigma,\lambda}$ by $\widehat{f}_H(\pi_{\sigma,\lambda})$ and $\widehat{f}_H(\sigma,\lambda)$ interchangeably. Similarly Fourier transform of f with respect to the discrete series representation π_k is denoted either by $\widehat{f}(\pi_k)$ or simply by $\widehat{f}(k)$.

As an $n \in \mathbf{Z}$ can determine a $\sigma \in \widehat{M}$ by the relation $n \in \mathbf{Z}^\sigma$, for a function of right or left K-type n the Fourier transform of f at $\pi_{\sigma,\lambda}$ can be denoted by $\widehat{f}(\lambda)$ dropping σ without any risk of ambiguity.

The definitions and notation of this section are essentially reproduction from [Ba]. For any unexplained notation in the thesis we refer to the same source ([Ba]).

2 L^p-Harmonic Analysis on $SL^2(\mathbb{R})$

This section is a continuation of the last one and we continue to limit ourselves to the group $SL_2(\mathbb{R})$. Here we plan to put together some of the basic facts in a form suitable for our use and, on occasion, provide proofs where we could not locate a ready source for the results, though they must be known to the experts in the field. Some of the not-so-standard terminology used in the thesis are also developed here.

Let us recall some facts and definitions. Though we will work out everything in this section only for $SL_2(\mathbb{R})$, most of the definitions and results are relevant and valid in a more general context, especially so for groups of *real rank* one. In many places we indicate to what generality the corresponding results hold. As in the previous section, unless mentioned otherwise, G will denote $SL_2(\mathbb{R})$, p will be in $(0,2]$ and γ will always be related to p by the formulae $\gamma = \frac{2}{p} - 1$. A representation of G is tempered if its character as a distribution is tempered (i.e. can be extended to the *Schwartz space* C^2). Similarly L^p-tempered representations are those whose characters as distributions can be extended to the *L^p-Schwartz space* C^p. Like irreducible tempered representations, irreducible L^p-tempered representations are also irreducible admissible representations.

For $p, q \in \mathbb{R}$ we will say that q is conjugate to p if $\frac{1}{p} + \frac{1}{q} = 1$.

Some Standard Inequalities:

1. Let $\Xi(x) = \Phi^{0,0}_{\sigma+,0}$. Then

$$1 \leq \Xi(a_t)e^{|t|} \leq M(1 + |t|) \tag{12}$$

   for all $t \in \mathbb{R}$, where M is a positive number.

2. For any $\varepsilon > 0$ and $r \in \mathbb{R}$, there exists an $M < \infty$ such that for all $x \in G$

$$\Xi(x)^{\varepsilon}(1 + \sigma(x))^{r} < M \tag{13}$$

   where $\sigma(x) = |t|$, x having the Cartan decomposition $x = k_1 a_t k_2$.

3. There exists an $r > 0$ such that

$$\int_G \Xi(x)^2(1+\sigma(x))^{-r}dx < \infty. \tag{14}$$

For $SL_2(\mathbb{R})$, this r can be taken as any real number greater than 3.

4. (Harish-Chandra) For fixed $g_1, g_2 \in \mathcal{U}$, $s \in \mathbb{N}$ and $\gamma \geq 0$ there exists a $C > 0$ and $r_1, r_2 \geq 0$ such that

$$\begin{aligned}&|(\tfrac{d}{d\lambda})^s \Phi^{n,m}_{\sigma,\lambda}(g_1; x; g_2)| \\ &\quad \leq C(1+|m|)^{r_1}(1+|n|)^{r_2}(1+|\lambda|)^{r_1+r_2}(1+\sigma(x))^{s+\gamma}\Xi(x)^{1-\gamma}\end{aligned} \tag{15}$$

for all $\sigma \in \widehat{M}$, $n, m \in \mathbb{Z}^\sigma$, $|\Re\lambda| \leq \gamma$ and $x \in G$. Further, for $j = 1, 2$, r_j can be chosen so that $r_j \leq \deg(g_j)$.

5. For each $\sigma \in \widehat{M}$, define a meromorphic function $\mu(\sigma, \lambda)$ on $\mathbb{C}$ by

$$\mu(\sigma,\lambda) = \begin{cases} (\lambda\pi i/2)\tan(\lambda\pi/2) & \text{if } \sigma = \sigma_+ \\ (-\lambda\pi i/2)\cot(\lambda\pi/2) & \text{if } \sigma = \sigma_-. \end{cases} \tag{16}$$

Then there is a constant c such that for all $\sigma \in \widehat{M}$ and $\lambda \in i\mathbb{R}$,

$$|\mu(\sigma,\lambda)| \leq c(1+|\lambda|). \tag{17}$$

This $\mu(\sigma, \lambda)$ is the meromorphic extension of the Plancherel measure on $\mathbb{C}$.

For $SL_2(\mathbb{R})$ these results are available in [Ba, (3.2), (3.3), (3.4), Theorem 4.1 and (10.2)]. However, these (or some analogous results) are true in a more general context. See [T, Section 4, Proposition 1, Section 6, equations (ii) and (v)] for the groups of real rank one; see [V, pp.349-340, Theorem 30, Prop 31]; [H-C, Lemma 17.1] for more general groups.

Definition 2.1 A measurable function f on G is said to satisfy *weak inequality* if for suitable constants $c > 0$ and $r \geq 0$

$$|f(x)| \leq c\,\Xi(x)(1+\sigma(x))^r \tag{18}$$

for all $x \in G$. It satisfies *strong inequality* if for every $r \geq 0$ there is a $c_r > 0$ such that

$$|f(x)| \leq c_r \, \Xi(x)(1+\sigma(x))^{-r} \tag{19}$$

for all $x \in G$.

Definition 2.2 A measurable function f on G is said to satisfy *L^p-weak inequality* if for suitable constants $c > 0$ and $r \geq 0$

$$|f(x)| \leq c \, \Xi(x)^{\frac{2}{q}}(1+\sigma(x))^{r}. \tag{20}$$

where q is conjugate to p.

It satisfies *L^p-strong inequality* if for every $r \geq 0$ there is a constant $c_r > 0$ such that

$$|f(x)| \leq c_r \, \Xi(x)^{\frac{2}{p}}(1+\sigma(x))^{-r}. \tag{21}$$

Thus, weak and strong inequalities are particular cases ($p = 2$) of L^p-weak and L^p-strong inequalities. For functions on a group in the Harish-Chandra class $\mathcal{H}$ having values in a complete locally convex space, weak and strong inequalities are defined in [V, p. 341]. The first inequality in [V, Prop 10] and the seminorm defined in [V, p. 342, (5)] may be considered as definitions of L^p-weak and strong inequalities respectively for such functions.

The *L^p-Schwartz space* for any $p \in (0,2]$, denoted by $C^p(G)$ can also be defined (compare with Definition 1.3) as the space of C^∞ functions f so that for every $g_1, g_2 \in \mathcal{U}$, $f(g_1; x; g_2)$ satisfies L^p-strong inequality.

Proposition 2.3 *If a function f satisfies L^p-weak inequality for $1 \leq p \leq 2$ then f is in $L^{q+\varepsilon}$ for every $\varepsilon > 0$, where q is conjugate to p.*

Proof. $|f(x)| \leq c(1+\sigma(x))^r(\Xi(x))^{\frac{2}{q}}$. So, $|f(x)|^{q+\varepsilon} \leq c'\Xi(x)^{2+\varepsilon'}(1+\sigma(x))^{r'}$ for some $r' \geq 0$, $\varepsilon' > 0$ and $c' > 0$. Now from inequalities (13) and (14) above the result follows. ∎

For fixed m, n, the inequlaity (15) above reduces to

$$|\Phi_{\sigma,\lambda}^{m,n}(x)| \leq c(1+\sigma(x))^{\frac{2}{p}-1}(\Xi(x))^{\frac{2}{q}} \qquad \lambda \in S^{\gamma},$$

since $1-\gamma = \frac{2}{q}$ and since r_1, r_2 can be taken to be zero. This shows that for fixed m, n, $\Phi_{\sigma,\lambda}^{m,n}(x)$ satisfies the L^p-weak inequality with a constant c independent of $\lambda \in S^{\gamma}$ (but depending on m, n). Therefore, by Proposition 2.3, we have:

Proposition 2.4 *For $\sigma \in \widehat{M}$, $1 \leq p \leq 2$ and $m, n \in \mathbb{Z}^{\sigma}$, and for λ in the closed strip S^{γ}, the function $x \longrightarrow \Phi_{\sigma,\lambda}^{m,n}(x)$ is in $L^{q+\varepsilon}$ for every $\varepsilon > 0$. Moreover, for fixed m, n and ε, $\|\Phi_{\sigma,\lambda}^{m,n}\|_{q+\varepsilon} < K$ for all $\lambda \in S^{\gamma}$ where $K < \infty$. Also for every λ in $\mathring{S}^{\gamma}$ these functions are in L^q.*

Discrete series and L^p-discrete series: Discrete series are so named because the Fourier coefficients with respect to them constitute the discrete direct summand in the Fourier inversion formula of the Schwartz space ($C^2(G)$) functions. A necessary precondition for this is the fact that discrete series representations are those whose K-finite matrix coefficients are in $C^2(G)$. Now, if a K-finite matrix coefficient of a discrete series is not in $C^p(G)$, then Fourier transfom with respect to it can not be a part of the discrete direct summand in the inversion formula of C^p functions. The following definition is made keeping this in view:

Definition 2.5 An admissible representation is called an L^p-discrete series representation if every K-finite matrix coefficient of it is in $C^p(G)$.

As every C^p function is in $C^2(G)$, the L^p-discrete series form a subset of the discrete series. We will presently see (Theorem 2.8) that any discrete series outside the strip S^{γ} (i.e. not embedded in any principal series representation on the strip) is an L^p-discrete series representation and they are the only discrete series of this kind. Let us quote an inequality from [Ba, Section 5] analogous to (15) satisfied by the matrix coefficients of discrete series.

Theorem 2.6 (*Trombi-Varadarajan*) *Fix a $p \in (0,2]$. There exists an $\varepsilon_p > 0$, and, for each $u_1, u_2 \in \mathcal{U}$, constants $c, r_1, r_2, r_3 \geq 0$ such that*

$$|\Psi_k^{m,n}(u_1; x; u_2)| \leq c(1+|m|)^{r_1}(1+|n|)^{r_2}(1+|k|)^{r_3}\Xi(x)^{\frac{2}{p}+\varepsilon_p} \tag{22}$$

for all $k \in \mathbb{Z}^*$ for which $|k| > \gamma$, and for all $m, n \in \mathbb{Z}(k)$ and $x \in G$.

From this we have

Proposition 2.7 *For $1 \leq p \leq 2$, every matrix coefficient of the L^p-discrete series is in L^q, where q is conjugate to p.*

Proof. For $1 \leq p \leq 2$, $q \in [2, \infty)$ and therefore $(\frac{2}{p} + \varepsilon)q = 2(q-1) + \varepsilon.q = 2q - 2 + \varepsilon' > 2$, where $\varepsilon' = \varepsilon.q$. For fixed $m, n, k \in \mathbb{Z}$ such that $m, n \in \mathbb{Z}(k)$ and $k > \gamma$ we obtain from the above thorem: $|\Psi_k^{m,n}(x)| \leq c\Xi(x)^{\frac{2}{p}+\varepsilon}$ for some constant $c > 0$, which further implies $|\Psi_k^{m,n}(x)|^q \leq c'\Xi(x)^{2+\varepsilon'}$ for some constant $c' > 0$ and $\varepsilon' > 0$. Now the result follows by inequalities (13) and (14) above. ∎

Theorem 2.8 (*Barker*) *Let $p \in (0,2]$ and $k \in \mathbb{Z}^*$. For a K-finite matrix coefficient $\Psi_k^{m,n}$ of π_k the following conditions are equivalent:*

(i) $|k| > \gamma = \frac{2}{p} - 1$,

(ii) $\Psi_k^{m,n} \in C^p(G)$,

(iii) $\Psi_k^{m,n} \in L^p(G)$.

See [Ba, Theorem 5.3 and Corollary 5.5] for a proof of this theorem. For the corresponding result in more general context we refer to [T, Section 5, Proposition 2].

Thus, it follows from the above proposition and theorem that matrix coefficients of an L^p-discrete series representation are both L^p-functions and L^q-functions (and hence in L^2 !). Theorem 2.8 provides the most convenient description of these representations as those discrete series representations which are not embedded in any of the principal series representations in $\mathcal{S}^\gamma$, the tube domain corresponding to p.

Spaces of Fourier transforms: (see [Ba, Section 9])

Definition 2.9 Suppose that $m, n \in \mathbb{Z}^\sigma$ for $\sigma \in \widehat{M}$. The principal part of the Fourier transforms space for $C^p(G)_{m,n}$, denoted by $C^p_H(\widehat{G})_{m,n}$, consists of continuous maps

$$F : \mathcal{S}^\gamma \longrightarrow \mathbb{C}$$

satisfying the following properties [Ba, p.39]:

1. F is holomorphic on $\mathring{\mathcal{S}}^\gamma$, the interior of $\mathcal{S}^\gamma$,
2. $$F(-\lambda) = \varphi^{n,m}_{\sigma,\lambda}\, F(\lambda) \quad \text{for all } \lambda \in \mathcal{S}^\gamma, \tag{23}$$
   where $\varphi^{m,n}_{\sigma,\lambda}$ is described as (see [Ba 7.1]):
   $$\varphi^{n,m}_{\sigma,\lambda} = \begin{cases} \frac{(|m|-1+\lambda)(|m|-3+\lambda)\cdots(|n|+1+\lambda)}{(|m|-1-\lambda)(|m|-3-\lambda)\cdots(|n|+1-\lambda)} & \text{if } |m| > |n|, \\ (-1)^{(n-m)/2} & \text{if } |m| = |n|, \\ \frac{(|n|-1-\lambda)(|n|-3-\lambda)\cdots(|m|+1-\lambda)}{(|n|-1+\lambda)(|n|-3+\lambda)\cdots(|m|+1+\lambda)} & \text{if } |m| < |n|, \end{cases} \tag{24}$$
3. $\widehat{\rho}_{H,l,r}(F) < \infty$ for all $l \in \mathbb{N}$, $r \in \mathbb{R}^+$, where
   $$\widehat{\rho}_{H,l,r}(F) = \sup_{\lambda \in \mathring{\mathcal{S}}^\gamma} |(\frac{d}{d\lambda})^l F(\lambda)|(1+|\lambda|)^r,$$
4. $F(k) = 0$ if $n.m < 0$, k is of parity opposite to that of m, n and $|k| \leq \min\{|m|,\ |n|,\ \gamma\}$,

We quote the following proposition from Barker[Ba, Proposition 7.2] .

Proposition 2.10 *Suppose $\sigma \in \widehat{M}$ and $n, m \in \mathbb{Z}^\sigma$. Then*

(i) $\lambda \mapsto \varphi_{\sigma,\lambda}^{n,m}$ *is a meromorphic function of* λ *whose only singularities are first order poles at* $\lambda = k \in \mathbb{Z}^{-\sigma}$ *such that* $|m| > k > |n|$ *or* $-|m| > k > -|n|$;

(ii) $\lambda \mapsto \varphi_{\sigma,\lambda}^{n,m}$ *has zeroes only at the points* $\lambda = k \in \mathbb{Z}^{-\sigma}$ *such that* $|n| > k > |m|$ *or* $-|n| > k > -|m|$;

(iii) $\varphi_{\sigma,-\lambda}^{n,m} = (\varphi_{\sigma,\lambda}^{n,m})^{-1} = \varphi_{\sigma,\lambda}^{m,n}$.

(iv) $\Phi_{\sigma,\lambda}^{n,m} = \varphi_{\sigma,\lambda}^{n,m}\Phi_{\sigma,-\lambda}^{n,m}$ *for all* $\lambda \in \mathbb{C}$ *which are not poles of* $\varphi_{\sigma,\lambda}^{n,m}$

Definition 2.11 The discrete part $C_B^p(\widehat{G})_{m,n}$ of the space of Fourier transforms of $C^p(G)_{m,n}$ is the set of all functions $F : L_\sigma^{m,n}(\gamma)^c \longrightarrow \mathbb{C}$.

Wave packet:

Definition 2.12 For $\varphi \in C_H^2(\widehat{G})_{m,n}$ the wave packet $\Phi_{\sigma,\varphi}^{m,n}$ is given by

$$\Phi_{\sigma,\varphi}^{m,n}(x) = \int_{\Re\lambda=0} \varphi(\lambda)\Phi_{\sigma,\lambda}^{m,n}(x)\mu(\sigma,\lambda)d\lambda,\ x \in G. \tag{25}$$

The existence of the integral is clear from the inequalities (15), (17) and property (3) of $C^p(\widehat{G})_{m,n}$. It can be shown that $\Phi_{\sigma,\varphi}^{m,n} \in C^2(G)$ ([Ba, Theorem 18.2]).

Analogously we define L^p-wave packet.

Definition 2.13 For $\varphi \in C_H^p(\widehat{G})_{m,n}$, the L^p-wave packet $S_{H,p}^{m,n}\varphi$ is given by,

$$\begin{aligned}(S_{H,p}^{m,n}\varphi)(x) &= (\tfrac{1}{2\pi})^2 \textstyle\int_{\Re\lambda=0} \varphi(\lambda)\Phi_{\sigma,\lambda}^{n,m}(x)\mu(\sigma,\lambda)d\lambda \\ &+ \tfrac{1}{2\pi}\Sigma_{l\in L_\sigma^{m,n}(\gamma)}\varphi(l)\Phi_{\sigma,l}^{n,m}(x)|l|.\end{aligned} \tag{26}$$

Again this L^p-wave packet, $S_{H,p}^{m,n}\varphi$ falls in $C^p(G)$ ([Ba, Theorem 18.2]). But note that the the integral term on the right side alone need not be even a L^p function as that would imply that the sum on the right side is also in $L^p(G)$. Now in case the sum consists of a single term (as will be in the case of a L^1-function of (m,n)-type with $m.n > 0$), this would contradict the fact that the matrix coefficients of a discrete series are L^p if and only if they are parametrized by points outside the strip (Theorem 2.8).

Also define,

$$(\mathcal{S}_{B,p}^{m,n}\varphi)(x) \quad = \quad \tfrac{1}{2\pi}\Sigma_{l\in L_\sigma^{m,n}(\gamma)^c}\varphi(l)\Psi_l^{n,m}(x)|l|. \tag{27}$$

The space $C^p(G)$ has a smooth splitting into topological direct sum of $C_B^p(G)$ and $C_H^p(G)$ ([Ba] Section 11) which may now be identified with the images of the spaces $C_H^p(\widehat{G})$ and $C_B^p(\widehat{G})$ under $\mathcal{S}_{H,p}$ and $\mathcal{S}_{B,p}$ respectively. Furthermore, for $F_H^{m,n}$, $F_B^{m,n}$, defined in Section 1, we have ([Ba, Theorem 18.2])

Theorem 2.14 (*Barker*)

$$F_H^{m,n} : C_H^p(G)_{m,n} \longrightarrow C_H^p(\widehat{G})_{m,n}$$

and

$$F_B^{m,n} : C_B^p(G)_{m,n} \longrightarrow C_B^p(\widehat{G})_{m,n}$$

are topological isomorphisms. Also $(F_H^{m,n})^{-1} = \mathcal{S}_{H,p}^{m,n}$ *and* $(F_B^{m,n})^{-1} = \mathcal{S}_{B,p}^{m,n}$.

As mentioned earlier we need to deal only with the *matrix coefficients* of the Fourier transforms and their spaces, namely $C^p(\widehat{G})_{m,n}$ for $m, n \in \mathbb{Z}$. And therefore the only isomorphism theorem we use is the above one.

We end this section with the following discussions:

1. A principal series representation π_{σ,λ_0} is not irreducible if and only if $\lambda \mapsto \mu(\sigma, \lambda)$ has a pole at the point λ_0.

2. We recall from Section 1 that the representations D_+ and D_- are known as mock discrete series as their matrix coefficients are not in $L^2(G)$ and hence they do not occur in the discrete part of the inversion formula for $C^2(G)$. Their matrix coefficients are infact $L^{2+\epsilon}$ for any $\epsilon > 0$. Similarly for $p < 2$ the discrete series representations which are *inside the strip* S^γ are mock in the context of $C^p(G)$. Because their matrix coefficients are not in $L^p(G)$ and therefore have no contribution to the discrete part of a function in C^p. These matrix coefficients are in $L^{q+\epsilon}$ for every $\varepsilon > 0$ (while L^p-discrete series are L^q).

Note that if for a function $f \in C^p(G)$, $\widehat{f}_H \equiv 0$ on the imaginary axis and hence on the strip S^γ, then $\widehat{f}_B$ is automatically zero on these mock L^p-discrete series. This points out that for L^p harmonic analysis the discrete series which are inside the strip are detached from the discrete part and, are actually integrated to the principal part.

It is possible to change the domain of integration in formula (26) for $\mathcal{S}^{m,n}_{H,p}\varphi$ from the imaginary axis (i.e. the unitary principal series) to any vertical line in S^γ which does not pass through any point of reducibility of the representation (equivalently, where the *Plancherel measure* $\mu(\sigma, \lambda)$ has a pole). Recall that when $m.n < 0$ there is no relevant discrete series *for a function of type* (m, n). *When both* m, n are positive and γ is not an integer of parity opposite to m, n, then by this change of contour (26) can be reduced to

$$(\mathcal{S}^{m,n}_{H,p}\varphi)(x) \quad = \quad (\tfrac{1}{2\pi})^2 \int_{\Re\lambda=\gamma} \varphi(\lambda)\Phi^{m,n}_{\sigma,\lambda}(x)\mu(\sigma, \lambda)d\lambda \qquad (28)$$

For m, n both negative and γ as above, the domain of integration $\Re\lambda = \gamma$ in the above formula is to be replaced by $\Re\lambda = -\gamma$. Note that the residue of a pole risen from the shift of contour cancels the component in the inversion formula corresponding to the discrete series parametrized by that pole.

3. It is known that unlike waves on $\mathbb{R}^n$ $(\chi_\lambda(t) = e^{i\lambda.t})$ as well as on any abelian groups, the matrix coeficients of the principal and discrete series representations of $SL_n(\mathbb{R})$, $n \geq 2$ which do not contain the trivial representation vanish at infinity. ((15) and Theorem 2.6.) But the matrix coefficients of most of the L^p-tempered representations do not have the required decay to be a member of $C^p(G)$. Only the matrix coefficients of the discrete series $\Psi^{m,n}_k$ which are embeded outside the strip S^γ are in C^p (Theorem 2.8). *These* $\Psi^{m,n}_k$ *decay faster as* $k \longrightarrow \infty$ (*see* Theorem 2.6). But then the zero-Schwartz space $C^0 = \cap_{p\in(0,2]}C^p$ (see [Ba, Section 19]) does not contain any $\Psi^{m,n}_k$ either. Note that for C^0 there is no *outside*

of the *strip*, because the *strip* is now the whole of $\mathbb{C}$.

3 Waves and Wave Packets

This section, aimed towards an exact analogue of the *original* Wiener Tauberian (W-T) theorem ([E-M]) for the case of non-spherical L^p-functions, somewhat digresses from the main theme of the thesis. We begin by explaining how the questions we address here have natural connection with the W-T theorem. However, we may stress that the rest of the thesis is independent of this section, and in the later sections we will develop W-T theorems, which are more general.

Recall that though the waves (*i.e* the matrix coefficients of the principal series representations) are not Schwartz space functions, the wave packets with Schwartz space functions on the strip S^γ belong to the corresponding Schwartz space $C^p(G)$. A possible question is: what condition on a function (on the strip S^γ) ensures that the wave packet with it is in the Schwartz space $C^p(G)$? This is a key step towards the W-T theorem for biinvariant functions in [E-M]. Now to tackle the non-biinvariant situation we need good answer to the same question in a more general context. We obtain a non-biinvariant quantitative version (Theorem 3.8) of Proposition 3.3 of [G-W] for the group $SL_2(\mathbb{R})$. For this the estimates come out of the work of [Ba]. This makes it possible to obtain a W-T theorem for $L^p(SL_2(\mathbb{R}))_{n,n}$, $1 \leq p \leq 2$ and $n \in \mathbb{Z}$.

In this section $G = SL_2(\mathbb{R})$ and $\gamma = \frac{2}{p} - 1$.

Definition 3.1 For a fixed $p \in (0, 2]$, let $T = (F_H, F_B) \in C^p_H(\widehat{G}) \oplus C^p_B(\widehat{G}) = C^p(\widehat{G})$ and $\gamma = \frac{2}{p} - 1$. T is said to be *K-finite* if it is the Fourier transform of a K-finite function in $C^p(G)$, *i.e.* if only finitely many components of F_H and F_B are nonzero.

Definition 3.2 Let T be a K-finite element of $C^p(\widehat{G})$. T is said to have a *pole neutralizing boundary* if for every $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$, $F_H(\sigma, \pm\gamma)_{m,n}$ and $F_B(\pm\gamma)_{m,n}$ are both zero whenever either $+\gamma$ or $-\gamma$ is in $L^{m,n}_\sigma$.

Asymptotic expansion of the matrix coefficients: For $\sigma \in \widehat{M}$, $m, n \in$

$\mathbb{Z}^\sigma$ and $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, there exist power series $\Sigma a_k x^k$ and $\Sigma b_k x^k$, convergent for $|x| < 1$, such that $\Phi^{m,n}_{\sigma,\lambda}$ has the following expansion ([Ba, 12.1]):

$$\Phi^{m,n}_{\sigma,\lambda}(a_t) = (-1)^{(m-n)/2} e^{-t}[e^{\lambda t}\Sigma_{k=0}^{\infty} b_k e^{-2kt} + e^{-\lambda t}\Sigma_{k=0}^{\infty} a_k e^{-2kt}] \tag{29}$$

for $t > 0$. Here $a_k = a^{m,n}_{\sigma,k}(\lambda)$ and $b_k = a^{m,n}_{\sigma,k}(\lambda)$ satisfy

$$a^{m,n}_{\sigma,k}(\lambda)\varphi^{m,n}_{\sigma,-\lambda} = b^{m,n}_{\sigma,k}(-\lambda). \tag{30}$$

And the constant terms in the expansions are the c-functions described below.

c-functions : (see [Ba, Section 6] for details.)

$$a^{m,n}_{\sigma,0}(\lambda) = c^{m,n,+}_{\sigma}(\lambda) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(-\lambda/2)\Gamma((1-\lambda)/2)}{\Gamma((1-\lambda-|n|)/2)\Gamma((1-\lambda+|n|)/2)} \tag{31}$$

and

$$b^{m,n}_{\sigma,0}(\lambda) = c^{m,n,-}_{\sigma}(\lambda) = \frac{(-1)^{(m-n)/2}}{\sqrt{\pi}} \cdot \frac{\Gamma(\lambda/2)\Gamma((1+\lambda)/2)}{\Gamma((1+\lambda-|m|)/2)\Gamma((1+\lambda+|m|)/2)}, \tag{32}$$

where $\lambda \in \mathbb{C} - \mathbb{Z}$ and Γ denotes the gamma function. From (31) and (32) it follows that [Ba; p.24, p.66]:

$$c^{n,m,-}_{\sigma}(\lambda)^{-1} = -\frac{\lambda+n-1}{\lambda-n+1} c^{n-2,m,-}_{\sigma}(\lambda)^{-1}. \tag{33}$$

$$c^{m,n,+}_{\sigma} = c^{m',n,+}_{\sigma}, \quad c^{m,n,-}_{\sigma} = (-1)^{(n-n')/2} c^{m,n',-}_{\sigma} \quad \text{for } m, m', n, n' \in \mathbb{Z}^\sigma; \tag{34}$$

$$c^{0,0,-}_{\sigma+}(\lambda) = \left[\frac{\lambda}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{1+\lambda/2}}\right]^{-1} \quad \text{for } \Re\lambda > -1; \tag{35}$$

$$c^{1,1,-}_{\sigma-}(\lambda) = \left[\frac{\lambda}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^{(1+\lambda)/2}}\right]^{-1} \quad \text{for } \Re\lambda > 0 \tag{36}$$

Also (see [Ba, p.64])

$$\mu(\sigma,\lambda) = -i(-1)^{(m-n)/2}(c^{m,n,+}_{\sigma}(\lambda)c^{n,m,-}_{\sigma}(\lambda))^{-1} \tag{37}$$

These c-functions are meromorphic as a function in λ and we will use the fact that the zeros of $c^{m,n,-}_{\sigma}$ occur at $\lambda \in \mathbb{Z}^{-\sigma}$ such that $\lambda < -|m| < 0$ or $0 < \lambda < |m|$ (see [Ba, Proposition 6.1 (iv)]).

Truncation of the asymptotic expansion of matrix coefficients : For each $j \in \mathbb{N}$ the j-th truncation of the matrix coefficients $\Phi^{m,n}_{\sigma,\lambda}$ are defined as

$${}^{j}\Lambda^{m,n}_{\sigma,\lambda}(a_t) = (-1)^{(m-n)/2} e^{-t}[e^{\lambda t}\Sigma_{k=0}^{j} b_k e^{-2kt} + e^{-\lambda t}\Sigma_{k=0}^{j} a_k e^{-2kt}],\ t > 0 \tag{38}$$

Definition 3.3 The j-th truncation wave packet of a function φ is defined by

$$ {}^j\Lambda^{m,n}_{\sigma,\varphi}(a_t) = \int_{\Re\lambda=0} \varphi(\lambda)\, {}^j\Lambda^{m,n}_{\sigma,\lambda}(a_t)\mu(\sigma,\lambda)d\lambda,\ t>0 \tag{39} $$

The range of p**:** Unlike the rest of the thesis, in this section p will lie in the range $\frac{2}{3} < p \leq 2$ (unless mentioned otherwise). We may stress that our decision to restrict p in the above interval is really a matter of convenience, firstly that for $p \in (\frac{2}{3}, 2)$ there is no other discrete series inside the strip S^γ except those which are already there for $p = 1$ and more importantly, in that range of p we need only the constant term (0-th truncation) of the expansion (29) in our analysis. In fact $\frac{2}{3}$ appears to be the *natural bound* for using only the constant term, as is evident from the proof of Theorem 3.8.

Now we shall closely examine some results of [Ba] and for our range of p, will compute the *exact* degrees of the polynomials (in m, n, t, λ etc), involved in these results. The same method, however, will work for any $p \in (0,1)$ to give analogous results. It is clear from [Ba] that the difference will be in the degrees of the polynomials involved, and in the order of the derivatives of a function on S^γ and its wave packet.

Lemma 3.4 *Let λ be such that either $\Re\lambda = 0$ or $\lambda = \pm 1$. Then there exist $r_1, r_2 \geq 0$ and a positive constant C such that the inequality*

$$ |\Phi^{m,n}_{\sigma,\lambda}(a_t) - {}^0\Lambda^{m,n}_{\sigma,\lambda}(a_t)| \leq C(1+|m|)^{r_1}(1+|n|)^{r_2}(1+|\lambda|)^2(1+t)^{1+\zeta}e^{-(3+\zeta)t}, $$

holds for all $t > 0$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$, with $\zeta = 0$ when $\Re\lambda = 0$ and for some $\zeta > 0$ when $\lambda = \pm 1$.

Proof. Let $m, n \in \mathbb{Z}^\sigma$, $t > 0$ and $\lambda = \pm 1$ or $\Re\lambda = 0$. Then

$$ \begin{aligned} |\Phi^{m,n}_{\sigma,\lambda}(\check{Y}; a_t; Y)| &\leq C_1 (1+|m|)^{r_1} (1+|n|)^{r_2} (1+|\lambda|)^2\, \Xi(a_t)^{1+\zeta} \\ &\leq C_2 (1+|m|)^{r_1} (1+|n|)^{r_2} (1+|\lambda|)^2 (1+t)^{1+\zeta} e^{-(1+\zeta)t}, \end{aligned} $$

for suitable r_1, r_2 and where ζ is as given in the statement of the lemma. For the first case the first inequality is a consequence of (15). And for the second case it is a consequence of Theorem 2.6 (with $p = 2$) and Proposition 1.2.

The second inequality follows from (12) above. Now,

$$\begin{aligned}\Phi^{m,n}_{\sigma,\lambda}(a_t; H^2 + H - (\lambda^2 - 1)/4) &= \Phi^{m,n}_{\sigma,\lambda}(a_t; \Omega + \overline{Y}Y - (\lambda^2 - 1)/4) \\ &= \Phi^{m,n}_{\sigma,\lambda}(a_t; \overline{Y}Y) \\ &= e^{-2t}\, \Phi^{m,n}_{\sigma,\lambda}(\overline{Y}; a_t; Y).\end{aligned}$$

Therefore,

$$\Phi^{m,n}_{\sigma,\lambda}(a_t; H^2+H-(\lambda^2-1)/4) \leq C\,(1+|m|)^{r_1}\,(1+|n|)^{r_2}\,(1+|\lambda|)^2\,(1+t)^{1+\zeta}\,e^{-(3+\zeta)t}.$$

Hence from Proposition 12.3 (iii) of [Ba], the result follows. ∎

Lemma 3.5

$$\begin{aligned} \textit{Let } P(\lambda) &= \lambda - 1 \quad \textit{when } \sigma = \sigma_+,\ p \leq 1,\ \textit{and } |n| > 1, \\ &= 1 \qquad\quad \textit{otherwise.}\end{aligned}$$

Then there exist real numbers M, r and $\varepsilon > 0$ such that for $m, n \in \mathbb{Z}^\sigma$ and $s \in \mathbb{N}$,

$$|(\frac{d}{d\lambda})^s (P(\lambda) a^{m,n}_{\sigma,0}(\lambda)\mu(\sigma,\lambda))| \leq M(1+|n|)^r(1+|\lambda|)^r$$

for $-\varepsilon \leq \Re\lambda \leq \gamma + \varepsilon$, and

$$|(\frac{d}{d\lambda})^s (P(-\lambda) b^{m,n}_{\sigma,0}(\lambda)\mu(\sigma,\lambda))| \leq M(1+|m|)^r(1+|\lambda|)^r$$

for $-\gamma - \varepsilon \leq \Re\lambda \leq \varepsilon$, where $r = 2$ when $P(\lambda) = \lambda - 1$ and $r = 1$ when $P(\lambda) = 1$.

Proof. Let us first consider the first inequality in the case $s = 0$. From (31) above it is clear that the value of m is not relevant here and n can be taken to be ≥ 0. So we may assume that $m = 0$ or $m = 1$ according as $\sigma = \sigma_+$ or $\sigma = \sigma_-$. Using (33) and (37) we have,

$$P(\lambda) a^{m,n}_{\sigma,0}\mu(\sigma,\lambda) = \begin{cases} -i(-1)^{m/2} \frac{P(\lambda)}{c^{0,0,-}_{\sigma_+}(\lambda)} \prod_{j=1}^{n/2} \frac{\lambda+2j-1}{\lambda-2j+1} & \text{if } \sigma = \sigma_+ \\ -i(-1)^{(m-1)/2} \frac{P(\lambda)}{c^{1,1,-}_{\sigma_-}(\lambda)} \prod_{j=1}^{(n-1)/2} \frac{\lambda+2j}{\lambda-2j} & \text{if } \sigma = \sigma_- \end{cases}$$

When the product is taken over empty set our convention is to take the product term equal to 1. From the discussion about the c-functions above, we know that the c-functions in the above expression have no zeros in the region $\Re\lambda > -1$. And a careful choice of ε_1 to avoid any integer in $(\gamma, \gamma+\varepsilon_1)$ ensures that any possible zero in the range $-\varepsilon_1 < \lambda < \gamma+\varepsilon_1$ in the denominator of the product term will be cancelled by a zero in $P(\lambda)$.

Consider the case $\sigma = \sigma_+$. Let

$$S = \prod_{j=1}^{n/2} \frac{\lambda+2j-1}{\lambda-2j+1} = \prod_{j=1}^{n/2} \frac{\lambda+n-2j+1}{\lambda-2j+1}$$

and let η_0 denote the first integer which is greater than or equal to $\Re\lambda$. Then for $\eta_0 < j \leq \frac{n}{2}$ the absolute value of the j-term in the second expression is less than or equal to 1. Let $\eta_1 = \min(\eta_0, n/2)$. Then

$$|S| \leq \prod_{j=1}^{\eta_1} |\frac{\lambda+n-2j+1}{\lambda-2j+1}|.$$

We now consider the specific cases.

When $P(\lambda) = (\lambda-1)$: Here $p \leq 1$ and $|n| > 1$. Hence $\eta_0 = 2$ and only possible values of η_1 are 1 and 2. When $\eta_1 = 2$, then $|P(\lambda)S| \leq |\frac{(\lambda+n-1)(\lambda+n-3)}{\lambda-3}|$ and when $\eta_1 = 1$, then $|P(\lambda)S| \leq |\lambda+n-1|$.

Next we consider the case when $P(\lambda) = 1$. Then either $p > 1$ or $|n| = 0$ or both (as we have assumed $\sigma = \sigma_+$). Also $\eta_0 = 1$ or 2 according as $p > 1$ or $p \leq 1$. But if $p \leq 1$ then $n = 0$ in this case. Then only possible values of η_1 are 0 and 1. When $\eta_1 = 0$ then clearly $n = 0$ and hence $S = 1$ (as $m = n = 0$). When $\eta_1 = 1$ then $p > 1$ and $n \neq 0$. Hence $|P(\lambda)S| \leq |\frac{(\lambda+n-1)}{(\lambda-1)}|$.

Combining all the cases we have for $\sigma = \sigma_+$,

$$|P(\lambda)a^{m,n}_{\sigma,0}\mu(\sigma,\lambda)| \leq C(1+|n|)^2(1+|\lambda|)^r |\, c^{0,0,-}_{\sigma_+}(\lambda)|^{-1} \tag{40}$$

where $r = 1$ or 0 according as $P = (\lambda-1)$ or $P = 1$ for all $n, m \in \mathbf{Z}^\sigma$ and $-\varepsilon_1 \leq \Re\lambda \leq \gamma+\varepsilon_1$. Here C is a constant depending on γ. In exactly the same way we can also obtain the relation involving $b^{m,n}_{\sigma,0}$, replacing σ_+ by σ_- and $c^{0,0,-}_{\sigma_+}$ by $c^{1,1,-}_{\sigma_-}$. Now with a carefully chosen $\varepsilon_1 > 0$, we have from (35) and (36)

$$|c^{0,0,-}_{\sigma_+}|^{-1} \leq C'(1+|\lambda|) \text{ and } |c^{1,1,-}_{\sigma_-}|^{-1} \leq C'(1+|\lambda|). \tag{41}$$

From (40) and (41) we have, for all $\sigma \in \widehat{M}$ and $\gamma > 0$,

$$|P(\lambda)a^{m,n}_{\sigma,0}\mu(\sigma,\lambda)| \leq M(1+|n|)^2(1+|\lambda|)^{r+1} \tag{42}$$

for some constant M, where $r = 1$ or 0 according as $P = (\lambda - 1)$ or $P = 1$. This proves the lemma for $s = 0$.

Now $\lambda \longrightarrow P(\lambda)a^{m,n}_{\sigma,0}\mu(\sigma,\lambda)$ is holomorphic in the strip $-\varepsilon_1 \leq \Re\lambda \leq \gamma+\varepsilon_1$. Take any λ_0 in the strip $-\varepsilon_1/2 \leq \Re\lambda \leq \gamma+\varepsilon_1/2$ and apply Cauchy's estimate on the ball $||\lambda - \lambda_0|| \leq \varepsilon/4$. Then from (42) we get the required inequality, namely

$$|(\tfrac{d}{d\lambda})^s(P(\lambda)a^{m,n}_{\sigma,0}(\lambda)\mu(\sigma,\lambda))| \leq M(1+|n|)^r(1+|\lambda|)^r$$

in the strip $-\varepsilon_1/2 \leq \Re\lambda \leq \gamma + \varepsilon_1/2$ for $s > 0$. An exactly similar proof will work for $b^{m,n}_{\sigma,0}$. ∎

Also from [Ba, Lemma 15.1] we get:

Lemma 3.6 *Let η be $+1$ or -1 and $F(\lambda)$ be a continuous function on S^1 which is analytic in the interior. Let $F(\eta) = 0$. Then for any $r, s \in \mathbb{N}$ there are polynomials $p_0, \ldots, p_l$ in the variable λ such that,*

$$\sup_{|\Re\lambda|<1} (1+|\lambda|)^r|(\frac{d}{d\lambda})^s\frac{F(\lambda)}{\lambda-\eta}| \leq M\Sigma^l_{k=0} \sup_{|\Re 1|<\gamma} p_k(|\lambda|)|F^{(k)}(\lambda)| \tag{43}$$

where the degree of each p_k is at most $s+r$ and $l \leq 2s+1$. Moreover, $\lambda \longrightarrow F(\lambda)/(\lambda-\eta)$ defines a continuous function on S^1.

Proof. The inequality follows from Lemma 15.1 of [Ba]. To find the degrees of the polynomials p_k and the order of the derivatives of $F(\lambda)$ one just needs to follow the proof carefully. Here we will avoid reproducing the proof from [Ba] and instead refer to the equation numbers whenever necessary.

Let $\mathring{S}^\gamma = S_1 \cup S_2$, where

$$S_1 = \{\lambda \in \mathring{S}^\gamma : |\Im\lambda| \geq 1\}$$
$$S_2 = \{\lambda \in \mathring{S}^\gamma : |\Im\lambda| < 1\}$$

For S_1 we apply Leibniz's rule to obtain the existence of a constant M such that

$$\sup_{\lambda\in S_1}(1+|\lambda|)^r|(d/d\lambda)^s\left(\frac{F(\lambda)}{\lambda-\eta}\right)| \leq M\sup_{\lambda\in S_1}|\lambda-\eta|^{-s-1}\sum_{k=0}^{s}(1+|\lambda|)^r|\lambda-\eta|^kF^{(k)}(\lambda)|,$$

for F as stated above. Since $|\lambda - \eta|^{-s-1} \leq 1$ for $\lambda \in S_1$, this establishes (43).

While considering S_2 we see that there exists an M such that

$$\sup_{\lambda \in S_2} (1 + |\lambda|)^r |(d/d\lambda)^s \left(\frac{F(\lambda)}{\lambda - \eta} \right)| \leq M \sup_{\lambda \in S_2} |(d/d\lambda)^s \left(\frac{F(\lambda)}{\lambda - \eta} \right)|$$

It then follows from [Ba, (15.9) (15.12), (15,13)] that

$$|(d/d\lambda)^s \left(\tfrac{F(\lambda)}{\lambda - \eta} \right)| \leq \sup_{\lambda \in S_2} |\psi_s^{(s+1)}(\lambda)|/s!,$$

where

$$\psi_s(\lambda) = s! \textstyle\sum_{k=0}^{s} (-1)^{s-k} (\lambda - \eta)^k F^{(k)}(\lambda)/k!$$

However, from the last equation, it follows that there exist polynomials p_k of degree $\leq s$ such that

$$\psi_s^{(s+1)}(\lambda) = \textstyle\sum_{k=0}^{2s+1} p_k(|\lambda|) F^{(k)}(\lambda).$$

Now from (43) it follows that $F(\lambda)/(\lambda - \eta)$ is uniformly continuous on $\mathring{S}^1$. Hence the second conclusion of the lemma. ∎

Examining [Ba, Proposition 13.4] for the cases we are dealing with, we get

Proposition 3.7 *(Barker) Consider the functions $\lambda \mapsto a_{\sigma,k}^{m,n}(\lambda)\mu(\sigma, \lambda)$ and $\lambda \mapsto b_{\sigma,k}^{m,n}(\lambda)\mu(\sigma, \lambda)$, where $m, n \in \mathbb{Z}^\sigma$, $\sigma \in \widehat{M}$ and $k \in \mathbb{N}$. Then*

1. *For $m = n = 0$ or $m = n = 1$ the above functions have no pole. More generally, for $n = 0$ or 1 the first function has no pole and for $m = 0$ or 1 the second function has no pole.*
2. *If $p > 1$ then none of the functions has any pole inside the strip S^γ.*
3. *If m, n are of odd parity then even for $\frac{2}{3} \leq p \leq 1$ they have no pole in the strip S^γ.*
4. *If m, n are of even parity and $\frac{1}{2} \leq p \leq 1$ then the the functions have at most one pole in the strip S^γ. The first function has a pole at $\lambda = 1$ if $n \neq 0$, and the second one has a pole at $\lambda = -1$ if $m \neq 0$.*

As p decreases below 1, more and more singularities of the above functions come inside the strip S^γ.

Proposition 3.7(1) is what simplifies the treatment of the K-biinvariant functions. It also points out that functions of type $(1,1)$ can be treated analogously. Proposition 3.7 also shows that the case when $p = 1$ and the K-types involved are even (but not zero), is the most delicate one. Because when $p = 1$ (and hence $\gamma = 1$) then the Fourier transform of a function with only even K-types m, n does not have, in general, pole neutralizing boundary and the above fuctions have singularities at $k = \pm 1$ (see Proposition 3.7(4)).

Now we are in a position to state the main result of this section, which is proved by a step by step adaptation of the arguments in [Ba, Theorem 15.2]. In writing the proof, whenever no more than an exact reproduction of the argument from [Ba] is necessary, we merely refer to the appropriate step in [Ba].

Theorem 3.8 *Fix any $p \in (\frac{2}{3}, 2]$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^{\sigma}$. Let $F(\lambda) = F(\sigma, \lambda)$ be a continuous function on S^{γ} which is analytic on $\mathring{S}^{\gamma}$ and satisfies the relation $F(\lambda) = \varphi^{m,n}_{\sigma,\lambda} F(-\lambda)$ on S^{γ}. Assume further that $F(0) = 0$ when m, n are odd and $m.n < 0$.*

Let there exist an $s \in \mathbb{N}$ such that $\sup_{\lambda \in \mathring{S}^{\gamma}} |(\frac{d}{d\lambda})^{l} F(\sigma, \lambda)|(1 + |\lambda|)^{u} < \infty$ for all $u > 2$ and $l \leq s$.

Then the wave packet $S^{m,n}_{H,p} F$ exists and

$$\sup_{x \in G} (1 + \sigma(x))^{r} \Xi(x)^{-\frac{2}{p}} |(S^{m,n}_{H,p} F)(g_1; x; g_2)| < \infty,$$

and also

$$\sup_{x \in G} (1 + \sigma(x))^{r} e^{\frac{2}{p} t} |(S^{m,n}_{H,p} F)(g_1; x; g_2)| < \infty$$

for $g_1, g_2 \in \mathcal{U}$ and for all $r \in \mathbb{R}$ such that $2r+1 \leq s$ and $\max\{2+r, 4+\varepsilon\} + d \leq u$ for some $\varepsilon > 0$, where a_t is the A-part in the Cartan decomposition of x, $\sigma(x) = |t|$ and $d = \deg g_1 + \deg g_2$. When either $p \neq 1$ or m, n are odd or $m = n = 0$, it is sufficient to take $r \leq s$.

Proof. We first note that in view of the relation $(\Xi(a_t))^{-1} \leq e^{t}$ for $t > 0$ (equation (12) in Section 2), it is enough to prove the second inequality, namely

$$\sup_{x\in G}(1+\sigma(x))^r e^{\frac{2}{p}t}|(\mathcal{S}^{m,n}_{H,p}F)(g_1;x;g_2)| < \infty$$

where x, t, g_i etc are described as in the statement of the theorem. To prove the above inequality, first consider the case when F has pole neutralizing boundary (see Definition 3.2). For our range of p it means that $F(\pm 1) = 0$ when $p = 1$, $\sigma = \sigma_+$ and $m.n > 0$.

It is clear from inequalities (15) and (17) of Section 2, that $u > 2$ is enough for the existence of the wave packet $\mathcal{S}^{m,n}_{H,p}F$. Again from the inequality (17) and Lemma 3.4 we get for all $\sigma \in \widehat{M}$, $m,n \in \mathbb{Z}^\sigma$ and $t > 0$,

$$\begin{aligned}
&(\tfrac{1}{2\pi})^2|\Phi^{n,m}_F(a_t) - {}^0\Lambda^{n,m}_F(a_t)| \\
&\quad\leq\ (\tfrac{1}{2\pi})^2 \textstyle\int_{\Re\lambda=0}|F(\lambda)|\,|\Phi^{n,m}_\lambda(a_t) - {}^0\Lambda^{n,m}_\lambda(a_t)|\,|\mu(\lambda)|\,d\lambda \\
&\quad\leq\ C\,(1+|m|)^{r_1}\,(1+|n|)^{r_2}\,(1+t)\,e^{-3t}\textstyle\int_{\Re\lambda=0}|F(\lambda)|(1+|\lambda|)^3 d\lambda \\
&\quad\leq\ C'\,(1+|m|)^{r_1}\,(1+|n|)^{r_2}\,(1+t)\,e^{-3t}\sup_{\Re\lambda=0}(1+|\lambda|)^{4+\varepsilon}|F(\lambda)|,
\end{aligned}$$

for constants C and C', $\varepsilon > 0$ and positive real numbers r_1, r_2. Now as $\frac{2}{3} < p < 2$, $\frac{2}{p} - 3 = -\varepsilon_p$, for some $\varepsilon_p > 0$, we have

$$\begin{aligned}
(\star)\ \ &(1+t)^r e^{\frac{2}{p}t}|(\mathcal{S}^{m,n}_{H,p}F)(a_t)| \\
&\qquad\leq (1+t)^r e^{(\frac{2}{p})t}|(1/2\pi)^2\,{}^0\Lambda^{n,m}_F(a_t) + [(\tfrac{1}{2\pi})F(l)\Phi^{n,m}_{\sigma,l}]| \\
&\qquad\quad + (1+t)^{1+r}e^{-\varepsilon_p t}(1+|m|)^{r_1}(1+|n|)^{r_2}\sup_{\Re\lambda=0}(1+|\lambda|)^{4+\varepsilon}|F(\lambda)|,
\end{aligned}$$

for all $\sigma \in \widehat{M}$, $m,n \in \mathbb{Z}^\sigma$ and $t > 0$. Here, the part of the first term on the right hand side which is within the square bracket, occurs only when $p \in (\frac{2}{3}, 1)$ and m,n are even with $m.n > 0$. Also $l = +1$ if $m,n > 0$ and $l = -1$ if $m,n < 0$. We note that since $t > 0$, the last term on the right side of $(\star)$ is bounded.

To estimate the term involving ${}^0\Lambda^{n,m}_{\sigma,F}(a_t)$ on the right side of $(\star)$, consider

$$\begin{aligned}
{}^0\Lambda^{n,m}_{\sigma,F}(a_t) &= (-1)^{(m-n)/2}e^{-t}\textstyle\int_{\Re\lambda=0}F(\lambda)e^{-\lambda t}a^{n,m}_{\sigma,0}(\lambda)\mu(\sigma,\lambda)d\lambda \\
&\quad + (-1)^{(m-n)/2}e^{-t}\textstyle\int_{\Re\lambda=0}F(\lambda)e^{\lambda t}b^{n,m}_{\sigma,0}(\lambda)\mu(\sigma,\lambda)d\lambda.
\end{aligned}$$

We will shift the integrations involving $a^{n,m}_{\sigma,0}$ and $b^{n,m}_{\sigma,0}$ respectively to $\Re\lambda = \gamma$ and $\Re\lambda = -\gamma$. These shifts will be valid for the following reasons: (i) by the estimate on F given in the hypothesis and by Lemma 3.5 above, the contour integral over the horizontal line segments will tend to zero as their height goes to infinity and *(ii)* again by Proposition 3.7 together with the fact F has pole neutralizing boundary, the integrand has no pole on $\Re\lambda = -\gamma$ and on $\Re\lambda = \gamma$.

Again by Proposition 3.7 the only possible singularities to be encountered in these shifts are poles of order one which occur only when $p \in (\frac{2}{3}, 1)$ and m, n are even. More precisely, the singularity for the integral involving $a^{n,m}_{\sigma,0}$ occurs at 1 if $|m| > 1$, while the singularity for the $b^{n,m}_{\sigma,0}$ integral occurs at -1 if $|n| > 1$.

From [Ba, Proposition 14.1, Theorem 14.3] we see that the residue is zero unless m, n, l are of the same sign, where $l = \pm 1$ and the residue in the a_k or b_k shifts (according as m, n are positive or negative) is given by:

$$-2\pi F(l)^0\Lambda^{n,m}_{\sigma,l}(a_t).$$

Therefore, we have, after the shifts,

$$\begin{aligned}
&(1+t)^r e^{\frac{2}{p}t}|(S^{m,n}_{H,p}F)(a_t)| \\
&\quad \le (1+t)^r e^{(\frac{2}{p})t}(1/2\pi)^2\,\{e^{-t}|\textstyle\int_{\Re\lambda=\gamma} F(\lambda)e^{-\lambda t}a^{n,m}_{\sigma,0}(\lambda)\mu(\sigma,\lambda)d\lambda| \\
&\quad + e^{-t}|\textstyle\int_{\Re\lambda=-\gamma} F(\lambda)e^{\lambda t}b^{n,m}_{\sigma,0}(\lambda)\mu(\sigma,\lambda)d\lambda|\} \\
&\quad + [(1/2\pi)(1+t)^r e^{(\frac{2}{p})t}|F(l)(\Phi^{n,m}_{\sigma,l}(a_t) - {}^0\Lambda^{n,m}_{\sigma,l}(a_t))|] \\
&\quad + C\,(1+|m|)^{r_1}\,(1+|n|)^{r_2}\sup\nolimits_{\Re\lambda=0}(1+|\lambda|)^{4+\varepsilon}|F(\lambda)|
\end{aligned} \tag{44}$$

for some constant C, where $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $t > 0$. As before, the third term in the above inequality is kept within the square bracket to indicate that it occurs only when $p \in (\frac{2}{3}, 1)$, m, n are even and $m.n > 0$. Also, $l = +1$ or -1 according as m, n are positive or negative.

By Lemma 3.4 there exists $\zeta > 0$ such that

$$|\Phi^{n,m}_{\sigma,l}(a_t) - {}^0\Lambda^{n,m}_{\sigma,l}(a_t)| \le (1+|m|)^{r_1}(1+|n|)^{r_2}(1+|l|)^2(1+t)^{1+\zeta}e^{-(3+\zeta)t},$$

for $l = \pm 1$ and $n, m \in \mathbb{Z}(l)$. Thus the third term in (44) is bounded by

$$M(1+t)^{r+1+\zeta}e^{(\frac{2}{p}-3-\zeta)t}\sup_{|\Re\lambda|<\gamma}(1+|m|)^{r_1}(1+|n|)^{r_2}|F(\lambda)|,$$

for some constant $M > 0$ and positive real numbers r_1, r_2. Now as $\frac{2}{p} - 3 - \zeta$ is strictly negative for any fixed p in the range $(\frac{2}{3}, 2]$, the above expression is bounded as required in the theorem. Now as in Lemma 3.5 we define

$$\begin{aligned} P(\lambda) &= \lambda - 1 \quad \text{if } p = 1,\ \sigma = \sigma_+ \text{ and } |m| > 1 \\ &= 1 \qquad \text{otherwise} \end{aligned}$$

We absorb $(1+t)^r$ in the first term of (44) within the integral and make use of the above polynomial to write:

$$\begin{aligned} &\int_{\Re\lambda=\gamma} F(\lambda)(1+t)^r e^{-\lambda t} a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda \\ &\quad = \int_{\Re\lambda=\gamma} (1 - \tfrac{d}{d\lambda})^r \{e^{-\lambda t}\} F(\lambda) a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda \\ &\quad = \int_{\Re\lambda=\gamma} (1 - \tfrac{d}{d\lambda})^r \{e^{-\lambda t}\} \tfrac{F(\lambda)}{P(\lambda)}(P(\lambda) a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda))d\lambda. \end{aligned} \tag{45}$$

When $p = 1$ then $F(1) = 0$ and $P(1) = 0$. By Lemma 3.6, $\frac{F(\lambda)}{P(\lambda)}$ is continuous on $\Re\lambda = \gamma$, and by Proposition 3.7 $P(\lambda)a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)$ is well defined on $\Re\lambda = \gamma$. Lemma 3.5 and Lemma 3.6 show that we can perform integration by parts on the right hand side of (45) and obtain

$$\begin{aligned} &\int_{\Re\lambda=\gamma} F(\lambda)(1+t)^r e^{-\lambda t} a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda \\ &= \int_{\Re\lambda=\gamma} e^{-\lambda t}(1 + d/d\lambda)^r \{\tfrac{F(\lambda)}{P(\lambda)}(P(\lambda)a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda))\}d\lambda. \end{aligned}$$

Using Leibnitz rule on the right hand side we get,

$$\begin{aligned} &|\int_{\Re\lambda=\gamma} F(\lambda)(1+t)^r e^{-\lambda t} a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda| \\ &\le e^{-\gamma t}\Sigma_{j=0}^{r}\Sigma_{s=0}^{j}\binom{r}{j}\binom{j}{s}\int_{\Re\lambda=\gamma} |(\tfrac{d}{d\lambda})^s(\tfrac{F(\lambda)}{P(\lambda)})(\tfrac{d}{d\lambda})^{j-s}(P(\lambda)a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda))|.|d\lambda|. \end{aligned}$$

Using Lemma 3.5 and the Proposition 3.7 we see that the second term of the integrand on the right side is bounded by $M(1+|m|)^2(1+|\lambda|)^2$ for some constant M. Therefore

$$\begin{aligned} &|\int_{\Re\lambda=\gamma} F(\lambda)(1+t)^r e^{-\lambda t} a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda| \\ &\qquad \le M e^{-\gamma t}(1+|m|)^2 \sup_{0\le s\le r, \Re\lambda=\gamma} |(1+|\lambda|)^2 (d/d\lambda)^s \left(\tfrac{F(\lambda)}{P(\lambda)}\right)| \end{aligned}$$

for all $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $t > 0$. Also as $\frac{F(\lambda)}{P(\lambda)}$ is continuous we replace $\Re\lambda = \gamma$ by $|\Re\lambda| < \gamma$. Then using Lemma 3.6 we get

$$\begin{aligned} &|\int_{\Re\lambda=\gamma} F(\lambda)(1+t)^r e^{-\lambda t} a_{\sigma,0}^{n,m}(\lambda)\mu(\sigma,\lambda)d\lambda| \\ &\qquad \le M e^{-\gamma t}(1+|m|)^2 \sup_{|\Re\lambda|<\gamma,\ 0\le s\le v}(1+|\lambda|)^{2+r}\,|F^{(s)}(\lambda)| \end{aligned}$$

for all $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $t > 0$ where $v = 2r + 1$ or r according as $P(\lambda) = \lambda - 1$ or $P(\lambda) = 1$.

For the $b^{n,m}_{\sigma,0}$ integral we get a similar result and finally arrive at,

$$(1+t)^r e^{\frac{2}{p}t} |(S^{m,n}_{H,p} F)(a_t)| \leq M q(m,n) \sup_{|\Re\lambda|<\gamma, 0\leq s\leq v} (1+|\lambda|)^u |F^{(s)}(\lambda)| \tag{46}$$

for all $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $t > 0$, where v is as above and $u = \max\{4 + \varepsilon, 2 + r\}$ and $q(m, n)$ is a polynomial in m, n.

For $g_1 = g_2 = 1$, our theorem follows trivially from (46) using Cartan decomposition on G if F has pole neutralizing boundary.

Notice that for general $g_1, g_2 \in \mathcal{U}$, we justify differentiation under the integral sign in (25) by appealing to (15) and (17), and get:

$$\begin{aligned}(S^{m,n}_{H,p} F)(g_1, x, g_2) &= (\tfrac{1}{2\pi})^2 \textstyle\int_{\Re\lambda=0} F(\lambda) \Phi^{m,n}_{\sigma,\lambda}(g_1, x, g_2) \mu(\sigma, \lambda) d\lambda \\ &+ \tfrac{1}{2\pi} \Sigma_{l\in L^{m,n}_\sigma(\gamma)} F(l) \Phi^{m,n}_{\sigma,l}(g_1, x, g_2)|l|.\end{aligned} \tag{47}$$

Using relations (9) in Section 1, one can show that there is a finite collection of polynomials $\{p^\sigma_{i,j} | \sigma \in \widehat{M}, i, j \in I\}$ in m, n, λ such that

$$\Phi^{n,m}_{\sigma,\lambda}(g_1, x, g_2) = \Sigma_{i,j\in I} p^\sigma_{i,j}(\lambda, m, n) \Phi^{n+i,m+j}_{\sigma,\lambda}(x)$$

for all $\lambda \in \mathbb{C}, \sigma \in \widehat{M}, m, n \in \mathbb{Z}^\sigma$ and $x \in G$, where the degree of the polynomial $p^\sigma_{i,j}$ in λ is less than or equal to the sum of the degrees of g_1 and g_2 while the degrees in terms of m and n are respectively less than or equal to the degree of g_2 and g_1. Now one can replace $F(\lambda)$ in (46), by $F_{i,j}(\lambda) = p^\sigma_{i,j}(\lambda, m, n) F(\lambda)$. Because $F_{i,j}$ satisfies $F_{i,j}(\lambda) = \varphi^{m+j,n+i}_{\sigma,\lambda} F_{i,j}(-\lambda)$ on S^γ, it also has *pole neutralizing boundary* and it satisfies all the other properties of F stated in the hypothesis with u replaced by $u - d$.

For F with pole neutralizing boundary, we may leave the proof at this point. From here the arguments in [Ba, Theorem 15.1] will directly lead to the desired result.

Now we remove the restriction that F has pole neutralizing boundary and deal with the general case.

When $p \neq 1$ or m, n are odd or $m.n \leq 0$, then the above discussion itself settles the result as any function on the strip S^γ trivially has pole neutralizing boundary. To deal with the case $p = 1$ and m, n even with $m.n > 0$, we have to go through the methods developed by Campoli, Trombi, Ragozin and Barker (see [C], [Ba] and [T]). Define H^1 by:

$$H^1(\lambda) = e^{\lambda^2 - 1}(\varphi^{m,n}_{\sigma_+,\lambda}/\varphi^{m,n}_{\sigma_+,1})^{1/2} \tag{48}$$

for $\lambda \in S^1$ (see [Ba, Section 17]). Then the branch of the square root in (48) can be chosen such that:

(i) $H^1(1) = 1$ and $H^1(-1) = \varphi^{m,n}_{\sigma,-1}$,

(ii) $H^1(\lambda) = \varphi^{m,n}_{\sigma_+,\lambda}H^1(-\lambda)$ for all $\lambda \in \mathbb{C}$,

(iii) $H^1 \in C^{p'}_H(\widehat{G})_{m,n}$ for all $p' \in (0, 2]$ (see [Ba, Lemma 17.1]).

Let $F_0(\lambda) = F(\lambda) - F(1)H^1(\lambda)$. Then F_0 will satisfy all the conditions of the theorem and moreover, it has pole neutralizing boundary. Hence the theorem is true for F_0. On the other hand, $S^{m,n}_{H,1}H^1 \in C^1(G)_{m,n}$ since $H^1 \in C^1(\widehat{G})_{m,n}$. This proves the theorem for F as $F(\lambda) = F_0(\lambda) + F(1)H^1(\lambda)$.∎

Corollary 3.9 *Let $p \in [1, 2]$. If F is as in Theorem 3.8 with $s \geq 5$ and $u > 4$, then $S^{m,n}_{H,p}F$ is in $L^p(G)_{m,n}$. In fact, if m, n are odd or $m = n = 0$, or if $p > 1$ then $s \geq 2$ and $u > 4$ will serve the purpose.*

Proof Take $r = 2$ in the second inequality of the previous theorem. Then it is clear that $(S^{m,n}_{H,p}F)(a_t)$ is of order $\frac{1}{(1+t^2)}e^{-\frac{2}{p}t}$ and hence is in $L^p(A, e^{2t}dt)$. The result now follows from the decomposition of Haar measure on G. ∎

Theorem 3.10 *If F is as in Theorem 3.8 with $p = 1$, $s \geq 5$ and $u > 4$, then $(S^{m,n}_{H,1}F)^\wedge = F$.*

Proof. For $H \in C^1(\widehat{G})$, we claim that

$$\int_{\mathfrak{a}^*} F(\lambda)H(\lambda)\mu(\lambda)d\lambda = \int_{\mathfrak{a}^*} (\mathcal{S}_{H,p}^{m,n}F)^\frown(\lambda)H(\lambda)\mu(\lambda)d\lambda. \qquad (49)$$

The existence of the first integral is obvious from the given growth rate of F and H and from the inequality (17), while the existence of the second one follows from Riemann-Lebesgue lemma ([E-M, Thoerem 5]) for L^p functions and again by definition of $C^1(\widehat{G})$ and (17). Now as

(i) F is integrable with respect to λ,

(ii) $\mathcal{S}_{H,p}^{m,n}F$ is in $L^1(G)$ (by the previous corollary),

(iii) $\mathcal{S}_{H,p}^{m,n}H$ is in $L^1(G)$ and

(iv) $(\mathcal{S}_{H,p}^{m,n}H)^\frown = H$,

the equality of two integrals follows from Fubini's theorem. This implies that $F(\lambda)\mu(\lambda) = (\mathcal{S}_{H,p}^{m,n}F)^\frown\mu(\lambda)$ *a.e.* on $\mathfrak{a}^*$ with respect to Lebesgue measure. As $\mu(\lambda)$ has exactly one zero on $\mathfrak{a}^*$ (at 0) and the functions are continuous the result follows. ∎

An analogue of Theorem 3.10, when F is K-finite (Definition 3.1) with more than one nonzero matrix coefficient, is a consequence of the same inequalities we used above because all the inequalities remain valid for finitely many fixed m, n. We have:

Theorem 3.11 *Suppose $F : \widehat{M} \times S^\gamma \longrightarrow \bigcup \mathcal{B}(H_\sigma)$ satisfies the following conditions:*

1. *$F(\sigma, \lambda) \in \mathcal{B}(H_\sigma)$ for all $\sigma \in \widehat{M}$ and $\lambda \in S^\gamma$,*
2. *$F(\sigma, \lambda)$ has only finitely many non zero matrix coefficients,*
3. *$\lambda \longrightarrow F(\sigma, \lambda)$ is continuous on S^γ and analytic on $\mathring{S}^\gamma$ for each $\sigma \in \widehat{M}$,*
4. *$F_{m,n}(\lambda) = \varphi_{\sigma,\lambda}^{m,n} F_{m,n}(-\lambda)$, for all m, n, σ such that $m, n \in \mathbb{Z}^\sigma$ and $\lambda \in S^\gamma$,*

5. $F_{m,n}(\sigma, k) = 0$ *if* $m, n \in \mathbb{Z}^{\sigma}$ *are of opposite sign,* $k \in \mathbb{Z}^{-\sigma}$ *and* $|k| \leq \min\{|k|, |n|, \gamma\}$,

6. *For some* $s \in \mathbb{N}$ *and* $r_1 \in \mathbb{R}^+$, $\sup |(\frac{d}{d\lambda})^s F_{m,n}(\sigma, \lambda)|(1 + |\lambda|)^{r_1} < \infty$, *where the supremum is taken over all* $\lambda \in \overset{\circ}{S^{\gamma}}$, $\sigma \in \widehat{M}$ *and* $m, n \in \mathbb{Z}^{\sigma}$.

Then $\rho^p_{g_1,g_2;r}(\Sigma_{m,n} S^{m,n}_{H,p} F) < \infty$ *for all* $g_1, g_2 \in \mathcal{U}$ *and* $r \in \mathbb{R}$ *such that* $2r + 1 \leq s$ *and* $\max\{2 + r, 4 + \varepsilon\} + d \leq r_1$ *for some* $\varepsilon > 0$, *where* $d = \deg g_1 + \deg g_2$.

Corollary 3.12 *If* F *is as in Theorem* 3.11 *with* $s \geq 5$ *and* $r_1 \geq 4 + \varepsilon$, *then the sum of the wave packets* $\Sigma_{m,n} S^{m,n}_{H,p} F$ *is in* $L^p(G)$.

Proof. Follows from Corollary 3.9 in view of the fact that the estimate on the wave packet obtained in Theorem 3.11 is *uniform with respect to* m, n.■

When $F(\lambda)$ is not K-finite, it needs summing over m, n and so in that case the function F will have to be rapidly decreasing with respect to m, n also.

Application : Wiener Tauberian Theorems

For biinvariant functions in $L^1(SL_2(\mathbb{R}))$ Ehrenpreis and Mautner proved two versions of Wiener Tauberian (W-T) theorems ([E-M], Theorem 6 and Theorem 7). Here we use the analysis done in this section to provide an analogue of Theorem 7 of [E-M] for L^p-functions with K-types (n, n) (essentially the same method will also work for obtaining an analogue of Theorem 6 of [E-M]). In the remainder of this section, $p \in [1, 2)$.

Theorem 3.13 *Let* $f \in L^p(G)_{n,n} \cap L^1(G)_{n,n}$. *Suppose that the Fourier transform* $(\widehat{f}_H, \widehat{f}_B)$ *of* f *satisfies the following conditions:*

1. $\widehat{f}_H \in C^r(S^{\gamma})$

2. $\widehat{f}_H$ *never vanishes on the strip and* $\widehat{f}_B$ *never vanishes on the discrete series which are relevant for an* (n, n) *type function.*

3. $\sup_{\lambda \in S^\gamma} |\frac{d^j}{d\lambda^j}((\widehat{f}_H(\lambda))^{-1} e^{-\lambda^{4l}})| < \infty$ *for some* $l \in \mathbb{N}$ *and for* $j = 0, 1, \ldots, r$, *where* $r = 5$ *when* $p = 1$ *and* m, n *are nonzero even numbers, and* $r = 2$ *otherwise.*

Then the $L^p(G)_{n,n}$ *bimodule generated by* f *is dense in* $L^p(G)_{n,n}$.

Note that if in this theorem $p > 1$ then hypothesis 1 is redundant.

Lemma 3.14 *For any* δ *and* $k > 0$, *let* $\mathcal{D}_H(\widehat{G})$ *be the space of functions* $X(\lambda) : \mathbb{C} \longrightarrow \mathbb{C}$ *which satisfy the following conditions:*

1. $X(\lambda)$ *is holomorphic on* S_δ^γ,
2. $X(\lambda).e^{\lambda^{4k}}$ *is bounded in* S_δ^γ,
3. $X(\lambda) = X(-\lambda)$.

Then $\mathcal{D}_H(\widehat{G})$ *is dense in* $C_H^p(\widehat{G})_{n,n}$.

From the description of $C_H^p(\widehat{G})_{n,n}$ it is clear that there is no essential difference between this space and $C_H^1(\widehat{G})_{0,0}$ (see Observation in the next section). Therefore the Lemma follows from the proof of the corollary to Lemma 5.3 in [E-M], and hence the proof is omitted.

Proof of Theorem 3.13: As $\mathcal{D}_H(\widehat{G})$ is dense in $C_H^p(\widehat{G})_{n,n}$, $\mathcal{D}_H(\widehat{G}) \times C_B^p(\widehat{G})_{n,n}$ is dense in $C^p(\widehat{G})_{n,n}$. Take any $X(\lambda) \in \mathcal{D}_H(\widehat{G})$. Let $F(\lambda) = \widehat{f}_H(\lambda)$ and $\phi(\lambda) = F(\lambda)^{-1}X(\lambda) = (F(\lambda)^{-1}e^{-\lambda^{4l}})(e^{\lambda^{4l}}X(\lambda))$. From the properties of $F(\lambda)$ and $X(\lambda)$, we find that

(i) $\phi(\lambda) \in C^r(S^\gamma)$ (ii) $\phi(\lambda) = \phi(-\lambda)$ and

(iii) $\frac{d^j\phi(\lambda)}{d\lambda^j} = O(\frac{1}{1+|\lambda|^\alpha})$ uniformly in the strip S^γ for $\alpha \geq 0$ and $j = 0, 1, \ldots, r$.

So, by Corollary 3.9 above $\mathcal{S}_{H,p}^{n,n}\phi \in L^p(G)_{n,n}$.

Now let $Y \in C_B^p(\widehat{G})_{n,n}$. For $k_0 \in L_\sigma^{n,n}(\gamma)^c$ define $\phi_B(k_0) = \frac{Y(k_0)}{F(k_0)}$. Note that $F(k_0) \neq 0$ by the hypothesis of the theorem. Then $F(k_0)\phi_B(k_0) = Y(k_0)$. Defining ϕ_B in this way for all points of $L_\sigma^{n,n}(\gamma)^c$, which are only finitely many, we get in general $F(k)\phi_B(k) = Y(k)$ for all those points. And $h(x)$ defined by

$$h(x) = \mathcal{S}_{H,p}^{n,n}\phi(x) + \frac{1}{2\pi}\Sigma_{k \in L_{n,n}^\sigma(\gamma)^c}\,\phi_B(k)\Psi_k^{n,n}(x)|k|$$

is in $L^p(G)_{n,n}$ as for $k \in L^{\sigma}_{n,n}(\gamma)^c$, $\Psi^{n,n}_k$ belongs to $C^p(G)_{n,n}$ (see Theorem 2.8). So, $F(\lambda)$ will generate an ideal containing $\mathcal{D}_H(\widehat{G}) \times C^p_B(\widehat{G})_{n,n}$ in $\widehat{L^p(G)_{n,n}}$. The theorem is immediate from this step. ■

In the proof of the above theorem we have used Theorem 3.8 only for the case $p \in [1,2]$ and $m = n$.

By a skillful use of an 'extension of Corona Theorem' due to T. Wolff, W-T theorem was extended from $L^1(SL_2(\mathbb{R}))$ to $L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$ in [S2]. We will proceed in an analogous way to get a W-T theorem for $L^p(SL_2(\mathbb{R}))_n$. First let us quote Wolff's Theorem from [S2] :

Theorem 3.15 *(Wolff)Let S be a open vertical strip on $\mathbb{C}$ and let $H^\infty(S)$ denote the set of bounded holomorphic functions on S. Let $g_1, g_2, ... g_n$ and f belong to $H^\infty(S)$ such that $\Sigma_i |g_i(\lambda)| \geq |f(\lambda)|$ for all λ. Then $\alpha_1.g_1 + \alpha_2.g_2 + ... + \alpha_n.g_n = f^3$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in H^\infty(S)$.*

A consequence of this is:

Theorem 3.16 *Let $f_1, f_2, \ldots, f_n \in L^p(G)_{n,n}$ be such that their Fourier transforms can be extended to bounded holomorphic functions to an augmented strip S^γ_δ. Suppose further that for some positive constant K and a positive integer l, $\Sigma_i |\widehat{f}_i(\lambda)| \geq K e^{-\lambda^{4l}}$ for all $\lambda \in S^\gamma_\delta$, and $\widehat{f}_{iB}(k)$, $i = 1, \ldots, n$, do not vanish simultaneously at any $k \in \Gamma_n$. Then $f_1, f_2, \ldots, f_n$ generate $L^p(G)_{n,n}$ as an $L^1(G)_{n,n}$ module and hence generate $L^p(G)_n$ as an $L^1(G)$ module.*

Proof. By Theorem 3.15 above, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in H^\infty(\mathring{S}^\gamma_\delta)$ so that $\Sigma \alpha_i \widehat{f}_i = e^{-3\lambda^{4l}}$. As the $\widehat{f}_i$'s and $e^{-3\lambda^{4l}}$ are even function on the augmented strip, one can choose α_i's to be even. Now $\Sigma e^{-\lambda^2} \alpha_i(\lambda) \widehat{f}_i(\lambda) = e^{-\lambda^2}.e^{-3\lambda^{4l}}$ As α_i is bounded holomorphic, $e^{-\lambda^2}\alpha_i \in C^1(\widehat{G})_{n,n}$ by the characterization of the space $C^1(\widehat{G})_{n,n}$ (see Definitions 2.9, 2.11). Note that the definition of $C^1_B(\widehat{G})$ does not provide any restriction.

Now as there is an $l' > 0$ such that $e^{-\lambda^2}.e^{-3\lambda^{4l}} \geq e^{-3\lambda^{4l'}}$ it is clear that the function $e^{-\lambda^2}.e^{-3\lambda^{4l}}$ satisfies all the conditions of Theorem 3.13 and hence generate $L^1_H(G)_{n,n}$. Therefore $f_1, f_2, \ldots, f_n$ generate $L^p_H(G)_{n,n}$. Also as $f_1, f_2, \ldots, f_n$ do not vanish simultaneously on Γ_n it is clear from Theorem 3.13 that they generate the discrete part $L^p_B(G)_{n,n}$ as well. ■

Note that $e^{-\lambda^{4k}}$ is the Fourier transform (of some function) with the greatest decay which fits in Theorem 3.13 and generates $L^1(G)_{n,n}$.

Now if $f \in L^p(G)_n$ is left K-finite and is of right type n then $\widehat{f}_H$ is a coloumn vector with finitely many matrix coefficients. These matrix coefficients are Fourier transforms of projections of f in its left K-types, say $m_1, m_2, \ldots m_r$. Let us call these projections $f_1, f_2, \ldots, f_r$. For such an f we have,

Theorem 3.17 *Let f and $f_1, f_2, \ldots, f_r$ be as above. Suppose that the Fourier transforms $\widehat{f_1}_H, \widehat{f_2}_H, \ldots \widehat{f_r}_H$ can be extended to bounded holomorphic functions on S^{γ}_{δ}, and for all $\lambda \in S^{\gamma}_{\delta}$,*

$$\Sigma_i |\widehat{f_i}_H(\lambda)| \geq K e^{-\lambda^{4l}}$$

for some constants K and $l \in \mathbb{N}$. Suppose further that $\widehat{f_1}_B, \widehat{f_2}_B, \ldots \widehat{f_r}_B$ do not vanish simultaneously at any point $k \in \Gamma_n$. Then the left $L^1(G)$-module generated by f is dense in $L^p(G)_n$.

The theorem will follow from the proof of a more general theorem (Theorem 4.5) proved in the next section. To avoid repeatation we omit the proof here.

4 Wiener Tauberian Theorems for $SL_2(\mathbb{R})$

In this section $G = SL_2(\mathbb{R})$. We recall two estimates on the matrix coefficients:

(*i*) For $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $\lambda \in S^1$, $|\Phi_{\sigma,\lambda}^{m,n}(x)| \leq 1$ for all $x \in G$. ([E-M], 2.9)

(*ii*) For $p \in (1,2)$ and for arbitrary but fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\int_G |\Phi_{\sigma,\lambda}^{m,n}(x)|^q dx \leq C_\varepsilon$, for $\sigma \in \widehat{M}$, $m, n \in \mathbb{Z}^\sigma$ and $\lambda \in S^{\gamma-\varepsilon}$. Here C_ε depends on ε, and $\frac{1}{p} + \frac{1}{q} = 1$.

The second result can also be stated as: If an admissible representation is L^p tempered, then its matrix coefficients are in $L^{q+\delta}$ for any $\delta > 0$. (Follows from Proposition 2.4, which can be considered as the definition of L^p-temperedness)

Reviewing the definition of the space $C_H^p(\widehat{G})_{m,n}$ (given in Definition 2.9) we see that, as $\varphi_{\sigma,\lambda}^{n,n} = 1$, in the definition of $C_H^p(\widehat{G})_{n,n}$ property 2 reduces to $F(\lambda) = F(-\lambda)$ and property 4 is not relevant. Thus the space $C_H^p(\widehat{G})_{n,n}$ consists of the continuous maps

$$F : S^\gamma \longrightarrow \mathbb{C}$$

which has the properties:

1. F is holomorphic on $\mathring{S}^\gamma$, the interior of S^γ,

2. $F(\lambda) = F(-\lambda)$ for all $\lambda \in S^\gamma$,

3. $\widehat{\rho}_{H,l,r}(F) < \infty$ for all $l \in \mathbb{N}$, $r \in \mathbb{R}^+$, where

$$\widehat{\rho}_{H,l,r}(F) = \sup_{\lambda \in S^\gamma} |(\frac{d}{d\lambda})^l F(\lambda)|(1 + |\lambda|)^r,$$

Also recall from Definition 2.11 that $C_B^p(\widehat{G})_{n,n}$ is the set of all functions $F : \mathbb{Z}^\gamma(n) \longrightarrow \mathbb{C}$.

Observation.

1. Though $C_H^p(\widehat{G})_{n,n}$ is the image under Fourier transform of functions of $C_H^p(G)_{n,n}$ relative to the principal series reperesentation, the definition is *independent* of n.

2. This definition is in a sense independent of p. The only thing which changes with p is the width of the strip S^γ and so far we want to use properties of holomorphic functions on a vertical strip we are always in the same situation. For $p \in (1,2)$, our analysis is rather simpler for the fact that S_δ^γ does not contain any integer point parametrizing the discrete series, *i.e.* no discrete series is embedded in any of the principal series parametrized by that strip.

Following [B-W], we define $A_0^p(\delta)$ to be the space of continuous functions $F : S_\delta^\gamma \longrightarrow \mathbb{C}$ satisfying the following properties:

1. F is holomorphic on $\mathring{S}_\delta^\gamma$,

2. $F(\lambda) = F(-\lambda)$ for all $\lambda \in S_\delta^\gamma$,

3. $\lim_{|\lambda| \longrightarrow \infty} F(\lambda) = 0$ on $\lambda \in S_\delta^\gamma$.

There is a conformal map ψ from the strip S_δ^γ onto the unit disc $\mathbb{D}$ such that $i\mathbb{R}$ is mapped onto the line segment joining i and $-i$ under it and $\psi(-\lambda) = -\psi(\lambda)$, namely,

$$\psi(\lambda) = \frac{i(1 - e^{\pi i\lambda/2(\gamma+\delta)})}{(1 + e^{\pi i\lambda/2(\gamma+\delta)})},\ \lambda \in S_\delta^\gamma.$$

Let $A(\mathbb{D})$ be the algebra of all functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ which are analytic on $\mathbb{D}$ and continuous in its closure and let $A_0(\mathbb{D})$ be its subspace of all functions $f \in A(\mathbb{D})$ such that $f(z) = f(-z)$ for all $z \in \mathbb{D}$ and $f(i) = f(-i) = 0$. So if $f \in A_0(\mathbb{D})$, then $f \circ \psi \in A_0^p(\delta)$. Then ([B-W, Lemma 1.2]),

Lemma 4.1 (*Benyamini-Weit*) *For every ideal I in $A_0(\mathbb{D})$ there is an ideal J in $A(\mathbb{D})$, so that $I = J \cap A_0(\mathbb{D})$.*

(See [B-W, Lemma 1.2].) From the above lemma and the Beurling-Rudin Theorem ([Ho, last Corollary on p. 88]) we conclude (following [B-W]) that if the functions in an ideal I in $A_0(\mathbb{D})$ has no common zero other than $\pm i$, and if I contains a function whose decay is less than exponential, then $I = A_0(\mathbb{D})$.

Now one can extend Lemma 1.3 and Lemma 1.4 of [B-W] to our context.

Lemma 4.2 *Fix $\delta > 0$. Then the space*

$$\mathcal{F} = \left\{ f \in C^p(G)_{n,n} \mid \begin{array}{c} \widehat{f}_H \text{ can be extended holomorphically to } S_\delta^\gamma \\ \text{and } \widehat{f}_H(\lambda)e^{-K\lambda^2} \in A_o^p(\delta) \text{ for some } K > 0 \end{array} \right\}$$

is dense in $C^p(G)_{n,n}$ (and hence in $L^p(G)_{n,n}$).

Proof. Take $g \in C_c^\infty(G)_{n,n} \subset C^p(G)_{n,n}$. Then $F_m(\lambda) = \widehat{g}_H(\lambda)e^{\lambda^2/m} \in A_o^p(\delta)$ for all $m \in \mathbb{N}$, since by the Paley-Wiener theorem ([Ba, Theorem 10.5]) $\widehat{g}_H$ is an entire function of exponential type. Again from Paley-Wiener theorem, it follows that $e^{\lambda^2/m}$ is the Fourier transform of a function in $C_c^\infty(G) \subset C^1(G)_{n,n}$. Therefore, taking convolution, $F_m = \widehat{f}_m$ for some $f_m \in C^p(G)_{n,n}$, and hence $f_m \in \mathcal{F}$. Now for a fixed r, $|\lambda^r(F_m(\lambda) - \widehat{g}_H(\lambda))| = |\widehat{g}_H(\lambda)\lambda^r|\,|1 - e^{\lambda^2/m}|$. The first factor on the right hand side converges to zero on S_δ^γ as $|\lambda| \longrightarrow \infty$. The second factor is bounded and converges to zero uniformly as $m \longrightarrow \infty$ on compact subsets of S_δ^γ. Therefore $\lambda^r(F_m(\lambda) - \widehat{g}_H(\lambda))$ converges to zero uniformly on S_δ^γ as $m \longrightarrow \infty$. As $F_m(\lambda)$ and $\widehat{g}_H(\lambda)$ are analytic, from Cauchy's formula it follows that on the smaller strip S^γ, the same is true for all derivatives of $\lambda^r(F_m(\lambda) - \widehat{g}_H(\lambda))$. Therefore F_m converges to $\widehat{g}_H$ in $C^p(\widehat{G})_{n,n}$. Actually this proves that the set $\widehat{\mathcal{F}}_H$ of all $F \in C_H^p(\widehat{G})_{n,n}$ satisfying,

(*i*) F is holomorphically extendable to S_δ^γ and

(*ii*) $F(\lambda)e^{-K\lambda^2} \in A_o^p(\delta)$ for some $K > 0$,

is dense in $C_H^p(\widehat{G})_{n,n}$.

So $\widehat{\mathcal{F}}_H \oplus C_B^p(\widehat{G})_{n,n}$ is dense in $C^p(\widehat{G})_{n,n}$. But $\{\widehat{f} | f \in \mathcal{F}\} = \widehat{\mathcal{F}}_H \oplus C_B^p(\widehat{G})_{n,n}$ as for $f \in \mathcal{F}$, there is no restriction on its Fourier transform $\widehat{f}_B$, with respect to discrete series representations. ∎

Lemma 4.3 *Let $\delta > 0$. Let a sequence of functions $\{f_i\}$ and f be of type (n,n) such that $\widehat{f}_{iH}$, $\widehat{f}_H \in A_o^p(\delta)$. If there is a $K > 0$ such that*

$$\widehat{f}_H(\lambda)e^{-K\lambda^2} \in A_o^p(\delta)$$

and if the sequence $\widehat{f}_{iH}(\lambda)$ converges to $\widehat{f}_H(\lambda)e^{-K\lambda^2}$ in the topology of $A_o^p(\delta)$, then $\widehat{f}_{iH}(\lambda)e^{K\lambda^2}$ converges to $\widehat{f}_H$ in $C_H^p(\widehat{G})_{n,n}$ topology.

Proof. It is enough to prove that for each r, the sequence $\lambda^r(\widehat{f}_{iH}(\lambda)e^{K\lambda^2} - \widehat{f}_H(\lambda))$ converges uniformly to zero in the strip S_δ^γ as $i \longrightarrow \infty$. Because then the lemma will follow from Cauchy's formula. Now,

$$\lambda^r(\widehat{f}_{iH}(\lambda)e^{K\lambda^2} - \widehat{f}_H(\lambda)) = \lambda^r e^{K\lambda^2}(\widehat{f}_{iH}(\lambda) - \widehat{f}_H(\lambda)e^{-K\lambda^2}).$$

The first factor on the right hand side is a bounded function and the second factor is a sequence which converges uniformly to zero by the hypothesis. ∎

At this point we make an observation on the hypotheses of the main theorems of this section, Theorem 4.5 and Theorem 4.7. We assume in both of them that the operator Fourier transforms $\widehat{f}_H(\pi_{\sigma,\lambda})$ are defined for $\lambda \in S^{\gamma+\delta}$ for some $\delta > 0$, to keep the statement relatively simple. However, what we really need and make use of is only that the matrix coefficients of the transforms, $(\widehat{f}_H)_{m,n}(\sigma, \lambda)$, have analytic extensions on $S^{\gamma+\delta}$ beyond their natural domain S^γ. The extension of the operator transform is, in fact, a mere notational convenience.

We now prove a W-T theorem for $L^p(G)_{n,n}$ functions. This is an extension of Theorem 1.1 of [B-W] which proves the case $n = 0$, $p = 1$. As always, $\gamma = \frac{2}{p} - 1$. If a function f satisfies the folowing decay condition:

$$\limsup_{|t|\longrightarrow\infty} |(\widehat{f}_H)_{m,n}(it)e^{Ke^{|t|}}| > 0 \tag{50}$$

for all $K > 0$ then we will simply say that the (m,n)-th matrix coefficient of the Fourier transform of f is *not-too-rapidly-decreasing* or *does not decay too rapidly at* ∞. This is obtained from the Beurling-Rudin decay condition, mentioned above (see discussion following Lemma 4.1) on composition with $\psi(\lambda)$.

Theorem 4.4 *Let $\{f^\alpha | \alpha \in \Lambda\}$ be a subset of $L^p(G)_{n,n}$, where Λ is an index set. Suppose that, for some $\delta > 0$, each $\widehat{f}_H^\alpha$ extends holomorphically to S_δ^γ and satisfies $\lim_{|\lambda|\longrightarrow\infty} \widehat{f}_H^\alpha(\lambda) = 0$ in S_δ^γ. Let there exist an $\alpha_0 \in \Lambda$ such that some matrix coefficient of $f_H^{\alpha_0}$ does not 'decay too rapidly at ∞'. Moreover,* *(a) if the collections $\{\widehat{f}_H^\alpha\}$ and $\{\widehat{f}_B^\alpha\}$ do not have any common zeros on S_δ^γ and Γ_n respectively then the $L^1(G)_{n,n}$-module generated by $\{f^\alpha | \alpha \in \Lambda\}$ is dense in $L^p(G)_{n,n}$.*

(b) for the particular case of $p = 1$ and $n = 0$, if the Fourier transforms $\widehat{f}_H^\alpha$ do not have any common zeros on S_δ^1 except the points ± 1, then the $L^1(G)_{0,0}$-module generated by $\{f^\alpha | \alpha \in \Lambda\}$ is dense in $L^1(G)_{0,0}^0$, the space of functions with zero integral in $L^1(G)_{0,0}$.

Proof. Since $\widehat{f}_H^\alpha \in A_0^p(\delta)$ for all $\alpha \in \Lambda$ and f^{α_0} does not 'decay too rapidly at ∞', by Beurling-Rudin theorem (see discussion following Lemma 4.1), the algebraic ideal $\widehat{\mathcal{I}}$ generated by $\widehat{f}_H^\alpha$'s is dense in $A_0^p(\delta)$.

Fix $h \in \mathcal{F}$, where $\mathcal{F}$ is as in Lemma 4.2. Then $\widehat{h}_H(\lambda)e^{-K\lambda^2} \in A_0^p(\delta)$ for some $K > 0$. So there is a sequence $F_n \in \widehat{\mathcal{I}}$ such that $F_n \longrightarrow \widehat{h}_H(\lambda)e^{-K\lambda^2}$. Since $F_n \in \widehat{\mathcal{I}}$ and $e^{K\lambda^2} \in A_0^p(\delta)$, $F_n e^{K\lambda^2} \in \widehat{\mathcal{I}}$ and by Lemma 4.3 $F_n e^{K\lambda^2}$ converges to $\widehat{h}_H(\lambda)$ in the topology of $C_H^p(\widehat{G})_{n,n}$. We may assume that the sequence $F_n e^{K\lambda^2}$ is in the ideal generated by $\{\widehat{f}_H^\alpha e^{K'\lambda^2}\}_{\alpha\in\Lambda}$ in $C_H^p(\widehat{G})_{n,n}$ for some $K' < K$. Because for any function $a(\lambda) \in A_0^p(\delta)$, $a(\lambda).e^{(K-K')\lambda^2}$ is in $C_H^p(\widehat{G})_{n,n}$.

When $p = 1$ this shows that the ideal generated by $\{\widehat{f}_H^\alpha\}_{\alpha\in\Lambda}$ in $C_H^1(\widehat{G})_{n,n}$ is dense in $C_H^1(\widehat{G})_{n,n}$. If $p > 1$ then from the facts $C_H^1(\widehat{G})_{n,n}$ is dense in $C_H^p(\widehat{G})_{n,n}$ and $C_H^p(\widehat{G})_{n,n}$ is a Frechét algebra, it follows that the $C_H^1(\widehat{G})_{n,n}$-module generated by $\{\widehat{f}_H^\alpha\}_{\alpha\in\Lambda}$ in $C_H^p(\widehat{G})_{n,n}$ is dense in $C_H^p(\widehat{G})_{n,n}$.

Let $\{k_i | 1 \leq i \leq r\}$ be the natural enumeration of $\mathbb{Z}^\gamma(n)$. By hypothesis, for each k_i there exits an $s_i \in \Lambda$ such that $\widehat{f}_B^{s_i}(k_i) \neq 0$. Set $\Lambda' = \{s_i | 1 \leq i \leq r\} \subset \Lambda$. Let

$$g^{s_i}(k_j) = \delta_{i,j} \frac{\widehat{h}_B(k_j)}{\widehat{f}_B^{s_j}(k_j)}$$

for $1 \leq i, j \leq r$, and

$$g^\alpha(k) = 0 \text{ for all } k \in \Gamma_n, \text{ and } \alpha \in \Lambda - \Lambda'.$$

Then $\widehat{h}_B(k) = \sum_\alpha \widehat{f}_B^\alpha(k) g^\alpha(k)$ for $k \in \Gamma_n$. This proves part (a) of the theorem in view of the isomorphism between $C^p(\widehat{G})_{n,n}$ and $C^p(G)_{n,n}$ and the injectiveness of the Fourier transform on $L^p(G)_{n,n}$. Part (b) is proved in [B-W Theorem 1.1]. ∎

Theorem 4.5 *Let $n \in \mathbb{Z}$ be arbitrary but fixed. Let $\mathcal{F} = \{f^\alpha | \alpha \in \Lambda\}$ be a subset of $L^p(G)_n$, Λ being an index set, such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f}^\alpha_H$ of f^α has a holomorphic extension on S_δ^γ for some $\delta > 0$, and all the matrix coefficients of $\widehat{f}^\alpha_H$ vanish at infinity, i.e., $\lim_{|\lambda| \longrightarrow \infty} |(\widehat{f}^\alpha_H(\lambda))_{m,n}| = 0$ on S_δ^γ. Let there be an $\alpha_0 \in \Lambda$ such that one of the matrix coefficients, say $(\widehat{f}^{\alpha_0}_H)_{m_0,n}$ satisfies the not-too-rapidly decreasing consition stated in* (35).

Also assume that the collections $\{\widehat{f}^\alpha_H | \alpha \in \Lambda\}$ and $\{\widehat{f}^\alpha_B | \alpha \in \Lambda\}$ do not have any common zero on S_δ^γ and Γ_n respectively. Then the left $L^1(G)$ module generated by $\mathcal{F}$ is dense in $L^p(G)_n$.

Moreover, in the case $p = 1$ and $n = 0$, if the collection $\mathcal{F}$ does not have any common zero on S_δ^1 except at ± 1 then the left ideal generated by $\mathcal{F}$ is dense in $L^1(G)_0^0$.

Working towards a proof of this theorem we recall that if $\widehat{f} = (\widehat{f}_H, \widehat{f}_B) \in L^p(\widehat{G})$ then $(\widehat{f}_B(k))_{m,n} = \eta^{m,n}(k)(\widehat{f}_H(k))_{m,n}$ for $k \in S^\gamma$ where $\eta^{m,n}(k)$ is a positive number (see Proposition 1.2). Therefore $(\widehat{f}_B(k))_{m,n} \neq 0 \Leftrightarrow (\widehat{f}_H(k))_{m,n} \neq 0$.

Suppose that $\widehat{f}_B(k) \neq 0$ for all $k \in \Gamma_n$. Then it implies the following:

(a) If n is positive then f has at least one (nonzero component of) left type m such that $m \geq n$, because for every $m < n$, $(\widehat{f}_B(n-1))_{m,n} = 0$. Similarly, when n is negative f has at least one left type m for some $m \leq n$.

(b) Let $f \in L^1(G)_n$. If n is even and $\delta < 1$ then exatly one point in the strip S_δ^1, $+1$ or -1, parametrizes a discrete series representation relevant for f according as $n > 0$ or $n < 0$. So, when $n > 0$, by above hypothesis $\widehat{f}_B(1) \neq 0$. In fact, there is a K-type m such that $m \in \mathbb{Z}(1)$ and $(\widehat{f}_B(1))_{m,n} \neq 0$. This is equivalent to saying that $(\widehat{f}_H(1))_{m,n} \neq 0$. For

$n < 0$ one can have a similar statement. When n is odd neither the original strip S^1 nor the (carefully chosen) augmented strip S^1_δ has any point-parameter of discrete series representation relevant to this K-type n. For $p \in (1,2)$ the strip S^γ and its δ augmentation S^γ_δ can avoid points which parametrize discrete series representation.

Proof of Theorem 4.5. The fixed n in the hypothesis determines a unique $\sigma \in \widehat{M}$ by the relation $n \in \mathbb{Z}^\sigma$. Recall that when $m \in \mathbb{Z}^\sigma$ for this σ, then m is of the same parity as n, while integers in $Z^{-\sigma}$ are of parity opposite to n. Throughout the proof by σ we mean this σ determined by n. Instead of (σ, λ) we may use only λ as a parameter of the Fourier transform.

We will first consider the case when $p = 1$. For any function $f \in \mathcal{F}$ and for any $m \in \mathbb{Z}$, let $f_{m,-}$ be the projection of f in the left K-type m. That is,

$$f_{m,-}(x) = \int_K \bar{\chi}_m(k_\theta) f(k_\theta x) d\theta = \int_0^{2\pi} e^{-im\theta} f(k_\theta x) d\theta.$$

Then $f_{m,-}$ is clearly an (m,n) type function. Also the (m,n)-th matrix coefficient of $\widehat{f}_H$ is given by $(\widehat{f}_H)_{m,n} = (\widehat{f}_H)_{m,-} = (\widehat{f_{m,-}})_H$.

The main step of the proof is to associate to each $m \in \mathbb{Z}^\sigma$, a polynomial $P_m(\lambda)$ in λ which has the property that $\mathcal{G}_m(\lambda) = e^{-\lambda^4} P_m \in C^1_H(\widehat{G})_{n,m}$. Let $p_{n,m}$ denote the numerator of the rational function $\varphi^{n,m}_{\sigma,\lambda}$. Then

$$\varphi^{n,m}_{\sigma,\lambda} = p_{n,m}(\lambda)/p_{n,m}(-\lambda) \tag{51}$$

(see (24), Section 2), and

$$p_{n,m} = \begin{cases} (|m| - 1 + \lambda)(|m| - 3 + \lambda) \cdots (|n| + 1 + \lambda) & \text{if } |m| > |n|, \\ (-1)^{(n-m)/2} & \text{if } |m| = |n|, \\ (|n| - 1 - \lambda)(|n| - 3 - \lambda) \cdots (|m| + 1 - \lambda) & \text{if } |m| < |n|, \end{cases} \tag{52}$$

We define the polynomials P_m as follows:

1. When $m.n \geq 0$ let $P_m = p_{n,m}$.

2. When $m.n < 0$ and m, n are odd, let $P_m(\lambda) = p_{n,m}(\lambda).\lambda^2$.

3. When $m.n < 0$ and m, n are even then $P_m(\lambda) = p_{n,m}(\lambda).(1-\lambda^2)$.

Then in all the above cases $e^{-\lambda^4} P_m(\lambda) = \varphi_{\sigma,\lambda}^{n,m} e^{-\lambda^4} P_m(-\lambda)$ by (51). Therefore from the definition of $C^p(\widehat{G})_{n,m}$ (Definition 2.9) it is clear that $\mathcal{G}_m(\lambda) = e^{-\lambda^4} P_m(\lambda) \in C_H^1(\widehat{G})_{n,m}$. Also,

$$\begin{aligned}
\mathcal{G}_m(\lambda)(\widehat{f}_H)_{m,-}(\lambda) &= e^{-\lambda^4} P_m(\lambda)(\widehat{f}_H)_{m,-}(\lambda) \\
&= e^{-\lambda^4} P_m(-\lambda)\varphi_\lambda^{n,m}\varphi_\lambda^{m,n}(\widehat{f}_H)_{m,-}(-\lambda) \\
&= \mathcal{G}_m(-\lambda)(\widehat{f}_H)_{m,-}(-\lambda)
\end{aligned}$$

since $(\widehat{f}_H)_{m,-}(\lambda) = \varphi^{m,n}(\lambda)(\widehat{f}_H)_{m,-}(-\lambda)$, f_m being an (m,n) type function and $\varphi_\lambda^{m,n} = (\varphi_\lambda^{n,m})^{-1}$ (see Proposition 2.10 and equation (23) in Section 2). This shows that for all m, $\mathcal{G}_m(\widehat{f}_H)_{m,-}$ is the Fourier transform of an (n,n) type function with respect to the principal series representation $\pi_{\sigma,\lambda}$. It is obvious that they can be holomorphically extended to the strip $\overset{\circ}{S}{}_\delta^1$ and that $\lim_{|\lambda| \longrightarrow \infty} |\mathcal{G}_m(\lambda)(\widehat{f}_H)_{m,-}(\lambda)| = 0$ as the claims hold for f_m's.

Next we want to show that for each $\lambda \in S_\delta^1$ there is an $m \in \mathbb{Z}$ and an $\alpha \in \Lambda$ such that $\mathcal{G}_m(\lambda)(\widehat{f}_H^\alpha)_{m,-}(\lambda) \neq 0$.

It is clear that the only possible zeros of the polynomials $p_{n,m}$ and P_m in the strip S_δ^1 are ± 1 and 0 (when δ is chosen carefully). Concentrating on them we find:

(i) $p_{n,m}(0) \neq 0$;

(ii) if $m \neq 0$ then $p_{n,m}(1) \neq 0$ for all n;
if $n \neq 0$ then $p_{n,0}(1) = 0$
and $p_{0,0} = 1$.

(iii) $p_{n,m}(-1) = 0$ if and only if $n = 0$ and $m \neq 0$.

Therefore the only possible zeros of P_m's within the strip S^1 are:

Case (a) if $n = 0$ then $P_m(-1) = 0$ for all $m \neq 0$.

Case (b) if $n \neq 0$ then $P_0(1) = 0$.

Case (c) if $n.m < 0$ and n, m are even then $P_m(\pm 1) = 0$

Case (d) if $n.m < 0$ and n, m are odd then $P_m(0) = 0$.

By hypothesis, for any $\lambda \in \mathcal{S}_\delta^1$ there is an $\alpha \in \Lambda$ and an $m \in \mathbb{Z}$ such that $(\widehat{f_H^\alpha})_{m,-}(\lambda) \neq 0$. If n, m and λ does not fall in one of the cases mentioned above then clearly $\mathcal{G}_m(\lambda)(\widehat{f_H^\alpha})_{m,-}(\lambda) \neq 0$.

Now consider the point $\lambda = 0$. This is a possible zero for P_m when n is *nonzero* and odd; But in that case $n.m > 0$. Because otherwise, $\Phi_{\sigma_-,0}^{m,n} \equiv 0$ (see Proposition 1.1, Section 1) which would imply that $(\widehat{f_H^\alpha})_{m,-}(0) = 0$ contradicting our choice of m. So case (d) is not relevant here and hence the function $\mathcal{G}_m$ satisfies the relation $\mathcal{G}_m(0)(\widehat{f_H^\alpha})_{m,-}(0) \neq 0$.

Next consider the points 1 and -1. It is enough to deal with the case when n is even.

First consider that n is even and positive. By hypothesis, there exists an $\alpha \in \Lambda$ such that $(\widehat{f_B^\alpha})_{m,-}(1) \neq 0$ for some $m \in \mathbb{Z}(1)$. Now by the discussion (b) preceeding this proof, this implies that $(\widehat{f_H^\alpha})_{m,-}(1) \neq 0$. Also, as $n \geq 2$ and $m \geq 2$ in this case, $(\widehat{f}_H)_{m,-}(-1) = \varphi_{\sigma_+,1}^{n,m}(\widehat{f}_H)_{m,-}(+1)$ and $\varphi_{\sigma_+,\lambda}^{n,m}$ has no zero at $\lambda = 1$ (see Proposition 2.10). Therefore $(\widehat{f}_H)_{m,-}(-1) \neq 0$. Also, since $m > 0$ and $n > 0$, case (b) and case (c) do not occur. Hence $\mathcal{G}_m(\pm 1)(\widehat{f_H^\alpha})_{m,-}(\pm 1) \neq 0$.

Similarly we can tackle the case when n is even and negative.

Finally, if $n = 0$ then for every $\alpha \in \Lambda$, $(\widehat{f_H^\alpha})_{m,-}(1) = 0$ for all $m \neq 0$ as $\Phi_{\sigma_+,1}^{m,0}(x) \equiv 0$ (see Proposition 1.1, Section 1). But, then the hypothesis that $\widehat{f_H^\alpha}(1) \neq 0$ for some $\alpha \in \Lambda$ forces $(\widehat{f_H^\alpha})_{0,-}(-1) = (\widehat{f_H^\alpha})_{0,-}(1)$ to be nonzero. (Note that $f_{0,-}^\alpha$ is a (0,0) type function in this case.) Therefore, Case (a) will not concern us.

To deal with the discrete part, let us take $\mathcal{G}_m(k) = e^{-k^4} P_m(k)$ for all $k \in \Gamma_n$. Now let for $k' \in \Gamma_n$, $(\widehat{f_B^\alpha})_{m',-}(k') \neq 0$ for some $\alpha \in \Lambda$. Then $m' \in \mathbb{Z}(k')$. Therefore $P_{m'}(k') \neq 0$ as all the zeros of the polynomial are either between m' and n or between $-m'$ and $-n$.

We will now show that $\mathcal{G}_{m_0}.\widehat{f}_{m_0,,H}$ does not 'decay too rapidly at ∞'. Let $K > 0$ be fixed. Take a K' such that $0 < K' < K$. Then

$$
\begin{aligned}
|\mathcal{G}_{m_0}(it)(\widehat{f}_H)_{m_0,-}(it)e^{Ke^{|t|}}| &= |(\widehat{f}_H)_{m_0,-}(it)P_{m_0}(it)e^{Ke^{|t|}-(it)^4}| \\
&= |(\widehat{f}_H)_{m_0,-}(it)P_{m_0}(it)e^{(K-K')e^{|t|}-t^4}e^{K'e^{|t|}}|
\end{aligned}
$$

Hence,

$$
\limsup_{|t|\longrightarrow\infty} |\mathcal{G}_{m_0}(it)(\widehat{f}_H)_{m_0,-}(it)e^{Ke^{|t|}}| > 0
$$

as $\limsup_{|t|\longrightarrow\infty} |(\widehat{f}_H)_{m_0,-}(it)e^{(K-K')e^{|t|}-t^4}| > 0$, $|e^{K'e^{|t|}}| \longrightarrow \infty$ as $|t| \longrightarrow \infty$. Note that on the imaginary axis only at the point 0, P_{m_0} may have a possible zero which clearly has no effect on the limsup above.

Now for every m, the isomorphism between $C^1(G)_{n,m}$ and its image under Fourier transform, $C^1(\widehat{G})_{n,m}$ tells us that there exists a $g_m \in C^1(G)_{n,m}$ such that $\widehat{g}_{m\,H}(\lambda) = \mathcal{G}_m(\lambda)$ for $\lambda \in S_\delta^1$ and $\widehat{g}_{m\,B}(k) = \mathcal{G}_m(k)$ for $k \in \Gamma_n$. Thus we have established that the set of $L^1(G)_{n,n}$ functions $\{g_m * f^\alpha_{m,-} | m \in \mathbb{Z}^\sigma,\ \alpha \in \Lambda\}$ satisfy all the conditions of Theorem 4.4 and hence the ideal generated by them is dense in $L^1(G)_{n,n}$. But $g_m * f^\alpha_{m,-} = g_m * f^\alpha$. Therefore the result follows from the fact that the smallest left $L^1(G)$ module of $L^1(G)_n$ containing $L^1(G)_{n,n}$ is all of $L^1(G)_n$. This completes the proof for L^1 case.

The proof for $p > 1$ will almost follow the above word for word. In fact, the case $p > 1$ is simpler as the troublesome points ± 1 are not in the (carefully chosen) strip S_δ^γ. Note that for every $p \in [1, 2)$, we always get a $C_H^1(\widehat{G})$-function, *e.g.* $P(\lambda)e^{-\lambda^4}$, to change the K-type of the Fourier transforms. So arguments similar to that of the previous theorem will take care of this situation.

The last part of the theorem, which deals with the special case $n = 0$ and $p = 1$, is a consequence of the application of the same argument as above on the corresponding result for the biinvariant functions (part (*b*) of Theorem 4.4). ■

Remark. The functions on G of right K-type n in the above theorem may also be viewed as sections of a certain line bundle over G/K. In fact, for each character χ_n of K we can construct a line bundle in the following way. Let $\sim_n$ be an equivalence relation on $G \times \mathbb{C}$: $(g, z) \sim_n (g', z')$ if and only if there is a $k \in K$ such that $g' = gk$ and $z' = \chi_n(k^{-1})z$. Now if we define

$G \times_{\chi_n} \mathbb{C} = G \times \mathbb{C}/\sim_n$ and $p : G \times_{\chi_n} \mathbb{C} \longrightarrow G/K$ by $p(gz) = gK$, then $G \times_{\chi_n} \mathbb{C}$ with the projection p becomes a line bundle over G/K. Now, if s is a section of this line bundle then it has the representation, $s(gK) = [g, z_g]$, where $z_g \in \mathbb{C}$ is uniquely determined by $g \in G$. Hence s determines a unique function $f_s : G \longrightarrow \mathbb{C}$ by $g \mapsto z_g$, and it is clear from the defintion of the line bundle that f_s is of right K-type n. Moreover, $s \mapsto f_s$ is a 1-1 correspondence between the space of sections of the line bundle $G \times_{\chi_n} \mathbb{C}$ and the space of maps on G of right K-type n. Borrowing terminology from function spaces through the correspondence, let $\Gamma^p(G \times_{\chi_n} \mathbb{C})$ denote the space of L^p sections of $G \times_{\chi_n} \mathbb{C}$. Then Theorem 4.5 can be restated as:

Theorem 4.6 *Let $\mathcal{F}$ be a subset of $\Gamma^p(G \times_{\chi_n} \mathbb{C})$, such that for each section in $\mathcal{F}$, the Fourier transform of it has a holomorphic extension on S_δ^γ for some $\delta > 0$ and all the matrix coefficients of that Fourier ransform vanish at infinity, on S_δ^γ. Let the Fourier transform of one of the section in $\mathcal{F}$ has a matrix coefficient which satisfy the not-too-rapidly-decreasing condition at ∞. Also assume that the Fourier transforms of sections in $\mathcal{F}$ do not vanish simultaneously on S_δ^γ and Γ_n. Then the left $L^1(G)$ module generated by $\mathcal{F}$ is dense in $\Gamma^p(G \times_{\chi_n} \mathbb{C})$.*

Now we are in a position to consider the final result of this section. Before stating it let us note that the trivial representation is an irreducible L^1-tempered representation. It is a subrepresentation of the principal series reprsentation $\pi_{\sigma_+,-1}$ (See **Subrepresentation of** $\pi_{\sigma,\lambda}$ in Section 1). The Fourier transform of $f(x)$ with respect to the trivial representation is $\int_G f(x)\,dx$. In fact, since $\Phi^{0,0}_{\sigma_+,-1} \equiv 1$,

$$\int_G f(x)\,dx = \int_G f(x)\,\Phi^{0,0}_{\sigma_+,-1}(x)\,dx = (\widehat{f}(-1))_{0,0} = (\widehat{f}(1))_{0,0} \qquad (53)$$

So, in the theorem below the hypothesis $\int_G f(x)\,dx \neq 0$ means that the Fourier transform of f with respect to the trivial representation is nonzero. Also note that for a function of type (m, n) with either m or n nonzero, $\int_G f(x)dx = 0$.

Theorem 4.7 *Let $\mathcal{F} = \{f^\alpha | \alpha \in \Lambda\}$ be a collection of functions in $L^p(G)$, $p \in (1,2)$, such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f}^\alpha_H$ has a holomorphic extension on $\widehat{M} \times S^\gamma_\delta$ for some $\delta > 0$. Let all matrix coefficients $\widehat{f}^\alpha_H(\sigma, .)_{m,n}$, $f \in \mathcal{F}, s \in \widehat{M}$, $m, n \in \mathbb{Z}$ satisfy $\lim_{|\lambda| \longrightarrow \infty} |(\widehat{f}^\alpha_H(\sigma, \lambda))_{m,n}| = 0$ on S^γ_δ; and let two of the matrix coefficients, one from each parity, not decay too rapidly at ∞. If $\{\widehat{f}^\alpha_H\}$ and $\{\widehat{f}^\alpha_B\}$ do not have any common zero on $\widehat{M} \times S^\gamma_\delta \cup \{D_+, D_-\}$ and $\mathbb{Z}^*$ respectively, where D_+ and D_- are the mock discrete series, then the $L^1(G)$-bimodule generated by $\mathcal{F}$ is dense in $L^p(G)$.*

Moreover, for the case $p = 1$, if, in addition to above, there is at least one f^α with non-vanishing integral then the ideal generated by $\mathcal{F}$ is dense in $L^1(G)$. Otherwise, the ideal is dense in $L^1(G)^0$.

Proof. As we have seen in the proof of the previous theorem, it is enough to consider the case when $p = 1$. For any function $f \in \mathcal{F}$ let $f_{-,i}$ be the projection of f to $L^1(G)_i$ for every $i \in \mathbb{Z}$.

For $i, m \in \mathbb{Z}$ let $p_{i,m}$ be the numerator of the rational function $\varphi^{i,m}_{\sigma,\lambda}$ and $P_{i,m}$ a polynomial in λ as described below: (Compare Theorem 4.5)

1. When $i.m \geq 0$, let $P_{i,m} = p_{i,m}$.

2. When $i.m < 0$ and i, m are odd integers, $P_{i,m}(\lambda) = \lambda^2 p_{i,m}(\lambda)$.

3. When $i.m < 0$ and i, m are even integers, $P_{i,m}(\lambda) = (1 - \lambda^2).p_{i,m}(\lambda)$.

Then for all $i, m \in \mathbb{Z}$, $P_{i,m}(\lambda)e^{-\lambda^4} \in C^1_H(\widehat{G})_{i,m}$. By the isomorphism of $C^1_H(G)_{i,m}$ with $C^1_H(\widehat{G})_{i,m}$ (Theorem 2.14) there exists a $g_{i,m} \in C^1(G)_{i,m}$ such that $\widehat{g}_{i,m\,H}(\lambda) = P_{i,m}(\lambda)\, e^{-\lambda^4}$ for $\lambda \in S^1_\delta$ and $\widehat{g}_{i,m\,B}(k) = P_{i,m}(k)e^{-k^4} \in C^1_B(G)_{i,m}$ for $k \in \Gamma_i$.

Now for each $m \in \mathbb{Z}$ we construct a collection of functions

$$\mathcal{F}_m = \{\, f * g_{i,m} \,|i \in \mathbb{Z}, f \in \mathcal{F}\} = \{\, f_{-,i} * g_{i,m} \,|i \in \mathbb{Z}, f \in \mathcal{F}\}$$

contained in $L^1(G)_m$. We will show that the collection $\mathcal{F}_m$ satisfies the conditions of Theorem 4.5 and hence generates $L^1(G)_m$.

For each $\lambda \in \mathcal{S}_\delta^1$ we will find a function in $\mathcal{F}_m$ which has nonzero Fourier transform at (σ_+, λ) or at (σ_-, λ) according as m is even or odd. As it has been observed in the proof of the previous theorem, the zeros of the polynomials $P_{i,m}$ inside $\mathcal{S}_\delta^1$ are as follows:

Case (a) if $i = 0$ and $m \neq 0$ then $P_{0,m}(-1) = 0$,

Case (b) if $i \neq 0$ and $m = 0$ then $P_{i,m}(1) = 0$

Case (c) if $i.m < 0$ and i, m are even then $P_{i,m}(\pm 1) = 0$,

Case (d) if $i.m < 0$ and i, m are odd then $P_{i,m}(0) = 0$,

Therefore, for any point λ on the strip $\mathcal{S}_\delta^\gamma$ other than 0, 1 and -1, $P_{i,m}$ and consequently the Fourier transform of $g_{i,m}$ is nonzero at λ. Hence for such a λ we easily get a function in $\mathcal{F}_m$ which has nonvanishing Fourier transform at (σ_+, λ) or at (σ_-, λ) according as m is even or odd. In fact, if $f \in \mathcal{F}$ be such that $f_{-,i}$ has nonvanishing Fourier transform at λ, then the function $f * g_{i,m}$ serves the purpose.

Now we deal with the point 0. We shall show that there is a function $f_{-,i} * g_{i,m}$ in the collection $\mathcal{F}_m$ with nonzero Fourier transform at $(\sigma_+, 0)$ or at $(\sigma_-, 0)$ according as m is even or odd. When m is even we choose an $f \in \mathcal{F}$ and even integers r, s such that the (r, s)-th matrix coefficient of $\widehat{f}(\sigma_+, 0)$ is nonzero. Therefore the Fourier transform of $f_{-,s} * g_{s,m}$ is nonzero at $(\sigma_+, 0)$ and the purpose is served; because, here s, m are even and hence $P_{s,m}(0) \neq 0$.

If m is odd and positive, we have to appeal to the hypothesis: $\widehat{f}_H(D_+) \neq 0$ for some $f \in \mathcal{F}$. This implies that there is a nonzero matrix coefficient $(\widehat{f}_H(\sigma_-, 0))_{u_1, v_1}$ of the Fourier transform of f, where both u_1 and v_1 are positive odd integers. Then $P_{v_1,m}$ does not vanish at 0 as both v_1 and m are positive. Therefore the Fourier transform of $f_{-,v_1} * g_{v_1,m}$ will be nonzero at $(\sigma_-, 0)$.

Again as $\widehat{f}_H(D_-) \neq 0$ for some $f \in \mathcal{F}$, there are K-finite vectors e_{u_2} and e_{v_2}, such that $(\widehat{f}_H)_{u_2,v_2}(\sigma_-, 0) \neq 0$, where u_2 and v_2 are both negative odd integers. If m is odd and negative then $P_{v_2,m}$ does not have a zero at 0. Consequently the Fourier transform of $f_{-,v_2} * g_{v_2,m}$ will be nonzero at 0.

Next we consider the points 1 and -1. First note that for odd i, m, $P_{i,m}$ has no zero at ± 1. So for odd m we can easily find a function from $\mathcal{F}_m$ with nonzero Fourier transform at $(\sigma, \pm 1)$. So it remains to prove the case when m is even.

Let m be nonzero positive even. By hypothesis there exists $f \in \mathcal{F}$ such that $\widehat{f}_B(1) \neq 0$. This implies that there is a nonzero matrix coefficient $(\widehat{f}_B(1))_{r_1,s_1}$ of $\widehat{f}_B$, where r_1, s_1 are even integers and > 1 as $r_1, s_1 \in \mathbb{Z}(1)$. Then $(\widehat{f}_H(1))_{r_1,s_1} \neq 0$, $(\widehat{f}_H(1))_{r_1,s_1}$ being a nonzero mutiple of $(\widehat{f}_B(1))_{r_1,s_1}$. Now as $(\widehat{f}_H(-1))_{r_1,s_1} = \varphi^{r_1,s_1}_{\sigma_+,-1}(\widehat{f}_H(1))_{r_1,s_1}$ and $\varphi^{r_1,s_1}_{\sigma_+,-1}$ is nonzero (see Proposition 2.10), $(\widehat{f}_H(-1))_{r_1,s_1} \neq 0$. Now as both m and s are positive $f_{-,s_1} * g_{s_1,m}$ is a function in $\mathcal{F}_m$ such that its Fourier transform is nonzero at $(\sigma_+, \pm 1)$. And thus we can bypass the difficulties in cases (a) and (c) above when m is positive.

Let now m be nonzero negative even. As $\widehat{f}_B(-1) \neq 0$, for some $f \in \mathcal{F}$ there is a matrix coefficient of its Fourier transform, $(\widehat{f}_B)_{r_2,s_2}$ which is nonzero at -1. Then clearly r_2 and s_2 are even integers and < -1 as $r_2, s_2 \in \mathbb{Z}(-1)$. Therefore $(\widehat{f}_H)_{-,s_2}(\sigma_+, -1) \neq 0$. Also, $(\widehat{f}_H(1))_{r_2,s_2} = \varphi^{r_2,s_2}_{\sigma_+,1}(\widehat{f}_H(-1))_{r_2,s_2}$ and $\lambda \mapsto \varphi^{r_2,s_2}_{\sigma_+\lambda}$ has no zero at 1 (see Proposition 2.10), $(\widehat{f}_H(+1))_{r_2,s_2} \neq 0$. Since both m and s are negative, the cases (a) and (c) above will not concern us and for we find $f_{-,s} * g_{s,m}$, a function in the collection $\mathcal{F}_m$ which has nonzero Fourier transform at $(\sigma_+, \pm 1)$.

We will now treat the case $m = 0$. Here the point -1 can be easily dealt with as for any (even) $i \neq 0, P_{i,0}(-1) \neq 0$. So we concentrate on the point 1. By hypothesis, there is a function $f \in \mathcal{F}$, such that $\int_G f(x)\,dx = (\widehat{f}(1))_{0,0} = \widehat{f}_{-,0}(1) \neq 0$ (see discussion preceeding this proof). So $f_{-,0} * g_{0,0}$ is a function in the collection $\mathcal{F}_0$ which has nonzero Fourier transform at $+1$.

Thus we have shown that Fourier transform of the functions in $\mathcal{F}_m$ do not vanish simultaneously at any point of $\{\sigma\} \times S^\gamma_\delta$ where σ is σ_+ or σ_- according as m is even or odd.

To find a function in $\mathcal{F}_m$ whose Fourier transform does not 'decay too rapidly at ∞' we get the matrix coefficient of the parity of m which has that property. Let that matrix coefficient be the (α, β)-th one. Then the (α, m)-th coefficient of the Fourier transform of $f_{-,\beta} * g_{\beta m}$ also will not 'decay too rapidly ∞'. Note that the only zero of the polynomial $P_{i,m}$ on the imaginary axis is at 0.

The collection $\mathcal{F}_m$ for $m \neq 0$ satisfies the conditions of Theorem 4.5 and hence generates $L^1(G)_m$ under left convolution.

Now as $f_i * g_{i,m} = f * g_{i,m}$, for every m, elements of $\mathcal{F}_m$ are right convolutions of functions in $\mathcal{F}$. So the two sided (closed) ideal generated by $\mathcal{F}$ contains $L^1(G)_m$ for all m. The smallest closed right G-invariant subspace of $L^1(G)$ containing $L^1(G)_m$ for all $m \in \mathbb{Z}$ is $L^1(G)$ itself. Hence the first part of the theorem follows.

If we omit the condition $\int_G f(x)\,dx \neq 0$, there is no effect on the collection $\mathcal{F}_m$ for $m \neq 0$. But it follows from the last part of Theorem 4.5 that, in this case $\mathcal{F}_0$ will generate $L^1(G)_0^0$, the space of $L^1(G)_0$ functions with integral zero. Note that $L^1(G)_m$ is contained in $L^1(G)^0$ for any $m \neq 0$; *i.e.*, $L^1(G)_m = L^1(G)_m^0$ (see discussion preceeding this theorem). Hence, in this case the function f under left and right convolution generates an ideal which contains $L^1(G)_m^0$ for all $m \in \mathbb{Z}$. Since the smallest closed right G-invariant subspace of $L^1(G)^0$ containing all the $L^1(G)_m^0$'s is $L^1(G)^0$, the the second part of the theorem follows. ∎

It is clear from the description of $C^p(\widehat{G})$, $0 < p \leq 2$, that following exactly the same steps we can prove:

Theorem 4.8 *Let $\{f^\alpha | \alpha \in \Lambda\}$ be a subset of $C^p(G)$, $0 < p \leq 2$, such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f^\alpha_H}$ has holomorphic extension on $\widehat{M} \times S^\gamma_\delta$ for some $\delta > 0$ and all matrix coefficients $(\widehat{f^\alpha_H}(\sigma, .))_{m,n}$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}$, satisfy $\lim_{|\lambda| \longrightarrow \infty} |(\widehat{f^\alpha_H}(\sigma, \lambda))_{m,n}| = 0$ on S^γ_δ. Let two of the matrix coefficients, one from each parity, not decay too rapidly at ∞. If $\{\widehat{f^\alpha_H}\}$ and $\{\widehat{f^\alpha_B}\}$ do not have common zeros on the principal series representations parametrized by $\widehat{M} \times S^\gamma_\delta$ and on the discrete series* *respectively and if $\int_G f dx \neq 0$ then the ideal generated by $\{f^\alpha | \alpha \in \Lambda\}$ is dense in $C^p(G)$.*

This is an extension of Proposition 5.1 in [E-M] which deals with $C^1(G)_{0,0}$.

Now for the zero-Schwartz space, $C^0 = \cap\{C^p(G) | 0 < p \leq 2\}$ (see [Ba, Section 19.] for detail) we can show:

Theorem 4.9 *Let $\mathcal{F} = \{f^\alpha | \alpha \in \Lambda\}$ be a subset of $C^0(G)$. Let two of the matrix coefficients, one from each parity, not decay too rapidly at ∞. If $\{\widehat{f^\alpha_H}\}$ do not have any common zero on any irreducible subrepresentations of the principal series representations, then the ideal generated by $\mathcal{F}$ is dense in $C^0(G)$.*

Proof. It is clear that the ideal generated by $\mathcal{F}$ in C^0 is dense in C^p for all $p \in (0, 2]$ because $\mathcal{F}$ satisfies conditions of Theorem 4.8 for all p and C^0 is dense in all C^p. Now as $C^0 = \cap\{C^p(G) | 0 < p \leq 2\}$ the theorem is proved. ∎

5 Rank one Symmetric spaces

In this section and in the next one we work with the Riemannian symmetric spaces G/K of non compact type, where G is one of the following semi simple Lie groups of *real rank* one: $SU(n,1)$, $SO(n,1)$, $SP(n,1)$ or the connected Lie group of real type F_4; and K is a fixed maximal compact subgroup of G in each case. Here we extend the standard notation for $SL_2(\mathbb{R})$ as in Section 1 and 2 to the context of real rank one groups where the meaning is unambiguous, and add some more which are required for this case.

Let θ be the Cartan involution corresponding to K. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G and K respectively and $\mathfrak{k} + \mathfrak{p} = \mathfrak{g}$ be the Cartan decomposition with respect to θ. For $x \in G$ let $\sigma(x) = \|X\|$ when $x = k \exp X$ ($k \in K$, $X \in \mathfrak{p}$). Here $\|.\|$ is the norm given by the Killing form. Also for $x \in G$, let

$$\Xi(x) = \int_K e^{-\rho h(xk)} dk,$$

where ρ is the half sum of positive roots.

Let $\mathfrak{a}$ be a fixed maximal abelian subspace of $\mathfrak{p}$ and $A = \exp(\mathfrak{a})$. Then $\dim \mathfrak{a} = 1$. Consider the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Due to the one dimensionality of $\mathfrak{a}^*$ all roots will give rise to the same reflection. In fact, only possible roots in this case are $\pm\frac{1}{2}\lambda$, $\pm\lambda$, $\pm 2\lambda$ (see [G-V], p. 62) of which only one is simple, and the Weyl group $\mathcal{W}(A) \cong \mathbb{Z}_2$. Let $G = KAN$ be the corresponding Iwasawa decomposition. We denote by M (resp. $\widetilde{M}$) the centralizer (resp. normalizer) of A in K. Then $\mathcal{W}(A) \cong \frac{\widetilde{M}}{M}$. Let $\mathcal{P}(A)$ stand for the set of parabolic subgroups of G with split part A. Conjugation by elements of $\widetilde{M}$ on N induces a transitive group action on $\mathcal{P}(A)$. Now, since M normalizes N, the Weyl group $\mathcal{W}(A)$ acts (transitively) on $\mathcal{P}(A)$. Let ω be the only non-trivial element of $\mathcal{W}(A)$ which takes the positive roots to the negative ones, and let $x_w \in \widetilde{M}$ be such that $\pi(x_w) \equiv \omega \in \mathcal{W}(A)$, π being the quotient map $\pi : \widetilde{M} \longrightarrow \frac{\widetilde{M}}{M}$. Then, for $P = MAN \in \mathcal{P}(A)$, $P^\omega = PAN^\omega = PA\bar{N} = \bar{P}$, where $\bar{N} = \theta(N) = x_\omega N x_\omega^{-1}$. Thus $\mathcal{P}(A)$ consists of two minimal parabolic subgroups, namely P and $\bar{P}$. Also recall that the only nonminimal parabolic subgroup in our case is G itself.

The representations $\pi(P, \sigma, \lambda)$ and $\pi(\bar{P}, \sigma, \lambda)$ are the principal series representations induced from P and $\bar{P}$ respectively, where $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. The Fourier transform of any right K-invariant function $f \in L^p(G)$ with respect to the nonspherical principal series is zero (see Proposition 5.1 below). Therefore, only the spherical principal series representations $\pi(P, \sigma_0, \lambda)$ (σ_0 being the trivial representation of M) are relevant here. We will denote the spherical representation $\pi(P, \sigma_0, \lambda)$ simply by π_λ. This is an L^p tempered when $\lambda \in S^\gamma = \{\lambda \in \mathbb{C} \,|\, |\Re\lambda| \leq \gamma\rho\}$, where $\gamma = \frac{2}{p} - 1$ and ρ is the half sum of positive roots (see [T, Sec. 4, Definition 1]). The strip S^γ augmented by ε will be denoted by $S^\gamma_\varepsilon = S^{\gamma+\varepsilon}$.

The spherical principal series representations π_λ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ are realised on the same subspace, say H_{σ_0}, of the Hilbert space $L_2(K)$ (compact picture). Let us fix an *orthonormal basis* $\{e_\alpha\}_{\alpha \in \mathbb{Z}}$ of H_{σ_0} of K-finite vectors among which e_0 is the K-fixed vector. By matrix coefficients of a representation π_λ we will always mean matrix coefficients with respect to this $\{e_\alpha\}$. The (e_r, e_s)-th matrix coefficient of π_λ will be denoted by $\Phi^{r,s}_\lambda$. For $f \in L^p(G/K)$ and for λ in the corresponding strip S^γ, the matrix coefficients of the Fourier transforms, $(\widehat{f}(\pi_\lambda))_{e_i, e_j} = \int \langle \pi_\lambda(x^{-1})e_i, e_j \rangle f(x)dx$ exist and constitute the formal *matrix Fourier transform* $\widehat{f}(\pi_\lambda)$. For $f, g \in L^p(G/K)$, if at least one of $\widehat{f}(\pi_\lambda)$ and $\widehat{g}(\pi_\lambda)$ has finitely many nonzero entries and if $f * g$ is defined, then $(\widehat{f * g})(\pi_\lambda)$ is given by the matrix multiplication $(\widehat{f * g})(\pi_\lambda) = \widehat{f}(\pi_\lambda)\widehat{g}(\pi_\lambda)$.

For $\delta \in \widehat{K}$ let $\alpha(\delta) = d(\delta)\overline{\chi_\delta}$, where $d(\delta)$ = degree of δ and χ_δ = character of δ. Let dk denote the normalized Haar measure of K. Define $\alpha(\delta) * f$ and $f * a(\delta)$ by

$$(\alpha(\delta) * f)(x) = \int_K \alpha(\delta)(k) f(kx) dk$$

and

$$(f * \alpha(\delta))(x) = \int_K f(xk)\alpha(\delta)(k) dk,$$

where $x \in G$. f is said to be of left (resp. right) type δ when $\alpha(\delta) * f \equiv f$ (resp. $f * \alpha(\delta) \equiv f$). On the other hand by *f has no left (resp. right) K-type δ* we mean $\alpha(\delta) * f \equiv 0$ (resp. $f * \alpha(\delta) \equiv 0$). Here $\alpha(\delta) * f$ (resp. $f * \alpha(\delta)$)

is the projection of f in the left (resp. right) K-type δ and we shall denote it by ${}_\delta f$ (resp. by f_δ). A function is of type $(\delta,0)$ (resp. of type $(0,\delta)$) when it is right invariant (resp. left invariant) and its left type (resp. right type) is δ.

We shall now prove two propositions which we need in the next section. They will extend some of the facts for $SL_2(\mathbb{R})$ mentioned towards the end of Section 1 to more general groups, in particular for groups of real rank one.

Proposition 5.1 *Let $f \in L^1(G)$, $f \neq 0$, be of type $(\delta,0)$, where $\delta \in \widehat{K}$ is a nontrivial K-type. Then δ restricted to M contains the trivial representation of M, and the Fourier transform of f with respect to a nonspherical principal series representation is zero.*

Proof. Since f is right K-invariant we can find a bounded biinvariant function $h \in L^1(G)$ such that $f * h \neq 0$. It is clear that $f * h$ is right K-invariant and transforms according to δ under left K-action. But $f * h$ is a continouous function, so we may assume f to be continuous. Further, we may translate f from the left by an element of K and hence assume that $f(a) \neq 0$ for some $a \in A$ in the KAK decomposition of G. Now, let $g(x) = \int_M f(mx)\,dm$, $x \in G$, where dm is the normalised Haar measure on M. Then g is again of type $(\delta,0)$, and $g(a) = \int_M f(ma)\,dm = \int_M f(am)\,dm = f(a) \neq 0$. The first assertion of the proposition now follows by noting that g is left M invariant.

For the second part of the proposition let $\pi = (\pi_{\sigma,\lambda}, H_{\sigma,\lambda})$, $\sigma \in \widehat{M}$ and $\sigma \neq 0$, be a non-spherical principal series representation of G, and let $u, v \in H_{\sigma,\lambda}$ be nonzero K-finite vectors, of which v is of K-type μ_1. Then,

$$\begin{aligned}
\langle \pi_{\sigma,\lambda}(f)u, v\rangle &= \int_G f(x)\langle \pi(x^{-1})u, v\rangle dx \\
&= \int_{K\overline{A}^+K} f(k_1ak_2)\langle \pi(k_2^{-1}a^{-1}k_1^{-1})u, v\rangle J(a)dk_1\,da\,dk_2 \qquad (54)\\
&= \int_{K\overline{A}^+K} f(k_1a)\langle \pi(k_2^{-1}a^{-1}k_1^{-1})u, v\rangle J(a)dk_1\,da\,dk_2.
\end{aligned}$$

Here $J(a)$ is the Jacobian of the transformation $x = k_1ak_2$. Let $\pi(a^{-1}k_1^{-1})u = w$; then $w \in H_{\sigma,\lambda}$. Since $\pi_{\sigma,\lambda}|K$ is unitary, we have:

$$\langle \pi(k_2^{-1}a^{-1}k_1^{-1})u, v\rangle \;=\; \langle \pi(k_2^{-1})w, v\rangle \;=\; \langle w, \pi(k_2)v\rangle.$$

Now $H_{\sigma,\lambda} = \oplus_{\mu \in \widehat{K}} H_\mu$, where H_μ is the K-isotypic subspace of type μ and $\oplus$ denotes the orthogonal direct sum of Hilbert spaces. Note that $\pi(k_2)v \in H_{\mu_1}$ for all $k_2 \in K$. Writing $w = \Sigma_\mu w_\mu$ (not necessarily a finite sum), $w_\mu \in H_\mu$, we get $\langle \pi(k_2)v, w\rangle = \langle \pi(k_2)v, w_{\mu_1}\rangle$. Here again $w_{\mu_1} = \Sigma_{i=1}^{\alpha_1} a_i w_{\mu_1,i}$, where $\alpha_1 = (\deg \mu_1)^2$ and $\{w_{\mu_1,i} | i = 1, \cdots \alpha_1\}$ is an *orthonormal basis* of H_{μ_1}. Therefore, the integral over k_2 in (54) is equal to

$$\Sigma_{i=1}^{\alpha_1} a_i \int_K \langle w_{\mu_1,i}, \pi(k_2)v\rangle dk_2 = \Sigma_{i=1}^{\alpha_1} a_i \langle w_{\mu_1,i}, e_0\rangle = 0,$$

by Schur's orthogonality relations (since μ_1 is a nontrivial representation of K). ∎

Note that in the above situation all the K-types have multiplicity one as (K, M) is a Gelfand pair and the set of right K-types are precisely those $\mu \in \widehat{K}$ which possess a nonzero M-fixed vector.

Proposition 5.2 *Let f, $g \in C_c^\infty(G)$ and $\delta \in \widehat{K}$. Suppose that f is of right type δ and g has no component with left K-type δ. Then $f * g \equiv 0$.*

In particular if $f \in L^1(G/K)$ and g has no component which is K-fixed on the left, then $f * g \equiv 0$

Proof. For $x \in G$,

$$\begin{aligned}
(f * g)(x) &= \textstyle\int_G f(y) g(y^{-1}x) dy \\
&= \textstyle\int_G (\int_K f(yk)\alpha(\delta)(k) dk\,) g(y^{-1}x) dy \\
&= \textstyle\int_K \alpha(\delta)(k)\, (\int_G f(yk) g(y^{-1}x) dy\,)\, dk \\
&= \textstyle\int_K \alpha(\delta)(k)\, (\int_G f(z) g(kz^{-1}x) dz\,)\, dk \\
&= \textstyle\int_G (\int_K \alpha(\delta)(k) g(kz^{-1}x) dk\,) f(z) dz.
\end{aligned}$$

Since g has no component of left K-type δ,

$$\int_K \alpha(\delta)(k) g(kz^{-1}x) dk = (\alpha(\delta) * g)(z^{-1}x) = 0,$$

which implies that $f * g(x) = 0$. Since x is arbitrary this proves that $f * g \equiv 0$. ∎

Let us now add some more notation, definitions and preliminaries esentially from [T]. Instead of giving full details (which is available in [T, Section 6 and 8]) we include only some specific parts which will be necessary in the next section.

For $p \in [1,2]$, $C^p(G)_{0,\delta}$ is the space of all C^∞ functions f of type $(0,\delta)$ such that $\rho^p_{g,r}(f) < \infty$ for all $g \in \mathcal{U}$ and $r \in \mathbb{R}$, where $\mathcal{U}$ denotes the universal enveloping algebra of $\mathfrak{g}$ and $\rho^p_{g,r}$ is the seminorm given by

$$\rho^p_{g,r}(f) = \sup_{x \in G}(1+\sigma(x))^r \Xi(x)^{-\frac{2}{p}}|f(g;x)|. \tag{55}$$

$C^p(G)_{0,\delta}$ is the L^p-schwartz space of $(0,\delta)$ type functions.

Let $F = \{\delta, 0\}$ be the set of two K-types 0 and δ, of which '0' is the trivial one and also, $\delta|M$ contains the trivial representation σ_0 of M. Let the basis elements $e_1, e_2, \ldots, e_l$ of the orthonormal basis $\{e_\alpha\}$ of H_{σ_0} transform according to δ, and let e_0 be as before the K-fixed vector. We denote by $H_{\sigma_0,F}$ the subspace of H_{σ_0} spanned by e_0 and $\{e_1, e_2, \ldots, e_l\}$. Fix a minimal parabolic P and consider the principal series representation $\pi(P, \sigma_0, \lambda) = \pi_\lambda$ for $\lambda \in \mathbb{C}$. Let U be the set of points in $\mathfrak{a}^*_{\mathbb{C}} = \mathbb{C}$ at which there are singularities of the asymptotic expansions of the matrix coefficients corresponding to vectors e_0 and $e_1, e_2 \ldots, e_l$. A detailed description of U is available in [T, p. 103]. Let $\mathcal{E}$ consist of the

(i) matrix coefficients of the principal series representations π_λ with respect to vectors in $H_{\sigma_0,F}$ at $w\zeta$ and

(ii) the derivatives of these matrix coefficients with respect to λ at the points $w\zeta$, where the order of derivatives is less than the order of singularity $o(\zeta)$ of the asymptotic expansion of matrix coefficients at ζ,

where $w \in \mathcal{W}$ and $\zeta \in U$.

There are linear relations among the elements of $\mathcal{E}$ and one can construct a basis for it in a way demonstrated in [T, p. 104]. $\mathcal{E}$ can be indexed by the set

$$I = \bigcup_{\zeta \in U} \{0, 1, \ldots, o(\zeta) - 1\}^\zeta \times \{e_1, \ldots, e_l\} \times \mathcal{W}.$$

Let I' be the subset of I which indexes the basis of $\mathcal{E}$. Now if we denote the matrix coefficients and their derivatives in $\mathcal{E}$ corresponding to the index i, by $\pi(i)$ then

$$\pi(i) = \Sigma_{i' \in I'} C(i : i')\pi(i'),$$

for some constants $C(i : i') \in \mathbb{C}$.

Let $U_p = S^\gamma \cap U$ and $L(H_{\sigma_0,F})$ be the linear endomorphisms of $H_{\sigma_0,F}$.

Then $C^p(\widehat{G})_{0,\delta}$ can be defined (see [T, section 8, Definition 1]) as the space of continuous functions $F : S^\gamma \longrightarrow L(H_{\sigma_0,F})$ such that the only nonzero matrix coefficients of $F(\lambda)$ are $\langle F(\lambda)e_0, e_i \rangle$ for $i = 1, \ldots, l$ (i.e. $F(\lambda)$ is a row vector) and which satisfy the following:

1. F is holomorphic in $\mathring{S}^\gamma$, the interior of S^γ.

2. $\rho^p_{u,r}(F) < \infty$ for all $r \in \mathbb{R}^+$ and for all differential operators u on functions defined over $\mathbb{C}$, where

   $$\rho^p_{u,r}(F) \;=\; \sup_{\lambda \in \mathring{S}^\gamma} ||uF(\lambda)||(1 + |\lambda|)^r,$$

   $||.||$ being the norm of the matrix.

3. In the notation explained above

   $$F(i) = \Sigma_{i' \in I'_p} C(i : i')F(i')$$

   for $i \in I_p = I \cap S^\gamma$. Also $I'_p = I' \cap S^\gamma$. Here if $i = (w, \zeta, j, e_k) \in \mathcal{W} \times U \times \{0, 1, \ldots, o(\zeta) - 1\} \times \{e_1, \ldots, e_l\}$, then $F(i) = \left(\frac{d}{d\lambda}\right)^j \langle F(\lambda)e_0, e_k \rangle|_{w\zeta}$.

4. For each $\lambda \in S^\gamma$, there is a relation between $F(\lambda)$ and $F(w\lambda)$ ([T, Section 8, Definition 1(2)]) due to the equivalence of principal series representations π_λ and $\pi_{w\lambda}$. (Details omitted as it will not be required for the proof of our result in the next section).

With the topology induced by the seminorms $\rho^p_{u,r}$, $C^p(\widehat{G})_{0,\delta}$ becomes a Fréchet space. It is isomorphic with $C^p(G)_{0,\delta}$ under the Fourier transfrom (see [T, Section 11, Theorem 1]). Note that for a function which is either left or right K-invariant the discrete part of the Fourier transform is absent and hence the principal part $C^p_H(\widehat{G})_{0,\delta} = C^p(\widehat{G})_{0,\delta}$.

The isomorphic image of $C^\infty_c(G)_{0,\delta}$ under the Fourier transform is denoted by $C^\infty_c(\widehat{G})_{0,\delta}$. By the Paley-Wiener theorem, $C^\infty_c(\widehat{G})_{0,\delta}$ consists of entire functions F from S^γ to $L(H_{\sigma_0,F})$ of exponential type (i.e. there exists $C > 0$, and for each $N \geq 0$, $C_N > 0$ such that, $||F(\lambda)|| \leq C_N(1+|\lambda|)^{-N}e^{C|\Re\lambda|}$), which satisfy relations 3 and 4 above, with S^γ, I_p and I'_p replaced by $\mathbb{C}$, I and I' respectively (see [T, Section 7]).

6 Wiener Tauberian Theorem for rank one Symmetric Spaces

In this section G is one of the semisimple Lie groups of real rank one listed in the previous section. We start with the W-T theorem for biinvariant functions on such a G:

Theorem 6.1 *Let $\{f^\alpha|\alpha \in \Lambda\}$ be a family of functions in $L^p(G//K)$, Λ being an index set, such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f^\alpha}$ extends holomorphically on the strip $\overset{\circ}{S^\gamma_\varepsilon}$ for some $\varepsilon > 0$, and vanish at infinity , that is, $\lim_{|\lambda|\longrightarrow\infty} |\widehat{f^\alpha}(\lambda)| = 0$ on S^γ_ε. Suppose that the functions $\widehat{f^\alpha}$, $\alpha \in \Lambda$, do not vanish simultaneously on any point of S^γ_ε. Moreover let there be an $\alpha_0 \in \Lambda$ such that $\widehat{f^{\alpha_0}}$ satisfies the not-too-rapidly-decreasing condition at infinity :*

$$\limsup_{|t|\longrightarrow\infty} ||(\widehat{f^{\alpha_0}})(it)||.|e^{Ke^{|t|}}| > 0 \quad \textit{for all } K > 0.$$

Then the $L^1(G//K)$ module generated by $\{f^\alpha|\alpha \in \Lambda\}$ is dense in $L^p(G//K)$.

We omit the proof of this theorem as it runs entirely along the lines of the corresponding proof for $L^1(SL_2(\mathbb{R}))_{0,0}$ in [B-W]. The crux of the matter is that the space $C^p(\widehat{G})_{0,0}$ of Fourier transforms of the Schwartz space functions is indistinguishable as a function space from the corresponding space for $SL_2(\mathbb{R})$. And the only difference between $C^1(\widehat{G})_{0,0}$ and $C^p(\widehat{G})_{0,0}$ is in the width of the strip S^γ, which is the domain of the Fourier transforms.

As in the previous section let $F = \{\delta, 0\}$ be the set of K-types, where $\delta|M$ contains σ_0 and 0 is the trivial representation of K. Also $e_0, e_1, e_2, \ldots, e_l$ and $H_{\sigma_0,F}$ are as defined in the previous section.

On our way to the main theorem we need the following :

Observation. For every $\lambda \in \mathbb{C}$, the K-fixed vector e_0 is cyclic in at least one of the spherical principal series representations among $\{\pi_{w\lambda} | w \in \mathcal{W}\}$. (see

Johnson and Wallach [J-W] Theorem 5.1 (2),(3),(4) and Johnson [J] Theorem 5.2).

For a fixed p let us fix a $p' < p$. Then it is known that $C_c^\infty \subset C^{p'} \subset C^p$. When $p > 1$ we will take $p' = 1$. Then we have:

Lemma 6.2 *Let $\lambda \in S_\epsilon^\gamma$, and let $\pi_{w\lambda}$ be a spherical principal series representation in which e_0 is a cyclic vector. Suppose that for some $f \in L^p(G/K)$, $\widehat{f}(\pi_{w\lambda}) \neq 0$. Then, there is a $g \in C^1(G) \cap C^{p'}(G)$ such that $g * f$ is a biinvariant L^p-function and $\widehat{g * f}(\pi_{w\lambda}) \neq 0$.*

Proof. For any K-type δ, let ${}_\delta f$ be the projection of f in the left K-type δ. Then ${}_\delta f$ is a $(\delta, 0)$ type function and its Fourier transform is a coloumn vector.

Now, the condition $\widehat{f}(\pi_{w\lambda}) \neq 0$ implies that there is a K type δ so that ${}_\delta\widehat{f}$ is nonzero at $\pi_{w\lambda}$. If $e_1, \ldots, e_l$ form a basis for the space of vectors transforming according to δ in the representation space H_{σ_0} of $\pi_{w\lambda}$, then this means that for some e_r in the above set the (e_r, e_0)-th matrix coefficient of the Fourier transform of f at $\pi_{w\lambda}$ is nonzero. Now as the K-fixed vector e_0 is cyclic in $\pi_{w\lambda}$, the matrix coefficient $\langle \pi_{w\lambda}(x)e_0, e_r\rangle$ can not be indentically zero, since otherwise the closed linear span of $\{\pi_{w\lambda}(x)e_0 \mid x \in G\}$ will be a subrepresentation orthogonal to e_r, contradicting the fact that e_0 is cyclic in $\pi_{w\lambda}$. If $\lambda \in S^\gamma - \{w.U_p | w \in \mathcal{W}\}$ then it is clear from the description of $C_H^1(\widehat{G})_{0,\delta}$ and its isomorphism with $C_H^1(G)_{0,\delta}$ (see Section 5 and also [T, Section 8, Definition. 1] and [T, Section 11, Theorem 1]), that there exists a function $g \in C^1(G)$ of type $(0, \delta)$ such that only the (e_0, e_r)-th matrix coefficient of its Fourier transform is nonzero at $\pi_{w\lambda}$. It readily follows that $g * f$ is a biinvariant function with $\widehat{g * f}(\pi_{w\lambda}) \neq 0$.

Now when $\lambda \in S^\gamma \cap \{w.U_p | w \in \mathcal{W}\}$, it is not clear how to find a g as above which will have only one chosen matrix coefficient nonzero, as this time the matrix coefficients have dependencies among themselves (see Section 5 and [T, Sec. 8, Definition. 1(4)]). We need a more careful argument here to show that such a g is available. Since e_0 is cyclic for $\pi_{w\lambda}$, its (e_0, e_r)-

th matrix coefficient, $\Phi^{0,r}_{w\lambda}$ can not be identically zero. Also for the same reason, for linearly independent vectors $e_1, e_2, \ldots, e_l \in H_{\sigma_0,F}$ it can not happen that $\Sigma_{i=1}^k a_i \langle \pi_{w\lambda}(x)e_0, e_i\rangle = \langle \pi_{w\lambda}(x)e_0, \Sigma_{i=1}^l a_i e_i\rangle = 0$ for all $x \in G$, unless $a_1 = a_2 = \ldots = a_l = 0$. Thus the matrix coefficients $\Phi^{0,i}_{w\lambda}$, $i = 1, \ldots, l$, are linearly independent functions in $\mathcal{E}$. (However, they may depend on some of the derivatives of the others; but this will not concern us.) Now, as $\Phi^{0,i}_{w\lambda}$ are linearly independent elements of $(C^1(G)_{0,\delta})^*$, the dual space of the Frechét space $C^1(G)_{0,\delta}$, an application of Hahn-Banach thorem gives us a $g \in C^1(G)_{0,\delta}$ such that only the (e_0, e_r)-th matrix coefficient of its Fourier transform is non-zero at $w\lambda$. One can also appeal directly to isomorphism of $C^1(G)$ with $C^1(\widehat{G})$ [T, Sec. 11, Theorem 1] to get such a g. This proves the lemma for $p > 1$ as in this case p' can be taken to be 1.

When $p = 1$ we proceed through the same steps; only instead of appealing to the isomorphism theorem of schwartz spaces $C^{p'}(G)$, we use the Paley-Wiener Theorem (see Section 5) for getting a g as above. By the Paley-Wiener Theorem functions in $C_c^\infty(G)$ are entire and of exponential type. Also $C_c^\infty(G) \subset C^{p'}(G) \subset C^1(G)$ for any $p' < 1$. From the description of $C_c^\infty(\widehat{G})_{0,\delta}$, it is clear that a g, with nonvanishing Fourier transform at a given point λ is always available. Hence the lemma follows. ∎

Note that in the above proof the choice of the function g depends on λ.

Theorem 6.3 *Let $\{f^\alpha | \alpha \in \Lambda\}$ be a Λ-indexed family of functions in $L^p(G/K)$, such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f^\alpha}$ of f^α has a holomorphic extension on $\mathring{S}_\varepsilon^\gamma$ for some $\varepsilon > 0$, and all the matrix coefficients of $\widehat{f^\alpha}$ vanish at infinity, i.e. $\lim_{|\lambda| \longrightarrow \infty} |(\widehat{f^\alpha}(\lambda))_{m,n}| = 0$ on S_ε^γ. Suppose further that the collection $\{\widehat{f^\alpha} | \alpha \in \Lambda\}$ does not have common zero on any representation (containing the K-fixed vector) parametrized by S_ε^γ. Let there be an $\alpha_0 \in \Lambda$ such that $\widehat{f^{\alpha_0}}$ further satisfies the condition:*

$$\limsup_{|t| \longrightarrow \infty} ||_\delta(\widehat{f^{\alpha_0}})(it)||.|e^{Ke^{|t|}}| > 0 \quad \textit{for all } K > 0, \tag{56}$$

for some $\delta \in \widehat{K}$. Then the left $L^1(G)$ module generated by $\{f^\alpha | \alpha \in \Lambda\}$ is dense in $L^p(G/K)$.

Proof. We look at the collection of biinvariant functions:

$$\mathcal{F} = \{g * f^\alpha | \, g \in C^1(G) \cap C^{p'}(G) \text{ and } g \text{ is left invariant, } \alpha \in \Lambda\},$$

where p' is as in Lemma 6.2. Without loss of generality we assume that the strip $S_\varepsilon^\gamma = S^{\gamma+\varepsilon}$ corresponds to p', *i.e.* $\frac{2}{p'} - 1 = \gamma + \varepsilon$. Because otherwise $\gamma + \varepsilon$ can be replaced by $\min\{\gamma + \varepsilon, \frac{2}{p'} - 1\}$. Let $\lambda \in S_\varepsilon^\gamma$. Then there exists a $w_0 \in \mathcal{W}$ such that the K-fixed vector e_0 is cyclic in $\pi_{w_0\lambda}$. By hypothesis there is an $\alpha \in \Lambda$ such that $\widehat{f^\alpha}(w_0\lambda) \neq 0$. Therefore by the lemma above there is a member $g * f^\alpha$ in $\mathcal{F}$ for which $\widehat{g * f^\alpha}(w_0\lambda) \neq 0$. But $g * f^\alpha$ being a biinvariant function $\widehat{g * f^\alpha}(\lambda) = \widehat{g * f^\alpha}(w_0\lambda)$ (recall that $C^p(\widehat{G})_{0,0} = C^p(\widehat{SL_2(\mathbb{R})})_{0,0}$) and hence $\widehat{g * f^\alpha}(\lambda) \neq 0$. Thus the collection $\mathcal{F}$ satisfies the nonvanishing condition of Theorem 6.1.

The not-too-rapidly-decreasing condition in the hypothesis implies that there is a vector $e_n \in H_{\sigma_0}$ which transforms according to δ and satisfies the condition

$$\limsup_{|t| \longrightarrow \infty} \| {}_\delta(\widehat{f^{\alpha_0}})(it)_{n,0} \| \, |e^{Ke^{|t|}}| > 0$$

for all $K > 0$, where ${}_\delta(\widehat{f^{\alpha_0}})(it)_{n,0}$ is the matrix coefficient with respect to the pair (e_n, e_0). We find a function $g \in C^1(G)_{0,\delta}$ which is left invariant and the only nonzero component of its Fourier transform is $\widehat{g}_{0,n}$, and further $|\widehat{g}_{0,n}(it)|$ is nonvanishing almost everywhere and is of order e^{-t^2} for $t \in \mathbb{R}$. Such a choice is possible because, except for $t = 0$, all the representations parametrized by $\lambda = it$, $t \in \mathbb{R}$, are irreducible representations (see Knapp [K], Theorem 14.15). Therefore, the matrix coefficients are linearly independent since any linear relation between two matrix coefficients say, $\langle \pi(x)e_0, u\rangle$ and $\langle \pi(x)e_0, v\rangle$ for two linearly independent vectors u, v in the irreducible representation π would mean $\langle \pi(x)e_0, u - kv\rangle \equiv 0$. This implies that the closed linear span of $\pi(x)e_0$ for $x \in G$ is a subrepresentation orthogonal to $u - kv$, contradicting the irreducibility of π. Then $g * f^{\alpha_0}$ is biinvariant and $\widehat{g * f^{\alpha_0}}$ satisfies the decay condition of Theorem 6.1. As $g \in C^{p'}$ for $p' < p$, all other conditions of Theorem 6.1 are clearly satisfied. Therefore by that theorem the $L^1(G)$-module generated by $\mathcal{F}$ is dense in $L^p(G)_{0,0}$. Now as the smallest closed left $L^1(G)$-invariant subspace of $L^p(G/K)$ containing $L^p(G)_{0,0}$ is $L^p(G/K)$ itself, the theorem follows. ■

7 About augmented strip and exact strip

All the W-T theorems we have encountered or proved so far suffer from a common restriction of using a *slightly augmented strip* as the space of representation, over which the Fourier transform of the generators ought to be holomorphic and nonvanishing. This *more-than-necessary* analyticity and nonvanishing condition in the hypothesis is, in disguise, an assumption of some extra smoothness and decay on the generators. It has yet another incarnation as the ε in the nonvanishing condition on all $L^{p-\varepsilon}$ representation for the W-T theorem of L^p functions. This is a technical necessity, but not easily removable.

Only recently Ben Natan et al. has provided in [B-B-W-H 2] (announced in [B-B-W-H]) the following exact strip version of the W-T theorem for biinvariant L^1 functions of $SL_2(\mathbb{R})$:

Theorem 7.1 (*Ben Natan et al.*) *Let $\mathcal{F} \subset L^1(SL_2(\mathbb{R}))_{0,0}$. Suppose that the the Fourier transforms of elements of $\mathcal{F}$ has no common zero on S^1 and*

$$\delta_\infty(\mathcal{F}) = \inf\{-\lim_{t \longrightarrow \infty} \sup e^{-\pi t} \log|\widehat{f}(it)| : f \in \mathcal{F}\} = 0.$$

Then the ideal generated by $\mathcal{F}$ is dense in $L^1(SL_2(\mathbb{R}))_{0,0}$.

In this section we extend this result to prove an *exact* W-T theorem for $PSL_2(\mathbb{R})$ which is free from the extra restriction cited above. We shall presently see that the spherical principal series representations and the discrete series representations parametrized by the odd integers of $SL_2(\mathbb{R})$ are the only relevant representations for $PSL_2(\mathbb{R})$. All disrete series representations and the spherical principal series representations are L^1-tempered.

Theorem 7.2 *Let $\mathcal{F} \subset L^1(PSL_2(\mathbb{R}))$. Suppose that the Fourier transforms of the functions in $\mathcal{F}$ do not vanish simultaneously on any relevant L^1-tempered irreducible representation and $\delta_\infty(\mathcal{F}) = 0$. Then the ideal generated by $\mathcal{F}$ is dense in $L^1(PSL_2(\mathbb{R}))$.*

Let $\mathbb{H}$ be the upper half plane $\{z \in \mathbb{C} | \Im z > 0\}$. Then the group of all Möbius transformations preserving $\mathbb{H}$ is isomorphic to the group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I_2\}$, where I_2 is the 2×2 identity matrix. This group acts transitively on $\mathbb{H}$ and its subgroup $SO_2(\mathbb{R})$ stabilizes the point i on $\mathbb{H}$. The unit tangent bundle of $\mathbb{H}$ is also identifiable with $PSL_2(\mathbb{R})$.

Let $L^1(SL_2(\mathbb{R}))_{even}$ be the set of those L^1-functions which have no components of odd parity, *i.e.*, if $f \in L^1(SL_2(\mathbb{R}))_{even}$ then $f_{m,n} \equiv 0$ for any odd $m, n \in \mathbb{Z}$. Then $L^1(SL_2(\mathbb{R}))_{even}$ is the set of even functions in $L^1(SL_2(\mathbb{R}))$. Therefore $L^1(PSL_2(\mathbb{R}))$ can be realized as $L^1(SL_2(\mathbb{R}))_{even}$. In fact,

$$L^1(SL_2(R)) = L^1(SL_2(R))_{even} \oplus L^1(SL_2(R))_{odd},$$

where $L^1(SL_2(R))_{odd}$ has similar connotation. As there is no non-zero function with even parity on one side and odd on the other, and since convolution of two functions of opposite parity is zero, $L^1(SL_2(R))_{even}$, equivalently $L^1(PSL_2(R))$, is a two-sided ideal in $L^1(SL_2(R))$. We will denote by $L^1(PSL_2(R))^H$ the space of functions in $L^1(PSL_2(R))$ having zero Fourier transform at every discrete series. It is clear form the above realization of the functions of $PSL_2(\mathbb{R})$ as the functions of $SL_2(\mathbb{R})$ with only even parity, that for $PSL_2(\mathbb{R})$ the only relevant principal series representations are $\pi_{\sigma,\lambda}$, where σ is the trivial representation of M (i.e. the spherical principal series representations). As there is no ambiguity, these $\pi_{\sigma,\lambda}$'s will be denoted by π_λ. Also for the above realisation, the only relevant discrete series for $PSL_2(\mathbb{R})$ are those parametrized by odd integers.

Instead of the method developed in Section 4, here we will be guided by the following observations. As a result we get a shorter proof which replaces the long constructive arguments there.

Observations.

1. Fix a K type n of the group $SL_2(\mathbb{R})$. It determines an M type σ by $n \in \mathbb{Z}^\sigma$. Then, for every $\lambda \in S^\gamma$, the orbit $\{\pi_{\sigma,w\lambda} \,|\, w \in \mathcal{W}\}$ of λ (correspoding to the action of the Weyl group $\mathcal{W} = \mathbb{Z}_2$) has at least one element which has an irreducible subrepresentation π_λ^n containing a vector which

transforms according to the K-type n (see **Subrepresentation of** $\pi_{\sigma,\lambda}$, Section 1).

2. If π is an irreducible representation of $SL_2(\mathbb{R})$ then the matrix coefficient $\langle\pi(x)e_r, e_s\rangle$ for vectors $e_r, e_s \in \pi$ can not be identically zero over $SL_2(\mathbb{R})$; since in that case the closed linear span of $\{\pi(x)e_r \mid x \in G\}$ will be a subrepresentation orthogonal to e_s, contradicting the irreducibility of π.

Before we sketch the proof of Theorem 7.2, let us seperate out the basic argument employed in the main theorems of Section 4.

There are *trouble points* in the strip, which are singularities of the asymptotic expansions of the matrix coefficients. Except for these points the proof is rather simple. Suppose that, at a generic point λ_0 in the strip S^γ, a function f of right K-type n has only one component with nonzero Fourier transform and it is the (m,n)-th component of left K-type m. The proof requires a function g of type (n,m), preferably in $C^1(SL_2(R))$ so that $\widehat{g}(\lambda_0) \neq 0$. But if λ_0 is one of those trouble points, then it is possible that there are m, n so that $\Phi_{\lambda_0}^{m,n}(x) \neq 0$ for some $x \in G$ but $\Phi_{\lambda_0}^{n,m} \equiv 0$. This removes all hopes of getting a g as required.

But then the trouble really comes from the reducibility of the representation parametrized by that point, and the situation is saved by the Observation 1 above which tells us that e_n sits inside an irreducible subrepresentation of either the trouble point itself or of its Weyl group image. In view of the Observation 2 and the fact that for a function h in $L^p(SL_2(R))_{n,n}$, $n \in \mathbb{Z}$, $\widehat{h}(\lambda) = \widehat{h}(-\lambda)$ for any λ in the strip S^γ, it is good enough for our purpose if we start with a function which has non zero Fourier transform at every irreducible subrepresentation of the representations in the strip. We will make these points more precise in the proof below.

Proof of Theorem 7.2. Let $\mathbb{Z}^{\sigma+}$ be the set of even integers. For $\lambda \in S^1$ let π_λ^0 be an irreducible subrepresentation of $\pi_{w\lambda}$ containing e_0 for some $w \in \mathcal{W}$ (see Observation 1 above). Suppose that for $f \in \mathcal{F}$, $\widehat{f}(\pi_\lambda^0) \neq 0$. Then there exist r and s in $\mathbb{Z}^{\sigma+}$ such that $e_r, e_s \in \pi_\lambda^0$ and $\widehat{f}_{r,s}(\pi_\lambda^0) \neq 0$. We find a $g^{left} \in$

$C^1_{0,r}(PSL^2(\mathbb{R}))$ and a $g^{right} \in C^1_{s,0}(PSL^2(\mathbb{R}))$ such that both $\widehat{g}^{left}$ and $\widehat{g}^{right}$ are nonzero at $w\lambda$, which definitely exist by definitions of $C^1_H(\widehat{G})_{0,r}$ and $C^1_H(\widehat{G})_{s,0}$ (see Definitions 2.9 and 2.11). Now $g^{left} * f * g^{right}$ becomes an $(0,0)$ function and $(g^{left} * f * g^{right}_H)\widehat{\ }(w\lambda) \neq 0$. Therefore $h = (g^{left} * f * g^{right}_H)\widehat{\ }(\lambda) \neq 0$.

Let I be the two sided ideal generated by $\mathcal{F}$ in $L^1(PSL_2(\mathbb{R}))$ and let $\mathcal{F}'$ be the set of biinvariant functions in I. The only point on the imaginary axis where the principal series representation is not irreducible is 0. Therfore for any given K-type, there exist C^1 functions with Fourier transforms, nonvanishing everywhere except 0 and of order e^{-t^4} (see Theorem 4.7for construction of such function). Hence it is clear that $\delta_\infty(\mathcal{F}') = 0$ when the same is true for $\mathcal{F}$.

Therefore I contains a set of biinvariant functions which satisfy the conditions in Theorem 7.1. Hence $I \supset L^1(PSL_2(\mathbb{R}))_{0,0}$. But the smallest two sided ideal in $L^1(PSL_2(\mathbb{R}))$ containing $L^1(PSL_2(\mathbb{R}))_{0,0}$ is $L^1(PSL_2(\mathbb{R}))^H$. Hence $I \supset C^1_H(PSL_2(\mathbb{R}))$. In particlar $I \supset C^1_H(PSL_2(\mathbb{R}))_{n,n}$ for all even n.

Now fix an $n \in \mathbb{Z}^{\sigma+}$. There are only finitely many discrete series representations (parametrized by Γ_n) relevant to n-type functions and by hypothesis there are functions in $\mathcal{F}$ which has non zero Fourier transforms on them. We use similar arguments to tackle them. Let $f \in \mathcal{F}$ be such that $(\widehat{f}_B)_{r,s}(k) \neq 0$ for $k \in \Gamma_n$, and $r, s \in \mathbb{Z}(k)$. Then we can always find C^1-functions g_1, g_2 of type (n,r) and (s,n) respectively such that they have nonzero Fourier transform at k. Thus we get $g_1 * f * g_2$, a function in $L^1(PSL_2(\mathbb{R}))_{n,n}$ having nonzero Fourier transform at $k \in \Gamma_n$. As Γ_n is finite, from definition of $C^1_B(\widehat{PSL_2(\mathbb{R})})_{n,n}$ and its isomorphism with $C^1_B(PSL_2(\mathbb{R}))_{n,n}$, it is clear that I contains $C^1_B(PSL_2(\mathbb{R}))_{n,n}$. (Details omitted as this part of the argument is same as that of Theorem 4.4)

Hence I contains $C^1(PSL_2(\mathbb{R}))_{n,n}$ for all even n. The theorem now follows as $C^1(PSL_2(\mathbb{R}))_{n,n}$ is dense in $L^1(PSL_2(\mathbb{R}))_{n,n}$ and the smallest subspace of $L^1(PSL_2(\mathbb{R}))$ containing $L^1(PSL_2(\mathbb{R}))_{n,n}$ for all even n is $L^1(PSL_2(\mathbb{R}))$ itself. ∎

Using exactly the same arguments we can also prove:

Theorem 7.3 *Let $\mathcal{F} \subset L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$. Suppose that the Fourier transforms of the functions in $\mathcal{F}$ do not vanish simultaneously on relevant L^1-tempered irreducible representations and $\delta_\infty(\mathcal{F}) = 0$. Then the left $L^1(SL_2(\mathbb{R}))$ module generated by $\mathcal{F}$ is dense in $L^1(SL_2(\mathbb{R})/SO_2(\mathbb{R}))$.*

Remarks.

1. The only points λ in S^1 for which π_λ is not irreducible are ± 1. The representation π_1 has two subrepresentations which are the two discrete series representations parametrized by 1 and -1 and the representation π_{-1} has the trivial representation, say $\mathring{\pi}$ as a subrepresentation. Also note that $\widehat{f}(\mathring{\pi}) = \int_G f(x)dx$. In view of this and Observations 1 above the nonvanishing condition in the Theorem 7.2 is same as the familiar one: Fourier transforms of the functions in $\mathcal{F}$ do not vanish simultaneously on any point on the strip S^1 and on the discrete series parametrized by odd integers and also $\int_G f(x)dx \neq 0$ (compare with Theorem 4.7). The conditions in Theorem 7.3 imply that they do not vanish simultaneously on any points of S^1.

2. Methods developed in [S2] involving Corona theorem can not be used here. There the use of the Corona theorem needs an extended domain essentially. Even if we start from this exact strip version of the W-T theorem for biinvariant functions, during extension Corona theorem adds the restriction of bigger strip to it.

3. The Observations and the discussion preceding the proof indicate that similar proofs are also possible for the W-T theorems for the whole of $SL_2(\mathbb{R})$ (cited in Section 4) which would be much shorter but only at the cost of constructive and elementary argument.

Status of the problem: If it becomes possible to find an analogue of Theorem 7.1 for fuctions of type $(1,1)$ of $SL_2(\mathbb{R})$, then we can prove a replica of

Theorem 7.2 for $L^1(SL_2(\mathbb{R}))_{odd}$ by similar arguments, and together they will immediately lead to the following exact strip version of the W-T theorem of $SL_2(\mathbb{R})$:

If $\mathcal{F} \subset L^1(SL_2(\mathbb{R}))$ is such that the Fourier transforms of the functions in $\mathcal{F}$ do not vanish simultaneously on any relevant L^1-tempered irreducible rpresentations and that $\delta_\infty(\mathcal{F}_{even}) = 0$ and $\delta_\infty(\mathcal{F}_{odd}) = 0$. Then the ideal generated by $\mathcal{F}$ is dense in $L^1(SL_2(\mathbb{R}))$.

Note that no discrete series, when restricted to K, contains e_1. Therefore, like biinvariant functions, a $(1,1)$ type function also has no relevant discrete series.

8 Not-too-rapidly-decreasing conditions and Hardy's Theorem

In this section we will deal with the following question: What condition on a function on G ensures that its Fourier transform is not-too-rapidly-decreasing at ∞ ?

If f is in $C_c^\infty(G)$ then by the Paley-Wiener theorem, $\widehat{f}$ is an entire function of exponential type (see [Ba,Theorem 10.5]). Hence, using an application of Phragmen-Lindelöf theorem due to Carlson one can show that if $\widehat{f}(\lambda) = O(e^{-k|\lambda|})$ on the imaginary axis for some $k > 0$, then $\widehat{f} \equiv 0$ (see [Ti, p. 185, section 5.8]). Therefore, it is possible to replace the not-too-rapidly decreasing condition of W-T theorems by the putting a C_c^∞-function in the generating set.

We use a recent result due to Sitaram and Sundari [S-S], a Hardy's theorem for $SL_2(\mathbb{R})$ and for noncompact symmetric spaces, to replace the *not-too-rapidly-decreasing at* ∞ condition on the Fourier transform of the generator by a decay condition on the generator itself. First let us quote the theorems from [S-S]:

In the proofs of the W-T theorems in this thesis only the matrix coefficients of the Fourier transforms are used. Therefore, a Hardy's theorem for functions with arbitrary but fixed K-types is enough for our purpose, and hence instead of the exact statement in [S-S], we provide here the matrix coefficient version of the Hardy's theorem for $SL_2(\mathbb{R})$.

Theorem 8.1 (*Sitaram-Sundari*) *Let f be a measurable function on $SL_2(\mathbb{R})$. Suppose that, for some m, n, σ with $m, n \in \mathbb{Z}^\sigma$, $|f_{m,n}(x)| \le C_{mn} e^{-\alpha\sigma(x)^2}$ and $|\widehat{f}_{H\,m,n}(\lambda)| \le C_\sigma e^{-\beta|\lambda|^2}$ for $\lambda \in i\mathbb{R}$, where C_{mn} and C_σ are positive constants depending on m, n and σ respectively. If $\alpha\beta > \frac{1}{4}$ then $f_{m,n} \equiv 0$*

And for symmetric spaces we have:

Theorem 8.2 (*Sitaram-Sundari*) *Let G be a connected, noncompact semi-*

simple Lie group G with finite centre. Suppose that f is a measurable, right K-invariant function on G satisfying the following estimates for some positive constants C, C', α and β: $|f(x)| \leq Ce^{-\alpha\sigma(x)^2}$, $x \in G$ and $\|\pi_\lambda(f)e_0\| \leq C'e^{-\beta\|\lambda\|^2}$, $\lambda \in \mathfrak{a}^*$. If $\alpha\beta > \frac{1}{4}$, then $f = 0$ a.e.

From these we get for $SL_2(\mathbb{R})$:

Theorem 8.3 *Let $\mathcal{F}$ be a subset of $L^p(SL_2(\mathbb{R}))$. Suppose that $\mathcal{F}$ has nonzero functions f^1 and f^2 (not necessarily distinct) so that for $i = 1, 2$, $|f^i_{m_i,n_i}(x)| \leq C_i e^{-\alpha_i\sigma(x)^2}$, where α_i are positive constants and m_i, n_i are even or odd integers according as $i = 1$ or 2. Also suppose that the Fourier transforms $\{\widehat{f}_H \,|\, f \in \mathcal{F}\}$ have holomorphic extensions on $S^{0\gamma}_\delta$ for some $\delta > 0$ and they vanish at infinity, that is, $\lim_{|\lambda|\longrightarrow\infty} \|(\widehat{f}_H(\lambda))\| = 0$ on S^γ_δ for $f \in \mathcal{F}$.*

Now, if the Fourier transforms of the functions in $\mathcal{F}$ do not vanish simultaneously on any of the irreducible $L^{p'}$-tempered representations for some $p' \in (0, p)$, then the $L^1(SL_2(\mathbb{R}))$ module generated by $\mathcal{F}$ is dense in $L^p(SL_2(\mathbb{R}))$.

Proof. Take $\beta_i = \frac{1}{3\alpha_i}$. Then, by Theorem 8.1 above, there exist positive constants C_1, C_2 such that $\limsup_{|\lambda|\longrightarrow\infty} |\widehat{f}^i_{m_i,n_i,H}(\lambda)| > C_i e^{-\beta_i|\lambda|^2}$ for $i = 1, 2$ and $\lambda \in i\mathbb{R}$. Therefore, $\limsup_{|\lambda|\longrightarrow\infty} |\widehat{f}^i_{m_i,n_i,H}(\lambda)| > e^{-Ke^{|\lambda|}}$ for all $K > 0$. Now, as $\mathcal{F}$ satisfies all the conditions of Theorem 4.7, the theorem follows. ∎

Let G be one of the connected, noncompact semisimple Lie groups of real rank one, described in Section 6. Then from Theorem 6.3 and Theorem 8.2 and using similar arguments, we can prove the following:

Theorem 8.4 *Let $\mathcal{F}$ be a subset of $L^p(G/K)$, such that for each f in $\mathcal{F}$, the Fourier transform $\widehat{f}$ has a holomorphic extension on $S^{0\gamma}_\delta$ for some $\delta > 0$ and all the matrix coefficients of $\widehat{f}$ vanish at infinity on S^γ_δ. Also suppose that for some $f \in \mathcal{F}$,*

$$|f(x)| \leq Ce^{-\alpha\sigma(x)^2}, \ x \in G,$$

and the collection $\{\widehat{f} | f \in \mathcal{F}\}$ does not have common zero on S_δ^γ. Then, the left $L^1(G)$ module generated by $\mathcal{F}$ is dense in $L^p(G/K)$.

Remark. We know that $C_c^\infty(G)$ is densely embedded in every $C^p(G)$ for $p \in (0, 2]$. There is a distinguished space of functions on G containing $C_c^\infty(G)$, which also (densely) sits inside $C^p(G)$ for every $p \in (0, 2]$. This space is known as Zero-Schwartz space (see [Ba, Section 19] and Wallach [W, Section 2.5]) and is denoted by $C^0(G)$. In fact, $C^0(G) = \cap\{C^p(G) | p \in (0, 2]\}$. It follows from the definition that for $f \in C^0(G)$, $|f(x)| \leq e^{-k\sigma(x)}$ for all $k > 1$. Here $\sigma(x)$ is equal to $|t|$, where t comes from the Cartan decomposition, $x = k_1 a_t k_2$. One wonders at this point if a function from the zero-Schwartz space (instead of one from C_c^∞) in the generating set can substitute the not-too-rapidly-decreasing condition.

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