The Pompeiu problem and analogues of the Wiener-Tauberian theorem for certain homogeneous spaces

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Introduction

Let G be a connected locally compact unimodular group acting transitively on a locally compact space X. For a function f on X and $g \in G$, define ${}^g f$ by ${}^g f(x) = f(g.x)$, $x \in X$. One of the recurring themes in analysis is the question of when a function f in a given function space $\mathcal{F}(X)$ will have property that $Span\{{}^g f : g \in G\}$ is dense in $\mathcal{F}(X)$. If $X = \boldsymbol{R}$ and $G = \boldsymbol{R}$, the celebrated Wiener-Tauberian theorem answers this question completely for the space $L^1(\boldsymbol{R})$: The span of the translates of $f \in L^1(\boldsymbol{R})$ is dense in $L^1(\boldsymbol{R})$ if and only if the Fourier transform $\hat{f}$ of f is nowhere vanishing on $\boldsymbol{R}$.

If $x_0 \in X$ and $H_0 = \{g \in G : g.x_0 = x_0\}$, then H_0 is a closed subgroup of G and the homogeneous space G/H_0 can be identified with X under the identification: $gH_0 \leftrightarrow g.x_0$. If, further, H_0 is compact and the algebra of compactly supported functions on G which are bi-invariant under H_0 is commutative, then the pair (G, H_0) is called a *Gelfand pair*. In this thesis, we analyse the basic problem stated above for three well known Gelfand pairs:

Case (1): Let $X = \boldsymbol{R}^n$, $n \geq 2$. For G we take the group $M(n)$ of orientation preserving rigid motions of $\boldsymbol{R}^n$. Any element σ of $M(n)$ is given by (T, v_0), $T \in SO(n)$, $v_0 \in \boldsymbol{R}^n$, its action on $\boldsymbol{R}^n$ being given by $\sigma.v = Tv + v_0$, $v \in \boldsymbol{R}^n$. Here $SO(n)$, the special orthogonal group, is the collection $\{A : A\, an\, n \times n\, real\, matrix, AA^t = I, detA = 1\}$. The group law in G is given as follows:

$$(T, v)(S, w) = (TS, Tw + v), \ (T, v), (S, w) \in G.$$

For $x_0 = \underline{0}$, the origin $(0, 0, ..., 0)$ in $\boldsymbol{R}^n$, the corresponding $H_0 = SO(n)$. Hence the homogeneous space $M(n)/SO(n)$ can be identified with $\boldsymbol{R}^n$. It is a well known and easy fact that $(M(n), SO(n))$ is a Gelfand pair. In Chapter 2, for a function

$f \in L^1(X) \cap L^p(X)$, $1 \leq p < \infty$, we give conditions on the Fourier transform of f which ensure that the span of G-translates of f is dense in $L^p(X)$.

Case (2): In Chapter 3, we consider the space $X = H^n$ with the action of the group $G = HM(n)$. Here H^n denotes the n-dimensional Heisenberg group and $HM(n)$ is the Heisenberg motion group, a semi-direct product of H^n and $U(n)$, the group of $n \times n$ unitary matrices with entries in $\mathbb{C}$. A typical element in H^n will be denoted by (z, t), $z \in \mathbb{C}^n$, $t \in \mathbb{R}$ and a typical element in $HM(n)$ will be denoted by (σ, z, t), $\sigma \in U(n)$, $(z, t) \in H^n$. The group law in H^n is given by:

$$(z,t)(w,s) = (z + w, t + s + \frac{1}{2} Im\, z.\overline{w}),\ (z,t),\ (w,s) \in H^n.$$

The group law in $HM(n)$ is given by:

$$(\sigma, z, t)(\tau, w, s) = (\sigma\tau, \sigma w + z, s + t + \frac{1}{2} Im\, \sigma w.\overline{z}),\ (\sigma, z, t),\ (\tau, w, s) \in HM(n).$$

The group $HM(n)$ acts on H^n in the following way

$$(\sigma, z, t).(w, s) = (\sigma w + z, s + t + \frac{1}{2} Im\, \sigma w.\overline{z}).$$

If $x_0 = (0,0)$, the identity element in H^n, then $H_0 = U(n)$. Thus we can identify the space $HM(n)/U(n)$ with H^n in this case. It is a well known fact that the pair $(HM(n), U(n))$ forms a Gelfand pair ([2]).

For $f \in L^1(H^n)$, we have the notion of the group-theoretic Fourier transform (-see Section 3.3). (In the case of $\mathbb{R}^n$ one considers the usual (Euclidean) Fourier transform. However for a non-abelian group, the natural generalization of the traditional Fourier transform is the operator valued "group-theoretic" Fourier transform.) In the spirit of results in Chapter 2, we give conditions on the group-theoretic Fourier transform of f which guarantee that $\overline{Span\{{}^g f : g \in HM(n)\}} = L^1(H^n)$.

Case (3): Here we take X to be a symmetric space of the noncompact type and of real rank 1 and G to be the connected component of the group of isometries of

X. In this case G turns out to be a connected noncompact semi-simple Lie group with finite centre and of real rank 1. Further if K is the subgroup of G that leaves a given point $x_0 \in X$ fixed, then K is a maximal compact subgroup of G and the homogeneous space G/K can be identified with X. Again, the pair (G, K) is a Gelfand pair ([16]).

For functions on X, there is a notion of the Helgason-Fourier transform (-see Section 4.2). Given a function $f \in L^2(X)$, we find conditions in terms of this transform that ensure that the G-translates of f span a dense subspace of $L^2(X)$. For a non-trivial function $f \in L^p(X) \cap L^2(X)$, $2 < p < \infty$, it turns out that the G-translates of f always span a dense subspace of $L^p(X)$. We also briefly review some older results for $L^p(X)$, $1 \leq p < 2$, in terms of the Helgason-Fourier transform.

In Chapter 1, we consider certain questions related to the *Pompeiu problem*- the Pompeiu problem can be thought of as a special case of (1) above in the setting when the function f in question is the indicator function of a set of positive finite measure in X.

Chapter 1

The Pompeiu transform

1.1 Introduction

Let X be a locally compact Hausdorff space and μ a fixed non-negative Radon measure on X. Let G be a group of homeomorphisms of X with the further property that the G-action is transitive and leaves μ invariant. Fix E, a relatively compact measurable subset of X, with positive measure.

For a function $f \in L^1_{loc}(X)$, define the Pompeiu transform $P_E(f)$ of f as the following function on G:

$$(P_E(f))(g) = \int_{g.E} f \, d\mu, \; g \in G.$$

A natural question to ask is: *Under what conditions is f uniquely determined from the knowledge of $P_E(f)$? To put it more mathematically, determine those function spaces on which P_E is injective.*

(The question above is very similar to the basic problem in the theory of the Radon transform - can a function f on $\boldsymbol{R}^n$ be recovered from the knowledge of its integrals on all hyperplanes? See [20].)

This problem was initiated by Pompeiu in 1929 ([27], [28]), in the setup when X is $\boldsymbol{R}^2$, μ is the Lebesgue measure and G is the group of translations of R^2. Pompeiu conjectured that for the unit disc D in $\boldsymbol{R}^2$, P_D is injective on $C(R^2)$, the space of continuous functions on $\boldsymbol{R}^2$. This is *not* true as can be seen by considering the following: Let $\hat{1}_D$ denote the Euclidean Fourier transform of the characteristic function 1_D of D. Let $x_0 \in \boldsymbol{R}^2$ be chosen such that $\hat{1}_D(x_0) = 0$ (such an x_0 exists). Then the non-trivial continuous function $f(x) = e^{i\langle x, x_0\rangle}$ is in $Ker P_D$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\boldsymbol{R}^2$. In fact, for an arbitary spherically symmetric, relatively compact measurable subset E of $\boldsymbol{R}^n$, with positive Lebesgue measure, P_E is never injective on $L^1_{loc}(\boldsymbol{R}^n)$ ([36]).

We shall study the Pompeiu transform in the following cases:

Case (1): $X = \boldsymbol{R}^n$, $n \geq 2$, μ the Lebesgue measure on $\boldsymbol{R}^n$. For G we take the group $M(n)$ of orientation preserving rigid motions of $\boldsymbol{R}^n$. Any element σ of $M(n)$ is given by (T, v_0), $T \in SO(n)$, $v_0 \in \boldsymbol{R}^n$, its action being $\sigma \cdot v = Tv + v_0, v \in \boldsymbol{R}^n$. Here $SO(n)$, the special orthogonal group, is the collection $\{A : A \, an \, n \times n \, real \, matrix, AA^t = I, det A = 1\}$. The homogeneous space $M(n)/SO(n)$ can be identified with $\boldsymbol{R}^n$ via the identification $\sigma SO(n) \leftrightarrow \sigma \cdot \underline{0}$, where $\underline{0}$ is the origin $(0, 0, ..., 0)$ in $\boldsymbol{R}^n$.

Case (2): X is a symmetric space of the noncompact type and G is the connected component of the group of isometries of X. For μ, we take the canonical G-invariant measure on X. In this case G turns out to be a connected noncompact semi-simple Lie group with finite centre. Further if K is the subgroup of G that leaves a given point $x_0 \in X$ fixed, then K is a maximal compact subgroup of G. Also the map $gK \mapsto g \cdot x_0$ gives an identification of G/K with X.

Thus, in both cases that we are interested, X can be realised as a homogeneous space G/K, for a suitable compact subgroup K of G and so we can bring in the machinery of harmonic analysis on the group G in order to study the Pompeiu transform.

An important result for case (1) is the following theorem of Brown, Schrieber and Taylor ([11]):

Theorem 1.1.1 *Let E be a relatively compact subset of $\mathbb{R}^n$ of positive Lebesgue measure. Then P_E is injective on $L^1_{loc}(\mathbb{R}^n)$ if and only if for each $\alpha \in \mathbb{C} \setminus \{0\}$, $\hat{1}_E$, the Euclidean Fourier transform of 1_E, considered as a function on $\mathbb{C}^n$, does not vanish identically on $C_\alpha = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + ... + z_n^2 = \alpha^2\}$. (Here 1_E is the characteristic function of E and since E is relatively compact, $\hat{1}_E$ extends to an entire function on $\mathbb{C}^n$.)*

Actually, Brown, Schrieber and Taylor ([11]) proved the result for $C(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$. However by using convolution with continuous approximate identities in $L^1(\mathbb{R}^n)$, the result can easily be extended to $L^1_{loc}(\mathbb{R}^n)$.

It is proved in [10] that if the set E is of the form $E_1 \times E_2 \times \cdots \times E_n$, then P_E is injective on $C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at ∞. However, it is shown in [36] that if E is spherically symmetric then P_E is *not* injective on $C_0(\mathbb{R}^n)$. The example given in [36] for $f \in Ker P_E$ actually belongs to $C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for every $p > \frac{2n}{n-1}$. On the other hand, it is easy to show that P_E is injective on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Thus, it is natural to ask: *In general, what decay conditions on $f \in C_0(\mathbb{R}^n)$ will force $f \equiv 0$ when $P_E f = 0$?* Motivated by a result of Thangavelu ([44]) on spherical means, *we show that if $f \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n}{n-1}$, then indeed $f \equiv 0$ if $P_E f = 0$. In view of what has been said earlier, this is the best possible result in general (i.e. without assuming anything about the 'shape' of E).*

There is a result analogous to the Theorem 1.1.1 for symmetric spaces of the noncompact type and of real rank 1. This result is implicit in the work of Berenstein and Zalcman ([9]) and Berenstein and Shahshahani ([8]). It has also been recorded by Bagchi and Sitaram in [3]. Here we state it in a slightly different fashion.

Theorem 1.1.2 *Let X, μ, G be as in the case (2). Further assume that X is of real rank 1. Let E be a relatively compact measurable subset of X of positive measure. Then P_E is injective on $L^1_{loc}(X)$ if and only if for each $\lambda \in \mathbb{C}$, the function $b \mapsto \tilde{1}_E(\lambda, b)$ is a non-trivial function on K/M. (Here $\tilde{1}_E$ denotes the Helgason-Fourier transform of 1_E, the characteristic function of E).*

(For unexplained terminology, see Section 4.2.)

In the spirit of the result for case (1), *we show that in case (2), P_E is injective on $L^p(X)$, $1 \leq p \leq 2$.* (See [31].) This is a generalization of the main result in [37]. *In view of the counter example in [35], this is the best possible result in general i.e. without assuming anything about the shape of E.*

1.2 Notation, terminology and preliminary results

Most of the notation and terminology we follow is fairly standard - see, for example, [32]. $\mathcal{D}(\boldsymbol{R}^n)$ will denote the space of C^∞-functions of compact support, $\mathcal{S}(\boldsymbol{R}^n)$ the Schwartz space of rapidly decreasing functions, $\mathcal{E}'(\boldsymbol{R}^n)$ the space of compactly supported distributions, $S'(\boldsymbol{R}^n)$ the space of tempered distributions. For each $\lambda \geq 0$, define ϕ_λ as follows:

$$\phi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda\langle x,\omega\rangle}\, d\omega, \; x \in \boldsymbol{R}^n.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product, S^{n-1} is the unit sphere in $\boldsymbol{R}^n$ and $d\omega$ is the canonical (probability) measure on S^{n-1}. Then for $\lambda > 0$,

$$\phi_\lambda(x) = C_n\,(\lambda|x|)^{-n/2+1}\,J_{n/2-1}(\lambda|x|),$$

(-see [39]) where J_ν denotes the Bessel function of order ν on $\boldsymbol{R}$. (See [25].)

Define $\phi_{\lambda,k}$ by:

$$\phi_{\lambda,k}(x) = \frac{d^k}{dr^k}\phi_r(x)\,|_{r=\lambda}\,.$$

(Thus $\phi_{\lambda,0} = \phi_\lambda$.)

Note that if $f \in L^p(\boldsymbol{R}^n), 1 \leq p \leq \infty$, then f can be viewed as a tempered distribution and hence $\hat{f}$, sometimes also written as $f^\wedge$, its Fourier transform, makes sense as a tempered distribution. If T is a radial tempered distribution, then so is $\hat{T}$ and if further T is compactly supported, then $\hat{T}$ is given by a smooth function and $\hat{T}(v) = T(\phi_{\|v\|})$. For a distribution T, Supp T will denote the (closed) support of T. For any function g, Z_g denotes the set $\{x : g(x) = 0\}$. If g is a continuous function on $\boldsymbol{R}^n$ define $g^\#$ by $g^\#(x) = \int_{SO(n)} g(kx)\,dk$, $x \in \boldsymbol{R}^n$. Here dk is the normalized Haar measure on $SO(n)$. Then $g^\#$ is a continuous radial function, and further, if $g \in L^p(\boldsymbol{R}^n)$, $g^\#$ is also in $L^p(\boldsymbol{R}^n)$. For $\alpha > 0$, let M_α denote the sphere $\{v \in \boldsymbol{R}^n : \|v\| = \alpha\}$.

Next we record three lemmas that will be needed in the next section ([31]).

Lemma 1.2.1 *Let T be a non-trivial radial distribution of compact support such that $\hat{T}$ vanishes on M_α, for some $\alpha > 0$. Then there exists an annulus $A_\epsilon = \{x \in \boldsymbol{R}^n : \alpha - \epsilon < \|x\| < \alpha + \epsilon\}, \epsilon > 0$, such that $\hat{T}$ has no other zeros in this annulus.*

Proof : Let $h(\lambda) = \hat{T}(v) = \langle T, \phi_\lambda\rangle$ where $\|v\| = \lambda$. Then from the compactness of the support of T, it follows easily that h extends to an even, entire function of λ

with $h(\alpha) = 0$, the lemma follows easily from this observation.

Lemma 1.2.2 *Let $\lambda > 0$ and let $f = \sum_{k=0}^{N} a_k \phi_{\lambda,k}$. If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n}{n-1}$, then $a_0 = a_1 = \cdots = a_N = 0$.*

Proof : Recall that $\phi_{r,0}(x) = \phi_r(x) = C_n\,(r|x|)^{-\nu} J_\nu(r|x|)$, where $\nu = \frac{n}{2} - 1$. Using the formula $\frac{d}{dt}(t^{-\nu}J_\nu(t)) = -t^{-\nu}J_{\nu+1}(t)$, for $t \in \mathbb{R}$, we can explicitly compute $\phi_{\lambda,k}(x) = \frac{d^k}{dr^k}\phi_r(x)|_{r=\lambda}$. From the asymptotic behaviour of the Bessel functions (-see [25]):

$$J_\nu(t) \approx \left(\frac{2}{\pi t}\right)^{1/2} \cos\left(t - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$$

as $t \to \infty$, we get that, when $k > 0$ and $|x| \to \infty$,

$\phi_{\lambda,k}(x) \approx C\frac{1}{|x|^{\nu-k+\frac{1}{2}}} \cos\left(|x| - \frac{1}{2}(\nu+k)\pi - \frac{1}{4}\pi\right)$ + terms involving higher powers of $\frac{1}{|x|}$.

From this it easily follows, using polar coordinates on $\mathbb{R}^n$, that $\sum_{k=0}^{N} a_k \phi_{\lambda,k}$ cannot be in $L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n}{n-1}$, unless each $a_k = 0$.

Lemma 1.2.3 *Let T be a tempered radial distribution. Suppose $Supp\,\hat{T} = M_\lambda$, for some $\lambda > 0$. Then $T = \sum_{k=0}^{N} a_k \phi_{\lambda,k}$, for some constants $a_0, a_1, \cdots, a_N$.*

(It is easy to see that the Fourier transform of $\sum_{k=0}^{N} a_k \phi_{\lambda,k}$ considered as a tempered distribution is supported on M_λ. The above lemma is the converse of this statement.)

Proof : Define a distribution S on $\mathbb{R}^+ = (0,\infty)$ by $\langle S, \psi\rangle = \langle \hat{T}, \phi\rangle$, where $\phi(x) = \psi(|x|)$, $\psi \in \mathcal{D}(\mathbb{R}^+)$. Since $\hat{T}$ is radial and has support away from the origin, S is well defined. Also $Supp\,S = \{\lambda\}$. Therefore there exist constants a_k such that $S = \sum_{k=0}^{N} a_k\, \delta_\lambda^k$. Here δ_λ^k denotes the k-th distributional derivative of Dirac distribution δ_λ supported at λ. Now as T is radial, it is determined by its values on radial

Schwartz class functions on $\boldsymbol{R}^n$. For such an f,

$$\langle T, f\rangle = \langle \hat{T}, \hat{f}\rangle = \langle S, h\rangle,$$

where $h(r) = \hat{f}(x)$, $|x| = r$. Hence $\langle T, f\rangle = \langle \sum_{k=0}^{N} a_k\, \delta_\lambda^k, h\rangle$, using the fact that $S = \sum_{k=0}^{N} a_k\, \delta_\lambda^k$. Also, for x with $|x| = r$, $h(r) = \hat{f}(x) = \int f(y)\, \phi_r(y)\, dy$. Therefore

$$\langle \delta_\lambda^k, h\rangle = (-1)^k \frac{d^k h}{dr^k}|_{r=\lambda} = (-1)^k \int f(y)\, \phi_{\lambda,k}(y)\, dy = (-1)^k \langle \phi_{\lambda,k}, f\rangle$$

and as a consequence $T = \sum_{k=0}^{N} a'_k\, \phi_{\lambda,k}$, for some constants a'_k.

1.3 The Pompeiu transform for $\boldsymbol{R}^n$

We start with a generalization of Thangavelu's result in [44], (-see also [1]), from which our main result for case (1) will be deduced ([31]) :

Proposition 1.3.1 *Let $T \in \mathcal{E}'(\boldsymbol{R}^n)$ be non-trivial and radial. Let $1 \leq p \leq \frac{2n}{n-1}$. If $f \in L^p(\boldsymbol{R}^n)$ and $f * T = 0$, then $f = 0$ a.e.*

Proof : By convolving f against an approximate identity, if necessary, we can assume that f is continuous or even smooth. Suppose now $f \not\equiv 0$. We will show that this leads to a contradiction. Since translates of f are also 'killed' by convolution against T, we may assume that $f(0) \neq 0$. Hence $f^\#$ is also a continuous function, $f^\#(0) \neq 0$, and since T is radial, one can easily verify that $f^\# * T = 0$. Also, $f^\#$ is in $L^p(\boldsymbol{R}^n)$. Thus we may assume, by replacing f by $f^\#$, that f is a non-trivial, continuous, radial function (in $L^p(\boldsymbol{R}^n)$). Since f is non-trivial, $\hat{f}$ is a non-trivial tempered distribution. Hence $Supp\hat{f}$ is non-empty. Also, there exists $0 \neq v_0 \in Supp\hat{f}$. (Otherwise, if $Supp\hat{f} = \{0\}$, it will follow that f is a non-trivial polynomial and this contradicts the fact that $f \in L^p(\boldsymbol{R}^n)$ and $p < \infty$.) Since $\hat{f}$ is also radial, if $\alpha = \|v_0\|$, $M_\alpha \subseteq Supp\hat{f}$. Since T is compact, $\hat{T}$ is given by a smooth function.

Thus $(f * T)^{\wedge} = \hat{T}\hat{f}$ and since $(f * T)^{\wedge} = 0$, it follows that $Supp\hat{f} \subseteq Z_{T^{\wedge}}$. Thus $M_\alpha \subseteq Supp\hat{f} \subseteq Z_{T^{\wedge}}$. By Lemma 1.2.1, there exists $\epsilon > 0$ such that the only zeros of $\hat{T}$ in the annulus $A_\epsilon = \{v \in \boldsymbol{R}^n : \alpha - \epsilon < \|v\| < \alpha + \epsilon\}$ lie on M_α. Thus we have $Supp\hat{f} \cap A_\epsilon = M_\alpha$. Now choose a non-trivial, radial $\psi \in \mathcal{D}$ which is 1 in a neighbourhood of M_α and zero outside $A_{\frac{\epsilon}{2}}$. Then $\psi\hat{f}$ is a non-trivial radial distribution and *Supp* $\psi\hat{f} = M_\alpha$. But $\psi\hat{f} = (\hat{\psi} * f)^{\wedge}$ and $\hat{\psi} * f$ is therefore non-trivial. Further it is in $L^p(\boldsymbol{R}^n)$. (Since $\psi \in \mathcal{D}, \hat{\psi} \in \mathcal{S}$ and hence $\hat{\psi} * f \in L^p(\boldsymbol{R}^n)$.) Using Lemma 1.2.2 and Lemma 1.2.3 it follows that $\hat{\psi} * f \equiv 0$. This gives us the desired contradiction because $\hat{\psi} * f$ is non-trivial !

Coming to the question of injectivity of the Pompeiu transform in case (1), i.e.$X = \boldsymbol{R}^n$, $G = M(n)$, $n \geq 2$, we have the following result ([31]):

Theorem 1.3.2 *Let E be a bounded Borel set in $\boldsymbol{R}^n$, with positive Lebesgue measure. Then P_E is injective on $L^p(\boldsymbol{R}^n)$, if $1 \leq p \leq \frac{2n}{n-1}$.*

Proof : Let $1 \leq p \leq \frac{2n}{n-1}$ and let $X = \{f \in L^p(\boldsymbol{R}^n) : P_E f = 0\}$. Then it is easy to show that $f \in X$ if and only if $f * \check{1}_{TE} = 0$ for all $T \in SO(n)$, where $\check{1}_A(x) = 1_A(-x) = 1_{-A}(x)$. From this it follows easily that X is a closed subspace which is moreover closed under translations and rotations. Suppose $X \neq (0)$. Using the above observations it is easy to show that there exists a non-trivial $f \in X, f$ continuous. Thus $f * \check{1}_{TE} = 0$ for all $T \in SO(n)$ and it will follow that $f * \check{1}_E^{\#} = 0$. But $\check{1}_E^{\#}$ is a non-trivial, compactly supported, radial distribution and hence by Proposition 1.3.1, $f \equiv 0$, which gives us a contradiction. Thus $X = (0)$ and the proof of the theorem is complete.

As pointed out in the introduction to this chapter, this is the best possible result in general (i.e. without assuming anything about the 'shape' of E).

1.4 The Pompeiu transform for symmetric spaces of the noncompact type

Turning to the case (2), i.e. X is a symmetric space of the noncompact type and G the connected component of the group of isometries of X, we can modify the main result in [37] to show the following, which appears in [31] without a proof. (The case $p = 1$ had been considered in [37].)

Theorem 1.4.1 *If E is a Borel subset of X of finite positive measure (with respect to the canonical measure on X), then P_E is injective on $L^p(X)$, $1 \leq p \leq 2$.*

(Note that in this theorem we are not assuming that E is relatively compact.)

We postpone the proof of this theorem as it will follow from a more general result, Theorem 4.2.6, in Chapter 4. In view of the results in [35], this is the best possible statement in general (i.e. without assuming anything about the 'shape' of E). Thus the behaviour for symmetric spaces is slightly different from that for Euclidean spaces - the main reason for this being the difference in the asymptotic behaviour of the corresponding 'elementary' spherical functions.

Chapter 2

$\mathbb{R}^n$ with the Euclidean motion group action

2.1 Introduction

The question of injectivity of the Pompeiu transform, as considered in Chapter 1, for the case of $R^n, n \geq 2$, is closely related to theorems of the Wiener-Tauberian type. For any function f on $\mathbb{R}^n$ and any $g \in M(n)$, let ${}^g f$ be the function ${}^g f(x) = f(g.x), x \in \mathbb{R}^n$. Then the injectivity of P_E on $L^p(\mathbb{R}^n)$, $1 < p \leq \frac{2n}{n-1}$, $n \geq 2$, is equivalent, by duality, to the condition that $Span\{{}^g 1_E : g \in M(n)\}$ is dense in $L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$. Motivated by this observation, and in view of the important role played by the Fourier transform in the proof of Theorem 1.3.2, we consider the following question in this chapter:

If $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 \leq q < \infty$, what conditions on the Fourier transform of f will be equivalent to the condition that $Span\{{}^g f : g \in M(n)\}$ is dense in $L^q(\mathbb{R}^n)$?

Note that for $q = 1$, and for $\mathbb{R}^n$ acting on itself by translations, this question is completely answered by the celebrated Wiener-Tauberian theorem (- see theorem 9.4 in [32]) : For a function f in $L^1(\mathbb{R}^n)$, the closed subspace spanned by the translates of f is all of $L^1(\mathbb{R}^n)$ if and only if the function $\hat{f}$ *never vanishes on* $\mathbb{R}^n$.

(Later on, in Chapter 4, we consider a similar question for symmetric spaces of the noncompact type.)

Analogues of Wiener's theorem, even in the one dimensional case i.e. $\boldsymbol{R}$ acting on itself by translations, when $p \neq 1$ or 2 are quite hard (-see [13], Section 58). Therefore it is surprising that for $L^p(\boldsymbol{R}^n), n \geq 2$, and *with the action of the group of rigid motions, instead of the group of translations*, we are able to get reasonably complete results !

2.2 L^p-analogues of the Wiener-Tauberian theorem for $\boldsymbol{R}^n$ with the $M(n)$ action

Throughout this section we assume that $n \geq 2$. For a given function f in $L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n)$, $1 \leq q \leq \infty$, let $S = \{r > 0 : \hat{f} \equiv 0 \text{ on } M_r\}$, where $M_r = \{x \in \boldsymbol{R}^n : \|x\| = r\}$. Let $X = Span\{{}^g f : g \in M(n)\}$. Clearly S is a closed subset of $\boldsymbol{R}^+$. The rest of the notation and terminology we follow here is as in the Section 1.2. Then we have ([31]):

Theorem 2.2.1 *(1) Let $f \in L^1(\boldsymbol{R}^n)$. Then X is dense in $L^1(\boldsymbol{R}^n)$ if and only if $\hat{f}(0) \neq 0$ and S is empty.*

(2) Let $f \in L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n)$, $1 < q < \frac{2n}{n+1}$. Then X is dense in $L^q(\boldsymbol{R}^n)$ if and only if S is empty.

(3) Let $f \in L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n)$, $\frac{2n}{n+1} \leq q < 2$. If every point of S is an isolated point, then X is dense in $L^q(\boldsymbol{R}^n)$.

(4) Let $f \in L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n)$, $2 \leq q \leq \frac{2n}{n-1}$. If S is of zero measure (with respect to the Lebesgue measure on $\boldsymbol{R}^+$), then X is dense in $L^q(\boldsymbol{R}^n)$.

(5) Let $f \in L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n)$, $\frac{2n}{n-1} < q < \infty$. Then X is dense in $L^q(\boldsymbol{R}^n)$ if and only if S is nowhere dense.

Proof : (1) The proof follows easily from the classical Wiener-Tauberian theorem. (See Proposition 9.4 in [32].)

(2) Suppose X is dense in $L^q(\boldsymbol{R}^n)$, $1 < q < \frac{2n}{n+1}$. If S is nonempty then consider ϕ_r for some $r \in S$. (Here ϕ_r is the function introduced in Section 1.2.) Then $\phi_r \in L^p(\boldsymbol{R}^n)$ where $1/p + 1/q = 1$, since $p > \frac{2n}{n-1}$. Using the fact that $\hat{f}$ vanishes on M_r and that ϕ_r is radial, it can be proved that $\int {}^g f \phi_r = 0$ for all $g \in M(n)$, which contradicts the fact that X is dense in L^q. Hence S is empty.

Conversely, assume S is empty. Suppose X is *not* dense in L^q. Then there exists a non-trivial $h \in L^p(\boldsymbol{R}^n)$, $1/p + 1/q = 1$, $\frac{2n}{n-1} < p < \infty$ such that $\int {}^g f h = 0, \forall g \in M(n)$. Then arguing exactly as in Proposition 1.3.1, we can assume h to be *smooth* and radial. It will follow that $h * f \equiv 0$. Convolving h with a smooth compactly supported approximate identity we can even assume that h is *bounded.* Since $\hat{f}$ may not be smooth, $\hat{f}\hat{h}$ may not make sense! However by Theorem 9.3 of [32], which is essentially the Wiener-Tauberian Theorem in disguise, we can still conclude that $Supp\, \hat{h} \subseteq Z_{f^\wedge}$. (See also Proposition 6.1 in [11].) Since $\hat{h}$ is a radial distribution, if $x \in Supp\, \hat{h}$, then $\{y : \|y\| = \|x\|\} \subseteq Supp\, \hat{h}$. Thus, since S is empty, 0 can be the only possible point in $Supp\, \hat{h}$. If $Supp\, \hat{h} = \{0\}$, then h is a non-trivial polynomial and this is impossible since h is also in L^p with $p < \infty$. Therefore, $Supp\, \hat{h}$ is empty. Hence $h = 0$ a. e., a contradiction.

(3) Suppose X is not dense in $L^q(\boldsymbol{R}^n)$, $\frac{2n}{n+1} \leq q < 2$. Then arguing exactly as in (2) above, there exists a non-trivial smooth radial bounded $h \in L^p(\boldsymbol{R}^n)$, $1/p+1/q = 1$ with $2 < p \leq \frac{2n}{n-1}$, such that $h * f = 0$. As before this implies $Supp\, \hat{h} \subseteq Z_{f^\wedge}$. If $Supp\, \hat{h} = \{0\}$, then h must be a polynomial and this contradicts that $h \in L^p$ and $p < \infty$. Hence as in (2), there exists $r > 0$ such that $M_r \subseteq Supp\, \hat{h}$. But then $r \in S$. Since each point of S is an isolated point, there exists an $\epsilon > 0$ such that

$(r-\epsilon, r+\epsilon) \cap S = \{r\}$. Consider the annulus $A_\epsilon = \{x \in \boldsymbol{R}^n : r-\epsilon < \|x\| < r+\epsilon\}$. Choose $\psi \in \mathcal{D}(\boldsymbol{R}^n)$ as in the proof of Proposition 1.3.1. Then $Supp\, \psi \hat{h} = M_r$. Equivalently $Supp\, (\hat{\psi} * h)^\wedge = M_r$. Also since $\hat{\psi} \in \mathcal{S}(\boldsymbol{R}^n)$ and $h \in L^p, \hat{\psi} * h \in L^p$. Using Lemma 1.2.2 and Lemma 1.2.3, and the fact that $\hat{\psi} * h \in L^p(\boldsymbol{R}^n)$, $2 < p \leq \frac{2n}{n-1}$, we conclude that $\hat{\psi} * h \equiv 0$ exactly as in the proof of Proposition 1.3.1. But then this contradicts that $Supp\, (\hat{\psi} * h)^\wedge = M_r$.

(4) Suppose X is not dense in $L^q(\boldsymbol{R}^n)$, $2 \leq q \leq \frac{2n}{n-1}$. Then, as before, there exists $h \in L^p(\boldsymbol{R}^n)$, $1/p + 1/q = 1$, a non-trivial radial function such that $h * f = 0$. Since $\frac{2n}{n+1} \leq p \leq 2$, $\hat{h}$ is defined as a function. Therefore $\hat{f}\hat{h} = 0$. This together with the fact that $\hat{h}$ is radial and S is of zero measure in $\boldsymbol{R}^+$, implies that $\hat{h}$ is zero a.e. in $\boldsymbol{R}^n$. But then $h = 0$ a.e., a contradiction.

(5) Assume X is dense in $L^q(\boldsymbol{R}^n)$, $\frac{2n}{n-1} < q < \infty$. Since S is closed in $\boldsymbol{R}^+$, the fact that S is nowhere dense is equivalent to saying that S does not contain any non-empty open interval. So if S is not nowhere dense, then there is some annulus $A_{r_1,r_2} = \{x \in \boldsymbol{R}^n : r_1 < \|x\| < r_2\}, r_2 > r_1 > 0$ such that each $\hat{f}$ is zero on A_{r_1,r_2}. Choose $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ non-trivial and radial such that $Supp\, \hat{\psi} \subseteq A_{r_1,r_2}$. Then $\hat{f}\hat{\psi} = 0$ i.e., $f * \psi = 0$. Also $\psi \in L^p(\boldsymbol{R}^n)$, $1/p + 1/q = 1$. Since ψ is radial, this implies that $\int {}^g f \psi = 0$ for all $g \in M(n)$. This contradicts the fact X is dense in L^q.

To prove the 'if' part we need the following observation : For a *radial* $h \in L^p(\boldsymbol{R}^n)$, $1 \leq p < \frac{2n}{n+1}$, $\hat{h}$ is given *pointwise* on $\boldsymbol{R}^n \setminus \{0\}$ by the following expression:

$$\hat{h}(y) = \int_{\boldsymbol{R}^n} h(x)\, \phi_{\|y\|}(x)\, dx, \quad y \in \boldsymbol{R}^n \setminus \{0\}.$$

As for each $\lambda > 0$, $\phi_\lambda \in L^q(\boldsymbol{R}^n), \forall q > \frac{2n}{n-1}$, the integral on the right hand side makes sense. Using the asymptotic behaviour of $\phi_\lambda, \lambda \in \boldsymbol{R}^+$, and the dependence of the behaviour with respect to the parameter λ (-see the explicit formula for ϕ_λ given in

Section 1.2), we can show that $\lambda \to \int_{R^n} h(x)\, \phi_\lambda(x)\, dx$ is a continuous function on $\boldsymbol{R}^+$. Hence $\hat{h}$ is a continuous function on $\boldsymbol{R}^n \setminus \{0\}$.

Coming back to the proof of 'if' part in (5), suppose X is not dense in L^q. Then as before there exists a non-trivial radial $h \in L^p(\boldsymbol{R}^n)$, $1/p + 1/q = 1, 1 < p < \frac{2n}{n+1}$, with $h * f = 0$. Since f is also in L^1, $\hat{f}$ is continuous on $\boldsymbol{R}^n$. Also $f * h$ is in L^p (since $L^1 * L^p \subseteq L^p$) and the Fourier transform of $f * h$ is given by the continuous function $\hat{f}\hat{h}$ on $\boldsymbol{R}^n \setminus \{0\}$. Therefore, $\hat{f}\hat{h} \equiv 0$ on $\boldsymbol{R}^n \setminus \{0\}$. Again by assumption, S is nowhere dense, for any $M_a, a > 0$, there are points as close to M_a as we like where $\hat{f}$ does not vanish. Therefore $\hat{h}$ vanishes on these points. Since $\hat{h}$ is radial this implies that it vanishes on spheres arbitrarily close to M_a. By continuity of $\hat{h}$ on $\boldsymbol{R}^n \setminus \{0\}$ we conclude that $\hat{h} = 0$ on $M_a, a > 0$ i.e., $\hat{h} \equiv 0$ on $\boldsymbol{R}^n \setminus \{0\}$. But then $h = 0$ a.e., a contradiction.

Remark 2.2.2 *In the case $p = 2$, the condition 'S is of zero measure' is both a necessary and sufficient condition in (4) of Theorem 2.2.1. This follows easily from the Plancherel Theorem. Also in this case, it is enough to assume that $f \in L^2$, instead of the more restrictive condition $f \in L^1 \cap L^2$.*

Remark 2.2.3 *The most general formulation of the classical Wiener-Tauberian theorem is actually for a family of functions. Since in the later chapters we state analogous results for a single function only, to keep the exposition uniform, we have restricted ourselves to treating a single function in the Theorem 2.2.1. However Theorem 2.2.1 can be formulated for a family of functions in the following way:*

For a family $\{f_\alpha\}_{\alpha \in I}$ in $L^1(\boldsymbol{R}^n) \cap L^q(\boldsymbol{R}^n), 1 \leq q \leq \infty$, let $S = \cap_{\alpha \in I}\{r > 0 : \hat{f}_\alpha \equiv 0$ on $M_r\}$. Let $X = Span\{{}^g f_\alpha : g \in M(n),\, \alpha \in I\}$. As before, S is a closed *subset of $\mathbb{R}^+$. The arguments given in Theorem 2.2.1 actually work for the family f_α, $\alpha \in I$ and we have the following:*

(1) Let $f_\alpha \in L^1(\mathbb{R}^n)$, $\alpha \in I$. Then X is dense in $L^1(\mathbb{R}^n)$ if and only if there exsits an $\alpha_0 \in I$ such that $\hat{f}_{\alpha_0}(0) \neq 0$ and S is empty.

(2) Let $f_\alpha \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 < q < \frac{2n}{n+1}$, $\alpha \in I$. Then X is dense in $L^q(\mathbb{R}^n)$ if and only if S is empty.

(3) Let $f_\alpha \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $\frac{2n}{n+1} \leq q < 2$, $\alpha \in I$. If every point of S is an isolated point, then X is dense in $L^q(\mathbb{R}^n)$.

(4) Let $f_\alpha \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $2 \leq q \leq \frac{2n}{n-1}$, $\alpha \in I$. If S is of zero measure (with respect to Lebesgue measure on $\mathbb{R}^+$), then X is dense in $L^q(\mathbb{R}^n)$.

(5) Let $f_\alpha \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $\frac{2n}{n-1} < q < \infty$, for each $\alpha \in I$. Then X is dense in $L^q(\mathbb{R}^n)$ if and only if S is nowhere dense.

Since we were considering the injectivity of the Pompeiu transform on L^p-spaces, in Chapter 1, we need not have confined ourselves to bounded Borel sets E. We could equally well have considered Borel sets (bounded or unbounded) of finite positive measure. Note that on any L^p, even in this case, P_E can be defined exactly as before. As a corollary to Theorem 2.2.1 we have the following result :

Corollary 2.2.4 *Let E be a Borel subset of $\mathbb{R}^n$ of finite positive measure. Let $S = \{r > 0 : \hat{1}_E \equiv 0$ on $M_r\}$. Then*

(1) P_E is injective on $L^p(\mathbb{R}^n)$, $1 \leq p < \frac{2n}{n+1}$, if and only if S is nowhere dense.

(2) If S is of measure zero, then P_E is injective on $L^p(\mathbb{R}^n)$, $\frac{2n}{n+1} \leq p \leq 2$.

(3) If every point of S is an isolated point, then P_E is injective on $L^p(\mathbb{R}^n)$, $2 < p \leq \frac{2n}{n-1}$.

(4) P_E is injective on $L^p(\mathbb{R}^n)$, $\frac{2n}{n-1} < p \leq \infty$, if and only if S is empty.

Proof : All the statements above, except for $p = 1$, follow by duality. The case

$p = 1$ can be proved quite easily using the same ideas as in case (5), Theorem 2.2.1.

Remark 2.2.5 *For a bounded Borel set of positive measure conditions (1), (2) and (3) of Corollary 2.2.4 are automatically guaranteed.)*

Remark 2.2.6 *From the Remark 2.2.2, it follows that* in the case $p = 2$, *the condition 'S is of zero measure' is both a necessary and sufficient condition for P_E to be injective.*

Chapter 3

H^n with the Heisenberg motion group action

3.1 Introduction

In this chapter, we prove a Wiener-Tauberian type theorem for the n-dimensional Heisenberg group H^n with the Heisenberg motion group $HM(n)$ acting on H^n. For $f \in L^1(H^n)$, we have the notion of the group-theoretic Fourier transform. (In the case of $\boldsymbol{R}^n$ one considers the usual (Euclidean) Fourier transform. However for a non-abelian group, the natural generalization of the traditional Fourier transform is the operator valued group-theoretic Fourier transform -see Section 3.3.) In the spirit of Theorem 2.2.1, we would like to get conditions on the group-theoretic Fourier transform of f which guarantee that $\overline{Span\{{}^g f : g \in HM(n)\}} = L^1(H^n)$. In order to answer this question, we make crucial use of a theorem of Hulanicki-Ricci [23] about the ideals in the commutative Banach algebra of "radial" L^1- functions on H^n. Finally we should mention that the analogue of Wiener's theorem for the two sided action of H^n on itself has been known for sometime - see for example [26], [46].

Here we recall some facts about H^n, its representations, special Hermite functions etc.

Let $H^n = \mathbb{C}^n \times \boldsymbol{R}$ denote the n-dimensional Heisenberg group endowed with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Im\, z \cdot \overline{w}).$$

Here $z \cdot \overline{w} = \sum_{j=1}^{n} z_j \cdot \overline{w_j}$, for $z = (z_1, ..., z_n),\ w = (w_1, ..., w_n)$.

For each $\lambda \in \boldsymbol{R}^* = \boldsymbol{R} \setminus \{0\}$, we have an irreducible unitary representation π_λ of H^n realised on $L^2(\boldsymbol{R}^n)$, the action being

$$\pi_\lambda(z,t)\ \phi(\xi) = e^{i\lambda t} e^{i\lambda(\frac{x\cdot y}{2}+x\cdot\xi)}\,\phi(\xi+y),$$

for $z = x+iy$, $\phi \in L^2(\boldsymbol{R}^n)$, $\xi \in \boldsymbol{R}^n$. Upto unitary equivalence these π_λ give all the infinite dimensional irreducible unitary representations of H^n (see [15]). We also have another family of one-dimensional irreducible unitary representations χ_w, $w \in \mathbb{C}^n$, given by

$$\chi_w(z,t) = e^{iRe w.\overline{z}}, \qquad (z,t) \in H^n.$$

The representations π_λ for $\lambda \in \boldsymbol{R}^*$ together with χ_w for $w \in \mathbb{C}^n$ exhaust all the irreducible, pairwise inequivalent, unitary representations of H^n.

Throughout this chapter $\boldsymbol{N}$ denotes the set of non-negative integers. Consider the orthonormal basis $\{\Phi_\alpha : \alpha \in \boldsymbol{N}^n\}$ of $L^2(\boldsymbol{R}^n)$ consisting of the normalised Hermite functions. These Hermite functions can be given explicitly as follows: $\Phi_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j)$, for $x = (x_1, ..., x_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$ where $h_k(y) = (2^k k! \sqrt{\pi})^{\frac{-1}{2}} (-1)^k \frac{d^k}{dy^k}(e^{-y^2}) e^{\frac{y^2}{2}}$, $y \in \boldsymbol{R}$, $k = 0, 1, 2, ...$ Moreover Φ_α is an eigenfunction for the Hermite operator $H = -\Delta + |x|^2$ on $\boldsymbol{R}^n$ with the eigenvalue $(2|\alpha| + n)$. Here $|\alpha| = \alpha_1 + ... + \alpha_n$.

The special Hermite functions $\Phi_{\alpha\beta}$ are defined as follows:

$$\Phi_{\alpha\beta}(z) = (2\pi)^{\frac{-n}{2}} \langle \pi(z)\Phi_\alpha,\ \Phi_\beta \rangle_{L^2(\boldsymbol{R}^n)}$$

where $\pi(z) = \pi_1(z, 0)$. The system $\{\Phi_{\alpha\beta}\}_{\alpha,\beta}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. We also have the useful formulae : For multi-indices μ and m,

$$\Phi_{\mu+m,\mu}(z) = (2\pi)^{-\frac{n}{2}} \left(\frac{\mu!}{(\mu+m)!}\right)^{\frac{1}{2}} \left(\frac{i}{\sqrt{2}}\right)^m \bar{z}^m L_\mu^m(z)\, e^{-\frac{1}{4}|z|^2},$$

$$\Phi_{\mu,\mu+m}(z) = (2\pi)^{-\frac{n}{2}} \left(\frac{\mu!}{(\mu+m)!}\right)^{\frac{1}{2}} \left(-\frac{i}{\sqrt{2}}\right)^m z^m L_\mu^m(z)\, e^{-\frac{1}{4}|z|^2}.$$

Here we have used the notation

$$\begin{aligned} \mu! &= \mu_1!\, \mu_2! \,...\, \mu_n!, \\ z^m &= {z_1}^{m_1} {z_2}^{m_2} ... {z_n}^{m_n}, \\ {L_\mu}^m(z) &= \prod_{j=1}^{n} L_{\mu_j}^{m_j}\left(\frac{1}{2}|z_j|^2\right), \end{aligned}$$

where $L_k^\alpha(t)$ is the k-th Laguerre polynomial of type $\alpha > -1$. Therefore, $\Phi_{\mu\mu}(z_1, .., z_n) = \Phi_{\mu\mu}(|z_1|, .., |z_n|)$. Hence each $\Phi_{\mu\mu}$ is real-valued. Further let

$$\phi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)\, e^{-\frac{1}{4}|z|^2}$$

denote the k-th Laguerre function, where L_k^{n-1} denotes the k-th Laguerre polynomial of type $n-1$. Then

$$\phi_k(z) = (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \Phi_{\alpha\alpha}(z).$$

If $F_1, F_2 \in L^1(\mathbb{C}^n)$ and $\lambda \in \mathbb{R}^*$, we define $F_1 *_\lambda F_2$, the λ- twisted convolution of F_1 and F_2 by

$$F_1 *_\lambda F_2(z) = \int_{\mathbb{C}^n} F_1(z-w)\, F_2(w)\, e^{i\frac{\lambda}{2} Im z.\bar{w}}\, dw.$$

Then it can be seen that

$$\Phi_{\alpha\beta}^\lambda *_\lambda \Phi_{\mu\nu}^\lambda = (2\pi)^{\frac{n}{2}}\, \delta_{\beta\mu}\, \Phi_{\alpha\nu}^\lambda$$

where $\Phi_{\alpha\beta}^\lambda(z) = |\lambda|^{\frac{n}{2}}\, \Phi_{\alpha\beta}(|\lambda|^{\frac{1}{2}} z)$.

A reference for the results on Hermite functions, special Hermite functions, twisted convolutions etc. is [43].

3.2 The Gelfand pair $(HM(n), U(n))$

The compact group $U(n)$, of $n \times n$ unitary matrices with entries in $\mathbb{C}$, acts on H^n via the automorphism

$$\sigma(z,t) = (\sigma z, t), \;\; \sigma \in U(n), \; (z,t) \in H^n.$$

Therefore we can form the Heisenberg motion group $HM(n) = H^n \rtimes U(n)$, as a semi-direct product of H^n and $U(n)$. The group law in $HM(n)$ is given by:

$$(\sigma, z, t)(\tau, w, s) = (\sigma\tau, \sigma w + z, s + t + \frac{1}{2} Im\, \sigma w \cdot \overline{z}).$$

for (σ, z, t), $(\tau, w, s) \in HM(n)$. The group $HM(n)$ acts on H^n in the following way

$$(\sigma, z, t)(w, s) = (\sigma w + z, s + t + \frac{1}{2} Im\, \sigma w \cdot \overline{z}).$$

The group $U(n)$ is a maximal compact subgroup of $HM(n)$.

Henceforth we also write G for $HM(n)$ and K for $U(n)$.

Let $L^1(H^n)^\sharp$ be the closed subalgebra of K-invariant functions in $L^1(H^n)$. As shown in [2], $L^1(H^n)^\sharp$ is a *commutative* Banach $*$-algebra with respect to the usual convolution on H^n. (K-invariant functions on H^n are sometimes referred to as "radial functions".) Note that functions on H^n can be identified with functions on G that are right K-invariant. Thus $L^1(H^n)^\sharp$ can be identified with $L^1(K\backslash G/K)$, the subalgebra of $L^1(G)$ consisting of all K-bi-invariant functions on G. Further for $f, g \in L^1(H^n)^\sharp = L^1(K\backslash G/K)$, $f * g = f *_G g$ where $*$, $*_G$ denote the convolutions in H^n and G respectively. Hence $L^1(K\backslash G/K)$ is also a commutative Banach $*$-algebra and therefore (G, K) is a *Gelfand pair.* (See [21] for details about Gelfand pairs in general and [5], [6] and [7] for the Gelfand pairs associated with the Heisenberg group in particular.)

Remark 3.2.1 *For $f, g \in L^1(H^n)$, it is* not *in general true that $f * g = f *_G g$. However for functions $f, g \in L^1(H^n)^\natural$ (i.e. radial functions), it is indeed the case that $f * g = f *_G g$.*

Let N be any locally compact topological group and K_0 be a compact subgroup of N. Let $\pi : N \mapsto \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of N on a Hilbert space $\mathcal{H}$. We say that π is a class-1 representation for the pair (N, K_0) if the restriction of π to K_0 contains the trivial representation of K_0, i.e., the space $H_0 = \{v \in \mathcal{H} : \pi(k)v = v, \forall k \in K_0\} \neq (0)$.

In case (N, K_0) is a Gelfand pair, i.e., if the algebra $\{f \in L^1(N) : f(k_1 x k_2) = f(x),\ k_1,\ k_2 \in K_0,\ x \in N\}$, is commutative with respect to usual convolution on N, it is known that, for π, $\mathcal{H}$, H_0 as above, $dim\ H_0 = 1$. The function $x \mapsto \langle \pi(x) v_0, v_0 \rangle$, $x \in N$ where $v_0 \in H_0$ is such that $\|v_0\| = 1$, is called the elementary spherical function corresponding to π.

For more details on Gelfand pairs and elementary spherical functions etc. see [16], [21].

It is a well known fact that the vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n \\ Y_j &= \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n \\ T &= \frac{\partial}{\partial t} \end{aligned}$$

form a basis for the Lie algebra of right-invariant vector fields on H^n. The Heisenberg sublaplacian $\mathcal{L}$ is defined as follows: $\mathcal{L} = -\sum_{j=1}^{n} \ (X_j{}^2 + Y_j^2)$.

A family $\{\rho_{\lambda,k}\}_{\lambda\in\mathbb{R}^*,k\in\mathbb{N}}$ of class-1 representations for the pair (G, K) (see [40]) is defined as follows:

For $\lambda \in \mathbb{R}^*$ and $k \in \mathbb{N}$, define

$$\tilde{H}_{\lambda,k} = \{f : H^n \to \mathbb{C} \ \ smooth : \mathcal{L}f = |\lambda|(2k+n)f,\, Tf = i\lambda f,\, \int_{\mathbb{C}^n} |f(z,0)|^2\, dz < \infty\}.$$

An inner product $(\cdot,\cdot)$ on $\tilde{H}_{\lambda,k}$ is given as follows

$$(f,g) = (2\pi)^{-n}|\lambda|^n \int_{\mathbb{C}^n} f(z,0)\, \overline{g(z,0)}\, dz.$$

Let $H_{\lambda,k}$ be the completion of $\tilde{H}_{\lambda,k}$ with respect to $(\cdot,\cdot)$. Let $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\, \Phi_\alpha(|\lambda|^{\frac{1}{2}}x)$, $x \in \mathbb{R}^n$. Then the functions $E^\lambda_{\alpha\beta}(z,t) = \langle \pi_\lambda(z,t)\Phi^\lambda_\alpha,\ \Phi^\lambda_\beta\rangle$, $\alpha, \beta \in \mathbb{N}^n$, with $|\beta| = k$, form an orthonormal basis for $H_{\lambda,k}$. Define

$$\rho_{\lambda,k}(\sigma,z,t)\, f(w,s) = f((\sigma,z,t)^{-1}(w,s)),$$

for $(\sigma,z,t) \in G$, $f \in H_{\lambda,k}$, $(w,s) \in H^n$. Then $\rho_{\lambda,k}$ is a unitary representation of G. The following can be essentially found in [40] :

Theorem 3.2.2 *The representation $\rho_{\lambda,k}$ defined above is an irreducible unitary class-1 representation of G. The corresponding bounded elementary spherical function $e_{\lambda,k}$ is given as*

$$e_{\lambda,k}(\sigma,z,t) = \frac{k!(n-1)!}{(k+n-1)!}\, e^{-i\lambda t}\, \phi_k(|\lambda|^{1/2}z),$$

$(\sigma,z,t) \in G$. The restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $\frac{(k+n-1)!}{k!(n-1)!}$ irreducible representations, each of which is equivalent to the representation π_λ of H^n. Moreover, for λ, $\lambda_1 \in \mathbb{R}^*$, k, $k_1 \in \mathbb{N}$, $\rho_{\lambda,k}$ is equivalent to ρ_{λ_1,k_1} if and only if $\lambda = \lambda_1, k = k_1$.

The irreducibility and pairwise inequivalence of $\rho_{\lambda,k}$'s are proved in [40]. Also the fact that the restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $\frac{(k+n-1)!}{k!(n-1)!}$ irreducible representations, each of which is equivalent to the representation π_λ of H^n has been observed in [40]. To see that $\rho_{\lambda,k}$ is class-1 for each $\lambda \in \mathbb{R}^*, k \in \mathbb{N}$, note that the function

$$\begin{aligned}
E_k^\lambda(z,t) &= N_k^{-\frac{1}{2}} \sum_{|\beta|=k} E_{\beta\beta}^\lambda(z,t) \text{ where } N_k = \frac{(k+n-1)!}{k!(n-1)!} \\
&= N_k^{-\frac{1}{2}} e^{i\lambda t} (2\pi)^{\frac{n}{2}} \sum_{\beta=k} \Phi_{\beta\beta}(|\lambda|^{\frac{1}{2}} z) \\
&= N_k^{-\frac{1}{2}} e^{i\lambda t} \phi_k(|\lambda|^{\frac{1}{2}} z) \\
&= N_k^{-\frac{1}{2}} e^{i\lambda t} L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right) e^{-\frac{1}{4}|\lambda||z|^2}
\end{aligned}$$

(using results quoted in Section 3.1) is the essentially unique K-fixed vector in $H_{\lambda,k}$. The corresponding elementary spherical function $e_{\lambda,k}$ is therefore given by

$$\begin{aligned}
e_{\lambda,k}(\sigma,z,t) &= \langle \rho_{\lambda,k}(\sigma,z,t)E_k^\lambda, E_k^\lambda\rangle_{H_{\lambda,k}} \\
&= \langle \rho_{\lambda,k}(e,z,t)E_k^\lambda, E_k^\lambda\rangle_{H_{\lambda,k}},
\end{aligned}$$

where e is the identity element in $U(n)$. Hence the above expression becomes

$$\begin{aligned}
&(2\pi)^{-n}|\lambda|^n \int_{\mathbb{C}^n} E_k^\lambda((e,z,t)^{-1}(w,0))\, \overline{E_k^\lambda(w,o)}\, dw \\
&= (2\pi)^{-n}|\lambda|^n \int_{\mathbb{C}^n} E_k^\lambda((w,0)(z,t)^{-1})\, \overline{E_k^\lambda(w,o)}\, dw \\
&= (2\pi)^{-n}|\lambda|^n N_k^{-1} \int_{\mathbb{C}^n} \sum_{|\alpha|=k} \langle \pi_\lambda((w,o)(z,t)^{-1})\, \Phi_\alpha^\lambda, \Phi_\alpha^\lambda\rangle \sum_{|\beta|=k} \overline{\langle \pi_\lambda(w,0)\Phi_\beta^\lambda, \Phi_\beta^\lambda\rangle}\, dw \\
&= N_k^{-1} e^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} e^{i\frac{\lambda}{2} Im\, z\cdot\bar{w}}\, \Phi_{\alpha\alpha}^\lambda(w-z)\, \overline{\Phi_{\beta\beta}^\lambda(w)}\, dw.
\end{aligned}$$

Since $\Phi_{\mu\mu}^\lambda(z) = \Phi_{\mu\mu}^\lambda(-z)$ and $\Phi_{\mu\mu}^\lambda$ is real-valued, the above is equal to

$$\begin{aligned}
&N_k^{-1} e^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} e^{i\frac{\lambda}{2} Im\, z\cdot\bar{w}}\, \Phi_{\alpha\alpha}^\lambda(z-w)\, \Phi_{\beta\beta}^\lambda(w)\, dw. \\
&= N_k^{-1} e^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \Phi_{\alpha\alpha}^\lambda *_\lambda \Phi_{\beta\beta}^\lambda(z)
\end{aligned}$$

$$
\begin{aligned}
&= N_k^{-1} e^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} (2\pi)^{\frac{n}{2}} \delta_{\alpha\beta} \Phi^{\lambda}_{\alpha\beta}(z) \\
&= N_k^{-1} e^{-i\lambda t} (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \Phi^{\lambda}_{\alpha\alpha}(z) \\
&= N_k^{-1} e^{-i\lambda t} \phi_k(|\lambda|^{\frac{1}{2}} z).
\end{aligned}
$$

Since $e_{\lambda,k}(\sigma, z, t)$ is independent of choice of σ, we also write $e_{\lambda,k}(z,t)$ for $e_{\lambda,k}(\sigma,z,t)$ for any $\sigma \in U(n)$.

We now describe another set of class-1 representations of (G, K). Consider the one dimensional representation $\chi_w(z,t) = e^{iRe\, w\cdot\bar{z}}, w \in \mathbb{C}^n \setminus \{0\}, (z,t) \in H^n$, of H^n. Let $K_0 = \{k \in K : k.w = w\}$. Then K_0 is a closed subgroup of K. Let ρ_w be the induced representation obtained by inducing $\chi_w \otimes 1$ from $H^n \rtimes K_0$ to $H^n \rtimes K = G$. Here 1 denotes the trivial representation of K_0 and $\chi_w \otimes 1(k,z,t) = e^{iRe\, w.\bar{z}}$, for $(k,z,t) \in H^n \rtimes K_0$. The representation space of ρ_w is described as follows: Let

$$\tilde{H}_w = \{f : G \to \mathbb{C} \text{ continuous } : f(g_0 g) = (\chi_w \bigotimes 1)(g_0) f(g), g_0 \in H^n \rtimes K_0, g \in G\}.$$

Therefore for $f \in \tilde{H}_w, (\sigma, z, t) \in G$,

$$
\begin{aligned}
f(\sigma,z,t) &= f((e,z,t)(\sigma,0,0)) \\
&= (\chi_w \bigotimes 1)(e,z,t) f(\sigma,0,0) \\
&= e^{iRe\, w.\bar{z}} f(\sigma,0,0),
\end{aligned}
$$

and hence f can be viewed as a function on K. Let H_w be the completion of $\tilde{H}_w$ with respect to the inner product

$$(f,g) = \int_K f(k) \overline{g(k)}\, dk, \quad f,g \in \tilde{H}_w.$$

The induced representation ρ_w is given by :

$$\rho_w(\sigma,z,t) f(\tau,w,s) = f((\tau,w,s)(\sigma,z,t)), \quad f \in H_w, \ (\sigma,z,t),(\tau,w,s) \in G.$$

Then ρ_w is an irreducible unitary representation of G, with $f_0(\sigma, z, t) = \Omega_{2n-1}^{-\frac{1}{2}} e^{iRe\, w\cdot\bar{z}}$ as the essentially unique K-fixed vector. Here Ω_{2n-1} is the total surface measure of the unit sphere in $\boldsymbol{R}^{2n}$. (See [5] for details.) The corresponding elementary spherical functions η_τ can be computed to be the following

$$\eta_\tau(\sigma, z, \tau) = \frac{2^{n-1}(n-1)!\, J_{n-1}(\tau|z|)}{(\tau|z|)^{n-1}},$$

where $\tau = |w| > 0$ and J_{n-1} is the Bessel function of order $n-1$. Moreover, ρ_w is equivalent to $\rho_{w'}$ if and only if $|w| = |w'|$. Also the trivial representation of G is clearly a class-1 representation with the elementary spherical function $\eta_0 \equiv 1$ on G.

We also write $\eta_\tau(z,t)$ for $\eta_\tau(\sigma, z, t)$ for any $\sigma \in U(n)$. Since we know that $e_{\lambda,k}$ with $\lambda \in \boldsymbol{R}^*$, $k \in \boldsymbol{N}$ and η_τ with $\tau \geq 0$ are all the bounded elementary spherical functions for the pair (G, K), (see, for example, [2]), the above discussion completes the description of these in terms of class-1 representations of (G, K). The connection between representations and elementary spherical functions for Gelfand pairs associated with solvable Lie groups has been studied in detail in [5].

3.3 A theorem of the Wiener-Tauberian type for $L^1(H^n)$ with the $HM(n)$ action

We first state the Wiener-Tauberian theorem for $L^1(H^n)^\natural$ due to Hulanicki and Ricci [23].

Theorem 3.3.1 *([23]) Let J be a closed ideal in $L^1(H^n)^\natural$ and suppose that*
(1) For any $\lambda \in \boldsymbol{R}^*, k \in \boldsymbol{N}$, there exists some $f \in J$ such that
$\int f(z,t)\, e_{\lambda,k}(z,t)\, dz\, dt \neq 0$.
(2) For any $\tau \geq 0$ there exists some $f \in J$ such that $\int f(z,t)\, \eta_\tau(z,t)\, dz\, dt \neq 0$.
Then $J = L^1(H^n)^\natural$.

To state the analogue of the Wiener-Tauberian theorem for the action of G on H^n, we set up some notation.

Let $\widehat{H^n}$ denote the equivalence classes of irreducible unitary representations of H^n. For $h \in L^1(H^n)$, we define the "group-theoretic" Fourier transform on $\widehat{H^n}$ as follows : Let π be in $\widehat{H^n}$ with $\mathcal{H}_\pi$ as the corresponding representation space. Then $\pi(h)$ is the bounded operator defined by

$$\pi(h) = \int_{H^n} h((z,t)^{-1})\, \pi(z,t)\, dz\, dt,$$

where the integral is to be interpreted suitably. The assignment $\pi \mapsto \pi(h)$, defined on $\widehat{H^n}$ is known as the "group theoretic" Fourier transform of h. Thus for each $\lambda \in \boldsymbol{R}^*$, $\pi_\lambda(h)$ acts on the Hilbert space $L^2(\boldsymbol{R}^n)$ and for each $w \in \mathbb{C}^n$, $\chi_w(h)$ (is a scalar and) acts on the 1-dimensional space $\mathbb{C}$.

For each $\lambda \in \boldsymbol{R}^*$ and $k \in N$, let $P_{\lambda,k}$ be the projection on the k-th eigenspace $M_{\lambda,k} = Span\{\Phi_\alpha^\lambda \,:\, |\alpha| = k\}$ of the scaled Hermite operator $H_\lambda = -\Delta + |\lambda|^2|x|^2$ on $\boldsymbol{R}^n$. Recall $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\, \Phi_\alpha(|\lambda|^{\frac{1}{2}}x),\ x \in \boldsymbol{R}^n$.

Remark 3.3.2 *If we take the Fock space model for describing the infinite dimensional representations of H^n (see [15]), then the λ-dilated Hermite function Φ_α^λ corresponds to a nonzero multiple of the polynomial $z^\alpha = {z_1}^{\alpha_1}{z_2}^{\alpha_2}...{z_n}^{\alpha_n}$. Hence the subspace $M_{\lambda,k}$ of $L^2(\boldsymbol{R}^n)$ can be identified with the space of homogeneous polynomials of degree k in n-variables $z_1, ..., z_n$. The natural action, $(u.p)(z) = p(u^{-1}.z)$, $u \in U(n)$, of $U(n)$ on this space is irreducible. Thus $L^2(\boldsymbol{R}^n) = \bigoplus M_{\lambda,k}$ can be thought of as the decomposition of the representation space of π_λ into irreducible subspaces for the K-action, after $L^2(\boldsymbol{R}^n)$ is identified with the Fock space model.*

For a function h on H^n, define ${}^g h(z,t) = h(\,g.(z,t))$, for $g \in G, (z,t) \in H^n$. We are now in a position to give conditions under which $\overline{Span\{{}^g f:\ g \in G\}} = L^1(H^n)$, for a given function $f \in L^1(H^n)$. These conditions are purely in terms of the group theoretic Fourier transform of f ([29]).

Theorem 3.3.3 *Let $f \in L^1(H^n)$. Then $\overline{Span\{{}^g f:\ g \in G\}} = L^1(H^n)$ if and only if :*

(1) $\pi_\lambda(f)\ P_{\lambda,k} \neq 0$ for each $\lambda \in \mathbb{R}^*$ and $k \in \mathbb{N}$.

(2) For each $r > 0$, there exists $w \in \mathbb{C}^n$ with $|w| = r$ such that $\chi_w(f) \neq 0$.

(3) $1(f) \neq 0$, where 1 is the trivial representation of H^n.

Remark 3.3.4 *If we define $f_0(z) = \int f(z,t)\,dt,\ z \in \mathbb{C}^n$ then the condition (2) above can be rewritten as follows: For each $r > 0$, $\hat{f}_0$, the Euclidean Fourier transform of f_0, does not vanish identically on S_r, the sphere of radius r in $\mathbb{R}^{2n}$. Also condition (3) is equivalent to $\hat{f}_0(0) \neq 0$.*

Proof of Theorem 3.3.3: (a) (Sufficiency of the conditions (1), (2) and (3).) Assume that the conditions (1), (2) and (3) of the theorem are satisfied. As observed in Section 3.2, the given function f on H^n can be thought of as a right K-invariant function on G via $f(\sigma, z, t) = f(z,t)$, $(\sigma, z, t) \in G$. Define $f^*(\sigma, z, t) = \overline{f(\sigma, z, t)^{-1}} = \overline{f(-\sigma^{-1}z, t)}$, $(\sigma, z, t) \in G$. Then f^* is a left K-invariant function on G. Hence $f^* *_G f$ is a K-bi-invariant function on G. Equivalently, it can be viewed as a K-invariant function on H^n.

We claim that the closed ideal generated by $f^* *_G f$ in $L^1(H^n)^\natural$ is the full algebra $L^1(H^n)^\natural$. Note that once we establish the claim, the theorem follows from the observations

1. $h * (f^* *_G f) \in \overline{Span\{{}^g f:\ g \in G\}}$, for $h \in L^1(H^n)^\natural$ and hence $L^1(H^n)^\natural =$

$$L^1(K\backslash G/K) \subsetneq \overline{Span\{^g f:\ g \in G\}} \subseteq L^1(G/K) = L^1(H^n).$$

2. The smallest closed subspace of $L^1(G/K)$ containing $L^1(K\backslash G/K)$ and invariant under the (left) G-action, is all of $L^1(G/K)$.

To prove the claim, consider

$$\begin{aligned}
& \int (f^* *_G f)(z,t)\, e_{\lambda,k}(z,t)\, dz\, dt \\
= & \int (f^* *_G f)(z,t)\, \langle \rho_{\lambda,k}(e,z,t) E_k^\lambda,\ E_k^\lambda\rangle\, dz\, dt \\
= & \int (f^* *_G f)(z,t)\, e^{-i\lambda t}\, N_k^{-1}\, \phi_k(|\lambda|^{\frac{1}{2}} z)\, dz\, dt \\
= & \int (f^* *_G f)(z,t)\, e^{-i\lambda t}\, N_k^{-1}\, (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \langle \pi_\lambda(z)\Phi_\alpha^\lambda,\ \Phi_\alpha^\lambda\rangle\, dz\, dt \\
= & \ N_k^{-1}\, (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \langle \pi_\lambda(f^* *_G f)\, \Phi_\alpha^\lambda,\ \Phi_\alpha^\lambda\rangle.
\end{aligned}$$

Again an easy computation shows that

$$\begin{aligned}
\pi_\lambda(f^* *_G f) &= \int (f^* *_G f)(z,t)^{-1}\, \pi_\lambda(z,t)\, dz\, dt \\
&= \int_{H^n}\int_{U(n)} (f * f^*)(-z,-t)\, \pi_\lambda(u.z,t)\, du\, dz\, dt \\
&= \int_{U(n)} \pi_{\lambda,u}(f * f^*)\, du,
\end{aligned}$$

where $\pi_{\lambda,u}(z,t) = \pi_\lambda(u.z,t),\quad (z,t) \in H^n,\ u \in U(n)$. Therefore,

$$\begin{aligned}
& \int (f^* *_G f)(z,t)\ e_{\lambda,k}(z,t)\, dz\, dt \\
&\qquad = N_k^{-1}\, (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \int_{U(n)} \langle \pi_{\lambda,u}(f * f^*)\, \Phi_\alpha^\lambda,\ \Phi_\alpha^\lambda\rangle\, du \\
&\qquad = N_k^{-1}\, (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \int_{U(n)} \langle \pi_{\lambda,u}(f^*)\ \pi_{\lambda,u}(f)\ \Phi_\alpha^\lambda,\ \Phi_\alpha^\lambda\rangle\, du \\
&\qquad = N_k^{-1}(2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \int_{U(n)} \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\|^2 du.
\end{aligned}$$

Hence $\int (f^* *_G f)(z,t)\, e_{\lambda,k}(z,t)\, dz\, dt = 0 \Leftrightarrow \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\| = 0$, for all α such that $|\alpha| = k$ and a.e. $u \in U(n)$. Now as the irreducible representation $\pi_{\lambda,u}$ has the same central character as π_λ, by the Stone-von Neumann theorem (see [15]), $\pi_{\lambda,u}$ is equivalent to π_λ. Also for each $u \in U(n), \lambda \in \boldsymbol{R}^*$ we can choose an intertwining unitary operator $m_\lambda(u) : L^2(\boldsymbol{R}^n) \mapsto L^2(\boldsymbol{R}^n)$ such that $m_\lambda(u)\,\pi_\lambda(z,t) = \pi_{\lambda,u}(z,t)m_\lambda(u)$, for all $(z,t) \in H^n$ and $u \mapsto m_\lambda(u) : U(n) \mapsto \mathcal{U}(L^2(\boldsymbol{R}^n))$ is a continuous projective representation of $U(n)$. Therefore, the condition $\int (f^* *_G f)(z,t)\, e_{\lambda k}(z,t)\, dz\, dt = 0$ is equivalent to $m_\lambda(u)\,\pi_\lambda(f)\, m_\lambda(u)^{-1}\,\Phi_\lambda^\alpha = 0$, for all $u \in U(n)$, and α such that $|\alpha| = k$. Now for each λ and k, by the Remark 3.3.2, $M_{\lambda,k}$ can be identified in the Fock space model with polynomials of degree k in n-variables $z_1, z_2, \cdots z_n$. Also in this case, $m_\lambda(u)$ can be chosen to be the natural action of u on polynomials. Since this action preserves the degree of a polynomial, $m_\lambda(u)$ sends $M_{\lambda,k}$ onto $M_{\lambda,k}$. Hence the above is equivalent to $\pi_\lambda(f)P_{\lambda,k} = 0$. The condition (1) in the hypothesis implies that this is not the case. Hence $\int (f^* *_G f)(z,t)\, e_{\lambda,k}(z,t)\, dz\, dt \neq 0$, for any $\lambda \in \boldsymbol{R}^*, k \in \boldsymbol{N}$. Also for $\tau > 0$, we have

$$\begin{aligned} & \int (f^* *_G f)(z,t)\, \eta_\tau(z,t)\, dz\, dt \\ = \; & \int_{H^n}\int_{U(n)} (f^* *_G f)(z,t)\, e^{iRe((u.w)\cdot\bar{z})} du\, dz\, dt, \end{aligned}$$

where $w \in \mathbb{C}^n$ is such that $|w| = \tau$. Again using the fact that $(f^* *_G f)(z,t) = \int_{U(n)} (f * f^*)(u.z, t) du$, we have $\int (f^* *_G f)(z,t)\eta_\tau(z,t)\, dz\, dt = const. \int_{U(n)} |\chi_{u.w}(f)|^2\, du$, for a non-zero constant. Therefore,

$$\int (f^* *_G f)(z,t)\, \eta_\tau(z,t)\, dz\, dt = 0$$

if and only if $\chi_{u.w}(f) = 0$, for all $u \in U(n)$. This in turn is equivalent to $\hat{f}_0 \equiv 0$ on the sphere S_τ of radius τ in $\mathbb{C}^n$. But the condition (2) in the hypothesis implies that this is not the case. Hence $\int (f^* *_G f)(z,t)\, \eta_\tau(z,t)\, dz\, dt \neq 0$, for $\tau > 0$. For $\tau = 0$, $\int (f^* *_G f)(z,t)\, dz\, dt = |\int f(z,t)\, dz\, dt|^2 \neq 0$, by the condition (3) in the hypothesis. Hence the Wiener-Tauberian theorem of Hulanicki and Ricci ([23]) holds for the closed ideal $\overline{\{h * (f^* *_G f) : h \in L^1(H^n)^\sharp\}}$ generated by $f^* *_G f$ in $L^1(H^n)^\sharp$, i.e., $\overline{\{h * (f^* *_G f) : h \in L^1(H^n)^\sharp\}} = L^1(H^n)^\sharp$.

(b) (Necessity of the conditions (1), (2) and (3).) The fact is that the conditions in Theorem 3.3.3 are also necessary for $f \in L^1(H^n)$ to have the property that $\overline{Span\{{}^g f:\ g \in G\}} = L^1(H^n)$. This is because if any of the condition (1), (2) or (3) is violated for f, then it is violated for every ${}^g f$ and hence for every function in $\overline{Span\{{}^g f:\ g \in G\}}$. On the other hand, the conditions (1), (2) and (3) do hold for the function $h(z,t) = e^{-|z|^2} e^{-t^2}$, $(z,t) \in H^n$, which is also in $L^1(H^n)$. Hence h cannot be in $\overline{Span\{{}^g f:\ g \in G\}}$.

Chapter 4

Symmetric spaces of the noncompact type

4.1 Introduction

Throughout this chapter X will denote a symmetric space of the noncompact type and of real rank 1 (though many of our results will be valid even for symmetric spaces whose rank is greater than 1). We refer the reader to [19], [22] for facts about symmetric spaces.

Let G be the connected component of the group of isometries of X and K the subgroup of G which leaves a given point p_0 fixed. Then K is a maximal compact subgroup of G and G/K is diffeomorphic to X under the mapping $gK \to g \cdot p_0, \ \ g \in G$. (Indeed this mapping is a real-analytic diffeomorphism.) Moreover as X is of the noncompact type and of real rank 1, G is a connected noncompact semisimple Lie group with finite centre (and of real rank 1). For functions on the symmetric space X, one has the notion of the Helgason-Fourier transform. In this chapter, motivated by the results in the Chapter 2, we consider the span of the G-translates of an L^p-function, for $2 \leq p < \infty$, using the Helgason-Fourier transform as our main tool. We also briefly indicate how some older results for $L^p, 1 \leq p < 2$, can be reformulated in terms of the Helgason-Fourier transform.

The main result of this chapter is the surprising fact that if $2 < p < \infty$, any non-trivial $f \in L^p \cap L^2$ has the property that $Span\{g^f : g \in G\}$ is dense in $L^p(X)$. Along the way, we also establish an L^2-analogue of this Wiener-Tauberian theorem in terms of the Helgason-Fourier transform. (Such an L^2-result is probably not surprising to the experts. A very general L^2-result for Gelfand pairs, appears in [42].) Elsewhere somewhat easier fact that *if* $2 \leq p < \infty$, *any non-trivial* $f \in L^p \cap L^1$ *generates* $L^p(X)$ (in the sense described above) has already been recorded ([38] and [31]) without proof. We also give a proof of this using the Helgason-Fourier transform as our main tool.

A function f on X can be thought of as a right K-invariant function on G i.e. $f(xk) = f(x)$, $x \in G$, $k \in K$. Since K is compact and G is unimodular, there is a canonical G-invariant measure dx on $X = G/K$ such that $\int_G f(g)dg = \int_X f(x)dx$ for right K-invariant f. Here dg is the Haar measure on G. Thus we can think of $L^p(X)(= L^p(G/K))$ as a closed subspace of $L^p(G)$. The set of L^p functions on G which are bi-invariant under K i.e. $f(k_1xk_2) = f(x)$, $k_1, k_2 \in K$, $x \in G$ will be denoted by $L^p(K\backslash G/K)$. Note that a K-bi-invariant function on G can also be thought of as a function on X invariant under the natural action of K on X. Note that if f and h are functions on X, we can form the function $f * h$ on X. To do this, we think of f and h as functions on G and perform the convolution on G, noting that if f and h are right K-invariant functions on G, then so is $f * h$. Finally, we continue with the notation that if h is a function on X and $g \in G$, then ${}^g h$ is the function defined by ${}^g h(x) = h(g.x)$, $x \in X$.

We briefly review some facts about semi-simple Lie groups and Lie algebras. For any unexplained notation, concept etc., we refer the reader to [18], [19]. Let $G = NAK$, $\mathcal{G} = \mathcal{N} \oplus \mathcal{A} \oplus \mathcal{K}$ be the Iwasawa decomposition for the group G and its Lie algebra $\mathcal{G}$ respectively. For each $g \in G$, let $A(g) \in \mathcal{A}, u(g) \in K, n(g) \in N$

denote the unique elements such that $g = n(g)expA(g)u(g)$. Let $\sum$ denote the set of roots of $\mathcal{G}$ with respect to $\mathcal{A}$ and $\sum^+$ denote the subset of positive roots. Let ρ be the half sum of positive roots counted according to the multiplicity. Let $\mathcal{A}^*$ denote the real dual of $\mathcal{A}$ and $\mathcal{A}^*_{\mathbb{C}}$ its complexification. Also, let M be the centralizer of A in K.

The "elementary spherical function" ϕ_λ, $\lambda \in \mathcal{A}^*_{\mathbb{C}}$, for the pair (G, K) is then given by the following expression:

$$\phi_\lambda(g) = \int_K e^{(i\lambda+\rho)A(kg)}\, dk, \;\; g \in G.$$

Since X is of real rank 1, $dim\, \mathcal{A}^* = 1$. Therefore $\mathcal{A}^* = \boldsymbol{R}\rho$ and $\mathcal{A}^*_{\mathbb{C}} = \mathbb{C}\rho$. We identify $\lambda\rho \in \mathcal{A}^*_{\mathbb{C}}$ with $\lambda \in \mathbb{C}$. Hence ρ is identified with $1 \in \mathbb{C}$. Thus, in the sequel, by ϕ_λ for $\lambda \in \mathbb{C}$ we actually mean $\phi_{\lambda\rho}$. Each ϕ_λ is a real analytic K-bi-invariant function on G with $\phi(e) = 1$, where e denotes the identity element of G. Also $\{\phi_\lambda\}_{\lambda\in\mathbb{C}}$ exhausts the collection of the "elementary spherical functions" on G. Further $\phi_\lambda = \phi_\nu$ if and only if $\nu = \pm\lambda$. (See [21] for details.)

For a K-bi-invariant function f on G, the spherical Fourier transform $\tilde{f}$ of f is defined as follows:

$$\tilde{f}(\lambda) = \int_G f(g)\, \phi_{-\lambda}(g)\, dg$$

for all $\lambda \in \mathbb{C}$ for which the integral exists. (Thus if f is a compactly supported continuous K-bi-invariant function on G, then $\tilde{f}(\lambda)$ is defined for all of $\mathbb{C}$.)

Remark 4.1.1 *It is a well known fact that ϕ_λ is bounded if and only if $\lambda \in S = \boldsymbol{R} + i[-1, 1]$ and in fact $|\phi_\lambda(g)| \leq 1, \forall\, \lambda \in S,\, g \in G$ (see [21]). Also for each fixed $g \in G$, $\lambda \mapsto \phi_\lambda(g)$ is a holomorphic function on $\mathbb{C}$. Hence, using all the above facts, it can be shown that for $f \in L^1(K\backslash G/K), \lambda \mapsto \tilde{f}(\lambda)$ is a bounded continuous even function on S which is holomorphic in the interior S^0 of S ([21]).*

Also for $1 < p < 2$, if λ is in the open strip $S_p^0 = \mathbb{R} + i(-\gamma_p, \gamma_p)$ where $\gamma_p = \frac{2}{p} - 1$, then it follows from the asymptotic behaviour of ϕ_λ ([18]) that $\phi_\lambda \in L^q$, where $1/p + 1/q = 1$. Therefore, if $f \in L^p(K\backslash G/K)$ and $\lambda \in S_p^0$, then the integral $\int f \phi_\lambda$ converges absolutely. Hence for $f \in L^p(K\backslash G/K)$, $\tilde{f}(\lambda)$ is defined in the open strip $S_p^0 = \mathbb{R} + i(-\gamma_p, \gamma_p)$ and one can show using the holomorphy of $\lambda \to \phi_\lambda(g)$, for each $g \in G$, that $\tilde{f}(\lambda)$ defines a holomorphic function in the strip S_p^0.

4.2 The Helgason-Fourier transform and L^p-results for $2 \leq p < \infty$

For suitable functions f on X, the Helgason-Fourier transform of f, which we again denote by $\tilde{f}$, is a function on $\mathbb{C} \times K/M$. Let $\dot{k} = kM \in K/M$ and $x = gK \in X$ be arbitrary. Then by $A(x, \dot{k})$ we mean $A(k^{-1}g)$. With this notation, we define $\tilde{f}$ as follows:

$$\tilde{f}(\lambda, \dot{k}) = \int_X f(x)\, e^{(-i\lambda+1)\rho A(x,\dot{k})}\, dx,$$

where $\dot{k} = kM$ for all $(\lambda, k) \in \mathbb{C} \times K/M$ for which the integral exists (-see [22]). (Thus if f is a compactly supported continuous function on X, then $\tilde{f}$ is defined for all of $\mathbb{C} \times K/M$.) Note that if f is a K-bi-invariant function on G, or what is same as a (left) K-invariant function on X, then the Helgason-Fourier transform $\tilde{f}(\lambda, \dot{k})$ of f is independent of k and in fact coincides with the spherical Fourier transform $\tilde{f}(\lambda)$ defined earlier. By abuse of notation, in future we write $\tilde{f}(\lambda, k)$ instead of $\tilde{f}(\lambda, \dot{k})$.

To prove the L^2 analogue of the Wiener-Tauberian theorem, we need to extend the notion of the Helgason-Fourier transform to L^2 functions on X. This is done using the following Plancherel theorem for X (-see [22]).

Theorem 4.2.1 *([22]) The Helgason-Fourier transform $f \to \tilde{f}$ is an isometry* *of $C_c(X)$, the space of continuous functions on X with compact support, into $L^2(\boldsymbol{R}^+ \times K/M, |c(\lambda)|^{-2}\, d\lambda\, dk)$. Further it extends to an isometry of $L^2(X)$ onto $L^2(\boldsymbol{R}^+ \times K/M, |c(\lambda)|^{-2}\, d\lambda\, dk)$. Moreover, for $f_1, f_2 \in L^2(X)$,*

$$\int_X f_1(x)\overline{f_2(x)}\, dx = \frac{1}{2}\int_R \int_{K/M} \tilde{f}_1(\lambda, k)\, \overline{\tilde{f}_2(\lambda, k)}\, |c(\lambda)|^{-2}\, dk\, d\lambda.$$

Remark 4.2.2 *The function $|c(\lambda)|^{-2}$ occuring in the Plancherel theorem above can be explicitly written down (-see [18]). The function $|c(\lambda)|^{-2}$ is a C^∞ function on $\boldsymbol{R}$ with at most polynomial growth. (See [18].) Further a subset E of $\boldsymbol{R}$ has zero Lebesgue measure if and only if E has zero measure with respect to $|c(\lambda)|^{-2}\, d\lambda$.*

For the rest of this section the measure $|c(\lambda)|^{-2}\, d\lambda\, dk$ will be denoted by $d\mu$.

We also need the following fact, which is a mild strengthing of Lemma 1.4, page 226, in [22].

Lemma 4.2.3 *Let $f, \phi \in L^2(X)$ be such that $f * \phi \in L^2(X)$. Suppose further that ϕ is K-bi-invariant. Then*

$$(f * \phi)^{\sim}(\lambda, k) = \tilde{\phi}(\lambda)\tilde{f}(\lambda, k) \;\; a.e.\, (\lambda, k) \in \boldsymbol{R} \times K/M.$$

Now we prove a Wiener-Tauberian theorem for $L^2(X)$ in terms of the Helgason-Fourier transform. A general L^2 result for Gelfand pairs is proved in [42]. However the result in [42] is stated in terms of the group-theoretic Fourier transform. See Section 4.3, for the connection between the Helgason-Fourier transform and the group-theoretic Fourier transform in our case.

Theorem 4.2.4 *Let $f \in L^2(X)$ be such that the set $\{\lambda \in \boldsymbol{R} : \tilde{f}(\lambda, k) = 0 \;\, a.e.\, k\}$*

is of zero Lebesgue measure on $\mathbb{R}$. Then the Span$\{{}^g f : g \in G\}$ is dense in $L^2(X)$.

(Note that by the Remark 4.2.2, the condition that a subset E of $\mathbb{R}$ is of zero Lebesgue measure is equivalent to the condition that it has zero measure with respect to $|c(\lambda)|^{-2}d\lambda$. Also the theorem above is an analogue to the part (4), case $q = 2$ of Theorem 2.2.1.)

Proof : Suppose $Span\{{}^g f : g \in G\}$ is not dense in $L^2(X)$. Then there exists a non-trivial function $h_0 \in L^2(X)$, such that $\int_X {}^g f(x)\, h_0(x)\, dx = 0$, for each $g \in G$. Set $U = \{\psi \in L^2(X) : \int_X {}^g f(x)\, \psi(x)\, dx = 0, \forall g \in G\} = \{\psi \in L^2(X) : f * \check{\psi} \equiv 0\}$ where $\check{\psi}(g) = \psi(g^{-1})$, $g \in G$. Then U is non-trivial since by assumption it contains the non-trivial function h_0. As U is closed and invariant under the left G-action, $\phi * h_0 \in U$ for any $\phi \in C_c(G)$. Further as $h_0 \in L^2(G/K)$ is non-trivial, we can choose a $\phi \in C_c(G)$, which is also left K-invariant and such that $\phi * h_0$ is again non-trivial. (Otherwise, $\phi * h_0(e) = 0$ for any such ϕ. This in turn implies that h_0 is orthogonal to $C_c(G/K)$ and therefore to $L^2(G/K) = L^2(X)$.) Hence there is a non-trivial $h_1 = \phi * h_0 \in U$ such that h_1 is also left K-invariant. But then $f * h \equiv 0$ where $h(g) = \check{h}_1(g) = h_1(g^{-1})$, $g \in G$ and h is also a non-trivial left K-invariant function in $L^2(X)$. Consequently $(f * h)^\sim(\lambda, k) = 0$ for almost all (λ, k) with respect to the Plancherel measure $d\mu = |c(\lambda)|^{-2} d\lambda\, dk$ on $\mathbb{R} \times K/M$. Also by Lemma 4.2.3, $(f * h)^\sim(\lambda, k) = \tilde{h}(\lambda)\tilde{f}(\lambda, k)$, *a.e.* $d\mu(\lambda, k)$. By assumption, the set $\{\lambda \in \mathbb{R} : \tilde{f}(\lambda, k) = 0 \text{ } a.e.\, k\}$ has zero Lebesgue measure and therefore by Remark 4.2.2, zero μ measure. As a consequence, $\tilde{h}(\lambda) = 0\, a.e. d\mu(\lambda)$. As $h \in L^2(X)$, the Plancherel theorem for $L^2(X)$ then implies that h is a trivial function, a contradiction.

(Note that the condition given in the Theorem 4.2.4 is also necessary for the $Span\{{}^g f : g \in G\}$ to be dense in $L^2(X)$. (See also Remark 2.2.2.) Suppose E is a set of positive measure with respect to the measure $|c(\lambda)|^{-2}d\lambda$ on $\boldsymbol{R}$ such that for $\lambda \in E$, $\tilde{f}(\lambda, k) = 0$ *a.e.* k. Then we can choose a left K-invariant function $h \in L^2(X)$ such that $\tilde{h}(\lambda)$ is supported in E. Hence $\tilde{h}(\lambda)\tilde{f}(\lambda, k) = 0\,a.e.$ By reversing the arguments given in the proof of the Theorem 4.2.4, we have a non-trivial $h \in L^2(X)$ which is orthogonal to the $Span\{{}^g f : g \in G\}$ in $L^2(X)$.)

Next we take up the case of L^p for $2 < p < \infty$.

Theorem 4.2.5 *Let $f \in L^p(X) \cap L^2(X), 2 < p < \infty$. If f is a non-trivial function, then $\overline{Span\{{}^g f : g \in G\}} = L^p(X)$.*

Proof : Suppose $h_0 \in L^q(X)$, $1/p + 1/q = 1$ is such that $\int_X {}^g f(x)\, h_0(x)\, dx = 0$, for each $g \in G$. As before, the space $U = \{\psi \in L^q(X) : \int_X {}^g f(x)\, \psi(x)\, dx = 0, \forall\, g \in G\} = \{\psi \in L^2(X) : f * \check{\psi} \equiv 0\}$ is closed and invariant under the left G-action. Moreover, by assumption, it contains the non-trivial function h_0. Therefore as in the proof of Theorem 4.2.4, there exists a non-trivial $h_1 \in U$ such that h_1 is also left K-invariant. Thus the fact that $\int_X {}^g f(x)\, h_1(x)\, dx = 0$, $\forall\, g \in G$, implies that $f * h \equiv 0$. Here $h(g) = \check{h}_1(g) = h_1(g^{-1})$, $g \in G$. Also h is a non-trivial left K-invariant function in $L^q(X), 1 < q < 2$. Now $f \in L^2(G)$. Therefore by Kunze-Stein phenomenon, (see [12]), $f * h \in L^2(X)$. Consequently, $(f * h)^{\sim}(\lambda, k) = 0$ *a.e.* $d\mu(\lambda, k)$. Also as h is K-bi-invariant, another mild strengthing of the Lemma 1.4, page 226, in [22] shows that $(f * h)^{\sim}(\lambda, k) = \tilde{h}(\lambda)\tilde{f}(\lambda, k)\, a.e. d\mu(\lambda, k)$. Hence $\tilde{h}(\lambda)\, \tilde{f}(\lambda, k) = 0\, a.e.\, d\mu(\lambda, k)$. As h is a non-trivial K-bi-invariant function in $L^q(G)$, $1 < q < 2$, by Remark 4.1.1, $\lambda \mapsto \tilde{h}(\lambda)$ is a non-trivial holomorphic function in the open strip $\boldsymbol{R} + i(-\gamma_q, \gamma_q)$ in the complex plane. Therefore its restriction to $\boldsymbol{R}$ can vanish only on a set of Lebesgue measure zero. But then, by Remark 4.2.2, the measure $|c(\lambda)|^{-2}\, d\lambda$ on $\boldsymbol{R}$ is absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{R}$. Consequently, $\tilde{f}(\lambda, k) = 0$ a.e. $d\mu(\lambda, k)$. But then the Plancherel theorem for X implies that $f = 0$

a.e., a contradiction.

The case of a function $f \in L^p(X) \cap L^1(X)$ is somewhat easier to handle. It is considered in [38] and [31], where the following result is recorded without a proof :

Theorem 4.2.6 *Let $f \in L^p(X) \cap L^1(X), 2 \leq p < \infty$. If f is a non-trivial function, then $\overline{Span\{^g f : g \in G\}} = L^p(X)$.*

(Note that the case $p = 2$ is also included in the above statement.)

Proof : Since $f \in L^p(X) \cap L^1(X)$, $p \geq 2$, f is also in $L^2(X)$. For any function ϕ on G define $\phi^*(g) = \overline{\phi(g^{-1})}$, $g \in G$. Then if ϕ is non-trivial function in $L^2(G)$, $\phi^* * \phi$ is a continuous non-trivial function on G. Indeed, $\phi^* * \phi(e) = \|\phi\|_2^2$. Hence $f^* * f$ is a non-trivial function. As f is a right K-invariant function on G, f^* is left K-invariant and $f^* * f$ is therefore a non-trivial K-bi-invariant function on G. Moreover as f is in L^1, so is $f^* * f$. Thus $(f^* * f)^{\sim}(\lambda)$ is a holomorphic function in the strip S^0, by Remark 4.1.1. Arguing exactly as in the Theorem 4.2.5, if $\overline{Span\{^g f : g \in G\}} \neq L^p(X)$, then we can find a non-trivial h in $L^q(K \backslash G/K), 1 < q \leq 2$, such that $f * h = 0$. As a consequence, $f^* * f * h = 0$ and so $(f^* * f)^{\sim}(\lambda)\tilde{h}(\lambda) = 0\, a.e.$ But $(f^* * f)^{\sim}(\lambda)$ is a non-trivial holomorphic function and this forces $\tilde{h}(\lambda) = 0\, a.e.\ d\mu(\lambda)$. Hence $h = 0$, a contradiction.

An immediate corollary of the above theorem is the following which appears in [31] without a proof.

Corollary 4.2.7 *If E is a Borel subset of X of finite positive measure, then P_E is injective on $L^p(X), 1 \leq p \leq 2$.*

Proof : The case when $p = 1$, has been proved in [37]. The case when $1 < p \leq 2$, follows by duality, using the Theorem 4.2.6.

Although we restricted ourselves to a symmetric space of real rank 1, the Helgason-Fourier transform is defined in a similar way for symmetric spaces of arbitrary rank. Also, the Plancherel theorem holds for a symmetric space of arbitrary rank (see [22]). By using arguments similar to the ones employed above, we can show that the Theorem 4.2.6 and the Corollary 4.2.7 hold for a symmetric space of arbitrary rank. However in the next section, the fact that rank X is 1 is needed and so for uniformity in exposition we have imposed this condition.

Again, in view of the counter example in [35], Corollary 4.2.7 is the best possible result in general i.e. without assuming anything about the shape of E. Also note that unlike in the Euclidean case, we only need to assume that E is of finite measure; we *do not* need the fact that it is relatively compact.

4.3 Some results for L^p-spaces, $1 \leq p < 2$

In this section, to give a flavour of results analogous to Theorem 4.2.4, for L^p-spaces, $1 \leq p < 2$, we reformulate an old result from [38], which is proved for the case when $p = 1$, in terms of the Helgason-Fourier transform. We also give an analogue of this result for the case when $1 < p < 2$, which follows from slight modifications in the proof for the L^1-case.

The exact analogue of the classical Wiener-Tauberian theorem for $L^1(\boldsymbol{R})$, is no longer true if one considers the commutative Banach algebra $L^1(K\backslash G/K)$ of K-bi-invariant L^1-functions on G. (G, K are as in the Section 4.1.) In [14], Ehrenpreis and Mautner gave an example of $f \in L^1(K\backslash G/K)$ such that $\tilde{f}$ never vanishes on the maximal ideal space $\boldsymbol{R} + i[-1, 1]$ of $L^1(K\backslash G/K)$ but still the algebra generated by f in $L^1(K\backslash G/K)$ is *not* dense in $L^1(K\backslash G/K)$. However, when $G = SL(2,\mathbb{R})$, they were able to show that if for $f \in L^1(K\backslash G/K)$, $\tilde{f}$ is nowhere vanishing on the maximal ideal space of $L^1(K\backslash G/K)$ and "$\tilde{f}$ does not go to zero too fast at ∞", then the ideal generated by f in $L^1(K\backslash G/K)$ is indeed dense in $L^1(K\backslash G/K)$. Since then several modified versions of Wiener's theorem have been obtained, see for example [4], [33].

Coming back to the result in [38], we first describe some more basic facts about harmonic analysis of functions on X.

Now, for the harmonic analysis on functions on X i.e. functions on G which are right K-invariant, only the so called class-1 principal series representations $\{\pi\}_{\lambda\in\mathbb{C}}$ of G are relevant. These representations are all realised on $L^2(K/M)$ in the following way:

$$(\pi_\lambda(g)v)(\dot{k}) = e^{(i\lambda+1)\rho A(k^{-1}g)}\, v(u(k^{-1}g)^{-1}),\ g \in G,\ v \in L^2(K/M),\ \dot{k} = kM \in K/M.$$

Let v_0 be the constant function 1 on K/M. Then v_0 is the essentially unique K-fixed vector for the representation π_λ, $\lambda \in \mathbb{C}$ i.e. $\pi_\lambda(k)v_0 = v_0\, \forall\, k \in K$.

For $\lambda \in \mathbb{R}$, the corresponding principal series representation π_λ is unitary and irreducible. Therefore, for $f \in L^1(X)$, $\pi_\lambda(f) = \int_G f(g)\, \pi_{-\lambda}(g)\, dg$, defines a bounded operator on $L^2(K/M)$. Let v be any K-finite vector in $L^2(K/M)$ for the representation π_λ, $\lambda \in \mathbb{R}$, (i.e. $Span\{\pi_\lambda(k)v : k \in K\}$ is finite dimensional), then $\pi_\lambda(f)v = 0$ if the projection of v on $\mathbb{C}v_0$ is trivial. Thus in this case, the group theoretic Fourier transform $\lambda \to \pi_\lambda(f)$, is completely determined by its action on the vector v_0. Hence for $f \in L^1(X)$, the group-theoretic Fourier transform can be thought of as a function from $\mathbb{R}$ into $L^2(K/M)$ in the following way: $\lambda \to \pi_\lambda(f)v_0$.

To keep the exposition simple, we further restrict ourselves to functions on X

which are also left K-finite. *A function f on $L^p(X)$ is called left K-finite if the (left) K-translates of f span a finite dimensional subspace.* Then for a function $f \in L^1(X)$, which is also left K-finite, it can be seen easily that $\tilde{f}(\lambda, k)$ exists for $\lambda \in \boldsymbol{R}$ and $k \in K/M$. Furthermore, $\tilde{f}(\lambda, .) \in L^2(K/M)$. It is then immediate from the definition of $\pi_{-\lambda}$ that

$$(\pi_\lambda(f)v_0)(\dot{k}) = \tilde{f}(\lambda, k),\ \dot{k} = kM \in K/M.$$

If f is left K-invariant i.e. if f is a K-bi-invariant function in $L^1(G)$, then as remarked earlier, $\tilde{f}(\lambda, k)$ is independent of k and equals $\tilde{f}(\lambda)$, the spherical Fourier transform of f at λ. Therefore, in this case we have

$$\pi_\lambda(f)v_0 = \tilde{f}(\lambda)v_0.$$

These relations establish the connection between the Helgason-Fourier transform of f and the group-theoretic Fourier transform of f.

Again for $f \in L^p(X)$, $1 \leq p < 2$, *which is also left K-finite*, it can be seen easily that $\tilde{f}(\lambda, k)$ exists for all $k \in K/M$ and λ in the open strip $S_p^0 = \boldsymbol{R}+i(-\gamma_p, \gamma_p)$, where $\gamma_p = \frac{2}{p}-1$. In fact, for each fixed k, $\lambda \to \tilde{f}(\lambda, k)$ defines a holomorphic function in the open strip S_p^0.

We now reformulate the result in [38], in terms of the Helgason-Fourier transform. From now onwards, we take X to be the upper half plane, i.e., $X = \{x + iy : x, y \in \boldsymbol{R}, y > 0\}$, equipped with the Poincare metric (see [24]). Then X can be realised as G/K where $G = SL(2, \boldsymbol{R})$ and $K = SO(2)$. Then the Theorem 4.1 in [38] can be stated as follows:

Theorem 4.3.1 *Let $f \in L^1(X)$ be also left $K-$finite. Assume that for some $\epsilon > 0$ and for some $k \in K$, the function $\lambda \to \tilde{f}(\lambda, k)$ extends to a bounded holomorphic function in $S^\epsilon = \boldsymbol{R}+i[-1-\epsilon, 1+\epsilon]$. Further assume that for each*

$\lambda \in S^\epsilon$,

$$\|\tilde{f}(\lambda, .)\|_{L^2(K/M)} \geq M|e^{-\lambda^{2l}}|$$

for some positive constant M and positive integer l. Then Span $\{{}^g f : g \in G\}$ *is dense in* $L^1(X)$.

As remarked before, a slight modification in the proof of Theorem 4.3.1, gives the following:

Theorem 4.3.2 *Let* $f \in L^p(X)$, $1 < p < 2$, *be also left K−finite. Assume that for some* $\epsilon > 0$ *and for some* $k \in K$, *the function* $\lambda \to \tilde{f}(\lambda, k)$ *extends to a bounded holomorphic function in* $S_p^\epsilon = \mathbb{R} + i[-\gamma_p - \epsilon, \gamma_p + \epsilon]$. *Further assume that for each* $\lambda \in S_p^\epsilon$,

$$\|\tilde{f}(\lambda, .)\|_{L^2(K/M)} \geq M|e^{-\lambda^{2l}}|$$

for some positive constant M and positive integer l. Then Span $\{{}^g f : g \in G\}$ *is dense in* $L^p(X)$.

In view of the connection between the Helgason-Fourier transform and the group-theoretic Fourier transform, we have not given the proof of Theorem 4.3.1. Finally we should point out that much stronger results have been obtained recently by R. P. Sarkar (-see [33] and [34]). However, to keep the exposition simple and as a sample of results obtained in this direction we have described the somewhat weaker result from [38].

Concluding remarks

In this thesis, we have been mainly interested in L^1 and L^p analogues of the classical Wiener-Tauberian theorem for certain homogeneous spaces. While in Chapter 2 and Chapter 4 we have also been able to deal with the case $p \neq 1$, in Chapter 3 we restricted ourselves to $p = 1$. Hence to complete the picture, it would be interesting to obtain L^p-analogues of Theorem 2.2.1 for the Gelfand pair $(HM(n), U(n))$ i.e. for the $HM(n)$ action on H^n. Another direction is to consider a locally compact group G acting on itself by both left and right translations. While a lot of work has gone on in the case $p = 1$ for various classes of groups (-see for example [17], [26], [33], [46]), very little has been done in the case $p \neq 1$. For the special case of $G = M(2)$, motivated by Theorem 2.2.1, we have been able to obtain such a result. We briefly describe it below - the details can be found in [30].

For $G = M(2)$, we have the principal series representations $\{\pi_a\}_{a>0}$. Each π_a, $a > 0$ is an irreducible unitary representation of G realised on $H = L^2(S^1, \frac{1}{2\pi}d\theta)$, where $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}$. Further the representations π_a's are pair-wise inequivalent and the Plancherel measure of G is "supported" on this series of representations. (See [41] for details.) Let $f \in L^1(G)$. For each $a > 0$, define the operator-valued Fourier transform $\hat{f}$, of f, by:

$$\hat{f}(a) = \int_G f(g)\pi_a(g)dg.$$

Set $S = \{a > 0 : \hat{f}(a) = 0\}$ and $X = Span\{{}^{g_1}f^{g_2} : g_1, g_2 \in G\}$. Then we have the following result ([30]):

Let $f \in L^p(G) \cap L^1(G), 1 < p < \infty$.

1. For $1 < p < \frac{4}{3}$, X is dense in $L^p(G)$ if and only if S is empty.

2. For $\frac{4}{3} \leq p < 2$, X is dense in $L^p(G)$ if each point of S is an isolated point.

3. For $2 \leq p \leq 4$, X is dense in $L^p(G)$ if S is of zero measure (with respect to Lebesgue measure on $\boldsymbol{R}^+$).

4. For $4 < p < \infty$, X is dense in $L^p(G)$ if and only if S is nowhere dense.

Note that the result is for $p > 1$. For the case when $p = 1$, the reader is referred to [17], [45] and [46].

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Theorem 4.3.1 and 4.3.2 should read as follows:

Theorem 4.3.1 Let $f \in L^1(X)$ be also left K−finite. Assume that for some $\epsilon > 0$ and for each $k \in K$, the function $\lambda \rightarrow \tilde{f}(\lambda, k)$ extends to a holomorphic function in $S^{\epsilon,0} = \boldsymbol{R} + i(-1-\epsilon, 1+\epsilon)$. Further assume that there are positive constants M_1, M_2 and a positive integer l such that the following "boundedness" and "not too rapid decay" conditions are satisfied:

$$sup_{\lambda \in S^{\epsilon,0}} \left| \int_K \tilde{f}(\lambda, k)\chi(k)\, dk \right| \leq M_1,$$

for all characters χ of K (i.e. $\chi \in \hat{K}$) and

$$\|\tilde{f}(\lambda, \cdot)\|_{L^2(K/M)} \geq M_2 \, |e^{-\lambda^{2l}}|, \; \forall \lambda \in S^{\epsilon,0}.$$

Then $Span\,\{{}^g f : g \in G\}$ is dense in $L^1(X)$.

Theorem 4.3.2 Let $f \in L^p(X)$, $1 < p < 2$, be also left K−finite. Assume that there exists $\epsilon > 0$ such that for each $k \in K$, the function $\lambda \rightarrow \tilde{f}(\lambda, k)$ extends to a holomorphic function in $S_p^{\epsilon,0} = \boldsymbol{R} + i(-\gamma_p - \epsilon, \gamma_p + \epsilon)$. Further assume that there are positive constants M_1, M_2 and a positive integer l such that the following conditions are satisfied:

$$sup_{\lambda \in S_p^{\epsilon,0}} \left| \int_K \tilde{f}(\lambda, k)\chi(k)\, dk \right| \leq M_1,$$

for all characters χ of K and

$$\|\tilde{f}(\lambda, \cdot)\|_{L^2(K/M)} \geq M_2 \, |e^{-\lambda^{2l}}|, \; \forall \lambda \in S_p^{\epsilon,0}.$$

Then $Span\,\{{}^g f : g \in G\}$ is dense in $L^p(X)$.