ON THE MOST EFFICIENT DESIGNS IN WEIGHING

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There are p objects to be weighed and n weighments can be made. In case there is a bias in the scales this may be considered as an additional weight which is used in all weighments. The observational equations are of the form $Y = TA'$, where $Y = (y_1\ y_2 \ldots y_p)$ is the row matrix of recorded weights, $T = (\tau_1, \tau_2, \ldots \tau_p)$ is the row matrix of weights to be determined and A is a matrix with n rows and p columns with elements of the form 0, +1 or −1 (0, +1, and −1 correspond to the cases where the object is not used, or is put in the first pan or second pan). If we require that all weights including bias, if any, should be determined with equal accuracy and that this should be the maximum possible, it follows that the matirx A should satisfy the equation $A'A = nI$ if the dispersion matrix of Y is of the form $\sigma^2 I$. This equation is satisfied if the elements of A are either +1 or −1 and any two of its rows are orthogonal.

We may take the first row to be (1, 1, ...1). If the second row contains k_1 elements with +1 and k_2 with −1, then the orthogonality condition reduces to $k_1 = k_2 = k$ whence n is of the form $2k$. If a third row exists which is orthogonal to the first two then we can by similar algebra show that n is of the form 2^2k and in this case one more row could be constructed by multiplying the corresponding elements in the second and third rows so that the four rows are mutually orthogonal. By introducing variables giving the number of positives and negatives common to a newly introduced row and the other rows and solving the system of equations due to orthogonality we arrive at the following solution. *If $n = 2^r(2m+1)$, then there can be at the most 2^r rows with elements of the form +1 or −1, which are mutually orthogonal to one another.* When n is a full power of 2 we have n such rows. The above method gives a procedure for getting at the matrix A' in all cases. The number of objects p including bias, for the best possible design under consideration is less than or equal to 2^r when the number of weighments n is of the form $2^r(2m+1)$. In this case the weight of each object including bias is estimable with the variance σ^2/n and the estimate of σ^2 can be based on $(n-p)$ degrees of freedom obtained from the sum of squares of residuals of the normal equations. Some designs of special type have been considered by Hotelling (1944) and Kishen (1945).

It is interesting to note that in the case the total weight of the objects is known (a contingency which may arise when an object of known weight breaks into peices and their wieghts are to be determined) the weighment corresponding to the first row need not be made while others are necessary for the best possible design. The methods of deriving the normal equations, variances and covariances of estimates in the case of observational equations with restrictions on the parameters have been discussed by the author in (Rao : 1945).

Another important observation is that all the combinations of weights may not have the same error in weighing. This however requires experimental verification.

References

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