NOTES ON TESTING OF COMPOSITE HYPOTHESES—II

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This paper is a continuation of an earlier paper by the author (1948) on the same subject and under the same title, and sets forth developments along certain lines of investigation indicated therein. The present paper consists of four sections of which section 1 gives a resumo (which is supposed to offer a convenient starting point) of what was stated in the earlier paper about a sufficient set of conditions for the availability of valid critical regions (or what Neyman calls similar regions); section 2 is concerned with the discovery of a sufficient set of conditions for the availability, as amongst these valid tests, of a most powerful test of any specific composite hypothesis with respect to a specific composite alternative, while section 3 attempts to get a sufficient set of conditions under which the most powerful test would happen to be uniformly most powerful, if not in the original sense of covering all alternatives, at any rate in the sense of covering a continuous class of alternatives, and lastly section 4 discusses, by way of illustration, certain simple well known concrete cases. Not merely the problems posed but also the solutions discussed in section 4 are very well known. Here the whole matter is put in the modified setting introduced in sections 1–3, thus serving, as just observed, the purpose of illustration of the principles in sections 2–3.

1. Availability of valid critical regions and mechanism of their generation

With n stochastic variables $(x_1, x_2, \ldots, x_n)$, a law of probability density of the form $\phi(x_1, \ldots, x_n; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})$ and a composite hypothesis $H(\theta_{k+1} = \theta_{k+1}^\circ, \theta_{k+2} = \theta_{k+2}^\circ, \ldots, \theta_{k+s} = \theta_{k+s}^\circ)$ the problem, as is well known, is to find a critical region $w(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)$—whose location in the sample space might depend upon the non-free parameters $(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)$ under hypothesis but must be independent of the free parameters $(\theta_1, \ldots, \theta_k)$ and which is accordingly put in the form $w(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)$ to bring into prominence this respective dependence and independence—such that

$$\int_{w(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)} \phi(x_1, \ldots, x_n; \theta_1, \ldots, \theta_k; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) dv = \alpha \qquad \ldots(1)$$

for all values of $(\theta_1, \ldots, \theta_k)$ and of $(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)$. Here α is supposed to be a preassigned level of significance. It was observed in the earlier paper that a sufficient set of conditions for this being available would be given by

$$\phi \equiv \phi_1(F_1, \ldots, F_m; \theta_1, \theta_2, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})\, \phi_2(x_1, \ldots, x_n; \theta_{k+1}, \ldots, \theta_{k+s}) \qquad \ldots(1.1)$$

where

$$F_i \equiv F_i(x_1, \ldots, x_n; \theta_{k+1}, \ldots, \theta_{k+s}) \qquad (i = 1, 2, \ldots, m) \qquad \ldots(1.2)$$

It may be noted that we should try to express (1.1) in terms of as few functions F_i as possible, consistently of course with the condition (1.2). What might happen if we did not, would be discussed in the next paper. This requirement of an irreducible minimum will be understood to be always there throughout the present paper.

Here m may be $>$, or $=$, or $<k$ but must be $<n$ and usually $<<n$. Under the hypothesis to be tested θ_{k+j}'s in (1.1) and (1.2) would, of course, be replaced by θ_{k+j}°'s $(j=1, 2, \ldots, s)$. Under the conditions (1.1) and (1.2) the mechanism for the generation of valid tests, as explained in the earlier paper, would be as follows.

Consider the m-fold family of subsurfaces $S(\mu_1, \ldots, \mu_m)$ generated by the intersection of the surface

$$F_i^{\circ} \equiv F_i(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}) = \mu_i \qquad (i=1, 2, \ldots, m) \quad ..(1.3)$$

This family $S(\mu_1, \ldots, \mu_m)$, as noted earlier, possesses two important properties:— (a) *The location (in the sample space) of any member of the family might depend upon* $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ *but is independent of the free parameters* $(\theta_1, \ldots, \theta_k)$. This, of course, would be evident from (1.3). (b) If from any shell bounded by $S(\mu_1, \ldots, \mu_m)$ and $S(\mu_1+d\mu_1, \ldots, \mu_m+d\mu_m)$ we cut off a portion whose mass (for any set of values of $\theta_1, \ldots, \theta_k; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}$) is α times the mass of the total shell then this means a constraint :

$$\int_{\text{Portion of the shell}} \phi_2(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})dv = \alpha \int_{\text{Total shell}} \phi_2(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})dv \quad ..(1.4)$$

This constraint being in every way independent of the free parameters $(\theta_1, \ldots, \theta_k)$, we can, if we like, cut off from the shell a portion which satisfies (1.4) and whose relative location within the shell is independent of the free parameters $(\theta_1, \ldots, \theta_k)$ but might depend upon the non-free parameters $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$. This constraint, as is evident, would not in any way define the portion vis-a-vis the shell but would merely impose a restriction on its relative location and size, leaving otherwise a good deal of latitude. *If now, while keeping* $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ *fixed, we vary the free parameters* $(\theta_1, \ldots, \theta_k)$ *the mass of the portion would continue to be equal to* α *times the mass of the total shell (both masses varying in an absolute sense, but maintaining the same ratio), the relative location of the portion within the shell remaining, of course, by definition, unchanged. If, however, we change* $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ *as well, the proportionality of mass would persist (if w were chosen in the proper manner) but the relative locations would change.*

These principles we would call, for brevity, (i) *the principle of invariance of locations, and* (ii) *the principle of invariance of mass proportionality of a portion of any shell, under change of the free parameters* $(\theta_1, \ldots, \theta_k)$. In any shell such portions are available in an infinite number of ways. Take any such portion from one shell, another from another shell and so on. Piecing together such portions from all possible shells we have a critical region whose location in the sample space, while independent of the free parameters $(\theta_1, \ldots, \theta_k)$, might depend upon the non-free parameters $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$, but whose mass is α for all values of the parameters, free or non-free. In other words, this would be a valid critical region of size α and it is evident that under this mechanism such regions could be generated in an infinite number of ways, that is, there is aninfinity of alternative valid critical regions of size α. It can be shown that if (1.1)–(1.2) hold, then the mechanism of generation of valid tests indicated in and after (1.3) would, under certain mild additional restrictions, exhaust all possibilities in respect of valid tests, that is, there could be no valid test which could not be generated by the mechanism indicated. How this can be shown and what those mild additional restrictions might be, need not, however, be discussed here. This property might be called (iii) *the principle of exhaustiveness with regard to valid tests.*

Going back to condition (1.4) we note that while under this we can, if we like, cut off a portion whose relative location within the shell is independent of the free parameters $(\theta_1, \ldots, \theta_k)$ it is not said that we could not select a portion whose relative location depended upon $(\theta_1, \ldots, \theta_k)$. We could, if we liked; but this would defeat our purpose in the sense that it would not provide us with a mechanism for generation of valid critical regions.

There is another point to be noted. Considering (1.1)–(1.2), we could have denoted ϕ_2 by F_{m+1} and added to (1.3) a further surface

$$F_{m+1}^{\circ} \equiv F_{m+1}(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}) = \mu_{m+1} \qquad \ldots(1.5)$$

such that the intersection of (1.3) together with (1.5) would define $(m+1)$-fold family of sub-surfaces $S(\mu_1, \ldots, \mu_m, \mu_{m+1})$ providing us with a mechanism for generation of valid tests, similar to what we have just considered. It can be shown, and will actually be demonstrated in a later paper, that the power of such valid tests with respect to alternative hypothesis would be less than that of the valid tests earlier considered.

2. Availability as amongst the valid tests, of a most powerful critical region with regard to a specific alternative

As in the earlier paper, attach to the hypothesis $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ a pseudo-hypothesis $(\theta_1^{\circ}, \ldots, \theta_k^{\circ})$ and to the alternative $(\theta_{k+1}, \ldots, \theta_{k+s})$ a pseudo-alternative $(\theta_1, \ldots, \theta_k)$. The problem now is to choose from amongst the valid regions w_j's $(j = 1, 2, \ldots)$ of size α a region w whose location in the sample space might depend upon the non-free parameters $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ and also perhaps on $(\theta_{k+1}, \ldots, \theta_{k+s})$ but must be independent of the free parameters $(\theta_1^{\circ}, \ldots, \theta_k^{\circ})$, and also of $(\theta_1, \ldots, \theta_k)$, and which would satisfy the additional condition of being most powerful. Denote $\phi(x_1, \ldots, x_n; \theta_1^{\circ}, \ldots, \theta_k^{\circ}; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ by ϕ° and $\phi(x_1, \ldots, x_n; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})$ by ϕ. In symbols the problem is to choose from amongst w_j's $(j = 1, 2, \ldots)$ depending upon $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ and thus being denoted by $w_j(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ a w which might depend upon both $(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})$ and $(\theta_{k+1}, \ldots, \theta_{k+s})$ but must be independent of $(\theta_1^{\circ}, \ldots, \theta_k^{\circ})$ and $(\theta_1, \ldots, \theta_k)$ and might hence be denoted by $w(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}; \theta_{k+1}, \ldots, \theta_{k+s})$ such that the following conditions are to be satisfied

$$\int_{w(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}; \theta_{k+1}, \ldots, \theta_{k+s})} \phi^{\circ} dv = \alpha \qquad \ldots(2)$$

and

$$\int_{w(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}; \theta_{k+1}, \ldots, \theta_{k+s})} \phi dx \geqslant \int_{w_j(\theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})} \phi dv \qquad \ldots(2.1)$$

for all w_j's satisfying (2).

This is to hold no matter what the pseudo-hypothesis $(\theta_1^{\circ}, \ldots, \theta_k^{\circ})$ and the pseudo-alternative $(\theta_1, \ldots, \theta_k)$ might be. Suppose, starting from the family of subsurfaces $S(\mu_1, \ldots, \mu_m)$ of (1.3), we proceed to construct such a region by considering the intersection of

$$\mu_i \leqslant F_i(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}) \leqslant \mu_i + d\mu_i \qquad (i = 1, 2, \ldots, m) \quad \ldots(2.11)$$

with

$$\phi_1(F_1, \ldots, F_m; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})\, \phi_2(x_1, \ldots, x_n; \theta_{k+1}, \ldots, \theta_{k+s})$$
$$\geqslant \lambda \phi_1(F_1, \ldots, F_m; \theta_1^{\circ}, \ldots, \theta_k^{\circ}; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ})\, \phi_2(x_1, \ldots, x_n; \theta_{k+1}^{\circ}, \ldots, \theta_{k+s}^{\circ}) \quad \ldots(2.2)$$

where λ is so chosen that

$$\underset{\text{over intersection of (2.11) and (2.2)}}{\int \phi^{*} dv} = \alpha \underset{\text{over (2.2)}}{\int \phi^{\circ} dv} \qquad ..(2.21)$$

Using the symbol Λ to denote intersection and remembering the forms of ϕ and ϕ°, we easily see that (2.11) Λ (2.2) is equivalent to (2.11) Λ (2.22) where (2.22) is given by

$$\phi_1(F_1, ..., F_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})\, \phi_2(x_1, ..., x_n; \theta_{k+1}, ..., \theta_{k+s})$$
$$\geqslant \lambda \phi_1(\mu_1, ..., \mu_m; \theta_1^{\circ}, ..., \theta_k^{\circ}; \theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ})\, \phi_2(x_1, ..., x_n; \theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ}) \quad ..(2.22)$$

or simply by

$$\phi_1 \phi_2 \geqslant \nu \phi_2^{\circ} \qquad ..(2.22)$$

where ν is to be so chosen that

$$\underset{\text{over (2.11) } \Lambda \text{ (2.22)}}{\int \phi_2^{*} dv} = \alpha \underset{\text{over (2.11)}}{\int \phi_2^{*} dv} \qquad ..(2.221)$$

The trouble is that (2.11) Λ (2.22) might not lead to a region whose location is independent of the free parameters—no matter whether they are put forward as the pseudo-hypothesis $(\theta_1^{\circ}, ..., \theta_k^{\circ})$ or pseudo-alternative $(\theta_1, ..., \theta_k)$. If the location of (2.11) Λ (2.22) happens to be so independent then in that case ν will evidently be of the form

$$\nu \equiv \nu(\alpha; \mu_1, ..., \mu_m; \theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ}; \theta_{k+1}, ..., \theta_{k+s}) \qquad ..(2.23)$$

and a valid and most powerful critical region $w(\theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ}; \theta_{k+1}, ..., \theta_{k+s})$, *i.e.* one which satisfies conditions (2)–(2.1), would be bounded by a surface obtained by eliminating $(\mu_1, ..., \mu_m)$ between the $(m+1)$ equations

$$F_i^{\circ} \equiv \mu_i \qquad (i = 1, 2, ..., m) \quad ..(2.24)$$

and

$$\phi_1 \phi_2 = \nu(\alpha; \mu_1, ..., \mu_m; \theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ}; \theta_{k+1}, ..., \theta_{k+s})\, \phi_2^{\circ} \qquad ..(2.25)$$

The region itself, both the *inside* and the *boundary*, would be obtained by eliminating $(\mu_1, \mu_2, ..., \mu_m)$ between (2.24) and (2.26) where (2.26) is given by

$$\phi_1 \phi_2 \geqslant \nu(\alpha; \mu_1, ..., \mu_m; \theta_{k+1}^{\circ}, ..., \theta_{k+s}^{\circ}; \theta_{k+1}, ..., \theta_{k+s})\phi_2^{\circ} \qquad ..(2.26)$$

But the success of the whole scheme depends upon the possibility of (2.11) Λ (2.22) being located independently of the free parameters. It is evident that a ϕ subject to the general restrictions (1.1)–(1.2) would not ensure this. The question is, what further restrictions on ϕ (in addition to those indicated in (1.1)–(1.2)) would be needed to ensure this. The following different alternative sets of sufficient conditions have been obtained, by way of further restrictions on ϕ, in addition to those already indicated in (1.1)–(1.2).

(i) $$F_i \equiv F_i(x_1, ..., x_n) \qquad (i = 1, 2, ..., m) \quad ..(2.3)$$

with an additional condition (2.3a) to be presently specified.

It is clear that in this case (2.11) and (2.2) would be replaced respectively by

$$\mu_i \leqslant F_i(x_1, ..., x_n) \leqslant \mu_i + d\mu_i \qquad (i = 1, 2, ..., m) \quad ..(2.31)$$

$$\phi_1(F_1, ..., F_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})\phi_2 \geqslant \lambda\phi_1(F_1, ..., F_m; \theta_1°, ..., \theta_k°; \theta_{k+1}°, ..., \theta_{k+s}°)\phi_2° \quad ..(2.32)$$

and hence (2.22) would be replaced by

$$\phi_2 \geqslant \nu\phi_2° \text{ or } \leqslant \nu'\phi_2° \quad ..(2.33)$$

In (2.33) we have to take the first or the second inequality sign according as $\phi_1(\mu_1, ..., \mu_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})$ on the left hand side of (2.32), after substitutions of $(\mu_1, ..., \mu_m)$ for $(F_1, ..., F_m)$ becomes positive or negative. Here is a further assumption, namely, that whether $\phi_1(\mu_1, ..., \mu_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})$ will be positive or negative might depend upon the values of $(\mu_1, ..., \mu_m)$ and $(\theta_{k+1}, ..., \theta_{k+s})$ but must be independent of $(\theta_1, ..., \theta_k)$.

This is the additional assumption just mentioned. ..(2.3a)

Hence ν or ν' is chosen so as to satisfy

$$\underset{\text{over (2.31)}\,\Lambda\,\text{(2.33)}}{\int \phi_2° dv} = \alpha \underset{\text{over (2.31)}}{\int \phi_2° dv} \quad ..(2{\cdot}34)$$

It is clear that (2.31) Λ (2.33) subject to (2.34) defines a region whose location is independent of the free parameters, and would provide us with a mechanism for generation of the most powerful and valid critical region, which would be exactly similar to what has been indicated in (2.24)–(2.26).

(ii) A second alternative would be

$$\left.\begin{aligned} &F_1 \equiv F_1(x_1, ..., x_n; \theta_{k+1}, ..., \theta_{k+s}) \\ &F_i \equiv F_i(x_1, ..., x_n) \qquad (i = 2, ..., m) \\ &\phi_2 \equiv \phi_2(x_1, ..., x_n) \end{aligned}\right\} \quad ..(2.4)$$

with an additional assumption (2.4a) to be presently specified.

In this case, corresponding to (2.31) and (2.32) we would have respectively

$$\left.\begin{aligned} &\mu_1 \leqslant F_1° \equiv F_1(x_1, ..., x_n; \theta_{k+1}°, ..., \theta_{k+s}°) \leqslant \mu_1 + d\mu_1 \\ &\mu_i \leqslant F_i \leqslant \mu_i + d\mu_i \qquad (i = 2, 3, ..., m) \end{aligned}\right\} \quad ..(2.41)$$

and

$$\phi_1(F_1, F_2, ..., F_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})\phi_2$$
$$\geqslant \lambda\, \phi_1(F_1°, F_2, ..., F_m; \theta_1°, ..., \theta_k°; \theta_{k+1}°, ..., \theta_{k+s}°)\phi_2 \quad ..(2.42)$$

Now (2.41) Λ (2.42) is equivalent to (2.41) Λ (2.43) where (2.43) is given by

$$\phi_1(F_1, \mu_2, ..., \mu_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})$$
$$\geqslant \lambda\, \phi_1(\mu_1, \mu_2, ..., \mu_m; \theta_1°, ..., \theta_k°; \theta_{k+1}°, ..., \theta_{k+s}°)$$

that is, by $\quad F_1 \geqslant \nu$ or $\leqslant \nu'$ ..(2.43)

This is obtained on two further simplifying assumptions.
The first is that the equation

$$\phi_1(F_1, \mu_2, ..., \mu_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s})\, \phi_2$$
$$= \lambda\phi_1(\mu_1, \mu_2, ..., \mu_m; \theta_1°, ..., \theta_k°; \theta_{k+1}°, ..., \theta_{k+s}°)\phi_2 \quad ..(2.431)$$

give only one root for F_1 which is :

$$F_1 = \nu \qquad ..(2.432)$$

and for the critical region we take that one of the alternative regions $F_1 \geq \nu$ or $\leq \nu'$ for which the left hand side of (2.431) is greater than the right hand side.

Here the second assumption is made, namely, that whether the left hand side of (2.431) will be greater than the right hand side might depend upon $(\mu_1, ..., \mu_m)$, $(\theta_{k+1}^\circ, ..., \theta_{k+s}^\circ)$ and $(\theta_{k+1}, ..., \theta_{k+s})$ but must be independent of $(\theta_1^\circ, ..., \theta_k^\circ)$ and $(\theta_1, ..., \theta_k)$ ..(2.433)

This (2.433) together with (2.432) is the additional assumption. ..(2.4a)

We shall presently consider a slightly more general case obtained by relaxing the condition of one real root given by (2.432) but retaining a condition similar to (2.433). Meanwhile let us obtain ν or ν'.
Here ν or ν' is so chosen as to satisfy

$$\underset{\text{over } (2.41)\,\Lambda\,(2.43)}{\int \phi_2^\circ dv} = \alpha \underset{\text{over } (2.41)}{\int \phi_2^\circ dv} \qquad ..(2.44)$$

It is clear that (2.41) Λ (2.43) subject to (2.44) defines a valid critical region and provides us with a mechanism for generation of the most powerful valid test, exactly similar to what was noted in the previous case.

Let us go back to equation (2.431). Suppose it has a number of real roots of F_1 to be called ν_1, ν_2, etc. These ν's *prima facie* would appear to be explicit functions of the

quantities : $(\lambda; \mu_1, ..., \mu_m; \theta_{k+1}^\circ, ..., \theta_{k+s}^\circ; \theta_{k+1}, ..., \theta_{k+s}; \theta_1^\circ, ..., \theta_k^\circ; \theta_1, ..., \theta_k)$,

but let us make the more restrictive assumption that the free parameters would not be involved. This means that

$$\nu_j \equiv \nu_j(\lambda;\ \mu_1, ..., \mu_m;\ \theta_{k+1}^\circ, ..., \theta_{k+s}^\circ;\ \theta_{k+1}, ..., \theta_{k+s}) \qquad (j = 1, 2, ...) \quad ..(2.441)$$

where the functional forms of the right hand side of (2.441) are supposed to be definitely known. We note that we have a number of cuts on the F_1-line breaking it up into segments. In these segments we should have alternately the left hand side of (2.431) greater or less than the right hand side. Collect those segments on the F_1-line for which the left hand side of (2.431) is greater than the right hand side. Call this assembly F_1^c. Then this F_1^c is delimited by $\nu_1, \nu_2, ...$, that is, ultimately by $(\lambda; \mu_1, ..., \mu_m; \theta_{k+1}^\circ, ..., \theta_{k+s}^\circ; \theta_{k+1}, ..., \theta_{k+s})$; hence it might be written as

$$F_1^c \equiv F_1^c(\lambda; \mu_1, ..., \mu_m; \theta_{k+1}^\circ, ..., \theta_{k+s}^\circ; \theta_{k+1}, ..., \theta_{k+s}) \qquad ..(2.45)$$

On the shell (2.41) the most powerful region would be given by (2.41) Λ (2.45) where λ is so chosen as to satisfy

$$\underset{\text{over } (2.41)\,\Lambda\,(2.45)}{\int \phi_2 dv} = \alpha \underset{\text{over } (2.41)}{\int \phi_2 dv} \qquad ..(2.46)$$

The generation of the most powerful valid critical region from this would follow along lines indicated earlier.

(iii) A third alternative would be

$$\left.\begin{aligned} F_1 &= F_1(x_1, \ldots, x_n; \theta_{k+1}, \ldots, \theta_{k+s}) \\ F_i &\equiv F_i(x_1, \ldots, x_n) \quad (i = 2, 3, \ldots, m) \\ \phi_2 &\equiv \phi_2(F_1; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s}) \end{aligned}\right\} \quad ..(2.5)$$

with an additional assumption (2.5a) similar to (2.4a).

Corresponding to (2.41) and (2.42) we would now have respectively

$$\left.\begin{aligned} \mu_1 \leqslant F_1^\circ &\equiv F_1(x_1, \ldots, x_n; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \leqslant \mu_1 + d\mu_1 \\ \mu_i \leqslant F_i &\leqslant \mu_i + d\mu_i \quad (i = 2, 3, \ldots, m) \end{aligned}\right\} \quad ..(2.51)$$

(which is the same as (2.41), and

$$\begin{aligned} &\phi_1(F_1, F_2, \ldots, F_m; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})\phi_2 \\ &\geqslant \lambda\phi_1(F_1, F_2, \ldots, F_m; \theta_1^\circ, \ldots, \theta_k^\circ; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)\phi_2^\circ \end{aligned} \quad ..(2.52)$$

Now (2.51) Λ (2.52) is equivalent to (2.51) Λ (2.53) where (2.53) is given by

$$\begin{aligned} &\phi_1(F_1, \mu_2, \ldots, \mu_m; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})\ \phi_2(F_1; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s}) \\ &\geqslant \lambda\ \phi_1(\mu_1, \mu_2, \ldots, \mu_m; \theta_1^\circ, \ldots, \theta_k^\circ; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)\ \phi_2(\mu_1; \theta_1^\circ, \ldots, \theta_k^\circ; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \end{aligned}$$

that is, by $$F_1 \geqslant \nu \text{ or } \leqslant \nu' \quad ..(2.53)$$

Here again as in the previous case we make in the first instance two simplifying assumptions. The first is that the equation

$$\begin{aligned} &\phi_1(F_1, \mu_2, \ldots, \mu_m; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s})\phi_2 \\ &\quad = \lambda\phi_1(\mu_1, \mu_2, \ldots, \mu_m; \theta_1^\circ, \ldots, \theta_k^\circ; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)\phi_2 \end{aligned} \quad ..(2.531)$$

gives only real root for F_1 which is

$$F_1 = \nu, \quad ..(2.532)$$

and we take that one of $F_1 \geqslant \nu$ or $\leqslant \nu'$ for which the left hand side of (2.531) is greater than the right hand side. The second assumption is made that whether the left hand side of (2.531) will be greater than the right hand side might depend upon $(\mu_1, \ldots, \mu_m)$, $(\theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)$ and $(\theta_{k+1}, \ldots, \theta_{k+s})$ but must be independent of $(\theta_1^\circ, \ldots, \theta_k^\circ)$ and $(\theta_1, \ldots, \theta_k)$. ..(2.533)

This (2.533) together with (2.532) constitute the additional assumption. ..(2.5a)

The more general case with a number of roots can be treated in a manner exactly similar to what was discussed at the end of case (ii).

As regards ν or ν' each is so chosen as to satisfy

$$\int\limits_{\text{over (2.51) } \Lambda \text{ (2.53)}} dv = \alpha \int\limits_{\text{over (2.51)}} dv \quad ..(2.54)$$

Exactly as in the previous cases so also here it is clear that (2.51) Λ (2.53) subject to (2.54) defines a region independent of the free parameters and enables us to generate the most powerful valid test.

In a sense it might be said that (2.3), (2.4) and (2.5) are all really comprehended in a condition of the general form

$$\phi \equiv \phi_1(F_1, F_2, ..., F_m; \theta_1, ..., \theta_k; \theta_{k+1}, ..., \theta_{k+s}) \qquad ..(2.6)$$

where

$$\left.\begin{array}{l} F_1 \equiv F_1(x_1, ..., x_n; \theta_{k+1}, ..., \theta_{k+s}) \\ F_i \equiv F_i(x_1, ..., x_n) \qquad (i = 2, 3, ..., m) \end{array}\right\} \qquad ..(2.61)$$

But it seems that for practical purposes it might be more convenient to split up into three separate cases (2.3), (2.4) and (2.5) even at some expense of formal elegance.

In the earlier paper (Roy, 1948), for any general m only (2.3) was considered, while in the special case of $m = 1$, (2.4) and (2.5) were also considered.

In practice, however, that is, if we look to the usual concrete tests of composite hypotheses, it will be noted that we have sometimes the situations given by (2.4) or (2.5) with additional condition that $F_1^\circ \equiv F_1(x_1, ..., x_n; \theta_{k+1}^\circ, ..., \theta_{k+s}^\circ)$ becomes independent of $(x_1, ..., x_n)$ that is, becomes a constant, but $F_1 \equiv F_1(x_1, ..., x_n; \theta_{k+1}, ..., \theta_{k+s})$ depends upon $(x_1, ..., x_n)$. This, of course, is as much a restriction on F_1 as on the particular hypothesis $(\theta_{k+1}^\circ, ..., \theta_{k+s}^\circ)$ to be tested. In fact, if we alter the hypothesis this independence would no longer hold. We note that when F_1° is independent of $(x_1, ..., x_n)$ we would have corresponding to (2.41) or (2.51) merely

$$\mu_i < F_i(x_1, ..., x_n) < \mu_i + d\mu_i \qquad (i = 2, 3, ..., m) \qquad ..(2.7)$$

That is, instead of an m-fold family of subsurfaces $S(\mu_1, ..., \mu_m)$ we would have an $(m-1)$-fold family of subsurfaces $S(\mu_2, ..., \mu_m)$. The rest will be exactly similar to what we had in the previous cases. It will be noted that in all the situations considered in this section the most powerful test will, in general, depend on the alternative $(\theta_{k+1}, ..., \theta_{k+s})$, besides depending, as it would always do, upon the hypothesis $(\theta_{k+1}^\circ, ..., \theta_{k+s}^\circ)$. The question is: under what circumstances, that is, under what additional restrictions on ϕ will that most powerful test be independent of the alternative $(\theta_{k+1}, ..., \theta_{k+s})$ or at any rate would be the same for one continuous class of $(\theta_{k+1}, ..., \theta_{k+s})$, the same (but differently defined from the previous case) for another class and so on. These different most powerful tests, each defined for one continuous class of $(\theta_{k+1}, ..., \theta_{k+s})$, might be called uniformly most powerful tests in a general sense. The classes for $(\theta_{k+1}, ..., \theta_{k+s})$ are to be defined vis-a-vis the null hypothesis $(\theta_{k+1}^\circ, ..., \theta_{k+s}^\circ)$.

3. Availability of uniformly most powerful valid tests (in the general sense)

To fix our ideas we shall first consider a simple hypothesis and with only one parameter. Let θ be the parameter and let the hypothesis be $\theta = \theta^\circ$ and the alternative $\theta \neq \theta^\circ$. Also, as in the previous case, let

$$\left.\begin{array}{l} \phi \equiv \phi(x_1, ..., x_n; \theta) \\ \phi^\circ \equiv \phi(x_1, ..., x_n; \theta^\circ) \end{array}\right\} \qquad ..(3)$$

Alternative sets of sufficient conditions for the existence (and the mechanism of construction) of a uniformly most powerful test (in the general sense) would in this case appear to be as follows:

(i) The first alternative is: $\phi \equiv \phi_1(F, \theta)\, \phi_2(x_1, ..., x_n)$ ..(3.1) where $$F \equiv F(x_1, \ldots, x_n) \quad ..(3.11)$$

with an additional condition (3.1a) to be presently specified. Now (3.1) taken with (3.11) implies that there is a sufficient statistic for θ and that is F. Consider the most powerful test of $\theta°$ with respect to θ which is

$$\phi \geqslant \mu\phi° \quad ..(3.12)$$

where μ is so chosen that

$$\int_{\text{over (3.12)}} \phi° dv = \alpha \quad ..(3.13)$$

Now, in the present case, remembering the form of ϕ, we note that $\phi = \mu\phi°$ implies $F = \nu_1, \nu_2, \ldots$ Assume that there is only one real root ν (at least one root being always available in view of the restrictions on ϕ and $\phi°$ and also the conditions (3.12)–(3.13).

This is the additional condition mentioned earlier. ..(3.1a)

This means that corresponding to (3.12)–(3.13) we have

either $F \geqslant \nu$ or $\leqslant \nu'$ (depending upon the value of θ vis-a-vis $\theta°$). ..(3.14)

In either case ν or ν' is to be so chosen that

$$\int_{\text{over (3.14)}} \phi° dv = \alpha \quad ..(3.15)$$

But this, while determining ν or ν' in the forms $\nu(\alpha; \theta°)$ or $\nu'(\alpha; \theta°)$, shows that ν or ν' is *nearly* independent of θ. In other words, it is possible here to obtain a function $f_1(\theta, \theta°)$ such that (α) for all values of θ subject to $f_1(\theta, \theta°) > 0$, we have a common most powerful test which is

either of the regions: $F \geqslant \nu(\alpha; \theta°)$ or $F \leqslant \nu(\alpha; \theta°)$ ..(3.16)

(β) for all values of θ subject to $f_1(\theta, \theta°) < 0$ we have a common most powerful test which is just that one of the two alternatives $F \geqslant \nu(\alpha; \theta°)$ or $\leqslant \nu'(\alpha; \theta°)$ which does not occur in (3.16). ..(3.17)

The alternative regions might each be called a one-sided uniformly most powerful test. In many of the well known cases $f_1(\theta, \theta°) \equiv \theta - \theta°$ (simply), that is, for all $\theta > \theta°$ we have one uniformly most powerful test and for all $\theta < \theta°$ we have another.

(ii) A second alternative would be

$$\phi \equiv \phi(\phi°; \theta; \theta°) \quad ..(3.2)$$

with an additional condition (3.2a) to be presently specified.

The most powerful test of $\theta°$ with respect to θ is

$$\phi \geqslant \mu\phi° \quad ..(3.21)$$

where μ is chosen so as to satisfy

$$\int_{\text{over (3.21)}} \phi° dv = \alpha \quad ..(3.22)$$

In this case remembering (3.2), we note that $\phi = \mu\phi^{\circ}$ would imply $\phi^{\circ} = \nu_1, \nu_2$, etc. Assume that there is only one real root ν (as before there will be at least one in any case).

This is the additional condition mentioned earlier. ..(3.2a)

As before, corresponding to (3.2) we have either $\phi^{\circ} \geqslant \nu$ or $\leqslant \nu'$ (depending on the value of θ vis-a-vis θ°). ..(3.23)

In either case, as before, ν and ν' are to be so chosen that

$$\int_{\text{over } (3.23)} \phi^{\circ} dv = \alpha \qquad ..(3.24)$$

giving ν or ν' in the forms $\nu(\alpha;\theta^{\circ})$ or $\nu'(\alpha;\theta^{\circ})$ and showing that they are *nearly* independent of θ. In other words, it is possible to choose a $f_2(\theta,\theta^{\circ})$ such that

(α) for all θ subject to $f_2(\theta,\theta^{\circ}) > 0$ we have a common most powerful test which is either of the critical regions : $\quad F \geqslant \nu(\alpha,\theta^{\circ})$ or $\leqslant \nu'(\alpha;\theta^{\circ})$ ..(3.25)

(β) for all θ subject to $f_2(\theta,\theta^{\circ}) > 0$ we have a common most powerful test which is just that one of the two alternatives

$$F \geqslant \nu(\alpha;\theta^{\circ}) \text{ or } F \leqslant \nu'(\alpha;\theta^{\circ}) \text{ which does not occur in (3.25)} \qquad ..(3.26)$$

(iii) A third alternative would be

$$\phi \equiv \phi^{\circ}\,\phi_1(F^{\circ};\theta;\theta^{\circ}) \qquad ..(3.3)$$

where

$$F^{\circ} \equiv F(x_1,\ldots,x_n;\theta^{\circ}) \qquad ..(3.31)$$

with the additional condition (3.3a) that the equation

$$\phi = \mu\phi^{\circ}\phi_1(F^{\circ};\theta;\theta^{\circ}) \qquad ..(3.32)$$

where μ is determined so as to satisfy

$$\int_{\text{over } \phi \geqslant \mu\phi^{\circ}\phi_1(F^{\circ};\theta;\theta^{\circ})} \phi^{\circ} dv = \alpha \qquad ..(3.33)$$

will lead to only one root $\quad F^{\circ} = \nu$ ..(3.3a)

The rest is similar to the cases (i) and (ii), and need not be considered after all the explanation given under (i) and (ii). What will happen if the respective conditions (3.1a), (3.2a) and (3.3a) of (i), (ii) and (iii) are given up, that is, whether (3.1)–(3.11) or (3.2) or (3.3)–(3.31) would still remain a sufficient condition is an important and intriguing point which will be discussed in the next paper.

The cases discussed under (i), (ii) and (iii) could no doubt be comprehended under one single abstract functional form, but it is considered more convenient for practical purposes to split up and treat them separately.

Against this background let us now proceed to obtain alternative sets of sufficient conditions for the existence of uniformly most powerful tests of composite hypotheses and also the mechanism of generating such tests.

(iv) For the case (i) of section 2, we should have corresponding to (3.1)–(3.11) of this section the condition

$$\phi_s \equiv \phi_3(F;\theta_{s+1},\ldots,\theta_{s+a})\phi_4(x_1,\ldots,x_n) \qquad ..(3.4)$$

where $$F \equiv F(x_1, \ldots, x_n) \qquad ..(3.41)$$

with the additional condition that if we consider

$$F_i(x_1, \ldots, x_n) \leqslant \mu_i \qquad (i = 1, 2, \ldots, m) \qquad ..(3.42)$$

and

$$\phi_2\phi_1(\mu_1, \ldots, \mu_m; \theta_1, \ldots, \theta_k; \theta_{k+1}, \ldots, \theta_{k+s}) \geqslant \lambda\phi_1(\mu_1, \ldots, \mu_m; \theta_1^\circ, \ldots, \theta_k^\circ; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ)\lambda\phi_2^\circ$$

that is, $$\phi_2 > \lambda' \phi_2^\circ \qquad ..(3.43)$$

when λ' is chosen so as to satisfy

$$\int\limits_{\text{over (3.42)}\,\Lambda\,(3.44)} \phi_2^\circ \, dv = \alpha \int\limits_{\text{over (3.42)}} \phi_2^\circ \, dv \qquad ..(3.44)$$

then the equation $$\phi_2 = \lambda\phi_2^\circ \qquad (3.45)$$

will lead to only one real solution

$$F = \nu \qquad ..(3.4a)$$

Similarly corresponding to (3.2) of this section we should have

$$\phi_2 \equiv \phi_2(\phi_2^\circ; \theta_{k+1}, \ldots, \theta_{k+s}; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \qquad (3.46)$$

with an additional condition (3.46a) exactly similar to the condition (3.4a) discussed just now, which, after the explanation given in course of (iv), the reader will be able to easily perceive.

Likewise corresponding to (3.3)–(3.31) of this section we should have

$$\phi_2 \equiv \phi_2^\circ\phi_2(F^\circ; \theta_{k+1}, \ldots, \theta_{k+s}; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \qquad ..(3.47)$$

with an additional condition (3.47a) which the reader will be able to construct without difficulty.

(v) For the case (ii) of section 2, it seems that corresponding to (3.1)–(3.11) of the section we have nothing and corresponding to (3.2) or (3.3)–(3.31) also nothing direct except trivial cases. The proper condition here seems to be

$$F_1 \equiv F_1(F_1^\circ; F^\circ; \theta_{k+1}, \ldots, \theta_{k+s}; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \qquad ..(3.5)$$

where $$F^\circ \equiv F(x_1, \ldots, x_n; \theta_{k+1}^\circ, \ldots, \theta_{k+s}^\circ) \qquad ..(3.51)$$

together with an additional condition (3.5a) exactly similar to (3.4a), or (3.46a) or (3.47a), which the reader will be able to construct without difficulty.

(vi) Lastly, for the case (iii) of section 2 we would have, as could be easily perceived, a set of conditions identical with (3.5)–(3.51) and with the additional condition (3.5a).

As with regard to the respective restrictions (3.1a), (3.2a) and (3.3a) of (i), (ii) and (iii) of this section so also in respect of the corresponding restrictions (3.4a), (3.46a) and (3.47a) of (iv), and (3.5a) of (v), and a similar one for (vi) of this section, the question arises as to what happens if we relax the respective restrictions (3.4a), (3.46a), (3.47a) and (3.5a). That is, would the conditions (3.4)–(3.41), (3.5)–(3.51) or the same in case (vi), still remain sufficient conditions, if not in their proposed form, at any rate with slight qualifications of a much less restrictive nature than would be implied by (3.4a), (3.46a), (3.47a) or (3.5a). As in the previous case so also here, this is quite an intriguing and rather difficult question which will be discussed in the next paper.

It will be observed throughout this paper that about the conditions introduced at different stages it has not been stated whether they are supposed to hold at points of the sample space or whether it would do if they hold *almost anywhere* in the sample space. Such considerations, however important from the point of view of rigour, have not been discussed in the present paper, meant as it is primarily for statisticians.

4. Some well known concrete cases by way of illustration

(i) For a univariate normal population with population mean and S. D. m and σ, sample mean and S. D. $\bar{x}$ and s, and sample size n, consider a hypothesis m_0, a pseudo-hypothesis σ_0, an alternative m and a pseudo-alternative σ.

$$\phi \equiv \frac{1}{(\sigma\sqrt{2\pi})^n} \text{ Exp.} \left[-\frac{n}{2\sigma^2} \left((\bar{x}-m)^2+s^2\right) \right] \qquad ..(4.1)$$

Here $\phi = \phi_1(F_1; \sigma)$

where $F_1 \equiv (\bar{x}-m)^2+s^2 \; ; \; \phi_2 = 1$ ..(4.11)

Since $F_1^\circ = (\bar{x}-m_0)^2+s^2$ ..(4.12)

we have $F_1 = F_1^\circ - 2(\bar{x}-m_0)(m-m_0)+(m-m_0)^2$ ..(4.13)

$= F_1(F_1^\circ, F^\circ; m_0, m)$

where $F^\circ \equiv (\bar{x}-m_0)$ ..(4.14)

Now the condition (4.11), in view of (1.1) and (2.4), shows that (a) we have valid tests available, and (b) among these there is a most powerful test. The condition (4.14), in view of (3.5), shows that we have a (in this case one-sided) uniformly most powerful test. The additional conditions referred to in (2.4) and (3.5) are here easily seen to be satisfied by considering the most powerful test for m_0 with respect to m. The actual mechanism would be provided in this case by (4.15) $\wedge$ (4.16) where (4.15) and (4.16) are given by

$$\mu_1 \leqslant F_1^\circ \equiv (\bar{x}-m_0)^2+s^2 \leqslant \mu_1+d\mu_1 \qquad ..(4.15)$$

$$\phi \geqslant \lambda\phi^\circ \text{ (with } \lambda \text{ properly chosen)} \qquad ..(4.16)$$

But (4.15) $\wedge$ (4.16) $\equiv$ (4.15) $\wedge$ (4.17) where (4.17) is given by

$$(\bar{x}-m_0)(m-m_0) \geqslant \lambda'$$

or $(\bar{x}-m_0) \geqslant \lambda''$ or $\leqslant \lambda'''$ (according as $m > m_0$ or $< m_0$) ..(4.17)

where λ'' or λ''' would be given by

$$\int_{(4.15)\wedge(4.17)} \phi^\circ dv = \alpha \int_{(4.15)} \phi^\circ dv \text{ or simply, } \int_{(4.15)\wedge(4.17)} dv = \alpha \int_{(4.15)} dv \qquad ..(4.171)$$

Now $dv \to$ Const. $s^{n-2}\, ds\, d\bar{x} \to$ Const. $s^{n-2}\, ds\, d(\bar{x}-m_0)$

$\to$ Const. $r^{n-1} dr (\cos\theta)^{n-2}\, d\theta$

where $r^2 \equiv (\bar{x}-m_0)^2+s^2$ and $\tan\theta \equiv (\bar{x}-m_0)/s$ ..(4.172)

$$\to \text{Const. } (\mu_1)^{\frac{n-2}{2}} d\mu_1 (\cos\theta)^{n-2} d\theta \qquad ..(4.173)$$

This suggests that we had better replace (4.15) $\wedge$ (4.17), as we easily could, by (4.15) $\wedge$ (4.18) where (4.18) is given by

$$\theta \geqslant \theta' \text{ or } \leqslant \theta'' \text{ or } t \equiv \sqrt{n-1}\tan\theta \geqslant t' \text{ or } \leqslant t'' \text{ (according as } m > m_0 \text{ or } < m_0) \qquad ..(4.18)$$

where t' or t'' would be given by

$$\underset{(4.15)\,\Lambda\,(4.18)}{\int dv} = \alpha \underset{(4.15)}{\int dv} \quad \text{or} \quad \underset{(4.15)\,\Lambda\,(4.18)}{\int (\cos\theta)^{n-2} d\theta} = \alpha \underset{(4.15)}{\int (\cos\theta)^{n-2} d\theta}$$

or
$$\int_{-\infty}^{t''} \frac{dt}{\left(1+\frac{t^2}{n-1}\right)^{\frac{n}{2}}} = \int_{t'}^{\infty} \frac{dt}{\left(1+\frac{t^2}{n-1}\right)^{\frac{n}{2}}} = \alpha \int_{-\infty}^{+\infty} \frac{dt}{\left(1+\frac{t^2}{n-1}\right)^{\frac{n}{2}}} \quad ..(4.19)$$

It so happens that in this case $t'' = -t'$.

Now (4.15) Λ (4.18) subject to (4.19) gives us the most powerful valid test on (4.15). This in ordinary course should have involved μ_1 and eliminating μ_1 between (4.15) and (4.18) we would have obtained the most powerful valid test. But here, partly for intrinsic reasons and partly by design, it so happens that (4.18), being in view of (4.19) free from μ_1, is itself the result of elimination and gives us the most powerful valid test. This is a feature which will be met with in all the following examples. In addition, since (4.18) subject to (4.19) is *nearly* free from the alternative m, being $t \geqslant t'$ for $m > m_0$ and $t \leqslant t''$ for $m \leqslant m_0$, we have also two one-sided uniformly most powerful tests. This feature will be met with in cases (ii) and (iii) but not in (iv).

(ii) For 2 univariate normal populations with population means (m_1, m_2), population S. D.'s (σ_1, σ_2), sample means $(\bar{x}_1, \bar{x}_2)$, sample S. D.'s (s_1, s_2), and sample sizes (n_1, n_2) consider the hypothesis : $\sigma_1/\sigma_2 \equiv q = 1$, the pseudo-hypothesis : (m_1, m_2, σ_2) ; the alternative : $q \neq 1$, the pseudo-alternative (m_1', m_2', σ_2').

Then
$$\phi \equiv \text{Const. Exp.}\left[-\frac{1}{2\sigma_2'^2}\left\{\frac{n_1 s_1^2}{q^2}+n_2 s_2^2+\frac{n_1}{q^2}(\bar{x}_1-m_1')^2+n_2(\bar{x}_2-m_2')^2\right\}\right]$$

$$\equiv \phi_1(F_1, F_2, F_3; m_1', m_2', \sigma_2'; q) \quad ..(4.2)$$

where
$$F_1 \equiv \frac{n_1 s_1^2}{q^2}+n_2 s_2^2;\ F_2 \equiv \bar{x}_1;\ F_3 \equiv \bar{x}_2 \quad ..(4.21)$$

Since
$$F_1^\circ \equiv n_1 s_1^2+n_2 s_2^2 \quad ..(4.211)$$

we have
$$F_1 = F_1^\circ + n_1 s_1^2\left(\frac{1}{q^2}-1\right) \equiv F_1(F_1^\circ, F^\circ; q)$$
where
$$F^\circ \equiv n_1 s_1^2 \quad ..(4.212)$$

(4.2)–(4.21) show that (a) valid tests are available, and (b) amongst these there is a most powerful test of $q = 1$ with respect to $q \neq 1$. (4.211)–(4.212) show that we have (one-sided) uniformly most powerful tests. The additional conditions referred to earlier can be easily seen to be fulfilled by actually constructing the most powerful test of $q = 1$ with respect to any $q \neq 1$.

Using the same technique as in the earlier case the mechanism in this case would be provided by (4.22) Λ (4.221) where (4.22) and (4.221) are given by

$$\left.\begin{array}{l}\mu_1 < F_1^* \equiv n_1 s_1^2 + n_2 s^2 < \mu_1 + d\mu_1 \\ \mu_2 < F_2 \equiv \bar{x}_1 < \mu_2 + d\mu_2;\ \mu_3 < F_3 \equiv \bar{x}_2 < \mu_3 + d\mu_3\end{array}\right\} \quad ..(4.22)$$

and

$$-n_1 s_1^2 \left(\frac{1}{q^2} - 1\right) \geqslant \lambda$$

$$i.e.,\ s_1^2 \geqslant \lambda' \text{ or } < \lambda'' \quad (\text{according as } q > 1 \text{ or } < 1) \quad ..(4.221)$$

where λ' or λ'' is to be determined by

$$\int_{(4.22)\Lambda(4.221)} \phi^* d\sigma = \alpha \int_{(4.22)} \phi^* d\sigma \quad \text{or} \quad \int_{(4.22)\Lambda(4.221)} d\sigma = \alpha \int_{(4.22)} d\sigma \quad ..(4.222)$$

But
$$d\sigma \to \text{Const. } d\bar{x}_1 d\bar{x}_2 s_1^{n_1-2} ds_1\, s_2^{n_2-2} ds_2$$

$$\to \text{Const. } d\mu_2 d\mu_3 \mu_1^{(n_1+n_2-4)/2} d\mu_1 (\sin\theta)^{n_1-2} (\cos\theta)^{n_2-2} d\theta \quad ..(4.223)$$

where
$$\mu_1 \equiv n_1 s_1^2 + n_2 s_2^2 \text{ and } \tan\theta \equiv \sqrt{n_1}s_1 / \sqrt{n_2}s_2 \quad ..(4.224)$$

This suggests that we had better replace, as we can, (4.22) Λ (4.221) by (4.22) Λ (4.23) where (4.23) is given by

$\theta \geqslant \theta'$ or $< \theta''$, i.e., $F \equiv s_1^2/s_2^2 \equiv \sqrt{n_2/n_1} \tan\theta \geqslant F'$ or $< F''$ (according as $q > 1$ or < 1),

F' or F'' being determined by $\int_{(4.22)\Lambda(4.23)} (\sin\theta)^{n_1-2}(\cos\theta)^{n_2-2} d\theta = \alpha \int_{(4.22)} (\sin\theta)^{n_1-2}(\cos\theta)^{n_2-2} d\theta$ ..(4.23)

that is, by

$$\int_0^{F} \frac{F^{\frac{n_1-3}{2}} dF}{\left(1+\frac{n_1-1}{n_2-1}F\right)^{\frac{n_1+n_2-2}{2}}} = \int_{F'}^{\infty} \frac{F^{\frac{n_1-3}{2}} dF}{\left(1+\frac{n_1-1}{n_2-1}F\right)^{\frac{n_1+n_2-2}{2}}} = \alpha \int_0^{\infty} \frac{F^{\frac{n_1-3}{2}} dF}{\left(1+\frac{n_1-1}{n_2-1}F\right)^{\frac{n_1+n_2-2}{2}}}$$

..(4.231)

Thus F' and F'' being free from (μ_1, μ_2, μ_3), we note that (4.23) itself gives the most powerful test, and (4.23) being *nearly* free from q, it provides us with two one-sided uniformly most powerful tests.

(iii) With the same set-up as in (ii), consider a hypothesis: $m_1 = m_2$ and $\sigma_1/\sigma_2 \equiv q = 1$; an alternative $m_1' \neq m_2'$ but with $q = 1$. The hypothesis, the pseudo-hypothesis, the alternative and the pseudo-alternative will be presently more clearly specified. The hypothesis $q = 1$, since it is not violated in the alternative, had better be called a basic assumption. Assuming a σ_2' in the alternative we have

$$\phi \equiv \text{Const. Exp.} \left[-\frac{1}{2\sigma_2'^2}\left\{n_1 s_1^2 + n_2 s_2^2 + n_1(\bar{x}_1 - m_1')^2 + n_2(\bar{x}_2 - m_2')^2\right\}\right] \quad ..(4.3)$$

The argument under the exponential (apart from the factor $-1/2\sigma_2'^2$)

$$= (n_1 s_1^2 + n_2 s_2^2) + \frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2 + \frac{n_1 n_2}{n_1+n_2}(m_1'-m_2')^2$$

$$-2\frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)(m_1'-m_2') + (n_1+n_2)(\bar{x}-\bar{m}')^2 \qquad ..(4.31)$$

where $\bar{x} \equiv (n_1\bar{x}_1+n_2\bar{x}_2)/(n_1+n_2)$; $\bar{m}' \equiv (n_1 m_1'+n_2 m_2')/(n_1+n_2)$ ..(4.311)

The form (4.31) suggests that we could express our hypothesis as $m_1-m_2 \equiv \Delta = 0$ with basic assumption $\sigma_1/\sigma_2 \equiv q = 1$, the pseduo-hypothesis as $(\bar{m}, \sigma_2)$; the alternative as $\Delta \neq 0$ with $q \neq 1$, the pseudo-alternative as $(\bar{m}', \sigma_2')$.

Hence
$$\phi \equiv \text{Const. Exp.}\left[-\frac{1}{2\sigma_2'^2}\left\{n_1 s_1^2+n_2 s_2^2+\frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2 - \frac{2n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)\Delta+(n_1+n_2)(\bar{x}-\bar{m}')^2+\frac{n_1 n_2}{n_1+n_2}\Delta^2\right\}\right] \qquad ..(4.312)$$

$$= \phi_1(F_1/F_2;\ \sigma_2',\ \bar{m}';\ \Delta) \qquad ..(4.313)$$

where
$$\left.\begin{aligned} F_1 &\equiv n_1 s_1^2+n_2 s_2^2+\frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2-\frac{2n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)\Delta \\ F_2 &\equiv \bar{x}\end{aligned}\right\} \qquad ..(4.314)$$

Since
$$F_1^\circ \equiv (n_1 s_1^2+n_2 s_2^2)+\frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2,$$

we have
$$\left.\begin{aligned} F_1 &\equiv F_1^\circ-\frac{2n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)\Delta \equiv F_1(F_1^\circ, F^\circ; \Delta) \\ \text{where}\quad F^\circ &\equiv (\bar{x}_1-\bar{x}_2)\end{aligned}\right\} \qquad ..(4.315)$$

As in the earlier case (4.314) shows that (a) valid tests are available, and (b) amongst these there is a uniformly most powerful test. The condition (4.315) shows that there are (one-sided) uniformly most powerful tests. The requisite additional conditions will be shortly found to be satisfied.

As in the earlier case, the mechanism here would be provided by (4.32) $\wedge$ (4.321) which are given by

$$\left.\begin{aligned} &\mu_1 < F_1^\circ \equiv n_1 s_1^2+n_2 s_2^2+\frac{n_1 n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2 < \mu_1+d\mu_1 \\ &\mu_2 \leqslant \bar{x} \leqslant \mu_2+d\mu_2\end{aligned}\right\} \qquad ..(4.32)$$

and $(\bar{x}_1-\bar{x}_2)\Delta \geqslant \lambda$ or $\bar{x}_1-\bar{x}_2 \geqslant \lambda'$ or $\leqslant \lambda''$ (according as $\Delta > 0$ or < 0) ..(4.321)

where λ' or λ'' is to be chosen so that

$$\int\limits_{(4.32)\,\wedge\,(4.321)} \phi_\alpha\, dv = \alpha \int\limits_{(4.32)} \phi_\alpha\, dv \quad \text{or} \quad \int\limits_{(4.32)\,\wedge\,(4.321)} dv = \alpha \int\limits_{(4.32)} dv \qquad ..(4.322)$$

Now $$dv \to \text{Const. } s_1^{n_1-2}\, ds_1\, s_2^{n_2-2}\, ds_2\, d\bar{x}_1\, d\bar{x}_2$$
$$\to \text{Const.} (\sqrt{n_1 s_1^2+n_2 s_2^2})^{n_1+n_2-3}\, d(\sqrt{n_1 s_1^2+n_2 s_2^2}) d(\bar{x}_1-\bar{x}_2) d\bar{x}$$
$$\to \text{Const. } (\mu_1)^{(n_1+n_2-4)/2}\, d\mu_1\, d\mu_2\, (\cos\theta)^{n_1+n_2-3}\, d\theta \quad ..(4.323)$$

where $$\sqrt{n_1 s_1^2+n_2 s_2^2} = \cos\theta\sqrt{\mu_1} \quad \text{and} \quad \bar{x}_1-\bar{x}_2 = \sin\theta\sqrt{\mu_1} \quad ..(4.324)$$

This suggests that we can (and should) replace (4.32) Λ (3.321) by (4.32) Λ (4.33) where (4.33) is

$$\left.\begin{array}{l} \theta \leqslant \theta' \text{ or } \geqslant \theta'' \text{ i.e., } t \geqslant t' \text{ or } < t'' \\ \text{where} \quad t = \left(\dfrac{n_1 n_2}{n_1+n_2}\right)^{\frac{1}{2}} (\bar{x}_1-\bar{x}_2)/\sqrt{(n_1 s_1^2+n_2 s_2^2)/(n_1+n_2-2)} \end{array}\right\} \quad ..(4.33)$$

and where t' or t'' is to be so chosen that

$$\int\limits_{(4.32)\,\Lambda\,(4.33)} (\cos\theta)^{n_1+n_2-3} d\theta = \alpha \int\limits_{(4.32)} (\cos\theta)^{n_1+n_2-3} d\theta$$

$$\int_{-\infty}^{t''} \frac{dt}{\left(1+\dfrac{t^2}{n_1+n_2-2}\right)^{\frac{n_1+n_2-1}{2}}} = \int_{t'}^{\infty} \frac{dt}{\left(1+\dfrac{t^2}{n_1+n_2-2}\right)^{\frac{n_1+n_2-1}{2}}}$$
$$= \alpha \int_{-\infty}^{+\infty} \frac{t^2}{\left(1+\dfrac{t^2}{n_1+n_2-2}\right)^{\frac{n_1+n_2-1}{2}}} \quad ..(4.331)$$

giving t' and t'' in a form free from (μ_1, μ_2) and also from Δ. Here $t'' = -t'$. Thus the most powerful valid test itself is given by (4.33) and it also happens to yield two (one-sided) uniformly most powerful tests for $\Delta > 0$ and for $\Delta < 0$.

(iv) For k populations with population means and S.D.'s (m_i, σ_i) $(i = 1, 2, ..., k)$ sample means and S.D.'s and sample sizes, $(\bar{x}_i, s_i, n_i)$ $(i = 1, 2, ..., k)$ consider the hypothesis $(m_1 = m_2 = ... = m_k)$ with a basic assumption $(\sigma_1 = \sigma_2 = ... = \sigma_k)$ and an alternative $(m_1, ..., m_k')$ with the same basic assumption. The exact nature of the hypothesis, pseudo-hypothesis, alternative and pseudo-alternative will, as in the previous case, be more clearly specified shortly. It is well known that not only is the general problem of analysis of variance an extension of the simpler problem considered here but the more general structure needed there could be replaced for formal purposes by the simplified structure given here for convenience. Meanwhile in the alternative consider for the k populations a common S.D. σ_1'. Then in this case

$$\phi = \text{Const. Exp.}\left[-\frac{1}{2\sigma_1'^2}\sum_{i=1}^{k}\left\{n_i s_i^2+(\bar{x}_i-m_i')^2\right\}\right] \quad ..(4.4)$$

The argument within the exponential, leaving out the factor $-1/2\sigma_1'^2$ is

$$\sum_{i=1}^{k} n_i s_i^2+\sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x}+\bar{x}-\bar{m}'+\bar{m}'-m_i')^2$$
$$= \sum_{i=1}^{k} n_i s_i^2+\sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x})^2+N(\bar{x}-\bar{m}')^2-2\sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x})(m_i'-\bar{m}')+\sum_{i=1}^{k} n_i(m_i'-\bar{m}')^2$$
$$= W^2+B^2+N(\bar{x}-\bar{m}')^2-2B\beta\cos\theta+\beta^2 \quad ..(4.41)$$

where $$N = \sum_{i=1}^{k} n_i;\ \bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i / N;\ \bar{m}' = \sum_{i=1}^{k} n_i m_i'/N$$

$$W^2 = \sum_{i=1}^{k} n_i s_i^2;\ B^2 = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})^2;\ \beta^2 = \sum_{i=1}^{k} n_i(m_i' - \bar{m}')^2 \quad ..(4.411)$$

and $$B\beta \cos\theta = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})(m_i' - \bar{m}')$$

Consider now a transformation connecting the variables $\sqrt{n_j}\,\bar{x}_j$'s with z_j's :

$$z_1 = \sum_{j=1}^{k} \sqrt{n_j}\,\sqrt{n_j}\,\bar{x}_j/\sqrt{N};\qquad z_i = \sum_{j=1}^{k} \nu_{ij}\sqrt{n_j}\,\bar{x}_j$$

$$\zeta_1' = \sum_{j=1}^{k} \sqrt{n_j}\,\sqrt{n_j} m_j'/\sqrt{N};\ \zeta_i' = \sum_{j=1}^{k} \nu_{ij}\sqrt{n_j} m_j'\ (i = 2, 3, \ldots, k) \quad ..(4.412)$$

such that $\begin{pmatrix} \sqrt{n_1/N} & \ldots & \sqrt{n_k/N} \\ \nu_{21} & \ldots & \nu_{2k} \\ \ldots & \ldots & \ldots \\ \nu_{k1} & \ldots & \nu_{kk} \end{pmatrix}$ is a unitary orthogonal matrix.

Then under such transformation

$$N(\bar{x} - \bar{m}')^2 = (z_1 - \zeta_1')^2;\ B^2 = \sum_{i=2}^{k} z_i^2;\ \beta^2 = \sum_{i=2}^{k} \zeta_i'^2;\ B\beta\cos\theta = \sum_{i=2}^{k} z_i\zeta_i' \quad ..(4.413)$$

We can now express the hypothesis as $\zeta_2 = \zeta_3 = \ldots = \zeta_k = 0$ (with $\sigma_1 = \sigma_2 = \ldots = \sigma_k$) the pseudo-hypothesis as $(\bar{m}, \sigma_1)$, the alternative as $\zeta_2' \neq \zeta_3' \neq \ldots \neq \zeta_k' \neq 0$ and the pseudo-alternative as $(\bar{m}', \sigma_1')$

Going back to (4.41) we notice that

$$\phi = \phi_1(F_1, F_2;\ \bar{m}', \sigma_1';\ \zeta_2', \ldots, \zeta_k')$$

where $$F_1 \equiv W^2 + B^2 - 2B\beta\cos\theta;\ F_2 \equiv z_1 \quad ..(4.42)$$

Here $$F_1^0 \equiv W^2 + B^2;\ \text{but}\ F_1 \neq F_1(F_1^0,\ F^0;\ \zeta_2', \ldots, \zeta_k') \quad ..(4.421)$$

Hence (4.42) shows that (a) valid tests are available, and (b) amongst these there is a most powerful test, the additional condition being shortly found to be satisfied. Condition (4.421), however, shows that sufficient conditions for (one-sided) uniformly most powerful tests are lacking. In fact, working out the most powerful test it will be seen here that uniformly most powerful tests are here actually lacking.

(iv.1) As in the earlier cases, the mechanism for a most powerful test would in this case be provided by (4.43) Λ (4.431) given by

$$\mu_1 < F_1^0 \equiv W^2 + B^2 \leqslant \mu_1 + d\mu_1;\ \mu_2 \leqslant z_1 \leqslant \mu_2 + d\mu_2 \quad ..(4.43)$$

and $$B\beta\cos\theta > \lambda \quad ..(4.431)$$

where λ is so chosen as to satisfy

$$\int_{(4.43)\Lambda(4.431)} \phi_0\, dv = \alpha \int_{(4.43)} \phi_0\, dv \quad \text{or} \quad \int_{(4.43)\Lambda(4.431)} dv = \alpha \int_{(4.43)} dv \qquad ..(4.432)$$

Now

$$dv \to \text{Const.} \prod_{i=1}^{k} s_i^{n_i-2}\, ds_i \;\; dz_1 \prod_{i=2}^{k} dz_i$$

$$\to \text{Const.} \left\{\left(\sum_{i=1}^{k} n_i s_i^2\right)^{\frac{1}{2}}\right\}^{N-k-1} d\left(\sum_{i=1}^{k} n_i s_i^2\right)^{\frac{1}{2}} dz_1$$

$$\times \left\{\left(\sum_{i=2}^{k} z_i^2\right)^{\frac{1}{2}}\right\}^{k-2} d\left(\sum_{i=2}^{k} z_i^2\right)^{\frac{1}{2}} (\sin\theta)^{k-3}\, d\theta$$

$$\to \text{Const.}\; W^{N-k-1}\, dW\, B^{k-2}\, dB\, (\sin\theta)^{k-3}\, d\theta dz_1$$

$$\to \text{Const.}\; (\sqrt{B^2+W^2})^{N-2}\, d(\sqrt{B^2+W^2})\, (\sin\psi)^{k-2} (\cos\psi)^{N-k-1}\, d\psi\, (\sin\theta)^{k-3}\, d\theta dz_1$$

(where $\tan\psi = B/W$)

$$\to \text{Const.}\; (\sqrt{B^2+W^2})^{N-2}\, d(\sqrt{B^2+W^2})\, (\sin\chi)^{N-3}\, d(\chi)\, dz_1$$

(where $\cos\chi = \cos\theta\cos\psi$, i.e., $\sqrt{B^2+W^2}\,\beta\cos\chi = B\beta\cos\theta = \sum_{i=2}^{k} z_i\zeta_i = \sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x})(m_i'-\bar{m}')$)

$$\to \text{Const.}\; \mu_2^{\frac{N-2}{2}}\, d\mu_1\, d\mu_2\, (\sin\chi)^{N-3}\, d\chi \qquad ..(4.433)$$

This suggests that we had better replace (4.43) Λ (4.431) by (4.43) Λ (4.434) where (4.434) is given by

$$\left.\begin{array}{l} \chi \geqslant \chi_0 \text{ or } t = \sqrt{N-2}\cot\chi \geqslant t_0, \\ \text{where } \cos\chi = \sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x})(m_i'-\bar{m}) \Big/ \left[\sum_{i=1}^{k} n_i\{s_i^2+(\bar{x}_i-\bar{x})^2\}\right]^{\frac{1}{2}} \left[\sum_{i=1}^{k} n_i(m_i'-\bar{m}')^2\right]^{\frac{1}{2}} \geqslant t_0 \end{array}\right\} (4.434)$$

where t_0 is so chosen as to satisfy

$$\int_{t_0}^{\infty} \frac{dt}{\left(1+\frac{t^2}{N-2}\right)^{\frac{N-1}{2}}} = \alpha \int_{-\infty}^{+\infty} \frac{dt}{\left(1+\frac{t^2}{N-2}\right)^{\frac{N-1}{2}}} \qquad ..(4.435)$$

This shows that (4.434) being free from (μ_1 and μ_2), is itself the most powerful valid test, but being dependent on $(\zeta_2', \ldots, \zeta_k')$, it does not yield (one-sided) uniformly most powerful tests, as in the previous cases.

(iv.2) Since (one-sided) uniformly most powerful tests are lacking in this case we might (as the next best thing) look about for a most powerful test on an average, the averaging being in the parametric space $(\zeta_2', \ldots, \zeta_k')$ belonging to the hypothesis and its alternatives.

Consider the ϕ given by (4.42), the constant being $1/(\sigma_1'/\sqrt{2\pi})^N$.

Then the average $\bar{\phi}$, more properly denoted by $\bar{\phi}_h$, is given by

$$\bar{\phi}_h = (1/(\sigma_1'\sqrt{2\pi})^N) \text{ Exp.} \left[\frac{1}{2\sigma_1'^2}\left\{W^2+B^2+(z_1-\zeta_1')^2\right\}\right]$$

$$\times \int \text{Exp.} \left[-\frac{1}{2\sigma_1'^2}\left\{ (\beta^2-2\beta B\cos\theta\right\}\right] h(\beta) \prod_{i=2}^{k} d\zeta_i' \qquad ..(4.44)$$

where $h(\beta)$ is an arbitrary steadily decreasing function of β of the nature of a probability density (subject to the condition : $h(\beta) \geqslant 0$ everywhere, and also other condition of probability density) such that

$$\int h(\beta) \prod_{i=2}^{k} d\zeta_i' = 1 \qquad ..(4.441)$$

We note that there is an element of arbitrariness not merely in h being an arbitrary function of β but also in its being a function of β alone, where

$$\beta^2 = \sum_{i=2}^{k} \zeta_i'^2 \qquad ..(4.442)$$

The whole of the following investigation will thus be subject to this arbitrariness.

Now $\prod_{i=2}^{k} d\zeta_i'$, subject to $\beta =$ Const., and $\theta =$ Const., leads to

$$\text{Const.}\ (\beta)^{k-2}\, d\beta\, (\sin\theta)^{k-3}\, d\theta \qquad ..(4.443)$$

Hence the integral on the right hand side of (4.44) becomes

$$\int_{\beta=0}^{\infty} \int_{\theta=0}^{\pi} \text{Exp.} \left[-\frac{1}{2\sigma_1'^2}(\beta^2-2\beta B\cos\theta)\right] h(\beta)\ \beta^{k-2}\, d\beta\, (\sin\theta)^{k-3}\, d\theta \qquad ..(4.444)$$

$\equiv F_h(B; k, \sigma_1') \equiv f_h(B; \sigma_1')$ (where f_h is a non-decreasing function of B)

$$\text{Thus } \bar{\phi}_h = (1/\sigma_1'\sqrt{2\pi})^N \text{ Exp.} \left[-\frac{1}{2\sigma_1'^2}\left\{W^2+B^2+(z_1-\zeta_1')^2\right\}\right] f_h(B; \sigma_1') \qquad ..(4.45)$$

It is to be noted that in ϕ° this $f_h(B; \sigma_1')$ becomes a a pure constant, since $\beta = 0$.

$$\text{Thus:}\quad \bar{\phi}_h \equiv \phi_1(F_1, F_2; \sigma_1', \zeta_1') f_h(B; \sigma_1'), \quad \text{where}\quad F_1 \equiv W^2+B^2;\ F_2 = z_1 \qquad ..(4.451)$$

$$\text{and}\quad \phi^\circ = \bar{\phi}_h^\circ = \phi_1(F_1^\circ, F_2^\circ; \sigma_1, \zeta_1) \times \text{Const.}, \quad \text{where}\quad F_1^\circ \equiv F_1;\ F_2^\circ \equiv F_2 \qquad ..(4.452)$$

This shows that (a) valid tests are available, and (b) amongst these there is a most powerful test (on an average, the averaging being dependent on h). The mechanism for the most powerful test would be provided by (4.46) A (4.461) given by

$$\begin{aligned} \mu_1 \leqslant F_1^\circ \equiv F_1 \equiv W^2+B^2 \leqslant \mu_1+d\mu_1 \\ \mu_2 \leqslant F_2^\circ \equiv F_2 \equiv z_1 \leqslant \mu_2+d\mu_2 \end{aligned} \qquad ..(4.46)$$

$$\text{and}\quad f_h(B; \sigma_1') \geqslant \lambda \text{ or } B \geqslant \lambda' \qquad ..(4.461)$$

where λ' is chosen so as to satisfy

$$\int_{(4.40)\,\Lambda\,(4.461)} \phi_v dv = \alpha \int_{(4.40)} \phi_0 dv \quad \text{or} \quad \int_{(4.40)\,\Lambda\,(4.461)} dv = \alpha \int_{(4.40)} dv \qquad ..(4.462)$$

Now $\quad dv \to \text{Const. } B^{k-2}dB\, W^{N-k-1}dW dz_1$

$$\to \text{Const. } (\sqrt{B^2+W^2})^{N-3} d(\sqrt{B^2+W^2})(\sin\psi)^{k-2}(\cos\psi)^{N-k-1} d\psi dz_1$$

(where $\tan\psi = B/W$)

$$\to \text{Const. } \mu_1^{\frac{N-3}{2}} d\mu_1\, d\mu_2 (\sin\psi)^{k-2} (\cos\psi)^{N-k-1}\, d\psi \qquad ..(4.463)$$

This suggests that we should (as we could) replace (4.40) Λ (4.461) by (4.40) Λ (4.464) where (4.464) is given by

$$\psi \geqslant \psi_0 \text{ or } F = \frac{B/\sqrt{k-1}}{W/\sqrt{N-k}} \geqslant F_0 \qquad ..(4.464)$$

where F_0 is determined by

$$\int_{(4.40)\,\Lambda\,(4.464)} (\sin\psi)^{k-2}(\cos\psi)^{N-k-1} d\psi = \alpha \int_{(4.40)} (\sin\psi)^{k-2}(\cos\psi)^{N-k-1} d\psi$$

or

$$\int_{F_0}^{\infty} \frac{F^{\frac{k-3}{2}} dF}{\left(1+\frac{k-1}{N-k}F\right)^{\frac{N-1}{2}}} = \alpha \int_{0}^{\infty} \frac{F^{\frac{k-3}{2}} dF}{\left(1+\frac{k-1}{N-k}F\right)^{\frac{N-1}{2}}} \qquad ..(4.465)$$

(4.464)–(4.465) show that (4.464) is free from μ_1 and μ_2, and is thus itself the valid most powerful test (on an average).

CONCLUDING REMARKS

It seems to the author that these investigations do not carry us to the bottom of the matter, but put us perhaps on the track towards the objective. For one thing, the necessary conditions have not been obtained for availability (i) of valid tests, (ii) amongst these, of a most powerful test for a specific (composite) alternative, and (iii) of uniformly most powerful tests, in the generalised sense. So far as this particular problem is concerned, it appears that we should have got to the root if we could express the sufficient conditions in each of the sections 1–3 (including the additional conditions considered in sections 2·3) in a more unified form under a single scheme preferably in the form of a differential (or perhaps a functional) equation with certain boundary constraints, and if furthermore the necessary conditions also, when obtained, could be expressible in this form. Then only perhaps could a satisfactory solution be said to have been reached.

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