SOME NEW RESULTS INVOLVING THE NBU(2) CLASS OF LIFE DISTRIBUTIONS

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Abstract

Some new results about the NBU(2) class of life distributions are obtained. Firstly, it is proved that the decreasing with time of the increasing concave ordering of the excess lifetime in a renewal process leads to the NBU(2) property of the interarrival times. Secondly, the NBU(2) class of life distributions is proved to be closed under the formation of series systems. Finally, it is also shown that the NBU(2) class is closed under convolution operation.

Keywords: Convolution; excess lifetime; NBU(2); series systems

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1. Introduction

Several extensions of the NBU (*new better than used*) class of life distributions have been proposed in the literature. For definitions of such classes of life distributions, like NBUE, HNBUE and their duals, see Bryson and Siddiqui (1969), Barlow and Proschan (1981) and Klefsjö (1982). Let X be a non-negative random variable, representing the life length of a piece of equipment with life distribution $F(t)$ (i.e. $F(0-) = 0$) and the survival function $\overline{F}(t) = 1 - F(t)$. Denote by X_t the residual life of the used equipment at time $t \geq 0$ given that it is working at time t, that is $X_t = (X - t \mid X > t)$. It is well known that F belongs to the NBU class if and only if X is stochastically larger than X_t (written as $X \geq_{\text{st}} X_t$), for all $t \geq 0$. As a generalization, Deshpande *et al.* (1986) suggest that the NBU(2) (*new better than used of second order*(2)) class, which emphasizes that X is larger than X_t in the increasing concave ordering (written as $X \geq_{\text{icv}} X_t$), for all $t \geq 0$. Cao and Wang (1991) propose the NBUC (*new better than used in convex ordering*) class, which states that X is larger than X_t in the increasing convex ordering (written as $X \geq_{\text{icx}} X_t$), for all $t \geq 0$. In economics, the stochastic ordering, the increasing convex ordering and the increasing concave ordering are called the *first order stochastic dominance* ordering, the *second order stochastic dominance*(1) ordering and the *second order stochastic dominance*(2) ordering, respectively. They are often used to make a comparison between two random prospects. For definitions and details regarding these orderings, see Shaked and Shanthikumar (1994, Chapter 3). Of course, both extensions include the NBU class as a subclass and are included in the NBUE class.

The NBU and NBUE classes of life distributions have proved to be very useful in performing analyses of life lengths and a lot of results have been obtained in the literature about them. There are also several interesting results about the NBUC class (see Cao and Wang (1991) and Li *et al.* (2000)), whereas fewer articles about the NBU(2) class can be found in the existing literature. In Section 2, a sufficient condition for the interarrival times of a renewal process to be NBU(2) is obtained. It is shown that the decreasing with time of the increasing concave ordering of the excess lifetime can lead to the NBU(2) property for the interarrival times. In Section 3, it is proved that the NBU(2) class of life distributions is closed under the formation of series structure. Finally, the NBU(2) class is also shown to be closed under convolution operation.

For the sake of completeness, we present the following notion and definition. In what follows by 'decreasing', 'increasing' we mean nonincreasing and nondecreasing respectively.

Definition 1. A life distribution F is said to be *new better than used of second order(2)* (NBU(2)) if, $X_t \leq_{\text{icv}} X$ for all $t \geq 0$, that is, for all t, $y \geq 0$,

$$\int_0^y \overline{F}(t+x)\,\mathrm{d}x \leq \overline{F}(t)\int_0^y \overline{F}(x)\,\mathrm{d}x. \tag{1}$$

The dual notion of a *new worse than used of second order(2)* (NWU(2)) life distribution function is defined by reversing the inequality in (1). Note that F is NBU if $X_t \leq_{\text{st}} X$ for all $t \geq 0$, that is, for all $x, t \geq 0$, $\overline{F}(t+x) \leq \overline{F}(t)\overline{F}(x)$. Direct integration on both sides of this will lead to (1), and the NBUE property can follow by letting $y \to \infty$ in (1). So we have the following chain of implications,

$$\text{NBU} \Longrightarrow \text{NBU(2)(NBUC)} \Longrightarrow \text{NBUE}.$$

Definition 2. A stochastic process $X(t)$ is said to be *stochastically decreasing* in $t \geq 0$ in increasing concave ordering, written as $X(t) \downarrow_{\text{icv}}$, if, for all $0 \leq s \leq t$ and $y \geq 0$,

$$\int_0^y \mathrm{P}(X(t) \geq x)\,\mathrm{d}x \leq \int_0^y \mathrm{P}(X(s) \geq x)\,\mathrm{d}x. \tag{2}$$

The process $X(t)$ is said to be *stochastically increasing* in the increasing concave ordering, written as $X(t) \uparrow_{\text{icv}}$, when the inequality of (2) is reversed. Correspondingly, a process $X(t)$ is said to be *stochastically decreasing* or *stochastically increasing* in the increasing convex ordering, written as $X(t) \downarrow_{\text{icx}}$ or $X(t) \uparrow_{\text{icx}}$, when we substitute $[y, \infty)$ for the interval $[0, y)$ in (2).

2. A sufficient condition characterized by excess life function

Suppose that $\{X_n,\ n = 1, 2, \dots\}$ is a sequence of mutually independent and identically distributed non-negative random variables with common life distribution F. For the renewal process with interarrival times X_i, denote, for $n = 1, 2, \dots, S_n = \sum_{i=1}^n X_i$ the time of the nth arrival, and $S_0 \equiv 0$. Let $N(t) = \sup\{n : S_n \leq t\}$ represent the number of arrivals that occur up to time t. Then $\{N(t),\ t \geq 0\}$ is called a renewal counting process. Define $\gamma(t)$ to be the excess lifetime at time t, that is,

$$\gamma(t) = S_{N(t)+1} - t.$$

Chen (1994) proved that if $\gamma(t)$ is stochastically decreasing in t, then F is NBU. It is also shown in that paper that, if $\mathrm{E}\gamma(t)$ is decreasing in t and $\mathrm{E}\gamma(0) = \mathrm{E}X_1 < \infty$, then F is NBUE.

Li *et al.* (2000) proved that, if $\gamma(t)$ is decreasing in t in the increasing convex ordering, then F is NBUC. In this section, an analogous result about the NBU(2) life distributions is obtained.

Theorem 3. *If $\gamma(t)$ is decreasing in t in increasing concave ordering, then F is NBU(2) and if $\gamma(t)$ is increasing in t, then F is NWU(2).*

Proof. Let $g(t,x) = \mathrm{P}(\gamma(t) \geq x)$, for $t \geq 0$ and $x \geq 0$. By conditional probability, it follows that

$$g(t,x) = \overline{F}(t+x) + \int_0^t g(t-s,x)\,\mathrm{d}F(s).$$

For more details see Karlin and Taylor (1975, p. 193). Integrating both sides gives, for all $y \geq 0$,

$$\begin{aligned}\int_0^y g(t,x)\,\mathrm{d}x &= \int_0^y \overline{F}(t+x)\,\mathrm{d}x + \int_0^y\int_0^t g(t-s,x)\,\mathrm{d}F(s)\,\mathrm{d}x \\ &:= \Delta_1 + \Delta_2.\end{aligned}$$

Since $\gamma(t)$ is decreasing in t according to increasing concave ordering, we get from (2)

$$\begin{aligned}\Delta_2 &= \int_0^t\int_0^y g(t-s,x)\,\mathrm{d}x\,\mathrm{d}F(s) \\ &\geq \int_0^t\int_0^y g(t,x)\,\mathrm{d}x\,\mathrm{d}F(s) \\ &= \int_0^y\int_0^t g(t,x)\,\mathrm{d}F(s)\,\mathrm{d}x \\ &= F(t)\int_0^y g(t,x)\,\mathrm{d}x.\end{aligned}$$

Thus, we have, for all $y \geq 0$,

$$\int_0^y \overline{F}(t+x)\,\mathrm{d}x \leq \overline{F}(t)\int_0^y g(t,x)\,\mathrm{d}x. \tag{3}$$

Because $X_1 = \gamma(0)$ and the distribution of the interarrival times of the corresponding renewal process is larger in the increasing concave ordering than $\gamma(t)$ for any $t \geq 0$, we have, for all $y \geq 0$ and $t \geq 0$,

$$\int_0^y g(t,x)\,\mathrm{d}x = \int_0^y \mathrm{P}(\gamma(t) \geq x)\,\mathrm{d}x \leq \int_0^y \overline{F}(x)\,\mathrm{d}x. \tag{4}$$

From (3) and (4), it follows that, for all $t \geq 0$, $y \geq 0$,

$$\int_0^y \overline{F}(t+x)\,\mathrm{d}x \leq \overline{F}(t)\int_0^y \overline{F}(x)\,\mathrm{d}x.$$

That is, F is NBU(2).

When $\gamma(t)$ is increasing in t in the increasing concave ordering, we can prove that F is NWU(2) by reversing the inequalities above.

Combining Theorems 1 and 2 of Chen (1994) and Theorem 7 of Li *et al.* (2000) with Theorem 3, we get the following implications:

$$
\begin{array}{ccc}
\gamma(t) \downarrow_{\mathrm{st}} \ \text{in } t \ge 0 & \Longrightarrow & \text{NBU interarrivals} \\
\Downarrow & & \Downarrow \\
\gamma(t) \downarrow_{\mathrm{icv}} (\downarrow_{\mathrm{icx}}) \text{ in } t \ge 0 & \Longrightarrow & \text{NBU(2) (NBUC) interarrivals} \\
\Downarrow & & \Downarrow \\
E\gamma(t) \downarrow \ \text{in } t \ge 0 & \Longrightarrow & \text{NBUE interarrivals.}
\end{array}
$$

Similar implications can be obtained for dual notions when we substitute '↑', 'NWU' for '↓' and 'NBU' respectively.

3. Closure properties under formation of series systems

Cao and Wang (1991) proved that the NBUC class of life distributions is not closed under series systems, and hence it is not closed under formation of coherent systems either. Hendi *et al.* (1993) show that the NBUC class is closed under parallel systems and Li *et al.* (2000) gave another proof for this closure property, which is quite brief and simple. In this section we will prove that the NBU(2) class of life distributions is closed under the formation of series systems. The proof of the following theorem can be found in Barlow and Proschan (1981, p. 121).

Theorem 4. *Suppose that $\overline{F}_i$, $\overline{G}_i$ $(i = 1, \ldots, n)$ are the survival functions of $2n$ mutually independent units, all of them with finite means. If, for every pair $\overline{F}_i$, $\overline{G}_i$, $i = 1, \ldots, n$, F_i is larger than G_i in the increasing concave ordering, then, for all $t \ge 0$,*

$$\int_0^t \prod_{i=1}^n \overline{G}_i(x)\,\mathrm{d}x \le \int_0^t \prod_{i=1}^n \overline{F}_i(x)\,\mathrm{d}x.$$

Now we prove the desired result.

Theorem 5. *Let $X_1, \ldots, X_n$ denote the times to failure of n mutually independent units. Suppose that the corresponding survival functions $\overline{F}_i$, $i = 1, \ldots, n$ are NBU(2) with finite means. Then the survival function of the series system of the units has the NBU(2) property as well.*

Proof. Denote by $G_{i,t}$ $(i = 1, \ldots, n)$ the residual life distribution at time $t \ge 0$ of the ith unit, that is,

$$\overline{G}_{i,t}(y) = \frac{\overline{F}_i(y+t)}{\overline{F}_i(t)}, \quad \text{for all } y \ge 0.$$

Since F_i is NBU(2), F_i is larger than $G_{i,t}$ in the increasing concave ordering for every $t \ge 0$. Let X denote the life of the series system of these n units. Note that

$$\min_{1 \le i \le n} (X_i)_t = \Big(\min_{1 \le i \le n} X_i\Big)_t = X_t,$$

where X_t represents the residual life of X at time t. It follows from Theorem 4 that, for all $x \geq 0$ and $t \geq 0$,

$$\begin{aligned}\int_0^x \mathrm{P}(X_t > y)\,\mathrm{d}y &= \int_0^x \mathrm{P}\Big(\min_{1\leq i\leq n}(X_i)_t > y\Big)\,\mathrm{d}y\\ &= \int_0^x \prod_{i=1}^n \overline{G}_{i,t}(y)\,\mathrm{d}y\\ &\leq \int_0^x \prod_{i=1}^n \overline{F}_i(y)\,\mathrm{d}y\\ &= \int_0^x \mathrm{P}\Big(\min_{1\leq i\leq n} X_i > y\Big)\,\mathrm{d}y\\ &= \int_0^x \mathrm{P}(X > y)\,\mathrm{d}y,\end{aligned}$$

from which we can conclude that X is also NBU(2) by recalling (1).

4. Closure properties under the convolution operation

As an important reliability operation, convolution of life distributions of certain class is often paid much attention. It has been shown that both the NBUE and NBUC class are closed under this operation (Barlow and Proschan (1981), Cao and Wang (1991)). In the next theorem we establish the closure property of the NBU(2) class under the convolution operation.

Theorem 6. *Suppose that F_1 and F_2 are two independent NBU(2) life distributions. Then their convolution is also NBU(2).*

Proof. The survival function of the convolution of two life distributions F_1 and F_2 is

$$\overline{F}(y) = \int_0^\infty \overline{F}_1(y-z)\,\mathrm{d}F_2(z).$$

By Fubini's theorem and the NBU(2) property, we have, for all $t \geq 0$ and $x \geq 0$,

$$\begin{aligned}\int_0^x \overline{F}(y+t)\,\mathrm{d}y &= \int_0^x\int_0^\infty \overline{F}_1(y+t-z)\,\mathrm{d}F_2(z)\,\mathrm{d}y\\ &= \int_0^\infty\int_0^x \overline{F}_1(y+t-z)\,\mathrm{d}y\,\mathrm{d}F_2(z)\\ &\leq \int_0^\infty \overline{F}_1(t)\int_0^x \overline{F}_1(y-z)\,\mathrm{d}y\,\mathrm{d}F_2(z)\\ &= \overline{F}_1(t)\int_0^x\int_0^\infty \overline{F}_1(y-z)\,\mathrm{d}F_2(z)\,\mathrm{d}y\\ &\leq \overline{F}(t)\int_0^x \overline{F}(y)\,\mathrm{d}y,\end{aligned}$$

the last inequality following from the fact that the convolution of two independent nonnegative random variables is stochastically larger than each of them. This proves that F is also NBU(2).

Remarks. (i) We do not know whether NBU(2) is closed under the formation of parallel systems.

(ii) NBU(2) class is not closed under mixtures, since mixtures of some exponential life distributions often belong to the DFR class (Barlow and Proschan, (1981)).

(iii) Since a parallel system of i.i.d. units with constant failure rate is IFR (Grosh, (1982)), the dual class NWU(2) is not closed under the formation of parallel systems.

(iv) The NWU(2) class is not closed under convolution. To show this fact, consider two independent units with common distribution $F(t) = 1 - e^{-t}$, which is exponential, and hence NWU(2). The distribution function of their convolution is given by

$$F^{(2)}(t) = 1 - (1 + t)e^{-t},$$

which has a strictly increasing failure rate $r_2(t) = t/(1 + t)$. Thus the life distribution of the convolution is not NWU(2).

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