# Bayes estimation for some stochastic partial differential equations

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**Abstract**

The analogues of the Bernstein–von Mises theorem for two type of parabolic stochastic partial differential equations were developed. Asymptotic properties of Bayes estimators for the parameters are investigated following the results on maximum likelihood estimators for such equations discussed by Huebner et al. (in: Stochastic Processes: A Festschrift in Honour of Gopinath Kallianpur. Springer, New York, 1993, pp. 149–160).

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## 1. Introduction

In their recent monograph, Kallianpur and Xiong (1995) discuss the properties of solutions of stochastic partial differential equations (SPDEs). They indicate that SPDEs are being used for stochastic modelling, for instance, in the study of neuronal behaviour in neurophysiology and in building stochastic models of turbulence. The theory of SPDE's is investigated in Ito (1984) and more recently in Rozovskii (1990) and Da Prato and Zabczyk (1992). Huebner et al. (1993) started the investigation of maximum likelihood estimation of parameters of two types of SPDE's and extended their results for a class of parabolic SPDE's in Huebner and Rozovskii (1995).

Our aim in this paper is to obtain Bernstein–von Mises-type theorems for some class of SPDE's and investigate the properties of Bayes estimators of parameters involved in SPDE's. Similar results for a class of diffusion processes have been obtained in Prakasa Rao (1981) and for diffusion fields in Prakasa Rao (1984). We will come back to the problem of Bayes estimation for a general class of parabolic SPDE's in a future publication.

## 2. Stochastic PDE with linear drift (absolutely continuous case)

### 2.1. Bernstein–von Mises theorem

Let $(\Omega,\mathscr{F},P)$ be a probability space and consider the process $u_\varepsilon(t,x)$, $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant T$ governed by the stochastic partial differential equation

$$\mathrm{d}u_\varepsilon(t,x)=(\Delta u_\varepsilon(t,x)+\theta u_\varepsilon(t,x))\,\mathrm{d}t+\varepsilon\,\mathrm{d}W_Q(t,x), \tag{2.1}$$

where $\Delta=\partial^2/\partial x^2$. Suppose that $\varepsilon\to 0$ and $\theta\in\Theta\subset R$. Suppose the initial and the boundary conditions are given by

$$\begin{aligned} &u_\varepsilon(0,x)=f(x),\quad f\in L_2[0,1],\\ &u_\varepsilon(t,0)=u_\varepsilon(t,1)=0,\quad 0\leqslant t\leqslant T \end{aligned} \tag{2.2}$$

and $Q$ is the nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$, so that

$$W_Q(t,x)=Q^{1/2}W(t,x)$$

and $W(t,x)$ is a cylindrical Brownian motion in $L_2[0,1]$. Then, it is known that (cf. Rozovskii, 1990)

$$W_Q(t,x)=\sum_{i=1}^{\infty}q_i^{1/2}e_i(x)W_i(t)\quad \text{a.s.}, \tag{2.3}$$

where $\{W_i(t),0\leqslant t\leqslant T\}$, $i\geqslant 1$ are independent one-dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system in $L_2[0,1]$ consisting of eigen vectors of $Q$ and $\{q_i\}$ eigenvalues of $Q$.

Let us consider a special covariance operator $Q$ with $e_k=\sin k\pi x, k\geqslant 1$ and $\lambda_k=(\pi k)^2, k\geqslant 1$. Then $\{e_k\}$ is a complete orthonormal system with eigen values $q_i=(1+\lambda_i)^{-1}$, $i\geqslant 1$ for the operator $Q$ and $Q=(I-\Delta)^{-1}$. Furthermore,

$$\mathrm{d}W_Q=Q^{1/2}\,\mathrm{d}W.$$

We define a solution $u_\varepsilon(t,x)$ of (2.1) as a formal sum

$$u_\varepsilon(t,x)=\sum_{i=1}^{\infty}u_{i\varepsilon}(t)e_i(x). \tag{2.4}$$

(cf. Rozovskii, 1990). It is known that the Fourier coefficient $u_{i\varepsilon}(t)$ satisfies the stochastic differential equation

$$\mathrm{d}u_{i\varepsilon}(t)=(\theta-\lambda_i)u_{i\varepsilon}(t)\,\mathrm{d}t+\frac{\varepsilon}{\sqrt{\lambda_i+1}}\mathrm{d}W_i(t),\quad 0\leqslant t\leqslant T, \tag{2.5}$$

with the initial condition

$$u_{i\varepsilon}(0)=v_i,\quad v_i=\int_0^1 f(x)e_i(x)\,\mathrm{d}x. \tag{2.6}$$

It is further known that $u_\varepsilon(t,x)$ as defined above belongs to $L_2([0,T]\times\Omega;\ L_2[0,1])$ together with its derivative in $t$. Furthermore, $u_\varepsilon(t,x)$ is the only solution to (2.1) under the boundary condition (2.2). Let $P_\theta^{(\varepsilon)}$ be the measure generated by $u_\varepsilon$ when $\theta$ is the true parameter. Suppose $\theta_0$ is the true parameter. It has been shown by Huebner et al. (1993) that the family of measures $\{P_\theta^{(\varepsilon)}, \theta \in \Theta\}$ are mutually absolutely continuous and

$$\log \frac{\mathrm{d}P_\theta^{(\varepsilon)}}{\mathrm{d}P_{\theta_0}^{(\varepsilon)}}(u_\varepsilon) = \sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\varepsilon^2} \left[ (\theta - \theta_0) \int_0^T u_{i\varepsilon}(t)\,\mathrm{d}u_{i\varepsilon}(t) \right.$$
$$\left. - \frac{1}{2}\{(\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2\} \int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t \right]. \tag{2.7}$$

It can be checked that the MLE $\hat{\theta}_\varepsilon$ of $\theta$ based on $u_\varepsilon$ satisfies the likelihood equation

$$\alpha_\varepsilon = \varepsilon^{-1}(\hat{\theta}_\varepsilon - \theta_0)\beta_\varepsilon \tag{2.8}$$

when $\theta_0$ is the true parameter, where

$$\alpha_\varepsilon = \sum_{i=1}^{\infty} \sqrt{\lambda_i + 1} \int_0^T u_{i\varepsilon}(t)\,\mathrm{d}W_i(t) \tag{2.9}$$

and

$$\beta_\varepsilon = \sum_{i=1}^{\infty} (\lambda_i + 1) \int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t. \tag{2.10}$$

Huebner et al. (1993) proved that the estimator $\hat{\theta}_\varepsilon$ is consistent and asymptotically $\mathrm{N}(0, I(\theta)^{-1})$ and asymptotically efficient in the Hajek–Le Cam sense. They proved that

$$\lim_{\varepsilon \to 0} \sup_{|\theta - \theta_0| < \delta} E_{\theta,\varepsilon} w(\varepsilon^{-1}(\theta_\varepsilon^* - \theta)) \geqslant Ew(\xi), \tag{2.11}$$

where $\xi$ is $\mathrm{N}(0, I(\theta)^{-1})$ for any estimator $\theta_\varepsilon^*$ based on $u_\varepsilon(t,x)$ for a class of loss functions $w(x)$ which are bounded, symmetric with $w(0) = 0$ and $w(x)$ monotone for $x \geqslant 0$. Here

$$I(\theta) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\lambda_i - \theta} v_i^2 (1 - \mathrm{e}^{-2(\theta - \lambda_i)\mathrm{T}}). \tag{2.12}$$

Suppose that $\Lambda$ is a prior probability measure on $(\Theta, \mathscr{B})$ where $\mathscr{B}$ is the $\sigma$-algebra of Borel subsets of an open set $\Theta \subset R$. Further suppose that $\Lambda$ has the density $\lambda(\cdot)$ with respect to the Lebesgue measure and the density $\lambda(\cdot)$ is continuous and positive in an open neighborhood of $\theta_0$, the true parameter. The posterior density of $\theta$ given $u_\varepsilon(t,x)$, $0 < x < 1$, $0 \leqslant t \leqslant T$ is

$$p(\theta|u_\varepsilon) = \frac{(\mathrm{d}P_\theta^{(\varepsilon)}/\mathrm{d}P_{\theta_0}^{(\varepsilon)})(u_\varepsilon)\lambda(\theta)}{\int_\Theta (\mathrm{d}P_\theta^{(\varepsilon)}/\mathrm{d}P_{\theta_0}^{(\varepsilon)})(u_\varepsilon)\lambda(\theta)\,\mathrm{d}\theta}. \tag{2.13}$$

Let $\tau = \varepsilon^{-1}(\theta - \hat{\theta}_\varepsilon)$ and

$$p^*(\tau|u_\varepsilon) = \varepsilon p(\hat{\theta}_\varepsilon + \varepsilon\tau|u_\varepsilon). \tag{2.14}$$

Then $p^*(\tau|u_\varepsilon)$ is the posterior density of $\varepsilon^{-1}(\theta-\hat{\theta}_\varepsilon)$. Let

$$v_\varepsilon(\tau) \equiv \frac{\mathrm{d}P^{(\varepsilon)}_{\hat{\theta}_\varepsilon+\varepsilon\tau}}{\mathrm{d}P^{(\varepsilon)}_{\theta_0}} \bigg/ \frac{\mathrm{d}P^{(\varepsilon)}_{\hat{\theta}_\varepsilon}}{\mathrm{d}P^{(\varepsilon)}_{\theta_0}} = \frac{\mathrm{d}P^{(\varepsilon)}_{\hat{\theta}_\varepsilon+\varepsilon\tau}}{\mathrm{d}P^{(\varepsilon)}_{\hat{\theta}_\varepsilon}} \quad \text{a.s.} \tag{2.15}$$

In view of (2.7), it follows that

$$\begin{aligned}\log v_\varepsilon(\tau) &= \tau\alpha_\varepsilon - \tau\frac{\alpha_\varepsilon}{\beta_\varepsilon}\beta_\varepsilon - \frac{\tau^2}{2}\beta_\varepsilon \quad \text{a.s.}\\ &= -\frac{\tau^2}{2}\beta_\varepsilon \quad \text{a.s.}\end{aligned} \tag{2.16}$$

from Eqs. (2.5) and (2.8). Let

$$C_\varepsilon = \int_{-\infty}^{\infty} v_\varepsilon(\tau)\lambda(\hat{\theta}_\varepsilon + \varepsilon\tau)\,\mathrm{d}\tau. \tag{2.17}$$

It can be seen that

$$p^*(\tau|u_\varepsilon) = C_\varepsilon^{-1} v_\varepsilon(\tau)\lambda(\hat{\theta}_\varepsilon + \varepsilon\tau). \tag{2.18}$$

Suppose the following conditions hold:

(C1) There exists a constant $\beta > 0$ such that $\beta_\varepsilon = \sum_{i=1}^{\infty}(\lambda_i+1)\int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t \to \beta > 0$ a.s. $[P_{\theta_0}]$ as $\varepsilon \to 0$.

(C2) The maximum likelihood estimator $\hat{\theta}_\varepsilon$ is strongly consistent, that is,

$$\hat{\theta}_\varepsilon \to \theta_0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon \to 0;$$

and

(C3) $K(\cdot)$ is a nonnegative function such that, for some $0 < \gamma < \beta$,

$$\int_{-\infty}^{\infty} K(\tau)\mathrm{e}^{-(1/2)\tau^2(\beta-\gamma)}\,\mathrm{d}\tau < \infty.$$

In view of the special shape of the function $v_\varepsilon(\tau)$ given by (2.16), the following lemma can be proved by an application of the dominated convergence theorem.

**Lemma 2.1.** *Suppose conditions* (C1)–(C3) *hold. Then*

$$\lim_{\varepsilon\to 0}\int_{|\tau|\leqslant\delta\varepsilon^{-1}} K(\tau)|v_\varepsilon(\tau)\lambda(\hat{\theta}_\varepsilon+\varepsilon\tau) - \lambda(\theta_0)\exp(-\tfrac{1}{2}\beta\tau^2)|\,\mathrm{d}\tau = 0 \text{ a.s. } [P_{\theta_0}]. \tag{2.19}$$

*for every* $\delta > 0$.

**Lemma 2.2.** *In addition to conditions* (C1) *to* (C3), *suppose that the following condition* (C4) *holds.*

(C4) *For every* $\eta > 0$ *and* $\delta > 0$,

$$\mathrm{e}^{-\eta\varepsilon^{-2}}\int_{|\tau|>\delta} K(\tau\varepsilon^{-1})\lambda(\hat{\theta}_\varepsilon+\tau)\,\mathrm{d}\tau \to 0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon \to 0.$$

*Then*

$$\lim_{\varepsilon\to 0}\int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)|v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)-\lambda(\theta_0)e^{-(1/2)\beta\tau^2}|\,d\tau=0 \quad \text{a.s. } [P_{\theta_0}] \tag{2.20}$$

*for every* $\delta > 0$.

**Proof.** For any $\delta > 0$, there exists some $\eta > 0$ depending on $\delta$ and $\beta$ such that

$$\begin{aligned}
&\int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)|v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)-\lambda(\theta_0)e^{-(1/2)\beta\tau^2}|\,d\tau\\
&\quad\leqslant \int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)\,d\tau\\
&\qquad+\int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)\lambda(\theta_0)e^{-(1/2)\beta\tau^2}\,d\tau\\
&\quad\leqslant e^{-\eta\varepsilon^{-2}}\int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)\,d\tau\\
&\qquad+\lambda(\theta_0)\int_{|\tau|>\delta\varepsilon^{-1}} K(\tau)e^{-(1/2)\beta\tau^2}\,d\tau\\
&\quad=F_\varepsilon+G_\varepsilon \quad \text{(say)}.
\end{aligned} \tag{2.21}$$

In view of condition (C4), it follows that

$$F_\varepsilon \to 0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon\to 0 \tag{2.22}$$

for every $\delta > 0$. Condition (C3) implies that

$$G_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.23}$$

Relations (2.21) to (2.23) prove the lemma.

Lemmas 2.1 and 2.2 together prove that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} K(\tau)|v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)-\lambda(\theta_0)e^{-(1/2)\beta\tau^2}|\,d\tau=0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.24}$$

Let $K(\tau)\equiv 1$. It is easy to see that conditions (C3) and (C4) hold and we have

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)-\lambda(\theta_0)e^{-(1/2)\beta\tau^2}|\,d\tau=0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.25}$$

Hence the term $C_\varepsilon$ defined by (2.17) satisfies the property

$$C_\varepsilon=\int_{-\infty}^{\infty} v_\varepsilon(\tau)\lambda(\hat\theta_\varepsilon+\varepsilon\tau)\,d\tau \to \lambda(\theta_0)\int_{-\infty}^{\infty} e^{-(1/2)\beta\tau^2}\,d\tau \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon\to 0. \tag{2.26}$$

We have now the following main theorem which is an analogue of the Bernstein–von Mises theorem in Borwanker et al. (1971) (cf. Basawa and Prakasa Rao, 1980, and Prakasa Rao, 1987). □

**Theorem 2.3.** *Suppose conditions* (C1) *to* (C4) *hold where* $\lambda(\cdot)$ *is a prior density which is continuous and positive in an open neighborhood of* $\theta_0$, *the true parameter. Then*

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} K(\tau)\left| p^*(\tau|u_\varepsilon) - \left(\frac{\beta}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta\tau^2}\right| \mathrm{d}\tau = 0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.27}$$

**Proof.** Note that

$$\begin{aligned}
&\int_{-\infty}^{\infty} K(\tau)\left|p^*(\tau|u_\varepsilon) - \left(\frac{\beta}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta\tau^2}\right| \mathrm{d}\tau \\
&\quad\leqslant C_\varepsilon^{-1}\int_{-\infty}^{\infty} K(\tau)|v_\varepsilon(\tau)\lambda(\hat{\theta}_\varepsilon+\varepsilon\tau) - \lambda(\theta_0)\mathrm{e}^{-(1/2)\beta\tau^2}|\,\mathrm{d}\tau \\
&\quad\quad+\int_{-\infty}^{\infty} K(\tau)\left|C_\varepsilon^{-1}\lambda(\theta_0) - \left(\frac{\beta}{2\pi}\right)^{1/2}\right| \mathrm{e}^{-(1/2)\beta\tau^2}\,\mathrm{d}\tau
\end{aligned}$$

and the two terms on the right-hand side tend to zero a.s. $[P_{\theta_0}]$ as $\varepsilon\to 0$ by relations (2.24) and (2.26).

**Remark.** Proof of Theorem 2.3 given above is similar to that of Borwanker et al. (1971). The Bernstein–von Mises theorem for a class of diffusion processes with linear drift was proved in Prakasa Rao (1981) and for diffusion fields in Prakasa Rao (1984).

As a consequence of Theorem 2.3, it is easy to see that the following result holds (cf. Borwanker et al., 1971).

**Theorem 2.4.** *Suppose the following conditions hold*:

(D1) $\hat{\theta}_\varepsilon \to \theta_0$ *a.s.* $[P_{\theta_0}]$ *as* $\varepsilon\to 0$;
(D1) $\beta_\varepsilon \to \beta > 0$ *a.s.* $[P_{\theta_0}]$ *as* $\varepsilon \to 0$;
(D3) $\lambda(\cdot)$ *is a prior density which is continuous and positive in an open neighborhood of* $\theta_0$, *the true parameter*; *and*
(D4) $\int_{-\infty}^{\infty}|\theta|^m\lambda(\theta)\,\mathrm{d}\theta < \infty$ *for some integer* $m \geqslant 0$.

*Then*

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |\tau|^m\left| p^*(\tau|u_\varepsilon) - \left(\frac{\beta}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta\tau^2}\right| \mathrm{d}\tau = 0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.28}$$

**Remark.** It is clear that condition (D4) holds for $m = 0$. Suppose conditions (D1)–(D3) hold. Then it follows that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \left| p^*(\tau|u_\varepsilon) - \left(\frac{\beta}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta\tau^2}\right| \mathrm{d}\tau = 0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.29}$$

This is the analogue of the Bernstein–von Mises Theorem in the classical statistical inference. As a special case of Theorem 2.4, we obtain that

$$E_{\theta_0}[\varepsilon^{-1}(\hat{\theta}_\varepsilon - \theta_0)]^m \to E[Z^m] \quad \text{as } \varepsilon \to 0 \tag{2.30}$$

where $Z$ is $\mathrm{N}(0,\beta^{-1})$.

### 2.2. Bayes estimation

We define a *Bayes estimator* $\tilde{\theta}_\varepsilon$ of $\theta$, based on the path $u_\varepsilon$ and the prior density $\lambda(\theta)$, to be a minimizer of the function

$$B_\varepsilon(\phi) = \int_\Theta \tilde{L}(\theta,\phi)p(\theta|u_\varepsilon)\,\mathrm{d}\theta, \quad \phi \in \Theta \tag{2.31}$$

where $\tilde{L}(\theta,\phi)$ is a given loss function defined on $\Theta \times \Theta$. Suppose there exists a Bayes estimator $\tilde{\theta}_\varepsilon$. Further suppose that the loss function satisfies the following conditions:

(E1) $\tilde{L}(\theta,\phi) = L(|\theta-\phi|) \geqslant 0$;

(E2) $L(t)$ is non decreasing for $t \geqslant 0$;

(E3) there exist nonnegative functions $R_\varepsilon$, $K(\tau)$ and $G(\tau)$ such that

(a) $R_\varepsilon L(\tau\varepsilon) \leqslant G(\tau)$ for all $\varepsilon \geqslant 0$,

(b) $R_\varepsilon L(\tau\varepsilon) \to K(\tau)$ as $\varepsilon \to 0$ uniformly on bounded intervals of $\tau$,

(c) the function $\int_{-\infty}^{\infty} K(\tau+m)\mathrm{e}^{-(1/2)\beta\tau^2}\,\mathrm{d}\tau$ achieves its minimum at $m=0$, and

(d) $G(\tau)$ satisfies the conditions akin to (C3) and (C4).

The following result can be proved by arguments similar to those given in Borwanker et al. (1971)).

**Theorem 2.5.** *Suppose conditions* (D1)–(D3) *of Theorem* 2.4 *hold. In addition suppose that the loss function* $\tilde{L}(\theta,\phi)$ *satisfies conditions* (E1)–(E3) *stated above. Then*

$$\varepsilon^{-1}(\hat{\theta}_\varepsilon - \tilde{\theta}_\varepsilon) \to 0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon \to 0 \tag{2.32}$$

*and*

$$\lim_{\varepsilon\to 0} R_\varepsilon B_\varepsilon(\tilde{\theta}_\varepsilon) = \lim_{\varepsilon\to 0} R_\varepsilon B_\varepsilon(\hat{\theta}_\varepsilon) = \left(\frac{\beta}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(\tau)\mathrm{e}^{-(1/2)\beta\tau^2}\,\mathrm{d}\tau \quad \text{a.s. } [P_{\theta_0}]. \tag{2.33}$$

Relations (2.8)–(2.10) and the central limit theorem for stochastic integrals (cf. Basawa and Prakasa Rao, 1980) prove that

$$\varepsilon^{-1}(\hat{\theta}_\varepsilon - \theta_0) \xrightarrow{\mathscr{L}} \mathrm{N}(0,\beta^{-1}) \quad \text{as } \varepsilon \to 0 \tag{2.34}$$

under the probability measure $P_{\theta_0}$. As a consequence of Theorem 2.5 and condition (D1), it follows that

$$\tilde{\theta}_\varepsilon \to \theta_0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } \varepsilon \to 0 \tag{2.35}$$

and

$$\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0) \xrightarrow{\mathscr{L}} \mathrm{N}(0,\beta^{-1}) \quad \text{as } \varepsilon \to 0. \tag{2.36}$$

In other words, the Bayes estimator of the parameter $\theta$ in the SPDE given by (2.1) is asymptotically normal and asymptotically efficient under the conditions stated in Theorem 2.5.

## 3. Stochastic PDE with linear drift (singular case)

### 3.1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and consider the process $u_\varepsilon(t,x)$, $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$ governed by the stochastic partial differential equation

$$\mathrm{d}u_\varepsilon(t,x) = \theta\,\Delta u_\varepsilon(t,x)\,\mathrm{d}t + \varepsilon(I-\Delta)^{-1/2}\,\mathrm{d}W(t,x) \tag{3.1}$$

where $\theta > 0$ satisfying the initial and the boundary conditions

$$\begin{aligned} &u_\varepsilon(0,x) = f(x), \quad 0 < x < 1, \quad f \in L_2[0,1], \\ &u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{3.2}$$

Here $I$ is the identity operator, $\Delta = \partial^2/\partial x^2$ as defined in Section 2 and the process $W(t,x)$ is the cylindrical Brownian motion in $L_2[0,1]$. In analogy with (2.5), the Fourier coefficients $u_{i\varepsilon}(t)$ satisfy the stochastic differential equations

$$\mathrm{d}u_{i\varepsilon}(t) = -\theta\lambda_i u_{i\varepsilon}(t)\,\mathrm{d}t + \frac{\varepsilon}{\sqrt{\lambda_i+1}}\,\mathrm{d}W_i(t), \quad 0 \leqslant t \leqslant T, \tag{3.3}$$

with

$$u_{i\varepsilon}(0) = v_i, \quad v_i = \int_0^1 f(x)e_i(x)\,\mathrm{d}x. \tag{3.4}$$

Let $P_\theta^{(\varepsilon)}$ be the measure generated by $u_\varepsilon$ when $\theta$ is the true parameter. It can be shown that the family of measures $\{P_\theta^{(\varepsilon)},\ \theta \in \Theta\}$ *do not* form a family of equivalent probability measures. In fact, $P_\theta^{(\varepsilon)}$ is singular with respect to $P_{\theta'}^{(\varepsilon)}$ whenever $\theta \neq \theta'$ in $\Theta$ (cf. Huebner et al., 1993).

Let $u_\varepsilon^{(N)}(t,x)$ be the projection of $u_\varepsilon(t,x)$ onto the subspace spanned by $\{e_1,\dots,e_N\}$ in $L_2[0,1]$. In other words,

$$u_\varepsilon^{(N)}(t,x) = \sum_{i=1}^N u_{i\varepsilon}(t)e_i(x). \tag{3.5}$$

Let $P_\theta^{(\varepsilon,N)}$ be the probability measure generated by $u_\varepsilon^{(N)}$ on the subspace spanned by $\{e_1,\dots,e_N\}$ in $L_2[0,1]$. It can be shown that the measures $\{P_\theta^{(\varepsilon,N)}, \theta \in \Theta\}$ from an equivalent family and

$$\begin{aligned} &\log \frac{\mathrm{d}P_\theta^{(\varepsilon,N)}}{\mathrm{d}P_{\theta_0}^{(\varepsilon,N)}}(u_\varepsilon^{(N)}) \\ &\quad = -\frac{1}{\varepsilon^2}\sum_{i=1}^N \lambda_i(\lambda_i+1)\left[(\theta-\theta_0)\int_0^T u_{i\varepsilon}(t)(\mathrm{d}u_{i\varepsilon}(t) + \theta_0\lambda_i u_{i,\varepsilon}(t)\,\mathrm{d}t)\right. \\ &\qquad \left. + \frac{1}{2}(\theta-\theta_0)^2\lambda_i\int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t\right]. \end{aligned} \tag{3.6}$$

It can be checked that the MLE $\hat{\theta}_{\varepsilon,N}$ of $\theta$ based on $u_\varepsilon^{(N)}$ satisfies the likelihood equation

$$\alpha_{\varepsilon,N} = -\varepsilon^{-1}(\hat{\theta}_{\varepsilon,N} - \theta_0)\beta_{\varepsilon,N} \tag{3.7}$$

when $\theta_0$ is the true parameter where

$$\alpha_{\varepsilon,N} = \sum_{i=1}^{N} \lambda_i \sqrt{\lambda_i + 1} \int_0^T u_{i\varepsilon}(t)\,\mathrm{d}W_i(t) \tag{3.8}$$

and

$$\beta_{\varepsilon,N} = \sum_{i=1}^{N} (\lambda_i + 1)\lambda_i^2 \int_0^T u_{i,\varepsilon}^2(t)\,\mathrm{d}t. \tag{3.9}$$

Huebner et al. (1993) prove that, for any fixed $N \geqslant 1$, the estimator $\hat{\theta}_{\varepsilon,N}$ is consistent and asymptotically $\mathrm{N}(0, I_N(\theta_0)^{-1})$ under $P_{\theta_0}^{(\varepsilon,N)}$ as $\varepsilon \to 0$, where

$$I_N(\theta) = \frac{1}{2\theta} \sum_{i=1}^{N} \lambda_i(\lambda_i + 1)v_i^2(1 - \mathrm{e}^{-2\theta\lambda_i T}). \tag{3.10}$$

They further prove that, for any fixed $\varepsilon > 0$,

$$\hat{\theta}_{\varepsilon,N} \xrightarrow{p} \theta_0 \quad \text{under } P_{\theta_0}^{(\varepsilon)} \text{ as } N \to \infty \tag{3.11}$$

and

$$Q_{N,\varepsilon}^{-1}(\theta_0)(\hat{\theta}_{\varepsilon,N} - \theta_0) \xrightarrow{\mathscr{L}} \mathrm{N}(0,1) \quad \text{under } P_{\theta_0}^{(\varepsilon)} \text{ as } N \to \infty \tag{3.12}$$

where

$$Q_{N,\varepsilon}(\theta) = \left( \sum_{i=1}^{N} \lambda_i^2(\lambda_i + 1)E \int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t \right)^{-1/2}. \tag{3.13}$$

In addition, they show that, for any fixed $\varepsilon$ and for any estimator $\theta_{\varepsilon,N}^*$ based on $u_\varepsilon^{(N)}(t,x)$,

$$\lim_{N\to\infty} \sup_{|\theta-\theta_0|<\delta} E_{\theta,\varepsilon}^N \{w(Q_{N,\varepsilon}^{-1}(\theta)(\theta_{\varepsilon,N}^* - \theta))\} \geqslant Ew(\zeta), \tag{3.14}$$

where $\zeta$ is $\mathrm{N}(0,1)$ for a class of loss functions $w(x)$ which are bounded, symmetric with $w(0) = 0$ and $w(x)$ monotone for $x \geqslant 0$. Hence $E_{\theta,\varepsilon}^N$ denotes the expectation under the probability measure $P_\theta^{(\varepsilon,N)}$.

We will now investigate the asymptotic behaviour of the Bayes estimators of $\theta$ as $\varepsilon \to 0$ for fixed $N$ and as $N \to \infty$ for fixed $\varepsilon > 0$. The former case is similar to the results discussed in Section 2.

### 3.2. Bernstein–von Mises Theorem (when $N$ is fixed as $\varepsilon \to 0$)

Suppose that $\Lambda$ is a prior probability measure on $(\Theta, \mathscr{B})$ where $\mathscr{B}$ is the $\sigma$-algebra of Borel subsets of an open set $\Theta \subset R$. Further suppose that $\Lambda$ has a density $\lambda(\cdot)$ with respect to the Lebesgue measure and the density $\lambda(\cdot)$ is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter.

The posterior density of $\theta$ given $u_\varepsilon^{(N)}$ is

$$p(\theta|u_\varepsilon^{(N)}) = \frac{(\mathrm{d}P_\theta^{(\varepsilon,N)}/\mathrm{d}P_{\theta_0}^{(\varepsilon,N)})(u_\varepsilon^{(N)})\lambda(\theta)}{\int_\Theta (\mathrm{d}P_\theta^{(\varepsilon,N)}/\mathrm{d}P_{\theta_0}^{(\varepsilon,N)})(u_\varepsilon^{(N)})\lambda(\theta)\,\mathrm{d}\theta}. \tag{3.15}$$

Let

$$\tau = \varepsilon^{-1}(\theta - \hat{\theta}_{\varepsilon,N}) \tag{3.16}$$

and

$$p^*(\tau|u_\varepsilon^{(N)}) = \varepsilon p(\hat{\theta}_{\varepsilon,N} + \varepsilon\tau|u_\varepsilon^{(N)}). \tag{3.17}$$

Then $p^*(\tau|u_\varepsilon^{(N)})$ is the posterior density of $\varepsilon^{-1}(\theta - \hat{\theta}_{\varepsilon,N})$. Let

$$v_{\varepsilon,N}(\tau) = \frac{\mathrm{d}P^{(\varepsilon,N)}_{\hat{\theta}_{\varepsilon,N}+\varepsilon\tau}}{\mathrm{d}P^{(\varepsilon,N)}_{\theta_0}} \Bigg/ \frac{\mathrm{d}P^{(\varepsilon,N)}_{\hat{\theta}_{\varepsilon,N}}}{\mathrm{d}P^{(\varepsilon,N)}_{\theta_0}} \quad \text{a.s. } [P^{(\varepsilon,N)}_{\theta_0}]. \tag{3.18}$$

It is easy to see that

$$\log v_{\varepsilon,N}(\tau) = -\frac{\tau^2}{2}\beta_{\varepsilon,N} \quad \text{a.s. } [P^{(\varepsilon,N)}_{\theta_0}] \tag{3.19}$$

in view of (3.7). Suppose the following conditions hold:

(C1)′ $\beta_{\varepsilon,N} = \sum_{i=1}^N (\lambda_i + 1)\lambda_i^2 \int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t \to \beta_N > 0$ a.s. under $\{P^{(\varepsilon,N)}_{\theta_0}\}$ as $\varepsilon \to 0$;

(C2)′ the maximum likelihood estimator $\hat{\theta}_{\varepsilon,N}$ is strongly consistent as $\varepsilon \to 0$, that is,

$$\hat{\theta}_{\varepsilon,N} \to \theta_0 \quad \text{a.s. under } \{P^{(\varepsilon,N)}_{\theta_0}\} \text{ as } \varepsilon \to 0;$$

(C3)′ $K(\cdot)$ is a nonnegative function such that, for some $0 < \gamma < \beta_N$,

$$\int_{-\infty}^{\infty} K(\tau)\mathrm{e}^{-(1/2)\tau^2(\beta_N - \gamma)}\,\mathrm{d}\tau < \infty;$$

and

(C4)′ for every $\eta > 0$ and $\delta > 0$

$$\mathrm{e}^{-\eta\varepsilon^{-2}} \int_{|\tau|>\delta} K(\tau\varepsilon^{-1})\lambda(\hat{\theta}_{\varepsilon,N} + \tau)\,\mathrm{d}\tau \to 0 \quad \text{a.s.}$$

under $\{P^{(\varepsilon,N)}_{\theta_0}\}$ as $\varepsilon \to 0$.

Under conditions (C1)′–(C4)′, the following theorems can be proved by arguments analogous to those given in the proofs of Theorems 2.3 and 2.4.

**Theorem 3.1.** *Suppose conditions* (C1)′–(C4)′ *hold where $\lambda(\cdot)$ is a prior density which is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter. Then*

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} K(\tau) \left| p^*(\tau|u_\varepsilon^{(N)}) - \left(\frac{\beta_N}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta_N\tau^2} \right| \mathrm{d}\tau = 0 \quad \text{a.s.} \tag{3.20}$$

*under* $\{P^{(\varepsilon,N)}_{\theta_0}\}$.

**Theorem 3.2.** *Suppose the following conditions hold*:

(D1)′ $\hat{\theta}_{\varepsilon,N} \to \theta_0$ *a.s. under* $P^{(\varepsilon,N)}_{\theta_0}$ *as* $\varepsilon \to 0$;

(D2)′ $\beta_{\varepsilon,N} \to \beta_N > 0$ *a.s. under* $\{P^{(\varepsilon,N)}_{\theta_0}\}$ *as* $\varepsilon \to 0$;

(D3)′ *$\lambda(\cdot)$ is a prior density which is continuous and positive in an open neighbourhood of $\theta_0$, the true parameter; and*

(D4)′ $\int_{-\infty}^{\infty} |\theta|^m \lambda(\theta)\,\mathrm{d}\theta < \infty$ *for some integer* $m \geqslant 0$.

*Then*

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |\tau|^m \left| p^*(\tau|u_\varepsilon^{(N)}) - \left(\frac{\beta_N}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\beta_N \tau^2}\right| \mathrm{d}\tau = 0 \quad a.s. \tag{3.21}$$

*under* $\{P_{\theta_0}^{(\varepsilon,N)}\}$.

### 3.3. Bayes estimation (when $N$ is fixed and $\varepsilon \to 0$)

We define a Bayes estimator $\tilde{\theta}_{\varepsilon,N}$ of $\theta$ based on the path $u_\varepsilon^{(N)}$ and the prior density $\lambda(\theta)$ as an estimator which minimizes

$$B_{\varepsilon,N}(\phi) = \int_{\Theta} \tilde{L}(\theta,\phi)\, p(\theta|u_\varepsilon^{(N)})\,\mathrm{d}\theta \tag{3.22}$$

where $\tilde{L}(\theta,\phi)$ is a loss function satisfying the properties (E1)–(E3) stated in Section 2. One can prove the following theorem as an application of Theorem 3.2.

**Theorem 3.3.** *Suppose conditions* (D1)′–(D3)′ *of Theorem* 3.2 *hold. In addition, suppose the loss function* $\tilde{L}(\theta,\phi)$ *satisfies conditions* (E1)–(E3) *stated above. Then*

$$\varepsilon^{-1}(\hat{\theta}_{\varepsilon,N} - \tilde{\theta}_{\varepsilon,N}) \to 0 \quad \text{a.s. } under\ \{P_{\theta_0}^{(\varepsilon,N)}\}\ as\ \varepsilon \to 0 \tag{3.23}$$

*and*

$$\lim_{\varepsilon\to 0} R_\varepsilon B_{\varepsilon,N}(\tilde{\theta}_{\varepsilon,N}) = \lim_{\varepsilon\to 0} R_\varepsilon B_{\varepsilon,N}(\hat{\theta}_{\varepsilon,N}) = \left(\frac{\beta_N}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(\tau) \mathrm{e}^{-(1/2)\beta_N \tau^2}\,\mathrm{d}\tau \quad a.s. \tag{3.24}$$

*under* $\{P_{\theta_0}^{(\varepsilon,N)}\}$.

*In particular, it follows that*

$$\tilde{\theta}_{\varepsilon,N} \to \theta_0 \quad \text{a.s. } under\ \{P_{\theta_0}^{(\varepsilon,N)}\}\ as\ \varepsilon \to 0 \tag{3.25}$$

*and*

$$\varepsilon^{-1}(\tilde{\theta}_{\varepsilon,N} - \theta_0) \xrightarrow{\mathscr{L}} \mathrm{N}(0, \beta_N^{-1})\ as\ \varepsilon \to 0 \tag{3.26}$$

*giving the asymptotic properties of the Bayes estimator* $\tilde{\theta}_{\varepsilon,N}$.

Let us now consider the problem of Bayes estimation for the stochastic PDE given by (3.1) as $N \to \infty$ for any fixed $\varepsilon > 0$.

### 3.4. Bernstein–von Mises theorem and Bayes estimation (when $\varepsilon$ is fixed and $N \to \infty$)

Let

$$Q_N^{(\varepsilon)}(\theta) = \left(\sum_{i=1}^{N} \lambda_i^2(\lambda_i + 1) E_{\varepsilon,N} \int_0^T u_{i\varepsilon}^2(t)\,\mathrm{d}t\right)^{-1/2} \tag{3.27}$$

and suppose that

$(D0)Q_N^{(\varepsilon)}(\theta) \to 0$ as $N \to \infty$ for any fixed $\varepsilon > 0$. Let

$$\tau = Q_N^{(\varepsilon)}(\theta)^{-1}(\theta - \hat{\theta}_{\varepsilon,N}), \tag{3.28}$$

$$\tilde{p}(\tau|u_\varepsilon^{(N)}) = Q_N^{(\varepsilon)}(\theta)\, p(\hat{\theta}_{\varepsilon,N} + Q_N^{(\varepsilon)}(\theta)\tau|u_\varepsilon^{(N)}), \tag{3.29}$$

and

$$\tilde{v}_{\varepsilon,N}(\tau) = \frac{\mathrm{d}P^{(\varepsilon,N)}_{\hat{\theta}_{\varepsilon,N}+Q_N^{(\varepsilon)}(\theta)\tau}}{\mathrm{d}P^{(\varepsilon,N)}_{\theta_0}} \Bigg/ \frac{\mathrm{d}P^{(\varepsilon,N)}_{\hat{\theta}_{\varepsilon,N}}}{\mathrm{d}P^{(\varepsilon,N)}_{\theta_0}} \quad \text{a.s. } [P^{(\varepsilon,N)}_{\theta_0}]\,. \tag{3.30}$$

It can be checked that

$$\log \tilde{v}_{\varepsilon,N}(\tau) = -\frac{Q_N^{(\varepsilon)}(\theta_0)^2}{2\varepsilon^2}\tau^2 \beta_{\varepsilon,N} \quad \text{a.s. } [P^{(\varepsilon,N)}_{\theta_0}] \tag{3.31}$$

in view of (3.7). Note that $\varepsilon$ is a fixed positive constant in the present discussion. Suppose the following conditions hold:

(C1)″ $\frac{Q_N^{(\varepsilon)}(\theta_0)^2 \beta_{\varepsilon,N}}{\varepsilon^2} \to 1$ a.s. under $\{P^{(\varepsilon,N)}_{\theta_0}\}$ as $N \to \infty$;

(C2)″ the minimum likelihood estimator $\hat{\theta}_{\varepsilon,N}$ is strongly consistent as $N \to \infty$, that is

$$\hat{\theta}_{\varepsilon,N} \to \theta_0 \quad \text{a.s. under } \{P^{(\varepsilon,N)}_{\theta_0}\} \text{ as } N \to \infty;$$

(C3)″ the function $K(\cdot)$ is a nonnegative function such that for some $0 < \gamma < 1$,

$$\int_{-\infty}^{\infty} K(\tau)\mathrm{e}^{-(1/2)\tau^2(1-\gamma)}\,\mathrm{d}\tau < \infty;$$

and

(C4)″ for every $\eta > 0$ and $\delta > 0$

$$\mathrm{e}^{-\eta Q_N^{(\varepsilon)}}(\theta_0)^{-2} \int_{-\infty}^{\infty} K(\tau Q_N^{(\varepsilon)^{-1}}(\theta_0))\lambda(\hat{\theta}_{\varepsilon,N} + \tau)\,\mathrm{d}\tau \to 0$$
$$\text{a.s. under } \{P^{(\varepsilon,N)}_{\theta_0}\} \text{ as } N \to \infty.$$

The following analogues of Theorems 3.1–3.3 hold under the conditions (C1)″–(C4)″. We omit the details.

**Theorem 3.4.** *Suppose conditions* (C1)″–(C4)″ *hold where* $\lambda(\cdot)$ *is a prior density which is continuous and positive in an open neighbourhood of* $\theta_0$, *the true parameter. Then, for any fixed* $\varepsilon > 0$,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} K(\tau) \left| \tilde{p}(\tau | u_\varepsilon^{(N)}) - \left(\frac{1}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\tau^2} \right| \mathrm{d}\tau = 0 \quad \text{a.s. } [P_{\theta_0}^{(\varepsilon)}]. \tag{3.32}$$

**Theorem 3.5.** *Suppose the following conditions hold for a fixed* $\varepsilon > 0$, *in addition to the condition* (D0) *stated above*:

(D1)″ $\hat{\theta}_{\varepsilon,N} \to \theta_0$ *a.s.* $[P_{\theta_0}^{(\varepsilon)}]$ *as* $N \to \infty$;
(D2)″ $\frac{\beta_{\varepsilon,N} Q_N^{(\varepsilon)}(\theta_0)^2}{\varepsilon^2} \to 1$ *a.s.* $[P_{\theta_0}^{(\varepsilon)}]$ *as* $N \to \infty$;
(D3)″ $\lambda(\cdot)$ *is a prior density which is continuous and positive in an open neighbourhood of* $\theta_0$, *the true parameter*; *and*
(D4)″ $\int_{-\infty}^{\infty} |\theta|^m \lambda(\theta)\,\mathrm{d}\theta < \infty$ *for some integer* $m \geqslant 0$.
*Then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\tau|^m \left| \tilde{p}(\tau | u_\varepsilon^{(N)}) - \left(\frac{1}{2\pi}\right)^{1/2} \mathrm{e}^{-(1/2)\tau^2} \right| \mathrm{d}\tau = 0 \ \ \textit{a.s. under } \{P_{\theta_0}^{(\varepsilon,N)}\}. \tag{3.33}$$

**Theorem 3.6.** *Suppose conditions* (D1)″–(D3)″ *of Theorem* 3.5 *hold. In addition suppose the loss function* $\tilde{L}(\theta, \phi)$ *satisfies conditions* (E1)–(E3) *stated above. Then, for any fixed* $\varepsilon > 0$,

$$Q_N^{(\varepsilon)^{-1}}(\theta_0)(\hat{\theta}_{\varepsilon,N} - \tilde{\theta}_{\varepsilon,N}) \to 0 \quad \textit{a.s. } [P_{\theta_0}^{(\varepsilon)}] \textit{ as } N \to \infty, \tag{3.34}$$

*and*

$$\begin{aligned}\lim_{n\to\infty} R_{Q_N^{(\varepsilon)}(\theta_0)} B_{\varepsilon,N}(\tilde{\theta}_{\varepsilon,N}) &= \lim_{N\to\infty} R_{Q_N^{(\varepsilon)}(\theta_0)} B_{\varepsilon,N}(\tilde{\theta}_{\varepsilon,N}) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(\tau) \mathrm{e}^{-(1/2)\tau^2}\,\mathrm{d}\tau \quad \textit{a.s. } [P_{\theta_0}^{(\varepsilon)}].\end{aligned} \tag{3.35}$$

As a consequence of Theorem 3.6 and relation (3.12) it follows that, for any fixed $\varepsilon > 0$,

$$\tilde{\theta}_{\varepsilon,N} \to \theta_0 \quad \text{a.s. under } \{P_{\theta_0}^{(\varepsilon,N)}\} \quad \text{as } N \to \infty \tag{3.36}$$

and

$$Q_N^{(\varepsilon)^{-1}}(\theta_0)(\tilde{\theta}_{\varepsilon,N} - \theta_0) \xrightarrow{\mathscr{L}} \mathrm{N}(0,1) \quad \text{as } N \to \infty. \tag{3.37}$$

## References

Basawa, I.V., Prakasa Rao, B.L.S., 1980. Statistical Inference for Stochastic Processes. Academic Press, London.

Borwanker, J.D., Kallianpur, G., Prakasa Rao, B.L.S., 1971. The Bernstein–von Mises theorem for Markov processes. Ann. Math. Statist. 42, 1241–1253.

Da Prato, G., Zabczyk, J., 1992. Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge.
Huebner, M., Khasminski, R., Rozovskii, B.L., 1993. Two examples of parameter estimation for stochastic partial differential equations. In: Cambanis, S., Ghosh, J.K., Karandikar, R.L., Sen, P.K. (Eds.), Stochastic Processes: A Festschrift in Honour of Gopinath Kallianpur. Springer, New York, pp. 149–160.
Huebner, M., Rozovskii, B.L., 1995. On asymptotic properties of maximum likelihood estimators for parabolic stochastic SPDE's. Probab. Theory Related Fields 103, 143–163.
Ito, K., 1984. Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces. CBMS Notes, Vol. 47. SIAM, Baton Rouge.
Kallianpur, G., Xiong, J., 1995. Stochastic Differential Equations in Infinite Dimensions. IMS Lecture Notes, Vol. 26. Hayward, California.
Prakasa Rao, B.L.S., 1981. The Bernstein–von Mises theorem for a class of diffusion processes. Tear. Sluch. Proc. 9, 95–101 (in Russian).
Prakasa Rao, B.L.S., 1984. On Bayes estimation for diffusion fields. In: Ghosh, J.K., Roy, J. (Eds.), Statistics: Applications and New Directions. Statistical Publishing Society, Calcutta, pp. 504–511.
Prakasa Rao, B.L.S., 1987. Asymptotic Theory of Statistical Inference. Wiley, New York.
Rozovskii, B.L., 1990. Stochastic Evolution Systems. Kluwer, Dordrecht.