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SOME REMARKS ON BOCHNER–RIESZ MEANS

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Abstract. We study L^p norm convergence of Bochner–Riesz means $S_R^\delta f$ associated with certain non-negative differential operators. When the kernel $S_R^m(x,y)$ satisfies a weak estimate for large values of m we prove L^p norm convergence of $S_R^\delta f$ for $\delta > n|1/p-1/2|$, $1<p<\infty$, where n is the dimension of the underlying manifold.

1. Introduction and main results. The aim of this note is to make some remarks concerning the L^p mapping properties of the Bochner–Riesz means associated with certain differential operators. To set up the notation, let Ω be a Riemannian manifold and P a differential operator of order d on Ω which is self-adjoint and formally non-negative. Let

$$Pf = \int_0^\infty \lambda\, dE_\lambda f$$

be the spectral resolution of P. The *Bochner–Riesz mean* of order $\delta \ge 0$ of a function f is defined by

$$S_R^\delta f = \int_0^R \left(1-\frac{\lambda}{R}\right)^\delta dE_\lambda f.$$

Our aim is to study the convergence of $S_R^\delta f$ to f in $L^p(\Omega)$ as R tends to infinity.

It is clear that $S_R^\delta f$ converges to f in the L^2 norm if $f \in L^2(\Omega)$. However, if $1 \le p < 2$ we can expect the convergence in the L^p norm only for large values of δ. In fact, there is a necessary condition: let $\delta(p) = \max\{n|1/p-1/2|-1/2, 0\}$ be the critical index for the L^p summability. Then by a transplantation theorem of Mityagin [16] (see Kenig, Stanton and Tomas [13]) it is known that $\delta > \delta(p)$ is a necessary condition for the convergence of $S_R^\delta f$ to f in the L^p norm.

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It is therefore natural to conjecture that $\delta > \delta(p)$ is also sufficient for the L^p convergence of $S_R^\delta f$. Let us call this the Bochner–Riesz conjecture. In some special cases the conjecture has been proved for a certain range of p. For example, when $\Omega = T^n$ is the n-torus and P is the standard Laplacian on T^n then $S_R^\delta f$ converges to f in $L^p(T^n)$ for $\delta > \delta(p)$ provided $|1/p - 1/2| \ge 1/(n+1)$ for $n \ge 3$ and all p if $n = 2$. The same is true for the standard Laplacian on $\mathbb{R}^n$. For these results see Carleson–Sjölin [2] and Fefferman [4].

In the general situation the best known results are due to Hörmander [8] and Peetre [19]. Their results are that we have L^p convergence when $\delta > 2(n-1)|1/p - 1/2|$ provided Ω is compact. In 1987, Hörmander's result was greatly improved by Sogge [21]; he showed that the Bochner–Riesz conjecture holds for compact Riemannian manifolds when P is of degree 2 and $|1/p - 1/2| \ge 1/(n+1)$. When $n = 2$ the conjecture has been proved for all p.

Once we leave the premises of compact manifolds and consider non-compact situations, not much is known. In the special cases of Hermite and special Hermite expansions which are associated with the operators $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ and

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

on $\mathbb{C}^n$ respectively, the conjecture has been settled for a certain range of p. See the works of the author [24, 25], Karadzhov [12] and Stempak–Zienkiewicz [23]. When P is the sublaplacian on a stratified nilpotent Lie group a weaker form of the conjecture is known to be true (see Mauceri [14], Mauceri–Meda [15], Müller–Stein [17] and the references there).

Returning to the general situation we recall the following estimate, due to Hörmander [8] and Peetre [19], on the kernel of the Riesz mean associated with a dth order elliptic differential operator. If $S_R^\delta(x,y)$ is the kernel of $S_R^\delta f$, that is, if

$$S_R^\delta f(x) = \int S_R^\delta(x,y)f(y)\,dy$$

then for x, y belonging to a compact subset B of Ω,

$$|S_R^\delta(x,y)| \le C_B R^{n/d}(1 + R^{1/d}|x-y|)^{-\delta-1}$$

where C_B is independent of R. From this estimate it follows that the operators $\chi_B S_R^\delta \chi_B$, where $\chi_B f(x) = \chi_B(x) f(x)$ is the operator of multiplication by the indicator function of B, are uniformly bounded on $L^p(\Omega)$ when $\delta > 2(n-1)|1/p - 1/2|$. This is the best one can get from the above kernel estimates which is local in nature.

Even in the case of compact manifolds where the above estimate is "global" it cannot be improved further and, therefore, it is not good enough to prove the Bochner–Riesz conjecture. In order to get around this difficulty, in [21] Sogge used a Fourier transform side argument to prove certain L^p-L^2 estimates for the projections associated with the spectral resolution. To be more specific, when P is a second order operator, let

$$P_k f = \int_{(k-1)^2}^{k^2} dE_\lambda f.$$

Then Sogge used the following estimates, known as the *restriction theorem*:

$$\|P_k f\|_2 \le C k^{\delta(p)} \|f\|_p, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| \ge \frac{1}{n+1}.$$

The above estimates were proved by Sogge in [20] for second order elliptic differential operators on compact Riemannian manifolds. By adapting an argument of Fefferman–Stein [4] and Bonami–Clerc [1], he was able to show that the weak kernel estimates and restriction theorems are sufficient to prove summation results.

Unfortunately, we do not have good restriction theorems even on $\mathbb{R}^n$ for general elliptic differential operators. Since it is difficult to establish restriction theorems, we look for an alternative which can be used in the study of Bochner–Riesz means. Consider fractional powers of the operator P given by the spectral theorem as

$$(1+P)^{-\alpha/2} f = \int_0^\infty (1+\lambda)^{-\alpha/2}\, dE_\lambda f.$$

Once we have the restriction theorem it then follows that

$$\|(1+P)^{-\alpha/2} f\|_2^2 \le \sum_{k=1}^\infty k^{-2\alpha} \|P_k f\|_2^2,$$

which is dominated by

$$C\Big(\sum_{k=1}^\infty k^{-2\alpha + 2n(1/p-1/2)-1}\Big) \|f\|_p^2$$

and therefore

$$\|(1+P)^{-\alpha/2} f\|_2 \le C \|f\|_p$$

provided $\alpha > n(1/p - 1/2)$.

In many situations, the L^p-L^2 estimate for the operator $(1+P)^{-\alpha/2}$ is easy to establish. We propose to use this in place of the restriction theorem. As we show below, the L^p-L^2 estimate for $(1+P)^{-\alpha/2}$ will follow from a weak estimate for the Riesz kernel. Then by using the method of Sogge we can establish a positive result for the Bochner–Riesz means which improves the known results, though falling short of being optimal.

Now we state our main results. Let $P = P(x, D)$ be a (not necessarily elliptic) differential operator of degree d on $\mathbb{R}^n$ with smooth coefficients. We assume that the Riesz kernel associated with P satisfies the estimate

$$|S_R^\delta(x,y)| \le CR^{n/d}(1 + R^{1/d}|x-y|)^{-\delta+\beta} \tag{1.1}$$

for all $x, y \in \mathbb{R}^n$ where β is a fixed constant. Without loss of generality, we assume that the spectral projection P_0 onto the kernel of P is trivial on $L^p(\Omega)$.

THEOREM 1.1. *Let P be as above with Riesz kernel satisfying* (1.1). *Then the Bochner–Riesz means S_R^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ whenever $\delta > \delta(p) + 1/2$.*

As we have remarked earlier, for an arbitrary differential operator on a non-compact manifold the estimate (1.1) is available only locally. However, the local estimate is good enough to prove a local estimate for the Bochner–Riesz means.

THEOREM 1.2. *Let $P(x, D)$ be an elliptic differential operator of degree d with smooth coefficients on $\mathbb{R}^n$ and let B be any compact subset of $\mathbb{R}^n$. Then $\chi_B S_R^\delta \chi_B$ are uniformly bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, whenever $\delta > \delta(p) + 1/2$.*

By using the transplantation theorem of Mityagin [16], we can deduce global estimates for constant coefficient differential operators.

COROLLARY 1.3. *Let $P(D)$ be a homogeneous elliptic differential operator on $\mathbb{R}^n$ and let S_R^δ be the associated Bochner–Riesz means. Then S_R^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ whenever $\delta > \delta(p) + 1/2$.*

Apart from the realm of constant coefficient differential opeartors there is at least one more class of differential operators, namely, Rockland operators on stratified nilpotent groups, for which global estimates for the Bochner–Riesz kernel can be proved. So, for such operators we obtain the following result.

COROLLARY 1.4. *Let L be a non-negative Rockland operator of homogeneous degree d on a stratified nilpotent Lie group G and let S_R^δ be the associated Bochner–Riesz means. Let Q be the homogeneous dimension of the group and define the critical index by $\delta(p) = \max\{Q|1/p - 1/2| - 1/2, 0\}$. Then S_R^δ are uniformly bounded on $L^p(G)$, $1 < p < \infty$, provided $\delta > \delta(p) + 1/2$.*

Suppose H is a connected normal subgroup of G and π a unitary representation of G induced from a unitary character of H. Then following an idea of Hulanicki and Jenkins [11] we can get summability results for operators of the form $\pi(L)$ where L is a Rockland operator. This covers some Schrödinger operators with polynomial potential (see Corollaries 3.3 and 3.4).

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2. Elliptic operators on $\mathbb{R}^n$. In this section we will prove Theorems 1.1, 1.2 and Corollary 1.3. We start with the following estimate for $(1+P)^{-\alpha/d}$.

PROPOSITION 2.1. *Let $P(x,D)$ be a differential operator of degree d whose Riesz kernel satisfies the estimate* (1.1). *Then for $0<\alpha<n$, $1<p<q<\infty$ and $1/q=1/p-\alpha/n$ we have*

$$\|(1+P)^{-\alpha/d}f\|_q \le C\|f\|_p.$$

Proof. By the spectral theorem

$$(1+P)^{-\alpha/d}f = \int_0^\infty (1+\lambda)^{-\alpha/d}\,dE_\lambda f$$

and so the kernel of $(1+P)^{-\alpha/d}$ is given by

$$K_\alpha(x,y) = \int_0^\infty (1+\lambda)^{-\alpha/d}\,dE_\lambda(x,y).$$

We want to make use of the estimate (1.1) for large values of δ. As $E_\lambda(x,y) = S_\lambda^0(x,y)$, integrating by parts and making use of the identity

$$\frac{d}{d\lambda}(\lambda^m S_\lambda^m(x,y)) = m\lambda^{m-1}S_\lambda^{m-1}(x,y)$$

we obtain the expression

$$K_\alpha(x,y) = C_{\alpha,m}\int_0^\infty (1+\lambda)^{-\alpha/d-m}\lambda^{m-1}S_\lambda^{m-1}(x,y)\,d\lambda.$$

If we use the estimate (1.1) we get

$$|K_\alpha(x,y)| \le C\int_0^\infty \lambda^{-\alpha/d+n/d-1}(1+\lambda^{1/d}|x-y|)^{-m+\beta+1}\,d\lambda,$$

which is easily seen to be bounded by

$$C|x-y|^{\alpha-n}\int_0^\infty \lambda^{-\alpha/d+n/d-1}(1+\lambda^{1/d})^{-m+\beta+1}\,d\lambda.$$

The last integral converges if m is large since $0<\alpha<n$ and we obtain the estimate

$$|K_\alpha(x,y)| \le C|x-y|^{\alpha-n}$$

for the kernel of the operator $(1+P)^{-\alpha/d}$. Now it is a routine matter to show that this operator has the required mapping properties. See, e.g., the proof of Theorem 1, Chapter V of Stein [22]. This completes the proof of the proposition.

We now proceed to the proof of Theorem 1.1. Since we closely follow Sogge [21] we will not give details. Choose $\varphi \in C_0^\infty(1/2,2)$ so that $\sum_{j=-\infty}^{\infty} \varphi(2^j t) = 1$ for $t \neq 0$. Let

$$\varphi_{R,j}^\delta(t) = \varphi(2^j(1-t/R))(1-t/R)^\delta$$

and for $j = 1, 2, \ldots$ define

$$S_{R,j}^\delta f = \int_0^\infty \varphi_{R,j}^\delta(\lambda)\, dE_\lambda f.$$

For $j = 0$ we define

$$S_{R,0}^\delta f = \int_0^\infty \varphi_0\left(1-\frac{\lambda}{R}\right)\left(1-\frac{\lambda}{R}\right)^\delta dE_\lambda f$$

where $\varphi_0(t) = 1 - \sum_{j=1}^\infty \varphi(2^j t)$.

We can easily handle $S_{R,0}^\delta$ in the following way.

Proposition 2.2. $\|S_{R,0}^\delta f\|_p \le C\|f\|_p$, $1 \le p \le \infty$.

Proof. As in the proof of Proposition 2.1 we can get

$$S_{R,0}^\delta(x,y) = \int_0^\infty \lambda^{m-1} S_\lambda^{m-1}(x,y)\partial_\lambda^m \varphi_{R,0}^\delta(\lambda)\, d\lambda.$$

Note that $\varphi_{R,0}^\delta$ is supported in $(-\infty, R/2)$ and satisfies the estimate

$$|\partial_\lambda^m \varphi_{R,0}^\delta(\lambda)| \le CR^{-m}.$$

Therefore,

$$|S_{R,0}^\delta(x,y)| \le CR^{-m} \int_0^{R/2} \lambda^{m-1}|S_\lambda^{m-1}(x,y)|\, d\lambda.$$

If m is large enough, $S_\lambda^{m-1}(x,y)$ is uniformly integrable and hence the proposition follows.

Proceeding with the proof of Theorem 1.1 we will show that given $\delta > \delta(p) + 1/2$ there exists an $\varepsilon > 0$ such that

$$\|S_{R,j}^\delta f\|_p \le C2^{-\varepsilon j}\|f\|_p$$

for $j = 1, 2, \ldots$ As in [21], using the kernel estimate we can show that for each $\gamma > 0$ there is an $\varepsilon > 0$ such that

$$\int\limits_{R^{1/d}|x-y|\geq 2^{(1+\gamma)j}} |S^\delta_{R,j}(x,y)|\, dy \leq C2^{-\varepsilon j}.$$

This will take care of the global part of the Riesz kernel. Then we prove the following.

PROPOSITION 2.3. $\|S^\delta_{R,j} f\|_2 \leq C2^{-j\delta}(R^{1/d})^{\delta(p)+1/2}\|f\|_p$.

Proof. By the spectral theorem

$$\|S^\delta_{R,j} f\|_2^2 = \int\limits_0^\infty |\varphi^\delta_{R,j}(\lambda)|^2\, d(E_\lambda f, f).$$

Since $\varphi^\delta_{R,j}$ is supported in

$$R(1-2^{-j+1}) \leq \lambda \leq R(1-2^{-j-1})$$

and bounded by $C2^{-j\delta}$ it follows that

$$\|S^\delta_{R,j} f\|_2^2 \leq C2^{-2j\delta}(R^{1/d})^{2\delta(p)+1} \int\limits_0^\infty (1+\lambda)^{-(2\delta(p)+1)/d}\, d(E_\lambda f, f),$$

which is dominated by

$$C2^{-2j\delta}(R^{1/d})^{2\delta(p)+1}\|(1+P)^{-\alpha/d} f\|_2^2$$

with $\alpha = n|1/p - 1/2|$. Using the result of Proposition 2.1 we obtain

$$\|S^\delta_{R,j} f\|_2 \leq C2^{-j\delta}(R^{1/d})^{\delta(p)+1/2}\|f\|_p,$$

which completes the proof of the proposition.

Finally, if V is any ball of radius $2^{(1+\gamma)j}R^{-1/d}$, then

$$\|S^\delta_{R,j} f\|_{L^p(V)} \leq C(2^{(1+\gamma)j}R^{-1/d})^{\delta(p)+1/2}\|S^\delta_{R,j} f\|_2,$$

which by the result of the previous proposition is dominated by

$$C2^{-j\delta}2^{j(1+\gamma)(\delta(p)+1/2)}\|f\|_p.$$

Therefore, if $\delta > \delta(p) + 1/2$ we can choose $\gamma > 0$ so that $\delta > (1+\gamma)(\delta(p) + 1/2)$, which will then show that

$$\|S^\delta_{R,j} f\|_p \leq C2^{-\varepsilon j}\|f\|_p$$

for some $\varepsilon > 0$. The rest of the proof proceeds as in Sogge [21].

We will now indicate how Theorem 1.2 is proved. Suppose $P = P(x, D)$ is an elliptic differential operator of order d with smooth coefficients. The following local estimate for the associated Riesz kernel has been proved in Peetre [19] (see also Hörmander [8]).

Proposition 2.4. *Let P be as above. Then*

$$|S_R^\delta(x,y)| \le CR^{n/d}(1+R^{1/d}|x-y|)^{-\delta+\beta}$$

where C is uniform on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$ and β is a universal constant.

From this proposition it follows that the kernel of $\chi_B S_R^\delta \chi_B$ satisfies a uniform estimate of the form (1.1) which can be used to take care of the "part at infinity" of the operator. To deal with the local part we need a local version of Proposition 2.1.

Proposition 2.5. *Let P be as above. Then for $0<\alpha<n$, $1<p<q<\infty$, and $1/q = 1/p - \alpha/n$ we have*

$$\|(1+P)^{-\alpha/d} f\|_{L^p(B)} \le C_B \|f\|_{L^p(B)}$$

for any compact subset B of $\mathbb{R}^n$.

Proof. We only have to show that the kernel of $\chi_B(1+P)^{-\alpha/d}\chi_B$ is bounded by a constant times $|x-y|^{\alpha-n}$. But this follows from the local estimate for the Riesz kernel given in Proposition 2.4.

To complete the proof Theorem 1.2 we only need to make the following observation. If V is any ball, then as before

$$\|\chi_B S_{R,j}^\delta \chi_B f\|_{L^p(V)} \le |V|^{1/p-1/2} 2^{-j\delta} (R^{1/d})^{\delta(p)+1/2} \|(1+P)^{-\alpha/d}\chi_B f\|_2$$

where $\alpha = n(1/p-1/2)$. But now, by the spectral theorem,

$$\begin{aligned}
\|(1+P)^{-\alpha/d}&\chi_B f\|_2^2 \\
&= ((1+P)^{-2\alpha/d}\chi_B f, \chi_B f) \\
&\le \Big(\int_B |f(x)|^p\,dx\Big)^{1/p} \Big(\int_B |(1+P)^{-2\alpha/d}\chi_B f(x)|^{p'}\,dx\Big)^{1/p'}.
\end{aligned}$$

Using Proposition 2.5 we get the estimate

$$\|(1+P)^{-\alpha/d}\chi_B f\|_2 \le C_B \|f\|_p.$$

This estimate can be used to complete the proof of Theorem 1.2.

To prove Corollary 1.3 we make use the following transplantation theorem due to Mityagin [16] a proof of which can be found in [13].

Theorem 2.6. *Let $P(x,D)$ be a self-adjoint differential operator whose principal symbol is $p(x,\xi)$. Suppose for some p, $1 \le p \le \infty$, and a set B of positive measure the operators $\chi_B S_R^\delta \chi_B$ are uniformly bounded on $L^p(\mathbb{R}^n)$ for a sequence of values of R tending to infinity. Let x_0 be a point of density of B. Then $\chi_{(-\infty,\lambda)}(p(x_0,\xi))$ is a Fourier multiplier on $L^p(\mathbb{R}^n)$.*

Given a homogeneous differential operator $P(D)$ we can apply the above theorem to $P(D)+|x|^2$ to obtain Corollary 1.3.

We conclude this section with an example. Consider the operator $P(D)$ where

$$P(\xi) = \xi_1^m + \ldots + \xi_n^m$$

with m an even integer. Then Peetre [18] has proved that the associated Riesz kernel satisfies the estimate

$$|S_R^\delta(x,y)| \le CR^{n/m}(1+R^{1/m}|x-y|)^{-\delta'-1}$$

where $\delta' = \delta + (n-1)/m$. He has also shown that this estimate is optimal.

Using Peetre's estimate we can prove that S_R^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ for $\delta > 2(n-1)(1-1/m)|1/p-1/2|$ whereas by Corollary 1.3 we get the same for $\delta > n|1/p-1/2|$. This is still far from the optimal result which is known only in the case when the "cospheres" $\{\xi : P(\xi) = 1\}$ are strictly convex.

3. Rockland operators on nilpotent groups. In this section we study Bochner–Riesz means associated with positive Rockland operators on a stratified group. We employ standard notations and terminology. A general reference for this section is the monograph of Folland and Stein [5].

Let G be a stratified group with a dilation structure δ_t, $t > 0$. The *homogeneous dimension* Q of G is defined by the requirement

$$\int f(\delta_t x)\,dx = t^{-Q}\int f(x)\,dx$$

where dx is the Haar measure on the group. By $|x|$ we mean a homogeneous norm on G. A left invariant differential operator L on G is called a *Rockland operator* if it is homogeneous of some degree $d > 0$, that is,

$$L(f(\delta_t x)) = t^d Lf(\delta_t x), \qquad f \in C^\infty(G),$$

and for every non-trivial unitary representation π of G the operator $\pi(L)$ is injective on C^∞ vectors.

A positive Rockland operator L satisfies the following subelliptic estimate proved by Helffer and Nourrigat [7]: for every multi-index I there are constants C and k such that

$$\|X^I f\|_2 \le C(\|L^k f\|_2 + \|f\|_2), \qquad f \in C_0^\infty(G).$$

Then L is essentially self-adjoint and its closure is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which is of the form $T_t f = f * p_t$, $t > 0$, where p_t is a Schwartz class function (see Theorem 4.25 of [5]). The homogeneity of L implies that

$$p_t(x) = t^{-Q/d} p_1(\delta_{t^{-1/d}} x).$$

In our analysis the following estimate established by Dziubański, Hebisch and Zienkiewicz [3] plays an important role.

THEOREM 3.1. *Let L be a positive Rockland operator of homogeneous degree d and p_t the associated heat kernel. Then there are positive constants a and C such that*

$$|p_t(x)| \leq Ct^{-Q/d}e^{-a|x|^{d/(d-1)}t^{-1/(d-1)}}.$$

Consider the Bochner–Riesz means S_R^δ associated with the operator L. In order to prove Corollary 1.4 we require the following estimate on the Riesz kernel.

THEOREM 3.2. *Let L be homogeneous of degree d. There is a constant β such that for large positive values of δ we have the estimate*

$$|S_R^\delta(x)| \leq CR^{Q/d}(1 + R^{1/d}|x|)^{-\delta+\beta}$$

where the constant C is independent of R.

When L is the sublaplacian on a stratified group, the above estimate has been proved in Hulanicki and Jenkins [10] by using the functional calculus for the commutative Banach subalgebra A of $L^1(G)$ generated by linear combinations of $p_t(x)$, $t > 0$. In a later paper [9] Hulanicki proved the weaker estimate

$$|S_R^\delta(x)| \leq CR^{Q/d}(1 + R^{1/d}|x|)^{-\delta/3+\beta}$$

for any Rockland operator. At that time the sharp estimates on the heat kernel given in [3] were not known for general Rockland operators. A close examination of the proof in [10] reveals that once we have the estimates of Theorem 3.2, the main result of [10] (Theorem 1.12) can be proved for any Rockland operator of homogeneous degree d. We refer to [10] for the details.

Using estimates on the Bochner–Riesz kernel the authors in [10] and [11] have obtained summability results for Rockland operators. Once we have Theorem 3.2 the Riesz kernel estimate can be used to prove Corollary 1.4 as in Section 1, which is a quantitative version of the corresponding theorems in [10] and [11]. However, our results are still not optimal (see e.g. Mauceri [14] for the case of the sublaplacian on the Heisenberg group).

Now let π be a representation of G induced from a unitary character of a normal connected subgroup H of G. Then the operators $\pi(x)$, $x \in G$, act on functions on G/H according to the formula

$$\pi(x)f(yH) = a(x, yH)f(yH.xH)$$

where a is a scalar function of modulus one. In [11] Hulanicki and Jenkins have shown that when L is a Rockland operator of the form

$$L = \sum_{j=1}^{k}(-1)^{n_j}X_j^{2n_j}$$

where X_j, $j = 1, \dots, k$, generate the Lie algebra of G and π is as above, then $\pi(L)$ is a positive self-adjoint operator. They have further shown that the kernel s_R^δ of the Riesz means associated with $\pi(L)$ can be expressed as an integral of the kernel S_R^δ for L and hence proved summability results for operators of the form $\pi(L)$.

We can combine Corollary 1.4 with their idea to get more precise quantitative versions of summability results for operators of the form $\pi(L)$. This is better explained in the case of Heisenberg group. Recall that the Heisenberg group $G = \mathbb{C}^n \times \mathbb{R}$ is a two-step nilpotent Lie group with group law

$$(z,t)(w,s) = \left(w + z, t + s + \frac{1}{2}\mathrm{Im}(z.\overline{w})\right).$$

The vector fields X_j, $j = 1, \dots, 2n$, and T defined by

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \qquad X_{j+n} = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}$$

for $j = 1, \dots, n$ and $T = \partial/\partial t$ form a basis for the Heisenberg Lie algebra. The dilation structure is given by the automorphism $\delta_r(z,t) = (rz, r^2 t)$, and $|(z,t)|^4 = |z|^4 + t^2$ defines a homogeneous norm.

Let $H = \{(0,t) : t \in \mathbb{R}^n\}$ be the center of the Heisenberg group which is a normal subgroup. The quotient G/H is then identified with $\mathbb{C}^n$. Consider the unitary representation π of G on $L^2(\mathbb{C}^n)$ given by

$$\pi(z,t)f(w) = e^{it} e^{(i/2)\mathrm{Im}(w.\overline{z})} f(z+w).$$

Suppose

$$L = \sum_{i,j} a_{ij} X_i X_j$$

is a positive Rockland operator on G. Then

$$\pi(L) = \sum_{i,j} a_{ij} \widetilde{X}_i \widetilde{X}_j$$

where $\widetilde{X}_i = \pi(X_i)$ which is obtained by replacing $\partial/\partial t$ in X_i by i. Note that L and $\pi(L)$ are related by

$$L(e^{it} f(z)) = e^{it} \pi(L) f(z).$$

The Bochner–Riesz mean $s_R^\delta f$ associated with $\pi(L)$ can be expressed in terms of the kernel of S_R^δ as

$$s_R^\delta f(w) = \int_G S_R^\delta(z,t)\pi(z,t)f(w)\,dz\,dt.$$

Thus, if $s_R^\delta(w,z)$ denotes the kernel of s_R^δ then it is given by

$$s_R^\delta(w,z) = e^{(i/2)\mathrm{Im}(w.\overline{z})} \int_G S_R^\delta(z-w,t) e^{it}\,dt.$$

Now we can use the estimate on $S_R^\delta(z,t)$ to get an estimate for $s_R^\delta(w,z)$. Indeed,

$$|s_R^\delta(w,z)| \le CR^{Q/2} \int_{-\infty}^{\infty} (1+R^2|z-w|^4+R^2t^2)^{(-\delta+\beta)/4}\,dt$$

where $Q=2n+2$. By a change of variables, we get the estimate

$$|s_R^\delta(w,z)| \le CR^{n+1}(1+R^2|z-w|^4)^{(-\delta+\beta+2)/4} \int_{-\infty}^{\infty} (1+R^2t^2)^{(-\delta+\beta)/4}\,dt.$$

The last integral converges if δ is large and we get the estimate

$$|s_R^\delta(w,z)| \le CR^n(1+R^{1/2}|z-w|)^{-\delta+\beta+2}$$

for such values of δ.

Thus we can get the following corollary to Corollary 1.4.

Corollary 3.3. *Let $\pi(L)$ be as above. Then s_R^δ are uniformly bounded on $L^p(\mathbb{C}^n)$, $1<p<\infty$, whenever $\delta > 2n|1/p-1/2|$.*

A similar transference technique can be used to study Bochner–Riesz means associated with certain Schrödinger operators on $\mathbb{R}^n$. To do this we consider the Schrödinger representation ϱ of the Heisenberg group on $L^2(\mathbb{R}^n)$ which is given by

$$\varrho(z,t)f(\xi) = e^{it}e^{i(x.\xi+(1/2)x.y)}f(\xi+y).$$

Under this representation, the vector fields X_j transform as

$$\varrho(X_j) = -\frac{\partial}{\partial\xi_j}, \qquad \varrho(X_{j+n}) = i\xi_j$$

for $j=1,\ldots,n$. Thus, if $L=\sum_{i,j} a_{ij}X_iX_j$ is a Rockland operator, then $\varrho(L)$ is an operator of the form

$$\sum_{i,j} b_{ij}\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_j c_j(\xi)\frac{\partial}{\partial\xi_j} + p(\xi)$$

where $c_j(\xi)$ and $p(\xi)$ are polynomials.

Let $\widetilde{S}_R^\delta$ be the Bochner–Riesz mean associated with the operator $\varrho(L)$. Then it is given in terms of $S_R^\delta(z,t)$ by the equation

$$\widetilde{S}_R^\delta f(\xi) = \int_G S_R^\delta(z,t)\varrho(z,t)f(\xi)\,dz\,dt.$$

If we let $z=x+iy$ the kernel $\widetilde{S}_R^\delta(\xi,y)$ of $\widetilde{S}_R^\delta$ is given by

$$\widetilde{S}_R^\delta(\xi,y) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} S_R^\delta(x+i(y-\xi),t)e^{it}e^{(i/2)x.(y+\xi)}\,dx\,dt.$$

Since

$$|S_R^\delta(x+iy,t)| \le CR^{n+1}(1+R^2|x|^4+R^2|y|^4+R^2t^2)^{(-\delta+\beta)/4}$$

when δ is large enough we obtain the estimate

$$|\widetilde{S}_R^\delta(\xi,y)| \le CR^{n/2}(1+R^{1/2}|y-\xi|)^{-\delta+\beta+n+2}.$$

Once we have this estimate the next corollary follows.

COROLLARY 3.4. *Let L and $\varrho(L)$ be as above. Then $\widetilde{S}_R^\delta$ are uniformly bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ provided $\delta > n|1/p-1/2|$.*

An important and more difficult problem is to prove the summability theorems for $\delta > \delta(p)$. In a subsequent paper we will show that the weak estimates on the Bochner–Riesz kernel can be used to prove certain multiplier theorems.

REFERENCES

[1] A. Bonami et J. L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. 183 (1973), 223–263.

[2] L. Carleson and P. Sjölin, *Oscillatory integrals and multiplier problem for the disc*, Studia Math. 44 (1972), 287–299.

[3] J. Dziubański, W. Hebisch and J. Zienkiewicz, *Note on semigroups generated by positive Rockland operators on graded homogeneous groups*, ibid. 110 (1994), 115–126.

[4] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math. 15 (1972), 44–52.

[5] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ., Princeton, 1982.

[6] W. Hebisch, *Almost everywhere summability of eigenfunction expansions associated to elliptic operators*, Studia Math. 96 (1990), 263–275.

[7] B. Helffer et J. Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes à gauche sur un groupe nilpotent gradué*, Comm. Partial Differential Equations 4 (1979), 899–958.

[8] L. Hörmander, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, in: Some Recent Advances in the Basic Sciences, Vol. 2, Yeshiva Univ., New York, 1969, 155–202.

[9] A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups*, Studia Math. 78 (1984), 253–266.

[10] A. Hulanicki and J. Jenkins, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc. 278 (1983), 703–715.

[11] —, —, *Nilpotent Lie groups and summability of eigenfunction expansions of Schrödinger operators*, Studia Math. 80 (1984), 235–244.

[12] G. Karadzhov, *Riesz summability of multiple Hermite series in L^p spaces*, C. R. Acad. Bulgare Sci. 47 (1994), 5–8.

[13] C. E. Kenig, R. Stanton and P. Tomas, *Divergence of eigenfunction expansions*, J. Funct. Anal. 46 (1982), 28–44.

[14] G. Mauceri, *Riesz means for the eigenfunction expansions for a class of hypoelliptic differential operators*, Ann. Inst. Fourier (Grenoble) 31 (1981), no. 4, 115–140.
[15] G. Mauceri and S. Meda, *Vector valued multipliers on stratified groups*, Rev. Mat. Iberoamericana 6 (1990), 141–154.
[16] B. S. Mitjagin [B. S. Mityagin], *Divergenz von Spektralentwicklungen in L^p-Räumen*, in: Linear Operators and Approximation II, Internat. Ser. Numer. Math. 25, Birkhäuser, Basel, 1974, 521–530.
[17] D. Müller and E. M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. 73 (1994), 413–440.
[18] J. Peetre, *Remarks on eigenfunction expansions for elliptic differential operators with constant coefficients*, Math. Scand. 15 (1964), 83–97.
[19] —, *Some estimates for spectral functions*, Math. Z. 92 (1966), 146–153.
[20] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Funct. Anal. 77 (1988), 123–134.
[21] —, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. 126 (1987), 439–447.
[22] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1971.
[23] K. Stempak and J. Zienkiewicz, *Twisted convolution and Riesz means*, J. Anal. Math. 76 (1998), 93–107.
[24] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Princeton Univ. Press, Princeton, 1993.
[25] —, *Hermite and special Hermite expansions revisited*, Duke Math. J. 94 (1998), 257–278.
[26] —, *Harmonic Analysis on the Heisenberg Group*, Progr. Math. 159, Birkhäuser, Boston, 1998.

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