On the classes of fully copositive and fully semimonotone matrices

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Abstract

In this paper we consider the class $C_0^{\rm f}$ of fully copositive and the class $E_0^{\rm f}$ of fully semimonotone matrices. We show that $C_0^{\rm f}$ matrices with positive diagonal entries are column sufficient. We settle a conjecture made by Murthy and Parthasarathy to the effect that a $C_0^{\rm f} \cap Q_0$ matrix is positive semidefinite by providing a counterexample. We finally consider $E_0^{\rm f}$ matrices introduced by Cottle and Stone (Math. Program. 27 (1983) 191–213) and partially address Stone's conjecture to the effect that $E_0^{\rm f} \cap Q_0 \subseteq P_0$ by showing that $E_0^{\rm f} \cap D^{\rm c}$ matrices are P_0, where $D^{\rm c}$ is the Doverspike class of matrices.

Keywords: Linear complementarity problem; Fully copositive matrices; Fully semimonotone matrices; Principal pivot transform

1. Introduction

For a given square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ the linear complementarity problem (denoted by LCP(q, A)) is to find vectors $w,\ z \in \mathbb{R}^n$ such that

$$w - Az = q, \quad w \geqslant 0, \quad z \geqslant 0, \tag{1.1}$$

$$w^{\rm t} z = 0. \tag{1.2}$$

A pair of vectors (w, z) satisfying (1.1) is said to be *a feasible solution* to LCP (q, A). If there is a feasible solution to LCP(q, A), then the problem LCP(q, A) is said to be *feasible*. A pair (w, z) satisfying (1.1) and (1.2) is called a *solution to the* LCP(q, A). For further details on the theory and applications of LCP see [2,16]. Several matrix classes are introduced in the LCP literature due to their various important properties or their various applications or from an algorithmic point of view. In this paper we use the following conventions. Suppose a class of matrices $\mathscr{G} \subseteq \mathbb{R}^{n\times n}$ is defined by specifying a property which is satisfied by each square matrix of order n in $\mathscr{G}$. We then say that A is a $\mathscr{G}$ matrix. Thus the symbol $\mathscr{G}$ is used for the class of matrices satisfying the specified property as well for the property itself. For the definition of various classes of matrices see Section 2. We say that A is positive semidefinite (PSD) if $x^tAx \geqslant 0\ \forall x \in \mathbb{R}^n$ and A is positive definite (PD) if $x^tAx > 0\ \forall 0 \neq x \in \mathbb{R}^n$. A is said to be a P (P_0)-matrix if all its principal minors are positive (nonnegative). A is said to be *column sufficient* if for all $x \in \mathbb{R}^n$ the following implication holds:

$$x_i(Ax)_i \leqslant 0\ \forall i \ \text{ implies } \ x_i(Ax)_i = 0\ \forall i.$$

A is *sufficient* if A and A^t are both column sufficient. For details on sufficient matrices see [1,4,20]. We say that a matrix A is a Q_0*-matrix* if for any $q \in \mathbb{R}^n$ (1.1) has a solution implies that LCP(q, A) has a solution. The class (E_0^{f}) of fully monotone matrices was introduced by Cottle and Stone [3]. Stone [17] studied various properties of E_0^{f} and conjectured that $E_0^{\mathrm{f}} \cap Q_0 \subseteq P_0$. In [12], the conjecture was resolved for E_0^{f} matrices of order 4 and for some subcases under various assumptions on A. In the same paper, E_0^{f} was replaced by fully copositive matrices (C_0^{f}) and the conjecture was shown true for C_0^{f}-matrices with positive diagonal entries. Murthy and Parthasarathy [13] proved that $C_0^{\mathrm{f}} \cap Q_0 \subseteq P_0$. Murthy et al. [15] proved that $C_0^{\mathrm{f}} \cap Q_0$ matrices are sufficient. The class of $C_0^{\mathrm{f}} \cap Q_0$ matrices are completely Q_0 matrices [13] and share many properties of PSD matrices. Symmetric $C_0^{\mathrm{f}} \cap Q_0$ matrices are PSD. The principal pivoting algorithm of Graves [11] for solving LCPs with PSD matrices also processes matrices in the class $C_0^{\mathrm{f}} \cap Q_0$. Murthy and Parthasarathy [13] proved that 2×2, $C_0^{\mathrm{f}} \cap Q_0$ matrices are PSD and a bisymmetric Q_0 matrix is PSD if and only if it is fully copositive. It is known that PSD matrices are sufficient. They conjectured that if $A \in C_0^{\mathrm{f}} \cap Q_0$, then it is PSD.

In this paper, we study C_0^{f} and E_0^{f} matrices. In Section 2, we present the required definitions and introduce the notations used in this paper. In Section 3, we present a different proof of the result that $C_0^{\mathrm{f}} \cap Q_0$ matrices are column sufficient. We also consider C_0^{f} matrices with positive diagonal entries [12] and show that such C_0^{f} matrices are sufficient. We provide an example to show that $C_0^{\mathrm{f}} \cap Q_0 \not\subseteq$ PSD and thus settle the conjecture made by Murthy and Parthasarathy [13]. Finally, we consider E_0^{f} matrices introduced by Cottle and Stone [3] and partially address Stone's conjecture [17] that $E_0^{\mathrm{f}} \cap Q_0 \subseteq P_0$ by showing that $E_0^{\mathrm{f}} \cap D^{\mathrm{c}}$ matrices are contained in P_0, where D^{c} is the Doverspike class of matrices for which all the strongly degenerate complementary cones of $(I, -A)$ are contained in the boundary of pos$(I, -A)$ (see Section 2 for details). This generalizes the result of Sridhar [18] to the effect that $E_0^{\mathrm{f}} \cap R_0 \subseteq P_0$.

2. Preliminaries

We begin by introducing some basic notations used in this paper. We consider matrices and vectors with real entries. For any matrix $A \in \mathbb{R}^{m\times n}$, a_{ij} denotes its ith row and jth column entry. For any positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. For any set $\alpha \subseteq \{1, 2, \ldots, n\}$, $\bar{\alpha}$ denotes its complement in $\{1, 2, \ldots, n\}$. If A is a matrix of order $n \times n$, $\alpha \subseteq \{1, 2, \ldots, n\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$, then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A, whose indices are in α and β, respectively. Any vector $x \in \mathbb{R}^n$ is a column vector unless otherwise specified, and x^{t} denotes the row transpose of x. For any index i, e_i stands for the vector of appropriate order whose ith entry is 1 and the other entries are 0. Given a matrix A and a vector q we define the feasible set $F(q, A) = \{z \geqslant 0 \mid Az + q \geqslant 0\}$ and the solution set $S(q, A) = \{z \in F(q, A) \mid z^{\mathrm{t}}(Az + q) = 0\}$. The notation pos$A$ represents the cone generated by taking the nonnegative linear combinations of columns of A. $C_A(\alpha)$ denotes the complementary matrix of A with respect to α, where $C_A(\alpha)_{.j} = -A_{.j}$ if $j \in \alpha$ and $C_A(\alpha)_{.j} = I_{.j}$ if $j \notin \alpha$. The associated cone pos$C_A(\alpha)$ is called *complementary cone relative to A* with respect to α. The complementary cone with respect to α is said to be *nondegenerate* if $\det(A_{\alpha\alpha}) \neq 0$. Otherwise it is said to be *degenerate*. A degenerate pos$C_A(\alpha)$ is said to be *strongly degenerate* if there exists $0 \neq x \geqslant 0$, $x \in \mathbb{R}^n$, such that $C_A(\alpha)x = 0$. Any solution (w, z) of LCP(q, A) is said to be *nondegenerate* if $w + z > 0$. Otherwise it is called a *degenerate solution*. A vector $q \in \mathbb{R}^n$ is said to be *nondegenerate with respect to A* if every solution to LCP(q, A) is nondegenerate. Let $\mathscr{C}(A)$ be the union of the strongly degenerate complementary cones of A and let $\mathscr{K}(A)$ denote the union of all facets of all the complementary cones of A. A *connected component* of a set S containing a point x is defined as the union of all connected sets C such that $x \in C \subseteq S$.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a Q-matrix if for every $q \in \mathbb{R}^n$, LCP(q, A) has a solution. It is a Q_0-matrix if and only if pos$(I, -A) = \{q \mid$ LCP(q, A) has a solution$\}$. A is said to be a *completely* Q (Q_0) *matrix* if A and all its principal submatrices are Q (Q_0) matrices. $A \in \mathbb{R}^{n\times n}$ is said to be a E_0-matrix if for every $0 \neq y \geqslant 0$, $y \in \mathbb{R}^n$, $\exists$ an i such that $y_i > 0$ and $(Ay)_i \geqslant 0$. The class of such matrices is called the class of *semimonotone matrices.* A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an L_2-matrix if for each $0 \neq \xi \geqslant 0$, $\xi \in \mathbb{R}^n$, satisfying $\eta = A\xi \geqslant 0$ and $\eta^{\mathrm{t}}\xi = 0$, $\exists$ a $0 \neq \hat{\xi} \geqslant 0$ such that $\xi \geqslant \hat{\xi}$ and $\eta \geqslant \hat{\eta} \geqslant 0$, where $\hat{\eta} = -A^{\mathrm{t}}\hat{\xi}$. $A \in \mathbb{R}^{n\times n}$ is said to be an L-matrix if it is in both E_0 and L_2. This class was introduced by Eaves [8] who shows that Lemke's algorithm processes LCP(q, A) when $A \in L$ and hence $L \subseteq Q_0$.

We say that A satisfies *Doverspike's condition* [7], if all the strongly degenerate complementary cones of $(I, -A)$ lie on the boundary of $\text{pos}(I, -A)$. We denote the class of matrices satisfying Doverspike's condition by D^c. Doverspike [7] proved that if $A \in E_0 \cap D^c$, then $A \in Q_0$ by showing that Lemke's algorithm processes LCP(q, A) if $A \in E_0 \cap D^c$.

Let $\alpha \subseteq \{1, 2, \ldots, n\}$ and let $A_{\alpha\alpha}$ be nonsingular. The principal pivot transform (PPT) of A with respect to the index set α is defined as the matrix given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where

$$M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}, \quad M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1} A_{\alpha\bar{\alpha}},$$
$$M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}, \quad M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} A_{\alpha\bar{\alpha}}.$$

Similarly the PPT of a vector q with respect to the same index set α is a vector q', where $q'_\alpha = -A_{\alpha\alpha}^{-1} q_\alpha$ and $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha} A_{\alpha\alpha}^{-1} q_\alpha$. The PPT of LCP$(q, A)$ with respect to α (obtained by pivoting on $A_{\alpha\alpha}$) is given by LCP(q', M). When $\alpha = \phi$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For a detailed discussion on PPT see [1] or [2].

We say that A is *copositive* (C_0) if $x^t Ax \geqslant 0\ \forall x \geqslant 0$ and A is strictly copositive (C) if $x^t Ax > 0\ \forall 0 \neq x \geqslant 0$.

We say that A is *fully semimonotone* (E_0^f) if every PPT of A is in E_0. The class of E_0^f matrices was introduced by Cottle and Stone [3]. For this class if $q \in \mathbb{R}^n$ is in the interior of a full complementary cone, then LCP(q, A) has a unique solution. This is a geometric characterization of the class E_0^f. We say that A is *fully copositive* (C_0^f) if every PPT of A is in C_0. Note that $P \subseteq P_0 \subseteq E_0^f \subseteq E_0$ and $C_0^f \subseteq E_0^f$. If A belongs to any one of the classes $E_0, C_0, E, C, E_0^f, C_0^f$ or the class of sufficient matrices, then so is

(i) any principal submatrix of A; and
(ii) any principal permutation of A.

A matrix A is *sufficient of order k* if all its $k \times k$ principal submatrices are sufficient. If $A \in Q(Q_0)$, then every PPT of $A \in Q(Q_0)$. For details on the class of fully copositive matrices see [12–15].

Stone [17] conjectured that if $A \in E_0^f \cap Q_0$, then it is in P_0. Murthy and Parthasarathy [12] proved that for $n = 4$, $E_0^f \cap Q_0$ matrices are in P_0. They also established Stone's conjecture under some additional assumptions.

We require the following theorems and lemma in Section 3.

Theorem 2.1 [13, Theorem 4.5]. *Suppose $A \in C_0^f \cap Q_0$. Then $A \in P_0$.*

Theorem 2.2 [13, Theorem 3.3]. *Let $A \in C_0^f$. The following statements are equivalent*:

(a) *A is a* Q_0 *matrix.*
(b) *For every PPT M of A*, $m_{ii} = 0 \Rightarrow m_{ij} + m_{ji} = 0,\ \forall i, j \in \{1, 2, \ldots, n\}$.
(c) *A is a completely* Q_0 *matrix.*

Theorem 2.3 [13, Theorem 4.9]. *If* $A \in \mathbb{R}^{2\times 2} \cap C_0^{\mathrm{f}} \cap Q_0$, *then A is a PSD matrix.*

Lemma 2.1 [15, Lemma 14]. *Let* $A \in P_0$ *and* $q \in \mathbb{R}^n$. *If* (w, z) *and* (y, x) *are two distinct solutions of LCP*(q, A). *Then there exists an index i*, $1 \leqslant i \leqslant n$, *such that either* $z_i = x_i = 0$ *or* $w_i = y_i = 0$.

2.1. Degree theory

Let $f_A : \mathbb{R}^n \to \mathbb{R}^n$ be the piecewise linear map for a given matrix $A \in \mathbb{R}^{n\times n}$ defined as $f_A(e_i) = e_i$ and $f_A(-e_i) = -Ae_i,\ i = 1, 2, \ldots, n$. For any $x \in \mathbb{R}^n$, let

$$f_A(x) = \sum_{i=1}^{n} f_A(xe_i) = \sum_{i=1}^{n} |x_i| f_A(\mathrm{sgn}(x_i)e_i) = \sum_{i=1}^{n} e_i x_i^+ + \sum_{i=1}^{n} |x_i|(-Ae_i)$$
$$= \sum_{i=1}^{n} e_i x_i^+ + \sum_{i=1}^{n} (-A)(|x_i| e_i) = x^+ - Ax^-,$$

where $x_i^+ = \max(0, x_i)$ and $x_i^- = \max(0, -x_i)\forall i = 1, 2, \ldots, n$. Note that LCP$(q, A)$ is equivalent to finding an $x \in \mathbb{R}^n$ such that $f_A(x) = q$. If x belongs to the interior of some orthants of $\mathbb{R}^n$ and $\det(A_{\alpha\alpha}) \neq 0$, where $\alpha = \{i \mid x_i < 0\}$, then the index of $f_A(x)$ at x is well defined and

$$\mathrm{ind}\, f_A(q, x) = \mathrm{sgn}\, \det(A_{\alpha\alpha}) = \frac{\det(A_{\alpha\alpha})}{|\det(A_{\alpha\alpha})|}.$$

Let $f_A^{-1}(q)$ stand for the set of all vectors $x \in \mathbb{R}^n$, such that $f_A(x) = q$. From the linear complementarity theory, it is clear that the cardinality of $f_A^{-1}(q)$ denotes the number of solutions of LCP(q, A). In particular, if q is nondegenerate with respect to A, each index of f_A is well defined and we can then define local degree of A at q, denoted by $\deg_A(q)$, to be equal to the local degree of f_A at q, i.e.,

$$\deg_A(q) = \sum_{x \in f_A^{-1}(q)} \mathrm{ind}\, f_A(q, x) = \sum_{x \in f_A^{-1}(q)} \frac{\det(A_{\alpha\alpha})}{|\det(A_{\alpha\alpha})|},$$

where the summation is taken over the index sets $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $q \in \mathrm{pos} C_A(\alpha)$.

If $q, q' \in \mathbb{R}^n \setminus \mathscr{K}(A)$ and lie in the same connected component of $\mathbb{R}^n \setminus \mathscr{C}(M)$, then $\deg_A(q) = \deg_A(q')$. See Theorem 6.1.17 in [2, p. 515]. More specifically when $\mathbb{R}^n \setminus \mathscr{C}(M)$ is made up of a single connected component, we have the degree of A at q defined and equal to the same constant for every $q \in \mathbb{R}^n$, except possibly for a set of vectors which has measure 0. Such a scalar is called the global degree of A and is denoted by $\deg A$. For further details on degree theory see [2, Chapter 6].

3. Main results

It is known that positive semidefinite matrices are sufficient. Murthy and Parthasarathy [15] proved that $C_0^{\mathrm{f}} \cap Q_0$ matrices are sufficient. Here, we show that this result is a consequence of the following result proved by Cottle and Guu [4].

Theorem 3.1. *$A \in \mathbb{R}^{n\times n}$ is sufficient if and only if every matrix obtained from it by means of a PPT operation is sufficient of order* 2.

As a consequence we have the following theorem.

Theorem 3.2. *Let $A \in C_0^{\mathrm{f}} \cap Q_0$. Then A is sufficient.*

Proof. Note that all 2×2 submatrices of A or its PPTs are $C_0^{\mathrm{f}} \cap Q_0$ matrices since A and all its PPTs are completely Q_0 matrices. Now, by Theorem 2.3, all 2×2 submatrices of A or its PPTs are positive semidefinite, and hence sufficient. Therefore A or every matrix obtained by means of a PPT operation is sufficient of order 2. Now by Theorem 3.1, A is sufficient. □

Remark 3.1. In [13], it is shown that Graves's principal pivoting algorithm [11] for solving LCP(q, A), where A is positive semidefinite also processes LCP(q, A) with $A \in C_0^{\mathrm{f}} \cap Q_0$. By Theorem 3.2, it follows that other principal pivoting methods also processes LCP(q, A) when $A \in C_0^{\mathrm{f}} \cap Q_0$. See [1] and other references cited therein.

Murthy and Parthasarathy [12] proved the following theorem.

Theorem 3.3. *Suppose $A \in \mathbb{R}^{n\times n} \cap C_0^{\mathrm{f}}$. Assume that $a_{ii} > 0\ \forall i \in \{1, 2, \ldots, n\}$. Then $A \in P_0$.*

In contrast to the above we observe that with the assumption of positive diagonal entries, a C_0^{f} matrix is a column sufficient matrix and that if a matrix A with positive diagonal entries, and its transpose are in C_0^{f}, then such a matrix is in Q_0 and hence it is a completely Q_0 matrix.

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n} \cap C_0^{\mathrm{f}}$. Assume that $a_{ii} > 0\ \forall i \in \{1, 2, \ldots, n\}$. Then $A \in Q_0$.*
(i) *A is column sufficient.*
(ii) *In addition, if $A^{\mathrm{t}} \in \mathbb{R}^{n\times n} \cap C_0^{\mathrm{f}}$, then A is a completely Q_0 matrix.*

Proof. We shall first show that A sufficient.

Let $q \in \mathbb{R}^n$ and consider the solution set $S(q, A)$ of the LCP(q, A). From Theorem 3.3, it follows that $A \in P_0$. From Theorem 4.3 in [20], it follows that A is sufficient if $n = 1$ or 2. Let us make the induction hypothesis that if $B \in \mathbb{R}^{(n-1)\times(n-1)} \cap C_0^{\mathrm{f}}$ with the assumption $b_{ii} > 0,\ \forall i = 1, \ldots, n-1$, then B is sufficient of order $(n-1)$. Let $A \in C_0^{\mathrm{f}}$ be of order n with $a_{ii} > 0,\ \forall i$. To show that A is column sufficient, it is enough to show that $S(q, A)$ is convex $\forall q \in \mathbb{R}^n$ by Theorem 6 in [6]. Let $(w, z), (y, x)$ be two solutions to LCP(q, A) and let $0 < \lambda < 1$ be given.

Now since $A \in P_0$, from Lemma 2.1 it follows that there is an index i, $1 \leqslant i \leqslant n$, such that either $x_i = z_i = 0$ or $w_i = y_i = 0$.

Case (i): $x_i = z_i = 0$.

In this case $x_\alpha \neq z_\alpha \in S(q_\alpha, A_{\alpha\alpha})$, where $\alpha = \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. From the induction hypothesis $\lambda x_\alpha + (1-\lambda) z_\alpha \in S(q_\alpha, A_{\alpha\alpha})$. Hence it follows that $\lambda x + (1-\lambda) z \in S(q, A)$.

Case (ii): $y_i = w_i = 0$.

Without loss of generality, we assume that $i = 1$. We have $a_{11} > 0$ by the hypothesis of the theorem. Let LCP$(\bar{q}, M)$ be the PPT of LCP(q, A) with respect to $\alpha = \{1\}$. Let $(\bar{y}, \bar{x}), (\bar{w}, \bar{z})$ be the solutions to LCP$(\bar{q}, M)$ corresponding to the solutions $(y, x), (w, z)$ of LCP(q, A), respectively. It follows that $\bar{x}_1 = 0$ and $\bar{z}_1 = 0$. From here it follows that $\lambda(\bar{y}, \bar{x}) + (1-\lambda)(\bar{w}, \bar{z}) \in S(\bar{q}, M)$ and hence $\lambda(y, x) + (1-\lambda)(w, z) \in S(q, A)$. Thus it follows that $S(q, A)$ is convex.

By the principle of induction, it follows that A is column sufficient for all n.

Now to conclude (ii) under the additional assumption that A^{t} is a C_0^{f} matrix, we proceed as follows. As $A^{\mathrm{t}} \in C_0^{\mathrm{f}}$, and has positive diagonal entries, from the proof of part (i) it follows that A^{t} is also column sufficient. Thus, A is sufficient and hence $A \in Q_0$. Since the above arguments apply to every principal submatrix of A it follows that A is a completely Q_0 matrix. $\square$

The following example shows that in the above theorem for the stronger conclusion in (ii), it is necessary to assume that A^{t} is also a C_0^{f} matrix.

Example 3.1.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that the above matrix is a C_0^{f} matrix but it is not a Q_0 matrix. For example, the vector

$$q = \begin{bmatrix} -8 \\ -5 \\ 2 \end{bmatrix}$$

is feasible but LCP (q, A) has no solution. It is also easy to verify that A^{t} is not a C_0^{f} matrix.

Murthy and Parthasarathy [13] proved that if $A \in \mathbb{R}^{2\times 2} \cap C_0^{\mathrm{f}} \cap Q_0$, then A is positive semidefinite. They conjectured that this may be true for all $n \times n$ matrices. However we present below a counterexample to this conjecture.

Example 3.2.

$$A = \begin{bmatrix} 1 & \frac{7}{4} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that principal submatrices of order 2 of $A + A^{\mathrm{t}}$ are positive definite but $\det(A + A^{\mathrm{t}}) < 0$. Therefore, A is not positive semidefinite.

We now show that $A \in C_0^{\mathrm{f}}$.

Note that there are four distinct PPTs of A, each of which happens to correspond to four choices of the index set α. The first of these PPTs is the strictly copositive matrix itself. It is the PPT of A corresponding to

$$\alpha = \emptyset,\ \alpha = \{3\}.$$

The other PPTs are

$$M_1 = \begin{bmatrix} 1 & -\frac{7}{4} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & \frac{7}{4} & -\frac{7}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 & -\frac{7}{4} & \frac{7}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The index set α to which these PPTs correspond are, respectively,

(1) $\{1\}$ and $\{1, 3\}$;
(2) $\{2\}$ and $\{2, 3\}$;
(3) $\{1, 2\}$ and $\{1, 2, 3\}$.

The copositivity of the matrices M_1, M_2 and M_3 can be demonstrated by determinantal criteria such as those given in [5] or by an analysis of the corresponding quadratic forms which can be rewritten as follows:

(1) $x^{\mathrm{t}}M_1x = (x_1 - \frac{7}{8}x_2)^2 + \frac{15}{64}x_2^2 + x_2x_3 + x_3^2$,

(2) $x^{\mathrm{t}}M_2x = (x_2 - \frac{1}{2}x_3)^2 + x_1^2 + \frac{7}{4}x_1x_2 - \frac{7}{4}x_1x_3 + \frac{3}{4}x_3^2$,

(3) $x^t M_3 x = (x_3 - \frac{1}{2}x_2)^2 + x_1^2 + \frac{7}{4}x_1x_3 - \frac{7}{4}x_1x_2 + \frac{3}{4}x_2^2$.

Hence $A \in C_0^f \cap Q_0$. But A is not positive semidefinite.

Stone [17] conjectured that within the class of Q_0 matrices, fully semimonotone matrices are P_0. Note that $E_0^f \cap D^c \subseteq Q_0$. We now prove a special case of Stone's conjecture [17] by showing that $E_0^f \cap D^c \subseteq P_0$. This generalizes the result $E_0^f \cap R_0 \subseteq P_0$, due to Sridhar [18].

Theorem 3.5. *Let A be a $n \times n$ real matrix. Let $\mathscr{K}(A)$ denote the union of all the facets of the complementary cones of $(I, -A)$. Consider $q \in \mathbb{R}^n \backslash \mathscr{K}(A)$, where q is nondegenerate with respect to A. Let $\beta \subseteq \{1, 2, \ldots, n\}$ be such that* $\det(A_{\beta\beta}) \neq 0$ *and let $\bar{M}$ be a PPT of A with respect to β. Then*

$$\deg_{\bar{M}}(\bar{q}) = \frac{\det(A_{\beta\beta})}{|\det(A_{\beta\beta})|} . \deg_A(q).$$

Proof. Note that this is a generalization of Theorem 6.6.23 in [2]. This theorem asserts the conditions of Theorem 6.6.23 without assuming that A is R_0 for the local degree when it is defined. The proof of this theorem is similar to Theorem 6.6.23 in [2, p. 595]. □

Let $\zeta = \text{pos}(I, -A)$ and let $\mathscr{C}(A)$ denote the union of all strongly degenerate cones of $(I, -A)$. Further suppose that $\mathscr{C}(A)$ is contained in the boundary of ζ. Then ζ, being convex, is a connected component of $\mathbb{R}^n \backslash \mathscr{C}(A)$. Hence by Theorem 6.1.17, [2, p. 515] it follows that if q and q' are two nondegenerate vectors in ζ, then

$$\deg_A(q) = \deg_A(q'). \tag{3.1}$$

We denote this common degree of A, restricted to ζ by $\deg_\zeta(A)$. Let $\bar{M}$ be any PPT of A with respect to a given index set $\beta \subseteq \{1, 2, \ldots, n\}$ such that $\det(A_{\beta\beta}) \neq 0$. Let $\text{pos}(I, -\bar{M}) = \bar{\zeta}$. We now have the following theorem.

Theorem 3.6. *Let* $A \in \mathbb{R}^{n \times n} \cap E_0^f \cap D^c$. *Then* $\deg_\zeta(A) = 1$, *where* $\zeta = \text{pos}(I, -A)$.

Proof. It is well known that if $A \in \mathbb{R}^{n \times n} \cap E_0^f \cap D^c$, then the strongly degenerate complementary cones of $(I, -A)$ are contained in the boundary of $\text{pos}(I, -A)$. See [10]. Further since A is a Q_0 matrix, $\text{pos}(I, -A) = \{q \mid \text{LCP}(q, A) \text{ has a solution}\} = \zeta$, is a convex set. Hence the interior of ζ is a connected component of $\mathbb{R}^n \backslash \mathscr{C}(A)$. Thus $\deg_\zeta(A)$ is well defined. Further if $q^* \in \mathbb{R}^n_+$, then $\text{LCP}(q^*, A)$ has a unique solution, which is $w = q$, $z = 0$. Hence $\deg_A(q^*) = 1$. It follows from (3.1) that $\deg_\zeta(A) = 1$. □

We now prove our main result.

Theorem 3.7. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^{\mathrm{f}} \cap D^{\mathrm{c}}$. Then A is a P_0-matrix.*

Proof. Suppose not. Then there is a set $\beta \subseteq \{1, \ldots, n\}$ such that $\det(A_{\beta\beta}) < 0$. Let $\bar{M}$ be the PPT of A with respect to β. Note that $\bar{M}$ is again a $E_0^{\mathrm{f}} \cap D^{\mathrm{c}}$ matrix and hence $\deg_{\bar{\zeta}}(\bar{M}) = 1$, by Theorem 3.6, where $\mathrm{pos}(I, -\bar{M}) = \bar{\zeta}$. Now, however, from Theorem 3.5, it follows that for any $q \in \zeta$, which is nondegenerate with respect to A,

$$\deg_{\bar{M}}(\bar{q}) \;=\; \frac{\det(A_{\beta\beta})}{|\det(A_{\beta\beta})|} \cdot \deg_A(q) = -1 \cdot \deg_A(q) = -1.$$

Therefore $\deg_{\bar{\zeta}}(\bar{M}) = -1$ which is a contradiction. □

Corollary 3.1. *Suppose $A \in \mathbb{R}^{n\times n} \cap E_0^{\mathrm{f}} \cap L$. Then A is a P_0 matrix.*

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