Some estimates for the symmetrized first eigenfunction of the Laplacian

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Abstract. In this paper a method is developed to study the first eigenfunction $u > 0$ of the Laplacian. It is based on a study of the distribution function for u. The distribution function satisfies an integro–differential inequality, and by introducing a maximal solution Z of the corresponding equation, bounds obtained for Z are then used to estimate u. These bounds come from a detailed study of Z, especially the basic identity derived in Theorem 3.1.

Key words: partial differential equations, eigenfunctions, eigenvectors, symmetrization

0. Introduction

In this work, we obtain estimates involving the first eigenfunction of the Laplacian on bounded planar domains. In order to state our results more precisely, let D be a bounded domain in $I\!R^2$, and let u satisfy

$$\begin{aligned} \Delta u + \lambda_1 u &= 0, \quad \text{in} \quad D, \\ u &= 0, \quad \text{on} \quad \partial D, \end{aligned} \tag{0.1}$$

where λ_1 is the first eigenvalue on D. Now, u has one sign in D, so we may take $u > 0$. Let $|S|$ denote the area of an open set S in $I\!R^2$ and let S^* stand for the disc, centered at the origin, whose area equals $|S|$. For a domain S, let $\lambda_1(S)$ be the first eigenvalue of the Laplacian on S. For the rest of our work, we take $|D| = 1$, $\sup\limits_D u = 1$,

$$D_t = \{x \in D : u(x) > t\},$$

and

$$\mu(t) = |D_t| \quad .$$

Define

$$u^*(x) = \inf\{t \geq 0 : \mu(t) < \pi |x|^2\}. \tag{0.2}$$

Here u^* is the radially nonincreasing rearrangement of u in (0.1), and

$$|D_t^*| = |\{x \in D^* : u^*(x) > t\}| = |\{x \in D : u(x) > t\}| = |D_t|.$$

Let $\lambda_1^* = \lambda_1(D^*)$. It is classical that $\lambda_1 > \lambda_1^*$ unless $D = D^*$. Let B be the disc, centered at the origin, such that $\lambda_1(D) = \lambda_1(B)$. Then $|B| \le |D^*|$. Actually, via a scaling argument one can easily see that $|B| = \lambda_1^*|D^*|/\lambda_1 = \lambda_1^*/\lambda_1$. Let v be the first eigenfunction of the Laplacian on B,

$$\begin{aligned} \Delta v + \lambda_1 v &= 0, \quad \text{in} \quad B, \\ v &= 0, \quad \text{on} \quad \partial B. \end{aligned} \tag{0.3}$$

Then v is radial. We take $v > 0$ in B and $v(0) = \sup\ v = 1$. Also, let

$$\begin{aligned} \Delta U + \lambda_1^* U &= 0, \quad \text{in} \quad D^*, \\ U &= 0, \quad \text{on} \quad \partial D^*. \end{aligned} \tag{0.4}$$

Again, U is radial and we take $U(0) = \sup\ U = 1$, and so $U > 0$ in D.

In this work, we shall develop a method for obtaining estimates on u^*. We achieve this by studying the distribution functions of the various functions involved. Our starting point is Talenti's inequality [8] which we derive in §1. This inequality is stated in terms of the distribution function of u. We construct a maximal solution Z to the corresponding integro-differential equation. Let $V(r)$ be the nonincreasing radial function whose distribution function is Z. From the construction of Z, it will follow that V is an upper bound for u^*. It is known that v in (0.3) is a lower bound for u^*.

To facilitate a better understanding of V, we carry out a detailed study of Z in §2 and §3 where we obtain qualitative and quantitative information. This may be of independent interest, especially since much of the analysis, in particular the existence of the maximal solution, can be carried out in greater generality. See, for example, Remark 4.1.

The estimates on V so obtained, and those known for v yield information about u^*. Thus the results of §1, §2, and §3 lead to the following

THEOREM 4.1. *Let u and U be as above. There exists a constant C such that*

$$\|u^* - U\|_{L^\infty(D^*)} \le C\sqrt{\lambda - \lambda_1^*}.$$

The proof of Theorem 4.1 will follow from the observation that $v - U \le u^* - U \le V - U$, and the estimates available for the two sides of the inequality.

We also prove the following stability result.

THEOREM 5.1. *Let $\lambda_1 \ge \lambda_1^*$, and u, u^*, and v be as in (0.1), (0.2) and (0.3). Let B be as in (0.3) and R be such that $|B| = \pi R^2$. There exists a constant $C = C_R$ such that if $u^*(R) = \varepsilon > 0$, then for sufficiently small ε,*

$$\|u^* - v\|_{L^\infty(B)} \le C\sqrt{\varepsilon}.$$

1. Construction of the maximal solution Z

We start with a formal derivation of Talenti's inequality for eigenfunctions. Recall that u is analytic, and thus by Sard's theorem and the coarea formula [5, p. 248] we have, for $0 < t < 1$,

$$\left(\int_{\partial D_t} 1 \right)^2 \leq \int_{\partial D_t} |Du| \int_{\partial D_t} \frac{1}{|Du|}, \text{ a.e. } t.$$

Thus,

$$L\{\partial D_t\}^2 \leq \left(\lambda_1 \int_{D_t} u \right) (-\mu'(t)), \quad \text{a.e.} \quad t,$$

where $L\{\partial D_t\}$ is the one-dimensional Hausdorff measure of the boundary of D_t. The right side follows from an application of the divergence theorem on the p.d.e. in (0.1) over the set D_t. Employing the usual isoperimetic inequality we obtain

$$\frac{4\pi}{\lambda_1} \mu(t) \leq (-\mu'(t)) \int_{D_t} u \quad . \tag{1.1}$$

Now using Fubini's theorem, we may write

$$\begin{aligned} \int_{D_t} u &= \int_{D_t} \int_t^{u(x)} d\tau \; dx + t\mu(t) \\ &= \int_t^1 \mu(\tau) d\tau + t\mu(t). \end{aligned} \tag{1.2}$$

Thus, (1.1) and (1.2) yield

$$\begin{aligned} \frac{4\pi}{\lambda_1} \mu(t) \; &\leq \; (-\mu'(t)) \left[\int_t^1 \mu(\tau) d\tau + t\mu(t) \right], \text{ a.e. } t \in [0, 1], \\ \mu(0) \; &= \; 1, \text{ and } \mu(1^-) = 0. \end{aligned} \tag{1.3}$$

The inequality (1.3) which is a consequence of Talenti's inequality, plays a key role in motivating our work. Based on this, we are led to consider, for $\lambda > 0$, the o.d.e.

$$\begin{aligned} \frac{4\pi}{\lambda} z(t) \; &= \; (-z'(t)) \left[\int_t^1 z(\tau) d\tau + t z(t) \right], \\ z(0) \; &= \; 1. \end{aligned} \tag{1.4}$$

Since (1.4) is nonstandard, we must formally define what we shall mean by solutions and subsolutions to (1.4).

DEFINITION 1.1. Let $Y(t) \geq 0$ be nonincreasing for $0 \leq t \leq 1$ and satisfy the conditions

$$\frac{4\pi}{\lambda}\, Y(t) \;\leq\; (-Y'(t)) \left[\int_t^1 Y(\tau) d\tau + tY(t) \right], \quad \text{a.e. } t \in [0,1],$$
$$Y(0) \;=\; 1.$$

Then $Y(t)$ is a *subsolution* to (1.4).

Note that $\mu(t)$ in (1.3) is then a subsolution to (1.4) with $\lambda = \lambda_1$. For emphasis, we shall sometimes refer to subsolutions satisfying Definition 1.1 as nonincreasing nonnegative subsolutions.

DEFINITION 1.2. By a *solution* of (1.4) we will mean a continuous function $z(t) \geq 0$ such that

$$z(t) = \exp \left[-\frac{4\pi}{\lambda} \int_0^t \frac{d\tau}{\int_\tau^1 z(s) ds + \tau\, z(\tau)} \right], \quad 0 \leq t \leq 1. \tag{1.5}$$

The right hand side of (1.5) is interpreted as 0 for any t for which the term in the exponential becomes $-\infty$.

Again, for emphasis, we sometimes refer to solutions satisfying Definition 1.2 as nonnegative solutions.

By simple bootstrapping, we see that a solution to (1.5) becomes C^∞ at points $t \in (0,1)$ where $z(t) \neq 0$.

Let $W(t)$ be the distribution function corresponding to U as in (0.4). Then $W(t) > 0$ for $0 \leq t < 1$, and is decreasing and satisfies (we have equality in (1.1)),

$$\frac{4\pi}{\lambda_1^*} W(t) \;=\; (-W'(t)) \left[\int_t^1 W(\tau) d\tau + tW(t) \right] \tag{1.6}$$
$$W(0) \;=\; 1, \;\text{ and }\; W(1) = 0.$$

Later we will show uniqueness for (1.6).

We observe that if z in (1.5) is positive then z is decreasing in t; and W in (1.6) satisfies (1.5) with λ replaced by λ_1^*. In what follows, λ will play the role of a parameter in (1.4). We now study certain kinds of solutions of (1.4), which we shall call maximal solutions. The analysis of this section considers only the case $\lambda \geq \lambda_1^*$. We shall observe in section 3 that there are no nonnegative solutions to (1.4) for $\lambda < \lambda_1^*$.

THEOREM 1.1. *For each $\lambda \geq \lambda_1^*$, there exists a unique C^1 solution Z_λ of (1.4), in the sense of Definition 1.2 such that* *(i)* $Z_\lambda(t)$ *is positive and hence decreasing in* t *;*

(ii) $Z_\lambda(t)$ *is maximal in the sense that if* $\bar{Z}_\lambda(t)$ *is any nonnegative solution of (1.4), then* $Z_\lambda(t) \geq \bar{Z}_\lambda(t)$*;*

(iii) furthermore, if $W(t)$ *is as in (1.6), then* $Z_\lambda(t) \geq W(t)$*, and if* $Y(t)$ *is a nonincreasing, nonnegative subsolution of (1.4) in the sense of Definition 1.1 for the given value* λ*, then* $Z_\lambda(t) \geq Y(t)$*.*

Proof. For simplicity, we shall write Z instead of Z_λ. We prove the existence of Z via an iteration process. Take $Z_0(t) \equiv 1$ on $[0,1]$, and for $n = 1, 2, ...$, set,

$$Z_n(t) = \exp\left[-\frac{4\pi}{\lambda}\int_0^t \frac{d\tau}{\int_\tau^1 Z_{n-1}(s)ds + tZ_{n-1}(t)}\right] . \tag{1.7}$$

Thus,

$$\frac{Z_n'(t)}{Z_n(t)} = -\frac{4\pi}{\lambda}\,\frac{1}{\int_t^1 Z_{n-1}(s)ds + tZ_{n-1}(t)} .$$

Clearly, $0 < Z_n \leq 1$ on $[0,1]$, $n = 0, 1, 2, ...$; set

$$A_n(t) = \frac{4\pi}{\lambda}\,\frac{1}{\int_t^1 Z_n(s)ds + tZ_n(t)} . \tag{1.8}$$

Then,

$$Z_{n+1}(t) = \exp\left(-\int_0^t A_n(\tau)d\tau\right) .$$

If $Z_n(t) \leq Z_{n-1}(t)$, then $A_n(t) \geq A_{n-1}(t)$,. Thus from (1.7),

$$Z_{n+1}(t) \leq Z_n(t).$$

Let us then check the hypothesis for $n = 0$; it is easy to see that $Z_1(t) = \exp(-4\pi t/\lambda) \leq Z_0(t) \equiv 1$. By induction, we see that $\{Z_n\}_{n=0}^\infty$ is a decreasing sequence. That these will converge is clear as $Z_n \geq 0$. Call

$$B(t) = \frac{4\pi}{\lambda}\,\frac{1}{\int_t^1 Y(s)ds + tY(t)} ,$$

and

$$C(t) = \frac{4\pi}{\lambda_1^*}\,\frac{1}{\int_t^1 W(s)ds + tW(t)} ,$$

where Y is as in (1.3) and W as in (1.6). Recalling that $0 \leq Y \leq 1$, $0 \leq W \leq 1$, we have that $A_0(t) \leq B(t)$ and $A_0(t) \leq C(t)$. Thus $Z_1(t) \geq W(t)$ and $Z_1(t) \geq Y(t)$; this follows as

$$W(t) = \exp\left(- \int_0^t C(\tau) d\tau \right) \text{ and } Y(t) \leq \exp\left(- \int_0^t B(\tau) d\tau \right). \qquad (1.9)$$

Regarding the proof of the inequality (1.9) for Y, since $-\log\ Y(t)$ is increasing where $Y(t) > 0$, then for those points

$$-\log\ Y(t) \geq \int_0^t \frac{-Y'(s)}{Y(s)} ds \geq \frac{4\pi}{\lambda} \int_0^t \frac{d\tau}{\int_\tau^1 Y(s) ds + \tau Y(\tau)}.$$

At points where $B(t) = +\infty$, we take $Y(t) = 0$. It is then easy to see that (1.9) holds.

Assume that for some n, $Z_n(t) \geq W(t)$; then $A_n(t) \leq C(t)$ implying that $Z_{n+1}(t) \geq W(t)$. We may thus conclude that $Z_n(t) \geq W(t), n = 0, 1, 2, ...$ A similar argument also yields that $Z_n(t) \geq Y(t)$. Clearly then, $\lim_{n\to\infty} Z_n(t) = Z(t)$, where $Z(t)$ satisfies (1.5) and hence (1.4). Furthermore, $Z(t) \geq Y(t)$ and $Z(t) \geq W(t)$. In particular, then $Z(t) > 0$ on $[0, 1)$, so as previously noted, Z must therefore be continuously differentiable there. Regarding the point $t = 1$, it follows from (1.5) that, whether or not $Z(1) = 0$, we have that $Z'(1)$ exists. From the mean value theorem it then follows that the one sided derivative exists at $t = 1$ and is continuous.

In order to see the maximal nature of Z, let $\bar{Z}$ be any other solution. Then clearly $Z_0 \geq \bar{Z}$; now employing arguments as before this implies $Z_n(t) \geq \bar{Z}$, $n = 1, 2, ...$ The conclusion follows. The uniqueness of Z also follows in a similar fashion. ■

DEFINITION 1.3. Let $Z = Z_\lambda$ be as in Theorem 1.1. Then Z will be called the *maximal solution* to (1.4) (corresponding to λ).

Remark 1.1. Let v be as in (0.3), $X(t)$ be its distribution function. Then $X(t)$ satisfies

$$\begin{aligned} \frac{4\pi}{\lambda} X(t) &= -X'(t) \left[\int_t^1 X(\tau) d\tau + t X(t) \right] \qquad (1.10) \\ X(0) &= \lambda_1^*/\lambda_1 \text{ and } X(1) = 0. \end{aligned}$$

By a result of Chiti [3], $u^* - v \geq 0$ implying thereby that $X(t) \leq Y(t)$. Since v and U (as in (0.4)) are related via scaling, we also have $X(t) \leq W(t)$.

The next theorem demonstrates that Z is monotone increasing in λ.

THEOREM 1.2. *Let $\lambda \geq \hat{\lambda} \geq \lambda_1^*, Z$ and $\hat{Z}$ be the maximal solutions corresponding to λ and $\hat{\lambda}$ respectively. Then $Z(t) \geq \hat{Z}(t)$. Furthermore, if $\{\lambda^m\}_{m=1}^{\infty}$ is a decreasing sequence converging to $\lambda \geq \lambda_1^*$, and if Z^m's are the corresponding maximal solutions and $\tilde{Z}$ that for λ then $\lim_{m\to\infty} Z^m(t) = \tilde{Z}(t)$.*

Proof. We prove the first part. Let $\lambda \geq \hat{\lambda} \geq \lambda_1^*$. Let $\{Z_n\}$ and $\{\hat{Z}_n\}$ be the sequences, corresponding to Z and $\hat{Z}$, as given by the iterative scheme of Theorem 1.1. Now $Z_0 = \hat{Z}_0 \equiv 1$; if $Z_n(t) \geq \hat{Z}_n(t)$ for some n, then

$$-\frac{4\pi}{\lambda}\frac{1}{\int_t^1 Z_n + tZ_n} \geq -\frac{4\pi}{\hat{\lambda}}\frac{1}{\int_t^1 \hat{Z}_n + t\hat{Z}_n}.$$

This implies $Z_{n+1} \geq \hat{Z}_{n+1}$; thus we need to check the hypothesis for $n = 1$. One can easily see that $Z_1 \geq \hat{Z}_1$. Thus

$$Z_n(t) \geq \hat{Z}_n(t), \quad n = 0, 1, 2, \dots \; . \tag{1.11}$$

Passing to the limit, we see $Z \geq \hat{Z}$. In order to prove the second part, we note that $Z^m(t) \geq Z^{m+1}(t) \geq \tilde{Z}$, $m = 1, 2, \dots$. Here,

$$Z^m(t) = \exp\left[-\frac{4\pi}{\lambda^m}\int_0^t \frac{d\tau}{\int_\tau^1 Z^m(s)ds + \tau Z^m(\tau)}\right].$$

Passing to the limit, we get

$$\zeta(t) = \exp\left[-\frac{4\pi}{\lambda}\int_0^t \frac{d\tau}{\int_\tau^1 \zeta(s)ds + \tau\zeta(\tau)}\right], \tag{1.12}$$

where $\lim_{m\to\infty} Z^m(t) = \zeta(t)$. Again, $Z^m(t) \geq \tilde{Z}(t)$, and thus $\zeta(t) \geq \tilde{Z}(t)$. But $\tilde{Z}(t)$ is the maximal solution of (1.12). Therefore, by Theorem 1.1, $\zeta(t) = \tilde{Z}(t)$. ■

The maximal solution Z, as given by Theorem 1.1, may be thought of as the distribution function of a radial function $V(r)$. It is this $V(r)$ that will serve as an upper bound for u^*. We also point out that as $Z(t)$ decreases with t, one may calculate $\lim_{t\to 1^-} Z(t) = Z(1)$. If $Z(1) = 0$, then $Z(t)$ is the distribution function of a radially decreasing function which will be the first eigenfunction of the Laplacian (with eigenvalue λ) on D^*. This can happen if and only if $\lambda = \lambda_1^*$. Thus, for $\lambda > \lambda_1^*$, $Z(1) > 0$. In section 3, we derive an expression that will, not only prove the assertion, but also provide us with an estimate for $Z(1)$ important for later work. We will also conclude that the maximal solution for $\lambda = \lambda_1^*$ vanishes at $t = 1$.

2. Properties of Z

THEOREM 2.1. *Let $\lambda > 0$ and $z(t)$ be a solution of (1.4) corresponding to λ in the sense of Definition 1.2, which is strictly positive for $0 \le t < 1$. Then,*
(i) $z'(t) \le -4\pi/\lambda$ and $z'(1) = -4\pi/\lambda$;
(ii) $z(t)$ is convex.

Proof. If $z(t)$ is such a solution of (1.4) then $z(t)$ is decreasing for $0 \le t < 1$, and hence

$$\begin{aligned} z'(t) &= -\frac{4\pi}{\lambda} \frac{z(t)}{\int_t^1 z(s)ds + tz(t)} \qquad (2.1) \\ &\le -\frac{4\pi}{\lambda} \frac{z(t)}{(1-t)z(t) + tz(t)} \\ &= -\frac{4\pi}{\lambda}. \end{aligned}$$

Again, from (1.4),

$$\begin{aligned} z'(t) &\ge -\frac{4\pi}{\lambda} \frac{z(t)}{tz(t)} \\ &= -\frac{4\pi}{\lambda t}. \end{aligned}$$

Taking limits, i.e., $t \to 1^-$ we get $z'(1) = -4\pi/\lambda$. To prove convexity, we differentiate (1.4) once for $0 < t < 1$ to get

$$\begin{aligned} z''(t) &= -\frac{4\pi}{\lambda} \left[\frac{z'(t) \int_t^1 z(s)ds + tz(t)z'(t) - tz(t)z'(t)}{(\int_t^1 z(s)ds + tz(t))^2} \right] \\ &= -\frac{4\pi}{\lambda} \frac{z'(t) \int_t^1 z(s)ds}{(\int_t^1 z(s)ds + tz(t))^2} \\ &> 0. \end{aligned}$$

■

We make a few observations regarding $z''(1)$. If $z(1) \ne 0$, clearly $z''(1) = 0$. If $z(1) = 0$ then one can show, via L'Hopital's rule, that $z''(1) = 2\pi/\lambda$. However, the value of $z'(1)$ is independent of $z(1)$.

Let us call $\delta(\lambda) = Z(1)$, where Z is the maximal solution corresponding to λ. We know from Theorem 1.2 that $\delta(\lambda)$ is nondecreasing in λ. We prove

THEOREM 2.2. *Let $\lambda \geq \lambda_1^*$. Then the value of $\delta(\lambda)$ is strictly increasing in λ.*

Proof. Let $\lambda > \bar{\lambda} \geq \lambda_1^*$, then $\delta(\lambda) \geq \delta(\bar{\lambda})$. Suppose that $\delta(\lambda) = \delta(\bar{\lambda})$. Let the corresponding maximal solutions be Z_λ and $Z_{\bar{\lambda}}$. Then $Z_\lambda \geq Z_{\bar{\lambda}}$. Now $Z'_\lambda(1) = -4\pi/\lambda$ and $Z'_{\bar{\lambda}}(1) = -4\pi/\bar{\lambda}$, so

$$Z'_{\bar{\lambda}}(1) < Z'_\lambda(1) < 0.$$

This, in turn, implies that $Z'_{\bar{\lambda}}(t) < Z'_\lambda(t)$ near $t = 1$. Since $\delta(\lambda) = \delta(\bar{\lambda}) = Z_\lambda(1) = Z_{\bar{\lambda}}(1)$, this implies that $Z_\lambda(t) < Z_{\bar{\lambda}}(t)$ near $t = 1$. This contradicts the fact that $Z_\lambda(t) \geq Z_{\bar{\lambda}}(t)$ on $[0, 1]$. ■

We now make some observations regarding solutions $z(t)$. Let $\hat{z}(t) = cz(t),\ c > 0$. Then from (1.4),

$$\frac{4\pi}{\lambda}\,\frac{\hat{z}(t)}{c} = \frac{1}{c^2}(-\hat{z}'(t))\left[\int_t^1 \hat{z}(\tau)d\tau + t\hat{z}(t)\right].$$

That is,

$$\begin{aligned} \frac{4\pi}{(\lambda/c)}\,\hat{z}(t) &= (-\hat{z}'(t)\left[\int_t^1 \hat{z}(\tau)d\tau + t\hat{z}(t)\right], \\ \hat{z}(0) &= c. \end{aligned} \tag{2.2}$$

Thus $\hat{z}(t)$ solves (1.4) with λ replaced by λ/c and $\hat{z}(0) = c$. In particular, if we take $c = \lambda/\lambda_1^*$, then $cX(t)$, with $X(t)$ as in (1.10), solves (1.6). Actually, it will follow from the estimate for $Z(1)$ that $cX(t) = W(t)$. The basic result that implies uniqueness in the case $\lambda = \lambda_1^*$, is contained in Theorem 3.2.

One can easily show a Payne-Rayner identity for solutions z of (1.4) which are positive for $0 \leq t < 1$. Now,

$$\frac{4\pi}{\lambda}\,z(t)t = (-tz'(t))\left[\int_t^1 z(s)ds + tz(t)\right].$$

Set $F(t) = \int_t^1 z(s)ds + tz(t)$. Then $F'(t) = tz'(t)$. Integrating we obtain

$$\frac{4\pi}{\lambda}\int_0^1 tz(t)dt = \frac{1}{2}\left[(\int_0^1 z(t)dt)^2 - (z(1))^2\right],$$

$$\left(\int_0^1 z(t)dt\right)^2 - (z(1))^2 = \frac{8\pi}{\lambda}\int_0^1 tz(t)dt. \tag{2.3}$$

For W, we have

$$\left(\int_0^1 W(t)dt\right)^2 = \frac{8\pi}{\lambda_1^*}\int_0^1 tW(t)dt.$$

Let $Z(t)$ be the maximal solution as in Theorem 1.1, $0 \le V(r) \le 1$ be the radially nonincreasing function whose distribution function corresponds to Z. One can show, by retracing the steps in (1.1)-(1.3), that $V(r)$ satisfies

$$\begin{aligned} \Delta V + \lambda V &= 0, \quad \bar{r} < r < 1/\sqrt{\pi}, \\ V(1/\sqrt{\pi}) &= 0, \quad V'(\bar{r}) = -\lambda\,\bar{r}/2 \text{ and } V(r) \equiv 1, \;\; 0 < r < \bar{r}. \end{aligned} \tag{2.4}$$

Here $\bar{r} = \sqrt{Z(1)/\pi}$, and the condition on V' at $r = \bar{r}$ follows from the fact that $Z'(1) = -4\pi/\lambda$ and $Z(V(r)) = \pi r^2$, for $r > \bar{r}$.

3. Estimates for Z

A readily available estimate for Z follows from Theorem 2.1, namely,

$$Z'(t) \le -\frac{4\pi}{\lambda};$$

integrating, we get

$$Z(t) \le 1 - \frac{4\pi}{\lambda}t, \;\; 0 \le t \le 1.$$

In particular, $Z(1) \le 1 - 4\pi/\lambda$. Noting that

$$Z'(0) = -\frac{4\pi}{\lambda}\,\frac{1}{\int_0^1 Z},$$

and that Z is convex, we find

$$Z(t) \ge 1 - \frac{4\pi}{\lambda}\,\frac{t}{\int_0^1 Z}, \;\; 0 < t < 1.$$

If $\lambda \to \infty$, then Z increases, and it follows that

$$\lim_{\lambda\to\infty} Z(1) = 1. \tag{3.1}$$

We now state and prove an expression for Z that will provide us with an estimate for $Z(1)$. We do not assume here that $\lambda \ge \lambda_1^*$.

THEOREM 3.1. *Let $\lambda > 0$, and $z(t)$ be a nonnegative C^1 solution of*

$$\begin{aligned} \frac{4\pi}{\lambda} z(t) &= (-z'(t)) \left[\int_t^1 z(s)ds + tz(t) \right], \quad 0 \le t \le 1 \\ z(0) &= 1. \end{aligned} \tag{3.2}$$

Then,

$$z(1) J_2 \left(\sqrt{\frac{\lambda z(1)}{\pi}} \right) = -J_0 \left(\sqrt{\frac{\lambda}{\pi}} \right) \int_0^1 z(t)dt,$$

where J_0 and J_2 are the Bessel functions of order 0 and 2 respectively.

Proof. We first multiply the o.d.e. in (3.2) by $z^{m-1}, m = 1, 2, ...$ Integrating both sides we get,

$$\begin{aligned} \int_0^1 z^m(t)dt &= -\frac{\lambda}{4\pi m} \int_0^1 (z^m(t))' \left(\int_t^1 z(s)ds + tz(t) \right) dt \\ &= -\frac{\lambda}{4\pi m} \left\{ z^m(t) \left(\int_t^1 z(s)ds + tz(t) \right) \Big|_0^1 - \int_0^1 tz^m(t)z'(t)dt \right\} \\ &= -\frac{\lambda}{4\pi m} \left\{ z^{m+1}(1) - \int_0^1 z(t)dt - \frac{z^{m+1}(1)}{m+1} + \frac{1}{m+1} \int_0^1 z^{m+1}(t)dt \right\} \\ &= -\frac{\lambda z^{m+1}(1)}{4\pi(m+1)} + \frac{\lambda}{4\pi m} \int_0^1 z(t)dt - \frac{\lambda}{4\pi m(m+1)} \int_0^1 z^{m+1}(t)dt. \end{aligned} \tag{3.3}$$

We intend to use (3.3) recursively. We start at $m = 1$. Then (3.3) yields

$$\int_0^1 z(t)dt = -\frac{\lambda z^2(1)}{4\pi \cdot 2} + \frac{\lambda}{4\pi} \int_0^1 z(t)dt - \frac{\lambda}{4\pi \cdot 1 \cdot 2} \int_0^1 z^2(t)dt.$$

Thus,

$$\left(\frac{\lambda}{4\pi} - 1 \right) \int_0^1 z(t)dt = \frac{\lambda z^2(1)}{4\pi \cdot 2} + \frac{\lambda}{4\pi \cdot 1 \cdot 2} \int_0^1 z^2(t)dt. \tag{3.4}$$

Taking $m = 2$ in (3.3) we have

$$\int_0^1 z^2(t)dt = -\frac{\lambda z^3(1)}{4\pi \cdot 3} + \frac{\lambda}{4\pi \cdot 2} \int_0^1 z(t)dt - \frac{\lambda}{4\pi \cdot 2 \cdot 3} \int_0^1 z^3(t)dt. \tag{3.5}$$

Substituting (3.5) in (3.4), we get

$$\left(\frac{\lambda}{4\pi} - (\frac{\lambda}{4\pi})^2 \ \frac{1}{2\cdot 2} - 1\right)\int_0^1 z(t)dt \ = \ \frac{\lambda z^2(1)}{4\pi\cdot 2} - (\frac{\lambda}{4\pi})^2 \ \frac{z^3(1)}{1\cdot 2\cdot 3} - \ (\frac{\lambda}{4\pi})^2 \ \frac{1}{1\cdot 2\cdot 2\cdot 3}\int_0^1 z^3(t)dt. \tag{3.6}$$

Let us assume that for some m, we have

$$\left\{\sum_{n=0}^{m}(-1)^{n+1}(\frac{\lambda}{4\pi})^n\frac{1}{(n!)^2}\right\}\int_0^1 z(t)dt = z(1)\left\{\sum_{n=1}^{m}(-1)^{n+1}(\frac{\lambda}{4\pi})^n \ \frac{z^n(1)}{((n-1)!)^2 n(n+1)}\right\} +(-1)^{m+1}(\frac{\lambda}{4\pi})^m\frac{1}{(m!)^2(m+1)}\int_0^1 z^{m+1}(t)dt. \tag{3.7}$$

We use (3.3) to compute the integral on the right side of (3.7), i.e.

$$\int_0^1 z^{m+1}(t)dt = -\frac{\lambda z^{m+2}(1)}{4\pi(m+2)}+\frac{\lambda}{4\pi(m+1)}\int_0^1 z(t)dt-\frac{\lambda}{4\pi(m+1)(m+2)}\int_0^1 z^{m+2}(t)dt$$

Thus,

$$(-1)^{m+1}(\frac{\lambda}{4\pi})^m\frac{1}{(m!)^2(m+1)}\int_0^1 z^{m+1}(t)dt = (-1)^{m+2}(\frac{\lambda}{4\pi})^{m+1}\frac{z^{m+2}(1)}{(m!)^2(m+1)(m+2)} +(-1)^{m+1}(\frac{\lambda}{4\pi})^{m+1}\frac{1}{((m+1)!)^2}\int_0^1 z(t)dt +(-1)^{m+2}(\frac{\lambda}{4\pi})^{m+1} \ \frac{1}{((m+1)!)^2(m+2)}\int_0^1 z^{m+2}(t)dt. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\left\{\sum_{n=0}^{m+1}(-1)^{n+1}(\frac{\lambda}{4\pi})^n\frac{1}{(n!)^2}\right\}\int_0^1 z(t)dt = z(1)\left\{\sum_{n=1}^{m+1}(-1)^{n+1}(\frac{\lambda}{4\pi})^n\frac{z^n(1)}{((n-1)!)^2 n(n+1)}\right\} +(-1)^{m+2}(\frac{\lambda}{4\pi})^{m+1}\frac{1}{((m+1)!)^2(m+2)}\int_0^1 z^{m+2}(t)dt. \tag{3.9}$$

Since we have shown that (3.7) holds for $m = 1, 2$, it now follows by induction that (3.7) holds for every m. Noting that $0 < z(t) \le 1$, letting $m \to \infty$, in (3.9), we have

$$\left\{ \sum_{n=0}^{\infty} (-1)^{n+1} (\frac{\lambda}{4\pi})^n \frac{1}{(n!)^2} \right\} \int_0^1 z(t)dt$$
$$= z(1) \left\{ \sum_{n=1}^{\infty} (-1)^{n+1} (\frac{\lambda}{4\pi})^n \frac{z^n(1)}{((n-1)!)^2 n(n+1)} \right\}.$$

Comparing the formulas for J_0 and J_2 [10], we get

$$z(1) J_2 \left(\sqrt{\frac{\lambda z(1)}{\pi}} \right) = -J_0 \left(\sqrt{\frac{\lambda}{\pi}} \right) \int_0^1 z(t)dt.$$

■

THEOREM 3.2. *Let* $Z(t)$ *be the maximal solution of (1.4) corresponding to* $\lambda = \lambda_1^*$, *and* $z(t)$ *be a solution of (1.4) also corresponding to* λ_1^* *which is positive for* $0 \le t < 1$. *Then,* $z(1) = Z(1) = 0$ *and* $z(t) \equiv Z(t) \equiv W(t)$ *where* $W(t)$ *is the function of (1.6).*

Proof. We first observe that $z(1) = Z(1) = 0$. In fact since $Z(t) > 0$ for $0 \le t < 1$ Theorem 3.1 applies to $Z(t)$ as well as $z(t)$. Now, $\lambda_1^* = \nu^2 \pi$, where ν is the first zero of J_0. Thus from Theorem 3.1, we have

$$Z(1) J_2 \left(\sqrt{\frac{\lambda_1^* Z(1)}{\pi}} \right) = 0.$$

Since the first nonzero zero of J_2 is greater than ν, it follows that $Z(1) = 0$. Similarly, $z(1) = 0$.

Now, $z(t) \le Z(t)$, and

$$\begin{aligned} \frac{z'(t)}{z(t)} &= -\frac{4\pi}{\lambda_1^*} \frac{1}{\int_t^1 z(s)ds + t\, z(t)} \\ &\le -\frac{4\pi}{\lambda_1^*} \frac{1}{\int_t^1 Z(s)ds + t\, Z(t)} \\ &= \frac{Z'(t)}{Z(t)} \ . \end{aligned}$$

Integrating from $\bar{t}$ to $t, 0 < \bar{t} < t$, we obtain

$$\frac{z(t)}{z(\bar{t})} \le \frac{Z(t)}{Z(\bar{t})} \ .$$

This, in turn, implies

$$\begin{aligned}\frac{Z(\bar{t})}{z(\bar{t})} &\le \frac{Z(t)}{z(t)}\\ &\le \lim_{t\to 1^-}\frac{Z(t)}{z(t)}\\ &\le \lim_{t\to 1^-}\frac{Z'(t)}{z'(t)}\\ &= 1.\end{aligned}$$

The last step follows from Theorem 2.1 and the fact that $z(1) = Z(1) = 0$. Hence,

$$z(t) \le Z(t) \le z(t);$$

uniqueness follows. ■

THEOREM 3.3. *If $\lambda < \lambda_1^*$, then (1.4) has no nonnegative solutions.*

Proof. Let $\lambda < \lambda_1^*$, and z be such a solution to (3.2). Suppose first that $z(t) > 0$ for $0 \le t < 1$. Then z is C^1. Since $\lambda < \nu^2\pi$ and the first nonzero zero of J_2 is greater than the first zero $\nu = \sqrt{\lambda_1^*/\pi}$ of J_0, we have that

$$J_2\left(\sqrt{\frac{\lambda z(1)}{\pi}}\right) \ge 0 \quad \text{and} \quad J_0\left(\sqrt{\frac{\lambda}{\pi}}\right) > 0.$$

Thus, both sides of the formula in Theorem 3.1 vanish implying immediately

$$\int_0^1 z(t)dt = 0.$$

The conclusion follows in this case.

If $z(t) = 0$ for some $0 < t < 1$, letting $a = \sup\{t : 0 < t < 1, z(t) > 0\}$, we may then define $\zeta(t) = z(at)$. Then, it follows readily from (1.5) that $\zeta(t)$ is again a solution with the same λ, which is positive on $[0,1)$ and hence C^1. Thus, applying Theorem 3.1 to $\zeta(t)$ we find that $\zeta(t) \equiv 0$ and so again $z(t) \equiv 0$. ■

THEOREM 3.4. *There exists an absolute constant C such that if z is a solution to (1.4) corresponding to $\lambda > \lambda_1^*$, then $z(1) \le C\sqrt{\lambda - \lambda_1^*}$.*

Proof. By Theorems 1.2, 2.2, and 3.2, we find that $Z(1)$ decreases to zero as $\lambda \downarrow \lambda_1^*$. Inspecting the series expressions for J_2 and J_0, we find that,

$$\begin{aligned} Z(1)J_2\left(\sqrt{\frac{\lambda Z(1)}{\pi}}\right) &\approx \frac{\lambda}{\pi}Z(1)^2, \\ J_0\left(\sqrt{\frac{\lambda}{\pi}}\right) = J_0\left(\sqrt{\frac{\lambda}{\pi}}\right) - J_0\left(\sqrt{\frac{\lambda_1^*}{\pi}}\right) &\approx C(\lambda-\lambda_1^*), \end{aligned} \tag{3.10}$$

as $\lambda \downarrow \lambda_1^*$. The conclusion now follows from Theorem 3.1. ∎

If, in Theorem 3.1, we integrate from 0 to t (instead of 0 to 1) we may derive the following expression for $z(t)$.

THEOREM 3.5. *Let $\lambda \geq \lambda_1^*$ and $z(t)$ be a solution of (1.4) which is positive for $0 \leq t < 1$. Then*

$$tz(t)J_2\left(\sqrt{\frac{\lambda z(t)}{\pi}}\right) = J_0\left(\sqrt{\frac{\lambda z(t)}{\pi}}\right)\int_t^1 z(s)ds - J_0\left(\sqrt{\frac{\lambda}{\pi}}\right)\int_0^1 z(s)ds.$$

∎

It is clear from Theorem 2.2 and Theorem 3.2 that $Z(1) = 0$ if and only if $\lambda = \lambda_1^*$. However, this does not imply the statement about $z(1)$. Although Theorem 3.2 implies that $z(1) = 0$ when $\lambda = \lambda_1^*$, in order to prove the converse we have to employ Theorem 3.5 . So let us then assume that $z(t)$ is a positive solution of (1.4) with $z(1) = 0$. Then from Theorem 3.1

$$J_0\left(\sqrt{\frac{\lambda}{\pi}}\right) = 0.$$

Let $\nu = \nu_1 < \nu_2 < ...$ be the zeros of J_0. Then $\lambda = \pi\ \nu_i^2$ for some $i \geq 1$. If $\lambda = \pi\ \nu_1^2 = \lambda_1^*$ then we are done. So let us assume that $\lambda = \pi\ \nu_i^2$ for some $i > 1$. We now observe that J_2 and J_0 do not vanish together. This follows from the recurrence formula $J_2(x) = (2/x)\ J_1(x) - J_0(x)$ and the fact that J_1 and J_0 have no common zeros [10]. Thus, $J_2(\nu_l) = (2/\nu_l)\ J_1(\nu_l) \neq 0,\ l = 0, 1, 2, ...$ Furthermore, $z(0) = 1, z(1) = 0$ and $z(t)$ is continuous. Thus, there are i numbers $0 = t_1 < t_2 < .... < t_i < 1$ such that $\nu_i^2\ z(t_j) = \nu_{i-j+1}^2,\ j = 1, 2, ..., i$. Upon substituting the $t_j's$ in the formula in Theorem 3.5 we see that

$$t_j z(t_j)\ J_2(\nu_{i-j+1}) = J_0(\nu_{i-j+1})\ \int_{t_j}^1 z(s)ds = 0.$$

Therefore, $z(t_j) = 0$ for $j = 2, ..., i$. This contradicts the positivity of $z(t)$. Thus we obtain

COROLLARY 3.1. *Let $z(t)$ be a solution of (1.4) such that $z(t) > 0$ for $0 \le t < 1$. Then $z(1) = 0$ if and only if $\lambda = \lambda_1^*$.* ■

Let $\lambda \ge \bar{\lambda} \ge \lambda_1^*, Z$ and $\bar{Z}$ be the corresponding maximal solutions of (1.4). We show that $Z' \ge \bar{Z}'$. This will provide us with pointwise estimates for $Z - W$. Recall from Theorem 3.2 that W is also maximal.

THEOREM 3.6. *Let $\lambda \ge \bar{\lambda} \ge \lambda_1^*, Z$ be the maximal solution corresponding to λ, and $\bar{z}$ be a solution to (1.4) corresponding to $\bar{\lambda}$ such that $\bar{z}(t) > 0$ for $0 \le t < 1$. Then $Z'(t) \ge \bar{z}'(t)$.*

Proof. Recall that we have

$$Z'(t) = -\frac{4\pi}{\lambda} \frac{Z(t)}{\int_t^1 Z(s)ds + tZ(t)}, \qquad Z(0) = 1, \tag{3.11}$$

and

$$\bar{z}'(t) = -\frac{4\pi}{\bar{\lambda}} \frac{\bar{z}(t)}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}, \qquad \bar{z}(0) = 1. \tag{3.12}$$

If $\bar{Z}$ is the maximal solution corresponding to $\bar{\lambda}$, then $Z(t) \ge \bar{Z}(t) \ge \bar{z}(t)$. Hence

$$\begin{aligned} Z'(t) - \bar{z}'(t) &= -\frac{4\pi}{\lambda} \frac{Z(t)}{\int_t^1 Z(s)ds + tZ(t)} \\ &\quad + \frac{4\pi}{\lambda} \frac{\bar{z}(t)}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)} + \left(\frac{4\pi}{\bar{\lambda}} - \frac{4\pi}{\lambda}\right) \frac{\bar{z}(t)}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}. \end{aligned}$$

Thus,

$$\begin{aligned} Z'(t) - \bar{z}'(t) &= \frac{4\pi}{\lambda} \left[\frac{\bar{z}(t) \int_t^1 Z(s)ds - Z(t) \int_t^1 \bar{z}(s)ds}{(\int_t^1 Z(s)ds + tZ(t))(\int_t^1 \bar{z}(s)ds + t\bar{z}(t))} \right] \\ &\quad + \frac{4\pi}{\lambda\bar{\lambda}}(\lambda - \bar{\lambda}) \frac{\bar{z}(t)}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}. \end{aligned} \tag{3.13}$$

Now set $F(t) = \bar{z}(t)\int_t^1 Z(s)ds - Z(t)\int_t^1 \bar{z}(s)ds$. Then $F(1) = 0$, and $F(0) = \int_0^1 Z(t) - \bar{z}(t)dt \geq 0$. Differentiating F,

$$\begin{aligned}
F'(t) &= \bar{z}'(t)\int_t^1 Z(s)ds - Z'(t)\int_t^1 \bar{z}(s)ds \\
&= -\frac{4\pi}{\bar{\lambda}}\frac{\bar{z}(t)\int_t^1 Z(s)ds}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)} + \frac{4\pi}{\lambda}\frac{Z(t)\int_t^1 \bar{z}(s)ds}{\int_t^1 Z(s)ds + tZ(t)} \\
&= -\frac{4\pi}{\bar{\lambda}}\left[\frac{\bar{z}(t)\int_t^1 Z(s)ds - Z(t)\int_t^1 \bar{z}(s)ds}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}\right] \\
&\quad + \left(\frac{4\pi}{\lambda} - \frac{4\pi}{\bar{\lambda}}\right)\frac{Z(t)\int_t^1 \bar{z}(s)ds}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)} \\
&\quad + \frac{4\pi}{\lambda}Z(t)\int_t^1 \bar{z}(s)ds\left[\frac{1}{\int_t^1 Z(s)ds + tZ(t)} - \frac{1}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}\right] \\
&= -\frac{4\pi}{\bar{\lambda}}\left(\frac{F(t)}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)}\right) + \frac{4\pi}{\lambda\bar{\lambda}}(\bar{\lambda} - \lambda)\frac{Z(t)\int_t^1 \bar{z}(s)ds}{\int_t^1 \bar{z}(s)ds + t\bar{z}(t)} \\
&\quad - \frac{4\pi}{\lambda}Z(t)\int_t^1 \bar{z}(s)ds\left[\frac{\int_t^1(Z(s) - \bar{z}(s))ds + t(Z(t) - \bar{z}(t))}{(\int_t^1 Z(s)ds + tZ(t))(\int_t^1 \bar{z}(s)ds + t\bar{z}(t))}\right]
\end{aligned}$$

Using (3.12), and observing that $\lambda \geq \bar{\lambda}$ and $Z(t) \geq \bar{z}(t)$, we have

$$F'(t) - \frac{F(t)\bar{z}'(t)}{\bar{z}(t)} \leq 0,$$

implying thereby ,

$$(F(t)/\bar{z}(t))' \leq 0.$$

Thus $F(t)/\bar{z}(t)$ is decreasing, and

$$\begin{aligned}
\frac{F(t)}{\bar{z}(t)} &\geq \lim_{t\to 1^-}\frac{F(t)}{\bar{z}(t)} \\
&= \lim_{t\to 1^-}\left[\int_t^1 Z(s)ds - Z(t)\frac{\int_t^1 \bar{z}(s)ds}{\bar{z}(t)}\right]
\end{aligned} \tag{3.14}$$

If $\bar{z}(1) \neq 0$, then (3.14) yields

$$F(t)/\bar{z}(t) \geq 0.$$

If $\bar{z}(1) = 0$, then by Corollary 3.1 $\bar{\lambda} = \lambda_1^*$, and again the right side can be easily shown to be zero. This follows from Theorem 2.1, i.e., $\bar{z}'(1) = 4\pi/\bar{\lambda} \neq 0$. Thus

$$F(t)/\bar{z}(t) \geq 0, \quad 0 < t < 1.$$

Since $\bar{z}(t) > 0$, this implies that

$$F(t) = \bar{z}(t)\int_t^1 Z(s)ds - Z(t)\int_t^1 \bar{z}(s)ds > 0. \tag{3.15}$$

Employing (3.15) in (3.13) and observing that $\lambda \geq \bar{\lambda}, Z(t) \geq \bar{z}(t) \geq 0$, we have

$$Z'(t) - \bar{z}'(t) \geq 0. \tag{3.16}$$

■

As an immediate consequence of Theorem 3.6 we have

COROLLARY 3.2. *Let Z and $\bar{z}$ be as in Theorem 3.6. Then*

$$0 \leq Z(t) - \bar{z}(t) \leq Z(1) - \bar{z}(1).$$

■

If, in Corollary 3.2, we take $\bar{\lambda} = \lambda_1^*, \ W = \bar{z}$, then

$$0 \leq Z(t) - W(t) \leq Z(1). \tag{3.17}$$

Thus, by Theorem 3.4 and Corollary 3.2, we have that for λ close to λ_1^*,

$$Z(t) - W(t) = O(\sqrt{\lambda - \lambda_1^*}).$$

Recall that $U(r)$ is the eigenfunction whose distribution function is W, $V(r)$ is the function whose distribution function is Z. Then

$$\|U\|_{L^1(D^*)} = \int_0^1 W(t)dt,$$

and

$$\|V\|_{L^1(D^*)} = \int_0^1 Z(t)dt,$$

Noting that $U \leq V$ and using (3.17),

$$\|V - U\|_{L^1(D^*)} = \int_0^1 (Z(t) - W(t))dt \leq Z(1)$$

Thus, (3.17) and Theorem 3.4 yield,

$$\|V - U\|_{L^1(D^*)} \leq C\sqrt{\lambda - \lambda_1^*} \tag{3.18}$$

for some constant C. That this is sharp follows from

THEOREM 3.7. *Let $\lambda \geq \lambda_1^*, Z$ the maximal solution in (1.4), and W as in (1.6). Then there exist constants $\bar{C}_1$ and $\bar{C}_2$ such that for λ sufficiently close to λ_1^*,*

$$\bar{C}_1\sqrt{\lambda-\lambda_1^*} \leq \int_0^1 Z(t) - W(t)dt \leq \bar{C}_2\sqrt{\lambda-\lambda_1^*}. \tag{3.19}$$

Proof. The right side of (3.19) follows from (3.17) and Theorem 3.4. Recall that $Z(0) = 1, W(0) = 1$ and $W(1) = 0$. From the o.d.e.'s for Z and W, we see, using integration by parts, that

$$\begin{aligned}
-\frac{4\pi}{\lambda} &= \int_0^1 \frac{Z'(t)}{Z(t)}\left(\int_t^1 Z(s)ds + tZ(t)\right)dt \\
&= \log\ Z(t)\left(\int_t^1 Z(s)ds + tZ(t)\right)\Bigg|_0^1 - \int_0^1 tZ'(t)\ \log\ Z(t)dt \\
&= Z(1)\ \log\ Z(1) - \int_0^1 t\left(Z(t)\ \log\ Z(t) - Z(t)\right)' dt \\
&= Z(1)\ \log\ Z(1) - Z(1)\ \log\ Z(1) + Z(1) \\
&\quad + \int_0^1 (Z(t)\ \log\ Z(t) - Z(t))dt \\
&= Z(1) + \int_0^1 (Z(t)\ \log\ Z(t) - Z(t))dt.
\end{aligned} \tag{3.20}$$

Similarly,

$$\int_0^1 (W(t)\ \log\ W(t) - W(t))dt = -\frac{4\pi}{\lambda_1^*}. \tag{3.21}$$

Combining (3.20) and (3.21), we see

$$\begin{aligned}
\int_0^1 (Z(t) - W(t))dt &= Z(1) - \frac{4\pi}{\lambda\lambda_1^*}(\lambda - \lambda_1^*) \\
&\quad + \int_0^1 (Z(t)\ \log\ Z(t) - W(t)\log\ W(t))dt.
\end{aligned} \tag{3.22}$$

We proceed with the integral on the right side as follows. Multiplying, the o.d.e. for $Z(t)$ by log $Z(t)$ and integrating, we obtain

$$\int_0^1 Z(t)\log\ Z(t)dt = -\frac{\lambda}{4\pi}\int_0^1 Z'(t)\log\ Z(t)\left(\int_t^1 Z(s)ds + tZ(t)\right)dt$$

$$
\begin{aligned}
&= -\frac{\lambda}{4\pi}\left\{\int_0^1 \{Z(t)\log\ Z(t)-Z(t)\}'\left(\int_t^1 Z(s)ds+tZ(t)\right)dt\right\}\\
&= -\frac{\lambda}{4\pi}\left\{\{Z(t)\log\ Z(t)-Z(t)\}\left(\int_t^1 Z(s)ds+tZ(t)\right)\Bigg|_0^1\right.\\
&\quad\left.-\int_0^1 tZ'(t)\{Z(t)\log\ Z(t)-Z(t)\}dt\right\}\\
&= -\frac{\lambda}{4\pi}\left\{Z^2(1)\log\ Z(1)-Z^2(1)+\int_0^1 Z(t)dt\right.\\
&\quad\left.-\int_0^1 t\left(\frac{Z^2(t)}{2}\log\ Z(t)-\frac{3}{4}Z^2(t)\right)'dt\right\}\\
&= -\frac{\lambda}{4\pi}\left\{\frac{Z^2(1)}{2}\log\ Z(1)-\frac{Z^2(1)}{4}+\int_0^1 Z(t)dt\right.\\
&\quad\left.+\int_0^1(\frac{Z^2(t)}{2}\log\ Z(t)-\frac{3}{4}Z^2(t))dt\right\}. \qquad (3.23)
\end{aligned}
$$

Similarly, we may show

$$
\begin{aligned}
\int_0^1 W(t)\log\ W(t)dt &= -\frac{\lambda_1^*}{4\pi}\left\{\int_0^1 W(t)dt\right.\\
&\quad\left.+\int_0^1(\frac{W^2(t)}{2}\log\ W(t)-\frac{3}{4}W^2(t))dt\right\}. \qquad (3.24)
\end{aligned}
$$

Set

$$
A = Z(1)+\frac{\lambda}{16\pi}Z^2(1)-\frac{\lambda}{8\pi}Z^2(1)\log\ Z(1)-\frac{4\pi}{\lambda\lambda^*}(\lambda-\lambda^*). \qquad (3.25)
$$

Now combining (3.22) with (3.23) and (3.25), (3.20) we obtain

$$
\begin{aligned}
\int_0^1 (Z(t)-W(t))dt &= A-\frac{\lambda}{4\pi}\int_0^1 Z(t)dt+\frac{\lambda_1^*}{4\pi}\int_0^1 W(t)dt\\
&\quad-\frac{\lambda}{4\pi}\int_0^1 I(Z(t))dt+\frac{\lambda_1^*}{4\pi}\int_0^1 I(W(t))dt\\
&= A-\frac{\lambda}{4\pi}\int_0^1 (Z(t)-W(t))dt
\end{aligned}
$$

$$+\frac{\lambda_1^* - \lambda}{4\pi}\left(\int_0^1 W(t)dt + \int_0^1 I(W(t))dt\right)$$
$$-\frac{\lambda}{4\pi}\int_0^1 (I(Z(t)) - I(W(t)))dt, \qquad (3.26)$$

where

$$I(f(t)) = \frac{f^2(t)}{2}\log\ f(t) - \frac{3}{4}f^2(t).$$

Now, $Z(0) = W(0) = 1$, and $0 \leq W(t) \leq Z(t) \leq 1$. Thus, $I(Z(0)) - I(W(0)) = 0$. Now the function $(x^2/2)\log\ x - 3x^2/4$ is decreasing for $0 < x < 1$. Thus, $I(Z(t)) \leq I(W(t))$. Finally, from (3.25) we obtain

$$(1 + \frac{\lambda}{4\pi})\int_0^1 (Z(t) - W(t))dt \geq A + \frac{\lambda_1^* - \lambda}{4\pi}\left(\int_0^1 W(t)dt + \int_0^1 I(W(t))dt\right). \quad (3.27)$$

Applying Theorem 3.1 to $Z(t)$, and again using (3.10), the result now follows from (3.25) and (3.27). ∎

4. Pointwise estimates on u^*

We now set $\lambda = \lambda_1$, and recall (0.1), (0.3), (0.4), (1.3), (1.6) and (2.4). If u is as in (0.1), u^* as in (0.2), and v as in (0.3), it follows from a result of Chiti [3] that $u^*(r) \geq v(r)$. Also , from Theorem 1.1 we have that $u^*(r) \leq V(r)$. Thus, if U is as in (0.4) we have

$$v(r) - U(r) \leq u^*(r) - U(r) \leq V(r) - U(r). \qquad (4.1)$$

With these preliminaries, we now prove

THEOREM 4.1. *Let u and U be as above. There exists a constant C such that*

$$\|u^* - U\|_{L^\infty(D^*)} \leq C\sqrt{\lambda - \lambda_1^*}.$$

Proof. We first estimate $V - U$. We state once again that $U(r)$ and $V(r)$ satisfy

$$\Delta U + \lambda_1^* U = 0, \qquad 0 < r < \sqrt{1/\pi},$$
$$U(0) = 1,\ \ U'(0) = 0,\ \text{and}\ \ U(\sqrt{1/\pi}) = 0;$$

and

$$\Delta V + \lambda_1 V = 0, \qquad \bar{r} < r < \sqrt{1/\pi},$$
$$V(r) \equiv 1,\ 0 < r \leq \bar{r}, V'(\bar{r}^+) = -\lambda_1\bar{r}/2,\ \text{and}\ \ V(\sqrt{1/\pi}) = 0.$$

Here, $\bar{r} = \sqrt{Z(1)/\pi}$; note U and V are both positive and radially decreasing. The function U is the first eigenfunction on D^*. Regarding $V'(\bar{r})$, observe that $Z(V(r)) = \pi r^2$, for $\bar{r} < r$. Thus $Z'(V(r))V'(r) = 2\pi r$, hence $V'(\bar{r}^+) = 2\pi\bar{r}/Z'(1) = -\lambda_1\bar{r}/2$. Let us first estimate U on $[0, \bar{r}]$. It is easily shown that the o.d.e for U yields

$$\begin{aligned} U(r) \;&=\; 1 - \lambda_1^* \int_0^r \frac{1}{t} \int_0^t sU(s)ds\; dt \\ &\geq\; 1 - \frac{\lambda_1^* r^2}{4}. \end{aligned} \tag{4.2}$$

Thus, for $0 < r < \bar{r}$, it follows from (4.2) that

$$\begin{aligned} V(r) - U(r) \;&\leq\; \frac{\lambda_1^*}{4}\, r^2, \\ &\leq\; \frac{\lambda_1^*}{4\pi} Z(1). \end{aligned} \tag{4.3}$$

Now consider the interval $\bar{r} < r < \sqrt{1/\pi}$. Set $t = U(r)$ and $t' = V(r)$. Then $Z(t') = W(t)$, and noting that W is one-one, decreasing and differentiable, (3.17) and Theorem 2.1 imply

$$\begin{aligned} V(r) - U(r) = t' - t \;&=\; W^{-1}(W(t')) - W^{-1}(Z(t')) \\ &\leq\; \|\frac{1}{W'}\|_{L^\infty}\{Z(t') - W(t')\} \\ &\leq\; \frac{\lambda_1^*}{4\pi}\{Z(t') - W(t')\} \\ &\leq\; \frac{\lambda_1^*}{4\pi} Z(1). \end{aligned} \tag{4.4}$$

Thus from (4.3) and (4.4), it follows from λ_1 close to λ_1^*,

$$u^*(r) - U(r) \leq V(r) - U(r) = O(\sqrt{\lambda_1 - \lambda_1^*}). \tag{4.5}$$

Now $v(r)$ and $U(r)$ are related via a scaling, i.e., $v(r) = U(cr)$ with $c = \sqrt{\lambda_1/\lambda_1^*}$. Thus

$$v(r) - U(r) = \begin{cases} -U(r), & R \leq r \leq \sqrt{1/\pi}, \\ U(cr) - U(r), & 0 < r \leq R, \end{cases} \tag{4.6}$$

where $R = \sqrt{|B|/\pi} = \sqrt{\lambda_1^*/\pi\lambda_1}$. Clearly,

$$\begin{aligned} |U(cr) - U(r)| \;&\leq\; \|U'\|_{L^\infty}(c-1)r \\ &\leq\; \|U'\|_{L^\infty}(\sqrt{\lambda_1/\lambda_1^*} - 1)\frac{1}{\sqrt{\pi}}. \end{aligned}$$

Recall that $W(U(r)) = \pi r^2$; hence $W'(U(r))U'(r) = 2\pi r$, implying by Theorem 2.1 that

$$|U'(r)| = \frac{2\pi r}{|W'(U(r))|} \leq \frac{\lambda_1^*}{2\sqrt{\pi}}.$$

A similar calculation in (4.6) for $R \leq r \leq 1/\sqrt{\pi}$, yields that

$$0 \leq U(r) - v(r) = O(\lambda_1 - \lambda_1^*). \tag{4.7}$$

Putting together (4.5) and (4.7) in (4.1), we deduce, for λ_1 close to λ_1^* and $0 < r < \sqrt{1/\pi}$,

$$|u^*(r) - U(r)| = O(\sqrt{\lambda_1 - \lambda_1^*}).$$

The Theorem now follows. ■

Remark 4.1. We mention here that Theorem 4.1 holds for uniformly elliptic p.d.e.'s. Consider the following eigenvalue problem. Let $u \in W_0^{1,2}(D)$ be such that,

$$\begin{gathered} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x)u = \lambda_1 u, \text{ in } D, \\ u = 0, \text{ on } \partial D. \end{gathered} \tag{4.8}$$

We will assume that $u \geq 0$ and that $\sup u = 1$. Here $a_{ij}(x)$ and $c(x)$ are bounded, real and measurable, and the a_{ij}'s satisfy ellipticity, i.e.

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \;\; \forall\, x \in D, \text{ and } \forall\, \xi \in I\!R^2.$$

We also assume that $c(x) \geq 0, \lambda_1$ is the first eigenvalue and u is the first eigenfunction of the elliptic operator on D. Let u^* be as in (0.2) and (0.4). By the work in [8], (1.1) and (1.2) continue to hold. Furthermore, by [3] and [4], $u^* - v \geq 0$, where v is as in (0.3). All our results regarding Z are applicable and hence Theorem 4.1 holds for the first eigenfunction of (4.8).

5. A Stability Result

We now apply our methods to derive another estimate on u^*. Let v be as (0.3) and u^* the radially decreasing rearrangement of u as in (0.1) and (0.2). We will prove the following

THEOREM 5.1. *Let $\lambda_1 \geq \lambda_1^*$, and u, u^*, and v be as in (0.1), (0.2) and (0.3). Let B be as in (0.3) and R be such that $|B| = \pi R^2$. There exists a constant $C = C_R$ such that if $u^*(R) = \varepsilon > 0$, then for sufficiently small ε,*

$$\|u^* - v\|_{L^\infty(B)} \leq C\sqrt{\varepsilon}. \tag{5.1}$$

The proof of Theorem 5.1 will follow from two lemmas. First recall that $|B| = \lambda_1^*/\lambda_1$. Let $Y(t) = \mu(t)$ be the subsolution of (1.3) and $X(t)$ be as in (1.10). Then

$$Y(\varepsilon) = X(0) = \lambda_1^*/\lambda_1. \tag{5.2}$$

We will construct an upper bound for $Y(t)$, say $G(t)$, much the same way as in Theorem 1.1. The function Z will not be useful here as $Z(\varepsilon)$ may be large compared to $Y(\varepsilon)$, especially if ε is very small. We again proceed via an iteration. For $\varepsilon < t < 1$, let $G(t)$ satisfy

$$\frac{4\pi}{\lambda_1} G(t) = (-G'(t)) \left[\int_t^1 G(s)ds + tG(t) \right], \text{ and } G(\varepsilon) = Y(\varepsilon) = \lambda_1^*/\lambda_1. \tag{5.3}$$

We introduce the following iterative scheme. Take $G_0(t) = \lambda_1^*/\lambda_1$ on $[\varepsilon, 1]$, and define $G_n(t)$ on $[\varepsilon, 1]$ by

$$G_n(t) = \frac{\lambda_1^*}{\lambda_1} \exp \left[-\frac{4\pi}{\lambda_1} \int_\varepsilon^t \frac{d\tau}{\int_\tau^1 G_{n-1}(s)ds + \tau G_{n-1}(\tau)} \right], \tag{5.4}$$

where $n = 1, 2, \ldots.$ As in Theorem 1.1, $G_n(t)$ are decreasing and $G_n(t) \geq Y(t) \geq X(t)$ on $[\varepsilon, 1], n = 1, 2, \ldots.$ Using the same procedure as in the proof of Theorem 1.1, one can easily show that

$$Y(t) \leq Y(\varepsilon) \exp \left[-\frac{4\pi}{\lambda_1} \int_\varepsilon^t \frac{d\tau}{\int_\tau^1 Y(s)ds + \tau Y(\tau)} \right],$$

and

$$X(t) = X(\varepsilon) \exp \left[-\frac{4\pi}{\lambda_1} \int_\varepsilon^t \frac{d\tau}{\int_\tau^1 X(s)ds + \tau X(\tau)} \right].$$

Here $X(\varepsilon) < Y(\varepsilon) = X(0)$, as X is decreasing. Passing to the limit, we obtain $\lim_{n\to\infty} G_n(t) = G(t)$, a maximal solution of

$$G(t) = G(\varepsilon) \exp \left[-\frac{4\pi}{\lambda_1} \int_\varepsilon^t \frac{d\tau}{\int_\tau^1 G(s)ds + \tau G(\tau)} \right]. \tag{5.5}$$

This, in turn, satisfies (5.3). We now calculate $G(1)$. We first state an easy upper bound. Since, as in Theorem 2.1, $G'(t) \le -4\pi/\lambda_1$ on $[\varepsilon, 1]$, then

$$G(t) \le \frac{\lambda_1^*}{\lambda_1} - \frac{4\pi}{\lambda_1}(t-\varepsilon). \tag{5.6}$$

Thus,

$$G(1) \le \frac{\lambda_1^*}{\lambda_1} - \frac{4\pi}{\lambda_1}(1-\varepsilon).$$

We now deduce an expression involving $G(1)$ which will aid us in estimating $G(1)$ for small values of ε. This is the basic estimate that will lead us to the proof of (5.1).

LEMMA 5.1. *Let $G(t)$ be as in (5.3), and J_2 be the Bessel function of order 2. Then*

$$G(1) J_2\left(\sqrt{\frac{\lambda_1 G(1)}{\pi}}\right) = \varepsilon\, \frac{\lambda_1^*}{\lambda_1}\, J_2\left(\sqrt{\frac{\lambda_1^*}{\pi}}\right). \tag{5.7}$$

Proof. We follow the proof of Theorem 3.1. In what follows, $G(t)$ is any positive, decreasing C^1 solution of (5.3). Then, integrating by parts and proceeding as before, for $m = 1, 2, ...,$

$$\begin{aligned}
\int_\varepsilon^1 G^m(t)dt &= -\frac{\lambda_1}{4\pi m}\int_\varepsilon^1 (G^m(t))' \left(\int_t^1 G(s)ds + tG(t)\right) dt \\
&= -\frac{\lambda_1}{4\pi m}\left\{G^m(t)\left(\int_t^1 G(s)ds + tG(t)\right)\Big|_\varepsilon^1 - \int_\varepsilon^1 tG^m(t)G'(t)dt\right\} \\
&= -\frac{\lambda_1}{4\pi m}\left\{G^{m+1}(1) - \varepsilon G^{m+1}(\varepsilon) - G^m(\varepsilon)\int_\varepsilon^1 G(t)dt\right. \\
&\quad \left. - \frac{G^{m+1}(1)}{m+1} + \frac{\varepsilon G^{m+1}(\varepsilon)}{m+1} + \frac{1}{m+1}\int_\varepsilon^1 G^{m+1}(t)dt\right\} \\
&= -\frac{\lambda_1 G^{m+1}(1)}{4\pi(m+1)} + \frac{\lambda_1 \varepsilon G^{m+1}(\varepsilon)}{4\pi(m+1)} + \frac{\lambda_1 G^m(\varepsilon)}{4\pi m}\int_\varepsilon^1 G(t)dt \\
&\quad - \frac{\lambda_1}{4\pi m(m+1)}\int_\varepsilon^1 G^{m+1}(t)dt.
\end{aligned} \tag{5.8}$$

We use (5.8) recursively; start at $m = 1$

$$\int_\varepsilon^1 G(t)dt = -\frac{\lambda_1 G^2(1)}{4\pi\cdot 2} + \frac{\lambda_1 \varepsilon G^2(\varepsilon)}{4\pi\cdot 2} + \frac{\lambda_1 G(\varepsilon)}{4\pi\cdot 1}\int_\varepsilon^1 G(t)dt$$

$$-\frac{\lambda_1}{4\pi\cdot 1\cdot 2}\int_\varepsilon^1 G^2(t)dt.$$

Setting $m=2$, we get

$$\int_\varepsilon^1 G^2(t)dt = -\frac{\lambda_1 G^3(1)}{4\pi\cdot 3} + \frac{\lambda_1\varepsilon G^3(\varepsilon)}{4\pi\cdot 3} + \frac{\lambda_1 G^2(\varepsilon)}{4\pi\cdot 2}\int_\varepsilon^1 G(t)dt$$

$$-\frac{\lambda_1}{4\pi\cdot 2\cdot 3}\int_\varepsilon^1 G^3(t)dt.$$

Comparing these formulas with those in Theorem 3.1 and employing induction we get for $m=1,2,...,$

$$\left[\sum_{n=0}^{m}(-1)^{n+1}\left(\frac{\lambda_1 G(\varepsilon)}{4\pi}\right)^n \frac{1}{(n!)^2}\right]\int_\varepsilon^1 G(t)dt$$
$$= G(1)S_m(\frac{\lambda_1 G(1)}{4\pi}) - \varepsilon G(\varepsilon)S_m\left(\frac{\lambda_1 G(\varepsilon)}{4\pi}\right)$$
$$+(-1)^{m+1}\left(\frac{\lambda_1}{4\pi}\right)^m \frac{1}{(m!)^2(m+1)}\int_\varepsilon^1 G^{m+1}(t)dt, \tag{5.9}$$

where $S_m(x) = \sum_{n=1}^{m}(-1)^{n+1}\dfrac{x^n}{((n-1)!)^2 n(n+1)}$. Passing to the limit in (5.9), we obtain

$$G(1)J_2(A(1)) - \varepsilon G(\varepsilon)J_2(A(\varepsilon)) = -J_0\left(\sqrt{\frac{\lambda_1 G(\varepsilon)}{\pi}}\right)\int_\varepsilon^1 G(t)dt, \tag{5.10}$$

where $A(t)=\sqrt{\lambda_1 G(t)/\pi}$. Now recall that $G(\varepsilon)=\lambda_1^*/\lambda_1$ and hence $\lambda_1 G(\varepsilon)/\pi = \lambda_1^*/\pi = \nu^2$. Thus (5.10) reduces to

$$G(1)J_2\left(\sqrt{\frac{\lambda_1 G(1)}{\pi}}\right) = \varepsilon\frac{\lambda_1^*}{\lambda_1}J_2(\nu), \tag{5.11}$$

where ν is the first zero of J_0. Thus (5.7) follows. ∎

COROLLARY 5.1. *Let $\lambda_1 \geq \lambda_1^*$ and $G(t)$ be the maximal solution of (5.3). Then,*

$$G(1) = O(\sqrt{\varepsilon}), \quad as\ \ \varepsilon\to 0^+. \tag{5.12}$$

Proof. Set $T = \sqrt{\lambda_1 G(1)/\pi}$. Then (5.11) reads

$$T^2 J_2(T) = \varepsilon \, \frac{\lambda_1^*}{\pi} J_2(\nu) = \varepsilon \nu^2 J_2(\nu), \tag{5.13}$$

where ν is the first zero of J_0. Let α be the second zero of J_1, the Bessel function of order 1. Then $\alpha > \nu$. This follows from the interlacing of zeros. Observe that

$$\frac{\lambda_1 G(1)}{\pi} \leq \frac{\lambda_1}{\pi} \, \frac{\lambda_1^*}{\lambda_1} = \frac{\lambda_1^*}{\pi} = \nu^2 < \alpha^2.$$

Thus, $0 \leq T < \alpha$. Now, [10],

$$\frac{d}{dx} x^2 J_2(x) = x^2 J_1(x).$$

Thus for $0 \leq x \leq \alpha$, $x^2 J_2(x)$ is positive, increasing, continuous and vanishes only at $x = 0$. Thus $T \to 0$ as $\varepsilon \to 0$. Recalling that $J_2(T) = O(T^2)$ as $T \to 0$, (5.13) yields that $T^4 = O(\varepsilon)$. Employing the definition of T, we have

$$G(1) = O(\sqrt{\varepsilon}), \quad \text{as} \quad \varepsilon \to 0. \qquad \blacksquare$$

Next we prove that $G(t) - X(t)$ is increasing. The proof is similar to that of Theorem 3.6. We provide details wherever necessary.

LEMMA 5.2. *Let $G(t)$ and $X(t)$ be as in (5.3) and (1.10). Then, for $\varepsilon < t < 1$, we have $G(t) - X(t) \geq 0$ and $G'(t) - X'(t) \geq 0$.*

Proof. By following the procedure of Theorem 3.6 and the construction of $G(t)$ (see (5.3) and (5.4)), we find quite easily that $G(t) \geq X(t)$. To prove that $G'(t) - X'(t) \geq 0$, we see (as in (3.13)) that for $\varepsilon < t < 1$,

$$G'(t) - X'(t) = \frac{4\pi}{\lambda_1} \left[\frac{X(t) \int_t^1 G(s)ds - G(t) \int_t^1 X(s)ds}{(\int_t^1 G(s)ds + tG(t))(\int_t^1 X(s)ds + tX(t))} \right] \tag{5.14}$$

Set $F(t) = X(t) \int_t^1 G(s)ds - G(t) \int_t^1 X(s)ds$. Then $F(1) = 0$. Differentiating F,

$$\begin{aligned} F'(t) &+ \frac{4\pi}{\lambda_1} \left(\frac{F(t)}{\int_t^1 X(s)ds + tX(t)} \right) \\ &= -\frac{4\pi}{\lambda_1} G(t) \int_t^1 X(s)ds \left[\frac{\int_t^1 G(s) - X(s)ds + t(G(t) - X(t))}{(\int_t^1 X(s)ds + tX(t))(\int_t^1 G(s)ds + tG(t))} \right]. \end{aligned} \tag{5.15}$$

Using (1.10) and $G(t) - X(t) \geq 0$ in $[\varepsilon, 1]$, we get

$$F'(t) - \frac{F(t) X'(t)}{X(t)} \leq 0.$$

Thus $F(t)/X(t)$ is decreasing and as $\lim_{t\to 1^-} F(t)/X(t) = 0$, we conclude that $F(t) \geq 0$ on $[\varepsilon, 1]$. Then (5.14) yields $G'(t) - X'(t) \geq 0$. ∎

It is possible to obtain a lower bound for $G'(t) - X'(t)$, namely,

$$G'(t) - X'(t) \geq \frac{\lambda_1^*}{\lambda_1 \bar{\lambda}_1}(\lambda_1 - \bar{\lambda}_1)|W'(t)|, \tag{5.16}$$

where $\bar{\lambda}_1$ is such that there is an eigenfunction $\phi(r)$ with the property that, for some $\bar{R}$,

$$\begin{aligned} &\Delta\phi + \bar{\lambda}_1\phi = 0, \;\; 0 < r < \bar{R}, \\ &\phi(0) = 1, \;\; \phi'(0) = 0, \;\; \phi(R) = \varepsilon, \;\; \phi(\bar{R}) = 0, \text{ and } \;\phi \geq 0. \end{aligned} \tag{5.17}$$

The eigenfunction ϕ is constructed by scaling U (see (0.4)). Let $W(t)$ be as in (1.6), $H(t)$ be given by

$$H(t) = \frac{\lambda_1^*}{\lambda_1}\frac{W(t)}{W(\varepsilon)}.$$

Then $H(\varepsilon) = \lambda_1^*/\lambda_1$. Let $\phi(r)$ be the radially decreasing function whose distribution function is $H(t)$. Thus $\bar{\lambda}_1 = \lambda_1 W(\varepsilon)$ and $\phi(r) = U(cr)$ with $c = \sqrt{\lambda_1 W(\varepsilon)/\lambda_1^*}$. Clearly, $\bar{\lambda}_1 < \lambda_1$ and proceeding as in Theorem 1.1, one shows $G(t) \geq H(t)$. Following Theorem 3.6, one may show that $G'(t) - H'(t) \geq 0$. Recalling that $X(t) = \lambda_1^* W(t)/\lambda_1$,

$$\begin{aligned} G'(t) - X'(t) &= G'(t) - H'(t) + H'(t) - X'(t) \\ &\geq \frac{\lambda_1^*}{\lambda_1\bar{\lambda}_1}(\lambda_1 - \bar{\lambda}_1)|W'(t)| \\ &> 0. \end{aligned}$$

Although (5.15) is a stronger result than Lemma 5.2, we will not be using this to prove Theorem 5.1.

Proof of Theorem 5.1. We now consider R (and hence λ_1) fixed and let ε vary. By Chiti's results [3], [4], and (5.3), (5.4), and Lemma 5.2, we see

$$0 \leq Y(t) - X(t) \leq G(t) - X(t) \leq G(1), \;\; \varepsilon < t < 1.$$

Employing the estimate in Corollary 5.1, we obtain a constant $A > 0$ such that,

$$0 \leq Y(t) - X(t) \leq A\sqrt{\varepsilon}, \quad \varepsilon < t < 1. \tag{5.18}$$

Let $g(r)$ be a radially decreasing function whose distribution function is $G(t)$. Then $G(g(r)) = \pi r^2$; $\hat{r} < r < R$, where $\pi\hat{r}^2 = G(1)$, and $\pi R^2 = \lambda_1^*/\lambda_1$. We now proceed as in Section 4. It is easily seen that from (0.3) that

$$\begin{aligned} v(r) &= 1 - \lambda_1 \int_0^r \frac{1}{t} \int_0^t v(s) ds\, dt \\ &\geq 1 - \frac{\lambda_1}{4} r^2. \end{aligned}$$

Thus,

$$0 \leq g(r) - v(r) \leq 1 - (1 - \frac{\lambda_1}{4} r^2) \leq \frac{\lambda_1 G(1)}{\pi}, \quad 0 < r < \hat{r}. \tag{5.19}$$

Now on $[\hat{r}, R]$ we set $t = v(r)$ and $t' = g(r)$. Then $G(t') = X(t)$, and

$$\begin{aligned} g(r) - v(r) = t' - t &= X^{-1}(G(t)) - X^{-1}(X(t)) \\ &\leq \|\frac{1}{X'}\|_{L^\infty} \{G(t) - X(t)\} \\ &\leq \frac{\lambda_1}{4\pi} G(1). \end{aligned} \tag{5.20}$$

Here X' is estimated much the same way as W'. We know X is convex, $X' < 0$ and $X'(1) = -4\pi/\lambda_1$. Thus $\|1/X'\|_{L^\infty} \leq \lambda_1/4\pi$. Combining (5.18), (5.19) and Corollary 5.1, for $0 < r < R$,

$$0 \leq u^*(r) - v(r) \leq g(r) - v(r) = O(\sqrt{\varepsilon}).$$

The result now follows. ■

6. Concluding Remarks

The maximal solution Z has provided a convenient method for estimating the symmetrized first eigenfunctions in §4 and §5. Although we showed in Theorem 3.7 that the inequality (3.18) is sharp, it would interesting to know if Theorems 4.1 and 5.1 can be improved. In particular, it would be useful to know if the square roots which appear in those theorems can be replaced by first powers.

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