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ON THE CONSTRUCTION OF ASYMMETRIC ORTHOGONAL ARRAYS

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Abstract: A general method for constructing asymmetric orthogonal arrays of arbitrary strength is proposed. Application of this method is made to obtain several new families of tight asymmetric orthogonal arrays of strength three. A procedure for replacing a column with 2^ν symbols in an orthogonal array of strength three by several 2-symbol columns, without disturbing the orthogonality of the array, leads to some new tight asymmetric orthogonal arrays of strength three. Some new families of asymmetric orthogonal arrays of strength four are also reported, and it is shown that these arrays accommodate the maximum number of columns for given values of other parameters of the array.

Key words and phrases: Arrays of strength three and four, Galois field, replacement procedure, tight arrays.

1. Introduction

Asymmetric orthogonal arrays, introduced by Rao (1973), have received much attention in recent years. These arrays play an important role in experimental design as universally optimal fractions of asymmetric factorials; see Cheng (1980) and Mukerjee (1982). Asymmetric orthogonal arrays have been used extensively in industrial experiments for quality improvement, and their use in other experimental situations has also been widespread.

Construction of asymmetric arrays of strength *two* have been studied extensively in the literature. However, relatively less work on the construction of asymmetric orthogonal arrays of strength greater than two is available. For a review on these methods of construction, see Dey and Mukerjee (1999, Chapter 4).

In this paper, we present a general method of construction of asymmetric orthogonal arrays of arbitrary strength with number of rows as well as the number of symbols in each column being a power of s where s is a prime or a prime power. The proposed method is essentially a modification of a method of construction of *symmetric* orthogonal arrays due to Bose and Bush (1952). Using the proposed method, several families of *tight* asymmetric orthogonal arrays of strength three are constructed (tight arrays are defined later in this section).

Some other asymmetric arrays of strength three, not belonging to the general families are also constructed. We propose a procedure for replacing a column with 2^ν symbols, $\nu \geq 2$ an integer, in an orthogonal array of strength three by several 2-symbol columns, without affecting the orthogonality of the array. This replacement procedure leads to a new family of tight asymmetric orthogonal arrays of strength three. Some asymmetric orthogonal arrays of strength four are also constructed and it is shown that the arrays so constructed accommodate the maximum number of columns for given values of other parameters of the array.

For completeness, we recall the definition of an asymmetric orthogonal array.

Definition 1.1. An orthogonal array $OA(N, n, m_1 \times m_2 \times \cdots \times m_n, g)$, having N rows, $n(\geq 2)$ columns and strength $g(\leq n)$, is an $N \times n$ array, with elements in the ith column from a set of m_i distinct symbols $(1 \leq i \leq n)$, in which all the possible combinations of symbols occur equally often as rows in every $N \times g$ subarray.

The special case $m_1 = \cdots = m_n (= m, \text{say})$ corresponds to a symmetric orthogonal array which will be denoted by $OA(N, n, m, g)$. Generalizing Rao's (1947) bound for symmetric orthogonal arrays, it can be shown that in an $OA(N, n, m_1 \times m_2 \times \cdots \times m_n, 3)$,

$$N \geq 1 + \sum_{i=1}^{n}(m_i - 1) + (m^* - 1)\{\sum_{i=1}^{n}(m_i - 1) - (m^* - 1)\}, \qquad (1.1)$$

where $m^* = max_{1 \leq i \leq n} m_i$. Arrays of strength three attaining these bounds are called *tight*.

2. The Method

We propose a method of construction of orthogonal arrays of the type $OA(s^t, n, m_1 \times \cdots \times m_n, g)$ where for $1 \leq i \leq n$, $m_i = s^{u_i}$, s is a prime or a prime power, the u_i's and t are positive integers and $2 \leq g < n$. This method is a modification of a method of construction of symmetric orthogonal arrays, due to Bose and Bush (1952). Throughout, following the terminology in factorial experiments, we find it convenient to call the columns of an arbitrary $OA(N, n, m_1 \times \cdots \times m_n, g)$ *factors*, and to denote these factors by $F_1, \ldots, F_n$. We shall also denote a Galois field of order s by $GF(s)$ with 0 and 1 denoting the identity elements of $GF(s)$ with respect to the operations of addition and multiplication, respectively.

For the factor F_i $(1 \leq i \leq n)$, define u_i columns, say $\boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{iu_i}$, each of order $t \times 1$ with elements from $GF(s)$. Thus for the n factors, we have in all $\sum_{i=1}^{n} u_i$ columns. Also, let B be an $s^t \times t$ matrix whose rows are all possible t-tuples over $GF(s)$. We then have the following result.

Theorem 2.1. *Consider a* $t \times \sum_{i=1}^{n} u_i$ *matrix* $C = [A_1 \vdots A_2 \vdots \cdots \vdots A_n]$, $A_i = [\boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{iu_i}]$, $1 \leq i \leq n$, *such that for every choice of* g *matrices* $A_{i_1}, \ldots, A_{i_g}$ *from* $A_1, \ldots, A_n$, *the* $t \times \sum_{j=1}^{g} u_{i_j}$ *matrix* $[A_{i_1}, \ldots, A_{i_g}]$ *has full column rank over* $GF(s)$. *Then an* $OA(s^t, n, (s^{u_1}) \times (s^{u_2}) \times \cdots \times (s^{u_n}), g)$ *can be constructed.*

Proof. For a fixed choice of $\{i_1, \ldots, i_g\} \in \{1, \ldots, n\}$, let $C_1 = [A_{i_1}, \ldots, A_{i_g}]$ and for this choice of $i_1, \ldots, i_g$, define $r = \sum_{j=1}^{g} u_{i_j}$. Consider the product BC_1. The rows of BC_1 are of the form $\boldsymbol{b}_i{}'C_1$ where $\boldsymbol{b}_i{}'$ is the ith row of B. By the stated rank condition, C_1 has full column rank and thus there are s^{t-r} choices of $\boldsymbol{b}_i{}'$ such that $\boldsymbol{b}_i{}'C_1$ equals any fixed $r-$component row vector with elements from $GF(s)$. Thus, in BC_1, each possible r-component row vector appears with frequency s^{t-r}. Next, for each $1 \leq j \leq g$, replace the $s^{u_{i_j}}$ distinct combinations under A_{i_j} by $s^{u_{i_j}}$ distinct symbols via a one-one correspondence. It follows that in the resultant $s^t \times g$ matrix, the i_jth column has $m_{i_j} = s^{u_{i_j}}$ symbols $(1 \leq j \leq g)$ and that each of the possible $\Pi_{j=1}^{g} m_{i_j}$ combinations of symbols occurs equally often as a row. This completes the proof.

Remark 2.1. Note that for Theorem 2.1 to hold, it is necessary that $t \geq \sum_{j=1}^{g} u_{i_j}$ for each choice of g indices $i_1, \ldots, i_g$ from $1, \ldots, n$.

Remark 2.2. A result similar to Theorem 2.1 has been obtained by Saha and Midha (1999), following a different technique.

Example 2.1. Let $s = 2, t = 4, n = 4, u_1 = 2, u_2 = u_3 = u_4 = 1, g = 3$. We can construct an $OA(16, 4, 4 \times 2^3, 3)$ provided we can find a 4×5 matrix C with elements from $GF(2)$ such that the rank condition of Theorem 2.1 is satisfied. Take

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where the first two columns correspond to the first factor F_1 while the other columns correspond to the remaining three factors. It can be checked that with $g = 3$, the rank condition of Theorem 2.1 is satisfied by the above matrix C. Hence, on computing BC where B is a 16×4 matrix with rows as 4-tuples over $GF(2)$, and replacing the four combinations $(0,0)$, $(0,1)$, $(1,0)$, (11) under the first two columns of BC by four distinct symbols 0,1,2,3, respectively, we get the following array which can be verified to be an $OA(16, 4, 4 \times 2^3, 3)$:

$$\begin{bmatrix} 0210 & 0322 & 1103 & 3213 \\ 0001 & 0010 & 1011 & 0111 \\ 0000 & 1001 & 0110 & 1111 \\ 0011 & 1111 & 0000 & 0011 \end{bmatrix}'.$$

In the next two sections we present methods for choosing C to satisfy the conditions of Theorem 2.1, and to produce orthogonal arrays of strength three or four. Theorem 2.1 can be used to construct orthogonal arrays of strength two as well; however, the arrays of strength two constructed via the proposed method are similar to those considered by Wu, Zhang and Wang (1992) and Hedayat, Pu and Stufken (1992).

3. Orthogonal Arrays of Strength Three

In this section, we construct several families of asymmetric orthogonal arrays of strength three. Most of these arrays are tight. We need the following well-known result.

Lemma 3.1. *Let α and β be two elements of $GF(s)$, such that $\alpha^2 = \beta^2$. Then (i) $\alpha = \beta$, if s is even and, (ii) either $\alpha = \beta$ or $\alpha = -\beta$, if s is odd.*

It follows from Lemma 3.1 that if $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are the elements of $GF(s)$ then $S = \{{\alpha_0}^2, {\alpha_1}^2, \ldots, {\alpha_{s-1}}^2\}$ contains all the elements of $GF(s)$ if s is even. If s is odd, then one element of S is 0 and there are $(s-1)/2$ distinct (nonzero) elements of $GF(s)$, each appearing twice in S.

We shall have occassion to refer to the following result, proved in Dey and Mukerjee (1999, p.71).

Theorem 3.1. *Suppose an orthogonal array A, $OA(N, n, m_1 \times \cdots \times m_n, 3)$, is available and let t be a positive integer such that $m_1|t$. Then there exists an array A^*, $OA(Nt/m_1, n, t \times m_2 \times \cdots \times m_n, 3)$. Furthermore, if A is tight and $m_1 = max_{1\le i\le n}\, m_i$, then A^* is also tight.*

3.1. Tight arrays of strength three

We first have the following result.

Theorem 3.2. *For every prime or prime power s, there exists a tight orthogonal array $OA(s^4, s+2, (s^2) \times s^{s+1}, 3)$.*

Proof. Let the factors of the array be denoted as before by $F_1, F_2, \ldots, F_{s+2}$, where F_1 has s^2 symbols and each of the remaining factors has s symbols. For $1 \le j \le s+2,\ u_1 = 2, u_2 = \cdots = u_{s+2} = 1$, let the $t \times u_j$ matrices corresponding to the factors be as follows (note that $t = 4$ here):

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}'; \quad A_2 = [0, 0, 0, 1]'; \quad A_i = [\beta_i, {\alpha_i}^2, 1, \alpha_i]', \quad i = 3, 4, \ldots, s+2,$$

where $\alpha_3, \alpha_4, \ldots, \alpha_{s+2}$ are distinct elements of $GF(s)$. The choice of $\beta_i \in GF(s)$ depends on whether s is even or odd, as indicated below.

(a) Let s be even. Then take $\beta_i = 0$ for all i.
(b) Let s be odd. Then take $\beta_i = 0$ if $\alpha_i = 0$, and take $\beta_i = 0$, $\beta_j = 1$, if ${\alpha_i}^2 = {\alpha_j}^2$ for $i \neq j$.

We show that for the above choices of the A_i the condition of Theorem 2.1 is met with $g = 3$.

(i) Let i, j, k be distinct in $\{3, \dots, s+2\}$. For i, j, k, the matrix $[A_i,\ A_j,\ A_k]$ must have rank 3, where

$$[A_i,\ A_j,\ A_k] = \begin{bmatrix} \beta_i & \beta_j & \beta_k \\ {\alpha_i}^2 & {\alpha_j}^2 & {\alpha_k}^2 \\ 1 & 1 & 1 \\ \alpha_i & \alpha_j & \alpha_k \end{bmatrix}.$$

This follows since the determinant of the 3×3 submatrix of the above matrix given by the last three rows is $(\alpha_k - \alpha_i)(\alpha_k - \alpha_j)(\alpha_j - \alpha_i)$.

(ii) Let $i = 2$ and $j, k \in \{3, \dots, s+2\}$. Then the matrix

$$[A_2,\ A_j,\ A_k] = \begin{bmatrix} 0 & \beta_j & \beta_k \\ 0 & {\alpha_j}^2 & {\alpha_k}^2 \\ 0 & 1 & 1 \\ 1 & \alpha_j & \alpha_k \end{bmatrix}.$$

If s is even, ${\alpha_j}^2 \neq {\alpha_k}^2$ whenever $\alpha_j \neq \alpha_k$. Since the determinant of the 3×3 submatrix of the matrix $[A_2, A_j, A_k]$ given by its last three rows equals ${\alpha_j}^2 - {\alpha_k}^2$, the rank condition is fulfilled. Let s be odd and ${\alpha_j}^2 = {\alpha_k}^2$. Then $\beta_j \neq \beta_k$ (one is zero and the other one is 1). Consider the 3×3 submatrix of $[A_2, A_j, A_k]$ given by its first, third and fourth rows. The determinant of this submatrix is $\beta_j - \beta_k \neq 0$.

(iii) Let $i = 1$ and $j, k \in \{3, \dots, s+2\}$. Then

$$[A_1,\ A_j,\ A_k] = \begin{bmatrix} 1 & 0 & \beta_j & \beta_k \\ 0 & 1 & {\alpha_j}^2 & {\alpha_k}^2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \alpha_j & \alpha_k \end{bmatrix}.$$

This matrix has rank 4, since the determinant of the matrix is $\alpha_k - \alpha_j$.

(iv) Let $i = 1, j = 2$ and $k \in \{3, \dots, s+2\}$. Then

$$[A_1,\ A_2,\ A_k] = \begin{bmatrix} 1 & 0 & 0 & \beta_k \\ 0 & 1 & 0 & {\alpha_k}^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha_k \end{bmatrix},$$

and this matrix has rank 4 since its determinant is -1.

In each case, the rank condition of Theorem 2.1 is met and the desired array can be constructed using Theorem 2.1. The tightness of the array follows from (1.1).

Remark 3.1. The array $OA(s^4, s+2, (s^2) \times s^{s+1}, 3)$, where s is even, can also be constructed from the (symmetric) $OA(s^3, s+2, s, 3)$ (*cf.* Bush (1952)), using Theorem 3.1. However, when s is odd, the array $OA(s^3, s+2, s, 3)$ does not exist.

We need the following result in the sequel.

Lemma 3.2. *For each integer $k \geq 1$, let D be a $(2k+1) \times s^k$ matrix whose columns are of the form $({\alpha_1}^2, \ldots, {\alpha_k}^2, \alpha_1, \ldots, \alpha_k, 1)'$, where $(\alpha_1, \ldots, \alpha_k)$'s are all possible $k-$tuples with elements from $GF(s)$. Then any three columns of D are linearly independent.*

Proof. Consider a $(2k+1) \times 3$ submatrix of D, say

$$D_1 = \begin{bmatrix} {\alpha_1}^2 \ldots {\alpha_k}^2 \; \alpha_1 \ldots \alpha_k \; 1 \\ {\beta_1}^2 \ldots {\beta_k}^2 \; \beta_1 \ldots \beta_k \; 1 \\ {\gamma_1}^2 \ldots {\gamma_k}^2 \; \gamma_1 \ldots \gamma_k \; 1 \end{bmatrix}'.$$

First suppose that $\alpha_i, \beta_i, \gamma_i$ are all distinct for some i, $1 \leq i \leq k$. Then the 3×3 submatrix of D_1, given by

$$\begin{bmatrix} {\alpha_i}^2 & {\beta_i}^2 & {\gamma_i}^2 \\ \alpha_i & \beta_i & \gamma_i \\ 1 & 1 & 1 \end{bmatrix},$$

has determinant equal to $(\gamma_i - \alpha_i)(\gamma_i - \beta_i)(\alpha_i - \beta_i) \neq 0$.

Next, suppose for each i, $\alpha_i, \beta_i, \gamma_i$ are not distinct. Then, there exists a $j \in \{1, \ldots, k\}$ such that $\alpha_j \neq \beta_j$, and either $\gamma_j = \alpha_j$ or $\gamma_j = \beta_j$. Assuming $\gamma_j = \beta_j$, there exists a $u \neq j$ such that $\gamma_u \neq \beta_u$. Then

$$\begin{bmatrix} \alpha_j & \beta_j & \gamma_j \\ \alpha_u & \beta_u & \gamma_u \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_j & \beta_j & \beta_j \\ \alpha_u & \beta_u & \gamma_u \\ 1 & 1 & 1 \end{bmatrix},$$

and the determinant of the second matrix above is $(\alpha_j - \beta_j)(\beta_u - \gamma_u) \neq 0$. Similarly, this determinant is nonzero if $\gamma_j = \alpha_j$. This completes the proof.

Theorem 3.3. *If s is a power of two, then a tight orthogonal array $OA(s^5, s^2 + s + 2, (s^2) \times s^{s^2+s+1}, 3)$ can be constructed.*

Proof. Let F_1 have s^2 symbols and the rest of the factors have s symbols each. Let the matrices A_i, $1 \leq i \leq s^2 + s + 2$, corresponding to the different factors be chosen as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}', \; A_2 = [0,0,0,0,1]',$$

$A_3, \ldots, A_{s+2}$ of the form $[0, \alpha^2, 0, 1, \alpha]'$, $\alpha \in GF(s)$, and $A_{s+3}, \ldots, A_{s^2+s+2}$ of the form $[\beta^2, \gamma^2, 1, \beta, \gamma]'$, $\beta, \gamma \in GF(s)$. We need to show that $[A_i, A_j, A_k]$, $i, j, k \in \{1, \ldots, s^2+s+2\}$ is of full column rank. This is done below by considering several cases.

(a) Let $i, j, k \in \{s+3, \ldots, s^2+s+2\}$. This case follows from Lemma 3.2.

(b) Let $i \in \{3, \ldots, s+2\}$, $j, k \in \{s+3, \ldots, s^2+s+2\}$. Then

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & \alpha^2 & 0 & 1 & \alpha \\ {\beta_1}^2 & {\gamma_1}^2 & 1 & \beta_1 & \gamma_1 \\ {\beta_2}^2 & {\gamma_2}^2 & 1 & \beta_2 & \gamma_2 \end{bmatrix}'.$$

If $\beta_1 \neq \beta_2$, the 3×3 submatrix formed by the first, third and fourth rows has a determinant equal to ${\beta_1}^2 - {\beta_2}^2 \neq 0$. If $\beta_1 = \beta_2$, then $\gamma_1 \neq \gamma_2$ and the 3×3 submatrix formed by the last three rows has determinant $\gamma_1 - \gamma_2 \neq 0$.

(c) Let $i = 2$, $j, k \in \{s+3, \ldots, s^2+s+2\}$. Then

$$[A_2, A_j, A_k] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ {\beta_1}^2 & {\gamma_1}^2 & 1 & \beta_1 & \gamma_1 \\ {\beta_2}^2 & {\gamma_2}^2 & 1 & \beta_2 & \gamma_2 \end{bmatrix}'.$$

If $\beta_1 \neq \beta_2$, the determinant of the 3×3 submatrix formed by the last three rows is $\beta_2 - \beta_1 \neq 0$. If $\beta_1 = \beta_2$ then $\gamma_1 \neq \gamma_2$ and the determinant of the 3×3 submatrix formed by the second, third and fifth rows equals ${\gamma_1}^2 - {\gamma_2}^2 \neq 0$.

(d) Let $i = 1$, $j, k \in \{s+3, \ldots, s^2+s+2\}$. Then

$$[A_1, A_j, A_k] = \begin{bmatrix} 1 & 0 & {\beta_1}^2 & {\beta_2}^2 \\ 0 & 1 & {\gamma_1}^2 & {\gamma_2}^2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & \gamma_1 & \gamma_2 \end{bmatrix},$$

must have rank 4.

If $\beta_1 \neq \beta_2$, the determinant of the 4×4 submatrix formed by the first four rows, equals $\beta_2 - \beta_1 \neq 0$. If $\beta_1 = \beta_2$ then $\gamma_1 \neq \gamma_2$ and the determinant of the 4×4 submatrix given by the first, second, third and fifth rows is $\gamma_2 - \gamma_1 \neq 0$.

(e) Let $i, j \in \{3, \ldots, s+2\}$, $k \in \{s+3, \ldots, s^2+s+2\}$. In this case,

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & {\alpha_1}^2 & 0 & 1 & \alpha_1 \\ 0 & {\alpha_2}^2 & 0 & 1 & \alpha_2 \\ \beta^2 & \gamma^2 & 1 & \beta & \gamma \end{bmatrix}'$$

where $\alpha_1 \neq \alpha_2$, so the 3×3 submatrix formed by the last three rows has determinant $\alpha_2 - \alpha_1 \neq 0$.

(f) Let $i = 2,\ j \in \{3, \ldots, s+2\},\ k \in \{s+3, \ldots, s^2+s+2\}$. Here we have

$$[A_i, A_j, A_k] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \alpha^2 & 0 & 1 & \alpha \\ \beta^2 & \gamma^2 & 1 & \beta & \gamma \end{bmatrix}'.$$

The determinant of the 3×3 submatrix formed by the last three rows is -1.

(g) Let $i = 1,\ j \in \{3, \ldots, s+2\},\ k \in \{s+3, \ldots, s^2+s+2\}$, so

$$[A_1, A_j, A_k] = \begin{bmatrix} 1 & 0 & 0 & \beta^2 \\ 0 & 1 & \alpha^2 & \gamma^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \alpha & \gamma \end{bmatrix}.$$

The determinant of the 4×4 submatrix formed by the first 4 rows is -1.

(h) Let $i = 1,\ j = 2,\ k \in \{s+3, \ldots, s^2+s+2\}$, so

$$[A_1, A_2, A_k] = \begin{bmatrix} 1 & 0 & 0 & \beta^2 \\ 0 & 1 & 0 & \gamma^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}.$$

The determinant of the 4×4 submatrix formed by the first, second, third and fifth rows is -1.

(i) Let $i, j, k \notin \{s+3, \ldots, s^2+s+2\}$. This case is similar to that considered in Theorem 3.2.

Thus the rank condition of Theorem 2.1 holds. The tightness of the array is a consequence of (1.1).

Theorem 3.4. *If s is a power of 2 then a tight orthogonal array $OA(s^{2k+1}, s^k + 2, (s^k)^2 \times (s)^{s^k}, 3)$ can be constructed for $k = 1, 2, \ldots$.*

Proof. Let F_1 and F_2 have s^k symbols each, and the remaining factors $F_3, \ldots, F_{s^k+2}$ have s symbols each. Define the following matrices corresponding to the factors: $A_1 = [I_k, O_{kk}, O_{k1}]'$, $A_2 = [O_{kk}, I_k, O_{k1}]'$ and, for $3 \leq j \leq s^k + 2$, A_j is of the form $[{\alpha_1}^2, \ldots, {\alpha_k}^2, \alpha_1, \ldots, \alpha_k, 1]'$, where α_i's are elements of $GF(s)$, I_k is the identity matrix of order k, and O_{mn} is an $m \times n$ null matrix. We need to show that for each choice of $i, j, l \in \{1, \ldots, s^k + 2\}$, the matrix $[A_i, A_j, A_l]$ has full column rank. We consider several cases to achieve this.

(a) Let $i, j, l \in \{3, \ldots, s^k + 2\}$. Then the matrix $[A_i, A_j, A_l]$ has column rank 3, by Lemma 3.2.

(b) Let $i = 2,\ j, l \in \{3, \ldots, s^k + 2\}$. Here

$$[A_2, A_j, A_l] = \begin{bmatrix} 0 & 0 & \cdots & 0 & {\alpha_1}^2 & {\beta_1}^2 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & {\alpha_k}^2 & {\beta_k}^2 \\ 1 & 0 & \cdots & 0 & \alpha_1 & \beta_1 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & \alpha_k & \beta_k \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix},$$

and this matrix must have rank $(k+2)$. Observe that there is a $u \in \{1, \ldots, k\}$ such that $\alpha_u \neq \beta_u$. For this u, the $(k+2) \times (k+2)$ submatrix formed by the uth row and the last $(k+1)$ rows has determinant $({\alpha_u}^2 - {\beta_u}^2) \neq 0$.

(c) Let $i = 1,\ j, l \in \{3, \ldots, s^k + 2\}$, so

$$[A_1, A_j, A_l] = \begin{bmatrix} 1 & 0 & \cdots & 0 & {\alpha_1}^2 & {\beta_1}^2 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & {\alpha_k}^2 & {\beta_k}^2 \\ 0 & 0 & \cdots & 0 & \alpha_1 & \beta_1 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \alpha_k & \beta_k \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Let $\alpha_u \neq \beta_u$ for some $u \in \{1, \ldots, k\}$. Then the determinant of the $(k+2) \times (k+2)$ submatrix given by the first k rows, the $(k+u)$th row, and the last row is equal to $\alpha_u - \beta_u \neq 0$.

(d) Let $i = 1,\ j = 2,\ l \in \{3, \ldots, s^k + 2\}$. Then

$$[A_1, A_2, A_l] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & {\alpha_1}^2 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & {\alpha_k}^2 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & \alpha_1 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & \alpha_k \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which is clearly a nonsingular matrix of order $(2k+1)$. This completes the proof of the theorem.

3.2. More arrays of strength three

We construct some more families of orthogonal arrays of strength three. These arrays are in general not tight. The first result relates to the construction of orthogonal arrays with s^5 rows, where s is a power of an *odd* prime.

Theorem 3.5. *If s is an odd prime or odd prime power, then an orthogonal array $OA(s^5, s^2+3, (s^2) \times s^{s^2+2}, 3)$ can be constructed.*

Proof. Take the following matrices corresponding to the factors F_i, $1 \le i \le s^2+3$:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}', A_2 = [1,0,0,0,1]', A_3 = [0,1,0,1,0]',$$

and the matrices corresponding to the factors $F_4, \ldots, F_{s^2+3}$ of the form $[\alpha^2, \beta^2, 1, \alpha, \beta]'$, $\alpha, \beta \in GF(s)$. We show that for distinct $i, j, k \in \{1, \ldots, s^2+3\}$ the matrix $[A_i, A_j, A_k]$ satisfies the rank condition of Theorem 2.1. As before, we consider several cases.

(a) If $i, j, k \in \{4, \ldots, s^2+3\}$, the result follows from Lemma 3.2.

(b) Let $i = 3$, $j, k \in \{4, \ldots, s^2+3\}$. Then

$$[A_3, A_j, A_k] = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ {\alpha_1}^2 & {\beta_1}^2 & 1 & \alpha_1 & \beta_1 \\ {\alpha_2}^2 & {\beta_2}^2 & 1 & \alpha_2 & \beta_2 \end{bmatrix}'.$$

If $\beta_1 \ne \beta_2$, then the 3×3 submatrix given by the last three rows of the above matrix is clearly nonsingular. If $\beta_1 = \beta_2$, then $\alpha_1 \ne \alpha_2$ and the 3×3 submatrix

$$\begin{bmatrix} 1 & {\beta_1}^2 & {\beta_2}^2 \\ 0 & 1 & 1 \\ 1 & \alpha_1 & \alpha_2 \end{bmatrix}$$

is seen to be nonsingular.

(c) Now let $i = 2$, $j, k \in \{4, \ldots, s^2+3\}$. Then

$$[A_2, A_j, A_k] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ {\alpha_1}^2 & {\beta_1}^2 & 1 & \alpha_1 & \beta_1 \\ {\alpha_2}^2 & {\beta_2}^2 & 1 & \alpha_2 & \beta_2 \end{bmatrix}'.$$

If $\alpha_1 \ne \alpha_2$, the 3×3 submatrix given by the last three rows of the above matrix is nonsingular. However if $\alpha_1 = \alpha_2$, then $\beta_1 \ne \beta_2$ and the 3×3 submatrix formed by the first, third and the last rows is nonsingular.

(d) Let $i = 1,\ j, k \in \{4, \dots, s^2 + 3\}$. Here

$$[A_1, A_j, A_k] = \begin{bmatrix} 1 & 0 & {\alpha_1}^2 & {\alpha_2}^2 \\ 0 & 1 & {\beta_1}^2 & {\beta_2}^2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \beta_1 & \beta_2 \end{bmatrix},$$

and this matrix must have rank 4. If $\alpha_1 \neq \alpha_2$ then the 4×4 submatrix given by the first four rows is seen to be nonsingular. If $\alpha_1 = \alpha_2$ then $\beta_1 \neq \beta_2$ and the 4×4 submatrix given by the first, second, third and last rows is nonsingular.

(e) Let $i = 2,\ j = 3,\ k \in \{4, \dots, s^2 + 3\}$. In this case

$$[A_2, A_3, A_k] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ \alpha^2 & \beta^2 & 1 & \alpha & \beta \end{bmatrix}',$$

and this matrix can be easily seen to have rank 3.

(f) Let $i = 1,\ j = 3,\ k \in \{4, \dots, s^2 + 3\}$. We have

$$[A_1, A_3, A_k] = \begin{bmatrix} 1 & 0 & 0 & \alpha^2 \\ 0 & 1 & 1 & \beta^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix},$$

and this matrix has rank 4.

(g) Let $i = 1,\ j = 2,\ k \in \{4, \dots, s^2 + 3\}$. Then

$$[A_1, A_2, A_k] = \begin{bmatrix} 1 & 0 & 1 & \alpha^2 \\ 0 & 1 & 0 & \beta^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & \beta \end{bmatrix}$$

and has rank 4.

(h) Let $i = 1,\ j = 2,\ k = 3$. Then it is easy to see that the 5×4 matrix $[A_1, A_2, A_3]$ has rank 4. This completes the proof.

Theorem 3.6. *If s is an odd prime or odd prime power then an orthogonal array $OA(s^{2k+1}, 2 + (\frac{s+1}{2})^k, (s^k)^2 \times s^{(\frac{s+1}{2})^k}, 3)$ can be constructed for $k = 1, 2, \dots$.*

Proof. Let $\beta_1, \dots, \beta_{\frac{s-1}{2}}$ be $(s-1)/2$ nonzero elements of $GF(s)$ such that ${\beta_i}^2 \neq {\beta_j}^2$ whenever $\beta_i \neq \beta_j$, see the discussion following Lemma 3.1. Let us choose the matrices corresponding to the factors $F_1, \ldots, F_{2+(\frac{s+1}{2})^k}$, where F_1, F_2 have s^k symbols each and the rest have s symbols each, as follows: $A_1 = [I_k, O_{kk}, O_{k1}]'$, $A_2 = [O_{kk}, I_k, O_{k1}]'$ and, for $3 \le j \le (\frac{s+1}{2})^k + 2$, A_j is of the form $[{\alpha_1}^2, \ldots, {\alpha_k}^2, \alpha_1, \ldots, \alpha_k, 1]'$ where $\alpha_i \in \{\beta_1, \ldots, \beta_{\frac{s-1}{2}}\}$. The proof that $[A_i, A_j, A_l]$ is of full column rank for every choice of $i, j, l \in \{1, \ldots, 2 + (\frac{s+1}{2})^k\}$ follows as it did in Theorem 3.4.

The arrays constructed in Theorems 3.5 and 3.6 are not tight. However for $s = 3$ we have the following.

Theorem 3.7. *Tight orthogonal arrays* (i) $OA(3^5, 14, 9 \times 3^{13}, 3)$ *and* (ii) $OA(3^5, 11, 9^2 \times 3^9, 3)$ *exist.*

Proof. The array in (i) can be obtained by choosing the matrix

$$C = \begin{bmatrix} 10 & 0001 & 0012 & 12001 \\ 01 & 0010 & 0210 & 22212 \\ 00 & 0000 & 1111 & 11111 \\ 00 & 0111 & 0001 & 11222 \\ 00 & 1012 & 0120 & 12012 \end{bmatrix},$$

where the first two columns correspond to the first factor giving rise to a 9-symbol column while the other thirteen columns correspond to the 3-symbol columns.

The tight orthogonal array $OA(3^5, 11, 9^2 \times 3^9, 3)$ can be constructed by choosing the matrix

$$C = \begin{bmatrix} 10 & 00 & 000 & 111 & 222 \\ 01 & 00 & 012 & 012 & 012 \\ 00 & 10 & 002 & 012 & 121 \\ 00 & 01 & 010 & 221 & 021 \\ 00 & 00 & 111 & 111 & 111 \end{bmatrix},$$

where the first two columns correspond to a 9-symbol column, the next two columns correspond to the second 9-symbol column and the rest columns correspond to 3-symbol columns.

3.3. A replacement procedure

We now take up the construction of tight asymmetric orthogonal arrays of the type $OA(2s^3, t+u+1, (2s) \times s^t \times 2^u, 3)$ where s is a power of two and t, u are integers. To that end, we first construct an array $OA(2s^3, s+2, (2s) \times s^{s+1}, 3)$ following the method proposed in this paper, and then replace a column with s symbols by several 2-symbol columns. Suppose $s = 2^k$, where $k (\ge 1)$ is an integer. First we construct a symmetric orthogonal array $OA(s^3, s+2, s, 3)$. This array can be constructed by choosing $s+2$ vectors, say $A_1, \ldots, A_{s+2}$, each of order 3×1 with elements from $GF(2^k)$, such that for any choice of distinct $i, j, k \in \{1, \ldots, s+2\}$ the matrix $[A_i, A_j, A_k]$ is nonsingular. Let us take these vectors as $A_1 = [1, 0, 0]'$, $A_2 = [0, 1, 0]'$, $A_3 = [0, 0, 1]'$, $A_4 = [1, w, w^2]'$, $A_5 = [1, w^2, w^4]', \ldots, A_{s+2} = [1, w^{s-1}, w^{2(s-1)}]'$, where w is a primitive element of $GF(2^k)$. It can be verified that these vectors satisfy the condition of Theorem 2.1 and hence lead to an $OA(s^3, s+2, s, 3)$. Note that for obtaining the symmetric $OA(s^3, s+2, s, 3)$, we work with the elements of $GF(2^k)$. However, in what follows, we find it convenient to work with elements in $GF(2)$ rather than those of $GF(2^k)$. To use elements in $GF(2)$ instead of $GF(2^k)$ we need a matrix representation of the elements of $GF(2^k)$, the entries of these matrices being the elements of $GF(2)$. As above, let w be a primitive element of $GF(2^k)$ and let the minimum polynomial of $GF(2^k)$ be $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w + \alpha_0$. The companion matrix of the minimum polynomial is

$$W = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\alpha_{k-1} \end{bmatrix}.$$

Recall that if w is a primitive element of $GF(2^k)$ then the set $\{0, w^0, w^1, w^2, \ldots, w^{s-2}\}$, where $s = 2^k$, contains all the elements of $GF(2^k)$ in some order. A typical element, w^i, of $GF(2^k)$ can be represented by a $k \times k$ matrix W^i with entries from $GF(2)$, where 0 (the additive identity of $GF(2^k)$) is represented by a $k \times k$ null matrix and 1 (the multiplicative identity) by I_k, the kth order identity matrix. Replacing each element in the vectors $A_1, \ldots, A_{s+2}$ by the corresponding $k \times k$ matrix, we get matrices ${A_1}^*, \ldots, {A_{s+2}}^*$, each of order $3k \times k$ with elements from $GF(2)$. Let the $3k \times k(s+2)$ matrix C^* be defined as $C^* = [{A_1}^*, \ldots, {A_{s+2}}^*]$. Then it can be verified that BC^*, where B is an $2^{3k} \times 3k$ matrix with rows as all possible $3k$-tuples over $GF(2)$, gives an $OA(2^{3k}, 2^k+2, 2^k, 3) \equiv OA(s^3, s+2, s, 3)$ after replacing the s distinct combinations under the columns of $B{A_j}^*$ by s distinct symbols for each j, $1 \le j \le s+2$. From this array, we can get the array $OA(2s^3, s+2, (2s) \times (s)^{s+1}, 3)$ in the following manner. Corresponding to the factors $F_1, \ldots, F_{s+2}$, where F_1 has $2s$ symbols and the other factors have s symbols each, define matrices $D_1, D_2, \ldots D_{s+2}$ where

$$D_1 = \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & {A_1}^* \end{bmatrix}, D_i = \begin{bmatrix} \mathbf{0}' \\ {A_i}^* \end{bmatrix}, \quad i = 2, \ldots, s+2,$$

and $\mathbf{0}$ is a null column vector of appropriate order. The array $OA(2s^3, s+2, (2s) \times s^{s+1}, 3)$ can be obtained via Theorem 2.1 by taking the product BF, where B is a

$2s^3 \times (3k+1)$ matrix of $(3k+1)$-tuples over $GF(2)$, $F = [D_1, D_2, \ldots, D_{s+2}]$, and replacing the $2s = 2^{k+1}$ distinct combinations under the $(k+1)$ columns in BD_1 by the $2s$ distinct symbols of F_1 as well as the s distinct combinations under the columns in BD_j by s distinct symbols of the factor F_j for each j, $2 \leq j \leq s+2$.

In order to get an array of the type $OA(2s^3, t+u+1, (2s) \times s^t \times 2^u, 3)$, we replace a column with $s = 2^k$ symbols by $2^k - 1$ columns, each having two symbols. The idea of replacement of the symbols in a 2^ν-symbol column of an orthogonal array of strength *two* by the rows of an orthogonal array $OA(2^\nu, n, 2, 2)$, without disturbing the orthogonality of the array, is originally due to Addelman (1962). However, it appears that no general technique of replacement of a 2^ν-symbol column by several 2-symbol columns in an orthogonal array of strength three or more is available. Here we propose one such replacement procedure in the context of the arrays $OA(2s^3, s+2, (2s) \times s^{s+1}, 3)$, where s is a power of two, constructed above. Our procedure is as follows.

Consider the matrix D_i defined above for some $i \in \{2, \ldots, s+2\}$. Let B' be a matrix of order $k \times (2^k - 1)$ whose columns are all possible k-tuples over $GF(2)$, excluding the null column. Let $E_i = D_i B'$ and G_i be a matrix obtained from E_i by replacing its first row of all zeros by a row of all ones. It can be seen that the factor F_i with s symbols, represented by the matrix D_i can be replaced by $2^k - 1 = s - 1$ factors, each having 2 symbols, represented by the matrix G_i, without disturbing the rank condition of Theorem 2.1. This replacement can be done for each F_i, $2 \leq i \leq s+2$. The array $OA(2s^3, s^2 - (s-2)t, (2s) \times s^t \times 2^{(s+1-t)(s-1)}, 3)$, $0 \leq t \leq s+1$, can now be obtained via Theorem 2.1 by choosing the matrix C defined there as $C = [D_1, D_2, \ldots, D_{t+1}, G_{t+2}, \ldots, G_{s+2}]$, where $G_i = [\boldsymbol{g}_{i1}, \ldots, \boldsymbol{g}_{i,s-1}]$ and $\boldsymbol{g}_{ij}$ is the column corresponding to the 2-symbol factor $F_{t+1+(i-t-2)(s-1)+j}$, $t+2 \leq i \leq s+2, 1 \leq j \leq s-1$. Summarizing, we have the following result.

Theorem 3.8. *If s is a power of two then a tight array $OA(2s^3, s^2 - (s-2)t, (2s) \times s^t \times 2^{(s+1-t)(s-1)}, 3)$ can be constructed for $0 \leq t \leq s+1$.*

Example 3.1. Let $k = 2$ so that $s = 4$ in Theorem 3.8. We start with the construction of a symmetric $OA(4^3, 6, 4, 3)$ using the following matrices corresponding to factors: $A_1 = [1,0,0]'$, $A_2 = [0,1,0]'$, $A_3 = [0,0,1]'$, $A_4 = [1,w,w^2]'$, $A_5 = [1,w^2,w]'$, $A_6 = [1,1,1]'$, where w is a primitive element of $GF(2^2)$, and a minimum polynomial of $GF(2^2)$ is taken as $w^2 + w + 1$. The companion matrix is

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and the elements of $GF(2^2)$ can be represented by the 2×2 matrices

$$0 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad w \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad w^2 \equiv \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Replacing the elements of $GF(2^2)$ in $A_1, \ldots, A_6$ by the above matrices, we arrive at matrices $A_i{}^*$ $(1 \le i \le 6)$ with elements over $GF(2)$ as

$$A_1{}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}', A_2{}^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}', A_3{}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}',$$

$$A_4{}^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}', A_5{}^* = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}', A_6{}^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}'.$$

From the matrices $A_i{}^*$ $(1 \le i \le 6)$, we get the matrices D_i, $1 \le i \le 6$, where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

and the matrices D_j are obtained by augmenting each $A_j{}^*$ by a single null row for $2 \le j \le 6$. Using the matrices D_i $(1 \le i \le 6)$, we get the array $OA(128, 6, 8 \times 4^5, 3)$ via Theorem 2.1. In order to get an $OA(128, 16-2t, 8 \times 4^t \times 2^{15-3t}, 3)$, $0 \le t \le 5$, we replace $5 - t$, $(0 \le t \le 5)$ of the 4-symbol columns in the array $OA(128, 6, 8 \times 4^5, 3)$ by $15-3t$ columns, each having 2-symbols. The replacement can be affected in the matrices $D_2, \ldots, D_6$ without actually constructing the full array $OA(128, 6, 8 \times 4^5, 3)$. As described above, this replacement procedure for the first 4-symbol column say, represented by D_2, can be exhibited as

$$D_2 = \begin{matrix} 00 \to 111 \\ 00 \to 000 \\ 00 \to 000 \\ 10 \to 101 \\ 01 \to 011 \\ 00 \to 000 \\ 00 \to 000 \end{matrix} = G_2.$$

The final array can now be constructed via Theorem 2.1, using the matrix C defined there as $C = [D_1,\ D_2,\ \ldots,\ D_{t+1},\ G_{t+2},\ \ldots,\ G_6]$, where the matrices G_i have three columns each, representing three 2-symbol factors.

Remark 3.2. The $OA(128, 16-2t, 8 \times 4^t \times 2^{15-3t}, 3)$ can also be constructed via Theorem 3.1 provided an $OA(64, 16-2t, 4^{t+1} \times 2^{15-3t}, 3)$ exists. However, the existence of the latter array is known only for $t = 0, 1, 5$. Hence the arrays in Example 3.1 appear to be new for $t = 2, 3, 4$.

4. Arrays of Strength Four

In this section we present methods of construction of certain asymmetric orthogonal arrays of strength four. We begin with the following result.

Theorem 4.1.

(i) *If s is a prime or prime power, then the array $OA(s^5, s+2, (s^2) \times s^{s+1}, 4)$ can be constructed.*

(ii) *In an $OA(s^5, m+1, (s^2) \times s^m, 4)$, with s odd, we have $m \leq s+1$, and the arrays in* (i) *attain this upper bound for odd s.*

Proof.

(i) Let the factors of the array be $F_1, \ldots, F_{s+2}$, where the first factor corresponds to the s^2-symbol column and the rest correspond to s-symbol columns. Define the following matrices corresponding to the factors:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}', \qquad A_2 = [0, 0, 0, 0, 1]',$$

and $A_i = [{\alpha_i}^2, {\alpha_i}^3, 1, \alpha_i, {\alpha_i}^2]'$, $i = 3, \ldots, s+2$, where $\alpha_3, \ldots, \alpha_{s+2}$ are distinct elements of $GF(s)$. We need to show that for $1 \leq i < j < k < l \leq s+2$, the matrix $[A_i, A_j, A_k, A_l]$ has full column rank. To that end, we consider several cases.

(a) Let $i = 1$, $j = 2$, $3 \leq k < l \leq s+2$. Then

$$[A_1, A_2, A_k, A_l] = \begin{bmatrix} 1 & 0 & 0 & {\alpha_k}^2 & {\alpha_l}^2 \\ 0 & 1 & 0 & {\alpha_k}^3 & {\alpha_l}^3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \alpha_k & \alpha_l \\ 0 & 0 & 1 & {\alpha_k}^2 & {\alpha_l}^2 \end{bmatrix}.$$

It is easy to see that this matrix is nonsingular.

(b) Let $i = 1$, $3 \leq j < k < l \leq s+2$. In this case, we have

$$[A_1, A_j, A_k, A_l] = \begin{bmatrix} 1 & 0 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \\ 0 & 1 & {\alpha_j}^3 & {\alpha_k}^3 & {\alpha_l}^3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \alpha_j & \alpha_k & \alpha_l \\ 0 & 0 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \end{bmatrix}.$$

The determinant of this matrix is $(\alpha_l - \alpha_j)(\alpha_l - \alpha_k)(\alpha_k - \alpha_j)$.

(c) Let $i = 2,\ 3 \le j < k < l \le s + 2$. Here

$$[A_2, A_j, A_k, A_l] = \begin{bmatrix} 0 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \\ 0 & {\alpha_j}^3 & {\alpha_k}^3 & {\alpha_l}^3 \\ 0 & 1 & 1 & 1 \\ 0 & \alpha_j & \alpha_k & \alpha_l \\ 1 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \end{bmatrix}.$$

The determinant of the 4×4 submatrix given by the first, third, fourth and fifth rows is $-(\alpha_l - \alpha_j)(\alpha_l - \alpha_k)(\alpha_k - \alpha_j)$.
(d) Let $3 \le i < j < k < l \le s + 2$. Then

$$[A_i, A_j, A_k, A_l] = \begin{bmatrix} {\alpha_i}^2 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \\ {\alpha_i}^3 & {\alpha_j}^3 & {\alpha_k}^3 & {\alpha_l}^3 \\ 1 & 1 & 1 & 1 \\ \alpha_i & \alpha_j & \alpha_k & \alpha_l \\ {\alpha_i}^2 & {\alpha_j}^2 & {\alpha_k}^2 & {\alpha_l}^2 \end{bmatrix}.$$

The 4×4 submatrix given by the second, third, fourth and fifth rows is nonsingular. Part (i) of the theorem is thus proved.

(ii) If possible let the number of s-symbol columns, m, in an $OA(s^5, m+1, (s^2) \times s^m, 4)$ be greater than $s + 1$. The existence of such an array will imply that of an $OA(s^3, m, s, 3)$ obtained by permuting the rows of the array $OA(s^5, m + 1, (s^2) \times s^m, 4)$ according to the symbols of the first column, having s^2 symbols and then deleting this column. But, by a result of Bush (1952), in an $OA(s^3, m, s, 3)$, $m \le s+1$ if s is odd and $m \le s+2$ if s is even. This proves part (ii) of the theorem. Note that in an $OA(s^5, m + n, (s^2)^n \times s^m, 4)$, n cannot exceed unity.

Remark 4.1. When s is a power of two, we have not been able to get a general method of construction of an $OA(s^5, s+3, (s^2) \times s^{s+2}, 4)$, which has the maximum number of s-symbol columns. However, the array $OA(32, 5, 4 \times 2^4, 4)$ exists (*cf.* Addelman (1972)). It can be constructed following the method proposed in this paper by choosing

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where the first two columns correspond to the 4-symbol column and the rest correspond to the 2-symbol columns.

Another array $OA(4^5, 7, 16 \times 4^6, 4)$ with maximum number of 4-symbol columns exists and can be constructed by choosing

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & w \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & w & w^2 \\ 0 & 0 & 0 & 0 & 1 & 1 & w^2 & w \end{bmatrix},$$

where $0, 1, w, w^2$ are the elements of $GF(2^2)$ with $w^2 = w + 1$, and the first two columns in C correspond to the 16-symbol column, the rest correspond to 4-symbol columns. Note that in this case, the matrix B defined in Section 2 is a $4^5 \times 5$ matrix with rows as 5-tuples over $GF(2^2)$. It is not hard to see that in an array $OA(4^5, m+1, 16 \times 4^m, 4)$, the maximum number of 4-symbol columns is six, attained by the array constructed above.

The following result can be proved on the lines of that of Theorem 4.1.

Theorem 4.2.

(i) *If s is a prime or a prime power, then an $OA(s^6, s+3, (s^2)^2 \times s^{s+1}, 4)$ can be constructed.*

(ii) *In an $OA(s^6, m+2, (s^2)^2 \times s^m, 4)$, we have $m \leq s+1$ and this upper bound is attained by the arrays in* (i) *above.*

Proof.

(i) Choose the following matrices, corresponding to the factors of the array:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}', A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}', A_3 = [0, 0, 0, 0, 0, 1]',$$

and $A_i = [\alpha_i, {\alpha_i}^2, {\alpha_i}^2, {\alpha_i}^3, 1, \alpha_i]'$, for $4 \leq i \leq s+3$, where $\alpha_4, \ldots, \alpha_{s+3}$ are distinct elements of $GF(s)$. The rest of the proof follows on lines of Theorem 4.1.

(ii) Consider the subarray obtained by arranging the rows of an $OA(s^6, m+2, (s^2)^2 \times s^m, 4)$ for any fixed combination of the symbols of the two s^2-symbol columns. Then this subarray is clearly an $OA(s^2, m, s, 2)$. But in such an array, $m \leq s+1$ and thus our claim is established.

Finally, we have the following result.

Theorem 4.3. *If s is a prime or a prime power then the following orthogonal arrays can be constructed:*

(i) $OA(s^6, s+2, (s^3) \times s^{s+1}, 4)$ *if s is odd.*

(ii) $OA(s^6, s+3, (s^3) \times s^{s+2}, 4)$ *if s is even.*

Proof. Let s be odd. Consider the matrices corresponding to the factors of the required array:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}', \qquad A_2 = [0,0,0,0,0,1]',$$

and $A_i = [\alpha_i, {\alpha_i}^2, {\alpha_i}^3, 1, \alpha_i, {\alpha_i}^2]'$, for $3 \le i \le s+2$, where $\alpha_3, \ldots, \alpha_{s+2}$ are distinct elements of $GF(s)$.

When s is even, another factor given by $A_{s+3} = [0,0,0,0,1,0]'$ can be added to those above. The rest of the proof can be done as in Theorem 4.1. Note that the number of s-symbol columns in an $OA(s^6, m+1, (s^3) \times s^m, 4)$ cannot exceed $s+1$ if s is odd and $s+2$ if s is even. These upper bounds on m are attained by the arrays constructed in this theorem.

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