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Fully copositive matrices

G.S.R. Murthy [a,*], T. Parthasarathy [b]

[a] SQC & OR Unit, Indian Statistical Institute, 110 Nelson Manickam Road, Aminjikarai, Madras 600 029, India

[b] Indian Statistical Institute, 7 SJS Sansanwal Marg, New Delhi 110 016, India

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Abstract

The class of fully copositive (C_0^f) matrices introduced in [G.S.R. Murthy, T. Parthasarathy, SIAM Journal on Matrix Analysis and Applications 16 (4) (1995) 1268–1286] is a subclass of fully semimonotone matrices and contains the class of positive semidefinite matrices. It is shown that fully copositive matrices within the class of $\boldsymbol{Q}_0$-matrices are $\boldsymbol{P}_0$-matrices. As a corollary of this main result, we establish that a bisymmetric $\boldsymbol{Q}_0$-matrix is positive semidefinite if, and only if, it is fully copositive. Another important result of the paper is a constructive characterization of $\boldsymbol{Q}_0$-matrices within the class of $\boldsymbol{C}_0^f$. While establishing this characterization, it will be shown that Graves's principal pivoting method of solving Linear Complementarity Problems (LCPs) with positive semidefinite matrices is also applicable to $\boldsymbol{C}_0^f \cap \boldsymbol{Q}_0$ class. As a byproduct of this characterization, we observe that a $\boldsymbol{C}_0^f$-matrix is in $\boldsymbol{Q}_0$ if, and only if, it is completely $\boldsymbol{Q}_0$. Also, from Aganagic and Cottle's [M. Aganagic, R.W. Cottle, Mathematical Programming 37 (1987) 223–231] result, it is observed that LCPs arising from $\boldsymbol{C}_0^f \cap \boldsymbol{Q}_0$ class can be processed by Lemke's algorithm.

Keywords: Linear complementarity problem; Incidence; Matrix classes; Principal pivoting

1. Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ the Linear Complementarity Problem (LCP) is to find a vector $z \in \mathbb{R}^n$ such that

$$Az + q \geqslant 0, \quad z \geqslant 0 \quad \text{and} \quad z^t(Az + q) = 0. \tag{1}$$

LCP has numerous applications, both in theory and practice, and is treated by a vast literature (see [1]). Let $F(q,A) = \{z \in \mathbb{R}^n_+ \colon Az + q \geqslant 0\}$ and $S(q,A) = \{z \in F(q,A) \colon (Az + q)^t z = 0\}$. A number of matrix classes have been defined in connection with LCP, the fundamental ones being $\boldsymbol{Q}$ qnd $\boldsymbol{Q}_0$. The class $\boldsymbol{Q}$ consists of all

* Corresponding author.

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real square matrices A such that $S(q,A) \neq \phi$ for every $q \in \mathbb{R}^n$ [2]; and $\boldsymbol{Q}_0$ consists of all real square matrices A such that $S(q,A) \neq \phi$ whenever $F(q,A) \neq \phi$ [3]. A matrix A is said to be completely $\boldsymbol{Q}_0$ if every principal submatrix of A is in Q_0.

Stone [4] conjectured that the class of fully semimonotone matrices ($\boldsymbol{E}_0^{\mathrm{f}}$) within the class of $\boldsymbol{Q}_0$ are $\boldsymbol{P}_0$-matrices (see Section 2 for definitions of matrix classes). In [5], the authors partially addressed the conjecture and introduced the class of fully copositive ($\boldsymbol{C}_0^{\mathrm{f}}$) matrices – a subclass of $\boldsymbol{E}_0^{\mathrm{f}}$ – and obtained some results on the same. In Section 3, we establish a constructive characterization of $\boldsymbol{Q}_0$-matrices within the class of $\boldsymbol{C}_0^{\mathrm{f}}$-matrices by showing that Graves's algorithm can process LCP (q,A) when A is a $\boldsymbol{C}_0^{\mathrm{f}}$-matrix. As a byproduct of this characterization, we observe that a $\boldsymbol{C}_0^{\mathrm{f}}$-matrix is in $\boldsymbol{Q}_0$ if, and only if, it is completely $\boldsymbol{Q}_0$. It may be noted that the algorithm uses only the single or double pivots while processing LCPs.

By introducing the concept of *incidence* of complementary cones, we prove in Section 4 that $\boldsymbol{C}_0^{\mathrm{f}}$-matrices that are also $\boldsymbol{Q}_0$ are $\boldsymbol{P}_0$-matrices. Furthermore, we prove that bisymmetric $\boldsymbol{E}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$-matrices as well as 2×2 $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$-matrices are positive semidefinite.

In the light of a result of Aganagic and Cottle [6], we observe that Lemke's algorithm processes LCPs (q,A) when $A \in \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$.

2. Notation and background

For any positive integer n, $\bar{n}$ stands for the set $\{1,2,\ldots,n\}$ and for any subset α of $\bar{n}$, $\bar{\alpha}$ denotes its complement with respect to $\bar{n}$. For any $A \in \mathbb{R}^{n\times n}$, $A_{\alpha\alpha}$ is obtained by dropping rows and columns corresponding to $\bar{\alpha}$ from A. For any $x \in \mathbb{R}^n, x_\alpha$ is obtained from x by dropping coordinates corresponding to $\bar{\alpha}$; and x_i denotes the ith coordinate of x.

For any $A \in \mathbb{R}^{n\times n}$, the set $\mathrm{pos}A = \{Ax\colon\ x \in \mathbb{R}^n,\ x \geqslant 0\}$ is the cone generated by columns of A, called the generators of the cone; the cone is said to be full or nondegenerate if A is nonsingular. Given $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq \bar{n}$, define the matrix B whose ith column is $-A_i$ (the ith column of $-A$) if $i \in \alpha$, and if $i \notin \alpha$, then the ith column of B is the ith column of I (the identity matrix). B is denoted by $C_A(\alpha)$ and is called the complementary matrix with respect to α. The cone $\mathrm{pos}\, C_A(\alpha)$ is called the complementary cone with respect to α. Note that, given q and A, solving (q,A) is equivalent to identifying a complementary cone $\mathrm{pos}\, C_A(\alpha)$ which contains q; also given $A \in \mathbb{R}^{n\times n}$, there are 2^n complementary cones (not necessarily distinct) and the union of all these cones is denoted by $K(A)$.

A solution z to (q,A) is said to be nondegenerate if $z + Az + q > 0$ (strictly positive). In the problem (q,A), q is said to be *nondegenerate* if every solution of (q,A) is nondegenerate.

A matrix A is said to be a $\boldsymbol{P}$-matrix ($\boldsymbol{P}_0$-matrix) if all its principal minors are positive (nonnegative). Cottle and Stone [7] introduced the class of fully semimonotone matrices ($\boldsymbol{E}_0^{\mathrm{f}}$) and its subclass $\boldsymbol{U}$. A matrix A is in $\boldsymbol{E}_0^{\mathrm{f}}$ if (q,A) has a unique solution for every nondegenerate q, and A is in $\boldsymbol{U}$ if (q,A) has a unique solution for every q in the interior of $K(A)$. Stone [4] showed that $\boldsymbol{U} \cap \boldsymbol{Q}_0$ is subset of $\boldsymbol{P}_0$ and conjectured that $\boldsymbol{E}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. The authors addressed this conjecture in [5] and showed that the conjecture is true for matrices of order up to 4×4 and $\boldsymbol{E}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$-matrices of general order which are either symmetric or nonnegative are in $\boldsymbol{P}_0$. Further, a subclass of $\boldsymbol{E}_0^{\mathrm{f}}$, the class fully copositive matrices ($\boldsymbol{C}_0^{\mathrm{f}}$, defined below) was introduced. It was shown that symmetric $\boldsymbol{E}_0^{\mathrm{f}}$-matrices are contained in $\boldsymbol{C}_0^{\mathrm{f}}$.

In this note we introduce the concept of incidence of complementary cones. Using this concept, we show that $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$.

A real square matrix A is said to be copositive if for every nonnegative real vector x (of appropriate order), $x^{\mathrm{t}}Ax$ is nonnegative. The class of semimonotone matrices ($\boldsymbol{E}_0$) introduced by Eaves [8] (he denoted it by L_1, see also [9]) consists of all real square matrices A such that (q, A) has a unique solution for every $q > 0$. The following inclusions are well known in the literature (see [1] for details).

$$\boldsymbol{P} \subseteq \boldsymbol{P}_0 \subseteq \boldsymbol{E}_0^{\mathrm{f}} \subseteq \boldsymbol{E}_0, \quad \boldsymbol{C}_0 \subseteq \boldsymbol{E}_0.$$

It is also known that symmetric $\boldsymbol{E}_0$-matrices are copositive.

Consider $A \in \mathbb{R}^{n \times n}$. If $\alpha \subseteq \bar{n}$ is such that $\det A_{\alpha\alpha} \neq 0$, then the matrix M defined by

$$M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}, \quad M_{\alpha\bar{\alpha}} = -M_{\alpha\alpha}A_{\alpha\bar{\alpha}}, \quad M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}M_{\alpha\alpha}, \quad M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - M_{\alpha\alpha}A_{\alpha\bar{\alpha}}$$

is known as the principal pivotal transform (PPT) of A with respect to α and will be denoted by $\wp_\alpha(A)$. Note that a PPT is defined only with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. By convention, when $\alpha = \emptyset$, $\det A_{\alpha\alpha} = 1$ and $M = A$ (see [1]). Whenever we refer to PPTs, we mean the ones which are well defined. One of the characterizations of $\boldsymbol{E}_0^{\mathrm{f}}$-matrices is that $A \in \boldsymbol{E}_0^{\mathrm{f}}$ if, and only if, every PPT of A is in $\boldsymbol{E}_0$. This characterization means that $\boldsymbol{E}_0^{\mathrm{f}}$-matrices are invariant under PPTs. A matrix $A \in \mathbb{R}^{n \times n}$, not necessarily symmetric, is said to be positive semidefinite (PSD) if $x^{\mathrm{t}}Ax \geqslant 0$ for all $x \in \mathbb{R}^n$. It is a well known fact that PPTs of a PSD matrix are also PSD. To see this, let $M = \wp_\alpha(A)$ and let $y = Ax$. It is easy to check that $x^{\mathrm{t}}Ax = z^{\mathrm{t}}Mz$ where $z^{\mathrm{t}} = (y_\alpha^{\mathrm{t}}, x_{\bar{\alpha}}^{\mathrm{t}})$. Since this holds for any arbitrary x, it immediately follows that M is a PSD matrix.

Definition 2.1. Let $A \in \mathbb{R}^{n \times n}$. Say that A is a fully copositive matrix if every PPT of A is a copositive matrix.

The class of fully copositive matrices is denoted by $\boldsymbol{C}_0^{\mathrm{f}}$. From the definition and the fact that $\boldsymbol{C}_0 \subseteq \boldsymbol{E}_0$, it is clear that $\boldsymbol{C}_0^{\mathrm{f}} \subseteq \boldsymbol{E}_0^{\mathrm{f}}$. In [5], it was shown that symmetric $\boldsymbol{E}_0^{\mathrm{f}}$-matrices are fully copositive. It was also shown that if a fully copositive matrix has at most one zero diagonal entry, then it is a $\boldsymbol{P}_0$-matrix. While $\boldsymbol{U}$ and $\boldsymbol{C}_0^{\mathrm{f}}$ are both subclasses of $\boldsymbol{E}_0^{\mathrm{f}}$, there is no relationship between $\boldsymbol{C}_0^{\mathrm{f}}$ and $\boldsymbol{U}$.

Example 2.2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Note that $A \in \boldsymbol{C}_0^{\mathrm{f}}$ but not a $\boldsymbol{U}$-matrix, and B is a $\boldsymbol{U}$-matrix but not a $\boldsymbol{C}_0^{\mathrm{f}}$-matrix.

3. Algorithmic aspects

Given a LCP (q, A), consider another LCP (p, M) where M is a PPT of A with respect to some $A_{\alpha\alpha}, p_\alpha = -(A_{\alpha\alpha})^{-1}q_\alpha$ and $p_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}q_\alpha$. We say that (p, M) is a PPT of (q, A). The two problems are equivalent in the sense that, given a solution to one of the problems, a solution to the other can easily be constructed (see p. 74 of [1]). When $|\alpha| = 1$ ($|\alpha| = 2$), we say (p, M) is obtained from (q, A) using a single (double) pivot. The principal pivoting methods for solving LCPs transform the original problem into its equivalent PPTs until a PPT is obtained for which zero is a solution. Graves's principal pivoting algorithm for solving LCPs with PSD matrices uses only single and/or double pivots. The following is a brief description of the algorithm. Complete details and proof of finiteness of the algorithm can be found in Section 4.2 of [10] (see also [11]).

3.1. Graves's algorithm

Step 0: Input $M = A$ and $p = q$.

Step 1: If $p \geqslant 0$, then $z = 0$ is a solution of (p, M); obtain a solution of (q, A) using this and stop.

Step 2: If there exists an index i such that $p_i < 0$ and $M_{i\cdot} \leqslant 0$, then conclude that the LCP has no solution and stop.

Step 3: Choose i with $p_i < 0$ using lexicographic rule. If $m_{ii} > 0$, then replace (p, M) by its PPT with respect to $\alpha = \{i\}$. If $m_{ii} = 0$, then choose j from $\{k\colon m_{ik} > 0\}$ using lexicographic rule and replace (p, M) by its PPT with respect to $\alpha = \{i, j\}$. Go to Step 1.

When A is a PSD matrix, Graves's algorithm will never get stuck in Step 3 and hence either produces a solution to the problem (termination in Step 1) or exhibits that the problem has no solution (Step 2 termination). To show that the algorithm applies to $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$, we establish the following result. The results of this section will use our main result that $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$ which is proved in Section 4.

Lemma 3.1. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$. Assume that $a_{ii} = 0$ and $a_{ij} \neq 0$ for some i and j. Then $a_{ij} + a_{ji} = 0$.*

Proof. Suppose

$$B = \begin{bmatrix} 0 & b \\ c & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \cap \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{P}_0.$$

If $bc \neq 0$, then bc must be negative and

$$B^{-1} = \begin{bmatrix} \frac{-a}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}.$$

Since B is copositive, $b+c \geqslant 0$ and since B^{-1} is copositive, $(b+c)/bc \geqslant 0$ or $b+c \leqslant 0$. Hence $b+c=0$. Consider the hypothesis of the theorem. By Theorem 4.5, $A \in \boldsymbol{P}_0$. If $a_{ij} < 0$, then as $a_{ii} = 0$ and A is copositive, we must have $a_{ji} > 0$ and from the above argument it follows that $a_{ij} + a_{ji} = 0$. On the other hand, if $a_{ij} > 0$, then there exists an index k such that $a_{ki} < 0$. This follows from Theorem 2.9 of [5], since $A \in \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{E}_0 \cap \boldsymbol{Q}_0$. Suppose $a_{ji} = 0$. Then $k \neq j$ and $a_{ik} > 0$ (as A is copositive). Let $\alpha = \{i, j, k\}$. Then

$$A_{\alpha\alpha} \simeq \begin{bmatrix} 0 & + & + \\ 0 & \star & \star \\ - & \star & \star \end{bmatrix} \quad \text{and} \quad M_{\alpha\alpha} \simeq \begin{bmatrix} \star & \star & - \\ \star & \star & 0 \\ + & - & 0 \end{bmatrix},$$

where M is the PPT of A with respect to $\{i, k\}$. Here '$\simeq$' stands for sign equivalence of left and right hand side matrices with $\star$ indicating the unknown sign of the corresponding entry. The sign pattern of $M_{\alpha\alpha}$ implies that $M_{\alpha\alpha}$ is not copositive. This contradicts that $A \in \boldsymbol{C}_0^{\mathrm{f}}$. It follows that $a_{ji} \neq 0$ and hence $a_{ij} + a_{ji} = 0$. □

Lemma 3.2. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$. For any index i if $a_{ii} = 0$, then $a_{ij} + a_{ji} = 0$ for all j.*

Proof. Suppose i is such that $a_{ii} = 0$. From Lemma 3.1, we only need to consider the case $a_{ij} = 0$. If possible, assume $a_{ji} \neq 0$. By copositivity of $A, a_{ji} > 0$. By Lemma 3.1, $a_{jj} > 0$. But then for $\alpha = \{i, j\}$, $[\wp_{\{j\}}(A)]_{\alpha\alpha}$ does not belong to $\boldsymbol{C}_0$. From this contradiction we conclude that $a_{ji} = 0$ and hence $a_{ij} + a_{ji} = 0$. □

The $\boldsymbol{Q}_0$ assumption in the above theorem is essential as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is an example of a $\boldsymbol{C}_0^{\mathrm{f}}$-matrix but it is not $\boldsymbol{Q}_0$ (see Theorem 2.5 of [5]). The above results yield a constructive characterization of $\boldsymbol{Q}_0$-matrices within the class of $\boldsymbol{C}_0^{\mathrm{f}}$-matrices. From this characterization, we deduce that a $\boldsymbol{C}_0^{\mathrm{f}}$-matrix is in $\boldsymbol{Q}_0$ if, and only if, it is a completely $\boldsymbol{Q}_0$-matrix. There is no characterization of completely $\boldsymbol{Q}_0$-matrices in general (see [5,12,13]).

Theorem 3.3. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_0^{\mathrm{f}}$. Then the following conditions are equivalent:*
(a) $A \in \boldsymbol{Q}_0$;
(b) for every PPT M of A, $m_{ii} = 0 \Rightarrow m_{ij} + m_{ji} = 0 \ \forall i, j \in \bar{n}$;
(c) A is completely $\boldsymbol{Q}_0$.

Proof. It is easy to see from Lemma 3.2 that (a) implies (b). Note that if A satisfies condition (b), then so does every principal submatrix of A. To see that (b) implies (c), let M be a principal submatrix of A, say of order k. Let $p \in \mathbb{R}^k$ be arbitrary. Note that Graves's algorithm when applied to (p, M), terminates either in Step 1 or Step 2 of Section 3.1 (follows from results of Section 4.2 of [10]). If the algorithm terminates in Step 2, then it is clear that (p, M) has no feasible solution. It follows that $M \in \boldsymbol{Q}_0$. As M is an arbitrary principal submatrix of A, it follows that A is completely $\boldsymbol{Q}_0$. The implication (c) implies (b) is obvious. □

Thus, to verify whether a given $\boldsymbol{C}_0^{\mathrm{f}}$-matrix A is in $\boldsymbol{Q}_0$, it suffices to check the condition (b) of Theorem 3.3. Another way of expressing the condition is: for every PPT M of A,

$$M + M^{\mathrm{t}} = \begin{bmatrix} 0 & 0 \\ 0 & M_{\bar{\alpha}\bar{\alpha}} + M_{\bar{\alpha}\bar{\alpha}}^{\mathrm{t}} \end{bmatrix}, \quad \text{where } \alpha = \{i \in \bar{n}:\ m_{ii} = 0\}. \tag{2}$$

4. Main result

Stone [4] showed that $\boldsymbol{U} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$ and conjectured that $\boldsymbol{E}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. In [5], it was shown that the conjecture is true for matrices of order up to 4×4. In this section we establish that $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. This is done by introducing the concept of incidence of complementary cones.

Definition 4.1. Let $A \in \mathbb{R}^{n\times n}$ and let $\alpha \subseteq \bar{n}$ be such that pos $C_A(\alpha)$ is full. Let $B = C_A(\alpha)$. Then pos $B_{\cdot\beta}$ is called a facet of pos $C_A(\alpha)$ provided $|\beta| = n - 1$.

Definition 4.2. Let $A \in \mathbb{R}^{n\times n}$ and let $\alpha, \beta \subseteq \bar{n}$ be such that pos $C_A(\alpha)$ and pos $C_A(\beta)$ are full cones. Say that the cones pos $C_A(\alpha)$ and pos $C_A(\beta)$ are *incident* to each other on a hyperplane H if the relative interior (with respect to H) S of $H \cap \operatorname{pos} C_A(\alpha) \cap \operatorname{pos} C_A(\beta)$ is nonempty.

Lemma 4.3. *Suppose* $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_0^{\mathrm{f}}$. *Suppose* α *is a nonempty subset of* $\bar{n}$ *such that* pos $C_A(\alpha)$ *is full and is incident to* $\mathbb{R}_+^n (= \operatorname{pos} C_A(\emptyset))$. *Then* det $A_{\alpha\alpha} > 0$.

Proof. We shall prove this by induction on n. When $n = 1$ the lemma is obvious. Assume that the lemma is valid for all matrices of order $n - 1$, $n > 1$. Let $A \in \mathbb{R}^{n\times n}$ satisfy hypothesis of the lemma along with a subset α of $\bar{n}$. Let $B = C_A(\alpha)$. Since $A \in \boldsymbol{C}_0$, pos $C_A(\alpha)$ and $\mathbb{R}_+^n$ cannot intersect in the interior. For simplicity, we assume that pos $C_A(\alpha)$ is incident to pos $[I_{\cdot 2}, I_{\cdot 3}, \ldots, I_{\cdot n}]$. Note that the common hyperplane containing the facets of pos I and pos $C_A(\alpha)$ is given by $H = \{x \in \mathbb{R}^n:\ x_1 = 0\}$. Let S denote the relative interior (with respect to H) of $H \cap \operatorname{pos} [I_{\cdot 2}, I_{\cdot 3}, \ldots, I_{\cdot n}] \cap \operatorname{pos} C_A(\alpha)$.

Choose $(n-1)$ linearly independent vectors $q^1, q^2, \ldots, q^{(n-1)}$ from S. Let $B_{.i_1}, B_{.i_2}, \ldots, B_{.i_{(n-1)}}$ be the generators of the facet (of pos $C_A(\alpha)$) containing S. Then there exists a nonsingular matrix X (strictly positive) of order $(n-1)$ such that $[q^1, q^2, \ldots, q^{(n-1)}] = [B_{.i_1}, B_{.i_2}, \ldots, B_{.i_{(n-1)}}]X$. From this it follows that the first coordinates of $B_{.i_1}, B_{.i_2}, \ldots, B_{.i_{(n-1)}}$ are equal to zero. Note that as $A \in \boldsymbol{C}_0^{\mathrm{f}}, I_{.1}$ cannot be a generator of pos $C_A(\alpha)$. Hence $1 \in \alpha$.

Case (i). $-A_{.1} \notin H$. Clearly, in this case, $-A_{.1}, q^1, q^2, \ldots, q^{(n-1)}$ are linearly independent, and their convex hull – which contains an open ball of $\mathbb{R}^n$ – is contained in pos $C_A(\alpha) \cap$ pos $[-A_{.1}, I_{.2}, I_{.3}, \ldots, I_{.n}]$. This implies, as $A \in \boldsymbol{C}_0^{\mathrm{f}}$, that the two complementary cones are one and the same and that $\alpha = \{1\}$. As pos $C_A(\alpha)$ is full and $A \in \boldsymbol{C}_0$, det $A_{\alpha\alpha} = a_{11} > 0$.

Case (ii). $-A_{.1} \in H$. Since pos $C_A(\alpha)$ is full, we must have a $k \in \bar{n}$ such that $-A_{.k} \notin H$. Without loss of generality assume $k = n$. Suppose $|\alpha| < n$, say $(n-1) \notin \alpha$. Let $\beta = \bar{n} \setminus \{n-1\}$ and let $M = A_{\beta\beta}$. It can be verified that M together with α satisfies the assumptions of the lemma. That is, pos $C_M(\alpha)$ is full and is incident to $\mathbb{R}_+^{n-1}$ on the hyperplane $\bar{H} = \{(x_1, \ldots, x_{(n-2)}, x_n)^{\mathrm{t}} \in \mathbb{R}^{n-1}\colon\ x_1 = 0\}$. By induction hypothesis, det $M_{\alpha\alpha} > 0$. But $M_{\alpha\alpha} = A_{\alpha\alpha}$ and hence det $A_{\alpha\alpha} > 0$.

Suppose $|\alpha| = n$. Since $S \subseteq$ pos $[-A_{.1}, \ldots, -A_{.(n-1)}]$, there exists a positive vector $(x_1, \ldots, x_{(n-1)})^{\mathrm{t}}$ such that

$$-\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{(n-1)} \end{bmatrix} > \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If $a_{i1} \geqslant 0$ for all $i \in \gamma = \{2, \ldots, (n-1)\}$, then it follows that $A_{\gamma\gamma}$ is not copositive which is a contradiction. Hence there must exist an index $k \in \gamma$ such that $a_{k1} < 0$. But then for $\theta = \{1, k\}$,

$$A_{\theta\theta} = \begin{bmatrix} 0 & 0 \\ a_{k1} & a_{kk} \end{bmatrix} \notin \boldsymbol{C}_0,$$

which contradicts that $A \in \boldsymbol{C}_0^{\mathrm{f}}$. It follows that $|\alpha|$ cannot be equal to n. This completes the proof of the lemma. □

Lemma 4.4. *Suppose* $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_o^{\mathrm{f}}$. *Assume that* $\alpha, \beta \subseteq \bar{n}$ *are such that* pos $C_A(\alpha)$ *and* pos $C_A(\beta)$ *are full. If* pos $C_A(\alpha)$ *and* pos $C_A(\beta)$ *are incident to each other (with respect to a common hyperplane containing the facets), then* det $A_{\alpha\alpha}$ *and* det $A_{\beta\beta}$ *have the same sign.*

Proof. Let $M = \wp_\alpha(A)$. Note that the PPT merely transforms the complementary cones of $K(A)$ to those of $K(M)$ through the nonsingular linear transformation q going to $C_A(\alpha)^{-1}q$. In particular, pos $C_A(\alpha)$ gets transformed to $\mathbb{R}_+^n$ and pos $C_A(\beta)$ to pos $C_M(\gamma)$ where $\gamma = \alpha \Delta \beta$. As pos $C_A(\alpha)$ and pos $C_A(\beta)$ are incident to each other, it follows that $\mathbb{R}^n_+$ and $\operatorname{pos} C_M(\gamma)$ are incident to each other. By Lemma 4.4, it follows that det is positive. From symmetric difference formula (see [1]) it follows that $\det A_{\alpha\alpha}$ and $\det A_{\beta\beta}$ have the same sign. □

Theorem 4.5. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$. Then $A \in \boldsymbol{P}_0$.*

Proof. Let α, any nonempty subset of $\bar{n}$, be such that $\operatorname{pos} C_A(\alpha)$ is full. We may assume that $\operatorname{pos} C_A(\alpha)$ is different from $\mathbb{R}^n_+$ for in this case we have nothing to prove. Let $q^0 \in$ interior $\operatorname{pos} C_A(\alpha)$. Let $r > 0$ be such that $B_r(q^0) \subseteq \operatorname{pos} C_A(\alpha)$. Since $A \in \boldsymbol{Q}_0$, $K(A)$ in convex. Define the set

$$P = \{q \in \mathbb{R}^n\colon\ q = \lambda p + (1-\lambda)e \text{ for some } \lambda \in [0,1] \text{ and some } p \in B_r(q^0)\},$$

where $e = (1,1,\ldots,1)^{\mathrm{t}} \in \mathbb{R}^n$. Clearly P is an open set and is contained in the interior of $K(A)$. Furthermore, if any full complementary cone of $K(A)$ intersects P, then it must be incident to another full complementary cone of $K(A)$ which also has a nonempty intersection with P. Let $\emptyset = \alpha_0, \alpha_1, \ldots, \alpha_m = \alpha \subseteq \bar{n}, m \geqslant 1$ be all the full complementary cones that have nonempty intersection with P. From Lemma 4.4 and Theorem 4.5, it follows that $\det A_{\alpha_i\alpha_i}$ is positive for $i = 0, 1, \ldots, m$. Thus $\det A_{\alpha\alpha} > 0$. As α was arbitrary, this completes the proof of the theorem. □

It may be observed that Lemma 4.3 is valid when $\mathbf{C}_0^{\mathrm{f}}$ is replaced by $\boldsymbol{U}$. This gives an alternative proof of Stone's result that $\boldsymbol{U} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. Unfortunately the lemma is valid for $\boldsymbol{E}_0^{\mathrm{f}}$-matrices only when $n \leqslant 3$. The following serves as a counter example.

Example 4.6. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be checked that $A \in \boldsymbol{E}_0^{\mathrm{f}}$ and $\operatorname{pos} A$ is incident to $\mathbb{R}^4_+$ (on the hyperplane $H = \{x \in \mathbb{R}^4_+\colon\ x_1 = 0\}$). However, $\det A < 0$. It may be worth noting that A is not a $\boldsymbol{Q}_0$-matrix. This can be seen as follows. Since $A_{1.} \geqslant 0$, if A is in $\boldsymbol{Q}_0$, then $A_{\alpha\alpha}, \alpha = \{2,3,4\}$, must also be in $\boldsymbol{Q}_0$ (see [5]). But it is easy to check that $A_{\alpha\alpha}$ is not in $\boldsymbol{Q}_0$ (this also follows from Theorem 2.5 of [5] which characterizes nonnegative $\boldsymbol{Q}_0$-matrices).

Since symmetric $\boldsymbol{P}_0$-matrices are PSD, if follows that symmetric $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$-matrices are PSD. In fact, we can marginally relax this condition of symmetry by replacing it with bisymmetry. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be bisymmetric if there exists an $\alpha \subseteq \bar{n}$, possibly empty, such that $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are symmetric, and $A_{\alpha\bar{\alpha}} = -A^{\mathrm{t}}_{\bar{\alpha}\alpha}$. We first show that if A is a bisymmetric $\boldsymbol{E}_0^{\mathrm{f}}$-matrix, then it is fully copositive. The authors established the equivalence of $\boldsymbol{E}_0^{\mathrm{f}}$ and $\boldsymbol{C}_0^{\mathrm{f}}$ under symmetry in [5].

Theorem 4.7. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{E}_0^{\mathrm{f}}$ is a bisymmetric matrix. Then A is fully copositive.*

Proof. Let M be any PPT of A. Since PPTs of bisymmetric matrices are bisymmetric (easy to check), M is bisymmetric. Let α be such that $M_{\alpha\alpha}$ and $M_{\bar\alpha\bar\alpha}$ are symmetric, and $M_{\alpha\bar\alpha} = -M^{\mathrm{t}}_{\bar\alpha\alpha}$. Then $M + M^{\mathrm{t}}$ is a symmetric $\boldsymbol{E}_0$-matrix and hence copositive (see pp. 177–178 of [9]). This proves that A is fully copositive. □

Theorem 4.8. *Suppose $A \in \mathbb{R}^{n\times n} \cap \boldsymbol{Q}_0$ is a bisymmetric matrix. Then the following conditions are equivalent:*
(a) A is PSD;
(b) A is fully copositive;
(c) A is fully semimonotone.

Proof. To prove the theorem we only need to show that (b) implies (a). So assume that A is a $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$-matrix. It suffices to show that $A + A^{\mathrm{t}}$ is positive semidefinitive. By Theorem 4.5, A is in $\boldsymbol{P}_0$. Let α be such that $A_{\alpha\alpha}$ and $A_{\bar\alpha\bar\alpha}$ are symmetric, and $A_{\alpha\bar\alpha} = -A^{\mathrm{t}}_{\bar\alpha\alpha}$. Obviously $A_{\alpha\alpha}$ and $A_{\alpha\bar\alpha}$ are positive semidefinite. Therefore,

$$A + A^{\mathrm{t}} = \begin{bmatrix} 2A_{\alpha\alpha} & 0 \\ 0 & 2A_{\bar\alpha\bar\alpha} \end{bmatrix}$$

is PSD. □

We believe that $\boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0$ is nothing but the class of PSD matrices. In the following theorem we show that this is true for 2×2 matrices.

Theorem 4.9. *Suppose $A \in \mathbb{R}^{2\times 2} \cap \boldsymbol{Q}_0$. Then A is PSD if, and only if, $A \in \boldsymbol{C}_0^{\mathrm{f}}$.*

Proof. The 'only if' part is obvious. We shall prove the 'if' part. If $a_{11} = 0$ or $a_{22} = 0$, then $a_{12} + a_{21} = 0$ and $x^{\mathrm{t}}Ax$ involves only a square term and hence A will be PSD. If A is singular, then a PPT of A will have a zero diagonal entry and hence A will be PSD. So assume that A is nonsingular and that $a_{11}a_{22} > 0$. Without loss of generality we may assume that $a_{11} = a_{22} = 1$ (this is because, if $A \in \boldsymbol{C}_0^{\mathrm{f}}$, then $DAD \in \boldsymbol{C}_0^{\mathrm{f}}$ for any positive diagonal matrix D). Suppose $x^{\mathrm{t}}Ax < 0$ for some x. As A is copositive, $x_1x_2 < 0$. Also

$$\begin{aligned} 0 > x^{\mathrm{t}}Ax = (x_1 - x_2)^2 + (a_{12} + a_{21} + 2)x_1x_2 \;\Rightarrow\; 0 \geqslant -(x_1 - x_2)^2 \\ > (a_{12} + a_{21} + 2)x_1x_2 \;\Rightarrow\; a_{12} + a_{21} + 2 > 0 \;\Rightarrow\; a_{12} + a_{21} > -2. \end{aligned}$$

Note that

$$A^{-1} = 1/(1 - a_{12}a_{21}) \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix} \in \boldsymbol{C}_0^{\mathrm{f}} \cap \boldsymbol{Q}_0.$$

Since A is not PSD, A^{-1} is also not positive semidefinite but copositive (hence $a_{12}a_{21} < 1$). So there exists a z such that

$$z^t \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix} z < 0$$

and $z_1 z_2 < 0$. Again

$$0 > z^t \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix} z = (z_1 + z_2)^2 - (a_{12} + a_{21} + 2) z_1 z_2$$

implies $a_{12} + a_{21} < -2$ which is a contradiction. It follows that A is PSD. □

Aganagic and Cottle [6] showed that if $A \in \boldsymbol{P}_0 \cap \boldsymbol{Q}_0$, then Lemke's algorithm processes (q, A) for any $q \in \mathbb{R}^n$ (with a suitable apparatus to resolve degeneracy). Since we have shown that $\boldsymbol{C}_0^f \cap \boldsymbol{Q}_0$ is a subclass of $\boldsymbol{P}_0 \cap \boldsymbol{Q}_0$, we conclude, in the light of the above result, that LCPs (q, A) can be processed by Lemke's algorithm when $A \in \boldsymbol{C}_0^f \cap \boldsymbol{Q}_0$.

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