A Unified Way of deriving LMP Rank Tests from Censored Data

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SUMMARY

In this note a unified way of deriving locally most powerful rank tests for censored data is given. A lemma is proven. This lemma is then used to derive LMP rank tests.

Keywords: CENSORED DATA; LOCALLY MOST POWERFUL TEST; RANK TEST; TYPE I AND II CENSORING

1. INTRODUCTION

In problems of reliability and survival analysis, censored data arise naturally so that one needs to obtain optimal rank tests based on censored data which may be Type I, Type II, arbitrarily censored, and so on. Several authors have considered the problem of obtaining locally most powerful rank (LMPR) tests. See, for example, Rao *et al.* (1960), Johnson and Mehrotra (1972), Shirahata (1975), among others. Basu (1967, 1968) and Lochner (1968) also obtained rank tests based on heuristic arguments. The purpose of this note is to provide a unified and alternate approach of obtaining LMPR tests under censoring and justify why the heuristic methods used by Basu and Lochner worked. In Section 2, we provide a useful lemma. This lemma is closely related to the EM algorithm of Dempster *et al.* (1977). LMPR tests for various censored/truncation models are obtained as applications of this lemma in Section 3.

2. A USEFUL LEMMA

As a basis of our proposed study, we consider first the following.

Lemma 2.1. Let T_1, T_2 be two (possibly vector valued) statistics and, for each θ, let $L_\theta(t_1, t_2)$, $L_{1\theta}(t_1)$ and $L_{2\theta}(t_2)$ be respectively the joint and marginal densities of T_1 and T_2 (with respect to some sigma-finite measure μ). Then

$$\frac{L_{1\theta}(t_1)}{L_{1\theta_0}(t_1)} = E_{\theta_0}\left\{\frac{L_\theta(T_1, T_2)}{L_{\theta_0}(T_1, T_2)}\;\middle|\; T_1 = t_1\right\}, \tag{2.1}$$

where the expectation (at $\theta = \theta_0$) is with respect to T_2, given $T_1 = t_1$.

Proof. The conditional density of T_2 given $T_1 = t_1$ is given by $L^*_\theta(t_2 \mid t_1) = L_\theta(t_1, t_2)/L_{1\theta}(t_1)$. Hence,

$$E_{\theta_0}\left\{\frac{L_\theta(T_1, T_2)}{L_{\theta_0}(T_1, T_2)}\;\middle|\; T_1 = t_1\right\}$$

$$= \int \cdots \int \{L_\theta(t_1, t_2)/L_{\theta_0}(t_1, t_2)\}\{L_{\theta_0}(t_1, t_2)/L_{1\theta_0}(t_1)\}\, d\mu(t_2)$$

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$$= L_{1\theta_0}(t_1)\}^{-1} \int \cdots \int L_\theta(t_1, t_2)\, d\mu(t_2)$$

$$= L_{1\theta}(t_1)/L_{1\theta_0}(t_1). \text{ q.e.d.} \tag{2.2}$$

In the context of locally most powerful (LMP) tests, one usually considers a null hypothesis $H_0: \theta = \theta_0$ against a family $H_\Delta = \{H_\lambda: \theta = \theta_0 + \lambda; 0 \leqslant \lambda \leqslant \Delta\}$ of alternatives, where $\Delta \to 0$ and intends to find a test which remains uniformly MP for H_Δ, when $\Delta \to 0$. If one assumes regularity conditions (on L_θ) ensuring that for all $\theta = \theta_0 + \lambda$, $0 \leqslant \lambda \leqslant \Delta$, as $\Delta \to 0$,

$$\frac{L_\theta(T_1, T_2)}{L_{\theta_0}(T_1, T_2)} = 1 + \lambda\psi(T_1, T_2) + o(\lambda) \text{ a.s.}, \tag{2.3}$$

then by (2.1), we have for $\theta = \theta_0 + \lambda$, $0 < \lambda \leqslant \Delta$, $\Delta \to 0$,

$$\frac{L_{1\theta}(T_1)}{L_{1\theta_0}(T_1)} = 1 + \lambda\psi_1(T_1) + o(\lambda) \text{ a.s.}, \tag{2.4}$$

where

$$\psi_1(t_1) = E_{\theta_0}\{\psi(T_1, T_2) : T_1 = t_1\}. \tag{2.5}$$

Note that $\psi(T_1, T_2)$ $(=(\partial/\partial\theta) \log L_\theta(T_1, T_2) \mid \theta = \theta_0)$ and $\psi_1(T_1)$ $(=(\partial/\partial\theta) \log L_{1\theta}(T_1) \mid \theta = \theta_0)$ are the test statistics corresponding to the LMP tests based on (T_1, T_2) and T_1 (alone), so that (2.5) provides an easy way of deriving $\psi_1(T_1)$ when $\psi(T_1, T_2)$ is known. A variant form of this result is due to Sen (1981, Theorem 2.1).

3. APPLICATIONS

We shall utilize Lemma 2.1 and (2.5) in the formulation of various LMP rank tests for various censored/truncation models.

3.1. *Linear Rank Statistics under Censoring/Truncation*

Let $X_1, \ldots, X_n$ be independent (real valued) random variables with densities $d(x; \theta c_i)$, $i = 1, \ldots, n$, where θ is a real parameter, the c_i are specified constants and $d(x; \gamma)$ stands for a p.d.f. indexed by the real parameter γ. The null hypothesis of interest is $H_0: \theta = 0$, so that under H_0, the X_i are all i.i.d. r.v. with the density $d(x; 0)$. Two-sample location or scale models as well as the simple regression model are special cases of the above one; see Hájek and Šidák (1967, pp. 67–74).

Let $W_1 < \ldots < W_n$ be the order statistics associated with $X_1, \ldots, X_n$, and define the *ranks* R_i and *anti-ranks* S_i by

$$W_i = X_{S_i} \quad \text{and} \quad X_i = W_{R_i} \quad \text{for } i = 1, \ldots, n, \tag{3.1}$$

so that $R_{S_i} = S_{R_i} = i$, $1 \leqslant i \leqslant n$. In what follows we consider two models.

(i) *Censored (type II) model*

For some specified $r (1 \leqslant r \leqslant n)$, $W_1, \ldots, W_r$ as well as $S_1, \ldots, S_r$ are observable, while the remaining $n - r$ observations are censored. In this case, we let

$$T_1 = (S_1, \ldots, S_r) \quad \text{and} \quad T_2 = (S_{r+1}, \ldots, S_n). \tag{3.2}$$

Then, by a general theorem in Hájek and Šidák (1967, p. 70), for testing $H_0: \theta = 0$ against $H_\Delta = \{H_\lambda: \theta = \lambda; 0 < \lambda \leqslant \Delta\}$, the LMPR test statistic (based on T_1, T_2) is

$$\psi(T_1, T_2) = \sum_{i=1}^{n} (c_i - \bar{c}_n)\, a_n(R_i; d) = \sum_{i=1}^{n} (c_{S_i} - \bar{c}_n)\, a_n(i; d), \tag{3.3}$$

where $\bar{c}_n = n^{-1} \Sigma_{i=1}^n c_i$ and

$$a_n(i;d) = E_0\{(\partial/\partial\theta) \log d(W_i;\theta) \mid_{\theta=0}\}, \quad 1 \leqslant i \leqslant n. \tag{3.4}$$

Note that the LMPR test statistic in Hájek and Šidák (1967, p. 71) differs from (3.3) by a non-stochastic constant $\bar{c}_n(\Sigma_{i=1}^n a_n(i;d))$. We may remark that $\Sigma_{i=1}^n a_n(i;d) = 0$, and hence, the two statistics are identical, and hence, we use (3.3).

Therefore, according to (2.5), for the censored case, the LMPR test statistic is

$$\psi_1(T_1) = \sum_{i=1}^{r} (c_{S_i} - \bar{c}_n) a_n(i;d) + \sum_{i=r+1}^{n} a_n(i;d) E_0\{c_{S_i} - \bar{c}_n \mid S_1, \ldots, S_r\}, \tag{3.5}$$

where for every $i > r$,

$$E\{c_{S_i} - \bar{c}_n \mid S_1, \ldots, S_r\} = (n-r)^{-1} \sum_{j=r+1}^{n} (c_{S_j} - \bar{c}_n) = -(n-r)^{-1} \sum_{j=1}^{r} (c_{S_j} - \bar{c}_n).$$

Hence,

$$\psi_1(T_1) = \sum_{i=1}^{r} (c_{S_i} - \bar{c}_n) \left[a_n(i;d) - \frac{1}{n-r} \sum_{j=r+1}^{n} a_n(j;d) \right]. \tag{3.6}$$

Note that (3.6) is properly defined for every $r \leqslant n-1$, while for $r = n-1$, $\psi_1(T_1) = \psi(T_1, T_2)$ and hence, for $r = n-1$ or n, $\psi_1(T_1)$ agrees with (3.3).

For the special cases of two-sample models, such LMPR statistics were obtained by heuristic arguments by Basu (1967, 1968) and Lochner (1968), among others, while for general linear rank statistics, such censored forms were obtained by Chatterjee and Sen (1973) by a martingale argument, implicit in (2.1) and (2.5).

(ii) *Truncation* (*type I censoring*) *models*

Here experimentation is terminated at a prefixed point τ and $W_{r(\tau)} \leqslant \tau < W_{r(\tau)+1}$, where $r(\tau)$ is a non-negative integer valued random variable. If $F_0(x)$ be the d.f. of the X_i under H_0, then

$$P\{r(\tau) = r \mid H_0\} = \binom{n}{r} [F_0(\tau)]^r [1 - F_0(\tau)]^{n-r}, 0 \leqslant r \leqslant n, \tag{3.7}$$

which depends on the unknown F_0 through $\pi = F_0(\tau)$. Then, the distribution of $\{S_j, j \leqslant r(\tau)\}$, even under H_0, will depend on F_0, through π, and hence, the rank tests are not genuinely distribution-free. However, under H_0, the ranks R_i (or the antiranks S_i), $1 \leqslant i \leqslant n$ are distributed independently of $\{W_1, \ldots, W_n\}$, and hence, the conditional distribution of $\{S_j, j \leqslant r(\tau)\}$, given $r(\tau) = r$, will not depend on F_0. With this in mind, we need to set up a LMPR (conditional) test, where we let

$$T_1 = \{S_j, j \leqslant r(\tau); r(\tau) = r\}, \quad T_2 = \{S_j, j > r\}. \tag{3.8}$$

Since $\psi(T_1, T_2)$ will be the same as in (3.3), we obtain by (2.1), (2.5) and (3.6) that here

$$\psi_1(T_1) = \sum_{i \leqslant r(\tau)} (c_{S_i} - \bar{c}_n) \left[a_n(i;d) - \frac{1}{n - r(\tau)} \sum_{j=r(\tau)+1}^{n} a_n(j;d) \right]. \tag{3.9}$$

The difference between (3.7) and (3.9) is that in (3.7), r is non-stochastic and $\psi_1(T_1)$ is genuinely distribution-free under H_0, while in (3.9), $r(\tau)$ is stochastic, and $\psi_1(T_1)$ is conditionally (given $r(\tau) = r$) distribution-free, so that here we have a LMPR conditional test.

If we denote the statistics in (3.3) and (3.6) by L_n and L_{nr}, respectively, and if $\mathscr{B}_{nr}$ stands for the sigma-field generated by $\{S_j, j \leqslant r\}$, then it follows from Chatterjee and Sen (1973, Lemma 4.1) that under H_0, $E(L_n | \mathscr{B}_{nr}) = L_{nr}$, $\forall\, r \in [1, n]$, so that $\{L_{nr}, B_{nr}; r \leqslant n\}$ forms a martingale sequence. Also, by (3.7), under H_0, $r(\tau)/n\pi \to 1$ a.s., as $n \to \infty$. Hence, using the Kolmogorov inequality for martingales in a neighbourhood of $r = n\pi$, we obtain that under H_0, as $n \to \infty$,

$$| L_{nr(\tau)} - L_{n[n\pi]} | / \sqrt{\{\mathrm{var}\,(L_{n[n\pi]})\}} \xrightarrow{P} 0, \tag{3.10}$$

so that under H_0, $L_{nr(\tau)}$ and L_{nr}, for $r \equiv [n\pi]$, are asymptotically equivalent, in probability. This asymptotic equivalence also holds under contiguous alternatives (viz. Chapter VI of Hájek and Šidák, (1967)), so that for such contiguous alternatives the (local) power of the conditional test based on $L_{nr(\tau)}$ will be asymptotically the same as the unconditional test based on L_{nr}, if $r = [n\pi]$ were known. This explains the asymptotic power equivalence of the type I (conditional) and type II (genuinely distribution-free) censored LMPR tests, for contiguous alternatives.

Remarks

(i) Let us consider a more general alternative considered by Capon (1961). Let $F_\theta(x) = F(x, \theta)$ and $G_\phi(x) = F(x; \phi)$. Instead of conditioning on $r(\tau)$ we can, of course, use the unconditional distribution of $R_1, \ldots, R_{r(\tau)}$ to get a locally most powerful test for testing

$$H_0: \theta = \phi = \theta_0 \text{ (say)}$$

against alternatives

$$H_1: \theta \neq \phi.$$

Let the likelihood for censored data be given by $P_{\theta\phi}$. The resulting test depends not only on F_θ but the direction in which the alternative approaches the null hypothesis, e.g. if $\theta, \phi \to \theta_0$ such that $c_m(\theta - \theta_0) \sim c_n(\phi - \theta_0)$ then the resulting test would depend on

$$c_m \left.\frac{\partial \log P_{\theta\phi}}{\partial\theta}\right|_{(\theta_0, \theta_0)} + c_n \left.\frac{\partial \log P_{\theta\phi}}{\partial\phi}\right|_{(\theta_0, \theta_0)}.$$

Here c_m and c_n are arbitrary positive numbers. Of them, there is only one which is locally unbiased for *all* alternatives $\theta > \theta_0$, $\phi > \phi_0$. It can be shown by direct but tedious arguments that asymptotically this test is equivalent to the test for type II censoring or the conditional test for type I censoring.

(ii) The LMPR test for multicensored problems relating to the general model treated earlier (and hence, containing the two-sample model, treated by Mehrotra *et al.*, 1977, as a particular case) can be obtained similarly. Towards this, define the order statistics W_i as in (3.1) and assume that for some positive integer q, only q blocks of order statistics

$$W_{k_i+1}, \ldots, W_{k_i+r_i}, \quad \text{for } i = 1, \ldots, q$$

are observed, where $0 \leqslant k_1 < k_1 + r_1 \leqslant k_2 < k_2 + r_2 < k_3 \leqslant \ldots \leqslant k_q < k_q + r_q \leqslant n = k_{q+1}$. Let

$$J = \{k_1 + 1, \ldots, k_1 + r_1, k_2 + 1, \ldots, k_2 + r_2, \ldots, k_q + 1, \ldots, k_q + r_q\} \quad \text{and}$$

$$J^c = \{1, \ldots, n\} \setminus J.$$

Define then

$$T_1 = \{S_i; i \in J\} \quad \text{and} \quad T_2 = \{S_i; i \in J^c\}.$$

Also, we define the scores $a_n(i, d)$ as in (3.4) and let

$$a_n^*(i,d) = \begin{cases} a_n(i,d) & i \in J, \\ (k_j - k_{j-1} - r_{j-1})^{-1} \sum_{r=k_{j-1}+r_{j-1}+1}^{k_j} a_n(r,d) & k_{j-1}+r_{j-1}+1 \leqslant i \leqslant k_j, \\ & \text{for } j = 1, \ldots, q+1, \end{cases}$$

where $k_0 = r_0 = 0$. Note that if we take $q = 1$ and $k_1 = 0$, $r_1 = r$, then, we may rewrite $\psi_1(T_1)$ in (3.6) as

$$\sum_{i=1}^{n} (c_i - \bar{c}_n)\, a_n^*(R_i, d).$$

This representation is generally true for the multi-censored case, and we have

$$\psi_1(T_1) = \sum_{i=1}^{n} (c_i - \bar{c}_n)\, a_n^*(R_i, d),$$

where the ranks R_i are observable only for the uncensored order statistics, while for the censored ones, it is known that each R_i belongs to one of the sets $\{k_{j-1}+1, \ldots, k_j\}$, for $j = 1, \ldots, q+1$, where $k_0 = 0$. $\psi_1(T_1)$ may also be rewritten in terms of the anti-ranks S_i, but the expression will be comparatively complicated. In the particular case of the two-sample problem, the c_i can only assume the values 0 and 1, and the expression for $\psi_1(T_1)$ reduces to the one obtained by Mehrotra *et al.* (1977).

(iii) LMPR tests for other censoring schemes such as those considered by Halperin and Ware (1974) and Young (1970), among others, can be obtained similarly.

(iv) Note that these LMPR test statistics (for the various problems) are still simple linear rank statistics with adjusted scores depending on the censoring pattern. Also, the distribution of the rank vector over the permutations of $(1, \ldots, n)$ is uniform under the null hypothesis. Hence, the usual formulae for the mean and variance of linear rank statistics under the null hypothesis (viz. Hájek and Šidák, 1967, p. 61) can be used to provide the exact expressions for the null mean and variance of these LMPR test statistics.

3.2. *Rank Tests for Independence*

Let (X_i, Y_i), $i = 1, \ldots, n$, be n i.i.d. random vectors with a continuous bivariate d.f. $H(x, y; \theta)$ having marginal d.f.'s $F(x)$ and $G(y)$, respectively. The parameter θ indexes the dependence of X, Y in a certain manner, and under $H_0: \theta = 0$, we have $H(x, y; 0) = F(x)G(y)$. The p.d.f. corresponding to $H(x, y; \theta)$ is denoted by $h(x, y; \theta)$, so that $h(x, y; 0) = f(x)g(y)$, where f and g are the p.d.f. corresponding to F and G, respectively. Assume that the regularity conditions of Shirahata (1975) hold.

Let $R_i(Q_i)$ be the rank of $X_i(Y_i)$ among $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$, for $i = 1, \ldots, n$. As in (3.1), define the anti-ranks by R_i^* and Q_i^*, so that

$$R_{R_i^*} = R^*_{R_i} = i \quad \text{and} \quad Q_{Q_i^*} = Q^*_{Q_i} = i \quad \text{for } i = 1, \ldots, n.$$

Now, corresponding to the (type II) censored case, we let here (for some predetermined $r, s: 1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n$),

$$\begin{aligned} T_1 &= (R_1^*, \ldots, R_r^*; Q_1^*, \ldots, Q_s^*), \\ T_2 &= (R_{r+1}^*, \ldots, R_n^*; Q_{s+1}^*, \ldots, Q_n^*). \end{aligned} \tag{3.11}$$

Also, let $X_{(1)} < \ldots < X_{(n)}$ (and $Y_{(1)} < \ldots < Y_{(n)}$) be the order statistics corresponding to $X_1, \ldots, X_n$ (and $Y_1, \ldots, Y_n$), respectively; ties neglected with probability one by virtue of the absolute continuity of F and G. Let the E^* be defined by

$$E^* = \{x_{(1)} < \ldots < x_{(n)}; y_{(1)} < \ldots < y_{(n)}\} \subset E^{2n}.$$

Then

$$L_\theta(t_1, t_2) = \int \ldots \int_{E^*} \prod_{i=1}^{n} h(x_{(r_i^*)}, y_{(q_i^*)}; \theta)\, dx_{(r_i^*)}\, dy_{(s_i^*)}, \tag{3.12}$$

so that by Lemma 2.1 and (2.5), we obtain that

$$\psi_1(T_1) = \sum_{i=1}^{n} a_{rs}(R_i, Q_i), \tag{3.13}$$

where on letting

$$a_n(i,j) = n^2 \binom{n-1}{i-1} \binom{n-1}{j-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial\theta} h(x, y; \theta) \Big|_{\theta=0} \right\}$$

$$[F(x)]^{i-1} [1 - F(x)]^{n-i} \times [G(y)]^{j-1} [1 - G(y)]^{n-j} f(x) g(y)\, dx\, dy, \tag{3.14}$$

for $i = 1, \ldots, n$, $j = 1, 2, \ldots, n$, we have for given $r, s \in [1, n]^2$,

$$a_{rs}(i,j) = \begin{cases} a_n(i,j) & \text{if } i \leqslant r, j \leqslant s, \\ \dfrac{1}{n-r} \displaystyle\sum_{l=r+1}^{n} a_n(l,j) & \text{if } i > r, j \leqslant s, \\ \dfrac{1}{n-s} \displaystyle\sum_{l=s+1}^{n} a_n(i,l) & \text{if } i \leqslant r, j > s, \\ \dfrac{1}{(n-r)(n-s)} \displaystyle\sum_{l=r+1}^{n} \sum_{l'=s+1}^{n} a_n(l,l') & \text{if } i > r, j > s. \end{cases} \tag{3.15}$$

For the truncation scheme (type I censoring), in T_1, r and s are random variables (non-negative integers between 0, n), so that the rank statistics are not genuinely distribution-free. Nevertheless, they will be conditionally (given r, s) distribution-free under H_0, and hence, the same modification as in the case of linear rank statistics will apply here.

Note that if we let

$$M_n = \sum_{i=1}^{n} a_n(R_i, Q_i), \tag{3.16}$$

where $a_n(i,j)$'s are defined by (3.14) and if $\mathscr{B}_{nrs}$ be the sigma field generated by the first r anti-ranks of the X's and the first s anti-ranks of the Y's, then under H_0,

$$E_0(M_n \mid \mathscr{B}_{nrs}) = M_{nrs} = \sum_{i=1}^{n} a_{rs}(R_i, Q_i), \tag{3.17}$$

for every $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n$. This quasi-martingale property can be incorporated with maximal inequalities to yield a result parallel to (3.10) for the M_{nrs} where $r/n \to \pi_1$, and $s/n \to \pi_2$ are small for some $\pi_1 \in (0, 1)$, $\pi_2 \in (0, 1)$, so that the asymptotic power equivalence of the censored and the truncated LMPR tests (for contiguous alternatives) follows as in Section 3.1.

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