THE APPLICABILITY OF LARGE SAMPLE TESTS FOR MOVING AVERAGE AND AUTOREGRESSIVE SCHEMES TO SERIES OF SHORT LENGTH—AN EXPERIMENTAL STUDY

CONTENTS

PART 1: MOVING AVERAGES

By ABRAHAM MATTHAI and M. B. KANNAN

Statistical Laboratory, Calcutta

1. Introduction

In assigning a moving average scheme to a given time series $\xi_{(t)}$, the main problem is firstly of fitting and secondly of testing the goodness of fit of a model of the form :

$$\xi_{(t)} = \eta_{(t)} + a_1\eta_{(t-1)} + \dots + a_h\eta_{(t-h)} \qquad (t = 0, \pm 1, \pm 2 \dots)$$

For fitting such a model no direct method is yet available of estimating the parameters $a_1, a_2, \dots, a_h$ from the given values. A general procedure of fitting, applicable to stationary time series, however, is to identify the series as conforming to a particular scheme from a knowledge of the correlogram. This in other words means using the sample serial correlation coefficients to estimate the theoretical serial coefficients and hence to estimate the parameters $a_1, a_2, \dots, a$.

Thus for instance the first h sample coefficients $r_1, r_2, \dots, r_h$ can be conveniently used to estimate the theoretical coefficients $\rho_1, \rho_2, \dots, \rho_h$. For such a fitting a test for goodness of fit was suggested by Herman Wold (1949). It is a χ^2 test which uses any number of serial coefficients from the $(h+1)$th onwards and is a test of the large sample type.

This Part of the paper deals with a project taken up during the Research Seminars, with the main object of exploring the suitability of Wold's test for moving averages, when applied to samples of small sizes. While it was the aim of this project to gain as much insight into this problem as possible, it was also its purpose to create exploratory material for further study. The plan of the project accordingly consisted primarily in constructing artificial series of different lengths corresponding to different moving average models known a priori; in obtaining sets of serial coefficients for the samples so constructed; and finally in arriving at test-results arising from tests of two kinds, namely one with the known correlograms and the other with fitted correlograms. Three models and two sample sizes were considered giving rise to 2100 serial coefficients and about 500 χ^2-values of goodness of fit for a total of 225 samples.

The large amount of numerical computation thus involved in this project (as also in a similar project presented in Part 2) was rendered possible through facilities of punched-card and other mechanical calculations available at the Indian Statistical Institute. In view of the large-scaleness of the numerical work, special care was taken to impose suitable checks wherever possible and it is believed that no systematic or other kind of error has crept in and that the figures presented in different tables in the following pages are of reliable accuracy.

In section 2 of this Part are given details of construction of the series, while section 3 deals with tests with a priori models and section 4 with fitting of correlograms. In the course of the study, certain incidental problems such as bias in small samples, methods of fitting correlograms, are also discussed.

2. Selection of samples

2.0. *Construction of models* : The three moving average schemes chosen for the investigation are

$$\xi_{(t)} = \eta_{(t)} + \eta_{(t-1)} \qquad \dots \quad (A)$$

$$\xi_{(t)} = \eta_{(t)} + \tfrac{1}{2}\eta_{(t-1)} \qquad \dots \quad (B)$$

$$\xi_{(t)} = \eta_{(t)} + \eta_{(t-1)} + \eta_{(t-2)} + \eta_{(t-3)} \qquad \dots \quad (C)$$

Two sample sizes have been considered, namely, $T = 35$ and $T = 15$. In each of the schemes A, B and C, 25 samples for $T = 35$ and 50 samples for $T = 15$ have been constructed. All the series were constructed having the η's normally distributed with zero mean and unit standard deviation and satisfying $E(\eta_{(t)}\eta_{(t+k)}) = 0$ for $k = \pm 1, \pm 2, \dots$.

The series were constructed from Wold's random normal deviates in Tracts for Computers No. 25. For scheme A deviates from pages 42 and 43, for scheme B from pages 12, 46 and 49, for scheme C from pages 25 and 26 of the Tracts were used. These deviates all satisfied tests of randomness referred to in the introduction of the Tracts. The deviates constituting each series were further tested for independence, on the basis of the first two circular correlation coefficients and the test was found to be satisfied without exception.

In each scheme the first 12 serial coefficients have been calculated for the 35-items series and the first 8 coefficients for the 15-items series. These coefficients are given in Tables 8.A.1 to 8.C.2 at the end of this Part.

The theoretical serial coefficients for schemes A to C work out respectively to

$$\rho_1 = 0.5, \qquad \rho_2 = \rho_3 = \dots = 0$$

$$\rho_1 = 0.4, \qquad \rho_2 = \rho_3 = \dots = 0$$

$$\rho_1 = 0.75, \qquad \rho_2 = 0.50, \ \rho_3 = 0.25, \ \rho_4 = \rho_5 = \dots = 0$$

2.1. *Bias in serial coefficients* : The serial coefficients calculated from our samples however will not exactly correspond with the above theoretical values, because a considerable element of bias will be present for small sample sizes. Denoting the rth serial coefficient as

$$r_k = \frac{\Sigma\, \xi_{(t)}\xi_{(t+k)} - \sum_{1}^{T-k} \xi_{(t)} \sum_{k+1}^{T} \xi_{(t)} \Big/ (T-k)}{\sqrt{\left\{ \sum_{1}^{T-k} \xi_{(t)}^2 - \left(\sum_{1}^{T-k} \xi_{(t)}\right)^2 \Big/ (T-k) \right\} \left\{ \sum_{k+1}^{T} \xi_{(t)}^2 - \left(\sum_{k+1}^{T} \xi_{(t)}\right)^2 \Big/ (T-k) \right\}}}$$

$$= N/\sqrt{D_1 D_2}$$

and putting

$$E(r_k) = \frac{E(N)}{E\sqrt{D_1 D_2}} = \frac{E(N)}{E(D)}$$

the following approximate result can be had :

$$E(r_k) = [(T-k)\rho_k - \frac{1}{T-k}\{(T-2k)\rho_0+2(T-2k)\rho_1+2(T-2k)\rho_2+\dots$$

$$+2(T-2k)\rho_{k-1}+2(T-2k)\rho_k+2(T-2k+1)\rho_{k+1}$$
$$+\dots+2(k+1)\rho_{T-2k-1}+2k\rho_{T-2k}+(2k-1)\rho_{T-2k+1}$$

$$+(2k-2)\rho_{T-2k+2}+\dots+\rho_{T-1}\}]\div[T-k-\frac{1}{T-k}\{(T-k)\rho_0+2(T-k-1)\rho_1+\dots+2(1)\rho_{T-k-1}\}]$$

The bias in the different serial coefficients calculated with this formula, for schemes A, B, and C and for both the sample sizes are also given in Tables 8.A.1 to 8.C.2. In the tests described in the subsequent section the serial coefficients have been corrected for bias. Figures 2.A.1 to 2.C.2 at the end of this Part show the average correlogram, with the expected correlogram for each scheme and sample size.

3. TESTS ON CORRELOGRAMS WITH KNOWN MODELS

3.0. *Wold's test* : Wold's test for moving averages is as follows. Suppose that the given series belongs to a moving average having $\rho_1, \rho_2, \dots, \rho_h$ for serial coefficients, ρ_{h+i} ($i=1, 2\dots$) being zero. Then out of the observed serial coefficients $r_{h+1}, r_{h+2}, \dots$, which for large sample size T would be following a known multivariate normal distribution, can be formed linear forms:

$$R_{h+i} = z_{i1}r_{h+1}+z_{i2}r_{h+2}+\dots+z_{ii}r_{h+i} \qquad (i = 1, 2,\dots)$$

that again for large T are distributed independently and normally with zero mean and variance $\frac{1}{T-h-i}$.

A criterion for testing whether a series belongs to a moving average with prescribed values for the coefficients $\rho_1, \rho_2, \dots, \rho_h$ and with ρ_{h+i} ($i=1, 2\dots$)$=0$, is therefore provided by

$$\chi^2_{(n)} = \sum_1^n \chi^2_{(1)} = \sum_{s=h+1}^{h+n} (T-s)R_s^2$$

This follows a χ^2 distribution with n degree of freedom and permits the use of any number n of the serial coefficients.

The above χ^2 tests were performed for all the samples constructed, having for hypothetical values of $\rho_1, \rho_2, \dots, \rho_h$, the known theoretical serial coefficients of the constructed models.

3.1. *Computational aspects of test* : The computational procedure (Wold 1949) consists in firstly forming the dispersion matrix $X=\{x_{ik}\}$ of the observed serial coefficients r_{h+i}. This is done by making use of Bartlett's (1947) formula for the second moments of serial coefficients for stationary time series, which in the present case reduces to

$$\text{Cov}\,(r_{h\cdot i}, r_{h\cdot k}) \sim \frac{1}{\sqrt{(T-h-i)(T-h-k)}} \sum_{r=-\infty}^{\infty} \rho_r \rho_{r+i-k} \quad (i, k = 1, 2, \ldots, n)$$

The next step is to split the matrix X into $X = YY'$ where Y is a triangular matrix. Now Y^{-1} would give $Z = \{z_{ik}\}$ providing the z_{ik} coefficients for the linear forms

$$R_{h\cdot i} = z_{i1} r_{h\cdot 1} + z_{i2} r_{h\cdot 2} + \ldots + z_{ii} r_{h\cdot i}$$

The matrix Z could be determined for each scheme and the test conducted by calculating R_i and χ^2 values from sample to sample.

This procedure for testing has the advantage of enabling the test to be conducted step by step including every time additional serial coefficients, without requiring to have a predetermined number of coefficients to start with.

An alternate method for obtaining the χ^2 values would be through a process of sweep out of the dispersion matrix X along with an adjacent matrix formed of the $\overline{r_{h\cdot i}}$ values for any number of samples together (Rao, 1949). For illustration, a set of calculations are reproduced below. On account of symmetry elements of the dispersion matrix below the diagonal are omitted.

Dispersion matrix					$r_{h\cdot i}$ Values ($i = 1, 2, \ldots, 5$)				row-sum for check
					Sample 1	Sample 2	..	..	
.2550	.2384	.1947	.1332	.0712	.7480	−.0584	..	..	−.1106
	.2750	.2635	.2166	.1494	.7483	.1653	..	..	.1100
		.3056	.2946	.2441	.5858	−.1042	..	..	−.1328
			.3438	.3341	.2472	.5622	..	..	−.2442
				.3929	−.7021	−.5193	..	..	−.1330
			1st $\chi^2_{(1)}$ =		(.7450 × 2.9333)	(.0584 × .2290)	..	..	
1	.9349	.7635	.5224	.2792	2.9333	−.2290	..	..	−.4337
	.0521	.0815	.0920	.0828	.0490	.2198	..	..	.2134
		.1569	.1929	.1807	.0147	−.0398	..	..	−.0484
			.2742	.2969	−.1435	.5927	..	..	−.1864
				.3730	−.0110	−.5030	..	..	−.0995
			2nd $\chi^2_{(1)}$ =		(.0490 × .9405)	(.2198 × 4.2188)	..	..	
	1	1.5643	1.7658	1.5893	.9405	4.2188	..	..	4.0960
		.0294	.0490	.0602	−.0620	−.4034	..	..	−.3822
			.1132	.1505	−.2301	.2041	..	..	−.5636
				.2414	−.9889	−.8523	..	..	−.4366
			3rd $\chi^2_{(1)}$ =		(.0620 × 2.1088)	(.4034 × 13.7211)	..	..	
		1	1.6667	2.0476	−2.1088	−13.7211	..	..	−13.0000
			.0319	.0506	−.1272	.8737	..	..	.0708
				.1181	−.8620	−.0263	..	..	.3440
			4th $\chi^2_{(1)}$ =		(.1272 × 3.9875)	(.8737 × 27.3887)	..	..	
			1	1.5862	−3.9875	27.3887	..	..	
				.0382	−.6610	−1.4067	..	..	.2321
			5th $\chi^2_{(1)}$ =		(.6610 × 17.3037)	(1.4067 × 36.8240)	..	..	
				1	−17.3037	−36.8240	..	.	6.0759

TABLE 1. DISTRIBUTION OF χ^2 (1d.f.)

expected frequency per cent	χ^2 upper class limit	scheme A size 35	scheme B size 35	scheme C size 35	scheme A size 15	scheme B size 15	scheme C size 15
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	.000157	3	6	5	4	2	5
1	.000628	0	0	0	0	0	0
3	.00393	7	7	4	7	12	6
5	.0158	7	15	16	12	13	12
10	.0642	30	13	10	27	24	10
10	.148	34	25	25	33	30	23
20	.455	45	48	46	63	66	49
20	1.074	65	52	28	74	73	35
10	1.642	22	24	26	20	42	22
10	2.706	24	29	22	34	35	21
5	3.841	11	12	16	22	15	16
3	5.412	5	8	8	12	14	10
1	6.635	2	7	6	4	3	3
1	—	5	4	14	29	12	38
100		250	250	225	350	350	250

TABLE 2. DISTRIBUTION OF TOTAL χ^2

expected frequency percent	(10 d.f.) upper class limit	scheme A size 35	scheme B size 35	(7 d.f.) upper class limit	scheme A size 15	scheme B size 15	(9 d.f.) upper class limit	scheme C size 35	(5 d.f.) upper class limit	scheme C size 15
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	2.558	1	1	1.239	2	0	2.088	0	.554	0
1	3.059	0	0	1.564	0	1	2.532	0	.752	0
3	3.940	1	1	2.167	2	0	3.325	1	1.145	0
5	4.865	4	2	2.833	2	3	4.168	0	1.610	1
10	6.179	1	2	3.822	2	5	5.380	3	2.343	2
10	7.267	4	2	4.671	3	4	6.393	2	3.000	0
20	9.342	5	5	6.346	6	10	8.343	3	4.351	9
20	11.781	4	5	8.383	6	9	10.656	3	6.064	4
10	13.442	1	2	9.803	6	5	12.242	0	7.289	2
10	15.987	2	2	12.017	6	5	14.684	1	9.236	8
5	18.307	0	0	14.067	5	3	16.919	4	11.070	6
3	21.161	0	2	16.622	2	1	19.679	2	13.388	2
1	23.209	0	1	18.475	1	0	21.666	0	15.086	4
1	—	2	0	—	7	4	—	6	—	12
100		25	25		50	50		25		50

TABLE 3. DISTRIBUTION ACCORDING TO LAG NUMBER OF SIGNIFICANT χ^2 VALUES (1d.f.).

	5 per cent						1 per cent					
	scheme A size 35	scheme B size 35	scheme C size 35	scheme A size 15	scheme B size 15	scheme C size 15	scheme A size 35	scheme B size 35	scheme C size 35	scheme A size 15	scheme B size 15	scheme C size 15
expected frequency per cell:	(1.25)	(1.25)	(1.25)	(2.5)	(2.5)	(2.5)	(.25)	(.25)	(.25)	(.5)	(.5)	(.5)
lag no.												
1	..	..	..	..	..	..	..	..	..	..	..	..
2	0	3	..	0	0	..	0	0	..	0	0	..
3	0	2	..	1	3	..	0	1	..	0	0	..
4	3	1	0	6	2	0	1	1	0	2	2	0
5	2	0	1	8	4	7	1	0	0	6	1	5
6	4	4	2	10	4	13	1	0	1	7	2	8
7	0	1	3	10	7	15	0	0	2	7	2	10
8	1	3	2	10	9	16	1	1	1	7	5	14
9	1	2	5	..	..	..	1	0	3	..	..	..
10	1	2	4	..	..	..	0	1	2	..	..	..
11	0	1	6	..	..	..	0	0	2	..	..	..
12	..	..	5	..	..	..	..	..	2	..	..	..
total frequency	12	19	28	45	29	51	5	4	13	29	12	37
total no. of values	250	250	225	350	350	250	250	250	225	350	350	250

In this investigation the former procedure has been adopted for schemes A and B and the latter for scheme C. The elements z_{it} for schemes A and B are given in Tables 7.A and 7.B at the end.

3.2. *Test results* : The results of the χ^2 test performed for the samples constructed are summarised in Tables 1 and 2. Of the individual χ^2 values with 1 degree of freedom (Table 1) about 5 to 12 percent for sample size 35 and 8 to 20 percent for size 15 are seen to be significant as against an expected 5 percent.

With regard to the sample overall χ^2 values (Table 2) while the percentages of significant values vary between 8 and 12 percent for schemes A and B for both the sample sizes, scheme C reveals 32 and 36 percent significant values respectively for sizes 35 and 15.

These rather high percentages are not very encouraging. But before final conclusions are made it will be worthwhile to make certain other observations regarding the test results. Table 3 which gives the distribution of significant $\chi^2_{(1)}$ values against lag numbers shows a tendency for the frequency to increase with lag number. This is against expectation. One possible explanation is that the variance of R_s might have been getting under-estimated and it may be that the variance $1/(T-s)$ requires further correction.

Another interesting feature about the test results was that significant values in most cases were found to occur together, giving an indication that the independence of the $\chi^2_{(1)}$ values has not been fully achieved.

Another investigation made in connection with the test is as to whether correction for bias is necessary or not in making the above tests. For this purpose tests with the average correlogram with and without correction for bias have been made and the results are given in Table 4.

TABLE 4. χ^2 VALUES OF TESTS FOR AVERAGE CORRELOGRAMS WITH AND WITHOUT BIAS

lag no.	corrected for bias		not corrected for bias	
	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$
(1)	(2)	(3)	(4)	(5)
		scheme A (size 35)		
1	—	—	—	—
2	.8800	.8800	.2310	.2310
3	.0008	.8808	.3520	.5839
4	1.8756	2.7564	5.8590*	6.4429
5	.3015	3.0577	.0139	6.4568
6	5.0682*	8.1259	10.0265**	16.4833**
7	1.6540	9.7709	3.7394	20.2227**
8	1.1198	10.8997	.2751	20.4978**
9	.8870	12.9074	2.3580	22.8558**
10	8.7527**	21.6601*	5.1246*	27.9804**
11	.0122	21.6723*	.1513	28.1317**
		scheme A (size 15)		
1	—	—	—	—
2	.8197	.8197	17.3252**	17.3252**
3	1.4198	2.2395	.7472	18.0724**
4	2.7420	4.9815	25.2249**	43.2973**
5	3.2142	8.1957	.0789	43.3762**
6	19.4037**	27.5994**	36.1428**	79.5190**
7	6.8757**	34.4751**	6.2726**	85.7916**
8	2.6698	37.1449**	3.7778	89.5694**
		scheme B (size 35)		
1	—	—	—	—
2	.0000	.0000	1.8497	1.8497
3	1.0133	1.0133	.1238	1.9735
4	1.8065	2.8198	5.3390*	7.3125
5	.0078	2.8276	.4065	7.7190
6	1.0119	3.8395	.3065	8.0255
7	.0003	3.8398	.0753	8.1008
8	3.1553	6.9951	6.4104**	14.5112
9	.4118	7.4069	.0058	14.5170
10	1.3549	8.7618	3.2113	17.7283
11	.0002	8.7620	.0385	17.7668
		scheme B (size 15)		
1	—	—	—	—
2	.8843	.8843	14.7443**	14.7443**
3	.4651	1.3494	2.2118	16.9561**
4	14.9714**	16.3208**	47.0801**	64.0362**
5	.9278	17.2486**	.6934	65.0296**
6	6.9235**	24.1721**	1.4070	67.0366**
7	.9182	25.0903**	.6340	67.6706**
8	5.6687*	30.7590**	7.9647**	75.6353**
		scheme C (size 35)		
1	—	—	—	—
2	—	—	—	—
3	—	—	—	—
4	3.2025	3.2025	.1150	.1150
5	.0325	3.3250	.0275	.1425
6	5.9550*	9.2800	12.0625**	12.2050
7	.0825	9.3625	1.4650	13.6700
8	.3225	9.6850	.0325	13.7025
9	1.5050	11.1900	.7900	14.4925
10	7.5775**	18.7675	12.0300**	26.5225
11	.2900	19.0575	1.3275	27.8500
12	6.1825*	25.2400**	4.2850	31.1350**

TABLE 4—*contd.*

(1)	(2)	(3)	(4)	(5)
		scheme C (size 15)		
1	—	—	—	—
2	—	—	—	—
3	—	—	—	—
4	2.0000	2.0000	9.9050**	9.9950**
5	9.8500**	12.4500**	12.0600**	22.0550**
6	63.2650**	75.7350**	61.0100**	83.0650**
7	.2508	75.5418**	5.2700*	88.3350**
8	16.5350**	92.0708**	18.7550**	107.0900**

From the above results, obviously correction for bias is an improvement. However, the effect of bias may not be so pronounced in tests with individual correlograms as it is in the above case of average correlograms.

4. Fitting of models

4.0. *Estimation of parameters* : The problem of estimation of parameters in time series schemes does not appear to have been satisfactorily solved yet. What lies nearest at hand is to determine the parameters, say h in number, by the condition that the h first coefficients should be the same in the theoretical as in the empirical correlogram. For autoregressive series this method was suggested by Yule in the classical paper (1927) where such series were first introduced. It was pointed out by Wold (1938) that the parameters given by this method may require correction, for two reasons, *viz.* (i) the parameters must fulfill a certain condition in order to define a genuine autoregressive series (ii) a correction may result in a better fit to the correlogram as a whole. Turning to moving averages, the same method was used by Wold (1938) in a study where such schemes were first treated systematically. Again it was shown by Wold that the parameters may need correction for the same reason as under (i) above. Further he pointed out that corresponding to an assigned correlogram $\rho_1, \rho_2, \ldots, \rho_h$ there may exist several, at most 2^h solutions for the parameters in the moving average scheme.

Even in fitting a moving average scheme to a given series by means of the correlogram, instead of estimating $\rho_1, \rho_2 \ldots, \rho_h$ by $r_1, r_2, \ldots, r_h$, it will be possible to get improved estimates of the theoretical correlogram. For instance the maximum likelihood estimates of $\rho_1, \rho_2, \ldots, \rho_h$, obtained on the assumption that the r's follow a multivariate normal distribution in which $r_1, r_2, \ldots, r_h$ have expectations $\rho_1, \rho_2, \ldots \rho_h$ and r_{h+i} $(i = 1, 2, \ldots)$, have expectations $=0$, are more accurate than the estimates r_i for ρ_i. A discussion of this and other methods of fitting is given in Part 3.

In the present investigation, fitting of both moving average and autoregresive models to some of the constructed samples have been attempted. To six of the samples, (size 35) in scheme C moving average models of three constants each have been fitted.

The χ^2 test for fit was made by forming the dispersion matrix in each case and then by the process of sweep out described in section 3. The fit is found to be excellent in each case, as revealed by the resulting χ^2 values reproduced in Table 5.

TABLE 5. χ^2 VALUES IN FITTING A THREE CONSTANTS MOVING AVERAGE SCHEME TO SCHEME C (6 samples only)

	sample no. 2		sample no. 3		sample no. 4		sample no. 10		sample no. 17		sample no. 18	
lag no	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$	$\chi^2_{(1)}$	$\Sigma\chi^2_{(1)}$
4	.3789	.3789	1.5183	1.5183	.2314	.2314	.8725	.8725	.0140	.0140	.5089	.5089
5	.6058	.9845	.2760	1.7940	.2327	.4641	.2711	1.1436	.2024	.2164	.1165	.6254
6	1.2370	2.2215	.0135	1.8084	.9414	1.4055	.7679	1.9115	.0811	.2975	.0006	.6260
7	.5443	2.7658	.2057	2.0141	.0003	1.4058	.2357	2.1472	1.6650	1.9625	.2610	.8870
8	.8804	3.6462	.1055	2.1196	.0291	1.4349	.0965	2.2437	.8583	2.8208	.0451	.9321
9	.1448	3.7910	1.7321	3.8517	.0408	1.4757	.0163	2.2600	1.0901	3.9109	.1203	1.0524
10	.1007	3.8917	.1305	3.9822	1.0435	2.5192	.6406	2.9006	.1893	4.1002	2.1019	3.1543
11	.2001	4.0918	.3282	4.3104	.4521	2.9713	.0631	2.9637	.0590	4.1592	2.6111	5.7654
12	.2218	4.3136	.6014	4.9118	.0017	2.9730	.3051	3.2688	3.9213*	8.0805	.8895	6.6549

4.1. *Fitting autoregressive schemes to moving averages* : To the samples, size 35, constructed for scheme C, also have been fitted autoregressive schemes with two constants of the type

$$\eta_{(t)} = \xi_{(t)} + a\xi_{(t-1)} + b\xi_{(t-2)}$$

Different methods of estimating the parameters a and b in such a scheme and the appropriateness in small samples of the test available for testing fit, namely Quenouille's (1947) have been investigated in detail in Part 2 of this paper.

The method used for estimating a and b in the present case of 25 samples of scheme C is by least squares from equations provided by the recurrence relationship.

$$r_s + ar_{s-1} + br_{s-2} = 0$$

and making use of all the calculated sample serial co-efficients *i.e.* up to r_{12}.

The χ^2 values resulting from Quenouille's test for these 25 samples are distributed as in Table 6.

TABLE 6. DISTRIBUTION OF χ^2 IN FITTING AN AUTOREGRESSIVE SCHEME WITH 2 CONSTANTS TO SCHEME C (size 35)

expected frequency per cent	observed frequency	
	χ^2 (1.d.f)	total χ^2 (10.d.f.)
1	2	..
1	3	..
3	1	..
5	3	..
10	8	..
10	8	..
20	29	..
20	22	..
10	15	..
10	16	2
5	10	1
3	18	2
1	13	1
1	102	10
100	250	25

From Table 6, about 48% of the individual χ^2 values and about 80% of the total χ^2 values are seen to be significant at 1% level. The fit therefore, of a two-constants autoregressive scheme to scheme C (a three-constants moving average) has not at all been good.

Apparently contradicting this finding, a moving average scheme with three constants fitted to samples belonging to an autoregressive scheme with two constants was found to give very good fit in the investigations in Part 2. One has in the first instance to doubt whether this is due to difference in power between Wold's and Quenouille's tests, Wold's test having a low power of discrimination between autoregressive and moving average schemes. Perhaps an alternate explanation would be that it will be always possible to find a correlogram of the moving average type to align with any given correlogram of the autoregressive type, the agreement becoming closer, the larger the number of constants in the moving average, whereas for a given correlogram of a moving average it may not be always possible to find an aligning correlogram of the autoregressive type.

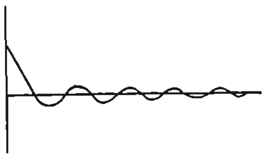

Fig 1 (a)

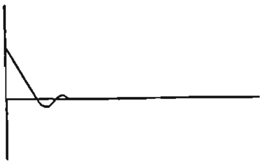

Fig 1(m_1)

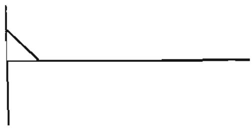

Fig 1(m_2)

Or in other words a correlogram of the autoregressive type which always would have form as in Fig. 1(a) can adequately be replaced by one of a moving average having the form in Fig. 1(m_1) and vice versa, while a moving average of the type in Fig. 1(m_2) may not be replaced by that of the autoregressive type having form in 1(a). Our scheme C has form 1(m_2).

5. SUMMARY OF CONCLUSIONS

1. Correction for bias in serial correlations of moving averages of short length is desirable. In making tests on correlogram the effect of bias may not be as much with individual correlograms as with average correlograms.

2. The large sample test for moving averages when applied to small samples reveal a considerable reduction in power. The test, therefore cannot be expected to give very exact results when applied to small samples, but it can nevertheless be used for a rough survey of the situation.

3. Possibilities of improvement of the test for application to small samples seem to be in the direction of achieving better orthogonalisation and better estimation of variances of the serial coefficients.

4. The method of fitting moving averages by means of sample correlograms is found to be satisfactory.

5. In the matter of fitting correlograms, (i) Wold's test, as it is to be expected, seems to reveal lesser power in discrimination between autoregressive and moving average types; (ii) also this discrimination appears to depend on whether the correlogram of the moving average is of the non-oscillatory type or not.

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TABLE 7.—z_{ik} VALUES

i	$10^4 z_{i1}$	$10^4 z_{i2}$	$10^4 z_{i3}$	$10^4 z_{i4}$	$10^4 z_{i5}$	$10^4 z_{i6}$	$10^4 z_{i7}$	$10^4 z_{i8}$	$10^4 z_{i9}$	$10^4 z_{i10}$	
					Table 7.A:—$\rho_1 = 0.5$; $\alpha_1 = 1.0$						
1	8165										1
2	—7303	10954									2
3	6325	—12649	12649								3
4	—5521	12421	—16562	13801							4
5	4880	—11711	17566	—19518	14639						5
6	—4364	10911	—17458	21822	—21822	15275					6
7	3944	—10142	16903	—22538	23555	—23064	15776				7
8	—3596	9439	—16181	22474	—26068	28317	—25170	16181			8
9	3303	—8808	15414	—22020	27524	—30827	30827	—26423	16514		9
10	—3054	8245	—14657	21375	—27482	32002	—34200	32978	—27482	16704	10
					Table 7.B:—$\rho_1 = 0.4$; $\alpha_1 = 0.5$						
1	8704										1
2	—6032	10043									2
3	4020	—9099	11878								3
4	—3054	7341	—11512	12204							4
5	1931	—4904	8588	—12135	12415						5
6	—1177	3091	—5741	9003	—12372	12471					6
7	697	—1870	3603	—6070	9278	—12437	12490				7
8	—403	1090	—2171	3801	—6180	9342	—12485	12496			8
9	229	—631	1270	—2283	3870	—6227	9361	—12491	12496		9
10	—128	357	—727	1333	—2322	3891	—6238	9364	—12492	12498	10

TABLES 8.A.1 to 8.C.2:—SERIAL CORRELATION COEFFICIENTS

TABLE 8.A.1—Scheme A ($\rho_1 = 0.5$, $\rho_2 \ldots = 0$), SAMPLE SIZE: 35

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1	.4574	.0360	−.0308	−.1882	−.1246	−.0483	.1360	.2054	−.0265	.1542	.4165	.2833
2	.4248	−.0202	.0021	.0060	.2017	.1042	.0118	−.1226	−.0524	.2291	.1542	−.3233
3	.2869	−.0749	−.0452	−.3770	−.3369	.0776	.0791	.2273	.0538	−.3082	−.2134	.0720
4	.3272	−.3150	−.2906	−.3071	−.0578	.3857	.3206	−.1515	−.3149	−.2031	−.0604	.2088
5	.3470	−.3252	−.1087	−.0550	−.0584	−.0105	−.0402	.0207	.1592	.1800	.0782	−.3401
6	.3821	−.3015	−.3089	−.0176	.0404	.0406	.1312	.0043	−.1038	.0311	.1806	.0564
7	.6434	.1079	.1042	.2280	.1340	.0507	−.0740	−.1377	−.1536	−.3256	−.3627	−.2984
8	.5756	.1074	.1773	.1100	.1631	−.1194	−.3265	−.2389	−.1555	−.0818	−.2092	−.1786
9	.4538	−.0405	.1788	−.0048	−.2595	−.1713	−.1564	−.2073	−.2056	−.1417	−.0570	−.1953
10	.6114	.0900	−.1092	−.2901	−.4272	−.5868	−.4807	−.0176	.2427	.3121	.4696	.5518
11	.4917	−.1210	−.2553	−.2809	−.3076	−.4585	−.1339	.2401	.2739	.2478	.2489	.1719
12	.5170	−.0059	−.0721	−.1557	−.1819	−.2073	−.2552	−.2016	−.0890	.0261	.0070	−.0203
13	.5483	−.0451	−.3062	−.3945	−.2250	−.0565	−.1032	−.1072	−.1305	−.0066	−.1321	−.0804
14	.4094	−.0727	.0077	.3719	.4024	.0870	−.1207	−.1660	.0050	.3430	.2242	−.2536
15	.4035	.1575	.2108	−.0266	−.1393	−.1431	−.0206	.0883	−.0847	−.2246	−.1705	−.1821
16	.5591	.1702	.1840	−.0443	−.3386	−.6035	−.4744	−.4770	−.3266	.0015	.3336	.2888
17	.5367	.1325	.1619	.1463	.1884	.0644	−.2147	−.2138	−.1715	−.1860	−.1939	−.2146
18	.5627	.0515	−.0905	−.1940	−.1995	−.3351	−.4189	−.1871	.0458	.2075	.3738	.4558
19	.4337	−.2007	−.2080	−.1133	−.0035	.0477	−.0699	−.1005	−.0044	−.0656	.1406	.1235
20	.3245	−.3412	−.1275	−.1003	−.2535	−.1508	.0138	.2118	.3299	.1120	−.1112	−.1084
21	.5756	.1282	.0250	−.0741	−.0217	−.0319	−.2185	−.2708	−.1631	−.0656	−.1389	−.1907
22	.2138	−.0888	.0182	−.3580	−.0074	−.0319	−.3307	−.1449	−.1501	−.0199	.1251	.1740
23	.5242	.0699	−.0262	−.1099	−.0423	−.0330	−.2691	−.2576	−.5039	.0391	.2760	.4953
24	.5444	−.0108	−.1220	−.1122	−.1547	−.1020	−.2338	−.0831	.0432	.0666	.2884	.2706
25	.0840	.3178	.2151	.0573	−.2703	−.0657	−.7003	−.5591	−.3901	−.3867	−.2700	.0573
average	.4815	−.0205	−.0329	−.0914	−.0914	−.1120	−.1041	−.1148	−.0651	−.0047	.0559	.0351
bias	−.0298	−.0605	−.0604	−.0600	−.0595	−.0587	−.0576	−.0562	−.0544	−.0521	−.0491	−.0455

Table 8.A.2—Scheme A ($\rho_1=0.5$, $\rho_2 \ldots\ldots=0$), sample size: 15

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1	.2739	−.3243	−.1828	−.2170	−.3575	−.2483	.3641	.4367
2	.1910	−.5121	−.4978	−.1252	.4318	.2802	.1787	−.1290
3	.2811	−.2413	−.0802	−.1947	−.0525	−.1752	−.6812	−.2531
4	.2528	−.1741	−.0113	−.1013	−.2710	−.5231	−.2043	.0827
5	.3110	−.0130	.4832	.2227	−.1568	−.0400	−.3892	−.1752
6	.3028	−.5553	−.0891	.3352	−.5287	−.7191	.1087	.2000
7	.5832	−.0028	−.1354	−.3701	−.5478	−.4580	−.3241	−.2007
8	.2701	−.2954	−.4212	−.2449	.4062	−.0634	−.7011	−.5813
9	.4917	−.1054	.1298	−.3763	−.4404	−.5100	−.8732	−.4093
10	.4215	−.0411	−.0048	−.2907	−.4282	−.7132	−.0687	.0051
11	.5689	.1298	−.2573	−.6046	−.5891	−.4794	.2402	.0806
12	.1527	−.6037	−.0015	.1512	.0840	.1874	−.1700	−.0524
13	.5088	−.2209	−.5180	−.4549	−.0271	.3449	.6036	−.2378
14	.4221	.0259	.4272	.4278	.4309	.3428	.2056	−.0647
15	.4007	.1393	.2362	−.4810	−.7036	−.4430	−.7453	−.4017
16	.3472	−.4093	−.4060	−.0229	.1303	−.1970	−.2958	−.2432
17	.5418	.0710	−.0727	−.2112	−.2406	−.3505	−.2269	−.4064
18	.4711	−.1078	−.2487	−.3311	−.1814	−.0937	−.3509	−.0138
19	.3810	−.0872	−.0729	−.1125	−.0873	−.3018	−.4162	.0320
20	.3304	−.5004	−.0017	.3473	−.1203	−.3761	.2233	.8399
21	.3751	−.3369	−.5209	−.0022	.6709	.0320	−.0027	−.5559
22	.0948	.0959	.2867	−.4780	.3100	.1457	−.7680	.2517
23	.4591	−.2111	−.3888	−.4290	.0880	.5340	−.2918	−.7603
24	.6578	.2485	.0468	−.2400	−.3678	−.5022	−.7497	−.8598
25	.5349	.1181	.2805	.2019	−.1295	−.6510	−.8171	.0889
26	.4158	.0640	−.1538	−.3842	−.1007	−.0725	−.1512	.0539
27	.4104	.0650	−.0228	−.5457	−.3829	−.0989	−.2041	.2674
28	.2734	−.2499	.0160	−.4902	−.5187	.3013	.6960	.1014
29	.3039	−.3223	−.2006	−.2988	−.1473	.5793	.3065	−.4775
30	.3147	−.5544	−.6013	−.0184	.1099	.0331	.2760	.5313
31	.2671	−.5042	.4585	.0870	.4032	−.4403	−.7055	.1245
32	.5522	.1545	.2077	.1156	.1291	.5714	.3512	.1335
33	.6566	.4798	.2962	.1866	.4032	.5125	.5052	−.6738
34	.2625	−.4665	−.0320	−.2117	−.8447	−.3087	.3211	.1777
35	.5496	−.0081	−.2300	−.4780	−.5101	−.6260	−.5712	.1645
36	.4057	−.3032	−.4601	−.0327	.0409	−.1205	.1595	.7100
37	.7838	.4247	−.0273	−.4069	−.6438	−.7736	−.7742	−.7060
38	.6585	.0372	−.2009	−.5667	−.4495	−.1728	−.1372	.3767
39	.4067	−.2524	−.1590	.4085	.0513	−.2360	−.7802	−.2797
40	.1270	−.2175	−.2231	−.1457	−.0317	−.1782	.3878	−.1351
41	.0508	.3620	.3885	.0537	−.4711	−.6493	−.3828	−.4910
42	.5338	.0256	.1502	.1550	.3734	.4879	−.1801	−.4038
43	.3432	.0274	−.2224	−.5516	−.2040	−.5880	−.3184	.7030
44	.4513	−.3082	−.4299	−.1722	.0167	−.0031	−.2802	−.8255
45	.2527	−.2076	.0312	.4822	−.7056	−.1250	.1503	−.0595
46	.3815	−.2104	−.1765	−.4541	−.4404	−.3059	−.3030	.0087
47	.2178	−.4880	−.2505	−.3307	.0060	.6198	.0827	−.2998
48	.3181	−.1344	−.0371	.2360	.4977	.1522	−.4287	−.2041
49	.4480	−.2183	−.2673	−.1027	−.1876	−.2809	−.1598	−.1144
50	.7451	.3939	.2026	.0310	−.4370	−.9150	−.8108	−.3262
average	.4030	−.1235	−.1034	−.1415	−.1217	−.1413	−.1020	−.1010
bias	−.0740	−.1528	−.1488	−.1100	−.1235	−.0937	−.0408	.0000

TABLE 8.B.1—Scheme B ($\rho_1 = 0.4$, $\rho_2, \ldots = 0$), SAMPLE SIZE: 35

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1	.4126	.0998	−.1452	−.4015	−.3107	−.1028	.1177	.1589	.1870	.0181	−.1040	−.2021
2	.2818	−.2340	−.0702	.1006	.1571	.0550	−.0102	−.2539	−.1293	.1204	−.0728	−.1784
3	−.0340	−.4550	.2071	−.1266	−.3039	.1630	.0101	−.0186	.4872	−.0095	−.4348	.2801
4	.2046	−.0862	−.1005	−.3466	−.2183	−.0588	−.0274	.3232	.1004	−.1664	−.0166	−.3118
5	.0701	−.2137	−.0512	.1065	.0584	−.0401	.2170	.0313	−.1756	−.2590	.0818	−.2318
6	.5340	.2100	−.2069	−.2041	−.2456	−.1037	.1121	.1343	.1395	−.1409	−.1386	−.1406
7	.3186	−.2118	−.1254	−.2103	−.2042	.2164	.4022	.2939	.0489	−.4029	−.5366	−.4303
8	.2829	.0680	−.0071	−.0615	−.2160	−.1125	−.0126	−.4173	−.2001	−.0948	.0083	−.1623
9	.3445	−.2100	−.2076	−.2480	−.0138	−.0114	−.1324	−.1704	.0962	.1743	−.0079	−.3189
10	.3234	−.0558	−.1418	−.1027	.0033	.1738	.1199	−.0755	.0421	.0604	.1821	.0850
11	.4029	.2707	.1445	.0411	.1108	−.1260	−.0346	−.2731	−.3448	−.3711	−.5148	−.3093
12	.1041	−.1297	.2279	−.0041	−.2987	.0355	−.0611	−.3020	−.0708	−.1110	−.4081	−.1836
13	.4707	.4114	.3992	.2407	.2657	−.0041	−.1024	−.1232	−.3173	−.2695	−.2354	−.3788
14	.4086	−.0521	−.1000	−.1416	−.1065	−.1442	.0307	−.0511	−.2409	−.2137	−.0100	−.2461
15	.3763	.1588	.2087	.2005	.3818	.3279	.1287	−.0032	.0506	.1599	.0465	−.1718
16	.4105	.0700	.1814	.0504	−.0033	.1370	−.2491	−.3249	−.2841	−.1535	.3128	.2445
17	.3421	−.1375	−.0405	−.2352	−.3005	.0508	.0769	−.1926	−.2082	−.1435	−.1430	.0293
18	.4441	−.1017	−.0180	.0898	−.1305	−.1081	.0821	.1007	.0110	.0463	−.0011	−.0309
19	.5260	.0805	−.0785	.1779	.1251	.0558	.1872	.2323	−.0345	−.1712	−.1404	.0823
20	.2078	−.0714	.2039	.2440	−.1967	−.3104	−.1016	−.2061	−.4778	−.1819	.1303	.0936
21	.2470	−.4648	−.3404	.0929	.1089	−.0570	.0210	−.1027	−.1803	.0230	.3439	.0211
22	.4250	−.1104	−.1252	−.2323	−.2824	−.2572	−.2212	.0035	.3094	.5005	.3838	.2357
23	.4469	.0042	−.1116	−.3411	−.3022	−.1230	−.1282	−.0054	−.1379	.0041	.2230	.1564
24	.3567	.1262	.2170	−.0615	.0598	−.2102	−.5122	−.3412	−.6000	−.2081	−.1543	−.1051
25	.2752	−.2074	−.2481	−.2028	−.0087	.2462	.0711	−.2050	−.2263	−.0580	.1437	.3279
average	.3371	−.0544	−.0216	−.0669	−.0824	−.0167	.0056	−.0808	−.0798	−.0773	−.0511	−.0793
bias	−.0322	−.0542	−.0540	−.0530	−.0531	−.0524	−.0515	−.0502	−.0480	−.0465	−.0437	−.0405

TABLE 8.B.2—SCHEME B ($\rho_1=0.4$, $\rho_2, \ldots = 0$), SAMPLE SIZE: 15

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1	.5709	.1421	.0219	−.3603	−.0031	−.4751	−.3113	−.4004
2	.3061	−.4447	−.4883	.1401	.5395	−.1015	−.5471	−.5177
3	−.1152	−.2418	.4076	−.4401	−.1517	.2995	−.5673	.2062
4	.0860	−.1059	−.2290	−.2306	−.4109	−.1436	.2284	.8414
5	−.4511	−.1729	−.0593	.3968	−.0582	−.3123	.2905	−.2112
6	.3548	.1410	−.2714	−.0030	−.0230	−.5127	.1160	.4452
7	.3938	−.2354	−.3662	−.5600	−.1418	.0078	.4300	.1361
8	.3235	.0685	−.0309	−.4251	−.6052	−.4867	−.0041	−.4207
9	.5239	−.3203	−.5984	−.3887	.2344	.1850	−.5423	−.7252
10	.5270	−.1930	−.6073	−.6108	−.2197	.2404	.4086	.0717
11	.5129	.2060	−.0463	−.1497	.4867	.3012	.1602	−.3860
12	−.1087	−.5590	.2731	−.0271	−.2147	.1964	−.3310	−.2551
13	.4083	.2702	.3279	.1841	.1051	−.1370	−.0540	.1022
14	.3990	.0600	.3188	−.1188	−.3408	−.3177	−.3070	−.3465
15	.3411	−.5701	−.6749	−.0470	.5098	.4834	−.4182	−.8860
16	.4495	.0310	.0913	.1629	−.2437	−.3396	−.2678	−.0713
17	.5832	.2475	.0833	−.2084	−.2893	−.4079	−.4540	−.4137
18	.1318	−.4120	−.0201	.1621	−.1712	−.1464	−.4314	−.3873
19	.4771	.3322	.3883	.4558	.0110	−.2432	−.1997	.0786
20	.3901	.1488	.4302	.2477	−.1584	−.1533	.1517	−.4032
21	.2844	−.0418	−.5831	.0546	.4738	.3036	−.1059	−.4713
22	.3384	−.1040	−.0342	−.2898	−.4008	−.2015	−.4117	−.2558
23	.2700	−.3193	−.1674	−.3206	−.1375	.4701	.3137	−.3519
24	.2964	.0870	−.0030	−.2770	−.1959	−.5070	−.2904	.0340
25	.1012	.0717	−.0707	−.0707	.2425	−.4385	−.0635	−.6526
26	.0836	.2231	−.4313	−.0039	−.1190	−.1608	.0648	−.1704
27	.0115	−.2301	.3297	−.1444	−.5032	.0001	−.0169	−.0022
28	.0060	−.4273	−.0223	−.2350	−.4298	.2097	.1914	.2505
29	.3198	−.1537	−.2137	−.4396	−.1019	−.0474	−.3798	−.0065
30	.2428	−.5070	−.2542	.0207	−.1714	−.1077	.5014	.7905
31	.6801	.2869	−.1541	−.3608	−.6334	−.5388	−.6010	−.5829
32	.1816	−.4208	.1056	.1437	−.3354	−.1535	.0602	.6233
33	.3212	−.1966	−.3209	−.2495	−.1737	−.1234	−.1114	−.7385
34	.3894	−.2327	−.1627	−.2892	−.4925	−.0240	.5858	.1560
35	−.0601	−.2233	−.0680	−.3294	−.2622	.0897	.3483	.4600
36	.0863	−.2130	.0117	.0990	−.0504	−.2521	.4494	−.2017
37	.4451	.1258	.3406	.1115	−.0902	−.1836	−.0222	.1595
38	−.2031	.2748	.1518	−.1018	.3802	.0316	−.0056	.5891
39	.3943	−.1164	−.3709	−.3874	−.2329	.0216	.1612	−.0117
40	−.5463	.2821	.0181	−.3428	.1500	.0569	−.3633	.2503
41	.3705	.0808	.2527	−.0303	.0112	.1532	−.3215	−.4001
42	.1147	−.4156	−.1830	−.2562	−.4248	.5005	.5838	−.0789
43	.4870	−.2153	−.4920	−.5133	−.6542	−.0913	.8001	.6074
44	.4792	−.3041	−.6501	−.3587	−.1524	.1587	.4691	.7300
45	−.1064	−.0526	.2211	.4224	−.2599	.5189	−.4308	.0009
46	.3465	−.4444	−.6345	−.1059	.1057	.4579	.1823	−.5206
47	.6108	.1170	−.3821	−.8160	−.7438	−.3034	.1079	.5694
48	.5841	.1341	−.1639	−.4137	−.5343	−.5131	−.3121	.0960
49	.3760	.0370	.0842	.1175	.4054	.1233	−.2983	−.4832
50	.1083	−.2988	.0075	−.0561	−.5397	.3521	.4781	.0797
average	.2635	−.1086	−.0098	−.1809	−.1631	−.0255	+.0036	−.0589
bias	−.0795	−.1352	−.1316	−.1235	−.1087	−.0823	−.0357	.0000

TABLE 8.C.1—SCHEME C ($\rho_1=0.75$, $\rho_2=0.50$, $\rho_3=0.25, \ldots \rho_s \ldots =0$), SAMPLE SIZE: 35

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1	.7787	.6667	.3548	.0535	−.0944	−.1035	−.2622	−.2075	−.1843	−.1433	−.0029	−.0827
2	.7709	.6331	.4763	.2000	.0788	−.1311	−.3133	−.5127	−.5295	−.5138	−.4877	−.3066
3	.7849	.6914	.5565	.4318	.4767	.4508	.4547	.3240	.4011	.3120	.2870	.4190
4	.6947	.3976	.1245	−.1314	−.1809	−.2744	−.2938	−.2743	−.1798	−.1815	−.2445	−.2845
5	.8067	.6424	.5119	.3778	.4140	.3520	.2702	.1351	−.0036	−.1310	−.1124	.0085
6	.7710	.5521	.2559	−.0309	−.0600	−.0972	−.0800	−.0798	−.1613	−.2550	−.2770	−.1550
7	.6250	.2046	−.2934	−.5802	−.5415	−.4397	−.3617	−.3144	−.0548	.0583	.5117	.7260
8	.5710	.3910	.2003	−.0831	−.0410	−.1077	−.2002	−.1660	−.2028	−.2103	−.0809	−.3497
9	.6745	.3054	.1070	−.1053	−.0420	−.1730	−.3693	−.4382	−.1538	−.1904	−.3431	−.4675
10	.5587	.0953	−.3234	−.4435	−.0816	−.0085	.0703	−.0169	−.0722	.0418	−.0033	.0855
11	.7013	.4225	.2438	−.1586	−.3747	−.4644	−.5789	−.0697	−.0058	−.3080	−.2053	.0909
12	.7014	.5541	.1920	−.1263	−.1075	−.2507	−.3718	−.4625	−.4332	−.3184	−.1035	.1830
13	.6890	.4013	.1423	−.2189	−.3589	−.4909	−.5706	−.5267	−.1865	.0502	.1325	.2751
14	.8555	.0868	.4671	.2369	.1207	−.0067	−.0776	−.1525	−.1023	−.1590	−.1700	−.1090
15	.7238	.0619	.5056	.2182	.2346	.0174	−.0781	−.2037	−.3171	−.3582	−.5015	−.3058
16	.6368	.3780	.1653	−.2322	−.1257	−.1094	−.1008	−.1232	−.1476	.0320	.0905	.2073
17	.7610	.4943	.2119	−.0351	.0247	.0015	.2367	.3934	.3009	.2694	.0602	−.2749
18	.8424	.0823	.4867	.2506	.1836	.1247	.0000	−.0400	−.1141	−.3233	−.5506	−.7587
19	.7879	.6921	.4782	.2707	.2415	.1176	.1780	−.0359	−.1414	−.3532	−.5887	−.4970
20	.5819	.4300	.3794	.0159	.0188	−.1186	−.3178	−.2629	−.1706	−.1536	−.0863	.0573
21	.7077	.4150	.2170	.0410	−.0516	−.2122	−.3287	−.5469	−.6725	−.6732	−.0306	−.3837
22	.3855	.2603	.2242	−.2336	.0087	.1132	.0222	.1948	−.0342	−.1061	−.3475	−.4323
23	.7078	.3631	−.1219	−.4102	−.2309	.2207	.2664	.3085	.2364	−.1106	−.4203	−.5742
24	.8007	.7093	.5350	.2322	.1679	−.0188	−.0393	.0418	.0646	.2920	.3320	.4880
25 [1]	.8003	.5405	.2848	−.0532	−.2166	−.3836	−.4090	−.5121	−.5438	−.5163	−.4536	−.4212
average	.7056	.4885	.2653	−.0202	−.0220	−.0842	−.1410	−.1711	−.1036	−.1043	−.1679	−.1237
bias	−.0310	−.0624	−.0046	−.1283	−.1274	−.1260	−.1241	−.1214	−.1170	−.1132	−.1072	−.0995

TABLE 8.C.2—SCHEME C (ρ_1=0.75, ρ_2=0.50, ρ_3=.25 ρ_4....=0) SAMPLE SIZE: 15

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1	.5613	.2132	−.1238	−.1400	.4820	.7590	.5520	.1197
2	.7937	.7114	.0269	.4070	.4403	.3458	.1302	−.7021
3	.1159	.1313	−.0871	−.3004	−.1427	−.3142	.1542	−.6193
4	.6268	.0723	−.2722	−.6805	−.6799	−.4107	−.0105	.3083
5	.7437	.6012	.3533	.1274	.0146	.0142	−.1308	.0212
6	.4724	.3141	−.0206	−.4397	−.3236	−.6271	−.1636	.2513
7	.4870	.1796	−.2504	−.3926	−.0532	−.0356	−.2741	−.8081
8	.4867	−.0417	−.3425	−.3598	−.3538	−.2814	−.2178	.4205
9	.7898	.6226	.2812	−.3105	−.4842	−.2272	−.4756	−.4190
10	.6170	.3604	−.0063	−.4475	−.4060	−.0307	−.7185	−.8158
11	.6548	.2209	.0092	−.1995	−.2788	.0202	.2283	−.0729
12	.5958	.0703	−.2520	−.4874	−.3986	−.3962	−.4895	−.7386
13	.2433	.0952	.4698	−.2041	.1063	.0339	−.6148	−.3150
14	.7184	.6450	.2527	−.0858	−.2560	−.8200	−.6800	−.7054
15	.6335	.4647	.1448	−.1006	−.3219	−.6716	−.4133	−.7020
16	.3215	.1868	.0957	−.6120	−.2452	−.2880	−.6776	−.6504
17	.0026	.0660	−.4098	−.7426	−.7091	−.4735	.0978	.6322
18	.8200	.6494	.3777	−.2015	−.3085	−.7175	−.8884	−.8396
19	.3359	.1370	−.2632	−.6854	−.0100	−.0591	.1498	−.0387
20	.6131	.3086	.0089	−.3472	−.3783	−.8192	−.8374	−.7077
21	.4411	−.0405	−.0000	.0804	−.0527	−.2518	.1816	.4090
22	.4758	.1805	.5859	−.8172	.2530	.2903	.6803	.6198
23	.8579	.7374	.6165	.0016	.5804	.4487	.4173	.6480
24	.4442	.4311	.6065	−.1293	.0830	−.6585	−.4070	.3038
25	.8345	.6920	.5928	.2538	.1146	−.2015	−.8399	−.3422
26	.6416	.6535	.3829	−.0279	−.4105	−.7522	−.9317	−.7449
27	.5256	.2527	.0384	−.3566	−.1693	−.4597	−.4720	−.6849
28	.6662	.4176	.0160	−.1975	.1510	.6022	.7680	.7148
29	.8636	.8347	.7316	.5970	.5710	.1207	−.0908	−.2054
30	.6792	.4650	.3474	.1718	.6085	.5142	.2878	.1300
31	.6201	.1589	−.3105	−.6854	−.6030	−.4432	−.1701	.5851
32	.7737	.3350	.2480	.5259	.5932	.7264	.0869	.7870
33	.6525	.4582	.3018	−.0725	.0538	−.1736	−.1046	.5191
34	.3406	.0267	.1594	.1205	.4396	−.2910	−.3954	.6157
35	.5394	−.0035	−.5345	−.5682	−.1090	−.1696	−.2558	−.8637
36	.8002	.6990	.0674	.5054	.2993	−.0864	−.1044	−.3246
37	.4428	.1144	−.1804	−.5194	.1331	.3027	.4753	.5438
38	.7044	.5006	.0211	−.3413	−.6091	−.8334	−.8878	−.5808
39	.7826	.4341	−.0387	−.5818	−.8098	−.8088	−.5810	−.4042
40	.6081	.7300	.0002	.2803	.5022	−.0078	.1958	.0872
41	.6138	.6220	.3010	−.0574	−.3243	−.3425	−.7353	−.2624
42	.6677	.3711	−.1000	−.6404	−.7591	−.7732	−.5335	−.1997
43	.6171	.3811	−.0005	−.5985	−.4853	−.2900	.1194	.5413
44	.5330	.3858	.0770	−.3352	−.3082	−.6027	−.5108	−.6821
45	.3832	.2770	.4818	−.2088	−.3257	.0865	−.7020	−.4891
46	.0315	.2890	−.1680	−.2901	−.1833	−.0706	−.2062	−.4860
47	−.0383	.0751	.1424	−.5496	.1049	−.3434	−.3232	.6480
48	.7242	.4376	−.0785	−.6618	−.2593	.1470	.7581	.8168
49	.7254	.4522	.1250	−.3091	−.4035	−.7326	−.4071	.6603
50	.0760	.3713	.1072	−.2024	−.2097	−.1210	.3320	.7248
average	.5040	.3494	.1215	−.2258	−.0090	−.1864	−.1701	−.0665
bias	−.0810	−.1639	−.2500	−.3415	−.3077	−.2400	−.1081	.0000

AVERAGE AND EXPECTED CORRELOGRAMS—Figs. 2. A. 1 to 2. C. 2

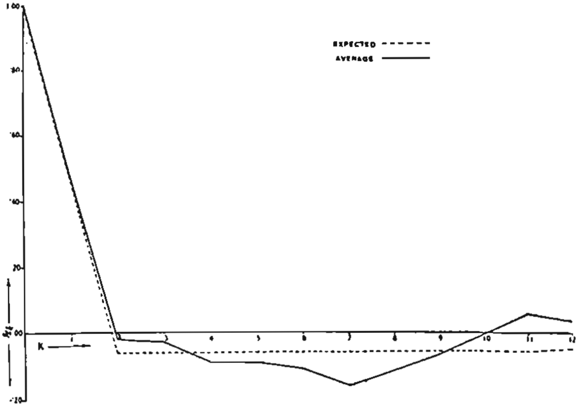

Fig. 2. A. 1: Scheme A, $T=35$

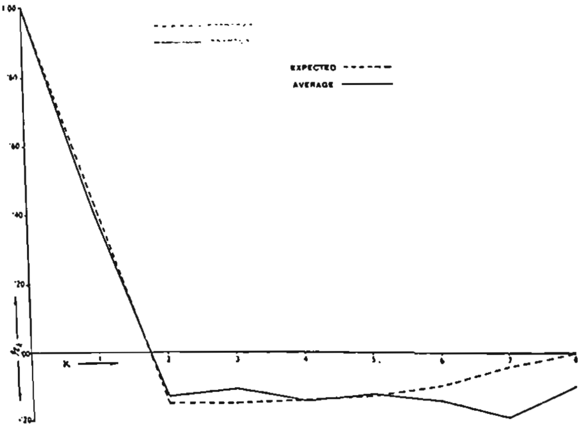

Fig. 2. A. 2: Scheme A, $T=15$

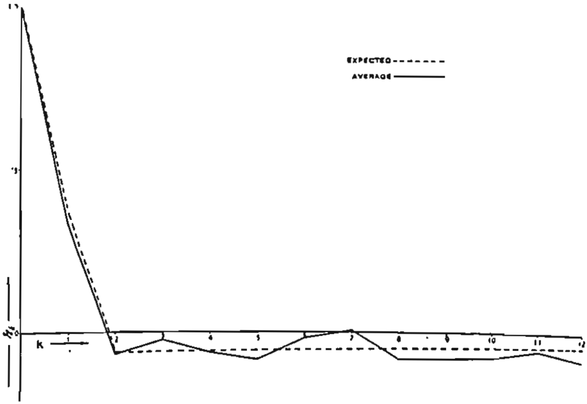

Fig. 2. B. 1: Scheme B, $T=38$

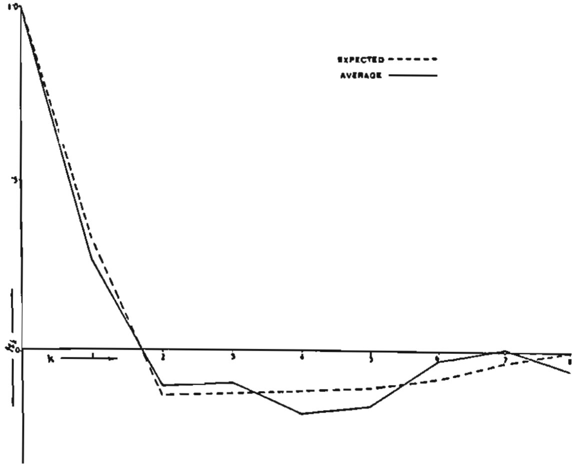

Fig. 2. B. 2: Scheme B, $T=13$

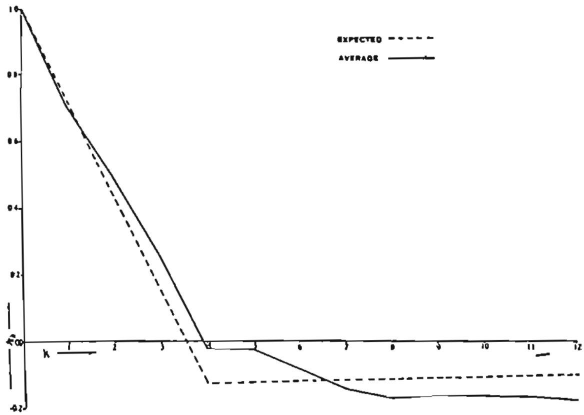

Fig. 2. C. 1: Scheme C, $T = 35$

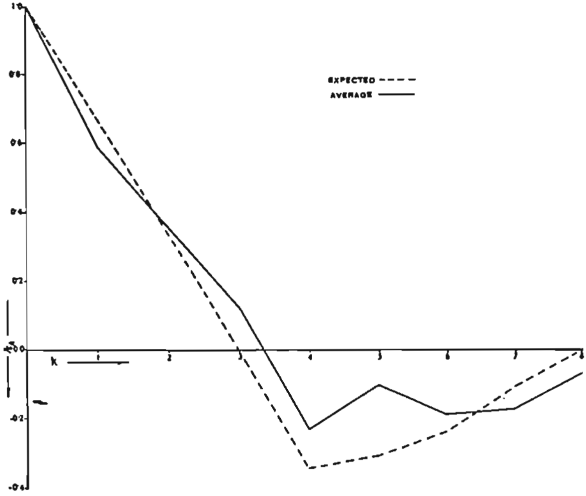

Fig. 2. C. 2. Scheme C, $T=15$